Chapter 1
Elements of Intuitionistic Fuzzy Propositional Calculus

1.1 Intuitionistic Fuzzification of the Validity of Propositions

In classical logic (e.g., [1–4]), to each proposition (sentence) we juxtapose its truth value: truth – denoted by 1, or falsity – denoted by 0. In the case of fuzzy logic [5], this truth value is a real number in the interval [0, 1] and it is called "truth degree" or "degree of validity". In the intuitionistic fuzzy case (see [6–9]) we add one more value – "falsity degree" or "degree of non-validity"– which is again in interval [0, 1]. Thus, to the proposition p, two real numbers, $\mu(p)$ and $\nu(p)$, are assigned with the following constraint:

$$\mu(p), \nu(p) \in [0, 1] \text{ and } \mu(p) + \nu(p) \leq 1. \tag{1.1.1}$$

Let

$$\pi(p) = 1 - \mu(p) - \nu(p).$$

This function determines the degree of uncertainty (indeterminacy).

In [10], the pair $\langle \mu(p), \nu(p) \rangle$ that satisfies condition (1.1.1) is called "Intuitionistic Fuzzy Pair" (IFP).

Let an evaluation function V be defined over a set of propositions $\mathcal{S}$, in such a way that for $p \in \mathcal{S}$:

$$V(p) = \langle \mu(p),\ \nu(p) \rangle.$$

Hence the function $V : \mathcal{S} \to [0, 1] \times [0, 1]$ gives the truth and falsity degrees of all elements of $\mathcal{S}$.

We assume that the evaluation function V assigns to the logical truth T

$$V(T) = \langle 1, 0 \rangle,$$

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and to the logical falsity F

$$V(F) = \langle 0, 1\rangle.$$

When

$$\nu(p) = 1 - \mu(p),$$

i.e.,

$$V(p) = \langle \mu(p), 1 - \mu(p)\rangle,$$

it coincides with the fuzzy case.

As it was discussed in the author's book [11], one of his major mistakes was that in the middle of 1980s, when he found the following two negations that can be defined over elements of $\mathcal{S}$, he did not study in details the properties of the second negation, because it was essentially more complex. For $p \in \mathcal{S}$ these negations are:

$$V(\neg_1 p) = \langle \nu(p), \mu(p)\rangle, \tag{1.1.2}$$

$$V(\neg_2 p) = \langle \overline{\text{sg}}(\mu(p)), \text{sg}(\mu(p))\rangle, \tag{1.1.3}$$

where here and below

$$\text{sg}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases},$$

and

$$\overline{\text{sg}}(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if} x \leq 0 \end{cases}.$$

Obviously, the first definition coincides with the formula in the ordinary fuzzy logic (see, e.g., [12]).

Immediately, we see that for each proposition $p \in \mathcal{S}$:

$$V(\neg_1\neg_1 p) = V(p),$$

which some colleagues interpreted as a contradiction with the idea of L. Brouwer's intuitionism (see, e.g., [13–15]). The author's opinion is that the words "intuitionistic fuzzy" correspond to the form of the elements of set $\mathcal{S}$, but not to the forms of the operations defined over the intuitionistic fuzzy propositions.

The second negation does not satisfy the equality $V(\neg_2\neg_2 p) = V(p)$, and as we see below, it exhibits truly intuitionistic behaviour. This is a confirmation of the author's assertion that intuitionistic fuzzy objects have intuitionistic properties. In the Sect. 1.4, we discuss not only the second, but more than 50 different negations. Here, we define only the operations "disjunction", "conjunction" and "implication", originally introduced in [6], that have classical logic analogues, as follows:

$$V(p \vee q) = \langle \max(\mu(p), \mu(q)), \min(\nu(p), \nu(q)) \rangle, \tag{1.1.4}$$

$$V(p \wedge q) = \langle \min(\mu(p), \mu(q)), \max(\nu(p), \nu(q)) \rangle, \tag{1.1.5}$$

$$V(p \rightarrow q) = \langle \max(\nu(p), \mu(q)), \min(\nu(p), \mu(q)) \rangle. \tag{1.1.6}$$

In some places below, we call them "standard" disjunction, conjunction and implication.

Similarly to [7, 16], several geometrical interpretations of the results of the function V will be discussed below.

In intuitionistic fuzzy propositional calculus, the formulas are defined in the manner of standard propositional calculus.

In [7], it is noted that the ordinary fuzzy sets have only one geometrical interpretation, while in [7, 16] several interpretations of IFSs are given. Here we show the most relevant interpretations for the logical case.

The first one (which is analogous to the standard fuzzy set interpretation) is shown on Fig. 1.1.

Its analogue is given in Fig. 1.2.

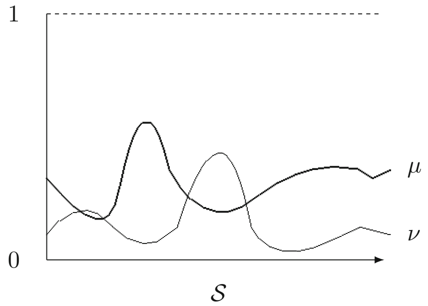


Fig. 1.1 First geometrical interpretation

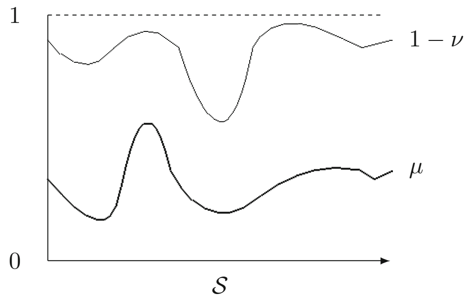
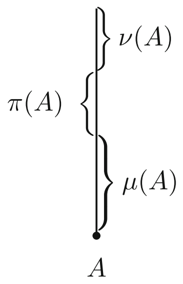


Fig. 1.2 Modified form of the first geometrical interpretation

Therefore, we can map to every formula A, a unit segment in the form:



Let a universe $\mathcal{S}$ be given and let us consider the figure F in the Euclidean plane with a Cartesian coordinate system (see Fig. 1.3). Then, we can construct an evaluation function V from $\mathcal{S}$ to F such that if $p \in \mathcal{S}$, then

$$x = V(p) \in F,$$

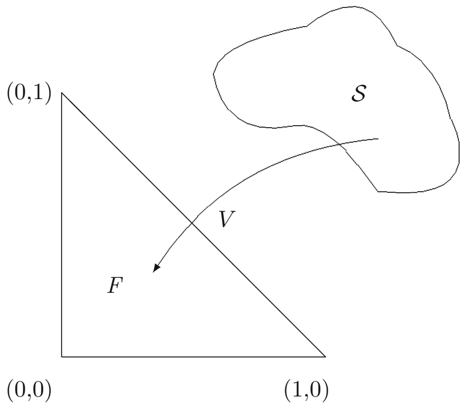
the point x has coordinates $\langle a, b\rangle$ for which: $0 \leq a + b \leq 1$ and these coordinates are such that $a = \mu(p), b = \nu(p)$.

We will note that there can exist two different elements $p, q \in S$ for which $\mu(p) = \mu(q)$ and $\nu(p) = \nu(q)$, i.e., for which $V(p) = V(q)$.

About the form and the methods of determining the functions μ and ν, we must repeat the same, as in [7]: everywhere below we will assume that these functions are either pre-determined or obtained as a result of the application of some operations or operators over pre-determined functions. In fuzzy set theory, there are three basic ways to construct membership functions:

(i) on the basis of expert knowledge;
(ii) explicitly—on the basis of observations collected in advance and processed appropriately (e.g., by statistical methods);
(iii) analytically—by suitably chosen functions (e.g. probabilistic distribution).

Fig. 1.3 Second geometrical interpretation



The two latter cases are treated in much the same way as ordinary fuzzy sets; however these methods are now used for the estimation of both the degree of membership and the degree of non-membership of a given element of a fixed universe to a subset of the same universe. It is clear that a correct method must respect the inequalities

$$0 \leq \mu(A) + \nu(A) \leq 1$$

for every formula A.

Following [7], we must also add, that the case when the functions values are calculated on the basis of expert knowledge is more complicated. In this case problems related to the correctness of the expert estimations arise. No such problems arise when dealing with ordinary fuzzy sets. These problems are discussed in Sect. 4.3 of [7], where five approaches of processing expert knowledge are proposed, regarding the construction of the degrees of membership and non-membership. These approaches are introduced in increasing order of complexity, and they reflect the assurance of the experts who estimate the corresponding events (objects, processes), their personal and collective opinion and their expert ratings. Similar approaches can be used for processing collected knowledge (observations), when incorrect data are suspected. Some of the methods from Sect. 4.3 of [7] and the methods introduced by P. Dworniczak in [17, 18], can help us locate the incorrect pieces of information.

Four other geometrical interpretations are shown in Figs. 1.4, 1.5 and 1.6.

The triangle from Fig. 1.4 has sides length of $\frac{2\sqrt{3}}{3}$ and therefore, the length of the altitude is equal to 1 $(= \mu(p) + \nu(p) + \pi(p))$.

The angles to the basis of the triangle from Fig. 1.5 have the values, respectively, $\alpha = \boldsymbol{\pi}.\mu(p)$ and $\beta = \boldsymbol{\pi}.\nu(p)$, where $\boldsymbol{\pi} = 3.14159\ldots$, and the length of the triangle basis is equal to 1, as well as the catheti of the rectangular triangle from Fig. 1.6.

In [19], a geometrical interpretation based on radar chart is proposed by V. Atanassova. In Fig. 1.7, the innermost zone corresponds to the membership degree, the outermost zone to the non-membership degree and the region between both zones to the degree of uncertainty. This interpretation can be especially useful for data in time series, multivaried data sets and other data with cyclic trait.

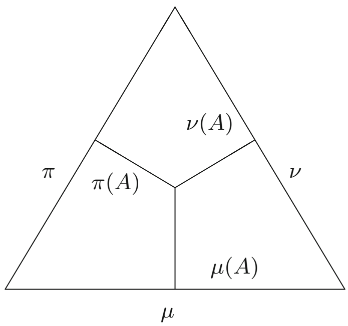


Fig. 1.4 Third geometrical interpretation

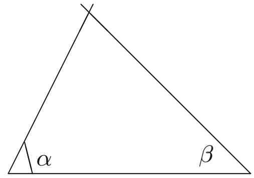


Fig. 1.5 Fourth geometrical interpretation

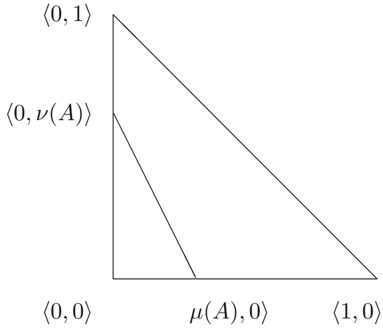


Fig. 1.6 Fifth geometrical interpretation

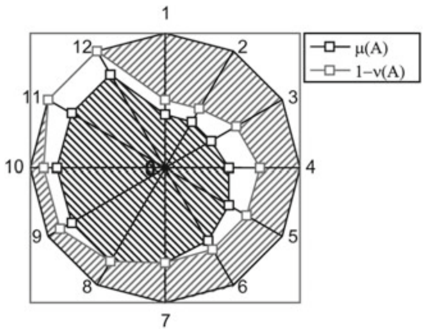


Fig. 1.7 Sixth geometrical interpretation

The way the two functions, μ and ν, are constructed will not be important for our considerations below.

The geometrical interpretations of the first two operations conjunction and disjunction are the following.

Let p and q be two propositions in $\mathcal{S}$. Let the evaluation function V assign to $p \wedge q \in \mathcal{S}$ a point $V(p \wedge q) \in F$ with coordinates

$$\langle \min(\mu(p), \mu(q)), \max(\nu(p), \nu(q)) \rangle.$$

There exist three geometrical cases (see Fig. 1.8a–c) - one general case (Fig. 1.8a) and two particular cases (Fig. 1.8b, c).

Now, the evaluation function V assigns to $p \vee q \in \mathcal{S}$ a point $V(p \vee q) \in F$ with coordinates

$$\langle \max(\mu(p), \mu(q)), \min(\nu(p), \nu(q)) \rangle.$$

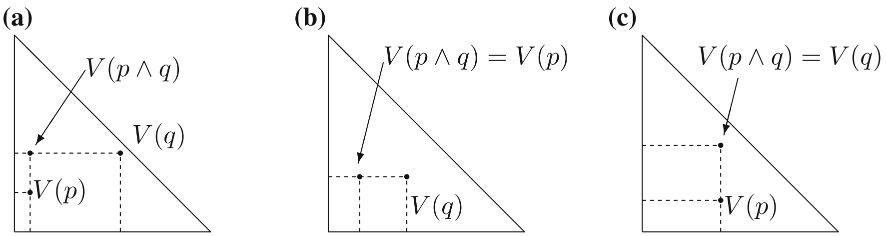


Fig. 1.8 Second geometrical interpretation of operation $\wedge$

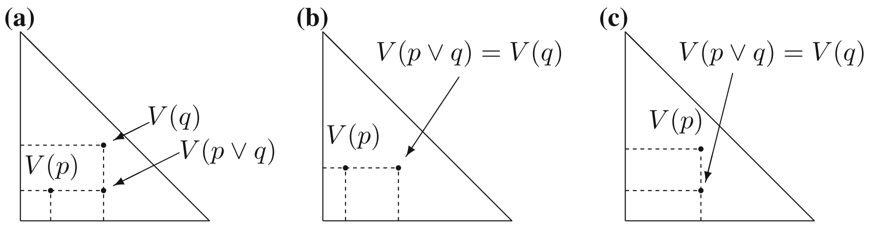


Fig. 1.9 Second geometrical interpretation of operation $\vee$

There exist also three geometrical cases as above, namely, one general case (Fig. 1.9a) and two particular cases (Fig. 1.9b, c).

In 1988 two different implications were defined (see [6, 16]). For the case of the first implication (the classical one, see (1.1.6)), function V assigns to $p \rightarrow q \in \mathcal{S}$ a point $V(p \rightarrow q) \in F$ with coordinates $\langle \max(b, c), \min(a, d) \rangle$ (Fig. 1.10a–h).

The second implication (non-classical) has the form

$$V(p \rightarrow q) = \langle 1 - (1 - c.\text{sg}(a - c)), d.\text{sg}(a - c)\text{sg}(d - b) \rangle.$$

For the case of the second implication, evaluation function V assigns to $p \rightarrow q \in \mathcal{S}$ a point $V(p \rightarrow q) \in F$ with coordinates $\langle 1-(1-c.\text{sg}(a-c)), d.\text{sg}(a-c)\text{sg}(d-b) \rangle$ (Fig. 1.11a–h).

In the beginning of this century, when the author started re-defining the multiple fuzzy implications in intuitionistic fuzzy form, he saw that the above implication coincides with implication $\rightarrow_3$ that we will discuss in the next section.

All this is valid for formulas, instead of propositions, too.

For the needs of the discussion below, we define the notions of Intuitionistic Fuzzy Tautology (IFT, see, e.g. [6, 11]) and tautology.

Formula A is an IFT if and only if (iff) for every evaluation function V, if $V(A) = \langle a, b \rangle$, then,

$$a \geq b, \tag{1.1.7}$$

while it is a (classical) tautology if and only if for every evaluation function V, if $V(A) = \langle a, b \rangle$, then,

$$a = 1, \; b = 0. \tag{1.1.8}$$

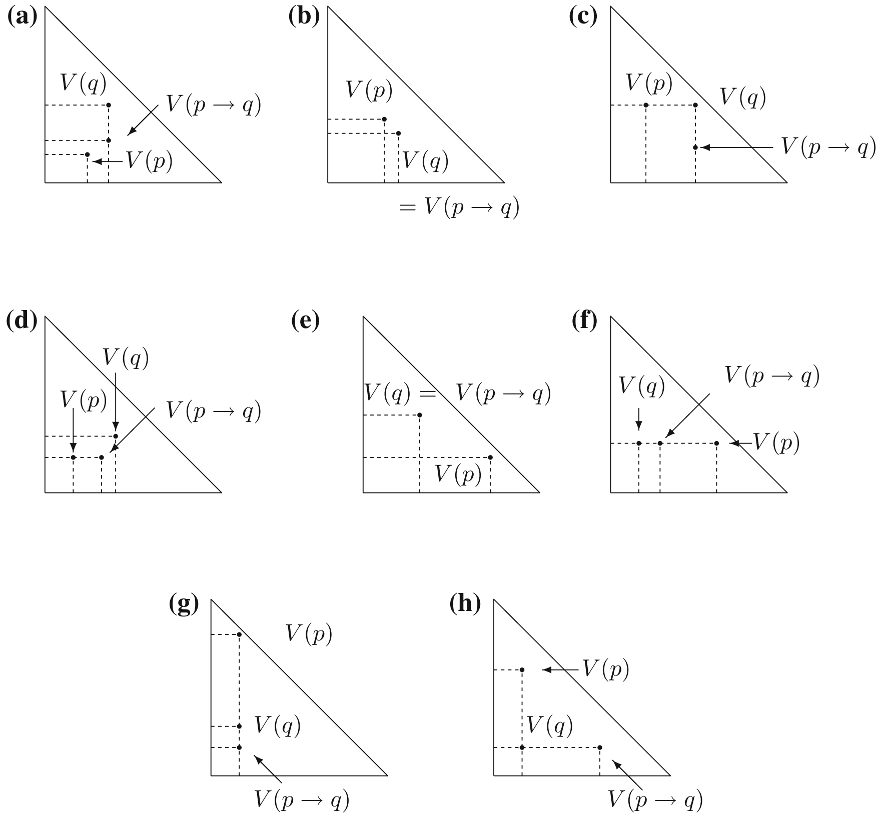


Fig. 1.10 Second geometrical interpretation of the classical operation →

For each evaluation function V and for each formula A such that $V(x) = \langle a, b\rangle$, let us say that A is "intuitionistic fuzzy sure" (IF-sure), iff $a \geq 0.5 \geq b$.

Below, when it is clear, we will omit notation "$V(A)$", using directly "A" of the intuitionistic fuzzy evaluation of A. Also, for brevity, in a lot of places, instead of the IFP $\langle \mu(A), \nu(A)\rangle$ we will use the IFP $\langle a, b\rangle$, where $a, b \in [0, 1]$ and $a + b \leq 1$.

It is also suitable, if $\langle a, b\rangle$ and $\langle c, d\rangle$ are IFPs, to have

$$\langle a, b\rangle \leq \langle c, d\rangle \text{ iff } a \leq c \text{ and } b \geq d \tag{1.1.9}$$

and

$$\langle a, b\rangle \geq \langle c, d\rangle \text{ iff } a \geq c \text{ and } b \leq d. \tag{1.1.10}$$

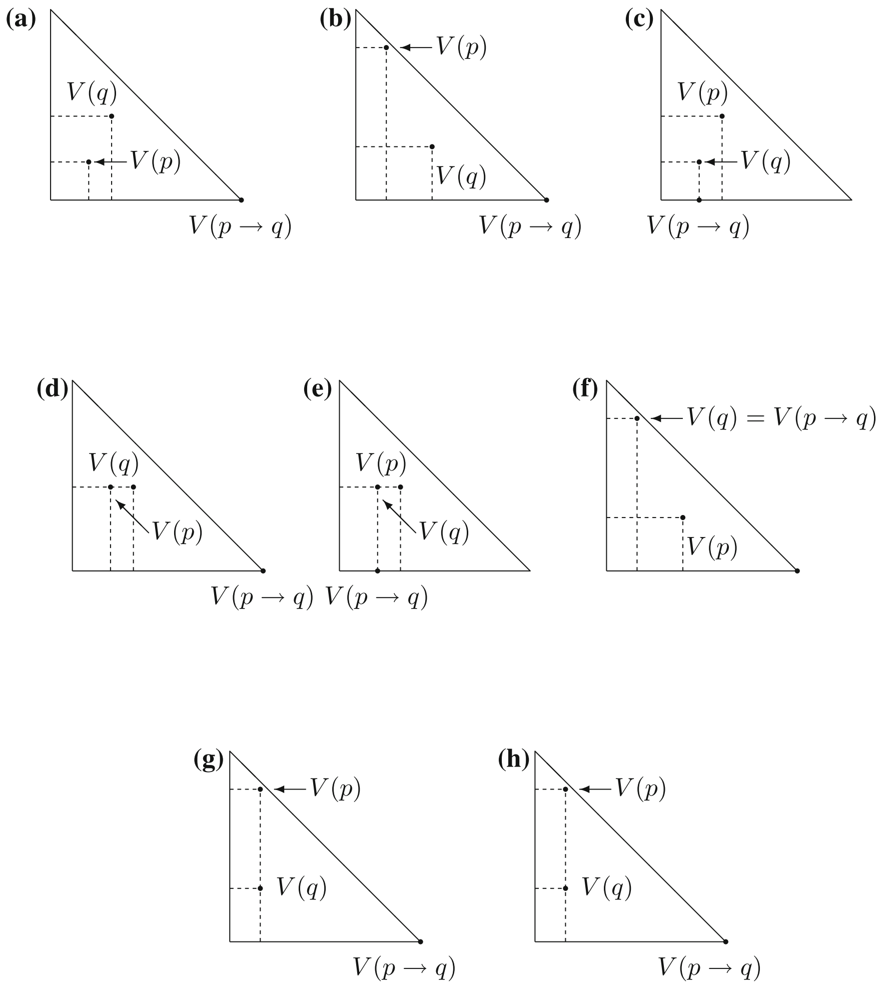


Fig. 1.11 Second geometrical interpretation of the first non-classical operation $\rightarrow$

1.2 Intuitionistic Fuzzy Implications

In the series of papers [20–55], 185 different implications were defined and some of their basic properties were studied. In some of these publications, some misprints in the formulas were found during the last years. Here, we give the full list of the corrected intuitionistic fuzzy implications.

The first 10 implications, described in [23] are intuitionistic fuzzy analogues of existing fuzzy implications in literature (see, e.g., [56]). The next five implications were introduced in author's publications [24, 31, 36, 37]. Two of them are given in papers of B. Kolev [31] ($\rightarrow_{13}$) and of T. Trifonov ($\rightarrow_{14}$) [36, 37] and the author.

From the so constructed 15 implications, using the formula

$$\neg A = A \to F, \tag{1.2.1}$$

five negations are formulated – the standard (classical) negation $\neg_1$ and four others $\neg_2, \ldots, \neg_5$, where F is defined above and $\to$ is each one of these 15 implications [57]. By formulas

$$A \to B = \neg A \vee B$$

and

$$A \to B = \neg A \vee \neg\neg B,$$

these 5 negations generate eight new implications [25]. Each of these implications generates four new implications, by formulas using intuitionistic fuzzy (standard, classical) modal operators, that we discuss in Chap. 3. The analogues of these implications are given in [11].

In [42–45], L. Atanassova introduced 11 new implications ($\to_{139}, \ldots, \to_{149}$). P. Dworniczak generalized them in [52–54] (implications $\to_{150}, \ldots, \to_{152}$) and L. Atanassova modified Dworniczak's implications in [49–51] (implications $\to_{154}$ $, \ldots, \to_{165}$).

The next five implications were introduced by the author in [58] as modifications of the first L. Zadeh's type intuitionistic fuzzy implication $\to_1$, implications $\to_{171}$ $, \to_{175}$ were introduced by the author in [29].

Implications $\to_{176}, \ldots, \to_{180}$ were proposed by E. Szmidt, J. Kacprzyk and the author in [33–35], while the last five implications $\to_{181}, \ldots, \to_{185}$ were introduced by the author in [29].

All currently existing implications were published in [29] and they are given in Table 1.1. In it, we keep the numeration from [11] up to number 138.

1.3 Discussion on Intuitionistic Fuzzy Implications

Let us have

$$\lambda \geq 1,\ \gamma \geq 1,\ \alpha \geq 1, \beta \in [1, \alpha],$$

$$\varepsilon, \eta \in [0, 1] \text{ and } \varepsilon \leq \eta < 1.$$

By direct check, we see that the following equalities hold.

Table 1.1 List of the intuitionistic fuzzy implications

$\to_1$	$\langle \max(b, \min(a, c)), \min(a, d)\rangle$
$\to_2$	$\langle \overline{\text{sg}}(a-c), d\text{sg}(a-c)\rangle$
$\to_3$	$\langle 1-(1-c)\text{sg}(a-c)), d\text{sg}(a-c)\rangle$
$\to_4$	$\langle \max(b, c), \min(a, d)\rangle$
$\to_5$	$\langle \min(1, b+c), \max(0, a+d-1)\rangle$
$\to_6$	$\langle b+ac, ad\rangle$
$\to_7$	$\langle \min(\max(b, c), \max(a, b), \max(c, d)),$ $\max(\min(a, d), \min(a, b), \min(c, d))\rangle$
$\to_8$	$\langle 1-(1-\min(b, c))\text{sg}(a-c), \max(a, d)\text{sg}(a-c), \text{sg}(d-b)\rangle$
$\to_9$	$\langle b+a^2c, ab+a^2d\rangle$
$\to_{10}$	$\langle c\overline{\text{sg}}(1-a)+\text{sg}(1-a)(\overline{\text{sg}}(1-c)+b\text{sg}(1-c)),$ $d\overline{\text{sg}}(1-a)+a\text{sg}(1-a)\text{sg}(1-c)\rangle$
$\to_{11}$	$\langle 1-(1-c)\text{sg}(a-c), d\text{sg}(a-c)\text{sg}(d-b)\rangle$
$\to_{12}$	$\langle \max(b, c), 1-\max(b, c)\rangle$
$\to_{13}$	$\langle b+c-bc, ad\rangle$
$\to_{14}$	$\langle 1-(1-c)\text{sg}(a-c)-d\overline{\text{sg}}(a-c)\text{sg}(d-b), d\text{sg}(d-b)\rangle$
$\to_{15}$	$\langle 1-(1-\min(b, c))\text{sg}(\text{sg}(a-c)+\text{sg}(d-b))$ $-\min(b, c)\text{sg}(a-c)\text{sg}(d-b),\ 1-(1-\max(a, d))\text{sg}(\overline{\text{sg}}(a-c)$ $+\overline{\text{sg}}(d-b))-\max(a, d)\overline{\text{sg}}(a-c)\overline{\text{sg}}(d-b)\rangle$
$\to_{16}$	$\langle \max(\overline{\text{sg}}(a), c), \min(\text{sg}(a), d)\rangle$
$\to_{17}$	$\langle \max(b, c), \min(ab+a^2, d)\rangle$
$\to_{18}$	$\langle \max(b, c), \min(1-b, d)\rangle$
$\to_{19}$	$\langle \max(1-\text{sg}(\text{sg}(a)+\text{sg}(1-b)), c), \min(\text{sg}(1-b), d)\rangle$
$\to_{20}$	$\langle \max(\overline{\text{sg}}(a), \text{sg}(c)), \min(\text{sg}(a), \overline{\text{sg}}(c))\rangle$
$\to_{21}$	$\langle \max(b, c(c+d)), \min(a(a+b), d(c^2+d+cd))\rangle$
$\to_{22}$	$\langle \max(b, 1-d), 1-\max(b, 1-d)\rangle$
$\to_{23}$	$\langle 1-\min(\text{sg}(1-b), \overline{\text{sg}}(1-d)), \min(\text{sg}(1-b), \overline{\text{sg}}(1-d))\rangle$
$\to_{24}$	$\langle \overline{\text{sg}}(a-c)\overline{\text{sg}}(d-b), \text{sg}(a-c)\text{sg}(d-b)\rangle$
$\to_{25}$	$\langle \max(b, \overline{\text{sg}}(a)\overline{\text{sg}}(1-b)), c\overline{\text{sg}}(d)\overline{\text{sg}}(1-c)), \min(a, d)\rangle$
$\to_{26}$	$\langle \max(\overline{\text{sg}}(1-b), c), \min(\text{sg}(a), d)\rangle$
$\to_{27}$	$\langle \max(\overline{\text{sg}}(1-b), \text{sg}(c)), \min(\text{sg}(a), \overline{\text{sg}}(1-d))\rangle$
$\to_{28}$	$\langle \max(\overline{\text{sg}}(1-b), c), \min(a, d)\rangle$
$\to_{29}$	$\langle \max(\overline{\text{sg}}(1-b), \overline{\text{sg}}(1-c)), \min(a, \overline{\text{sg}}(1-d))\rangle$
$\to_{30}$	$\langle \max(1-a, \min(a, 1-d)), \min(a, d)\rangle$
$\to_{31}$	$\langle \overline{\text{sg}}(a+d-1), d\text{sg}(a+d-1)\rangle$
$\to_{32}$	$\langle 1-d\text{sg}(a+d-1), d\text{sg}(a+d-1)\rangle$
$\to_{33}$	$\langle 1-\min(a, d), \min(a, d)\rangle$
$\to_{34}$	$\langle \min(1, 2-a-d), \max(0, a+d-1)\rangle$

(continued)

Table 1.1 (continued)

$\rightarrow_{35}$	$\langle 1-ad, ad\rangle$
$\rightarrow_{36}$	$\langle \min(1-\min(a,d), \max(a,1-a), \max(1-d,d)), \max(\min(a,d),$ $\min(a,1-a), \min(1-d,d))\rangle$
$\rightarrow_{37}$	$\langle 1-\max(a,d)\text{sg}(a+d-1), \max(a,d)\text{sg}(a+d-1)\rangle$
$\rightarrow_{38}$	$\langle 1-a+a^2(1-d), a(1-a)+a^2d\rangle$
$\rightarrow_{39}$	$\langle (1-d)\overline{\text{sg}}(1-a)+\text{sg}(1-a)(\overline{\text{sg}}(d)+(1-a)\text{sg}(d)),$ $d\overline{\text{sg}}(1-a)+a\text{sg}(1-a)\text{sg}(d)\rangle$
$\rightarrow_{40}$	$\langle 1-\text{sg}(a+d-1), 1-\overline{\text{sg}}(a+d-1)\rangle$
$\rightarrow_{41}$	$\langle \max(\overline{\text{sg}}(a), 1-d), \min(\text{sg}(a), d)\rangle$
$\rightarrow_{42}$	$\langle \max(\overline{\text{sg}}(a), \text{sg}(1-d)), \min(\text{sg}(a), \overline{\text{sg}}(1-d))\rangle$
$\rightarrow_{43}$	$\langle \max(\overline{\text{sg}}(a), 1-d), \min(\text{sg}(a), d)\rangle$
$\rightarrow_{44}$	$\langle \max(\overline{\text{sg}}(a), 1-d), \min(a, d)\rangle$
$\rightarrow_{45}$	$\langle \max(\overline{\text{sg}}(a), \overline{\text{sg}}(d)), \min(a, \overline{\text{sg}}(1-d))\rangle$
$\rightarrow_{46}$	$\langle \max(b, \min(1-b, c)), 1-\max(b, c)\rangle$
$\rightarrow_{47}$	$\langle \overline{\text{sg}}(1-b-c), (1-c)\text{sg}(1-b-c)\rangle$
$\rightarrow_{48}$	$\langle 1-(1-c)\text{sg}(1-b-c), (1-c)\text{sg}(1-b-c)\rangle$
$\rightarrow_{49}$	$\langle \min(1, b+c), \max(0, 1-b-c)\rangle$
$\rightarrow_{50}$	$\langle b+c-bc, 1-b-c+bc\rangle$
$\rightarrow_{51}$	$\langle \min(\max(b,c), \max(1-b,b), \max(c,1-c)),$ $\max(1-\max(b,c), \min(1-b,b), \min(c,1-c))\rangle$
$\rightarrow_{52}$	$\langle 1-(1-\min(b,c))\text{sg}(1-b-c), 1-\min(b,c)\text{sg}(1-b-c)\rangle$
$\rightarrow_{53}$	$\langle b+(1-b)^2c, (1-b)b+(1-b)^2(1-c)\rangle$
$\rightarrow_{54}$	$\langle c\overline{\text{sg}}(b)+\text{sg}(b)(\overline{\text{sg}}(1-c)+b\text{sg}(1-c)),$ $(1-c)\overline{\text{sg}}(b)+(1-b)\text{sg}(b)\text{sg}(1-c)\rangle$
$\rightarrow_{55}$	$\langle 1-\text{sg}(1-b-c), 1-\overline{\text{sg}}(1-b-c)\rangle$
$\rightarrow_{56}$	$\langle \max(\overline{\text{sg}}(1-b), c), \min(\text{sg}(1-b), 1-c)\rangle$
$\rightarrow_{57}$	$\langle \max(\overline{\text{sg}}(1-b), \text{sg}(c)), \min(\text{sg}(1-b), \overline{\text{sg}}(c))\rangle$
$\rightarrow_{58}$	$\langle \max(\overline{\text{sg}}(1-b), \overline{\text{sg}}(1-c)), 1-\max(b,c)\rangle$
$\rightarrow_{59}$	$\langle \max(\overline{\text{sg}}(1-b), c), 1-\max(b,c)\rangle$
$\rightarrow_{60}$	$\langle \max(\overline{\text{sg}}(1-b), \overline{\text{sg}}(1-c)), \min(1-b, \overline{\text{sg}}(c))\rangle$
$\rightarrow_{61}$	$\langle \max(c, \min(b,d)), \min(a,d)\rangle$
$\rightarrow_{62}$	$\langle \overline{\text{sg}}(d-b), a\text{sg}(d-b)\rangle$
$\rightarrow_{63}$	$\langle 1-(1-b)\text{sg}(d-b), a\text{sg}(d-b)\rangle$
$\rightarrow_{64}$	$\langle c+bd, ad\rangle$
$\rightarrow_{65}$	$\langle 1-(1-\min(b,c))\text{sg}(d-b), \max(a,d)\text{sg}(d-b)\text{sg}(a-c)\rangle$
$\rightarrow_{66}$	$\langle c+d^2b, bd+d^2a\rangle$
$\rightarrow_{67}$	$\langle b\overline{\text{sg}}(1-d)+\text{sg}(1-d)(\overline{\text{sg}}(1-b)+c\text{sg}(1-b)),$ $a\overline{\text{sg}}(1-d)+d\text{sg}(1-d)\text{sg}(1-b)\rangle$

(continued)

Table 1.1 (continued)

$\rightarrow_{68}$	$\langle 1-(1-b)\text{sg}(d-b), a\text{sg}(d-b)\text{sg}(a-c)\rangle$
$\rightarrow_{69}$	$\langle 1-(1-b)\text{sg}(d-b)-a\overline{\text{sg}}(d-b)\text{sg}(a-c), a\text{sg}(a-c)\rangle$
$\rightarrow_{70}$	$\langle \max(\overline{\text{sg}}(d), b), \min(\text{sg}(d), a)\rangle$
$\rightarrow_{71}$	$\langle \max(b, c), \min(cd+d^2, a)\rangle$
$\rightarrow_{72}$	$\langle \max(b, c), \min(1-c, a)\rangle$
$\rightarrow_{73}$	$\langle \max(1-\max(\text{sg}(d), \text{sg}(1-c)), b), \min(\text{sg}(1-c), a)\rangle$
$\rightarrow_{74}$	$\langle \max(\text{sg}(b), \overline{\text{sg}}(d)), \min(\overline{\text{sg}}(b), \text{sg}(d))\rangle$
$\rightarrow_{75}$	$\langle \max(c, b(a+b)), \min(d(c+d), a(b^2+a)+ab)\rangle$
$\rightarrow_{76}$	$\langle \max(c, 1-a), \min(1-c, a)\rangle$
$\rightarrow_{77}$	$\langle (1-\min(\overline{\text{sg}}(1-a), \text{sg}(1-c))), \min(\overline{\text{sg}}(1-a), \text{sg}(1-c))\rangle$
$\rightarrow_{78}$	$\langle \max(\overline{\text{sg}}(1-c), b), \min(\text{sg}(d), a)\rangle$
$\rightarrow_{79}$	$\langle \max(\overline{\text{sg}}(1-c), \text{sg}(b)), \min(\text{sg}(d), \overline{\text{sg}}(1-a))\rangle$
$\rightarrow_{80}$	$\langle \max(\overline{\text{sg}}(1-c), b), \min(d, a)\rangle$
$\rightarrow_{81}$	$\langle \max(\overline{\text{sg}}(1-b), \overline{\text{sg}}(1-c)), \min(d, \overline{\text{sg}}(1-a))\rangle$
$\rightarrow_{82}$	$\langle \max(1-d, \min(d, 1-a)), \min(d, a)\rangle$
$\rightarrow_{83}$	$\langle \overline{\text{sg}}(a+d-1), a\text{sg}(a+d-1)\rangle$
$\rightarrow_{84}$	$\langle 1-a\text{sg}(a+d-1), a\text{sg}(a+d-1)\rangle$
$\rightarrow_{85}$	$\langle 1-d+d^2(1-a), d(1-d)+d^2\rangle$
$\rightarrow_{86}$	$\langle (1-a)\overline{\text{sg}}(1-d)+\text{sg}(1-d)(\overline{\text{sg}}(a)+(1-d)\text{sg}(d)),$ $a\overline{\text{sg}}(1-d)+d\text{sg}(1-d)\text{sg}(a)\rangle$
$\rightarrow_{87}$	$\langle \max(\overline{\text{sg}}(d), 1-a), \min(\text{sg}(d), a)\rangle$
$\rightarrow_{88}$	$\langle \max(\overline{\text{sg}}(d), \text{sg}(1-a)), \min(\text{sg}(d), \overline{\text{sg}}(1-a))\rangle$
$\rightarrow_{89}$	$\langle \max(\overline{\text{sg}}(d), 1-a), \min(d, a)\rangle$
$\rightarrow_{90}$	$\langle \max(\overline{\text{sg}}(a), \overline{\text{sg}}(d)), \min(d, \overline{\text{sg}}(1-a))\rangle$
$\rightarrow_{91}$	$\langle \max(c, \min(1-c, b)), 1-\max(b, c)\rangle$
$\rightarrow_{92}$	$\langle \overline{\text{sg}}(1-b-c), \min(1-b, \text{sg}(1-b-c))\rangle$
$\rightarrow_{93}$	$\langle (1-\min(1-b, \text{sg}(1-b-c)), \min(1-b, \text{sg}(1-b-c))\rangle$
$\rightarrow_{94}$	$\langle c+(1-c)^2b, (1-c)c+(1-c)^2(1-b)\rangle$
$\rightarrow_{95}$	$\langle \min(b, \overline{\text{sg}}(c))+\text{sg}(c)(\overline{\text{sg}}(1-b)+\min(c, \text{sg}(1-b))),$ $\min(1-b, \overline{\text{sg}}(c))+\min(1-c, \text{sg}(c), \text{sg}(1-b))\rangle$
$\rightarrow_{96}$	$\langle \max(\overline{\text{sg}}(1-c), b), \min(\text{sg}(1-b), 1-c)\rangle$
$\rightarrow_{97}$	$\langle \max(\overline{\text{sg}}(1-c), \text{sg}(b)), \min(\text{sg}(1-c), \overline{\text{sg}}(b))\rangle$
$\rightarrow_{98}$	$\langle \max(\overline{\text{sg}}(1-c), b), 1-\max(b, c)\rangle$
$\rightarrow_{99}$	$\langle \max(\overline{\text{sg}}(1-c), \overline{\text{sg}}(1-b)), \min(1-c, \overline{\text{sg}}(b))\rangle$
$\rightarrow_{100}$	$\langle \max(b\text{sg}(a), c), \min(a\text{sg}(b), d)\rangle$
$\rightarrow_{101}$	$\langle \max(b\text{sg}(a), c\text{sg}(d)), \min(a\text{sg}(b), \text{sg}(c)d)\rangle$
$\rightarrow_{102}$	$\langle \max(b, c\text{sg}(d)), \min(a, \text{sg}(c)d)\rangle$
$\rightarrow_{103}$	$\langle \max(\min(1-a, \text{sg}(a)), 1-d), \min(a, \text{sg}(1-a), d)\rangle$

(continued)

Table 1.1 (continued)

$\rightarrow_{104}$	$\langle\max(\min(1-a, \mathrm{sg}(a)), \min(1-d, \mathrm{sg}(d))),$ $\min(a, \mathrm{sg}(1-a), d, \mathrm{sg}(1-d))\rangle$
$\rightarrow_{105}$	$\langle\max(1-a, \min(1-d, \mathrm{sg}(d))), \min(a, d, \mathrm{sg}(1-d))\rangle$
$\rightarrow_{106}$	$\langle\max(\min(b, \mathrm{sg}(1-b)), c), \min(1-b, \mathrm{sg}(b), 1-c)\rangle$
$\rightarrow_{107}$	$\langle\max(\min(b, \mathrm{sg}(1-b)), \min(c, \mathrm{sg}(1-c))),$ $\min(1-b, \mathrm{sg}(b), 1-c, \mathrm{sg}(c))\rangle$
$\rightarrow_{108}$	$\langle\max(b, \min(c, \mathrm{sg}(1-c))), \min(1-b, 1-c, \mathrm{sg}(c))\rangle$
$\rightarrow_{109}$	$\langle b+\min(\overline{\mathrm{sg}}(1-a), c), ab+\min(\overline{\mathrm{sg}}(1-a), d)\rangle$
$\rightarrow_{110}$	$\langle\max(b, c), \min(ab+\overline{\mathrm{sg}}(1-a), d)\rangle$
$\rightarrow_{111}$	$\langle\max(b, cd+\overline{\mathrm{sg}}(1-c)), \min(ab+\overline{\mathrm{sg}}(1-a),$ $d(cd+\overline{\mathrm{sg}}(1-c))+\overline{\mathrm{sg}}(1-d))\rangle$
$\rightarrow_{112}$	$\langle b+c-bc, ab+\overline{\mathrm{sg}}(1-a)d\rangle$
$\rightarrow_{113}$	$\langle b+cd-b(cd+\overline{\mathrm{sg}}(1-c)),$ $(ab+\overline{\mathrm{sg}}(1-a))(d(cd+\overline{\mathrm{sg}}(1-c))+\overline{\mathrm{sg}}(1-d))\rangle$
$\rightarrow_{114}$	$\langle 1-a+\min(\overline{\mathrm{sg}}(1-a), 1-d), a(1-a)+\min(\overline{\mathrm{sg}}(1-a), d)\rangle$
$\rightarrow_{115}$	$\langle 1-\min(a, d), \min(a(1-a)+\overline{\mathrm{sg}}(1-a), d)\rangle$
$\rightarrow_{116}$	$\langle\max(1-a, (1-d)d+\overline{\mathrm{sg}}(d)),$ $\min(a(1-a)+\overline{\mathrm{sg}}(1-a), d((1-d)d+\overline{\mathrm{sg}}(d))+\overline{\mathrm{sg}}(1-d))\rangle$
$\rightarrow_{117}$	$\langle 1-a-d+ad, (a(1-a)+\overline{\mathrm{sg}}(1-a))d\rangle$
$\rightarrow_{118}$	$\langle 1-a+(1-d)d-(1-a)((1-d)d+\overline{\mathrm{sg}}(d)),$ $(a(1-a)+\overline{\mathrm{sg}}(1-a))d((1-d)d+\overline{\mathrm{sg}}(d))+\overline{\mathrm{sg}}(1-d)\rangle$
$\rightarrow_{119}$	$\langle b+\min(\overline{\mathrm{sg}}(b), c), (1-b)b+\min(\overline{\mathrm{sg}}(b), 1-c)\rangle$
$\rightarrow_{120}$	$\langle\max(b, c), \min((1-b)b+\overline{\mathrm{sg}}(b), 1-c)\rangle$
$\rightarrow_{121}$	$\langle\max(b, c(1-c)+\overline{\mathrm{sg}}(1-c)),$ $\min((1-b)b+\overline{\mathrm{sg}}(b), (1-c)(c(1-c)+\overline{\mathrm{sg}}(1-c)))+\overline{\mathrm{sg}}(c)\rangle$
$\rightarrow_{122}$	$\langle b+c-bc, ((1-c)b+\overline{\mathrm{sg}}(b))(1-c)\rangle$
$\rightarrow_{123}$	$\langle b+c(1-c)-(b(c(1-c)+\overline{\mathrm{sg}}(1-c))),$ $((1-b)b+\overline{\mathrm{sg}}(b))(((1-c)(c(1-c)+\overline{\mathrm{sg}}(1-c)))+\overline{\mathrm{sg}}(c))\rangle$
$\rightarrow_{124}$	$\langle c+\min(\overline{\mathrm{sg}}(1-d), b), cd+\min(\overline{\mathrm{sg}}(1-d), a)\rangle$
$\rightarrow_{125}$	$\langle\max(b, c), \min(cd+\overline{\mathrm{sg}}(1-d), a)\rangle$
$\rightarrow_{126}$	$\langle\max(c, ab+\overline{\mathrm{sg}}(1-b)),$ $\min(cd+\overline{\mathrm{sg}}(1-d), a(ab+\overline{\mathrm{sg}}(1-b))+\overline{\mathrm{sg}}(1-a))\rangle$
$\rightarrow_{127}$	$\langle b+c-bc, (cd+\overline{\mathrm{sg}}(1-d))a\rangle$
$\rightarrow_{128}$	$\langle c+ab-c(ab+\overline{\mathrm{sg}}(1-b)),$ $(cd+\overline{\mathrm{sg}}(1-d))(a(ab+\overline{\mathrm{sg}}(1-b))+\overline{\mathrm{sg}}(1-a))\rangle$
$\rightarrow_{129}$	$\langle 1-d+\min(\overline{\mathrm{sg}}(1-d), 1-a), d(1-d)+\min(\overline{\mathrm{sg}}(1-d), a)\rangle$
$\rightarrow_{130}$	$\langle 1-\min(d, a), \min(d(1-d)+\overline{\mathrm{sg}}(1-d), a)\rangle$
$\rightarrow_{131}$	$\langle\max(1-d, (1-a)a+\overline{\mathrm{sg}}(a)),$ $\min(d(1-d)+\overline{\mathrm{sg}}(1-d), a((1-a)a+\overline{\mathrm{sg}}(a))+\overline{\mathrm{sg}}(1-a))\rangle$
$\rightarrow_{132}$	$\langle 1-ad, (d(1-d)+\overline{\mathrm{sg}}(1-d))a\rangle$

(continued)

Table 1.1 (continued)

$\rightarrow_{133}$	$\langle 1-d+(1-a)a-(1-d)((1-a)a+\overline{\text{sg}}(a)),$ $(d(1-d)+\overline{\text{sg}}(1-d))(a((1-a)a+\overline{\text{sg}}(a))+\overline{\text{sg}}(1-a))\rangle$
$\rightarrow_{134}$	$\langle c+\min(\overline{\text{sg}}(c), b), (1-c)c+\min(\overline{\text{sg}}(c), (1-b))\rangle$
$\rightarrow_{135}$	$\langle \max(b, c), \min((1-c)c+\overline{\text{sg}}(c), 1-b)\rangle$
$\rightarrow_{136}$	$\langle \max(c, b(1-b)+\overline{\text{sg}}(1-b)),$ $\min((1-c)c+\overline{\text{sg}}(c), (1-b)(b(1-b)+\overline{\text{sg}}(1-b))+\overline{\text{sg}}(b))\rangle$
$\rightarrow_{137}$	$\langle b+c-bc, ((1-c)c+\overline{\text{sg}}(c))(1-b)\rangle$
$\rightarrow_{138}$	$\langle c+b(1-b)-c(b(1-b)+\overline{\text{sg}}(1-b)),$ $((1-c)c+\overline{\text{sg}}(c))((1-b)(b(1-b)+\overline{\text{sg}}(1-b))+\overline{\text{sg}}(b))\rangle$
$\rightarrow_{139}$	$\langle \frac{b+c}{2}, \frac{a+d}{2}\rangle$
$\rightarrow_{140}$	$\langle \frac{b+c+\min(b,c)}{3}, \frac{a+d+\max(a,d)}{3}\rangle$
$\rightarrow_{141}$	$\langle \frac{b+c+\max(b,c)}{3}, \frac{a+d+\min(a,d)}{3}\rangle$
$\rightarrow_{142}$	$\langle \frac{3-a-d-\max(a,d)}{3}, \frac{a+d+\max(a,d)}{3}\rangle$
$\rightarrow_{143}$	$\langle \frac{1-a+c+\min(1-a,c)}{3}, \frac{2+a-c-\min(1-a,c)}{3}\rangle$
$\rightarrow_{144}$	$\langle \frac{1+b-d+\min(b,1-d)}{3}, \frac{2-b+d-\min(b,1-d)}{3}\rangle$
$\rightarrow_{145}$	$\langle \frac{b+c+\min(b,c)}{3}, \frac{3-b-c-\min(b,c)}{3}\rangle$
$\rightarrow_{146}$	$\langle \frac{3-a-d-\min(a,d)}{3}, \frac{a+d+\min(a,d)}{3}\rangle$
$\rightarrow_{147}$	$\langle \frac{1-a+c+\max(1-a,c)}{3}, \frac{2+a-c-\max(1-a,c)}{3}\rangle$
$\rightarrow_{148}$	$\langle \frac{1+b-d+\max(b,1-d)}{3}, \frac{2-b+d-\max(b,1-d)}{3}\rangle$
$\rightarrow_{149}$	$\langle \frac{b+c+\max(b,c)}{3}, \frac{3-b-c-\max(b,c)}{3}$
$\rightarrow_{150,\lambda}$	$\langle \frac{b+c+\lambda-1}{2\lambda}, \frac{a+d+\lambda-1}{2\lambda}$, where $\lambda \geq 1$
$\rightarrow_{151,\gamma}$	$\langle \frac{b+c+\gamma}{2\gamma+1}, \frac{a+d+\gamma-1}{2\gamma+1}$, where $\gamma \geq 1$
$\rightarrow_{152,\alpha,\beta}$	$\langle \frac{b+c+\alpha-1}{\alpha+\beta}, \frac{a+d+\beta-1}{\alpha+\beta}$ where $\alpha \geq 1, \beta \in [1, \alpha]$
$\rightarrow_{153,\varepsilon,\eta}$	$\langle \min(1, \max(c, b+\varepsilon)), \max(0, \min(d, a-\eta))\rangle$ where $\varepsilon, \eta \in [0, 1]$ and $\varepsilon \leq \eta < 1$
$\rightarrow_{154,\lambda}$	$\langle \frac{-a+c+\lambda}{2\lambda}, \frac{a-c+\lambda}{2\lambda}\rangle$, where $\lambda \geq 1$
$\rightarrow_{155,\lambda}$	$\langle \frac{1-a-d+\lambda}{2\lambda}, \frac{a+d+\lambda-1}{2\lambda}\rangle$, where $\lambda \geq 1$

(continued)

Table 1.1 (continued)

$\rightarrow_{156,\lambda}$	$\langle \frac{b+c+\lambda-1}{2\lambda}, \frac{1-b-c+\lambda}{2\lambda} \rangle$, where $\lambda \geq 1$
$\rightarrow_{157,\lambda}$	$\langle \frac{b-d+\lambda}{2\lambda}, \frac{-b+d+\lambda}{2\lambda} \rangle$, where $\lambda \geq 1$
$\rightarrow_{158,\gamma}$	$\langle \frac{1-a+c+\gamma}{2\gamma+1}, \frac{a-c+\gamma}{2\gamma+1} \rangle$, where $\gamma \geq 1$
$\rightarrow_{159,\gamma}$	$\langle \frac{2-a-d+\gamma}{2\gamma+1}, \frac{a+d+\gamma-1}{2\gamma+1} \rangle$, where $\gamma \geq 1$
$\rightarrow_{160,\gamma}$	$\langle \frac{b-d+\gamma+1}{2\gamma+1}, \frac{-b+d+\gamma}{2\gamma+1} \rangle$, where $\gamma \geq 1$
$\rightarrow_{161,\gamma}$	$\langle \frac{b+c+\gamma}{2\gamma+1}, \frac{1-b-c+\gamma}{2\gamma+1} \rangle$, where $\gamma \geq 1$
$\rightarrow_{162,\alpha,\beta}$	$\langle \frac{-a+c+\alpha}{\alpha+\beta}, \frac{a-c+\beta}{\alpha+\beta} \rangle$, where $\alpha \geq 1, \beta \in [1, \alpha]$
$\rightarrow_{163,\alpha,\beta}$	$\langle \frac{1-a-d+\alpha}{\alpha+\beta}, \frac{a+d+\beta-1}{\alpha+\beta} \rangle$, where $\alpha \geq 1, \beta \in [1, \alpha]$
$\rightarrow_{164,\alpha,\beta}$	$\langle \frac{b-d+\alpha}{\alpha+\beta}, \frac{-b+d+\beta}{\alpha+\beta} \rangle$, where $\alpha \geq 1, \beta \in [1, \alpha]$
$\rightarrow_{165,\alpha,\beta}$	$\langle \frac{b+c+\alpha-1}{\alpha+\beta}, \frac{1-b-c+\beta}{\alpha+\beta} \rangle$, where $\alpha \geq 1, \beta \in [1, \alpha]$
$\rightarrow_{166}$	$\langle \max(b, \min(a, c)), \min(a, \max(b, d)) \rangle$
$\rightarrow_{167}$	$\langle \max(1-a, \min(a, c)), \min(a, 1-\min(a, c)) \rangle$
$\rightarrow_{168}$	$\langle \max(1-a, \min(a, 1-d)), 1-\max(1-a, \min(a, 1-d)) \rangle$
$\rightarrow_{169}$	$\langle \max(b, \min(1-b, c)), 1-\max(b, \min(1-b, c)) \rangle$
$\rightarrow_{170}$	$\langle \max(b, \min(1-b, 1-d)), 1-\max(b, \min(1-b, 1-d)) \rangle$
$\rightarrow_{171}$	$\langle \overline{\text{sg}}(\max(a, d)-\max(b, c)), \text{sg}(\max(a, d)-\max(b, c)) \rangle$
$\rightarrow_{172}$	$\langle \overline{\text{sg}}(a-c), \text{sg}(a-c) \rangle$
$\rightarrow_{173}$	$\langle \overline{\text{sg}}(a+d-1), \text{sg}(a+d-1) \rangle$
$\rightarrow_{174}$	$\langle \overline{\text{sg}}(1-b-c), \text{sg}(1-b-c) \rangle$
$\rightarrow_{175}$	$\langle \overline{\text{sg}}(d-b), \text{sg}(d-b) \rangle$
$\rightarrow_{176}$	$\langle \overline{\text{sg}}(a-c)+\text{sg}(a-c)\max(b, c), \text{sg}(a-c)\min(a, d) \rangle$
$\rightarrow_{177}$	$\langle \overline{\text{sg}}(a-c)+\text{sg}(a-c)\max(1-a, c), \text{sg}(a-c)\min(a, 1-c) \rangle$
$\rightarrow_{178}$	$\langle \overline{\text{sg}}(a-1+d)+\text{sg}(a-1+d)(1-\min(a, d)),$ $\text{sg}(a-1+d)\min(a, d) \rangle$
$\rightarrow_{179}$	$\langle \overline{\text{sg}}(1-b-c)+\text{sg}(1-b-c)\max(b, c),$ $\text{sg}(1-b-c)(1-\max(b, c)) \rangle$
$\rightarrow_{180}$	$\langle \overline{\text{sg}}(d-b)+\text{sg}(d-b)\max(b, 1-d), \text{sg}(d-b)\min(1-b, d) \rangle$
$\rightarrow_{181}$	$\langle 1-\text{sg}(a).(1-c), d.\text{sg}(a) \rangle$
$\rightarrow_{182}$	$\langle 1-\text{sg}(a).(1-c), (1-c).\text{sg}(a) \rangle$
$\rightarrow_{183}$	$\langle 1-\text{sg}(a).d, d.\text{sg}(a) \rangle$
$\rightarrow_{184}$	$\langle 1-\text{sg}(1-b).d, d.\text{sg}(1-b) \rangle$
$\rightarrow_{185}$	$\langle 1-\text{sg}(1-b).(1-c), (1-c).\text{sg}(1-b) \rangle$

$$\langle 0,1\rangle \to_i \langle 0,1\rangle = \begin{cases} \langle 1,0\rangle, & \text{for } i = 1,\ldots,99,102,105,108,\ldots,127,\\ & 129,\ldots,132,134,\ldots,137,153,166,\ldots,185\\ \langle 0,0\rangle, & \text{for } i = 100,101,103,104,106,107,128,\\ & 133,138\\ \langle \frac{1}{2},\frac{1}{2}\rangle, & \text{for } i = 139\\ \langle \frac{1}{3},\frac{2}{3}\rangle, & \text{for } i = 140,142,\ldots,145\\ \langle \frac{2}{3},\frac{1}{3}\rangle, & \text{for } i = 141,146,\ldots,149\\ \langle \frac{\lambda+1}{2\lambda},\frac{\lambda-1}{2\lambda}\rangle, & \text{for } i = 150,154,\ldots,157\\ \langle \frac{\gamma+2}{2\gamma+1},\frac{\gamma-1}{2\gamma+1}\rangle, & \text{for } i = 151,158,\ldots,161\\ \langle \frac{\alpha+1}{\alpha+\beta},\frac{\beta-1}{\alpha+\beta}\rangle, & \text{for } i = 152,162,\ldots,165. \end{cases}$$

$$\langle 0,1\rangle \to_i \langle 1,0\rangle = \begin{cases} \langle 1,0\rangle, & \text{for } i = 1,\ldots,100,102,103,105,106,108,\\ & \ldots,112,114,\ldots,117,\ldots,121,123,\ldots,126,\\ & 128,\ldots,132,134,\ldots,137,139,\ldots,149,153,\\ & 166,\ldots,185\\ \langle 0,0\rangle, & \text{for } i = 101,104,107,113,118,122,127,\\ & 133,138\\ \langle \frac{\lambda+1}{2\lambda},\frac{\lambda-1}{2\lambda}\rangle, & \text{for } i = 150,154,\ldots,157\\ \langle \frac{\gamma+2}{2\gamma+1},\frac{\gamma-1}{2\gamma+1}\rangle, & \text{for } i = 151,158,\ldots,161\\ \langle \frac{\alpha+1}{\alpha+\beta},\frac{\beta-1}{\alpha+\beta}\rangle, & \text{for } i = 152,162,\ldots,165. \end{cases}$$

$$\langle 1,0\rangle \to_i \langle 0,1\rangle = \begin{cases} \langle 0,1\rangle, & \text{for } i = 1,\ldots,99,109,\ldots,149,166,\ldots,168,\\ & 170,\ldots,185\\ \langle 0,0\rangle, & \text{for } i = 100,\ldots,108,169\\ \langle \frac{\lambda-1}{2\lambda},\frac{\lambda+1}{2\lambda}\rangle, & \text{for } i = 150,154,\ldots,157\\ \langle \frac{\gamma}{2\gamma+1},\frac{\gamma+1}{2\gamma+1}\rangle, & \text{for } i = 151,158,\ldots,161\\ \langle \frac{\alpha-1}{\alpha+\beta},\frac{\beta+1}{\alpha+\beta}\rangle, & \text{for } i = 152,162,\ldots,165\\ \langle \varepsilon,1-\eta\rangle, & \text{for } i = 153. \end{cases}$$

$$\langle 1, 0\rangle \to_i \langle 1, 0\rangle = \begin{cases} \langle 1, 0\rangle, & \text{for } i = 1, \dots, 100, 103, 106, 109, \dots, 112,\\ & 114, \dots, 117, 119, \dots, 122, 124, \dots, 138,\\ & 153, 166, \dots, 185\\ \langle 0, 0\rangle, & \text{for } i = 101, 102, 104, 105, 107, 108, 113, 123\\ \langle 0, \frac{1}{2}\rangle, & \text{for } i = 139, 150, 155, \dots, 157\\ \langle \frac{1}{3}, \frac{2}{3}\rangle, & \text{for } i = 140, 142, \dots, 145\\ \langle \frac{2}{3}, \frac{1}{3}\rangle, & \text{for } i = 141, 146, \dots, 149\\ \langle \frac{\gamma+1}{2\gamma+1}, \frac{\gamma}{2\gamma+1}\rangle, & \text{for } i = 151, 158, \dots, 161\\ \langle \frac{\alpha}{\alpha+\beta}, \frac{\beta}{\alpha+\beta}\rangle, & \text{for } i = 152, 162, \dots, 165. \end{cases}$$

It is interesting to mention that the well-known axiom of the classical logic $A \to A$ is an IFT for implications $\to_1, \dots, \to_9, \to_{11}, \dots, \to_{15}, \to_{17}, \to_{18}, \to_{20}, \dots, \to_{24}, \to_{27}, \dots, \to_{38}, \to_{40}, \to_{42}, \to_{44}, \dots, \to_{53}, \to_{55}, \to_{57}, \to_{59}, \dots, \to_{66}, \to_{68}, \to_{69}, \to_{71}, \to_{72}, \to_{74}, \dots, \to_{77}, \to_{79}, \dots, \to_{85}, \to_{88}, \dots, \to_{94}, \to_{97}, \dots, \to_{139}, \to_{141}, \to_{146}, \dots, \to_{170}, \to_{176}, \dots, \to_{185}$, while it is just a tautology for implications $\to_2, \to_3, \to_5, \to_8, \to_{11}, \to_{14}, \to_{15}, \to_{20}, \to_{23}, \to_{24}, \to_{27}, \to_{31}, \to_{32}, \to_{34}, \to_{37}, \to_{40}, \to_{42}, \to_{47}, \dots, \to_{49}, \to_{52}, \to_{55}, \to_{57}, \to_{62}, \to_{63}, \to_{65}, \to_{68}, \to_{69}, \to_{74}, \to_{77}, \to_{79}, \to_{83}, \to_{84}, \to_{88}, \to_{92}, \to_{93}, \to_{97}, \to_{176}, \dots, \to_{185}$.

The intuitionistic fuzzy implications that satisfy the following equalities

$$\langle 0, 1\rangle \to_i \langle 0, 1\rangle = \langle 1, 0\rangle,$$

$$\langle 0, 1\rangle \to_i \langle 1, 0\rangle = \langle 1, 0\rangle,$$

$$\langle 1, 0\rangle \to_i \langle 0, 1\rangle = \langle 0, 1\rangle,$$

$$\langle 1, 0\rangle \to_i \langle 1, 0\rangle = \langle 1, 0\rangle,$$

as standard tautologies, will be called "implications of (fully) tautological type" (T-implications), while the implications that satisfy these equalities as IFTs, will be called "implications from IFT type" (I-implications), and the rest are incorrect and will be denoted as N-implications.

1.4 Intuitionistic Fuzzy Negations

The currently existing intuitionistic fuzzy implications generate the intuitionistic fuzzy negations [57, 59–62] using the formula

$$\neg\langle a, b\rangle = \langle a, b\rangle \to \langle 0, 1\rangle,$$

where $\to$ is any of the defined implications. Paper [62] is written together with D. Dimitrov and paper [60] – with N. Angelova. The full list of different negations is shown in Table 1.2. As it can be seen, a lot of different implications generate exactly one negation.

The relationships between the negations and implications are shown in Table 1.3.

1.5 Properties of Intuitionistic Fuzzy Implications and Negations

Here, we study some interesting properties of the intuitionistic fuzzy implications and negations. A part of them are published in [63].

For the well-known formula

$$(A \to B) \vee (B \to A), \tag{1.5.1}$$

in [46] the following two theorems are proved. Here, they are extended with check of the properties of the implications $\to_{153}, \ldots, \to_{185}$ (given after the symbol *).

Theorem 1.5.1 *For every two formulas A and B, (1.5.1) is an IFT for the intuitionistic fuzzy implications* $\to_1$, ..., $\to_6$, $\to_8$, $\to_9$, $\to_{11}$, $\to_{13}$, $\to_{14}$, $\to_{17}$, $\to_{18}$, $\to_{20}$, ..., $\to_{24}$, $\to_{27}$, ..., $\to_{38}$, $\to_{40}$, $\to_{42}$, $\to_{44}$, $\to_{45}$, $\to_{61}$, ..., $\to_{66}$, $\to_{68}$, $\to_{69}$, $\to_{71}$, $\to_{72}$, $\to_{74}$, ..., $\to_{77}$, $\to_{79}$, ..., $\to_{85}$, $\to_{88}$, ..., $\to_{90}$, $\to_{100}$, ..., $\to_{105}$, $\to_{109}$, ..., $\to_{118}$, $\to_{124}$, ..., $\to_{133}$, * $\to_{139}$, $\to_{141}$, $\to_{146}$, ..., $\to_{148}$, $\to_{150}$, ..., $\to_{155}$, $\to_{157}$, ..., $\to_{160}$, $\to_{162}$, ..., $\to_{164}$, $\to_{166}$, ..., $\to_{170}$, $\to_{176}$, ..., $\to_{178}$, $\to_{180}$, ..., $\to_{183}$, $\to_{185}$.

Theorem 1.5.2 *For every two formulas A and B, (1.5.1) is a tautology for the intuitionistic fuzzy implications* $\to_2$, $\to_3$, $\to_8$, $\to_{11}$, $\to_{20}$, $\to_{23}$, $\to_{31}$, $\to_{32}$, $\to_{34}$, $\to_{37}$, $\to_{40}$, $\to_{42}$, $\to_{62}$, $\to_{63}$, $\to_{65}$, $\to_{68}$, $\to_{74}$, $\to_{77}$, $\to_{83}$, $\to_{84}$, $\to_{88}$, * $\to_{153}$, $\to_{176}$, ..., $\to_{178}$, $\to_{180}$, ..., $\to_{183}$, $\to_{185}$.

G.F. Rose's formula [64, 65] has the form:

$$((\neg\neg A \to A) \to (\neg\neg A \vee \neg A)) \to (\neg\neg A \vee \neg A). \tag{1.5.2}$$

For it, the following two theorems are valid.

Theorem 1.5.3 *For each formula A, (1.5.2) is an IFT for the intuitionistic fuzzy implications* $\to_1, \ldots, \to_9$, $\to_{11}, \ldots, \to_{38}$, $\to_{40}, \ldots, \to_{53}$, $\to_{55}, \ldots, \to_{57}$, $\to_{61}$, $\to_{62}$, $\to_{64}, \ldots, \to_{67}$, $\to_{71}$, $\to_{72}$, $\to_{74}, \ldots, \to_{77}$, $\to_{79}, \ldots, \to_{83}$, $\to_{85}$, $\to_{86}$, $\to_{88}$, $\ldots$, $\to_{91}$, $\to_{94}$, $\to_{95}$, $\to_{97}, \ldots, \to_{107}$, $\to_{109}, \ldots, \to_{137}$, $\to_{151}$, $\to_{153}$, $\to_{158}$, $\ldots, \to_{161}$, $\to_{166}, \ldots, \to_{185}$.

Theorem 1.5.4 *For each formula A, (1.5.2) is a tautology for the intuitionistic fuzzy implications* $\to_2$, $\to_3$, $\to_8$, $\to_{11}$, $\to_{14}, \ldots, \to_{16}$, $\to_{19}$, $\to_{20}$, $\to_{23}$, $\to_{24}$, $\to_{31}$, $\to_{32}$, $\to_{37}$, $\to_{40}, \ldots, \to_{45}$, $\to_{47}$, $\to_{48}$, $\to_{55}, \ldots, \to_{57}$, $\to_{62}$, $\to_{65}$, $\to_{74}$, $\to_{77}$, $\to_{83}$, $\to_{88}$, $\to_{90}$, $\to_{97}$, $\to_{99}$, $\to_{153}$, $\to_{171}$, $\to_{180}$.

Table 1.2 List of the intuitionistic fuzzy negations

$\neg_1$	$\langle b, a\rangle$
$\neg_2$	$\langle \overline{\text{sg}}(a), \text{sg}(a)\rangle$
$\neg_3$	$\langle b, a.b + a^2\rangle$
$\neg_4$	$\langle b, 1 - b\rangle$
$\neg_5$	$\langle \overline{\text{sg}}(1 - b), \text{sg}(1 - b)\rangle$
$\neg_6$	$\langle \overline{\text{sg}}(1 - b), \text{sg}(a)\rangle$
$\neg_7$	$\langle \overline{\text{sg}}(1 - b), a\rangle$
$\neg_8$	$\langle 1 - a, a\rangle$
$\neg_9$	$\langle \overline{\text{sg}}(a), a\rangle$
$\neg_{10}$	$\langle \overline{\text{sg}}(1 - b), 1 - b\rangle$
$\neg_{11}$	$\langle \text{sg}(b), \overline{\text{sg}}(b)\rangle$
$\neg_{12}$	$\langle b.(b + a), a.(b^2 + a + b.a)\rangle$
$\neg_{13}$	$\langle \text{sg}(1 - a), \overline{\text{sg}}(1 - a)\rangle$
$\neg_{14}$	$\langle \text{sg}(b), \overline{\text{sg}}(1 - a)\rangle$
$\neg_{15}$	$\langle \overline{\text{sg}}(1 - b), \overline{\text{sg}}(1 - a)\rangle$
$\neg_{16}$	$\langle \overline{\text{sg}}(a), \overline{\text{sg}}(1 - a)\rangle$
$\neg_{17}$	$\langle \overline{\text{sg}}(1 - b), \overline{\text{sg}}(b)\rangle$
$\neg_{18}$	$\langle b.\text{sg}(a), a.\text{sg}(b)\rangle$
$\neg_{19}$	$\langle b.\text{sg}(a), 0\rangle$
$\neg_{20}$	$\langle b, 0\rangle$
$\neg_{21}$	$\langle \min(1 - a, \text{sg}(a)), \min(a, \text{sg}(1 - a))\rangle$
$\neg_{22}$	$\langle \min(1 - a, \text{sg}(a)), 0\rangle$
$\neg_{23}$	$\langle 1 - a, 0\rangle$
$\neg_{24}$	$\langle \min(b, \text{sg}(1 - b)), \min(1 - b, \text{sg}(b))\rangle$
$\neg_{25}$	$\langle \min(b, \text{sg}(1 - b)), 0\rangle$
$\neg_{26}$	$\langle b, a.b + \overline{\text{sg}}(1 - a)\rangle$
$\neg_{27}$	$\langle 1 - a, a.(1 - a) + \overline{\text{sg}}(1 - a)\rangle$
$\neg_{28}$	$\langle b, (1 - b).b + \overline{\text{sg}}(b)\rangle$
$\neg_{29}$	$\langle \max(0, b.a + \overline{\text{sg}}(1 - b)), \min(1, a.(b.a + \overline{\text{sg}}(1 - b)) + \overline{\text{sg}}(1 - a))\rangle$
$\neg_{30}$	$\langle a.b,\ a.(a.b + \overline{\text{sg}}(1 - b)) + \overline{\text{sg}}(1 - a)\rangle$
$\neg_{31}$	$\langle \max(0, (1 - a).a + \overline{\text{sg}}(a)), \min(1, a.((1 - a).a + \overline{\text{sg}}(a)) + \overline{\text{sg}}(1 - a))\rangle$
$\neg_{32}$	$\langle (1 - a).a,\ a.((1 - a).a + \overline{\text{sg}}(a)) + \overline{\text{sg}}(1 - a)\rangle$
$\neg_{33}$	$\langle b.(1 - b) + \overline{\text{sg}}(1 - b), (1 - b).(b.(1 - b) + \overline{\text{sg}}(1 - b)) + \overline{\text{sg}}(b))\rangle$
$\neg_{34}$	$\langle b.(1 - b),\ (1 - b).(b.(1 - b) + \overline{\text{sg}}(1 - b)) + \overline{\text{sg}}(b)\rangle$
$\neg_{35}$	$\langle \frac{b}{2}, \frac{1+a}{2}\rangle$
$\neg_{36}$	$\langle \frac{b}{3}, \frac{2+a}{3}\rangle$
$\neg_{37}$	$\langle \frac{2b}{3}, \frac{2a+1}{3}\rangle$
$\neg_{38}$	$\langle \frac{1-a}{3}, \frac{2+a}{3}\rangle$

(continued)

Table 1.2 (continued)

$\neg_{39}$	$\langle \frac{b}{3}, \frac{3-b}{3} \rangle$
$\neg_{40}$	$\langle \frac{2-2a}{3}, \frac{1+2a}{3} \rangle$
$\neg_{41}$	$\langle \frac{2b}{3}, \frac{3-2b}{3} \rangle$
$\neg_{42,\lambda}$	$\langle \frac{b+\lambda-1}{2\lambda}, \frac{a+\lambda}{2\lambda} \rangle$, where $\lambda \geq 1$
$\neg_{43,\gamma}$	$\langle \frac{b+\gamma}{2\gamma+1}, \frac{a+\gamma}{2\gamma+1} \rangle$, where $\gamma \geq 1$
$\neg_{44,\alpha,\beta}$	$\langle \frac{b+\alpha-1}{\alpha+\beta}, \frac{a+\beta}{\alpha+\beta} \rangle$, where $\alpha \geq 1, \beta \in [1, \alpha]$
$\neg_{45,\varepsilon,\eta}$	$\langle \min(1, b+\varepsilon), \max(0, a-\eta) \rangle$, where $\varepsilon, \eta \in [0, 1]$ and $\varepsilon \leq \eta < 1$
$\neg_{46,\lambda}$	$\langle \frac{\lambda-a}{2\lambda}, \frac{a+\lambda}{2\lambda} \rangle$, where $\lambda \geq 1$
$\neg_{47,\lambda}$	$\langle \frac{b+\lambda-1}{2\lambda}, \frac{1-b+\lambda}{2\lambda} \rangle$, where $\lambda \geq 1$
$\neg_{48,\gamma}$	$\langle \frac{1-a+\gamma}{2\gamma+1}, \frac{a+\gamma}{2\gamma+1} \rangle$, where $\gamma \geq 1$
$\neg_{49,\gamma}$	$\langle \frac{b+\gamma}{2\gamma+1}, \frac{1-b+\gamma}{2\gamma+1} \rangle$, where $\gamma \geq 1$
$\neg_{50,\alpha,\beta}$	$\langle \frac{b-1+\alpha}{\alpha+\beta}, \frac{a+\beta}{\alpha+\beta} \rangle$, where $\alpha \geq 1, \beta \in [1, \alpha]$
$\neg_{51,\alpha,\beta}$	$\langle \frac{b-1+\alpha}{\alpha+\beta}, \frac{1-b+\beta}{\alpha+\beta} \rangle$, where $\alpha \geq 1, \beta \in [1, \alpha]$
$\neg_{52}$	$\langle 1-a, \min(1, 1-a) \rangle$
$\neg_{53}$	$\langle \overline{\text{sg}}(a) + \text{sg}(a)b, a \rangle$

Now, we discuss the following new formulas, inspired by (1.5.2):

$$(A \vee \neg A) \rightarrow (A \rightarrow \neg\neg A) \tag{1.5.3}$$

$$(\neg\neg A \vee \neg A) \rightarrow (A \rightarrow \neg\neg A) \tag{1.5.4}$$

$$(A \rightarrow \neg\neg A) \rightarrow (A \vee \neg A) \tag{1.5.5}$$

$$(A \rightarrow \neg\neg A) \rightarrow (\neg\neg A \vee \neg A) \tag{1.5.6}$$

Obviously, in the classical propositional calculus, all these four formulas are tautologies. Now, we study their properties in the intuitionistic fuzzy case.

First, we mention that there are intuitionistic fuzzy implications for which

$$V((A \vee \neg A) \rightarrow (A \rightarrow \neg\neg A)) = V((\neg\neg A \vee \neg A) \rightarrow (A \rightarrow \neg\neg A)) \tag{1.5.7}$$

and others, for which (1.5.7) is not valid.

For example, we see that

$$V((A \vee \neg_8 A) \rightarrow_{34} (A \rightarrow_{34} \neg_8\neg_8 A))$$

Table 1.3 Relationships between negations and implications

$\neg_1$	$\rightarrow_1$, $\rightarrow_4$, $\rightarrow_5$, $\rightarrow_6$, $\rightarrow_7$, $\rightarrow_{10}$, $\rightarrow_{13}$, $\rightarrow_{61}$, $\rightarrow_{63}$, $\rightarrow_{64}$, $\rightarrow_{66}$, $\rightarrow_{67}$, $\rightarrow_{68}$, $\rightarrow_{69}$, $\rightarrow_{70}$, $\rightarrow_{71}$, $\rightarrow_{72}$, $\rightarrow_{73}$, $\rightarrow_{78}$, $\rightarrow_{80}$, $\rightarrow_{124}$, $\rightarrow_{125}$, $\rightarrow_{127}$, $\rightarrow_{166}$
$\neg_2$	$\rightarrow_2$, $\rightarrow_3$, $\rightarrow_8$, $\rightarrow_{11}$, $\rightarrow_{16}$, $\rightarrow_{20}$, $\rightarrow_{31}$, $\rightarrow_{32}$, $\rightarrow_{37}$, $\rightarrow_{40}$, $\rightarrow_{41}$, $\rightarrow_{42}$ $\rightarrow_{172}$, $\rightarrow_{173}$, $\rightarrow_{181}$, $\rightarrow_{182}$, $\rightarrow_{183}$
$\neg_3$	$\rightarrow_9$, $\rightarrow_{17}$, $\rightarrow_{21}$
$\neg_4$	$\rightarrow_{12}$, $\rightarrow_{18}$, $\rightarrow_{22}$, $\rightarrow_{46}$, $\rightarrow_{49}$, $\rightarrow_{50}$, $\rightarrow_{51}$, $\rightarrow_{53}$, $\rightarrow_{54}$, $\rightarrow_{91}$, $\rightarrow_{93}$, $\rightarrow_{94}$, $\rightarrow_{95}$, $\rightarrow_{96}$, $\rightarrow_{98}$, $\rightarrow_{134}$, $\rightarrow_{135}$, $\rightarrow_{137}$, $\rightarrow_{169}$, $\rightarrow_{170}$, $\rightarrow_{179}$, $\rightarrow_{180}$
$\neg_5$	$\rightarrow_{14}$, $\rightarrow_{15}$, $\rightarrow_{19}$, $\rightarrow_{23}$, $\rightarrow_{47}$, $\rightarrow_{48}$, $\rightarrow_{52}$, $\rightarrow_{55}$, $\rightarrow_{56}$, $\rightarrow_{57}$, $\rightarrow_{171}$, $\rightarrow_{174}$, $\rightarrow_{175}$, $\rightarrow_{184}$, $\rightarrow_{185}$
$\neg_6$	$\rightarrow_{24}$, $\rightarrow_{26}$, $\rightarrow_{27}$, $\rightarrow_{65}$
$\neg_7$	$\rightarrow_{25}$, $\rightarrow_{28}$, $\rightarrow_{29}$, $\rightarrow_{62}$
$\neg_8$	$\rightarrow_{30}$, $\rightarrow_{33}$, $\rightarrow_{34}$, $\rightarrow_{35}$, $\rightarrow_{36}$, $\rightarrow_{38}$, $\rightarrow_{39}$, $\rightarrow_{76}$, $\rightarrow_{82}$, $\rightarrow_{84}$, $\rightarrow_{85}$, $\rightarrow_{86}$, $\rightarrow_{87}$, $\rightarrow_{89}$, $\rightarrow_{129}$, $\rightarrow_{130}$, $\rightarrow_{132}$, $\rightarrow_{167}$, $\rightarrow_{168}$, $\rightarrow_{177}$, $\rightarrow_{178}$
$\neg_9$	$\rightarrow_{43}$, $\rightarrow_{44}$, $\rightarrow_{45}$, $\rightarrow_{83}$
$\neg_{10}$	$\rightarrow_{58}$, $\rightarrow_{59}$, $\rightarrow_{60}$, $\rightarrow_{92}$
$\neg_{11}$	$\rightarrow_{74}$, $\rightarrow_{97}$
$\neg_{12}$	$\rightarrow_{75}$
$\neg_{13}$	$\rightarrow_{77}$, $\rightarrow_{88}$
$\neg_{14}$	$\rightarrow_{79}$
$\neg_{15}$	$\rightarrow_{81}$
$\neg_{16}$	$\rightarrow_{90}$
$\neg_{17}$	$\rightarrow_{99}$
$\neg_{18}$	$\rightarrow_{100}$
$\neg_{19}$	$\rightarrow_{101}$
$\neg_{20}$	$\rightarrow_{102}$, $\rightarrow_{108}$
$\neg_{21}$	$\rightarrow_{103}$
$\neg_{22}$	$\rightarrow_{104}$
$\neg_{23}$	$\rightarrow_{105}$
$\neg_{24}$	$\rightarrow_{106}$
$\neg_{25}$	$\rightarrow_{107}$
$\neg_{26}$	$\rightarrow_{109}$, $\rightarrow_{110}$, $\rightarrow_{111}$, $\rightarrow_{112}$, $\rightarrow_{113}$
$\neg_{27}$	$\rightarrow_{114}$, $\rightarrow_{115}$, $\rightarrow_{116}$, $\rightarrow_{117}$, $\rightarrow_{118}$
$\neg_{28}$	$\rightarrow_{119}$, $\rightarrow_{120}$, $\rightarrow_{121}$, $\rightarrow_{122}$, $\rightarrow_{123}$
$\neg_{29}$	$\rightarrow_{126}$
$\neg_{30}$	$\rightarrow_{128}$
$\neg_{31}$	$\rightarrow_{131}$
$\neg_{32}$	$\rightarrow_{133}$
$\neg_{33}$	$\rightarrow_{136}$
$\neg_{34}$	$\rightarrow_{138}$
$\neg_{35}$	$\rightarrow_{139}$
$\neg_{36}$	$\rightarrow_{140}$

(continued)

Table 1.3 (continued)

$\neg_{37}$	$\rightarrow_{141}$
$\neg_{38}$	$\rightarrow_{142}, \rightarrow_{143}$
$\neg_{39}$	$\rightarrow_{144}, \rightarrow_{145}$
$\neg_{40}$	$\rightarrow_{146}, \rightarrow_{147}$
$\neg_{41}$	$\rightarrow_{148}, \rightarrow_{149}$
$\neg_{42}$	$\rightarrow_{150}$
$\neg_{43}$	$\rightarrow_{151}$
$\neg_{44}$	$\rightarrow_{152}$
$\neg_{45}$	$\rightarrow_{153}$
$\neg_{46,\lambda}$	$\rightarrow_{154,\lambda}, \rightarrow_{155,\lambda}$
$\neg_{47,\lambda}$	$\rightarrow_{156,\lambda}, \rightarrow_{157,\lambda}$
$\neg_{48,\gamma}$	$\rightarrow_{158,\gamma}, \rightarrow_{159,\gamma}$
$\neg_{49,\gamma}$	$\rightarrow_{160,\gamma}, \rightarrow_{161,\gamma}$
$\neg_{50,\alpha,\beta}$	$\rightarrow_{162,\alpha,\beta}, \rightarrow_{163,\alpha,\beta}$
$\neg_{51,\alpha,\beta}$	$\rightarrow_{164,\alpha,\beta}, \rightarrow_{165,\alpha,\beta}$
$\neg_{52}$	$\rightarrow_{167}$
$\neg_{53}$	$\rightarrow_{176}$

$$= (\langle a, b\rangle \vee \neg_8\langle a, b\rangle) \rightarrow_{34} (\langle a, b\rangle \rightarrow_{34} \neg_8\neg_8\langle a, b\rangle)$$

$$= (\langle a, b\rangle \vee \langle 1 - a, a\rangle) \rightarrow_{34} (\langle a, b\rangle \rightarrow_{34} \neg_8\langle 1 - a, a\rangle)$$

$$= \langle \max(a, 1 - a), \min(a, b)\rangle \rightarrow_{34} (\langle a, b\rangle \rightarrow_{34} \langle a, 1 - a\rangle)$$

$$= \langle \max(a, 1 - a), \min(a, b)\rangle \rightarrow_{34} \langle \min(1, 2 - a - 1 + a), \max(0, a + 1 - a - 1)\rangle$$

$$= \langle \max(a, 1 - a), \min(a, b)\rangle \rightarrow_{34} \langle 1, 0\rangle$$

$$= \langle \min(1, 2 - a), \max(0, a + 0 - 1)\rangle = \langle 1, 0\rangle,$$

and (using above calculated results)

$$V((\neg\neg A \vee \neg_8 A) \rightarrow_{34} (A \rightarrow_{34} \neg_8\neg_8 A))$$

$$= (\neg_8\neg_8\langle a, b\rangle \vee \neg_8\langle a, b\rangle) \rightarrow_{34} (\langle a, b\rangle \rightarrow_{34} \neg_8\neg_8\langle a, b\rangle)$$

$$= (\langle a, 1 - a\rangle \vee \langle 1 - a, a\rangle) \rightarrow_{34} \langle 1, 0\rangle$$

$$= \langle \max(a, 1 - a), \min(a, 1 - a)\rangle \rightarrow_{34} \langle 1, 0\rangle$$

$$= \langle \min(1, 2 - \max(a, 1-a)), \max(0, \max(a, 1-a) + 0 - 1)\rangle = \langle 1, 0\rangle,$$

i.e., the two formulas have equal values for implication $\rightarrow_{34}$ and for negation $\neg_8$ generated by it. Therefore, they satisfy (1.5.7).

On the other hand,

$$V((A \vee \neg_{10}A) \rightarrow_{59} (A \rightarrow_{59} \neg_{10}\neg_{10}A))$$

$$= (\langle a, b\rangle \vee \neg_{10}\langle a, b\rangle) \rightarrow_{59} (\langle a, b\rangle \rightarrow_{59} \neg_{10}\neg_{10}\langle a, b\rangle)$$

$$= (\langle a, b\rangle \vee \langle \overline{\text{sg}}(1-b), 1-b\rangle) \rightarrow_{59} (\langle a, b\rangle \rightarrow_{59} \neg_{10}\langle \overline{\text{sg}}(1-b), 1-b\rangle)$$

$$= \langle \max(a, \overline{\text{sg}}(1-b)), \min(b, 1-b)\rangle \rightarrow_{59} (\langle a, b\rangle \rightarrow_{59} \langle \overline{\text{sg}}(b), b\rangle)$$

$$= \langle \max(a, \overline{\text{sg}}(1-b)), \min(b, 1-b)\rangle \rightarrow_{59} \langle \max(\overline{\text{sg}}(1-b), \overline{\text{sg}}(b)), 1 - \max(b, b)\rangle$$

$$= \langle \max(\overline{\text{sg}}(1 - \min(b, 1-b)), \max(\overline{\text{sg}}(1-b), \overline{\text{sg}}(b))),$$

$$1 - \max(\min(b, 1-b), \max(\overline{\text{sg}}(1-b), \overline{\text{sg}}(b)))\rangle.$$

If $b = 1$ or $b = 0$, then

$$V((A \vee \neg_{10}A) \rightarrow_{59} (A \rightarrow_{59} \neg_{10}\neg_{10}A)) = \langle 1, 0\rangle.$$

If $0 < b < 1$, then

$$V((A \vee \neg_{10}A) \rightarrow_{59} (A \rightarrow_{59} \neg_{10}\neg_{10}A))$$

$$= \langle \max(\overline{\text{sg}}(1 - \min(b, 1-b)), 0), 1 - \max(\min(b, 1-b), 0)\rangle = \langle 0, 1\rangle.$$

Also,

$$V(\neg_{10}\neg_{10}A \vee \neg_{10}A) \rightarrow_{59} (A \rightarrow_{59} \neg_{10}\neg_{10}A))$$

$$= (\neg_{10}\langle \overline{\text{sg}}(1-b), 1-b\rangle \vee \langle \overline{\text{sg}}(1-b), 1-b\rangle) \rightarrow_{59} (\langle a, b\rangle \rightarrow_{59} \neg_{10}\langle \overline{\text{sg}}(1-b), 1-b\rangle)$$

$$= (\langle \overline{\text{sg}}(b), b\rangle \vee \langle \overline{\text{sg}}(1-b), 1-b\rangle) \rightarrow_{59} (\langle a, b\rangle \rightarrow_{59} \langle \overline{\text{sg}}(b), b\rangle)$$

$$= \langle \max(\overline{\text{sg}}(b), \overline{\text{sg}}(1-b)), \min(b, 1-b)\rangle \rightarrow_{59} \langle \max(\overline{\text{sg}}(1-b), \overline{\text{sg}}(b)), 1-b\rangle$$

$$= \langle \max(\overline{\text{sg}}(1 - \min(b, 1-b)), \max(\overline{\text{sg}}(1-b), \overline{\text{sg}}(b))),$$

$$1 - \max(\min(b, 1-b), \max(\overline{\text{sg}}(1-b), \overline{\text{sg}}(b)))\rangle$$

If $b = 1$ or $b = 0$, then

$$V((\neg_{10}(\neg_{10}A \vee \neg_{10}A) \rightarrow_{59} (A \rightarrow_{59} \neg_{10}\neg_{10}A)) = \langle 1, 0\rangle.$$

If $0 < b < 1$, then

$$V(\neg_{10}\neg_{10}A \vee \neg_{10}A) \rightarrow_{59} (A \rightarrow_{59} \neg_{10}\neg_{10}A))$$

$$= \langle \max(0, 0), 1 - \max(\min(b, 1 - b), 0)\rangle$$

$$= \langle 0, 1 - \min(b, 1 - b)\rangle.$$

Therefore, both formulas have different values and equality (1.5.7) is not valid.

The same holds for implications $\rightarrow_{58}, \dots, \rightarrow_{60}, \rightarrow_{70}, \rightarrow_{72}, \rightarrow_{73}, \rightarrow_{78}, \rightarrow_{80}, \rightarrow_{87}, \rightarrow_{89}, \rightarrow_{92}, \rightarrow_{96}, \rightarrow_{98}, \rightarrow_{120}, \rightarrow_{140}, \rightarrow_{142}, \dots, \rightarrow_{145}, \rightarrow_{157}, \rightarrow_{162}, \rightarrow_{163}, \rightarrow_{165}$.

On the other hand, formulas (1.5.3) and (1.5.4) have equal behaviour, as illustrated by the following two assertions.

Theorem 1.5.5 *Formula A satisfies (1.5.3) and (1.5.4) as a tautology for the intuitionistic fuzzy implications* $\rightarrow_{2}, \rightarrow_{3}, \rightarrow_{8}, \rightarrow_{11}, \rightarrow_{14}, \dots, \rightarrow_{16}, \rightarrow_{19}, \rightarrow_{20}, \rightarrow_{23}, \rightarrow_{24}, \rightarrow_{31}, \rightarrow_{32}, \rightarrow_{37}, \rightarrow_{40}, \dots, \rightarrow_{45}, \rightarrow_{47}, \rightarrow_{48}, \rightarrow_{52}, \rightarrow_{55}, \dots, \rightarrow_{57}, \rightarrow_{62}, \rightarrow_{65}, \rightarrow_{74}, \rightarrow_{77}, \rightarrow_{83}, \rightarrow_{88}, \rightarrow_{90}, \rightarrow_{97}, \rightarrow_{99}, \rightarrow_{153}, \rightarrow_{171}, \dots, \rightarrow_{180}$.

Theorem 1.5.6 *Formula A satisfies (1.5.3) and (1.5.4) as an IFT for the intuitionistic fuzzy implications* $\rightarrow_{1}, \dots, \rightarrow_{9}, \rightarrow_{11}, \dots, \rightarrow_{38}, \rightarrow_{40}, \dots, \rightarrow_{53}, \rightarrow_{55}, \dots, \rightarrow_{57}, \rightarrow_{61}, \dots, \rightarrow_{66}, \rightarrow_{68}, \rightarrow_{69}, \rightarrow_{71}, \rightarrow_{72}, \rightarrow_{74}, \dots, \rightarrow_{77}, \rightarrow_{79}, \dots, \rightarrow_{85}, \rightarrow_{88}, \dots, \rightarrow_{91}, \rightarrow_{93}, \rightarrow_{94}, \rightarrow_{97}, \rightarrow_{100}, \dots, \rightarrow_{107}, \rightarrow_{109}, \dots, \rightarrow_{121}, \rightarrow_{122}, \rightarrow_{137}, \rightarrow_{151}, \rightarrow_{153}, \rightarrow_{158}, \dots, \rightarrow_{161}, \rightarrow_{166}, \dots, \rightarrow_{185}$.

In all these cases, both formulas satisfy or do not satisfy each of these two formulas.

Analogously, we can check that equality

$$V((A \rightarrow \neg\neg A) \rightarrow (A \vee \neg A)) = V((A \rightarrow \neg\neg A) \rightarrow (\neg\neg A \vee \neg A)) \quad (1.5.8)$$

is not valid for implications $\rightarrow_{70}, \rightarrow_{72}, \rightarrow_{73}, \rightarrow_{78}, \rightarrow_{80}, \rightarrow_{84}, \rightarrow_{98}, \rightarrow_{155}, \rightarrow_{156}, \rightarrow_{163}$.

Now, we check the validity of the following two assertions.

Theorem 1.5.7 *Formula A satisfies (1.5.5) as a tautology for the intuitionistic fuzzy implications* $\rightarrow_{20}, \rightarrow_{23}, \rightarrow_{42}, \rightarrow_{74}, \rightarrow_{77}, \rightarrow_{88}, \rightarrow_{90}, \rightarrow_{153}$.

Theorem 1.5.8 *Formula A satisfies (1.5.5) as an IFT for the intuitionistic fuzzy implications* $\rightarrow_{1}, \rightarrow_{4}, \dots, \rightarrow_{7}, \rightarrow_{9}, \rightarrow_{13}, \rightarrow_{17}, \rightarrow_{19}, \rightarrow_{20}, \dots, \rightarrow_{23}, \rightarrow_{25}, \rightarrow_{27}, \dots, \rightarrow_{30}, \rightarrow_{33}, \dots, \rightarrow_{36}, \rightarrow_{38}, \rightarrow_{42}, \rightarrow_{44}, \rightarrow_{45}, \rightarrow_{61}, \rightarrow_{64}, \rightarrow_{66}, \rightarrow_{67}, \rightarrow_{71}, \rightarrow_{72}, \rightarrow_{74}, \dots, \rightarrow_{77}, \rightarrow_{279}, \dots, \rightarrow_{82}, \rightarrow_{85}, \rightarrow_{86}, \rightarrow_{88}, \dots, \rightarrow_{90}, \rightarrow_{100}, \dots, \rightarrow_{105}, \rightarrow_{117}, \rightarrow_{118}, \rightarrow_{124}, \dots, \rightarrow_{133}, \rightarrow_{151}, \rightarrow_{153}, \rightarrow_{158}, \dots, \rightarrow_{161}, \rightarrow_{166}, \dots, \rightarrow_{170}, \rightarrow_{181}, \dots, \rightarrow_{183}, \rightarrow_{185}$.

In comparison, formulas (1.5.5) and (1.5.6) behave differently. Now, Theorems 1.5.7 and 1.5.8 are changed to the following forms.

Theorem 1.5.9 *Formula A satisfies (1.5.6) as a tautology for the intuitionistic fuzzy implications* $\rightarrow_2$, $\rightarrow_3$, $\rightarrow_8$, $\rightarrow_{11}$, $\rightarrow_{14}, \dots, \rightarrow_{16}$, $\rightarrow_{19}$, $\rightarrow_{20}$, $\rightarrow_{23}$, $\rightarrow_{31}$, $\rightarrow_{32}$, $\rightarrow_{37}$, $\rightarrow_{40}, \dots, \rightarrow_{45}$, $\rightarrow_{47}$, $\rightarrow_{48}$, $\rightarrow_{52}$, $\rightarrow_{55}, \dots, \rightarrow_{57}$, $\rightarrow_{62}$, $\rightarrow_{65}$, $\rightarrow_{74}$, $\rightarrow_{77}$, $\rightarrow_{83}$, $\rightarrow_{88}$, $\rightarrow_{90}$, $\rightarrow_{97}$, $\rightarrow_{153}$, $\rightarrow_{171}, \dots, \rightarrow_{180}$.

Theorem 1.5.10 *Formula A satisfies (1.5.6) as an IFT for the intuitionistic fuzzy implications* $\rightarrow_1, \dots, \rightarrow_9$, $\rightarrow_{11}, \dots, \rightarrow_{38}$, $\rightarrow_{40}, \dots, \rightarrow_{53}$, $\rightarrow_{55}, \dots, \rightarrow_{57}$, $\rightarrow_{61}$, $\rightarrow_{62}$, $\rightarrow_{64}, \dots, \rightarrow_{67}$, $\rightarrow_{71}$, $\rightarrow_{72}$, $\rightarrow_{74}, \dots, \rightarrow_{77}$, $\rightarrow_{79}, \dots, \rightarrow_{83}$, $\rightarrow_{85}$, $\rightarrow_{86}$, $\rightarrow_{88}, \dots, \rightarrow_{91}$, $\rightarrow_{94}$, $\rightarrow_{95}$, $\rightarrow_{97}$, $\rightarrow_{98}$, $\rightarrow_{100}, \dots, \rightarrow_{107}$, $\rightarrow_{109}, \dots, \rightarrow_{137}$, $\rightarrow_{151}$, $\rightarrow_{153}$, $\rightarrow_{158}, \dots, \rightarrow_{161}$, $\rightarrow_{166}, \dots, \rightarrow_{185}$.

Having in mind the definitions of tautology and IFT (see (1.1.7) and (1.1.8)), let us define for every evaluation function V and for formulas A and B:

$$V(A) = V(B) \text{ if and only if } \mu(A) = \mu(B) \text{ and } \nu(A) = \nu(B).$$

Let $A \equiv_\rightarrow B$ denote $(A \rightarrow B) \wedge (B \rightarrow A)$ for any fixed implication $\rightarrow$.

For example, when $\rightarrow$ is $\rightarrow_4$, we prove the following lemma.

Lemma 1.5.1 *If* $V(A) = V(B)$, *then,* $A \equiv_\rightarrow B$ *is an IFT.*

Proof From $V(A) = V(B)$ and from:

$$V(A \equiv_\rightarrow B) = \langle \min(\max(\nu(A), \mu(B)), \max(\mu(A), \nu(B))),$$

$$\max(\min(\nu(A), \mu(B)), \min(\mu(A), \nu(B)))\rangle$$

it follows, that:

$$\begin{aligned} &\min(\max(\nu(A), \mu(B)), \max(\mu(A), \nu(B))) \\ &- \max(\min(\nu(A), \mu(B)), \min(\mu(A), \nu(B))) \\ = &\min(\max(\nu(A), \mu(A)), \max(\mu(A), \nu(A))) \\ &- \max(\min(\nu(A), \mu(A)), \min(\mu(A), \nu(A))) \\ = &\max(\nu(A), \mu(A)) - \min(\nu(A), \mu(A)) \geq 0, \end{aligned}$$

i.e., $A \equiv_\rightarrow B$ is an IFT.

The opposite assertion is not valid. For example, if $V(A) = \langle 0.4, 0.5\rangle$ and $V(B) = \langle 0.4, 0.3\rangle$ then $V(A \equiv B) = \langle 0.4, 0.4\rangle$, i.e., $A \equiv_\rightarrow B$ is an IFT, but obviously, $V(A) \neq V(B)$. □

Open Problem 1. Determine for which indices i and any two formulas A and B it is valid that:

$$A \equiv_{\rightarrow} B \text{ is a tautology, if and only if } V(A) = V(B)?$$

The weak form of this Problem is whether for the same conditions,

$$A \equiv_{\rightarrow} B \text{ is an IFT if and only if } V(A) = V(B).$$

Open Problem 2. Determine for which implications formulas (1.5.2)–(1.5.6) are tautologies (or IFTs).

In [47], inspired by [66], the following formulas are studied:

$$(A \wedge B) \rightarrow C \equiv_{\rightarrow} (A \rightarrow (B \rightarrow C)), \tag{1.5.9}$$

$$A \rightarrow B \equiv_{\rightarrow} (A \rightarrow (A \rightarrow B)), \tag{1.5.10}$$

and for them, the following two theorems are proved.

Theorem 1.5.11 *Implications* $\rightarrow_3, \rightarrow_4, \rightarrow_{11}, \rightarrow_{12}, \rightarrow_{14}, \rightarrow_{16}, \rightarrow_{18}, \ldots, \rightarrow_{20}, \rightarrow_{22}, \rightarrow_{23}, \rightarrow_{25}, \ldots, \rightarrow_{28}, \rightarrow_{31}, \ldots, \rightarrow_{33}, \rightarrow_{41}, \ldots, \rightarrow_{43}, \rightarrow_{48}, \rightarrow_{56}, \rightarrow_{57}, \rightarrow_{74}, \rightarrow_{76}, \rightarrow_{77}, \rightarrow_{79}, \rightarrow_{81}, \rightarrow_{88}, \rightarrow_{97}$ * $\rightarrow_{153}, \rightarrow_{171}, \ldots, \rightarrow_{175}$ *satisfy (1.5.9) as tautologies.*

Theorem 1.5.12 *Implications* $\rightarrow_1, \ldots, \rightarrow_4, \rightarrow_8, \rightarrow_{10}, \ldots, \rightarrow_{12}, \rightarrow_{14}, \rightarrow_{16}, \ldots, \rightarrow_{20}, \rightarrow_{22}, \rightarrow_{23}, \rightarrow_{25}, \ldots, \rightarrow_{28}, \rightarrow_{30}, \ldots, \rightarrow_{33}, \rightarrow_{36}, \rightarrow_{37}, \rightarrow_{39}, \ldots, \rightarrow_{43}, \rightarrow_{48}, \rightarrow_{51}, \rightarrow_{52}, \rightarrow_{54}, \ldots, \rightarrow_{57}, \rightarrow_{59}, \rightarrow_{61}, \rightarrow_{67}, \rightarrow_{72}, \ldots, \rightarrow_{74}, \rightarrow_{76}, \ldots \rightarrow_{81}, \rightarrow_{86}, \ldots, \rightarrow_{89}, \rightarrow_{91}, \rightarrow_{92}, \rightarrow_{95}, \ldots, \rightarrow_{97}, \rightarrow_{100}, \rightarrow_{105}, \rightarrow_{106}, \rightarrow_{109}, \rightarrow_{110}, \rightarrow_{114}, \rightarrow_{119}, \rightarrow_{120},$ * $\rightarrow_{153}, \rightarrow_{166}, \rightarrow_{168}, \rightarrow_{171}, \ldots, \rightarrow_{180}$ *satisfy (1.5.10) as tautologies.*

In Theorems 1.5.11–1.5.14, the lists of implications before symbol "*" are published by L. Atanassova and after this symbol they are added by the author.

In [48], the following formulas are also studied:

$$(A \vee B) \rightarrow C = (A \rightarrow C) \wedge (B \rightarrow C), \tag{1.5.11}$$

$$(A \wedge B) \rightarrow C = (A \rightarrow C) \vee (B \rightarrow C). \tag{1.5.12}$$

Theorem 1.5.13 *Implications* $\rightarrow_2, \rightarrow_3, \rightarrow_4, \rightarrow_5, \rightarrow_{12}, \rightarrow_{13}, \rightarrow_{14}, \rightarrow_{16}, \rightarrow_{18}, \rightarrow_{19}, \rightarrow_{20}, \rightarrow_{22}, \rightarrow_{23}, \rightarrow_{25}, \rightarrow_{26}, \rightarrow_{27}, \rightarrow_{28}, \rightarrow_{29}, \rightarrow_{31}, \rightarrow_{32}, \rightarrow_{33}, \rightarrow_{34}, \rightarrow_{35}, \rightarrow_{37}, \rightarrow_{40}, \rightarrow_{41}, \rightarrow_{42}, \rightarrow_{43}, \rightarrow_{44}, \rightarrow_{45}, \rightarrow_{47}, \rightarrow_{48}, \rightarrow_{49}, \rightarrow_{50}, \rightarrow_{52}, \rightarrow_{55}, \rightarrow_{56}, \rightarrow_{57}, \rightarrow_{58}, \rightarrow_{59}, \rightarrow_{60}, \rightarrow_{61}, \rightarrow_{64}, \rightarrow_{66}, \rightarrow_{67}, \rightarrow_{69}, \rightarrow_{70}, \rightarrow_{71}, \rightarrow_{72}, \rightarrow_{73}, \rightarrow_{74}, \rightarrow_{76}, \rightarrow_{77}, \rightarrow_{78}, \rightarrow_{79}, \rightarrow_{80}, \rightarrow_{81}, \rightarrow_{82}, \rightarrow_{83}, \rightarrow_{84}, \rightarrow_{85}, \rightarrow_{86}, \rightarrow_{87}, \rightarrow_{88}, \rightarrow_{89}, \rightarrow_{90}, \rightarrow_{91}, \rightarrow_{92}, \rightarrow_{93}, \rightarrow_{94}, \rightarrow_{95}, \rightarrow_{96}, \rightarrow_{97}, \rightarrow_{98}, \rightarrow_{99}, \rightarrow_{102}, \rightarrow_{105}, \rightarrow_{108}, \rightarrow_{124}, \rightarrow_{125}, \rightarrow_{127}, \rightarrow_{129}, \rightarrow_{130}, \rightarrow_{132}, \rightarrow_{134}, \rightarrow_{135}, \rightarrow_{137}$ *satisfy (1.5.11) as tautologies.*

Theorem 1.5.14 *Implications* $\rightarrow_2, \rightarrow_3, \rightarrow_4, \rightarrow_5, \rightarrow_8, \rightarrow_{11}, \rightarrow_{12}, \rightarrow_{13}, \rightarrow_{16}, \rightarrow_{18}, \rightarrow_{19}, \rightarrow_{20}, \rightarrow_{22}, \rightarrow_{23}, \rightarrow_{25}, \rightarrow_{26}, \rightarrow_{27}, \rightarrow_{28}, \rightarrow_{29}, \rightarrow_{31}, \rightarrow_{32}, \rightarrow_{33}, \rightarrow_{34}, \rightarrow_{35}, \rightarrow_{37}, \rightarrow_{40}, \rightarrow_{41}, \rightarrow_{42}, \rightarrow_{43}, \rightarrow_{44}, \rightarrow_{45}, \rightarrow_{47}, \rightarrow_{48}, \rightarrow_{49}, \rightarrow_{50}, 52, \rightarrow_{55}, \rightarrow_{56}, \rightarrow_{57}, \rightarrow_{58}, \rightarrow_{59}, \rightarrow_{60}, \rightarrow_{61}, \rightarrow_{62}, \rightarrow_{63}, \rightarrow_{64}, \rightarrow_{65}, \rightarrow_{66}, \rightarrow_{67}, \rightarrow_{68}, \rightarrow_{70}, \rightarrow_{71},$

$\to_{72}$, $\to_{73}$, $\to_{74}$, $\to_{76}$, $\to_{77}$, $\to_{78}$, $\to_{79}$, $\to_{80}$, $\to_{81}$, $\to_{82}$, $\to_{83}$, $\to_{84}$, $\to_{85}$, $\to_{86}$, $\to_{87}$, $\to_{88}$, $\to_{89}$, $\to_{90}$, $\to_{91}$, $\to_{92}$, $\to_{93}$, $\to_{94}$, $\to_{95}$, $\to_{96}$, $\to_{97}$, $\to_{98}$, $\to_{99}$, $\to_{102}$, $\to_{105}$, $\to_{108}$, $\to_{124}$, $\to_{125}$, $\to_{127}$, $\to_{129}$, $\to_{130}$, $\to_{132}$, $\to_{134}$, $\to_{135}$, $\to_{137}$ satisfy (1.5.12) as tautologies.

Now, we check which intuitionistic fuzzy implications and negations satisfy C.A. Meredith's axiom (see, e.g., [2]).

Theorem 1.5.15 *For every five formulas A, B, C, D and E, Meredith's axiom*

$$((((A \to B) \to (\neg C \to \neg D)) \to C) \to E) \to ((E \to A) \to (D \to A))$$

is valid as a tautology by implications $\to_{20}, \to_{23}, \to_{74}, \to_{77}, \to_{153}$.

Theorem 1.5.16 *For every five formulas A, B, C, D and E, Meredith's axiom is valid as an IFT by implications $\to_1$, $\to_4$, $\to_5$, $\to_6$, $\to_9$, $\to_{13}$, $\to_{17}$, $\to_{18}$, $\to_{20}$, $\dots$, $\to_{23}$, $\to_{25}$, $\to_{27}$, $\dots$, $\to_{29}$, $\to_{61}$, $\to_{64}$, $\to_{72}$, $\to_{74}$, $\dots$, $\to_{77}$, $\to_{79}$, $\dots$, $\to_{81}$, $\to_{100}$, $\dots$, $\to_{102}$, $\to_{109}$, $\dots$, $\to_{113}$, $\to_{124}$, $\dots$, $\to_{128}$, $\to_{133}$, $\to_{151}$, $\to_{153}$, $\to_{158}$, $\dots$, $\to_{161}$, $\to_{166}$, $\to_{167}$, $\to_{169}$, $\to_{170}$, $\to_{182}$, $\to_{185}$*

Proof Let $V(A) = \langle a, b\rangle$, $V(B) = \langle c, d\rangle$, $V(C) = \langle e, f\rangle$, $V(D) = \langle g, h\rangle$, $V(E) = \langle i, j\rangle$, where $a, b, \dots, j \in [0, 1]$ and $a+b \le 1, c+d \le 1, e+f \le 1, g+h \le 1$ and $i + j \le 1$. We check the validity of the assertion for the case when the implication is, for example, $\to_4$.

$$V(((((A \to_4 B) \to_4 (\neg C \to_4 \neg D)) \to_4 C) \to_4 E) \to_4 ((E \to_4 A) \to_4 (D \to_4 A)))$$

$$\begin{aligned}
&= ((((\langle a, b\rangle \to_4 \langle c, d\rangle) \to_4 (\langle f, e\rangle \to_4 \langle h, g\rangle)) \to_4 \langle e, f\rangle) \to_4 \langle i, j\rangle) \\
&\quad \to_4 ((\langle i, j\rangle \to_4 \langle a, b\rangle) \to_4 (\langle g, h\rangle \to_4 \langle a, b\rangle)) \\
&= (((\langle \max(b, c), \min(a, d)\rangle \to_4 \langle \max(e, h), \min(f, g)\rangle) \\
&\quad \to_4 \langle e, f\rangle) \to_4 \langle i, j\rangle) \to_4 ((\langle i, j\rangle \to_4 \langle a, b\rangle) \to_4 (\langle g, h\rangle \to_4 \langle a, b\rangle)) \\
&= ((\langle \max(e, h, \min(a, d)), \min(f, g, \max(b, c))\rangle \to_4 \langle e, f\rangle) \\
&\quad \to_4 \langle i, j\rangle) \to_4 ((\langle i, j\rangle \to_4 \langle a, b\rangle) \to_4 (\langle g, h\rangle \to_4 \langle a, b\rangle)) \\
&= (\langle \max(e, \min(f, g, \max(b, c))), \min(f, \max(e, h, \min(a, d)))\rangle \\
&\quad \to_4 \langle i, j\rangle) \to_4 (\langle \max(a, j), \min(b, i\rangle \to_4 \langle \max(a, h), \min(b, g)\rangle) \\
&= \langle \max(i, \min(f, \max(e, h, \min(a, d)))), \min(j, \max(e, \min(f, g, \\
&\quad \max(b, c))))\rangle \to_4 \langle \max(a, h, \min(b, i)), \min(b, g, \max(a, j))\rangle \\
&= \langle \max(a, h, \min(b, i), \min(j, \max(e, \min(f, g, \max(b, c))))), \\
&\quad \min(b, g, \max(a, j), \max(i, \min(f, \max(e, h, \min(a, d)))))\rangle.
\end{aligned}$$

Let

$$\begin{aligned}
X = {} & \max(a, h, \min(b, i), \min(j, \max(e, \min(f, g, \max(b, c)))))- \\
& \min(b, g, \max(a, j), \max(i, \min(f, \max(e, h, \min(a, d))))).
\end{aligned}$$

Obviously,

$$\max(a, b) \geq \min(a, b), \max(a, b) \geq \min(b, j), \max(a, i) \geq \min(a, b).$$

Let $\max(a, i) \geq \min(b, j)$. Then,

$$\begin{aligned} X &\geq \max(a, \min(b, i)) - \min(b, \max(a, j)) \\ &= \min(\max(a, b), \max(a, i)) - \max(\min(a, b), \min(b, j)) \geq 0. \end{aligned}$$

Let $\max(a, i) < \min(b, j)$. Then, $a < b, a < j, i < b$ and $i < j$.
If $j \leq \max(e, \min(f, g, \max(b, c)))$, then

$$\begin{aligned} X &= \max(a, h, i, \min(j, \max(e, \min(f, g, \max(b, c)))))- \\ &\quad \min(b, g, j, \max(i, \min(f, \max(e, h, \min(a, d))))) \\ &\geq \max(a, h, i, j) - \min(b, g, j) \geq 0. \end{aligned}$$

If $i \geq \min(f, \max(e, h, \min(a, d)))$, then

$$\begin{aligned} X &= \max(a, h, i, \min(j, \max(e, \min(f, g, \max(b, c)))))- \\ &\quad \min(b, g, j, \max(i, \min(f, \max(e, h, \min(a, d))))) \\ &\geq \max(a, h, i) - \min(b, g, i, j) \geq 0. \end{aligned}$$

Finally, let

$$j > \max(e, \min(f, g, \max(b, c)))$$

and

$$i < \min(f, \max(e, h, \min(a, d))).$$

Then, $j > e$ and

$$j > \min(f, g, \max(b, c))$$

and $i < f$ and

$$i < \max(e, h, \min(a, d)).$$

Therefore,

$$\begin{aligned} X &= \max(a, h, i.e., \min(f, g, \max(b, c))) \\ &\quad - \min(b, g, j, f, \max(e, h, \min(a, d))) \\ &\geq \min(f, g, \max(b, c))) - \min(b, g, j, f, \max(e, h, \min(a, d))) \\ &\geq \min(b, f, g) - \min(b, g, j, f, \max(e, h, \min(a, d))) \geq 0. \end{aligned}$$

Therefore, the Meredith's axiom is an IFT. □

The validity of the axioms of the intuitionistic logic (IL) (see, e.g., [67]) is checked for all implications in [28].

Below, we give the list of the implications for which all axioms of IL are valid. The checks are done both for the case of tautologies, as well as for the case of IFTs.

The IL axioms are the following.

(IL1) $A \rightarrow A$,

(IL2) $A \rightarrow (B \rightarrow A)$,

(IL3) $A \rightarrow (B \rightarrow (A \wedge B))$,

(IL4) $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$,

(IL5) $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$,

(IL6) $A \rightarrow \neg\neg A$,

(IL7) $\neg(A \wedge \neg A)$,

(IL8) $(\neg A \vee B) \rightarrow (A \rightarrow B)$,

(IL9) $\neg(A \vee B) \rightarrow (\neg A \wedge \neg B)$,

(IL10) $(\neg A \wedge \neg B) \rightarrow \neg(A \vee B)$,

(IL11) $(\neg A \vee \neg B) \rightarrow \neg(A \wedge B)$,

(IL12) $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$,

(IL13) $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$,

(IL14) $\neg\neg\neg A \rightarrow \neg A$,

(IL15) $\neg A \rightarrow \neg\neg\neg A$,

(IL16) $\neg\neg(A \rightarrow B) \rightarrow (A \rightarrow \neg\neg B)$,

(IL17) $(C \rightarrow A) \rightarrow ((C \rightarrow (A \rightarrow B)) \rightarrow (C \rightarrow B))$.

Theorem 1.5.17 *Implications* $\rightarrow_1, \rightarrow_3, \dots, \rightarrow_5, \rightarrow_9, \rightarrow_{11}, \rightarrow_{13}, \rightarrow_{14}, \rightarrow_{17}, \rightarrow_{18}, \rightarrow_{20}, \rightarrow_{21}, \rightarrow_{22}, \rightarrow_{23}, \rightarrow_{27}, \dots, \rightarrow_{29}, \rightarrow_{61}, \rightarrow_{66}, \rightarrow_{71}, \rightarrow_{74}, \dots, \rightarrow_{77}, \rightarrow_{79}, \rightarrow_{81}, \rightarrow_{100}, \dots, \rightarrow_{102}, \rightarrow_{109}, \dots, \rightarrow_{113}, \rightarrow_{118}, \rightarrow_{124}, \dots, \rightarrow_{128}, \rightarrow_{151}, \rightarrow_{153}, \rightarrow_{158}, \dots, \rightarrow_{160}, \rightarrow_{166}, \rightarrow_{167}, \rightarrow_{169}, \rightarrow_{170}, \rightarrow_{182}, \rightarrow_{185}$ *satisfy all IL axioms as IFTs.*

Proof Let us assume everywhere below that

$$V(A) = \langle a, b \rangle$$

$$V(B) = \langle c, d \rangle$$

$$V(C) = \langle e, f \rangle$$

Then, the validity of (IL5) for implication $\rightarrow_4$ is checked as follows. $V((A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)))$
$= (\langle a, b\rangle \rightarrow \langle \max(d, e), \min(c, f)\rangle) \rightarrow (\langle \max(b, c), \min(a, d)\rangle$
$\rightarrow \langle \max(b, e), \min(a, f)\rangle)$
$= \langle \max(b, d, e), \min(a, c, f)\rangle \rightarrow \langle \max(b, e, \min(a, d)), \min(a, f, \max(b, c))\rangle$
$= \langle \max(b, e, \min(a, d), \min(a, c, f)), \min(a, f, \max(b, c), \max(b, d, e))\rangle$
and
$\max(b, e, \min(a, d), \min(a, c, f)) \geq \max(b, e, \min(a, d))$
$\geq \min(a, \max(b, d, e)) \geq \min(a, f, \max(b, c), \max(b, d, e))$.
The validity of the other axioms is checked analogically. □

Theorem 1.5.18 *Only implication $\rightarrow_{153}$ satisfies all IL axioms as tautologies.*

The validity of Kolmogorov's and Łukasiewicz–Tarski's Axioms of Logic (see, e.g., [68]) are checked for all implications in [22] .

Now, we give the lists of the implications for which all Kolmogorov's and Łukasiewicz–Tarski's axioms are valid. The checks are done as for the case of tautologies, as well as for the case of IFTs.

The first group of axioms (of Kolmogorov) comprises

(K1) $A \rightarrow (B \rightarrow A)$,

(K2) $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B))$,

(K3) $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$,

(K4) $(B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$,

(K5) $(A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A)$.

Theorem 1.5.19 *Implications $\rightarrow_1, \rightarrow_3, \ldots, \rightarrow_5, \rightarrow_9, \rightarrow_{11}, \rightarrow_{13}, \rightarrow_{14}, \rightarrow_{17}, \rightarrow_{18}, \rightarrow_{20}, \rightarrow_{22}, \rightarrow_{23}, \rightarrow_{27}, \ldots, \rightarrow_{29}, \rightarrow_{61}, \rightarrow_{66}, \rightarrow_{71}, \rightarrow_{74}, \ldots, \rightarrow_{77}, \rightarrow_{79}, \rightarrow_{81}, \rightarrow_{100}, \ldots, \rightarrow_{102}, \rightarrow_{109}, \ldots, \rightarrow_{113}, \rightarrow_{118}, \rightarrow_{124}, \ldots, \rightarrow_{128}, \rightarrow_{151}, \rightarrow_{153}, \rightarrow_{158}, \ldots, \rightarrow_{160}, \rightarrow_{166}, \rightarrow_{167}, \rightarrow_{169}, \rightarrow_{170}, \rightarrow_{182}, \rightarrow_{185}$ satisfy all Kolmogorov's axioms as IFTs.*

Theorem 1.5.20 *Implications $\rightarrow_3, \rightarrow_{11}, \rightarrow_{14}, \rightarrow_{20}, \rightarrow_{23}, \rightarrow_{74}, \rightarrow_{77}, \rightarrow_{153}$ satisfy all Kolmogorov's axioms as tautologies.*

The second group of axioms (of Łukasiewicz and Tarski) is

(LT1) $A \rightarrow (B \rightarrow A)$,

(LT2) $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$,

(LT3) $\neg A \rightarrow (\neg B \rightarrow (B \rightarrow A))$,

(LT4) $((A \rightarrow \neg A) \rightarrow A) \rightarrow A$.

Theorem 1.5.21 *Implications* $\rightarrow_1, \rightarrow_4, \ldots, \rightarrow_6, \rightarrow_9, \rightarrow_{17}, \rightarrow_{18}, \rightarrow_{20}, \ldots, \rightarrow_{23}, \rightarrow_{27}, \ldots, \rightarrow_{29}, \rightarrow_{61}, \rightarrow_{64}, \rightarrow_{66}, \rightarrow_{71}, \rightarrow_{74}, \ldots, \rightarrow_{77}, \rightarrow_{79}, \rightarrow_{81}, \rightarrow_{100}, \ldots, \rightarrow_{102}, \rightarrow_{109}, \ldots, \rightarrow_{113}, \rightarrow_{124}, \ldots, \rightarrow_{128}, \rightarrow_{153}$ *satisfy all Łukasiewicz–Tarski's axioms as IFTs.*

Theorem 1.5.22 *Implications* $\rightarrow_{20}, \rightarrow_{23}, \rightarrow_{74}, \rightarrow_{77}, \rightarrow_{153}$ *satisfy all Łukasiewicz–Tarski's axioms as tautologies.*

Sixth, some variants of fuzzy implications (marked by $I(x, y)$) are described in the book of Klir and Yuan [56] and the following nine axioms are discussed, where $I(x, y)$ denotes $x \rightarrow y$ for any of the possible forms of the operation implication, N is the operation negation related with operation $\rightarrow$, and for $a, b, c, d \in [0, 1]$, $a+b \leq 1$, $c + d \leq 1$:

$$\langle a, b\rangle \leq \langle c, d\rangle \text{ iff } a \leq c \text{ and } b \geq d. \tag{1.5.13}$$

Axiom $A1$ $(\forall x, y)(x \leq y \rightarrow (\forall z)(I(x, z) \geq I(y, z)))$,
Axiom $A2$ $(\forall x, y)(x \leq y \rightarrow (\forall z)(I(z, x) \leq I(z, y)))$,
Axiom $A3$ $(\forall y)(I(0, y) = 1)$,
Axiom $A4$ $(\forall y)(I(1, y) = y)$,
Axiom $A5$ $(\forall x)(I(x, x) = 1)$,
Axiom $A6$ $(\forall x, y, z)(I(x, I(y, z)) = I(y, I(x, z)))$,
Axiom $A7$ $(\forall x, y)(I(x, y) = 1 \text{ iff } x \leq y)$,
Axiom $A8$ $(\forall x, y)(I(x, y) = I(N(y), N(x)))$,
Axiom $A9$ I is a continuous function.

For our research, having in mind the specific forms of the intuitionistic fuzzy implications, we modify five of these axioms, as follows.
Axiom $A3^*$ $(\forall y)(I(0, y)$ is an IFT),
Axiom $A4^*$ $(\forall y)(I(1, y) \leq y)$,
Axiom $A5^*$ $(\forall x)(I(x, x)$ is an IFT),
Axiom $A7^*$ $(\forall x, y)$ (if $x \leq y$, then, $I(x, y) = 1$),
Axiom $A8^*$ $(\forall x, y)(I(x, y) = N(N(I(N(y), N(x)))))$.

Here, we ignore Axiom 9, because, obviously, it is valid for all the implications that do not contain operations sg or $\overline{\text{sg}}$.

Following the paper of N. Angelova and the author [21], we formulate the following theorem.

Theorem 1.5.23 *The intuitionistic fuzzy implications that satisfy Klir and Yuan's axioms as (standard) tautologies, are marked in Table 1.4 by "•" and the implications that satisfy the same axioms (only) as IFTs – by "∘".*

Finally, following [8], we discuss the well-known Contraposition Law

$$(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A) \tag{1.5.14}$$

and its modified version

$$(\neg\neg A \to \neg\neg B) \to (\neg B \to \neg A). \tag{1.5.15}$$

For them, the following assertions are valid.

Theorem 1.5.24 *(a) Implications* $\to_1, \ldots, \to_9, \to_{11}, \ldots, \to_{29}, \to_{46}, \ldots, \to_{53}, \to_{55}, \ldots, \to_{57}, \to_{61}, \to_{62}, \to_{64}, \ldots, \to_{66}, \to_{71}, \to_{72}, \to_{74}, \ldots, \to_{77}, \to_{79}, \ldots, \to_{81}, \to_{91}, \to_{94}, \to_{99}, \to_{100}, \ldots, \to_{102}, \to_{104}, \to_{105}, \to_{109}, \ldots, \to_{113}, \to_{118}, \to_{120}, \ldots, \to_{122}, \to_{124}, \ldots, \to_{128}, \to_{133}, \ldots, \to_{137}, \to_{151}, \to_{153}, \to_{158}, \ldots, \to_{161}, \to_{166}, \to_{167}, \to_{169}, \ldots, \to_{172}, \to_{174}, \ldots, \to_{177}, \to_{180}, \ldots, \to_{182}, \to_{184}, \to_{185}$ *satisfy (1.5.14) as IFTs.*
(b) Implications $\to_2, \to_3, \to_8, \to_{11}, \to_{14}, \ldots, \to_{16}, \to_{19}, \to_{20}, \to_{23}, \to_{24}, \to_{47}, \ldots, \to_{49}, \to_{52}, \to_{55}, \ldots, \to_{57}, \to_{65}, \to_{74}, \to_{77}, \to_{97}, \to_{153}, \to_{171}, \to_{172}, \to_{174}, \ldots, \to_{177}, \to_{179}, \to_{180}, \to_{182}, \to_{184}, \to_{185}$ *satisfy (1.5.14) as tautologies.*

Theorem 1.5.25 *(a) Implications* $\to_1, \ldots, \to_9, \to_{11}, \ldots, \to_{38}, \to_{40}, \ldots, \to_{53}, \to_{55}, \ldots, \to_{57}, \to_{61}, \to_{62}, \to_{64}, \ldots, \to_{66}, \to_{71}, \to_{72}, \to_{74}, \ldots, \to_{77}, \to_{79}, \ldots, \to_{83}, \to_{85}, \to_{88}, \ldots, \to_{91}, \to_{94}, \to_{97}, \to_{100}, \ldots, \to_{107}, \to_{109}, \ldots, \to_{118}, \to_{120}, \ldots, \to_{122}, \to_{124}, \ldots, \to_{137}, \to_{151}, \to_{153}, \to_{158}, \ldots, \to_{161}, \to_{166}, \ldots, \to_{185}$ *satisfy (1.5.15) as IFTs.*
(b) Implications $\to_2, \to_3, \to_8, \to_{11}, \to_{14}, \ldots, \to_{16}, \to_{19}, \to_{20}, \to_{23}, \to_{24}, \to_{31}, \to_{32}, \to_{34}, \to_{34}, \to_{41}, \ldots, \to_{45}, \to_{47}, \ldots, \to_{49}, \to_{52}, \to_{55}, \ldots, \to_{57}, \to_{65}, \to_{74}, \to_{77}, \to_{83}, \to_{88}, \to_{90}, \to_{97}, \to_{153}, \to_{171}, \ldots, \to_{180}, \to_{182}, \ldots, \to_{185}$ *satisfy (1.5.15) as tautologies.*

Theorem 1.5.26 *(a) Implications* $\to_1, \to_4, \to_5, \to_7, \to_9, \to_{13}, \to_{18}, \to_{20}, \ldots, \to_{23}, \to_{25}, \to_{27}, \ldots, \to_{29}, \to_{61}, \to_{66}, \to_{71}, \to_{74}, \to_{76}, \to_{77}, \to_{79}, \to_{81}, \to_{100}, \ldots, \to_{102}, \to_{104}, \to_{105}, \to_{107}, \to_{109}, \ldots, \to_{113}, \to_{118}, \to_{124}, \ldots, \to_{128}, \to_{133}, \to_{151}, \to_{153}, \to_{158}, \ldots, \to_{161}, \to_{166}, \to_{167}, \to_{169}, \to_{170}, \to_{182}, \to_{185}$ *satisfy expression*

$$(\neg A \to \neg B) \to ((\neg A \to B) \to A)$$

as IFTs.
(b) Implications $\to_{20}, \to_{23}, \to_{74}, \to_{77}, \to_{153}$ *satisfy this expression as tautologies.*

Theorem 1.5.27 *(a) Implications* $\to_1, \to_4, \to_5, \to_7, \to_9, \to_{13}, \to_{18}, \to_{20}, \ldots, \to_{23}, \to_{25}, \to_{27}, \ldots, \to_{30}, \to_{33}, \ldots, \to_{36}, \to_{38}, \to_{42}, \to_{45}, \to_{61}, \to_{66}, \to_{71}, \to_{74}, \to_{76}, \to_{77}, \to_{79}, \to_{81}, \to_{82}, \to_{85}, \to_{88}, \to_{100}, \ldots, \to_{105}, \to_{107}, \to_{109}, \ldots, \to_{118}, \to_{124}, \ldots, \to_{133}, \to_{151}, \to_{153}, \to_{158}, \ldots, \to_{161}, \to_{166}, \ldots, \to_{170}, \to_{182}, \to_{183}, \to_{185}$ *satisfy expression*

$$(\neg A \to \neg B) \to ((\neg A \to \neg\neg B) \to A)$$

as IFTs.
(b) Implications $\to_{20}, \to_{23}, \to_{42}, \to_{45}, \to_{74}, \to_{77}, \to_{88}, \to_{153}$ *satisfy the same expression as tautologies.*

Table 1.4 The intuitionistic fuzzy implications that satisfy Klir and Yuan's axioms

	*A*1	*A*2	*A*3	*A*3*	*A*4	*A*4*	*A*5	*A*5*	*A*6	*A*7	*A*7*	*A*8	*A*8*
1		●	●	●	●	○		○					
2	●	●	●	●		●	●	●			●		
3	●	●	●	●	●	●	●	●	●		●		
4	●	●	●	●	●	●		○	●			●	●
5	●	●	●	●	●	●		○	●			●	●
6		●	●	●	●	●		○					
7				○	●	●		○				●	○
8	●	●	●	●		●	●	●			●		
9		●	●	●	●	●		○					
10		●	●	●	●	●							
11	●	●	●	●	●	●	●	●	●		●		
12	●	●	●	●		●			●				
13	●	●	●	●	●	●		○	●			●	●
14	●	●	●	●	●	●	●	●	●	●	●		
15	●	●	●	●		●	●	●		●	●		
16	●	●	●	●	●	●			●				
17		●	●	●	●	●		○	●				
18	●	●	●	●	●	●		○	●				
19	●	●	●	●	●	●			●				
20	●	●	●	●			●	●	●		●	●	●
21			●	●				○					
22	●	●	●	●				○	●			●	●
23	●	●	●	●			●	●	●		●	●	●
24	●	●	●	●		●	●	●		●	●		
25	●	●	●	●		●			●				
26	●	●	●	●	●	●			●				
27	●	●	●	○				○	●			●	●
28	●	●	●	●	●	●		○	●				
29	●	●	●	●				○					
30		●	●	●				○					
31	○	○	○	○			●	●	●		●		
32	●	●	●	●			●	●	●		●		
33	●	●	●	●				○	●				
34	●	●	●	●			●	●	●		●		
35	●	●	●	●				○	●				
36				○				○	●				
37	●	●	●	●			●	●			●		
38		●	●	●				○					
39		●	●	●									
40	●	●	●	●			●	●			●		

(continued)

Table 1.4 (continued)

	$A1$	$A2$	$A3$	$A3^*$	$A4$	$A4^*$	$A5$	$A5^*$	$A6$	$A7$	$A7^*$	$A8$	$A8^*$
41	•	•	•	•					•				
42	•	•	•	•			•	•	•		•		
43	•	•	•	•					•				
44	•	•	•	•				○					
45	•	•	•	•				○					
46		•	•	•		•							
47	•	•	•	•		•							
48	•	•	•	•		•			•				
49	•	•	•	•		•			•				
50	•	•	•	•		•			•				
51				○		•			•				
52	•	•	•	•		•							
53		•	•	•		•							
54		•	•	•		•							
55	•	•	•	•		•							
56	•	•	•	•		○			•				
57	•	•	•	•					•				
58	•	•	•	•		•							
59	•	•	•	•		•							
60	•	•	•	•									
61	•			○	•	○		○					
62	•	•	•	•			•	•			•		
63	•	•	•	•			•	•			•		
64	•			○	•	•		○					
65	•	•	•	•			•	•			•		
66	•			○				○					
67	•		•	•	•	•							
68	•	•	•	•			•	•			•		
69	•	•	•	•		•	•	•		•	•		
70	•	•	•	•									
71	•		•	•				○					
72	•	•	•	•		•		○					
73	•	•	•	•		•							
74	•	•	•	•			•	•	•		•	•	•
75			•	•				○					
76	•	•	•	•		•		○	•			•	•
77	•	•	•	•		•	•	•	•		•	•	•
78	•	•	•	•		•							
79	•	•	•	•		•		○	•			•	•

(continued)

Table 1.4 (continued)

	*A*1	*A*2	*A*3	*A*3*	*A*4	*A*4*	*A*5	*A*5*	*A*6	*A*7	*A*7*	*A*8	*A*8*
80	●	●	●	●		●		○					
81	●	●	●	●		●		○	●				
82	●			○				○					
83	●	●	●	●			●	●			●		
84	●	●	●	●			●	●			●		
85	●			○				○					
86	●		●	●									
87	●	●	●	●									
88	●	●	●	●			●	●	●		●		
89	●	●	●	●				○					
90	●	●	●	●				○					
91	●			○		●							
92	●	●	●	●		●							
93	●	●	●	●		●							
94	●			○		●							
95	●		●	●		●							
96	●	●	●	●		●							
97	●	●	●	●		●			●				
98	●	●	●	●		●							
99	●	●	●	●		●							
100	○	○		○				○	●		○		○
101	○	○		○				○			○		○
102	○	○	●	○				○			○		○
103	○	○		○				○			○		○
104	○	○		○				○			○		○
105	○	○	●	○				○			○		○
106	○	○		○							○		○
107	○	○		○							○		○
108	○	○	●	●							○		○
109		●	●	●	●	●		○					○
110		●	●	●	●	●		○	●				○
111			●	●				○					○
112		●	●	●	●	●		○	●				○
113				○				○					○
114		●	●	●				○					○
115		●	●	○				○					○
116			●	●				○					○
117		●	●	●				○					○
118				○				○					○

(continued)

Table 1.4 (continued)

	$A1$	$A2$	$A3$	$A3^*$	$A4$	$A4^*$	$A5$	$A5^*$	$A6$	$A7$	$A7^*$	$A8$	$A8^*$
119		●	●	●		●							○
120		●	●	●		●							○
121			●	●									○
122		●	●	●		●							○
123				○									○
124	●			○				○					○
125	●		●	●				○					○
126			●	●				○					○
127	●		●	●				○					○
128				○				○					○
129	●			○				○					○
130	●		●	●				○					○
131			●	●				○					○
132	●		●	●				○					○
133				○				○					○
134	●			○									○
135	●		●	●									○
136			●	●									○
137	●		●	●									○
138				○									○
139	○	○		○		●		○					○
140						●							○
141	○	○		○		●		○					○
142													○
143						●							○
144													○
145						●							○
146	○	○		○				○					○
147	○	○		○		●		○					○
148	○	○		○				○					○
149	○	○		○		●							○
150	○	○		○				○					○
151	○	○		○				○					○
152	○	○		○				○					○
153	●	●	●	●			●	●	●		●	●	●
154	○	○		○				○					○
155	○	○		○				○					○
156	○	○		○									○
157	○	○		○				○					○
158	○	○		○				○					○

(continued)

Table 1.4 (continued)

	*A*1	*A*2	*A*3	*A*3*	*A*4	*A*4*	*A*5	*A*5*	*A*6	*A*7	*A*7*	*A*8	*A*8*
159	∘	∘		∘				∘					∘
160	∘	∘		∘				∘					∘
161	∘	∘		∘									∘
162	∘	∘		∘				∘					∘
163	∘	∘		∘				∘					∘
164	∘	∘		∘				∘					∘
165	∘	∘		∘									∘
166		•	•	•	•	•		∘					∘
167	∘	•	•	•				∘			∘		∘
168		•	•	•				∘					∘
169	∘	•	•	∘				∘			∘		∘
170		•	•	•				∘					∘
171	•	•	•	•	•	•			•				∘
172	•	•	•	•		•			•				∘
173	•	•	•	•					•				∘
174	•	•	•	•					•				∘
175	•	•	•	•		•			•				∘
176	•	•	•	•		•	•	•			•		∘
177	•	•	•	•		•	•	•			∘		•
178	•	•	•	•			•	•			•		∘
179	•	•	•	•		•							∘
180	•	•	•	•			•	•			•		∘
181	•	•	•	•	•	•	•	•			•		∘
182	•	•	•	•		•	•	•	•		•	•	•
183	•	•	•	•			•	•	•		•		∘
184	•	•	•	•		•			•				∘
185	•	•	•	•	∘	∘	•	•	•	∘	•	•	•

Theorem 1.5.28 *(a) Implications* $\rightarrow_1, \ldots, \rightarrow_5, \rightarrow_7, \ldots, \rightarrow_9, \rightarrow_{11}, \ldots, \rightarrow_{29}, \rightarrow_{46}, \ldots, \rightarrow_{53}, \rightarrow_{55}, \ldots, \rightarrow_{57}, \rightarrow_{61}, \rightarrow_{66}, \rightarrow_{71}, \rightarrow_{74}, \ldots, \rightarrow_{77}, \rightarrow_{79}, \rightarrow_{81}, \rightarrow_{91}, \rightarrow_{94}, \rightarrow_{97}, \rightarrow_{99}, \ldots, \rightarrow_{102}, \rightarrow_{104}, \ldots, \rightarrow_{107}, \rightarrow_{109}, \ldots, \rightarrow_{113}, \rightarrow_{118}, \rightarrow_{119}, \rightarrow_{121}, \rightarrow_{124}, \ldots, \rightarrow_{128}, \rightarrow_{133}, \ldots, \rightarrow_{137}, \rightarrow_{151}, \rightarrow_{153}, \rightarrow_{158}, \ldots, \rightarrow_{161}, \rightarrow_{166}, \rightarrow_{167}, \rightarrow_{169}, \ldots, \rightarrow_{172}, \rightarrow_{174}, \ldots, \rightarrow_{177}, \rightarrow_{179}, \rightarrow_{180}, \rightarrow_{182}, \rightarrow_{184}, \rightarrow_{185}$ *satisfy expression*

$$(\neg A \rightarrow \neg B) \rightarrow ((\neg A \rightarrow B) \rightarrow \neg\neg A)$$

as IFTs.
(b) Implications $\rightarrow_2, \rightarrow_3, \rightarrow_8, \rightarrow_{11}, \rightarrow_{14}, \rightarrow_{15}, \rightarrow_{16}, \rightarrow_{19}, \rightarrow_{20}, \rightarrow_{23}, \rightarrow_{47},$

$\rightarrow_{48}, \rightarrow_{52}, \rightarrow_{55}, \ldots, \rightarrow_{57}, \rightarrow_{74}, \rightarrow_{77}, \rightarrow_{97}, \rightarrow_{99}, \rightarrow_{153}, \rightarrow_{171}, \rightarrow_{172}, \rightarrow_{174}, \ldots, \rightarrow_{177}, \rightarrow_{179}, \rightarrow_{180}$ *satisfy the same expression as tautologies.*

Theorem 1.5.29 *(a) Implications* $\rightarrow_1, \ldots, \rightarrow_5, \rightarrow_7, \ldots, \rightarrow_9, \rightarrow_{11}, \ldots, \rightarrow_{38}, \rightarrow_{40}, \ldots, \rightarrow_{43}, \rightarrow_{45}, \ldots, \rightarrow_{53}, \rightarrow_{55}, \ldots, \rightarrow_{57}, \rightarrow_{61}, \rightarrow_{66}, \rightarrow_{71}, \rightarrow_{74}, \ldots, \rightarrow_{77}, \rightarrow_{79}, \rightarrow_{81}, \ldots, \rightarrow_{83}, \rightarrow_{85}, \rightarrow_{88}, \rightarrow_{91}, \rightarrow_{94}, \rightarrow_{97}, \rightarrow_{99}, \ldots, \rightarrow_{119}, \rightarrow_{121}, \rightarrow_{124}, \ldots, \rightarrow_{137}, \rightarrow_{151}, \rightarrow_{153}, \rightarrow_{158}, \ldots, \rightarrow_{161}, \rightarrow_{166}, \ldots, \rightarrow_{180}, \rightarrow_{182}, \ldots, \rightarrow_{185}$ *satisfy expression*

$$(\neg A \rightarrow \neg B) \rightarrow ((\neg A \rightarrow \neg\neg B) \rightarrow \neg\neg A).$$

as IFTs.

(b) Implications $\rightarrow_2, \rightarrow_3, \rightarrow_8, \rightarrow_{11}, \rightarrow_{14}, \ldots, \rightarrow_{16}, \rightarrow_{19}, \rightarrow_{20}, \rightarrow_{23}, \rightarrow_{31}, \rightarrow_{32}, \rightarrow_{37}, \rightarrow_{40}, \ldots, \rightarrow_{43}, \rightarrow_{45}, \rightarrow_{47}, \rightarrow_{48}, \rightarrow_{52}, \rightarrow_{55}, \ldots, \rightarrow_{57}, \rightarrow_{74}, \rightarrow_{77}, \rightarrow_{83}, \rightarrow_{88}, \rightarrow_{97}, \rightarrow_{99}, \rightarrow_{153}, \rightarrow_{171}, \ldots, \rightarrow_{180}$ *satisfy the same expression as tautologies.*

In [69, 70], the Hauber's Law is formulated by

$$((A \rightarrow B) \wedge (C \rightarrow D) \wedge (A \vee C) \wedge \neg(B \wedge D)) \rightarrow ((B \rightarrow A) \wedge (D \rightarrow C))$$

and it is proved that it is a standard tautology.

Here we shall prove the following theorem.

Theorem 1.5.30 *The Hauber's Law is an IFT for standard intuitionistic fuzzy implication ($\rightarrow_4$), negation ($\neg_1$), disjunction and conjunction.*

Proof Let the formulas A, B, C, D be given and let everywhere:

$$V(A) = \langle a, b\rangle,$$

$$V(B) = \langle c, d\rangle,$$

$$V(C) = \langle e, f\rangle,$$

$$V(D) = \langle g, h\rangle.$$

Then, following [71] we calculate sequentially

$$V(((A \rightarrow B) \wedge (C \rightarrow D) \wedge (A \vee C) \wedge \neg(B \wedge D)) \rightarrow ((B \rightarrow A) \wedge (D \rightarrow C)))$$

$$= ((\langle a, b\rangle \rightarrow \langle c, d\rangle) \wedge (\langle e, f\rangle \rightarrow \langle g, h\rangle) \wedge (\langle a, b\rangle \vee \langle e, f\rangle)$$

$$\wedge\neg(\langle c, d\rangle \wedge \langle g, h\rangle)) \rightarrow ((\langle c, d\rangle \rightarrow \langle a, b\rangle) \wedge (\langle g, h\rangle \rightarrow \langle e, f\rangle))$$

$$= (\langle \max(b, c), \min(a, d)\rangle \wedge \langle \max(f, g), \min(e, h)\rangle \wedge \langle \max(a, e), \min(b, f)\rangle$$

$$\wedge\langle \max(d, h), \min(c, g)\rangle) \rightarrow (\langle \max(a, d), \min(b, c)\rangle \wedge \langle \max(e, h), \min(f, g)\rangle)$$

$$= (\langle \min(\max(b, c), \max(f, g), \max(a, e), \max(d, h)), \max(\min(a, d), \min(e, h),$$

$$\min(b, f), \min(c, g))\rangle) \to \langle \min(\max(a,d), \max(e,h)), \max(\min(b,c), \min(f,g))\rangle$$

$$= \langle \max(\min(a, d), \min(e, h), \min(b, f), \min(c, g), \min(\max(a, d), \max(e, h))),$$

$$\min(\max(b, c), \max(f, g), \max(a, e), \max(d, h), \max(\min(b, c), \min(f, g)))\rangle.$$

Let

$$X \equiv \max(\min(a, d), \min(e, h), \min(b, f), \min(c, g), \min(\max(a, d), \max(e, h)))$$

$$- \min(\max(b, c), \max(f, g), \max(a, e), \max(d, h), \max(\min(b, c), \min(f, g)))$$

$$\geq \max(\min(a, d), \min(e, h), \min(\max(a, d), \max(e, h)))$$

$$- \min(\max(a, e), \max(d, h)).$$

If $a \geq d \geq e \geq h$, then $X \geq \max(d, h, \min(a, e)) - \min(a, d) = d - d = 0$.

If $a \geq d \geq h \geq e$, then $X \geq \max(d, e, \min(a, h)) - \min(a, d) = d - d = 0$.

If $a \geq e \geq d \geq h$, then $X \geq \max(d, h, \min(a, e)) - \min(a, d) = d - d = 0$.

If $a \geq e \geq h \geq d$, then $X \geq \max(d, h, \min(a, e)) - \min(a, h) = e - h \geq 0$.

If $a \geq h \geq d \geq e$, then $X \geq \max(d, e, \min(a, h)) - \min(a, h) = h - h = 0$.

If $a \geq h \geq e \geq d$, then $X \geq \max(d, e, \min(a, h)) - \min(a, h) = h - h = 0$.

If $d \geq a \geq e \geq h$, then $X \geq \max(a, h, \min(d, e)) - \min(a, d) = a - a = 0$.

If $d \geq a \geq h \geq e$, then $X \geq \max(a, e, \min(d, h)) - \min(a, d) = a - a = 0$.

If $d \geq e \geq a \geq h$, then $X \geq \max(a, h, \min(d, e)) - \min(e, d) = e - e = 0$.

If $d \geq e \geq h \geq a$, then $X \geq \max(a, h, \min(d, e)) - \min(e, d) = e - e = 0$.

If $d \geq h \geq a \geq e$, then $X \geq \max(a, e, \min(d, h)) - \min(a, d) = h - a \geq 0$.

If $d \geq h \geq e \geq a$, then $X \geq \max(a, e, \min(d, h)) - \min(e, d) = h - e \geq 0$.

If $e \geq a \geq d \geq h$, then $X \geq \max(d, h, \min(a, e)) - \min(e, d) = a - d \geq 0$.

If $e \geq a \geq h \geq d$, then $X \geq \max(d, h, \min(a, e)) - \min(e, h) = a - h \geq 0$.

If $e \geq d \geq a \geq h$, then $X \geq \max(a, h, \min(d, e)) - \min(e, d) = d - d = 0$.

If $e \geq d \geq h \geq a$, then $X \geq \max(a, h, \min(d, e)) - \min(e, d) = d - d = 0$.

If $e \geq h \geq a \geq d$, then $X \geq \max(d, h, \min(a, e)) - \min(e, h) = h - h = 0$.

If $e \geq h \geq d \geq a$, then $X \geq \max(a, h, \min(d, e)) - \min(e, h) = h - h = 0$.

If $h \geq a \geq e \geq d$, then $X \geq \max(d, e, \min(a, h)) - \min(a, h) = a - a = 0$.

If $h \geq a \geq d \geq e$, then $X \geq \max(d, e, \min(a, h)) - \min(a, h) = a - a = 0$.

If $h \geq d \geq a \geq e$, then $X \geq \max(a, e, \min(d, h)) - \min(a, h) = d - a \geq 0$.

If $h \geq d \geq e \geq a$, then $X \geq \max(a, e, \min(d, h)) - \min(e, h) = d - e \geq 0$.

If $h \geq e \geq a \geq d$, then $X \geq \max(d, e, \min(a, h)) - \min(e, h) = e - e = 0$.

If $h \geq e \geq d \geq a$, then $X \geq \max(a, e, \min(d, h)) - \min(e, h) = e - e = 0$. Therefore, the Hauber's Law is an IFT. □

This theorem generates the following open problem.

Open Problem 3. For which other intuitionistic fuzzy implications, negations, disjunctions and conjunctions the Hauber's Law is an IFT? Are there operations of these four types, for which the Law is a standard tautology?

The expression

$$((A \vee B) \wedge (\neg A \vee C)) \to (B \vee C) \tag{1.5.16}$$

is called "the basic axiom of the resolution" (see [72]). Obviously, it is a tautology in the sense of first-order logic. Here we discuss its intuitionistic fuzzy interpretation.

Theorem 1.5.31 *For every three formulas A, B and C,*
(a) (1.5.16) is an IFT for implications $\to_1, \ldots, \to_5, \to_8, \to_{11}, \to_{13}, \ldots, \to_{15}, \to_{17}, \to_{18}, \to_{20}, \ldots, \to_{24}, \to_{27}, \ldots, \to_{38}, \to_{40}, \to_{42}, \to_{44}, \to_{45}, \to_{61}, \to_{65}, \to_{68}, \to_{69}, \to_{71}, \to_{74}, \ldots, \to_{77}, \to_{79}, \to_{81}, \ldots, \to_{85}, \to_{88}, \ldots, \to_{90}, \to_{100}, \ldots, \to_{105}, \to_{109}, \ldots, \to_{118}, \to_{124}, \ldots, \to_{133}, \to_{151}, \to_{158}, \ldots, \to_{160}, \to_{166}, \ldots, \to_{170}, \to_{176}, \ldots, \to_{178}, \to_{180}, \ldots, \to_{183}, \to_{185}$.
(b) (1.5.16) is a tautology for implications $\to_2, \to_8, \to_{11}, \to_{14}, \to_{15}, \to_{20}, \to_{23}, \to_{31}, \to_{32}, \ldots, \to_{37}, \to_{40}, \to_{42}, \to_{74}, \to_{77}, \to_{83}, \to_{88}, \to_{176}, \to_{177}, \to_{178}, \to_{180}$.

Proof Following [73], we check the validity of (a) for implication $\to_4$. Let $V(A) = \langle a, b\rangle$, $V(B) = \langle c, d\rangle$, $V(C) = \langle e, f\rangle$, where $a, b, c, d, e, f \in [0, 1]$, $a + b \leq 1$, $c + d \leq 1$, $e + f \leq 1$. Then,

$$V(((A \vee B)) \wedge ((\neg A \vee C)) \to_4 (B \vee C))$$

$$= (\langle \max(a, c), \min(b, d)\rangle \wedge \langle \max(b, e), \min(a, f)\rangle) \to_4 \langle \max(c, e), \min(d, f)\rangle$$

$$= \langle \min(\max(a, c), \max(b, e)), \max(\min(b, d), \min(a, f))\rangle$$

$$\to_4 \langle \max(c, e), \min(d, f)\rangle$$

$$= \langle \max(c, e, \min(b, d), \min(a, f)), \min(d, f, \max(a, c), \max(b, e))\rangle.$$

Then,

$$\max(c, e, \min(b, d), \min(a, f)) - \min(d, f, \max(a, c), \max(b, e))$$

$$\geq \max(c, \min(a, f)) - \min(f, \max(a, c)) \geq 0,$$

i.e., (1.5.16) is an IFT. □

Here, we give a modification of the Conditional logic VW (see [74]), discussed in [75]. It uses two different implications, mentioned by $\rightarrow$ and $\supset$. Here, for implication $\rightarrow$ we use implication $\rightarrow_4$ and for implication $\supset$ – implication $\rightarrow_{11}$. In [75], operation "$\rightarrow$" is called "ordinary material conditional" and operation "$\supset$" we use "counterfactual conditional".

Conditional logic contains the following axioms:

Axiom VW1. $p \supset p$
Axiom VW2. $(p \supset (q \rightarrow r)) \rightarrow ((p \supset q) \rightarrow (p \supset r))$
Axiom VW3. $(p \supset q) \rightarrow (p \rightarrow q)$
Axiom VW4. $(\neg p \supset p) \rightarrow (q \supset p)$
Axiom VW5. $((p \supset q) \wedge \neg(p \supset \neg r)) \rightarrow ((p \wedge r) \supset q)$
Axiom VW6. $((p \wedge q) \supset r) \rightarrow (p \supset (q \rightarrow r))$
Axiom VW7. $((p \supset q) \wedge (q \supset p)) \rightarrow ((p \supset r) \equiv (q \supset r))$
and rules:
Rule VW1. From p and $p \rightarrow q$ to infer q
Rule VW2. From $(p_1 \wedge \ldots \wedge p_n) \rightarrow q$, $r \supset p_1, \ldots, r \supset p_n$ to infer $r \supset q$ (where $n \geq 0$).

Here we shall assume that for the propositions x and y:

$$x \equiv y \text{ iff } (x \supset y) \wedge (y \supset x).$$

Let everywhere below $V(p) = \langle a, b\rangle$, $V(p_i) = \langle a_i, b_i\rangle$ $(1 \leq i \leq n;\ n$ - natural number), $V(q) = \langle c, d\rangle$ and $V(r) = \langle e, f\rangle$.

It is seen directly, that Rule1 is valid, while Rule2 is valid in the following form: if $(p_1 \wedge \ldots \wedge p_n) \rightarrow q$, is true and $r \supset p_1, \ldots, r \supset p_n$ are IF-sure, then, $r \supset q$ is IF-sure (where $n \geq 0$).

Let $(p_1 \wedge \ldots \wedge p_n) \rightarrow q$ is true, and let $r \supset p_1, \ldots, r \supset p_n$ are IF-sure, i.e.,

$$V((p_1 \wedge \ldots \wedge p_n) \rightarrow q) = \langle 1, 0\rangle,$$

$$\mu(r \supset p_1) \geq \frac{1}{2},$$

$$\ldots$$

$$\mu(r \supset p_n) \geq \frac{1}{2}.$$

Therefore,

$$1 - (1 - c).\text{sg}(\min(a_1, \ldots, a_n) - c) = 1,$$

$$d.\text{sg}(\min(a_1, \ldots, a_n) - c).\text{sg}(d - \max(b_1, \ldots, b_n)) = 0,$$

$$\max(a_1, f) \geq \frac{1}{2},$$

$$\ldots$$

$$\max(a_n, f) \geq \frac{1}{2}.$$

Let us assume that $r \supset q$ is not an IS, i.e.,

$$\max(c, f) < \frac{1}{2}.$$

Therefore, $c < \frac{1}{2}$ and $f < \frac{1}{2}$. Therefore, for every i $(1 \leq i \leq n) : a_i \geq \frac{1}{2}$, i.e.,

$$\min(a_1, \ldots, a_n) \geq \frac{1}{2}.$$

Hence $\text{sg}(\min(a_1, \ldots, a_n) - c) = 1$, i.e., $1 - (1 - c) = 1$ and $c = 1$, which is a contradiction.

Therefore, $\max(c, f) \geq \frac{1}{2}$ and hence $r \supset q$ is an IF-sure.

Now, we shall check the validity of the above Axioms.

Axiom VW1 is an IFT, because:

$$V(p \supset p) = \langle 1 - (1 - a).\text{sg}(a - a), b.\text{sg}(a - a).\text{sg}(b - b)\rangle = \langle 1, 0\rangle.$$

Axiom VW2 is an IFT. This is valid for the following reason.

$$\begin{aligned}
&V((p \supset (q \rightarrow r)) \rightarrow ((p \supset q) \rightarrow (p \supset r)))\\
&= (\langle a, b\rangle \supset \langle 1 - (1 - e).\text{sg}(c - e), f.\text{sg}(c - e).\text{sg}(f - d)\rangle)\\
&\quad \rightarrow (\langle \max(b, c), \min(a, d)\rangle \rightarrow \langle \max(b, e), \min(a, f)\rangle)\\
&= \langle \max(b, 1 - (1 - e).\text{sg}(c - e)), \min(a, f.\text{sg}(c - e).\text{sg}(f - d))\rangle\\
&\quad \rightarrow \langle 1 - (1 - \max(b, e)).\text{sg}(\max(b, c) - \max(b, e)), \min(a, f)\\
&\quad .\text{sg}(\max(b, c) - \max(b, e)).\text{sg}(\min(a, f) - \min(a, d))\rangle\\
&= \langle 1 - (1 - (1 - (1 - \max(b, e)).\text{sg}(\max(b, c) - \max(b, e))))\\
&\quad .\text{sg}(\max(b, 1 - (1 - e).\text{sg}(c - e)) - 1 + (1 - \max(b, e))\\
&\quad .\text{sg}(\max(b, c) - \max(b, e))), \min(a, f).\text{sg}(\max(b, c) - \max(b, e))\\
&\quad .\text{sg}(\min(a, f) - \min(a, d)).\text{sg}(\max(b, 1 - (1 - e).\text{sg}(c - e)) - 1\\
&\quad +(1 - \max(b, e)).\text{sg}(\max(b, c) - \max(b, e)))
\end{aligned}$$

$.\text{sg}(\min(a, f).\text{sg}(\max(b, c) - \max(b, e))$
$.\text{sg}(\min(a, f) - \min(a, d)) - \min(a, f.\text{sg}(c - e).\text{sg}(f - d)))\rangle$.

Let

$A \equiv 1 - (1 - (1 - (1 - \max(b, e)).\text{sg}(\max(b, c) - \max(b, e))))$
$.\text{sg}(\max(b, 1 - (1 - e).\text{sg}(c - e)) - 1 + (1 - \max(b, e))$
$.\text{sg}(\max(b, c) - \max(b, e))) - \min(a, f).\text{sg}(\max(b, c) - \max(b, e))$
$.\text{sg}(\min(a, f) - \min(a, d)).\text{sg}(\max(b, 1 - (1 - e).\text{sg}(c - e)) - 1$
$+(1 - \max(b, e)).\text{sg}(\max(b, c) - \max(b, e))).\text{sg}(\min(a, f).\text{sg}(\max(b, c)$
$- \max(b, e)).\text{sg}(\min(a, f) - \min(a, d)) - \min(a, f.\text{sg}(c - e).\text{sg}(f - d)))$.

Then,

1. If $c \leq e$, or if $c > e$ and $b \geq \max(c, e)$, then, $\max(b, e) \geq \max(b, c)$ and
   $A = 1 - \text{sg}(\max(b, 1 - (1 - e).\text{sg}(c - e)) - 1) - \min(a, f)$
   $.\text{sg}(\max(b, c) - \max(b, e)).\text{sg}(\min(a, f) - \min(a, d)).\text{sg}(\max(b, 1 - (1 - e)$
   $.\text{sg}(c - e)) - 1).\text{sg}(\min(a, f).\text{sg}(\max(b, c) - \max(b, e))$
   $.\text{sg}(\min(a, f) - \min(a, d)) - \min(a, f.\text{sg}(c - e).\text{sg}(f - d)))$
   $= 1 - 0 = 1;$
2. If $c > \max(b, e)$, then
   $A = \text{sg}(\max(b, 1 - (1 - e).\text{sg}(c - e)) - 1 + (1 - \max(b, e)).\text{sg}(\max(b, c) - \max(b, e)))$
   $= \text{sg}(\max(b, 1 - (1 - e)) - 1 + (1 - \max(b, e)))$
   $= \text{sg}(\max(b, -e) - \max(b, e)) = 0$
   and
   $A = 0 - 0 = 0.$

   Therefore, in both cases $A \geq 0$, i.e.,

$$(p \supset (q \rightarrow r)) \rightarrow ((p \supset q) \rightarrow (p \supset r))$$

is an IFT.

For **Axiom VW3** we obtain

$V((p \supset q) \rightarrow (p \rightarrow q)) = \langle \max(b, c), \min(a, d)\rangle \rightarrow \langle 1 - (1 - c)$
$.\text{sg}(a - c), d.\text{sg}(a - c).\text{sg}(d - b)\rangle$
$= \langle 1 - (1 - (1 - (1 - c).\text{sg}(a - c))).\text{sg}(\max(b, c) - (1 - (1 - c).\text{sg}(a - c))),$
$d.\text{sg}(a - c).\text{sg}(d - b).\text{sg}(\max(b, c) - (1 - (1 - c)$
$.\text{sg}(a - c))).\text{sg}(d.\text{sg}(a - c).\text{sg}(d - b) - \min(a, d))\rangle$.

Let

$A \equiv 1 - (1 - (1 - (1 - c).\text{sg}(a - c))).\text{sg}(\max(b, c) - (1 - (1 - c).\text{sg}(a - c)))$
$-d.\text{sg}(a - c).\text{sg}(d - b).\text{sg}(\max(b, c) - (1 - (1 - c).\text{sg}(a - c)))$.
$\text{sg}(d.\text{sg}(a - c).\text{sg}(d - b) - \min(a, d))$.

1. If $a \leq c$, then,
   $A = 1 - 0 = 1.$
2. If $a > c$, then
   $A = 1 - (1 - c).\text{sg}(\max(b, c) - c) - d.\text{sg}(d - b).\text{sg}(\max(b, c) - c).\text{sg}(d.\text{sg}(d - b) - \min(a, d))$

2.1. If $b \le c$, then,
$A = 1 - 0 = 1$;
2.2. If $b > c$, then,
$A = c - d.\text{sg}(d - b).\text{sg}(d.\text{sg}(d - b) - \min(a, d))$
2.2.1. If $d \le b$, then,
$A = c - 0 = c$;
2.2.2. If $d > b$, then,
$A = c - d.\text{sg}(d - \min(a, d))$
2.2.2.1. If $a \ge d$, then,
$A = c - 0 = c$;
2.2.2.2. If $a < d$, then,
$A = c - d < 0$,

because $d > b > c$. Therefore, in this case the Axiom VW3 is not valid.

On the other hand, the following form of Axiom VW3 is valid:

AxiomVW3′. $(p \to q) \supset (p \supset q)$.

In this case:

$V((p \to q) \supset (p \supset q))$
$= \langle 1 - (1 - c).\text{sg}(a - c), d.\text{sg}(a - c).\text{sg}(d - b)\rangle \supset \langle \max(b, c), \min(a, d)\rangle$
$= \langle \max(b, c, d.\text{sg}(a - c).\text{sg}(d - b)), \min(a, d, 1 - (1 - c).\text{sg}(a - c))\rangle$.

Let

$$A \equiv \max(b, c, d.\text{sg}(a - c).\text{sg}(d - b)) - \min(a, d, 1 - (1 - c).\text{sg}(a - c))\rangle.$$

If $a \le c$, then,
$A = \max(b, c, 0) - \min(a, d, 1)) \ge c - a \ge 0$;
If $a > c$, then,
$A = \max(b, c, d.\text{sg}(d - b)) - \min(a, d, c) \ge c - c = 0$,
i.e., Axiom VW3′ is an IFT.

Axiom VW4 is an IFT, because:
$V((\neg p \supset p) \to (q \supset p))$
$= \langle \max(a, a), \min(b, b)\rangle \to \langle \max(a, d), \min(b, c)\rangle$
$= \langle a, b\rangle \to \langle \max(a, d), \min(b, c)\rangle$
$= \langle \max(a, b, d), \min(a, b, c)\rangle$
and obviously,

$$\max(a, b, d) - \min(a, b, c) \ge 0.$$

For **Axiom VW5** we obtain
$V(((p \supset q) \wedge \neg(p \supset \neg r)) \to ((p \wedge r) \supset q))$
$= (\langle \max(b, c), \min(a, d)\rangle \wedge \langle \min(a, e),$
$\max(b, f)\rangle) \to (\langle \min(a, e), \max(b, f)\rangle \to \langle c, d\rangle)$
$= \langle \min(a, e, \max(b, c)), \max(b, f, \min(a, d))\rangle \to \langle \max(b, c, f), \min(a, d, e)\rangle$
$= \langle 1 - (1 - \max(b, c, f)).\text{sg}(\min(a, e, \max(b, c)) - \max(b, c, f)), \min(a, d, e)$.

$\text{sg}(\min(a, e, \max(b, c)) - \max(b, c, f)).\text{sg}(\min(a, d, e)$
$- \max(b, f, \min(a, d)))\rangle$
$= \langle 1, 0\rangle$,
because

$$\min(a, e, \max(b, c)) \leq \max(b, c,) \leq \max(b, c, f))$$

and, therefore,

$$\text{sg}(\min(a, e, \max(b, c)) - \max(b, c, f)) = 0.$$

Therefore,

$$((p \supset q) \wedge \neg(p \supset \neg r)) \rightarrow ((p \wedge r) \supset q))$$

is an IFT.

For **Axiom VW6** we obtain
$V(((p \wedge q) \supset r) \rightarrow (p \supset (q \rightarrow r)))$
$= (\min(a, c), \max(b, d)\rangle \supset \langle e, f\rangle)$
$\rightarrow (\langle a, b\rangle \supset \langle 1 - (1 - e).\text{sg}(c - e), f.\text{sg}(c - e).\text{sg}(f - d)\rangle)$
$= \langle \max(b, d, e), \min(a, c, f)\rangle \rightarrow \langle \max(b, 1 - (1 - e).\text{sg}(c - e)),$
$\min(a, f.\text{sg}(c - e).\text{sg}(f - d))\rangle$
$= \langle 1 - (1 - \max(b, 1 - (1 - e).\text{sg}(c - e))).\text{sg}(\max(b, d, e) - \max(b,$
$1 - (1 - e).\text{sg}(c - e))), \min(a, f.\text{sg}(c - e).\text{sg}(f - d)).\text{sg}(\max(b, d, e)$
$- \max(b, 1 - (1 - e).\text{sg}(c - e))).\text{sg}(\min(a, f.\text{sg}(c - e)$
$.\text{sg}(f - d)) - \min(a, c, f))\rangle.$

Let
$A \equiv 1 - (1 - \max(b, 1 - (1 - e).\text{sg}(c - e))).\text{sg}(\max(b, d, e) - \max(b,$
$1 - (1 - e).\text{sg}(c - e))) - \min(a, f.\text{sg}(c - e).\text{sg}(f - d)).\text{sg}(\max(b, d, e)$
$- \max(b, 1 - (1 - e).\text{sg}(c - e))).\text{sg}(\min(a, f.\text{sg}(c - e).\text{sg}(f - d))$
$- \min(a, c, f)).$

1. Let $c \leq e$, then
   $A = 1 - (1 - \max(b, 1)).\text{sg}(\max(b, d, e) - \max(b, 1)) - \min(a, 0)$
   $.\text{sg}(\max(b, d, e) - \max(b, 1)).\text{sg}(\min(a, 0) - \min(a, c, f))$
   $= 1 - 0 = 1;$
2. Let $c > e$. Then,
   $A = 1 - (1 - \max(b, e)).\text{sg}(\max(b, d, e) - \max(b, e)) - \min(a, f.\text{sg}(f - d))$
   $.\text{sg}(\max(b, d, e) - \max(b, e)).\text{sg}(\min(a, f.\text{sg}(f - d)) - \min(a, c, f))$

   2.1. If $d \leq \max(b, e)$, then,
   $A = 1 - 0 = 1.$

   2.2. If $d > \max(b, e)$, then,
   $A = \max(b, e) - \min(a, f.\text{sg}(f - d)).\text{sg}(\min(a, f.\text{sg}(f - d)) - \min(a, c, f))$

   2.2.1. If $f \leq d$, then,
   $A = \max(b, e) - \min(a, 0) = \max(b, e) \geq 0$

2.2.2. If $f > d$, then,
$A = \max(b, e) - \min(a, f).\text{sg}(\min(a, f) - \min(a, c, f))$
2.2.2.1. If $c \geq \max(a, f)$, then,
$A = \max(b, e) - 0 \geq 0$;
2.2.2.2. If $c < \max(a, f)$, then,
$A = \max(b, e) - \min(a, f)$
2.2.2.2.1. If $a \leq \max(b, e)$, form $f > d > \max(b, e)$, it follows that $f > a$ and $A = \max(b, e) - a \geq 0$.
2.2.2.2.2. If $a > \max(b, e)$, form $f > d > \max(b, e)$, it follows that $\min(a, f) > \max(b, e)$

and, therefore, $A < 0$, i.e., the expression

$$((p \wedge q) \supset r) \rightarrow (p \supset (q \rightarrow r))$$

is valid in all cases without the last one. Hence, it is not an IFT.

On the other hand, the following form of it is valid:

AxiomVW6′. $((p \wedge q) \supset r) \rightarrow (p \supset (q \supset r))$ because
$V(((p \wedge q) \supset r) \rightarrow (p \supset (q \supset r)))$
$= \langle \min(a, c), \max(b, d)\rangle \supset \langle e, f\rangle) \rightarrow (\langle a, b\rangle \supset \langle \max(d, e), \min(c, f)\rangle)$
$= \langle \max(b, d, e), \min(a, c, f)\rangle \rightarrow \langle \max(b, d, e)), \min(a, c, f)\rangle$
$= \langle 1 - (1 - \max(b, d, e)).\text{sg}(\max(b, d, e) - \max(b, d, e)), \min(a, c, f)$
$.\text{sg}(\max(b, d, e) - \max(b, d, e)).\text{sg}(\min(a, c, f) - \min(a, c, f))\rangle$
$= \langle 1, 0\rangle$,
i.e.,

$$((p \wedge q) \supset r) \rightarrow (p \supset (q \supset r))$$

is an IFT.

For **Axiom VW7** we obtain:
$V(((p \supset q) \wedge (q \supset p)) \rightarrow ((p \supset r) \equiv (q \supset r)))$
$= (\langle \max(b, c), \min(a, d)\rangle \wedge \langle \max(a, d), \min(b, c)\rangle) \rightarrow ((\langle \max(b, e), \min(a, f)\rangle$
$\supset \langle \max(d, e), \min(c, f)\rangle \wedge (\langle \max(d, e), \min(c, f)\rangle \supset \langle \max(b, e),$
$\min(a, f)\rangle))$
$= \langle \min(\max(b, c), \max(a, d)), \max(\min(a, d), \min(b, c))\rangle$
$\rightarrow (\langle \max(\min(a, f), d, e), \min(c, f, \max(b, e))\rangle$
$\wedge \langle \max(\min(c, f), b, e), \min(a, f, \max(d, e))\rangle)$
$= \langle \min(\max(b, c), \max(a, d)), \max(\min(a, d), \min(b, c))\rangle \rightarrow$
$\langle \min(\max(\min(a, f), d, e), \max(\min(c, f), b, e)), \max(\min(a, f, \max(d, e)),$
$\min(c, f, \max(b, e)))\rangle$
$= \langle \max(\min(a, d), \min(b, c), \min(\max(\min(a, f), d, e), \max(\min(c, f), b, e))),$
$\min(\max(a, d), \max(b, c), \max(\min(a, f, \max(d, e)), \min(c, f, \max(b, e)))))\rangle.$

Let
$A \equiv \max(\min(a, d), \min(b, c), \min(\max(\min(a, f), d, e), \max(\min(c, f), b, e)))$

$- \min(\max(a, d), \max(b, c), \max(\min(a, f, \max(d, e)),$
$\min(c, f, \max(b, e)))).$

From the equalities:

$$\min(\max(a, x), \max(b, x)) = \max(x, \min(a, b))$$

$$\max(\min(a, x), \min(b, x)) = \min(x, \max(a, b))$$

it follows that:
$A = \max(\min(a, d), \min(b, c), \max(e, \min(\min(a, f), d, \min(c, f), b)))$
$\min(\max(a, d), \max(b, c), \min(f, \max(a, \max(d, e), c, \max(b, e))))$
$= \max(\min(a, d), \min(b, c), \max(e, \min(a, b, c, d, f))) - \min(\max(a, d),$
$\max(b, c), \min(f, \max(a, b, c, d, e)))$

1. If $e \geq f$, then
   $A \geq \max(e, \min(a, b, c, d, f)) - \min(f, \max(a, b, c, d, e)) \geq e - f \geq 0;$

2. If $e < f$, then, e.g., for $a = b = f > c = d = e$ we obtain:
   $A = \max(c, d, e) - \min(a, b, f) < 0,$

i.e., in this case Axiom VW7 is not an IFT.

On the other hand, the following form of **Axiom VW7** is valid:
$V(((p \to q) \wedge (q \to p)) \supset ((p \to r) \equiv (q \to r)))$
$= (\langle 1 - (1 - c).\text{sg}(a - c), d.\text{sg}(a - c).\text{sg}(d - b)\rangle \wedge \langle 1 - (1 - a).\text{sg}(c - a),$
$b.\text{sg}(c - a).\text{sg}(b - d)\rangle) \supset ((\langle 1 - (1 - e).\text{sg}(a - e), f.\text{sg}(a - e).\text{sg}(f - b)\rangle$
$\supset \langle 1 - (1 - e).\text{sg}(c - e), f.\text{sg}(c - e).\text{sg}(f - d)\rangle) \wedge (\langle 1 - (1 - e).\text{sg}(c - e),$
$f.\text{sg}(c - e).\text{sg}(f - d)\rangle \supset \langle 1 - (1 - e).\text{sg}(a - e), f.\text{sg}(a - e).\text{sg}(f - b)\rangle))$
$= \langle \min(1 - (1 - c).\text{sg}(a - c), 1 - (1 - a).\text{sg}(c - a)), \max(d.\text{sg}(a - c).\text{sg}(d - b),$
$b.\text{sg}(c - a).\text{sg}(b - d)\rangle \supset (\langle \max(f.\text{sg}(a - e).\text{sg}(f - b), 1 - (1 - e).\text{sg}(c - e)),$
$\min(1 - (1 - e).\text{sg}(a - e), f.\text{sg}(c - e).\text{sg}(f - d))\rangle$
$= \langle \max(f.\text{sg}(c - e).\text{sg}(f - d)\rangle, 1 - (1 - e).\text{sg}(a - e)),$
$\min(1 - (1 - e).\text{sg}(c - e), f.\text{sg}(a - e).\text{sg}(f - b))\rangle)$
$= \langle \min(1 - (1 - c).\text{sg}(a - c), 1 - (1 - a).\text{sg}(c - a)), \max(d.\text{sg}(a - c).\text{sg}(d - b),$
$b.\text{sg}(c - a).\text{sg}(b - d)\rangle \supset \langle \min(\max(f.\text{sg}(a - e).\text{sg}(f - b), 1 - (1 - e).\text{sg}(c - e)),$
$\max(f.\text{sg}(c - e).\text{sg}(f - d), 1 - (1 - e).\text{sg}(a - e))), \max(\min(1 - (1 - e).\text{sg}(a - e),$
$f.\text{sg}(c - e).\text{sg}(f - d)), \min(1 - (1 - e).\text{sg}(c - e), f.\text{sg}(a - e).\text{sg}(f - b)))\rangle$
$= \langle \max(d.\text{sg}(a - c).\text{sg}(d - b), b.\text{sg}(c - a).\text{sg}(b - d), \min(\max(f.\text{sg}(a - e).\text{sg}(f - b),$
$1 - (1 - e).\text{sg}(c - e)), \max(f.\text{sg}(c - e).\text{sg}(f - d), 1 - (1 - e).\text{sg}(a - e)))),$
$\min(1 - (1 - c).\text{sg}(a - c), 1 - (1 - a).\text{sg}(c - a), \max(\min(1 - (1 - e)$
$.\text{sg}(a - e), f.\text{sg}(c - e).\text{sg}(f - d)), \min(1 - (1 - e).\text{sg}(c - e),$
$f.\text{sg}(a - e).\text{sg}(f - b)))\rangle.$

Let
$A \equiv \max(d.\text{sg}(a - c).\text{sg}(d - b), b.\text{sg}(c - a).\text{sg}(b - d), \min(\max(f.\text{sg}(a - e)$
$.\text{sg}(f - b), 1 - (1 - e).\text{sg}(c - e)), \max(f.\text{sg}(c - e).\text{sg}(f - d), 1 - (1 - e)$
$.\text{sg}(a - e)))) - \min(1 - (1 - c).\text{sg}(a - c), 1 - (1 - a).\text{sg}(c - a),$

$\max(\min(1-(1-e).\text{sg}(a-e), f.\text{sg}(c-e).\text{sg}(f-d)), \min(1-(1-e).\text{sg}(c-e), f.\text{sg}(a-e).\text{sg}(f-b)))$.

1. Let $a \leq c$, then:
$A = \max(0, b.\text{sg}(b-d), \min(\max(f.\text{sg}(a-e).\text{sg}(f-b), 1-(1-e).\text{sg}(c-e)), \max(f.\text{sg}(c-e).\text{sg}(f-d), 1-(1-e).\text{sg}(a-e)))) - \min(1, 1-(1-a).\text{sg}(c-a), \max(\min(1-(1-e).\text{sg}(a-e), f.\text{sg}(c-e).\text{sg}(f-d)), \min(1-(1-e).\text{sg}(c-e), f.\text{sg}(a-e).\text{sg}(f-b))))$
$= \max(b.\text{sg}(b-d), \min(\max(f.\text{sg}(a-e).\text{sg}(f-b), 1-(1-e).\text{sg}(c-e)), \max(f.\text{sg}(c-e).\text{sg}(f-d), 1-(1-e).\text{sg}(a-e)))) - \min(1-(1-a).\text{sg}(c-a), \max(\min(1-(1-e).\text{sg}(a-e), f.\text{sg}(c-e).\text{sg}(f-d)), \min(1-(1-e).\text{sg}(c-e), f.\text{sg}(a-e).\text{sg}(f-b))))$.

1.1. If $a \leq e$, then
$A = \min(\max(0, 1-(1-e).\text{sg}(c-e)), \max(f.\text{sg}(c-e).\text{sg}(f-d), 1)) - \min(1-(1-a).\text{sg}(c-a), \max(\min(1, f.\text{sg}(c-e).\text{sg}(f-d)), \min(1-(1-e).\text{sg}(c-e), 0)))$
$= \min(1-(1-e).\text{sg}(c-e), 1) - \min(1-(1-a).\text{sg}(c-a), \max(f.\text{sg}(c-e).\text{sg}(f-d), 0))$
$= 1-(1-e).\text{sg}(c-e) - \min(1-(1-a).\text{sg}(c-a), f.\text{sg}(c-e).\text{sg}(f-d))$

1.1.1. If $c \leq e$, then
$A = 1 - \min(1-(1-a).\text{sg}(c-a), 0) = 1 - 0 = 1$;

1.1.2. If $c > e$, then, $c > a$ and $A = e - \min(a, f.\text{sg}(f-d)) \geq e - a \geq 0$;

1.2. If $a > e$, then, $c > e$ and
$A = \max(b.\text{sg}(b-d), \min(\max(f.\text{sg}(f-b), e), \max(f.\text{sg}(f-d), 1-e))) - \min(1-(1-a).\text{sg}(c-a), \max(\min(1-e, f.\text{sg}(f-d)), \min(1-e, f.\text{sg}(f-b)))$

1.2.1. If $f \leq b$, then,
$A = \max(b.\text{sg}(b-d), \min(\max(0, e), \max(f.\text{sg}(f-d), 1-e))) - \min(1-(1-a)$
$.\text{sg}(c-a), \max(\min(1-e, f.\text{sg}(f-d)), \min(1-e, 0))$
$= \max(b.\text{sg}(b-d), \min(e, \max(f.\text{sg}(f-d), 1-e))) - \min(1-(1-a).\text{sg}(c-a),$
$\max(\min(1-e, f.\text{sg}(f-d)), 0))$
$= \max(b.\text{sg}(b-d), \min(e, \max(f.\text{sg}(f-d), 1-e))) - \min(1-(1-a).\text{sg}(c-a),$
$1-e, f.\text{sg}(f-d))$

1.2.1.1. If $f \leq d$, then,
$A = \max(b.\text{sg}(b-d), \min(e, \max(0, 1-e))) - \min(1-(1-a).\text{sg}(c-a), 1-e, 0)$
$= \max(b.\text{sg}(b-d), \min(e, 1-e)) - 0 \geq 0$;

1.2.1.2. If $f > d$, then, from $e + f \leq 1$
$A = \max(b, \min(e, \max(f, 1-e)) - \min(1-(1-a).\text{sg}(c-a), 1-e, f)$

$= \max(b, \min(e, 1-e)) - \min(1-(1-a).\text{sg}(c-a), f)$

1.2.1.2.1. If $c = a$ (below we assume that $c \geq a$) then,

$A = \max(b, \min(e, 1-e)) - \min(1, f)$

$= \max(b, \min(e, 1-e)) - f \geq b - f \geq 0;$

1.2.1.2.2. If $c > a$, then,

$A = \max(b, \min(e, 1-e)) - \min(a, f) \geq b - f \geq 0;$

1.2.2. If $f > b$, then,

$A = \max(b.\text{sg}(b-d), \min(\max(f, e), \max(f.\text{sg}(f-d), 1-e)))$
$- \min(1-(1-a).\text{sg}(c-a), \max(\min(1-e, f.\text{sg}(f-d)), \min(1-e, f)))$

1.2.2.1. If $f \leq d$, then, $d > b$ and

$A = \max(0, \min(\max(f, e), \max(0, 1-e))) - \min(1-(1-a).\text{sg}(c-a), \max(\min(1-e, 0), \min(1-e, f)))$

$= \min(\max(f, e), 1-e) - \min(1-(1-a).\text{sg}(c-a), \max(0, \min(1-e, f)))$

$= \min(\max(f, e), 1-e) - \min(1-(1-a).\text{sg}(c-a), f)$

$\geq \min(\max(f, e), 1-e) - f \geq 0,$

because as $\max(f, e) \geq f$, as well as $1-e \geq f$;

1.2.2.2. If $f > d$, then,

$A = \max(b.\text{sg}(b-d), \min(\max(f, e), \max(f, 1-e))) - \min(1-(1-a).\text{sg}(c-a),$
$\max(\min(1-e, f), \min(1-e, f)))$

$= \max(b.\text{sg}(b-d), \min(\max(f, e), 1-e)) - \min(1-(1-a).\text{sg}(c-a), \max(f, f))$

$= \max(b.\text{sg}(b-d), \min(\max(f, e), 1-e)) - \min(1-(1-a).\text{sg}(c-a), f)$

$\geq \min(\max(f, e), 1-e)) - f \geq 0;$

2. Let $a > c$, then

$A = \max(d.\text{sg}(d-b), 0, \min(\max(f.\text{sg}(a-e).\text{sg}(f-b), 1-(1-e).\text{sg}(c-e)),$
$\max(f.\text{sg}(c-e).\text{sg}(f-d), 1-(1-e).\text{sg}(a-e)))) - \min(c, 1, \max(\min(1-(1-e).\text{sg}(a-e), f.\text{sg}(c-e).\text{sg}(f-d)), \min(1-(1-e).\text{sg}(c-e), f.\text{sg}(a-e).\text{sg}(f-b))))$

$= \max(d.\text{sg}(d-b), \min(\max(f.\text{sg}(a-e).\text{sg}(f-b), 1-(1-e).\text{sg}(c-e)),$
$\max(f.\text{sg}(c-e).\text{sg}(f-d), 1-(1-e).\text{sg}(a-e)))) - \min(c, \max(\min(1-(1-e).\text{sg}(a-e), f.\text{sg}(c-e).\text{sg}(f-d)), \min(1-(1-e).\text{sg}(c-e), f.\text{sg}(a-e).\text{sg}(f-b))))$

2.1. If $a \leq e$, then, $e > c$ and

$A = \max(d.\text{sg}(d-b), \min(\max(0, 1), \max(0, 1)))) - \min(c,$
$\max(\min(e, 0), \min(1, 0)))$

$= \max(d.\text{sg}(d-b), \min(1, 1)) - \min(c, \max(0, 0))$

$= \max(d.\text{sg}(d-b), 1) - \min(c, 0)$

$= 1 - 0 = 1;$

2.2. If $a > e$, then,

$A = \max(d.\text{sg}(d-b), \min(\max(f.\text{sg}(f-b), 1-(1-e).\text{sg}(c-e)),$

$\max(f.\text{sg}(c-e).\text{sg}(f-d), e))) - \min(c, \max(\min(e,$
$f.\text{sg}(c-e).\text{sg}(f-d)), \min(1-(1-e).\text{sg}(c-e), f.\text{sg}(f-b))))$

2.2.1. If $f \leq b$, then,

$A = \max(d.\text{sg}(d-b), \min(\max(0, 1-(1-e).\text{sg}(c-e)), \max(f.\text{sg}(c-e).\text{sg}(f-d),$
$e))) - \min(c, \max(\min(e, f.\text{sg}(c-e).\text{sg}(f-d)), \min(1-(1-e).\text{sg}(c-e), 0)))$
$= \max(d.\text{sg}(d-b), \min(1-(1-e).\text{sg}(c-e), \max(f.\text{sg}(c-e).\text{sg}(f-d), e))) - \min(c,$
$\max(\min(e, f.\text{sg}(c-e).\text{sg}(f-d)), 0))$
$= \max(d.\text{sg}(d-b), \min(1-(1-e).\text{sg}(c-e), \max(f.\text{sg}(c-e).\text{sg}(f-d), e)))$
$- \min(c, \min(e, f.\text{sg}(c-e).\text{sg}(f-d)))$

2.2.1.1. If $f \leq d$, then,

$A = \max(d.\text{sg}(d-b), \min(1-(1-e).\text{sg}(c-e), \max(0, e))) - \min(c, \min(e, 0))$
$= \max(d.\text{sg}(d-b), \min(1-(1-e).\text{sg}(c-e), e)) - \min(c, 0)$
$= \max(d.\text{sg}(d-b), \min(1-(1-e).\text{sg}(c-e), e)) - 0 \geq 0;$

2.2.1.2. If $f > d$, then, $b > d$ and

$A = \max(0, \min(1-(1-e).\text{sg}(c-e), \max(f.\text{sg}(c-e), e)))$
$- \min(c, \min(e, f.\text{sg}(c-e)))$
$= \min(1-(1-e).\text{sg}(c-e), \max(f.\text{sg}(c-e), e))) - \min(c, \min(e, f.\text{sg}(c-e)))$

2.2.1.2.1. If $c \leq e$, then,

$A = \min(1, \max(0, e)) - \min(c, \min(e, 0))$
$= \min(1, e) - \min(c, 0) = e - 0 \geq 0;$

2.2.1.2.2. If $c > e$, then,

$A = \min(e, \max(f, e)) - \min(c, \min(e, f))$
$= e - \min(c, e, f) \geq 0;$

2.2.2. If $f > b$, then,

$A = \max(d.\text{sg}(d-b), \min(\max(f, 1-(1-e).\text{sg}(c-e)), \max(f.\text{sg}(c-e).\text{sg}(f-d), e))) - \min(c, \max(\min(e, f.\text{sg}(c-e).\text{sg}(f-d)), \min(1-(1-e).\text{sg}(c-e), f)))$

2.2.2.1. If $f \leq d$, then,

$A = \max(d.\text{sg}(d-b), \min(\max(f, 1-(1-e).\text{sg}(c-e)), \max(0, e)))$
$- \min(c, \max(\min(e, 0), \min(1-(1-e).\text{sg}(c-e), f)))$
$= \max(d.\text{sg}(d-b), \min(\max(f, 1-(1-e).\text{sg}(c-e)), e))$
$- \min(c, \max(0, \min(1-(1-e).\text{sg}(c-e), f)))$
$= \max(d.\text{sg}(d-b), \min(\max(f, 1-(1-e).\text{sg}(c-e)), e))$
$- \min(c, 1-(1-e).\text{sg}(c-e), f)$

2.2.2.1.1. If $c \leq e$, then,

$A = \max(d.\text{sg}(d-b), \min(\max(f, 1), e)) - \min(c, 1, f)$

$= \max(d.\mathrm{sg}(d-b), \min(1, e)) - \min(c, f)$
$= \max(d.\mathrm{sg}(d-b), e) - \min(c, f) \geq e - c \geq 0;$

2.2.2.1.2. If $c > e$, then,
$A = \max(d.\mathrm{sg}(d-b), \min(\max(f, e), e)) - \min(c, e, f)$
$= \max(d.\mathrm{sg}(d-b), e) - \min(c, e, f) \geq e - \min(c, e, f)$
$\geq 0;$

2.2.2.2. If $f > d$, then,
$A = \max(d.\mathrm{sg}(d-b), \min(\max(f, 1-(1-e).\mathrm{sg}(c-e)),$
$\max(f.\mathrm{sg}(c-e), e))) - \min(c, \max(\min(e, f.\mathrm{sg}(c-e)),$
$\min(1-(1-e).\mathrm{sg}(c-e), f)))$

2.2.2.2.1. If $c \leq e$, then,
$A = \max(d.\mathrm{sg}(d-b), \min(\max(f, 1), \max(0, e)))$
$- \min(c, \max(\min(e, 0), \min(1, f)))$
$= \max(d.\mathrm{sg}(d-b), \min(1, e)) - \min(c, \max(0, f))$
$= \max(d.\mathrm{sg}(d-b), e) - \min(c, f) \geq e - c \geq 0;$

2.2.2.2.2. If $c > e$, then,
$A = \max(d.\mathrm{sg}(d-b), \min(\max(f, e), \max(f, e)))$
$- \min(c, \max(\min(e, f), \min(e, f)))$
$= \max(d.\mathrm{sg}(d-b), \max(f, e)) - \min(c, \min(e, f))$
$\geq \max(f, e) - \min(e, f) \geq 0,$

i.e., in this case Axiom VW7′ is an IFT. □

So, we checked the validity of original or modified Conditional Logic Axioms for pair $(\rightarrow_4, \rightarrow_{11})$.

Open Problem 4. Which pairs of implications $(\rightarrow_i, \rightarrow_j)$ for $1 \leq i, j \leq 185$ satisfy these axioms in original or modified forms?

For example, we mention that the original Axioms are valid only for pairs $(\rightarrow_{20}, \rightarrow_{20})$, $(\rightarrow_{23}, \rightarrow_{23})$, $(\rightarrow_{74}, \rightarrow_{74})$ and $(\rightarrow_{77}, \rightarrow_{77})$, i.e., for the case, when the implications $\supset$ and $\rightarrow$ in the seven axioms coincide. If we change the condition for the Axioms to be true with condition for these axioms to be IFTs, then, these pairs with equal components are generated by implications $\rightarrow_{20}, \rightarrow_{23}, \rightarrow_{27}, \rightarrow_{29}, \rightarrow_{74}, \rightarrow_{77}, \rightarrow_{81}, \rightarrow_{101}, \rightarrow_{102}, \rightarrow_{111}, \ldots, \rightarrow_{113}, \rightarrow_{118}, \rightarrow_{126}, \rightarrow_{128}, \rightarrow_{167}, \rightarrow_{169}$.

1.6 De Morgan Laws and Law for Excluded Middle

Following and extending [59], first, we give the Law for Excluded Middle (LEM) in the forms:

$$V(A \vee \neg A) = \langle 1, 0 \rangle$$

(standard tautology) and

$$V(A \vee \neg A) = \langle p, q \rangle,$$

(IFT), where $1 \geq p \geq q \geq 0$ and $p + q \leq 1$.

Second, we give the Modified Law for Excluded Middle (MLEM) in the forms:

$$\neg\neg\langle a, b \rangle \vee \neg\langle a, b \rangle = \langle 1, 0 \rangle$$

(standard tautology) and

$$\neg\neg\langle a, b \rangle \vee \neg\langle a, b \rangle = \langle p, q \rangle,$$

(IFT), where $1 \geq p \geq q \geq 0$ and $p + q \leq 1$ and $\vee$ is the disjunction from Sect. 1.1.

Theorem 1.6.1 *Only negation $\neg_{13}$ satisfies the LEM in the tautological form.*

Proof Let $V(A) = \langle a, b \rangle$ and $a, b, a + b \in [0, 1]$. Then,

$$V(x \vee \neg_{13} x) = \langle a, b \rangle \vee \langle \text{sg}(1 - a), \overline{\text{sg}}(1 - a) \rangle$$

$$= \langle \max(a, \text{sg}(1 - a)), \min(b, \overline{\text{sg}}(1 - a)) \rangle.$$

Now, we see that

$$\max(a, \text{sg}(1 - a)) = \begin{cases} \max(1, \text{sg}(0)), & \text{if } a = 1 \\ \max(a, \text{sg}(1 - a)), & \text{if } a < 1 \end{cases}$$

$$= \begin{cases} \max(1, 0), & \text{if } a = 1 \\ \max(a, 1), & \text{if } a < 1 \end{cases} = 1$$

and

$$\min(b, \overline{\text{sg}}(1 - a)) = \begin{cases} \min(0, \overline{\text{sg}}(0)), & \text{if } a = 1 \\ \min(0, \text{sg}(1 - a)), & \text{if } a < 1 \end{cases}$$

$$= \begin{cases} \min(0, 1), & \text{if } a = 1 \\ \min(0, 1), & \text{if } a < 1 \end{cases} = 0.$$

Therefore,

$$V(x \vee \neg_{13} x) = \langle 1, 0 \rangle,$$

i.e., $x \vee \neg_{13} x$ is a tautology. □

Theorem 1.6.2 *Only negations $\neg_2$, $\neg_5$, $\neg_9$, $\neg_{11}$, $\neg_{13}$, $\neg_{16}$ satisfy the MLEM in the tautological form.*

Theorem 1.6.3 *Negations $\neg_1$, $\neg_3$, $\neg_4$, $\neg_7$, ..., $\neg_9$, $\neg_{11}$, ..., $\neg_{34}$, $\neg_{40}$, $\neg_{43}$, $\neg_{48}$, $\neg_{49}$, $\neg_{52}$, $\neg_{53}$ satisfy the LEM in the IFT form.*

Theorem 1.6.4 *Negations* $\neg_1, \ldots, \neg_9, \neg_{11}, \ldots, \neg_{34}, \neg_{43}, \neg_{48}, \neg_{49}, \neg_{52}, \neg_{53}$ *satisfy the MLEM in the IFT form.*

The checks of the last three assertions are similar to the proof of Theorem 1.6.1. By this reason we will prove only Theorems 1.6.2 and 1.6.4 for the case of negation $\neg_5$.

$$\neg_5\neg_5\langle a, b\rangle \vee \neg_5\langle a, b\rangle$$

$$= \langle 1 - \text{sg}(a) - \overline{\text{sg}}(a).\text{sg}(1-b), \text{sg}(1-b)\rangle \vee \langle 1 - \text{sg}(1 - \text{sg}(a) - \overline{\text{sg}}(a).\text{sg}(1-b))$$

$$-\overline{\text{sg}}(1 - \text{sg}(a) - \overline{\text{sg}}(a).\text{sg}(1-b)).\text{sg}(1 - \text{sg}(1-b)), \text{sg}(1 - \text{sg}(1-b))\rangle$$

$$= \langle \max(1 - \text{sg}(a) - \overline{\text{sg}}(a).\text{sg}(1-b), 1 - \text{sg}(1 - \text{sg}(a) - \overline{\text{sg}}(a).\text{sg}(1-b))$$

$$-\overline{\text{sg}}(1-\text{sg}(a)-\overline{\text{sg}}(a).\text{sg}(1-b)).\text{sg}(1-\text{sg}(1-b))), \min(\text{sg}(1-b), \text{sg}(1-\text{sg}(1-b)))\rangle.$$

Let

$$X \equiv \max(1 - \text{sg}(a) - \overline{\text{sg}}(a).\text{sg}(1-b), 1 - \text{sg}(1 - \text{sg}(a) - \overline{\text{sg}}(a).\text{sg}(1-b))$$

$$-\overline{\text{sg}}(1 - \text{sg}(a) - \overline{\text{sg}}(a).\text{sg}(1-b)).\text{sg}(1 - \text{sg}(1-b)))$$

$$- \min(\text{sg}(1-b), \text{sg}(1 - \text{sg}(1-b))).$$

Let $a = 0$. Then, $\text{sg}(a) = 0$, $\overline{\text{sg}}(a) = 1$ and

$$X = \max(1-\text{sg}(1-b), 1-\text{sg}(1-\text{sg}(1-b)) - \overline{\text{sg}}(1-\text{sg}(1-b)).\text{sg}(1-\text{sg}(1-b)))$$

$$- \min(\text{sg}(1-b), \text{sg}(1 - \text{sg}(1-b))).$$

If $b = 1$, then, $\text{sg}(1-b) = 0$ and

$$X = \max(1, 1-\text{sg}(1)-\overline{\text{sg}}(1).\text{sg}(1)) - \min(0, \text{sg}(1)) = \max(1, 0) - \min(0, 1) = 1.$$

If $b < 1$, then, $\text{sg}(1-b) = 1$ and

$$X = \max(1-1, 1 - \text{sg}(1-1) - \overline{\text{sg}}(1-1).\text{sg}(1-1)) - \min(1, \text{sg}(1-1))$$

$$= \max(0, 1) - \min(1, 0) = 1.$$

Let $a > 0$. Then, $\text{sg}(a) = 1$, $\overline{\text{sg}}(a) = 0$, $\text{sg}(1-b) = 1$ and

$$X \equiv \max(1-1, 1 - \text{sg}(1-1) - \overline{\text{sg}}(1-1).\text{sg}(1-1)) - \min(1, \text{sg}(1-1))$$

$$= \max(0, 1) - \min(1, 0) = 1.$$

Therefore, negation $\neg_5$ satisfies the Modified LEM in the IFT-form. On the other hand, in all cases the evaluation of the expression is equal to $\langle 1, 0\rangle$, i.e., this negation satisfies the Modified LEM in the tautological form. □

Third, following and extending [76], we study which negations satisfy De Morgan Laws (DMLs). Usually, they have the forms:

$$\neg x \wedge \neg y = \neg(x \vee y),$$

$$\neg x \vee \neg y = \neg(x \wedge y),$$

where $\wedge$ and $\vee$ are the conjunction and disjunction from Sect. 1.1, see (1.1.4) and (1.1.5).

Theorem 1.6.5 *For every two formulas A and B:*

$$\neg_i A \wedge \neg_i B = \neg_i (A \vee B),$$

$$\neg_i A \vee \neg_i B = \neg_i (A \wedge B)$$

for $i = 1, 2, 4, \ldots, 11, 13, \ldots, 17, 20, 23, 35, \ldots, 51, 53.$

We shall illustrate only the fact that the DMLs are not valid for $i = 3$. For example, if $a = b = 0.5$, $c = 0.1$, $d = 0$, then

$$V(\neg_3 A \wedge \neg_3 B) = 0.5,$$

$$V(\neg_3 (A \vee B) = 0.25.$$

The above mentioned change of the Law for Excluded Middle inspired the idea to study the validity of De Morgan's Laws, which the classical negation $\neg$ (here it is negation $\neg_1$) satisfies. Indeed, it can be easily proved that the expressions

$$\neg_1(\neg_1 A \vee \neg_1 B) = A \wedge B$$

and

$$\neg_1(\neg_1 A \wedge \neg_1 B) = A \vee B$$

are IFTs, while the other negations do not satisfy these equalities. For them the following assertion is valid.

Theorem 1.6.6 *For every two formulas A and B, it holds that*

$$\neg_i(\neg_i A \vee \neg_i B) = \neg_i \neg_i A \wedge \neg_i \neg_i B,$$

$$\neg_i(\neg_i A \wedge \neg_i B) = \neg_i \neg_i A \vee \neg_i \neg_i B$$

for $i = 1, 2, 4, \dots, 11, 13, \dots, 17, 19, 20, 23, 25, 35, \dots, 51, 53.$

1.7 New Intuitionistic Fuzzy Conjunctions and Disjunctions

From classical logic, it is well-known that for any two formulas A and B:

$$A \vee B = \neg A \rightarrow B, \tag{1.7.1}$$

$$A \wedge B = \neg(A \rightarrow \neg B). \tag{1.7.2}$$

Therefore, having the above intuitionistic fuzzy implications and negations, we can construct 185 disjunctions and 185 conjunctions. In Sect. 1.2 we saw that implication (1.1.6), which is denoted there by $\rightarrow_4$, and negation (1.1.3), which in Sect. 1.4 is denoted by $\neg_1$, are connected by equality (1.2.1). So, using formulas (1.7.1) and (1.7.2) we can construct from them a disjunction and a conjunction, and they exactly coincide with these from (1.1.4) and (1.1.5), respectively. All other disjunctions and conjunctions can be constructed in the same manner. For example, if $V(A) = \langle a, b\rangle$, $V(B) = \langle c, d\rangle$, $a, b, c, d \in [0, 1]$, $a + b \leq 1$, $c + d \leq 1$ and if we use negation $\neg_2$ and its related implication $\rightarrow_2$, we obtain sequentially:

$$V(A \vee_2 B) = \langle a, b\rangle \vee_2 \langle c, d\rangle$$

$$= \neg_2\langle a, b\rangle \rightarrow_2 \langle c, d\rangle$$

$$= \langle \overline{\text{sg}}(a), \text{sg}(a)\rangle \rightarrow_2 \langle c, d\rangle$$

$$= \langle \overline{\text{sg}}(\overline{\text{sg}}(a) - c), d\text{sg}(\overline{\text{sg}}(a) - c)$$

and

$$V(A \wedge_2 B) = \langle a, b\rangle \wedge_2 \langle c, d\rangle$$

$$= \neg_2(\langle a, b\rangle \rightarrow_2 \neg_2\langle c, d\rangle)$$

$$= \neg_2(\langle a, b\rangle \rightarrow_2 \langle \overline{\text{sg}}(c), \text{sg}(c)\rangle)$$

$$= \neg_2\langle \overline{\text{sg}}(a - \overline{\text{sg}}(c)), \text{sg}(c)\text{sg}(a - \overline{\text{sg}}(c))\rangle$$

$$= \langle \overline{\text{sg}}(\overline{\text{sg}}(a - \overline{\text{sg}}(c))), \text{sg}(\overline{\text{sg}}(a - \overline{\text{sg}}(c)))\rangle$$

$$= \langle \text{sg}(a - \overline{\text{sg}}(c)), \overline{\text{sg}}(a - \overline{\text{sg}}(c))\rangle.$$

As we saw in Sect. 1.6, the disjunctions and conjunctions can have two forms. Therefore, formulas (1.7.1) and (1.7.2) can be changed with the new ones:

$$A \vee B = \neg A \to \neg\neg B, \tag{1.7.3}$$

$$A \wedge B = \neg(\neg\neg A \to \neg B). \tag{1.7.4}$$

For example, for the above formulas A and B, negation $\neg_2$ and implication $\to_2$, we obtain sequentially:

$$V(A \vee_2 B) = \neg_2\langle a, b\rangle \vee_2 \neg_2\neg_2\langle c, d\rangle$$

$$= \neg_2\langle a, b\rangle \to_2 \neg_2\langle \overline{\text{sg}}(c), \text{sg}(c)\rangle$$

$$= \neg_2\langle a, b\rangle \to_2 \langle \overline{\text{sg}}(\overline{\text{sg}}(c)), \text{sg}(\overline{\text{sg}}(c))\rangle$$

(For every $x \in [0, 1]$:

$$\overline{\text{sg}}(\overline{\text{sg}}(x)) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases} = \text{sg}(x)$$

and

$$\text{sg}(\overline{\text{sg}}(x)) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x > 0 \end{cases} = \overline{\text{sg}}(x).$$

$$= \langle \overline{\text{sg}}(a), \text{sg}(a)\rangle \to_2 \langle \text{sg}(c), \overline{\text{sg}}(c)\rangle$$

$$= \langle \overline{\text{sg}}(\overline{\text{sg}}(a) - \text{sg}(c)), \overline{\text{sg}}(c)\text{sg}(\overline{\text{sg}}(a) - \text{sg}(c))\rangle$$

and

$$V(A \wedge_2 B) = \langle a, b\rangle \wedge_2 \langle c, d\rangle$$

$$= \neg_2(\neg_2\neg_2\langle a, b\rangle \to_2 \neg_2\langle c, d\rangle)$$

$$= \neg_2(\neg_2\langle \overline{\text{sg}}(a), \text{sg}(a)\rangle \to_2 \langle \overline{\text{sg}}(c), \text{sg}(c)\rangle)$$

$$= \neg_2(\langle \overline{\text{sg}}(\overline{\text{sg}}(a)), \text{sg}(\overline{\text{sg}}(a))\rangle \to_2 \langle \overline{\text{sg}}(c), \text{sg}(c)\rangle)$$

$$= \neg_2(\langle \text{sg}(a), \overline{\text{sg}}(a)\rangle \to_2 \langle \overline{\text{sg}}(c), \text{sg}(c)\rangle)$$

$$= \neg_2\langle \overline{\text{sg}}(\text{sg}(a) - \overline{\text{sg}}(c)), \text{sg}(c)\text{sg}(\text{sg}(a) - \overline{\text{sg}}(c))\rangle$$

$$= \langle \overline{\text{sg}}(\overline{\text{sg}}(\text{sg}(a) - \overline{\text{sg}}(c))), \text{sg}(\overline{\text{sg}}(\text{sg}(a) - \overline{\text{sg}}(c)))\rangle$$

$$= \langle \text{sg}(\text{sg}(a) - \overline{\text{sg}}(c)), \overline{\text{sg}}(\text{sg}(a) - \overline{\text{sg}}(c))\rangle.$$

Having in mind that a part of the disjunctions and conjunctions are generated by the classical negation $\neg_1$, we call all these *"semiclassical disjunctions and conjunctions",* excluding only the disjunction $\vee_4$ and conjunction $\wedge_4$, i.e., the disjunction $\vee$

and conjunction $\wedge$ from Sect. 1.1, that we call *"classical disjunctions and conjunctions"*. We call the rest disjunctions and conjunctions *"non-classical disjunctions and conjunctions"*.

Therefore, formulas (1.7.1)–(1.7.4) must be rewritten to

$$A \vee_{i,1} B = \neg_{\varphi(i)} A \rightarrow_i B, \tag{1.7.5}$$

$$A \wedge_{i,1} B = \neg_{\varphi(i)}(A \rightarrow_i \neg_{\varphi(i)} B), \tag{1.7.6}$$

$$A \vee_{i,2} B = \neg_{\varphi(i)} A \rightarrow_i \neg_{\varphi(i)}\neg_{\varphi(i)} B, \tag{1.7.7}$$

$$A \wedge_{i,2} B = \neg_{\varphi(i)}(\neg_{\varphi(i)}\neg_{\varphi(i)} A \rightarrow_i \neg_{\varphi(i)} B), \tag{1.7.8}$$

where $\varphi(i)$ is the number of the negation that corresponds to the i-th implication (cf. Table 1.3).

Now, we see a possibility for constructing a third group of disjunctions and conjunctions. They have the forms

$$A \vee_{i,3} B = \neg_1 A \rightarrow_i B, \tag{1.7.9}$$

$$A \wedge_{i,3} B = \neg_1(A \rightarrow_i \neg_1 B). \tag{1.7.10}$$

Therefore, in all of them only the classical negation $\neg_1$ is used and by this reason, we call them again *"semiclassical disjunctions and conjunctions"*, excluding as above only disjunction $\vee_{4,3}$ and conjunction $\wedge_{4,3}$.

For example, for the above formulas A and B, negation $\neg_2$ and implication $\rightarrow_2$, we obtain sequentially:

$$\begin{aligned} V(A \vee_{2,3} B) &= \neg_1\langle a, b\rangle \vee_2 \neg_1\neg_1\langle c, d\rangle \\ &= \langle b, a\rangle \vee_2 \langle c, d\rangle \\ &= \langle \overline{\text{sg}}(b - c), d.\text{sg}(b - c)\rangle \end{aligned}$$

and

$$\begin{aligned} V(A \wedge_{2,3} B) &= \neg_1(\neg_1\neg_1\langle a, b\rangle \rightarrow_2 \neg_1\langle c, d\rangle) \\ &= \neg_1(\langle a, b\rangle \rightarrow_2 \langle d, c\rangle) \\ &= \neg_1\langle \overline{\text{sg}}(a - d), c.\text{sg}(a - d)\rangle \\ &= \langle c.\text{sg}(a - d), \overline{\text{sg}}(a - d)\rangle. \end{aligned}$$

Three open problems arise here.

Open Problem 5. Construct all new disjunctions and conjunctions. It is interesting to check whether some disjunctions and conjunctions will coincide.

Open Problem 6. Study the behaviour of the new disjunctions and conjunctions. For example, which of them will satisfy De Morgan's Laws and in which form of these Laws?

Open Problem 7. Study the properties of the disjunctions and conjunctions from (1.7.5)–(1.7.10). It is very important to check the validity of the separate axioms – of the intuitionistic logic, of Kolmogorov, of Łukasiewicz and Tarski, of Klir and Yuan, and the other ones, discussed in Sect. 1.5.

We finish this chapter with a remark from the area of group theory.

Here, for the first time, the author described an idea generated by him about 45 year ago, when he was a schoolboy in the secondary school. Only in the last months, after discussions with colleagues, he collected enthusiasm to formulate it (of course, in essentially better form than in the beginning).

Let for a fixed set X, $P(X) = \{Y | Y \subseteq X\}$.

Obviously, if $X \neq \emptyset$ is a fixed set, then, $\langle \mathcal{P}(X), \wedge, X\rangle$ and $\langle \mathcal{P}(X), \vee, \emptyset\rangle$ are commutative monoids, but for every set $A \in \mathcal{P}(X)$ there is no element B such that $A \wedge B = X$ and $A \vee B = \emptyset$. By this reason, $\langle \mathcal{P}(X), \wedge, X\rangle$ and $\langle \mathcal{P}(X), \vee, \emptyset\rangle$ are not (commutative) groups.

We call $\langle M, *, e_*, e_\circ\rangle$ a "multi unitary group" (shortly, μ-group) if and only if

$$(\forall a, b \in M)(a * b \in M); \qquad (1.7.11)$$

$$(\forall a, b, c \in M)((a * b) * c = a * (b * c)); \qquad (1.7.12)$$

$$(\forall a \in M)(a * e_* = a = e_* * a); \qquad (1.7.13)$$

$$(\forall a \in M)(\exists a_\circ \in M)(a * a_\circ = e_\circ = a_\circ * a). \qquad (1.7.14)$$

The μ-group is commutative if and only if

$$(\forall a, b \in M)(a * b = b * a). \qquad (1.7.15)$$

For example, $\langle P(X), \wedge, X, \emptyset\rangle$ and $\langle P(X), \vee, \emptyset, X\rangle$ are μ-groups.

In the particular case, when $e_* = e_\circ$, the (commutative) μ-group is reduced to a standard (commutative) group.

Two μ-groups MG_1 and MG_2 are dual, if and only if they have the forms

$$MG_1 = \langle M, *, e_*, e_\circ\rangle \quad \text{and} \quad MG_2 = \langle M, \circ, e_\circ, e_*\rangle$$

for some given operations $*$ and $\circ$, and for the unitary elements e_* and $e_\circ$.

For example, $\langle P(X), \wedge, X, \emptyset\rangle$ and $\langle P(X), \vee, \emptyset, X\rangle$ are dual μ-groups.

Formulas (1.7.13) and (1.7.14) can be written in more details, if we like to define left- and right-μ-group. All formulas (1.7.11)–(1.7.15) have analogues in group theory.

In the intuitionistic fuzzy case, we must modify condition (1.7.14) so that for a relation R:

$$(\forall a \in M)(\exists a_\circ \in M)(e_\circ R(a * a_\circ) R e_*). \tag{1.7.16}$$

For example, let $*$ be either the operation $\wedge$ or the operation $\vee$, R be the relation $\subset$, $e_\circ$ be $\emptyset$ and e_* be X. Then, (1.7.16) obtains either the form

$$(\forall a \in M)(\exists a_\circ \in M)(\emptyset \subset (a \wedge a_\circ) \subset X),$$

or

$$(\forall a \in M)(\exists a_\circ \in M)(\emptyset \subset (a \vee a_\circ) \subset X).$$

Now, the following open problem arises.

Open Problem 8. For which numbers i ($1 \leq i \leq 185$), the objects $\langle \mathcal{S}, \vee_i, F, T \rangle$ and $\langle \mathcal{S}, \wedge_i, T, F \rangle$ are (commutative) dual μ-groups, satisfying condition (1.7.16), where $\mathcal{S}$, T, F are described in Sect. 1.1, relation R is $\leq$ or $\geq$, and the disjunctions and conjunctions are defined by (1.7.5)–(1.7.10)?

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