Adaptive Fuzzy Control-Based Projective Synchronization Scheme of Uncertain Chaotic Systems with Input Nonlinearities

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Abstract. In this paper, a projective synchronization scheme for a class of master–slave chaotic systems subject to dynamic disturbances and input nonlinearities (dead-zone and sector nonlinearities) is investigated. To practically achieve this synchronization, an adaptive fuzzy variable-structure control system is designed. The fuzzy systems are used to appropriately approximate the uncertain nonlinear functions. A Lyapunov approach is employed to prove the boundedness of all signals of the closed-loop control system as well as the exponential convergence of the synchronization errors to an adjustable region. Simulations results are presented to illustrate the effectiveness of the proposed projective synchronization scheme.

Keywords: Projective synchronization · Adaptive control · Fuzzy control · Dead-zones · Uncertain chaotic system

1 Introduction

Chaos synchronization is an important topic in nonlinear science. It has received increasing attention thanks to their applications in information processing, secure communications, pattern recognition, power convertors, chemical reactions, laser systems, ecological and biological systems, and so on [1–5]. The initial configuration of chaos synchronization consists of master-slave systems. The master system drives the slave one via the transmitted signals. In the past two decades, various types of the chaos synchronization have been revealed, such as complete synchronization (CS) [6], phase synchronization (PHS) [7], projective synchronization (PS) [8, 9], and so on. In PS, the state vectors of two synchronized systems evolve in a proportional scale.

Based on the universal approximation theorem [10], many adaptive fuzzy control systems have been incorporated in the synchronization schemes [11–16] to solve the problem of uncertainties. The problem of the input nonlinearities has been also considered in [17–19] in the designing of the control-based synchronization systems for a class of uncertain chaotic systems. However, the class of chaotic systems considered in these works is relatively simple, affine and free of the dynamical disturbances.

Therefore, in this paper, we aim at addressing the projective synchronization problem of a class of multivariable nonaffine chaotic systems subject to both dynamic

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disturbances and input nonlinearities. This synchronization can be achieved via a suitable fuzzy adaptive variable-structure controller. The main difficulties of this work are how to deal with unknown nonlinear functions, nonaffine multivariable control, the uncertain input nonlinearities and the combined effect of the uncertain dynamic disturbances, fuzzy approximation errors together with the higher-order terms (HOT) issued from the use of the Taylor series expansion. In this study, these difficulties can be, respectively, solved by fuzzy system approximation, Taylor series expansion, variable structure control and robust dynamical control. A Lyapunov approach is adopted to carry out the parameter adaptation design, the convergence of the synchronization error and the stability analysis involved in this proposed synchronization scheme. The main contributions of this paper lie in the following:

(1) A new projective synchronization scheme based on fuzzy adaptive controller is proposed for uncertain perturbed chaotic systems with input nonlinearities (i.e. dead-zone and sector nonlinearities).
(2) The model of the considered chaotic systems is assumed to be completely unknown (except its relative degree), multivariable, nonaffine in control, subject to both input nonlinearities and dynamical disturbances. To our best knowledge, such a class of chaotic systems with all these features has not been already studied in the literature.

The rest of the paper is organized in the following manner. Section 2 presents the problem formulation and preliminaries, followed by the design of fuzzy adaptive controller to practically achieve a projective synchronization in Sect. 3. The simulation results are presented to demonstrate the effectiveness of proposed synchronization scheme in Sect. 4. Section 5 contains the conclusion.

2 Problem Statements and Preliminaries

Consider the following class of uncertain chaotic master systems:

$$\dot{Y} = H_1(Y) \tag{1}$$

where $Y = [y_1, \ldots, y_n]^T \in R^n$ is the overall state vector of the master system which is assumed to be measureable. $H_1(Y) = [h_{11}(Y), \ldots, h_{1n}(Y)]^T \in R^n$ is a vector of smooth unknown nonlinear functions.

The uncertain chaotic multivariable slave system affected by unknown dynamic disturbances can be given as:

$$\dot{X} = H_2(X, v) + \wedge(X) \tag{2}$$

Where $X = [x_1, \ldots, x_n]^T \in R^n$ is the overall state vector of the slave system which is assumed to be measureable. $H_2(X, v) = [h_{21}(X, v), \ldots, h_{2n}(X, v)]^T \in R^n$ denotes unknown nonaffine functions vector, with $v = \varphi(u) = \varphi_1(u_1), \ldots, \varphi_n(u_n)]^T$ is a nonlinear input functions vector satisfying some properties which will be given later and $\wedge(X) = [\wedge_1(X), \ldots, \wedge_n(X)]^T$ is the unknown external disturbance vector.

Assumption 1: *The matrix $\partial H_2(X, v)/\partial v$ is non-singular and its sign is assumed to be known.*

Design Objective: *Determine an adaptive fuzzy variable- structure law u_i (for all i = 1, …,n) which achieves a practical projective synchronization between the master system* (1) *and the slave one* (2) *and ensures the boundedness of all the signals in the derived closed-loop system remain.*

To quantify this objective, we define the *synchronization error* between systems (1) and (2) for this PS, as follows

$E = X - BY$, with $E = (e_1, \ldots, e_n)^T$, $B = Diag(b_1, \ldots, b_n)$
and $e_i = x_i - b_i y_i$ $(i = 1, 2, \ldots, n)$.

Definition 1: *For the master system* (1) *and the slave system* (2)*, there is said to be projective synchronization if there exists a nonzero constant B such that $lim_{t\to\infty}\|X - BY\| = 0$, where $\|.\|$ represents the Euclidean norm.*

Definition 2: *The PS is said to be practically achieved, if there exists a strictly positive constant ε such that $lim_{t\to\infty}\|X - BY\| \le \varepsilon$, where ε depends on the design parameters.*

The dynamics of the synchronization error vector are

$$\dot{E} = H_2(X, v) + \wedge(X) - BH_1(Y) \tag{3}$$

Let us define a PI sliding surface as follows:

$$S = [S_1, S_2, \ldots, S_n]^T = \frac{d\left[\int_0^t E(\tau)d\tau\right]}{dt} + \lambda\left[\int_0^t E(\tau)d\tau\right] = E + \lambda\int_0^t E(\tau)d\tau \tag{4}$$

where λ is a positive design constant. The time derivative of S is given by

$$\dot{S} = H_2(X, v) + \wedge(X) - BH_1(Y) + \lambda E \tag{5}$$

By means of Taylor series expansion, the nonaffine system (5) can be transformed into an affine system in control, around an unknown ideal control $v = v^*(X)$ as follows:

$$H_2(X, v) = F(X) + G(X)v + HOT(X, v) \tag{6}$$

with

$$G(X) = \left[g_{ij}(X)\right] = \left[\frac{\partial H_2(X, v)}{\partial v}\right]_{v=v^*(X)}$$

and

$$F(X) = H_2(X, v^*(X)) - \left[\frac{\partial H_2(X, v)}{\partial v}\right]_{v=v^*(X)} v^*(X)$$

where $HOT(X, v)$ is the higher order terms (HOT) of the expansion, and $v = v^*(X)$ is an unknown smooth function minimizing the HOT.

Since the matrix $G(X)$ is not necessarily symmetric, the following important lemma will be used in the control design and the stability analysis [20–23]:

Lemma 1: *Any real matrix $G(X) \in R^{n\times n}$ with non-zero leading principal minors can be factorized as follows:*

$$G(X) = G_s(X)DT(X) \tag{7}$$

where $G_s(X) \in R^{n\times n}$ is a symmetric and positive-definite matrix, $D \in R^{n\times n}$ is a diagonal matrix with +1 or −1 on its diagonal, and $T(X) \in R^{n\times n}$ is a unity upper-triangular matrix. The diagonal elements of Dare nothing else than the ratios of the signs of the leading principal minors of $G(X)$.

Using the matrix factorization (7) and the expression (6), the dynamics of S can be expressed as follows:

$$\dot{S} = F(X) + G_s(X)DT(X)\varphi(u) + HOT(X, \varphi(u)) + \wedge(X) - H_3(Y, E) \tag{8}$$

where

$$H_3(Y, E) = BH_1(Y) - \lambda E \tag{9}$$

Assumption 2: *The matrix $G(X)$ is of class C^1 and satisfies the following property:*

$$\frac{1}{2}\left\|\frac{dG_s(X)}{dt}\right\| = \frac{1}{2}\left\|\frac{\partial G_s(X)}{\partial X}\dot{X}\right\| \le \bar{g}(X),$$

where $\bar{g}(X)$ is an unknown positive function.

A. Input Nonlinearity. The mathematical model of the input nonlinearity $\varphi(u) = [\varphi_1(u_1), \ldots, \varphi_n(u_n)]^T$ under consideration (i.e. sector nonlinearity and dead-zone) is given by [17]:

$$\varphi_i(u_i) = \begin{cases} \varphi_{i+}(u_i)(u_i - u_{i+}), & u_i > u_{i+} \\ 0, & -u_{i-} \le u_i \le u_{i+} \\ \varphi_{i-}(u_i)(u_i + u_{i-}), & u_i < -u_{i-} \end{cases} \tag{10}$$

where $\varphi_{i+}(u_i) > 0$ and $\varphi_{i-}(u_i) > 0$ are nonlinear functions of u_i, and $u_{i+} > 0$ and $u_{i-} > 0$.

We can show that $\varphi_i(u_i)$ satisfies the following properties:

$$\begin{aligned}(u_i - u_{i+})\varphi_i(u_i) &\geq m^*_{i+}(u_i - u_{i+})^2,\ u_i > u_{i+},\\ (u_i + u_{i-})\varphi_i(u_i) &\geq m^*_{i-}(u_i + u_{i-})^2,\ u_i < -u_{i-},\end{aligned} \tag{11}$$

where m^*_{i+} and m^*_{i-} are strictly positive constants witch called *"gain reduction tolerances"*.

Assumption 3: *Assume that:*

(a) *The functions $\varphi_{i+}(u_i)$ and $\varphi_{i-}(u_i)$ and the constants m^*_{i+} and m^*_{i-} are uncertain,*
(b) *The constants u_{i+} and u_{i-} are known and strictly positive.*

B. Description of the Fuzzy Logic System. The fuzzy system is based on particular knowledge of four main modules, namely: the rule base, fuzzifier, the inference engine and defuzzifier, as shown in Fig. 1.

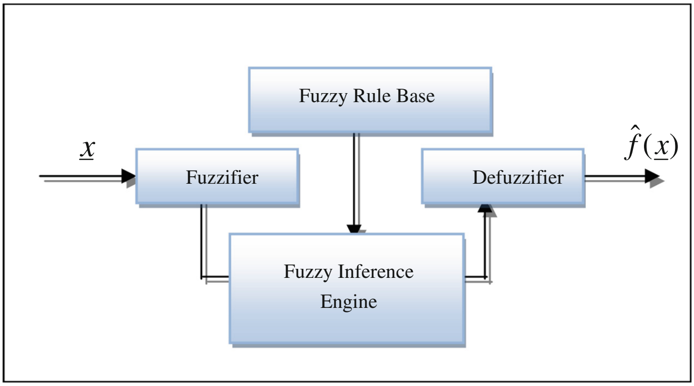

Fig. 1. Basic configuration of a fuzzy logic system.

The fuzzy inference engine uses the IF–THEN rules to achieve a mapping from an input vector $\underline{x}^T = [x_1, x_2, \ldots.., x_n] \in R^n$ to an output scalar $\hat{f} \in R$. The ith fuzzy rule can be written as:

$$R^{(i)} : if\ x_1\ is\ A^i_1\ and\ldots and\ x_n\ is\ A^i_n\ then\ \hat{f}\ is\ f^i \tag{12}$$

where $A^i_1, A^i_2, \ldots..$ and A^i_n are fuzzy sets and f^i is the fuzzy singleton for the output in the ith rule. The fuzzy-logic system can be expressed in the following form:

$$\hat{f}(\underline{x}) = \frac{\sum_{i=1}^{m}\left(f^i \prod_{j=1}^{n}\mu_{A^i_j}(x_j)\right)}{\sum_{i=1}^{m}\prod_{j=1}^{n}\mu_{A^i_j}(x_j)} = \theta^T\psi(\underline{x}) \tag{13}$$

where $\mu_{A_j^i}(x_j)$ is the membership function of A_j^i, m is the number of fuzzy rules, $\theta^T = [f^1, f^2, \ldots.., f^m]$ is the adjustable parameter vector (composed of consequent parameters), and $\psi^T = [\psi^1 \psi^2 \ldots \psi^m]$ with

$$\psi^i(\underline{x}) = \frac{\prod_{j=1}^{n} \mu_{A_j^i}(x_j)}{\sum_{i=1}^{m} \left(\prod_{j=1}^{n} \mu_{A_j^i}(x_j) \right)} \tag{14}$$

being the fuzzy basis function (FBF). Throughout the paper, it is assumed that the FBFs are properly chosen so that there is always at least one active rule, i.e.$\sum_{i=1}^{m} (\prod_{j=1}^{n} \mu_{A_j^i}(x_j)) > 0$, [10].

3 Design of Fuzzy Adaptive Controller

Multiplying the Eq. (8) by $G_s^{-1}(X)$ and by posing $\bar{S} = D^{-1}S$ or $\bar{S}_i = d_{ii}S_i$ (as $D^{-1} = D^T = D$ and $d_{ii} = +1 \; or \; -1$), we get

$$G_1(X)\dot{\bar{S}} = D^{-1}F_1(X, Y) + \varphi(u) + D^{-1}P(X, \varphi(u)) \tag{15}$$

where $G_1(X) = D^{-1}G_s^{-1}(X)D$,
$F_1(X, Y) = G_s^{-1}(X)[F(X) - H_3(Y, E)] + [DT(X) - D]\varphi(u)$,
$P(X, \varphi(u)) = G_s^{-1}(X)HOT(X, \varphi(u)) + G_s^{-1}(X)\Lambda(X)$.

The dynamics (15) can be rewritten as follows

$$\frac{1}{2}\dot{G}_1(X)\bar{S} + G_1(X)\dot{\bar{S}} = \alpha(z) + \varphi(u) + \frac{1}{2}\dot{G}_1(X)\bar{S} - \bar{g}(X)\bar{S} + R(X, \varphi(u)) \tag{16}$$

with $R(X, \varphi(u)) = D^{-1}P(X, \varphi(u))$,

$$\alpha(z) = [\alpha_1(z_1), \ldots, \alpha_n(z_n)]^T = \bar{g}(X)\bar{S} + D^{-1}F_1(X, Y) \tag{17}$$

where $z = [z_1^T, \ldots, z_n^T]^T$. The vectors z_i will be determined later.

Assumption 4: *There exists an unknown continuous positive function $\bar{\alpha}_i(z_i)$ such that: $|\alpha_i(z_i)| \le \eta\bar{\alpha}_i(z_i), \forall z_i \in \Omega_{z_i}$ with $\eta = min\{m_{i+}^*, m_{i-}^*\}$ for $i = 1, \ldots, n$.*

By examining the expressions of $F_1(X, Y, u)$ and $\alpha(z)$, and because the state vector of the master system $Y$is always bounded, the vectors z_i can be determined as follows:

$$\begin{gathered} z_1 = [X^T, u_2, \ldots.., u_n]^T \\ z_2 = [X^T, u_3, \ldots.., u_n]^T \\ \vdots \\ z_{n-1} = [X^T, u_n]^T \\ z_n = X \end{gathered} \tag{18}$$

The corresponding operating compact sets are defined as follows:

$$\Omega_{z_i} = \left\{[X^T, u_{i+1}, \ldots, u_n]^T | X \in \Omega_X \subset R^n, Y \in \Omega_Y\right\},$$
$$\Omega_{z_n} = \{X | X \in \Omega_X \subset R^n\}.$$

It is clear that z_1 depends on control inputs $u_2, \ldots, u_n, z_2$ depends on $u_3, \ldots, u_n$, and so on. In fact, the structure of the nonlinearities $\alpha(z)$ is known under the name *“upper triangular control structure”*.

The unknown nonlinear function $\bar{\alpha}_i(z_i)$ can be approximated, on the compact set Ω_{z_i}, by the fuzzy systems (13) as follows:

$$\hat{\bar{\alpha}}_i(z_i, \theta_i) = \theta_i^T \psi_i(z_i), \quad with\, i = 1, \ldots, n \tag{19}$$

where $\psi_i(z_i)$ is the fuzzy basis function (FBF) vector, which is fixed a priori by the designer, and θ_i is the adjustable parameter vector of the fuzzy system. Then, we define:

$\theta_i^* = \arg min\left[sup|\bar{\alpha}_i(z_i) - \widehat{\bar{\alpha}}_i(z_i, \theta_i)|\right]$ as the optimal value of θ_i which is mainly introduced for analysis purposes as its value is not needed when implementing the controller.

Define

$$\begin{aligned} \tilde{\theta}_i &= \theta_i - \theta_i^* \\ \varepsilon_i(z_i) &= \bar{\alpha}_i(z_i) - \widehat{\bar{\alpha}}_i\left(z_i, \theta_i^*\right) = \bar{\alpha}_i(z_i) - \theta_i^{*T}\psi_i(z_i) \end{aligned} \tag{20}$$

as the parameter estimation error and the fuzzy approximation error, respectively. As in [10, 17–22], the fuzzy approximation error is assumed to be bounded for all $z_i \in \Omega_{z_i}$, i.e.:

$$|\varepsilon_i(z_i)| \leq \bar{\varepsilon}_i \ , \ \forall z_i \in \Omega_{z_i} \tag{21}$$

where $\bar{\varepsilon}_i$ is an unknown constant.

Now, let us denote

$$\begin{aligned} \hat{\bar{\alpha}}(z, \theta) &= \left[\hat{\bar{\alpha}}_1(z_1, \theta_1), \ldots, \hat{\bar{\alpha}}_n(z_n, \theta_n)\right]^T = \left[\theta_1^T\psi_1(z_1), \ldots, \theta_n^T\psi_n(z_n)\right]^T \\ \varepsilon(z) &= [\varepsilon_1(z_1), \ldots, \varepsilon_n(z_n)]^T, \\ \bar{\varepsilon}(z) &= [\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_n]^T. \end{aligned}$$

Then, we have

$$\begin{aligned} \hat{\bar{\alpha}}(z, \theta) - \bar{\alpha}(z) &= \hat{\bar{\alpha}}(z, \theta) - \hat{\bar{\alpha}}(z, \theta^*) + \hat{\bar{\alpha}}(z, \theta^*) - \bar{\alpha}(z) \\ &= \hat{\bar{\alpha}}(z, \theta) - \hat{\bar{\alpha}}(z, \theta^*) - \bar{\varepsilon}(z) = \tilde{\theta}^T\psi(z) - \bar{\varepsilon}(z) \end{aligned} \tag{22}$$

where $\tilde{\theta}^T\psi(z) = \left[\tilde{\theta}_1^T\psi_1(z_1), \ldots, \tilde{\theta}_n^T\psi_n(z_n)\right]$ and $\tilde{\theta}_i = \theta_i - \theta_i^*$, for $i = 1, \ldots, n$.

Assumption 5: *We assume that:*

$$|\varepsilon(z) + R(X, \varphi(u))| \leq \eta \bar{R}(X, u)\kappa^*$$

with $\bar{R}(X, u) = 1 + \|X\| + \|u\|$, *where* $\kappa^* = [\kappa_1^*, \ldots, \kappa_n^*]^T$ *is an unknown constant vector to be estimated.*

To practically achieve a projective synchronization between the master system (1) and the slave system (2), we can consider the following fuzzy adaptive variable-structure controller:

$$u_i = \begin{cases} -\rho_i(t) sign(\bar{S}_i) - u_{i-}, & \bar{S}_i > 0 \\ 0, & \bar{S}_i = 0 \\ -\rho_i(t) sign(\bar{S}_i) + u_{i+}, & \bar{S}_i < 0 \end{cases} \tag{23}$$

with

$$\rho_i(t) = |u_{ri}| + k_{0i} + k_{1i}|\bar{S}_i| + \theta_i^T \psi_i(z_i) \ \forall i = 1, \ldots, n \tag{24}$$

$$\dot{\theta}_i(t) = -\gamma_{\theta i}\sigma_{\theta i}\theta_i + \gamma_{\theta i}|\bar{S}_i|\psi_i(z_i), \ with \ \theta_{ij}(0) > 0 \tag{25}$$

where $\gamma_{\theta i}, \sigma_{\theta i}$, k_{0i} and k_{1i} for $i = 1, \ldots, n$ are free positive design constants and $u_r = [u_{r1}, \ldots, u_{rn}]$, $u_{ri}(0) > 0$ is an adaptive control term added in order to dynamically compensate for the uncertain nonlinearity.

$$\dot{u}_r = -\gamma_r u_r + \gamma_r \left[E_{ur}|\bar{S}| - \frac{sign(u_r)}{\sum_{i=1}^n |u_{ri}| + \delta^2} \bar{R}(X, u)\kappa^T |\bar{S}| \right] \tag{26}$$

$$\dot{\delta} = -\gamma_\delta \sigma_\delta \delta - \gamma_\delta \frac{\delta}{\sum_{i=1}^n |u_{ri}| + \delta^2} \bar{R}(X, u)\kappa^T |\bar{S}|, \ \delta(0) > 0 \tag{27}$$

$$\dot{\kappa} = -\gamma_\kappa \sigma_\kappa \kappa + \gamma_\kappa \bar{R}(X, u)|\bar{S}|, \kappa_i(0) \geq 0 \tag{28}$$

where $E_{ur} = diag[sign(u_{r1}), \ldots, sign(u_{rn})]$, γ_κ, σ_κ, γ_δ, σ_δ and γ_r are strictly positive design parameters.

By exploiting Eq. (22), Assumptions 2, 4 and 5, and control law (23)–(25), (16) can be rewritten as follows

$$\begin{aligned} \frac{d}{dt}\left[\frac{1}{2\eta}\bar{S}^T G_1(X)\bar{S}\right] &\leq \sum_{i=1}^n |\bar{S}_i|\bar{\alpha}_i(z_i) + \frac{1}{\eta}\bar{S}^T \varphi(u) \\ &\quad + \bar{R}(X, u)\kappa^{*T}|\bar{S}| - \bar{S}^T \varepsilon(z) \\ &\leq -\sum_{i=1}^n |\bar{S}_i|\left(|u_{ri}| + k_{1i}|\bar{S}_i| + \tilde{\theta}_i^T \psi_i(z_i)\right) \\ &\quad + \sum_{i=1}^n |\bar{S}_i|\left(|u_{ri}| + k_{1i}|\bar{S}_i| + \theta_i^T \psi_i(z_i)\right) \\ &\quad + \frac{1}{\eta}\bar{S}^T \varphi(u) + \bar{R}(X, u)\kappa^{*T}|\bar{S}| \end{aligned} \tag{29}$$

Theorem 1. *Consider the master-slave system (1) and (2) with Assumptions 1–5. Then, the control law given by (23)–(28) can guarantee the following properties:*

- *All the variables in the closed-loop control system are semi-globally uniformly ultimately bounded (SUUB).*
- *The synchronization errors S_i exponentially converge to an adjustable domain defined as:*

$$\Omega_{Si} = \left\{ S_i \| S_i| \leq \left(\frac{2\pi}{\sigma_{g1}\mu} \right)^{1/2} \right\} \tag{30}$$

where π, μ and σ_{g1} will be defined later.

Proof. Consider the following Lyapunov function candidate:

$$V = \frac{1}{2\eta}\bar{S}^T G_1(X)\bar{S} + \frac{1}{2}\sum\nolimits_{i=1}^{n} \frac{1}{\gamma_{\theta_i}}\tilde{\theta}_i^T\tilde{\theta}_i + \frac{1}{2\gamma_\kappa}\tilde{\kappa}^T\tilde{\kappa} + \frac{1}{2\gamma_\delta}\delta^2 + \frac{1}{2\gamma_r}u_r^T u_r \tag{31}$$

The time derivative of V is given by

$$\dot{V} = \frac{1}{\eta}\bar{S}^T G_1(X)\dot{\bar{S}} + \frac{1}{2\eta}\bar{S}^T\dot{G}_1(X)\bar{S} + \sum\nolimits_{i=1}^{n}\frac{1}{\gamma_{\theta_i}}\tilde{\theta}_i^T\dot{\theta}_i + \frac{1}{\gamma_\kappa}\tilde{\kappa}^T\dot{\kappa} + \frac{1}{\gamma_\delta}\delta\dot{\delta} + \frac{1}{\gamma_r}u_r^T\dot{u}_r \tag{32}$$

From (11) and Eq. (23), we can easily get the following expressions:

$$u_i < -u_{i-} for\, \bar{S}_i > 0 \Rightarrow$$

$$(u_i + u_{i-})\varphi_i(u_i) \geq m_{i-}^*(u_i + u_{i-})^2 \geq \eta(u_i + u_{i-})^2 \tag{33}$$

and

$u_i > u_{i+}$ *for* $\bar{S}_i < 0 \Rightarrow$

$$(u_i - u_{i+})\varphi_i(u_i) \geq m_{i-}^*(u_i - u_{i+})^2 \geq \eta(u_i - u_{i+})^2 \tag{34}$$

From the above analysis and (23), we can conclude that

$$\begin{aligned} \bar{S}_i > 0 \Rightarrow (u_i + u_{i-})\varphi_i(u_i) &= -\rho_i(t) sign(\bar{S}_i)\varphi_i(u_i) \\ &\geq m_{i-}^*\rho_i^2(t)[sign(\bar{S}_i)]^2 \geq \eta\rho_i^2(t) \end{aligned} \tag{35}$$

$$\begin{aligned} \bar{S}_i < 0 \Rightarrow (u_i - u_{i+})\varphi_i(u_i) &= -\rho_i(t) sign(\bar{S}_i)\varphi_i(u_i) \\ &\geq m_{i+}^*\rho_i^2(t)[sign(\bar{S}_i)]^2 \geq \eta\rho_i^2(t) \end{aligned} \tag{36}$$

Using the fact that $\bar{S}_i sign(\bar{S}_i) = |\bar{S}_i|$, for $\bar{S}_i > 0$ and $\bar{S}_i < 0$, we have

$$-\rho_i(t)\bar{S}_i^2 sign(\bar{S}_i)\varphi_i(u_i) \geq \eta\rho_i^2(t)\bar{S}_i^2 = \eta\rho_i^2(t)|\bar{S}_i|^2 \tag{37}$$

Finally, while $\rho_i(t) > 0$, for all $\bar{S}_i$ we have

$$\bar{S}_i\varphi_i(u_i) \leq -\eta\rho_i(t)|\bar{S}_i| \tag{38}$$

Using expressions (24)–(28), (29) and (38), (32) becomes

$$\begin{aligned}\dot{V} \leq & -\sum_{i=1}^{n}|\bar{S}_i|(|u_{ri}| + k_{0i} + k_{1i}|\bar{S}_i|) - \sum_{i=1}^{n}\sigma_{\theta i}\tilde{\theta}_i^T\theta_i + \bar{R}(X,u)|\bar{S}|^T\kappa^* + \frac{1}{\gamma_\kappa}\tilde{\kappa}^T\dot{\kappa} \\ & + \frac{1}{\gamma_\delta}\delta\dot{\delta} + \frac{1}{\gamma_r}u_r^T\dot{u}_r \\ \leq & -\sum\nolimits_{i=1}^{n}u_{ri}^2 - \sum\nolimits_{i=1}^{n}k_{1i}\bar{S}_i^2 - \sum\nolimits_{i=1}^{n}\sigma_{\theta i}\tilde{\theta}_i^T\theta_i - \sigma_\kappa\tilde{\kappa}^T\kappa - \sigma_\delta\delta^2\end{aligned} \tag{39}$$

Now, we can use the following inequalities

$$-\sigma_\kappa\tilde{\kappa}^T\kappa \leq -\frac{\sigma_\kappa}{2}\|\tilde{\kappa}\|^2 + \frac{\sigma_\kappa}{2}\|\kappa^*\|^2$$

$$-\sigma_{\theta_i}\tilde{\theta}_i^T\theta_i \leq -\frac{\sigma_{\theta_i}}{2}\left\|\tilde{\theta}_i\right\|^2 + \frac{\sigma_{\theta_i}}{2}\left\|\theta_i^*\right\|^2$$

Then, (39) becomes

$$\begin{aligned}\dot{V} \leq & -\sum\nolimits_{i=1}^{n}u_{ri}^2 - \sum\nolimits_{i=1}^{n}k_{1i}\bar{S}_i^2 - \sum\nolimits_{i=1}^{n}\frac{\sigma_{\theta_i}}{2}\left\|\tilde{\theta}_i\right\|^2 + \sum\nolimits_{i=1}^{n}\frac{\sigma_{\theta_i}}{2}\left\|\theta_i^*\right\|^2 \\ & -\frac{\sigma_\kappa}{2}\|\tilde{\kappa}\|^2 + \frac{\sigma_\kappa}{2}\|\kappa^*\|^2 - \sigma_\delta\delta^2\end{aligned} \tag{40}$$

Thanks to the property of $G_s(x)$, there exists a positive scalar σ_{gs} such that $G_s(x) \geq \sigma_{gs}I_n$ yields

$$\bar{S}^T G_1(x)\bar{S} = S^T G_s^{-1} S \leq \frac{1}{\sigma_{gs}}\|\bar{S}\|^2 \tag{41}$$

And using (40) and (41), we obtain

$$\dot{V} \leq -\mu V + \pi \tag{42}$$

with $= \sum_{i=1}^{n}\frac{\sigma_{\theta_i}}{2}\left\|\theta_i^*\right\|^2 + \sum_{i=1}^{n}\frac{\sigma_{\kappa_i}}{2}\|\kappa^*\|^2$, and

$$\mu = min\{min\{2\eta\sigma_{gs}k_{1i}\}, min\{\gamma_{\theta i}\sigma_{\theta i}\}, 2\gamma_\delta\sigma_\delta, 2\gamma_r, \gamma_k\sigma_k\}$$

Multiplying (42) by $e^{\mu t}$, we get

$$\frac{d(Ve^{\mu t})}{dt} \leq \pi e^{\mu t} \tag{43}$$

And integrating (43) over [0,t], we have

$$0 \leq V(t) \leq \frac{\pi}{\mu} + \left(V(0) - \frac{\pi}{\mu}\right)e^{-\mu t} \tag{44}$$

Thus, all signals in the closed-loop control system are SUUB. And hence the input u_i is bounded.

From (43) and (31), and using the properties (the symmetry and its sign) of $G_1(x)$ i.e. *there exists an unknown positive constant* σ_{g1} *such that:* $G_1(x) \geq \sigma_{g1} I_n$. Then the following inequality results:

$$|\bar{S}_i| = |S_i| \leq \left(\frac{2}{\sigma_{g1}}\left(\frac{\pi}{\mu} + \left(V(0) - \frac{\pi}{\mu}\right)e^{-\mu t}\right)\right)^{1/2} \tag{45}$$

i.e. the solution of S_i exponentially converges to a bounded adjustable domain defined as follows:

$\Omega_{Si} = \left\{ S_i \| S_i| \leq \left(\frac{2\pi}{\sigma_{g1}\mu}\right)^{1/2} \right\}$. This ends the proof.

4 Simulation Results

This section is carried out to show the effectiveness of the proposed synchronization scheme. For this end, we consider the following two identical chaotic satellites systems [2]:

The master system

$$\begin{cases} \dot{y}_1 = \frac{1}{3} y_2 y_3 - 0.4 y_1 + \frac{\sqrt{6}}{6} y_3 \\ \dot{y}_2 = -y_1 y_3 + 0.175 y_2 \\ \dot{y}_3 = y_1 y_2 - \sqrt{6} y_1 - 0.4 y_3 \end{cases} \tag{46}$$

The slave system controlled and being subject to input nonlinearities and dynamical external disturbances is described by:

$$\begin{cases} \dot{x}_1 = \frac{1}{3} x_2 x_3 - 0.4 x_1 + \frac{\sqrt{6}}{6} x_3 + \varphi_1(u_1) + \varphi_1(u_1)^3 + \wedge_1(X) \\ \dot{x}_2 = -x_1 x_3 + 0.175 x_2 + \varphi_2(u_2) + \wedge_2(X) \\ \dot{x}_3 = x_1 x_2 - \sqrt{6} x_1 - 0.4 x_3 + \varphi_3(u_3) + 2\varphi_3(u_3)^3 + \wedge_3(X) \end{cases} \tag{47}$$

where the external disturbances are selected as follows:$\wedge_1(X) = 0.5x_1$, $\wedge_2(X) = 0.5x_2^2$ and $\wedge_3(X) = 0.5x_3^3$. The input nonlinearities $\varphi_i(u_i)$ for $i = 1, 2, 3$ are described by:

$$\varphi_i(u_i) = \begin{cases} (u_i - 3)(1 - 0.3\sin(u_i)), & u_i > 3 \\ 0, & -3 \leq u_i \leq 3 \\ (u_i + 3)(0.8 - 0.3\cos(u_i)), & u_i < -3 \end{cases} \tag{48}$$

The initial conditions of master-slave systems and the adaptation laws are respectively selected as: $Y(0) = [5, 3, -1]$, $X(0) = [3, 4.1, 2]$, $u_{r1}(0) = u_{r2}(0) = u_{r3}(0) = 0$, $\delta(0) = 2$, $\kappa_1(0) = \kappa_2(0) = \kappa_3(0) = 10$ and $\theta_{1_j}(0) = \theta_{2_j}(0) = \theta_{3_j}(0) = 0.001$, for $j = 1,\ldots,m$, where m is the number of the fuzzy rules.

The design parameters are chosen as: $\gamma_{\theta_1} = \gamma_{\theta_2} = \gamma_{\theta_3} = 300$, $\sigma_{\theta 1} = \sigma_{\theta 2} = \sigma_{\theta 3} = 10^{-3}$, $\gamma_{r_1} = \gamma_{r_2} = \gamma_{r_3} = 200, \gamma_{\kappa_1} = \gamma_{\kappa_2} = \gamma_{\kappa_3} = 200$, $\sigma_{\delta 1} = \sigma_{\delta 2} = \sigma_{\delta 3} = 10^{-7}$, $\gamma_{\delta_1} = \gamma_{\delta_2} = \gamma_{\delta_3} = 10^{-5}$, $\sigma_{\kappa 1} = \sigma_{\kappa 2} = \sigma_{\kappa 3} = 2 \times 10^{-3}$.

The proposed synchronization scheme is simulated in several cases according to the value of the scaling factor B, as shown in Figs. 2, 3 and 4. Firstly, when $B = 2$, a projective synchronization is achieved as shown in Fig. 2. Furthermore, we choose $B = 1$, Fig. 3 shows a complete synchronization where the trajectories of the master system converge to those of the slave one. And finally, with $B = -1$, an anti-phase synchronization is effectively obtained in Fig. 4.

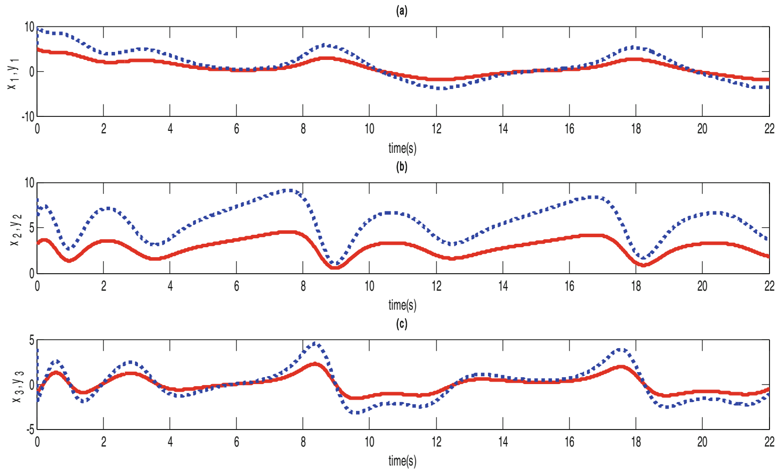

Fig. 2. Projective synchronization ($B = 2$): (a) x_1 (solid line) and y_1 (dotted line). (b) x_2 (solid line) and y_2 (dotted line). (c) x_3 (solid line) and y_3 (dotted line).

In summary, we can conclude that all simulation results demonstrate the effectiveness of the proposed projective synchronization scheme.

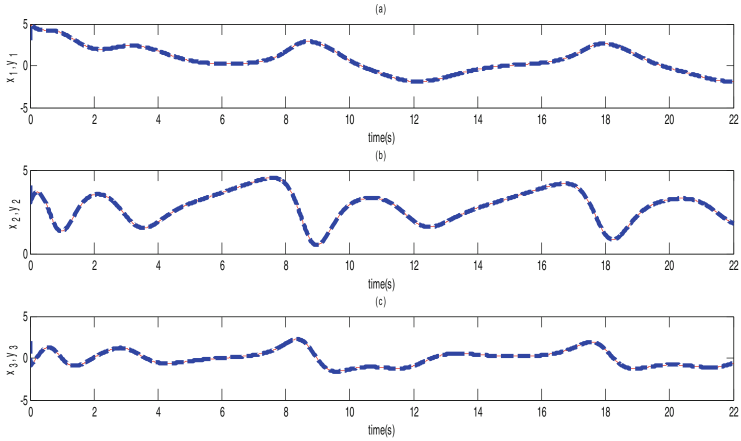

Fig. 3. Complete synchronization ($B = 1$): (a) x_1 (solid line) and y_1 (dotted line). (b) x_2 (solid line) and y_2 (dotted line). (c) x_3 (solid line) and y_3 (dotted line).

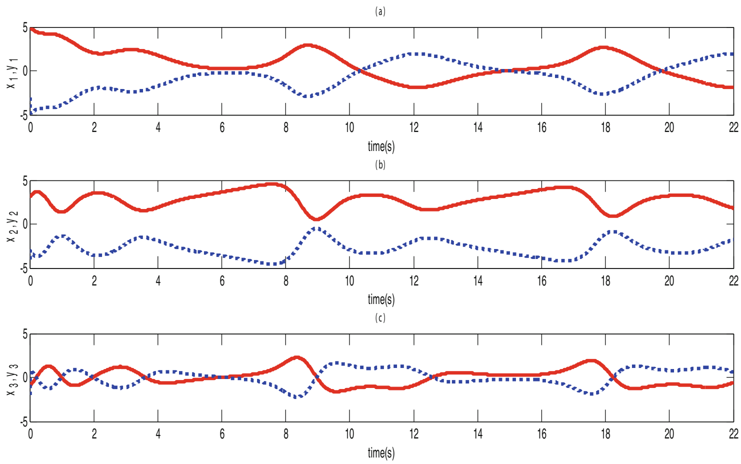

Fig. 4. Anti-phase synchronization ($B = -1$): (a) x_1 (solid line) and y_1 (dotted line). (b) x_2 (solid line) and y_2 (dotted line). (c) x_3 (solid line) and y_3 (dotted line).

5 Conclusion

In this paper, we have presented a projective synchronization scheme of two uncertain (chaotic or hyper-chaotic) systems subject to dynamic nonlinear disturbances and input nonlinearities (namely, dead-zone and sector nonlinearities). A fuzzy adaptive variable-structure controller has been designed to adequately achieve this projective synchronization. A Lyapunov based analysis has been carried out to conclude about the stability of the closed-loop system as well as the convergence of the synchronization error. Numerical simulations are presented to demonstrate the effectiveness of the proposed synchronization system.

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