

Chapter 2

Intuitionistic Fuzzy Predicate Logic

2.1 Short Remarks on Intuitionistic Fuzzy Predicate Logic

The idea for evaluation of the propositions was extended for predicates (see [1–6]) as follows (see, e.g., [7–10]).

Let x be a variable, obtaining values in set E and let $P(x)$ be a predicate with a variable x . Let

$$V(P(x)) = \langle \mu(P(x)), \nu(P(x)) \rangle.$$

The IF-interpretations of the (intuitionistic fuzzy) quantifiers *for all* (\forall) and *there exists* (\exists) are introduced in [7, 9, 10] by

$$V(\exists x P(x)) = \langle \sup_{y \in E} \mu(P(y)), \inf_{y \in E} \nu(P(y)) \rangle, \tag{2.1.1}$$

$$V(\forall x P(x)) = \langle \inf_{y \in E} \mu(P(y)), \sup_{y \in E} \nu(P(y)) \rangle. \tag{2.1.2}$$

If E is a finite set, then we can use the denotations

$$V(\exists x P(x)) = \langle \max_{y \in E} \mu(P(y)), \min_{y \in E} \nu(P(y)) \rangle, \tag{2.1.3}$$

$$V(\forall x P(x)) = \langle \min_{y \in E} \mu(P(y)), \max_{y \in E} \nu(P(y)) \rangle. \tag{2.1.4}$$

In general, below, we use the first forms of both quantifiers.

Their geometrical interpretations are illustrated in Figs. 2.1 and 2.2, respectively, where x_1, \dots, x_5 are possible values of variable x and $V(x_1), \dots, V(x_5)$ are their IF-evaluations.

The most important property of the two quantifiers is that each of them juxtaposes to predicate P a point (exactly one per quantifier) in the IF-interpretational triangle.

Fig. 2.1 Second geometrical interpretation of quantifier \forall

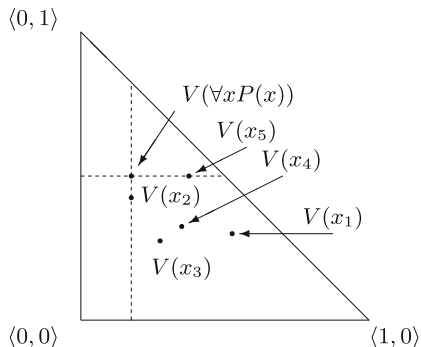
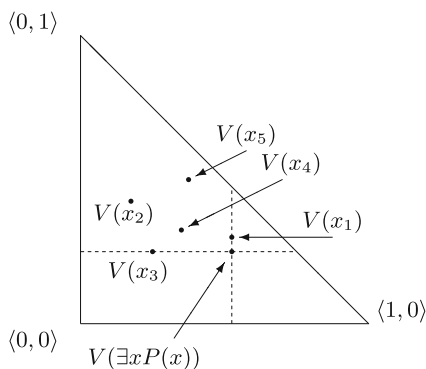


Fig. 2.2 Second geometrical interpretation of quantifier \exists



In [9, 10], for implication \rightarrow_4 the following two theorems are proved, where we used \rightarrow instead of \rightarrow_4 .

Theorem 2.1.1 *The logical axioms of the \mathcal{K} -theory (see [5]):*

- (a) $A \rightarrow (B \rightarrow A)$,
 - (b) $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$,
 - (c) $(\neg A \rightarrow \neg B) \rightarrow ((\neg A \rightarrow B) \rightarrow A)$,
 - (d) $\forall x A(x) \rightarrow A(t)$, for the fixed variable t ,
 - (e) $\forall x (A \rightarrow B) \rightarrow (A \rightarrow \forall x B)$
- are IFTs.

Proof Assertions (a)–(c) coincide with those in Theorem 1.5.17 (IL2), (IL5) and Theorem 1.5.26, respectively. We will here prove only assertions (d) and (e).

(d) Let the variable t be fixed.

Then,

$$\begin{aligned}
 & V(\forall x A(x) \rightarrow A(t)) \\
 &= \langle \inf_x \mu(A(x)), \sup_x \nu(A(x)) \rangle \rightarrow \langle \mu(A(t)), \nu(A(t)) \rangle
 \end{aligned}$$

$$= \langle \max(\sup_x \nu(A(x)), \mu(A(t))), \min(\inf_x \mu(A(x)), \nu(A(t))) \rangle$$

and

$$\begin{aligned} & \max(\sup_x \nu(A(x)), \mu(A(t))) - \min(\inf_x \mu(A(x)), \nu(A(t))) \\ & \geq \mu(A(t)) - \inf_x \mu(A(x)) \geq 0, \end{aligned}$$

i.e., (d) is an IFT.

For (e) we sequentially obtain:

$$\begin{aligned} V(\forall x(A \rightarrow B) \rightarrow (A \rightarrow \forall x B)) &= V(\forall x(A \rightarrow B)) \rightarrow V(A \rightarrow \forall x B) \\ &= \langle \inf_x \max(\mu(B), \nu(A)), \sup_x \min(\mu(A), \nu(B)) \rangle \\ &\rightarrow \langle \max(\nu(A), \inf_x \mu(B)), \min(\mu(A), \sup_x \nu(B)) \rangle \\ &= \langle \max(\nu(A), \inf_x \mu(B), \sup_x \min(\mu(A), \nu(B))), \\ & \quad \min(\mu(A), \sup_x \nu(B), \inf_x \max(\mu(B), \nu(A))) \rangle \end{aligned}$$

and

$$\begin{aligned} & \max(\nu(A), \inf_x \mu(B), \sup_x \min(\mu(A), \nu(B))) \geq \max(\nu(A), \inf_x \mu(B)) \\ & = \inf_x \max(\mu(B), \nu(A)) \geq \min(\mu(A), \sup_x \nu(B), \inf_x \max(\mu(B), \nu(A))), \end{aligned}$$

i.e., (e) also is an IFT. \square

Below, we list some assertions, which are theorems of the classical first order logic (see, e.g. [5]).

Theorem 2.1.2 *The following formulas are IFTs:*

- (a) $(\forall x A(x) \rightarrow B) \equiv \exists x(A(x) \rightarrow B)$,
- (b) $\exists x A(x) \rightarrow B \equiv \forall x(A(x) \rightarrow B)$,
- (c) $B \rightarrow \forall x A(x) \equiv \forall x(B \rightarrow A(x))$,
- (d) $B \rightarrow \exists x A(x) \equiv \exists x(B \rightarrow A(x))$,
- (e) $(\forall x A \wedge \forall x B) \equiv \forall x(A \wedge B)$,
- (f) $(\forall x A \vee \forall x B) \rightarrow \forall x(A \vee B)$,
- (g) $\neg \forall x A \equiv \exists x \neg A$,

- (h) $\neg\exists x A \equiv \forall x \neg A$,
 (i) $\forall x \forall y A \equiv \forall y \forall x A$,
 (j) $\exists x \exists y A \equiv \exists y \exists x A$,
 (k) $\exists x \forall y A \rightarrow \forall y \exists x A$,
 (l) $\forall x (A \rightarrow B) \rightarrow (\forall x A \rightarrow \forall x B)$.

Proof We shall use Lemma 1.5.1.

(a)

$$\begin{aligned}
 & V(\forall x A(x) \rightarrow B) \\
 &= \langle \max(\sup_x \nu(A(x)), \mu(B)), \min(\inf_x \mu(A(x)), \nu(B)) \rangle \\
 &= \langle \sup_x (\max(\nu(A(x)), \mu(B))), \inf_x (\min(\mu(A(x)), \nu(B))) \rangle \\
 &= V(\exists x (A(x) \rightarrow B));
 \end{aligned}$$

(b)

$$\begin{aligned}
 & V(\exists x A(x) \rightarrow B) \\
 &= \langle \max(\inf_x \nu(A(x)), \mu(B)), \min(\sup_x \mu(A(x)), \nu(B)) \rangle \\
 &= \langle \inf_x (\max(\nu(A(x)), \mu(B))), \sup_x (\min(\mu(A(x)), \nu(B))) \rangle \\
 &= V(\forall x (A(x) \rightarrow B));
 \end{aligned}$$

(c)

$$\begin{aligned}
 & V(B \rightarrow \forall x A(x)) \\
 &= \langle \max(\inf_x \mu(A(x)), \nu(B)), \min(\sup_x \nu(A(x)), \mu(B)) \rangle \\
 &= \langle \inf_x (\max(\mu(A(x)), \nu(B))), \sup_x (\min(\nu(A(x)), \mu(B))) \rangle \\
 &= V(\forall x (B \rightarrow A(x)));
 \end{aligned}$$

(d) is proved analogically;

(e)

$$\begin{aligned}
 & V(\forall x A \wedge \forall x B) \\
 &= \langle \min(\inf_x \mu(A), \min \mu(B)), \max(\sup_x \nu(A), \max \nu(B)) \rangle \\
 &= \langle \inf_x (\min(\mu(A), \mu(B))), \sup_x (\max(\nu(A), \nu(B))) \rangle
 \end{aligned}$$

$$= V(\forall x(A \wedge B));$$

(f) is proved analogically;

(g)

$$V(\neg\forall x A) = \langle \sup_x \nu(A), \inf_x \mu(A) \rangle = V(\exists x \neg A);$$

(h) is proved analogically;

(i)

$$\begin{aligned} & V(\forall x \forall y A) \\ &= \langle \inf_x \inf_y \mu(A), \sup_x \sup_y \nu(A) \rangle \\ &= \langle \inf_y \inf_x \mu(A), \sup_y \sup_x \nu(A) \rangle \\ &= V(\forall y \forall x A); \end{aligned}$$

(j) is proved analogically;

(k)

$$\begin{aligned} & V(\exists x \forall y A \rightarrow \forall y \exists x A) \\ &= \langle \sup_x \inf_y \mu(A), \inf_x \sup_y \nu(A) \rangle \rightarrow \langle \inf_y \sup_x \mu(A), \sup_y \inf_x \nu(A) \rangle \\ &= \langle \max_y(\inf_x \sup_y \nu(A), \inf_x \sup_y \mu(A)), \min_x(\sup_x \inf_y \mu(A), \sup_y \inf_x \nu(A)) \rangle \end{aligned}$$

and

$$\begin{aligned} & \max_y(\inf_x \sup_y \nu(A), \inf_x \sup_y \mu(A)) - \min_x(\sup_x \inf_y \mu(A), \sup_y \inf_x \nu(A)) \\ & \geq \sup_y \inf_x \nu(A) - \sup_y \inf_x \nu(A) = 0, \end{aligned}$$

i.e., $\exists x \forall y A \rightarrow \forall y \exists x A$ is an IFT;

(l) is proved analogically. □

Theorem 2.1.3 For a predicate P and for negation \neg_i , $\forall x P(x) \vee \exists x \neg_i P(x)$ is an IFT for $i = 1, 3, 4, 8, 9, 11, 12, 14, 15, 18, \dots, 23, 25, \dots, 32, 45, 52, 53$.

Proof Let for the variable x ,

$$V(\forall x P(x)) = \langle M, n \rangle,$$

$$V(\exists x P(x)) = \langle m, N \rangle,$$

where the pairs $\langle M, n \rangle$ and $\langle m, N \rangle$ are given by either (2.1.1) and (2.1.2), or (2.1.3) and (2.1.4). Below, we will discuss the proof for three of the negations: \neg_1 , \neg_{12} and \neg_{52} .

For \neg_1 we obtain:

$$\begin{aligned} \forall x P(x) \vee \exists x \neg_1 P(x) &= \langle m, N \rangle \vee V(\exists x \langle \nu(P(x)), \mu(P(x)) \rangle) \\ &= \langle m, N \rangle \vee \langle N, m \rangle = \langle \max(m, N), \min(m, N) \rangle, \end{aligned}$$

that is an IFT, because $\max(m, N) - \min(m, N) \geq 0$.

For \neg_{12} we obtain:

$$\begin{aligned} \forall x P(x) \vee \exists x \neg_{12} P(x) &= \langle m, N \rangle \vee \langle \sup_x (\nu(P(x))(\mu(P(x)) + \nu(P(x)))) \\ &\quad \inf_x (\mu(P(x))(\mu(P(x)) + \mu(P(x))\nu(P(x)) + \nu(P(x))^2)) \rangle \\ &= \langle \max(m, \sup_x (\nu(P(x))(\mu(P(x)) + \nu(P(x))))), \\ &\quad \min(N, \inf_x (\mu(P(x))(\mu(P(x)) + \mu(P(x))\nu(P(x)) + \nu(P(x))^2)) \rangle. \end{aligned}$$

Then,

$$\begin{aligned} &\max(m, \sup_x (\nu(P(x))(\mu(P(x)) + \nu(P(x)))) \\ &\quad - \min(N, \inf_x (\mu(P(x))(\mu(P(x)) + \mu(P(x))\nu(P(x)) + \nu(P(x))^2))) \\ &\geq m - \inf_x (\mu(P(x))(\mu(P(x)) + \mu(P(x))\nu(P(x)) + \nu(P(x))^2)) \geq 0, \end{aligned}$$

because for every two numbers $a, b \in [0, 1]$, such that $a + b \leq 1$: $a + ab + b^2 = a + b(a + b) \leq a + b \leq 1$, i.e., the expression is an IFT.

For \neg_{52} we obtain:

$$\begin{aligned} \forall x P(x) \vee \exists x \neg_{52} P(x) &= \langle m, N \rangle \vee \langle \sup_x (\text{sg}(\nu(P(x))) + \text{sg}(\mu(P(x))\nu(P(x)))) \\ &\quad \inf_x (\text{sg}(\mu(P(x))\nu(P(x)))) \rangle \\ &= \langle \max(m, \sup_x (\text{sg}(\nu(P(x))) + \text{sg}(\mu(P(x))\nu(P(x))))), \min(N, m) \rangle, \end{aligned}$$

which obviously is an IFT. The (sup – inf)-case is analogous.

All other checks are similar. \square

The link between the interpretations of quantifiers and the topological operators C (closure) and I (interior) defined over IFSs see [7] is obvious.

Open Problem 9. The basic problem which remains unsolved is related to the characterization of predicate IFTs by means of a calculus.

Following [9, 10], we mention that a partial solution of the problem of giving a calculus which generates all predicate IFTs is presented in the next theorem.

Theorem 2.1.4 *A prenex normal form A is an IFT if and only if it is a classical predicate tautology and its quantifier free matrix is a propositional IFT.*

Here, a prenex form means (see [9, 10]) a predicate formula in which all quantifiers are moved to the left. The proof is based on the fact that all predicate transformations leading to prenex forms in the classical logic are valid for the intuitionistic fuzzy case, too.

2.2 Extended Intuitionistic Fuzzy Quantifiers

In [8], we introduced the following six quantifiers and studied some of their properties.

$$V(\forall_{\mu}xP(x)) = \{\langle x, \inf_{y \in E} \mu(P(y)), \nu(P(x)) \rangle \mid x \in E\},$$

$$V(\forall_{\nu}xP(x)) = \{\langle x, \min(1 - \sup_{y \in E} \nu(P(y)), \mu(P(x))), \sup_{y \in E} \nu(P(y)) \rangle \mid x \in E\},$$

$$V(\exists_{\mu}xP(x)) = \{\langle x, \sup_{y \in E} \mu(P(y)), \min(1 - \sup_{y \in E} \mu(P(y)), \nu(P(x))) \rangle \mid x \in E\},$$

$$V(\exists_{\nu}xP(x)) = \{\langle x, \mu(P(x)), \inf_{y \in E} \nu(P(y)) \rangle \mid x \in E\},$$

$$V(\forall_{\nu}^*xP(x))$$

$$= \{\langle x, \min(1 - \sup_{y \in E} \nu(P(y)), \mu(P(x))), \min(\sup_{y \in E} \nu(P(y)), 1 - \mu(P(x))) \rangle \mid x \in E\},$$

$$V(\exists_{\mu}^*xP(x))$$

$$= \{\langle x, \min(\sup_{y \in E} \mu(P(y)), 1 - \nu(P(x))), \min(1 - \sup_{y \in E} \mu(P(y)), \nu(P(x))) \rangle \mid x \in E\}.$$

Fig. 2.3 Example of a second geometrical interpretation

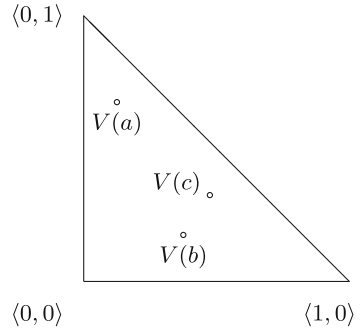


Fig. 2.4 Second geometrical interpretation of quantifier \exists_μ

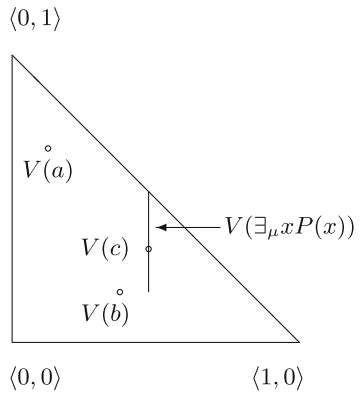
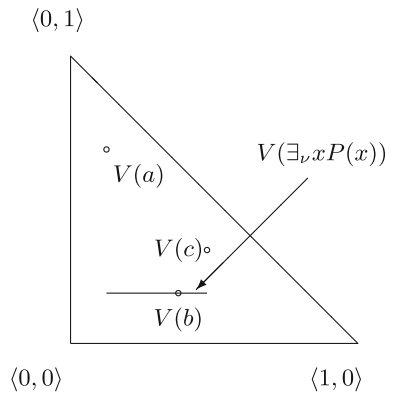


Fig. 2.5 Second geometrical interpretation of quantifier \exists_ν



Let the possible values of the variable x be a, b, c and let their IF-evaluations $V(a), V(b), V(c)$ be shown on Fig. 2.3. The geometrical interpretations of the new quantifiers are shown in Figs. 2.4, 2.5, 2.6, 2.7, 2.8 and 2.9.

Fig. 2.6 Second geometrical interpretation of quantifier \forall_μ

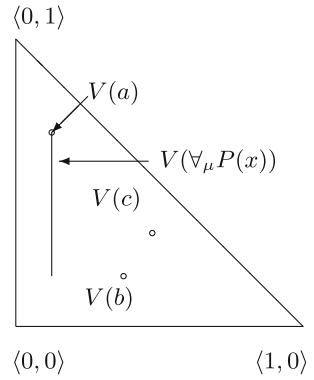


Fig. 2.7 Second geometrical interpretation of quantifier \forall_ν

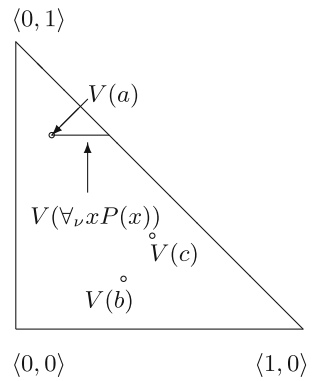


Fig. 2.8 Second geometrical interpretation of quantifier \exists_μ^*

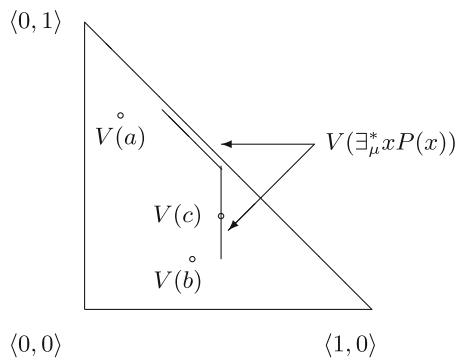
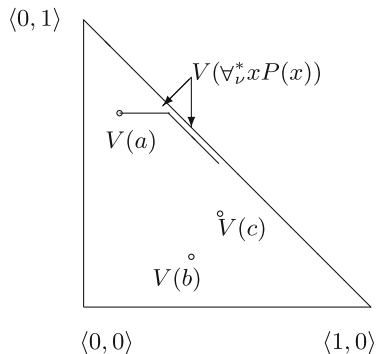


Fig. 2.9 Second geometrical interpretation of quantifier \forall_ν^*



Now, we see that we can change the forms of the first two quantifiers to the forms

$$V(\forall x P(x)) = \{ \langle x, \inf_{y \in E} \mu(P(y)), \sup_{y \in E} \nu(P(y)) \rangle \mid x \in E \},$$

$$V(\exists x P(x)) = \{ \langle x, \sup_{y \in E} \mu(P(y)), \inf_{y \in E} \nu(P(y)) \rangle \mid x \in E \}.$$

Obviously, for every predicate P ,

$$\begin{aligned} V(\forall x P(x)) &\subseteq V(\forall_{\mu} x P(x)) \subseteq V(\forall_{\nu} x P(x)) \subseteq V(\exists_{\nu} x P(x)) \\ &\subseteq V(\exists_{\mu} x P(x)) \subseteq V(\exists x P(x)) \end{aligned}$$

and

$$\begin{aligned} V(\forall x P(x)) &\subseteq V(\forall_{\nu} x P(x)) \subseteq V(\forall_{\nu}^* x P(x)) \\ &\subseteq V(\exists_{\mu}^* x P(x)) \subseteq V(\exists_{\mu} x P(x)) \subseteq V(\exists x P(x)). \end{aligned}$$

Open Problem 10. Which implications satisfy Theorem 2.1.1(d) and (e) in Sect. 2.1?

Now, we can modify the new six operators, so as to change their set form to the form of the first two operators.

Let a be one of the possible values for variable x . Then,

$$V(\forall_{\mu} x P(x), a) = \langle \inf_{y \in E} \mu(P(y)), \nu(P(a)) \rangle,$$

$$V(\forall_{\nu} x P(x), a) = \langle \min(1 - \sup_{y \in E} \nu(P(y)), \mu(P(a))), \sup_{y \in E} \nu(P(y)) \rangle,$$

$$V(\exists_{\mu} x P(x), a) = \langle \sup_{y \in E} \mu(P(y)), \min(1 - \sup_{y \in E} \mu(P(y)), \nu(P(a))) \rangle,$$

$$\begin{aligned}
V((\exists_\nu x P(x)), a) &= \langle \mu(P(a)), \inf_{y \in E} \nu(P(y)) \rangle, \\
V((\forall_\nu^* x P(x)), a) \\
&= \langle \min(1 - \sup_{y \in E} \nu(P(y)), \mu(P(a))), \min(\sup_{y \in E} \nu(P(y)), 1 - \mu(P(a))) \rangle, \\
V((\exists_\mu^* x P(x)), a) \\
&= \langle \min(\sup_{y \in E} \mu(P(y)), 1 - \nu(P(a))), \min(1 - \sup_{y \in E} \mu(P(y)), \nu(P(a))) \rangle.
\end{aligned}$$

We finish this section with an example.

Let the universe comprise the members of the European Union and let for each country the degree of government approval and disapproval be known. Let the predicate $P(x)$ be “The government of country x is widely approved by the people of country x ”. The first quantifier \forall will give the minimal degree of approval which exists in the countries of the EU, and the maximal degree of disapproval in the countries (not necessarily the same). Conversely, the second operator \exists will give us the maximal degree of approval in one of these countries and the minimal degree of disapproval.

Let us assume that for some reason we do not have complete information about either the approval or disapproval for a fixed country a from the EU (but we have such information about the rest). If we are missing information about the degree of approval for a , then, the third operator \forall_μ will give us a lower bound for this degree of approval for a . The fifth operator \exists_μ will give us an upper bound for the degree of approval for a .

Conversely, if we are missing information about the degree of disapproval, the fourth operator will give us \forall_ν will give us the upper bound and the sixth \exists_ν will give us the lower bound for the degree of disapproval for a .

The seventh and eighth operators act exactly like the fourth and the fifth operators, respectively, but provide a more precise evaluation for the respective degree.

2.3 Ideas for New Types of Quantifiers

It is well known from the classical logic that for each predicate P with argument x having a finite number of interpretations a_1, a_2, \dots, a_n :

$$V(\forall x P(x)) = V(P(a_1) \wedge P(a_2) \wedge \dots \wedge P(a_n)),$$

$$V(\exists x P(x)) = V(P(a_1) \vee P(a_2) \vee \dots \vee P(a_n)).$$

Now, following [11] and having in mind the ideas from Sect. 1.7, we can construct a lot of new quantifiers. For each new pair of conjunction and disjunction, we obtain a pair of quantifiers that have the forms

$$V(\forall_{i,j} P(x)) = V(P(a_1) \wedge_{i,j} P(a_2) \wedge_{i,j} \cdots \wedge_{i,j} P(a_n)),$$

$$V(\exists_{i,j} P(x)) = V(P(a_1) \vee_{i,j} P(a_2) \vee_{i,j} \cdots \vee_{i,j} P(a_n)),$$

where i ($1 \leq i \leq 185$) and j ($1 \leq j \leq 3$) are the indices of the respective pair of conjunction and disjunction that generates the new pair of quantifiers.

Obviously, $\forall_{4,1}$ coincides with the standard quantifier \forall , and $\exists_{4,1}$ coincides with the standard quantifier \exists .

One special case is the following: using implication \rightarrow_{139} and negation \neg_1 we obtain for $a, b, c, d \in [0, 1]$ and $a + b, c + d \leq 1$:

$$V(\langle a, b \rangle \vee_{139,3} \langle c, d \rangle) = \left\langle \frac{a+c}{2}, \frac{b+d}{2} \right\rangle = \langle a, b \rangle \wedge_{139,3} \langle c, d \rangle.$$

Therefore, if for each i : $V(P(x_i)) = \langle a_i, b_i \rangle$, then,

$$V(\forall_{139,3} P(x)) = \left\langle \frac{\sum_{i=1}^n a_i}{n}, \frac{\sum_{i=1}^n b_i}{n} \right\rangle = V(\exists_{139,3} P(x)).$$

In this case, we check directly, that

$$\neg_1 \forall_{139,3} \neg_1 P(x) = \forall_{139,3} P(x).$$

Hence, there exists a quantifier's interpretation for which both quantifiers “ \forall ” and “ \exists ” coincide.

It is very interesting that the topological weight-center operator W (see, e.g. [12]) is an exact analogue of quantifier $\forall_{139,3}$. So, we can denote it as $WxP(x)$.

The so defined quantifiers give us the possibility to classify all of them in two groups.

- Global quantifiers: \forall, \exists, W ,
- Local quantifiers: $\forall_\mu, \forall_\nu, \forall_\mu^*, \exists_\mu, \exists_\nu, \exists_\mu^*, U$.

Open Problem 11. Study in details the behaviour of these quantifiers.

References

1. Barwise J, editor. Handbook of mathematical logic., Studies in Logic and the Foundations of Mathematics, Amsterdam: North Holland; 1989.
2. Crossley JN, Ash CJ, Brickhill CJ, Stillwell JC, Williams NH. What is mathematical logic?. London: Oxford University Press; 1972.

3. van Dalen D. *Logic and structure*. Berlin: Springer; 2013.
4. Ebbinghaus H-D, Flum J, Thomas W. *Mathematical logic*. 2nd ed. New York: Springer; 1994.
5. Mendelson E. *Introduction to mathematical logic*. Princeton: D. Van Nostrand; 1964.
6. Shoenfield JR. *Mathematical logic*. 2nd ed. Natick: A.K. Peters; 2001.
7. Atanassov K. *Intuitionistic fuzzy sets*. Heidelberg: Springer; 1999.
8. Atanassov K. On intuitionistic fuzzy logics: Results and problems. In: Atanassov K, Baczynski M, Drewniak J, Kacprzyk J, Krawczak M, Szmidt E, Wygralak M, Zadrozny S, editors. *Modern approaches in fuzzy sets, intuitionistic fuzzy sets, generalized nets and related topics*, Volume 1: Foundations, SRI-PAS, Warsaw, 2014. P. 23–49.
9. Atanassov K, Gargov G. Elements of intuitionistic fuzzy logic I. *Fuzzy sets Syst.* 1998;95(1):39–52.
10. Gargov G, Atanassov K. Two results in intuitionistic fuzzy logic. *Comptes Rendus de l'Academie bulgare des Sciences Tome.* 1992;45(12):29–31.
11. Atanassov, K. On intuitionistic fuzzy quantifiers. *Notes Intuitionistic Fuzzy Sets.* 2016;22(2): 1–12.
12. Atanassov K. *On intuitionistic fuzzy sets theory*. Berlin: Springer; 2012.