Rational Function Approximation of a Fundamental Fractional Order Transfer Function

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Abstract. This paper introduces a rational function approximation of the fractional order transfer function $H(s) = \frac{(\tau_0 s)^\alpha}{[1 + (\tau_0 s)^{2\alpha}]}, \quad for\ 0 < \alpha \leq 0.5$. This fractional order transfer function is one of the fundamental functions of the linear fractional system of commensurate order corresponding to pure complex conjugate poles or eigenvalues, in s^α. Hence, the proposed approximation will be used in the solution of the linear fractional systems of commensurate order. Illustrative examples are given to show the exactitude and the efficiency of the approximation method.

Keywords: Fractional power zero · Linear fractional system · Irrational transfer function · Rational transfer function

1 Introduction

The theory of fractional order systems has gained some importance during the last decades (Miller et al. 1993), (Podlubny 1999), (Kilbas et al. 2006), (Monje et al. 2010), (Caponetto et al. 2010). Therefore, active research work to find accurate and efficient methods to solve linear fractional order differential equations is still underway to establish a clear linear fractional order system theory accessible to the general engineering community. More recently, a great deal of effort has been expended in the development of analytical techniques to solve them. The goal of these methods is to derive an explicit analytical expression for the general solution of the linear fractional differential equations (Charef 2006a), (Bonilla et al. 2007), (Oturanç et al. 2008), (Hu et al. 2008), (Arikoglu et al. 2009), (Odibat 2010), (Charef et al. 2011).

A linear single input single output (SISO) fractional system of commensurate order is described by the following linear fractional order differential equation:

$$\sum_{i=0}^{N} a_i D^{i\alpha} y(t) = \sum_{j=0}^{M} b_j D^{j\alpha} u(t) \quad (1)$$

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where u(t) is the input, y(t) is the output, α is a real number such that $0 < \alpha < 1$, a_i ($1 \leq i \leq N$) and b_i ($0 \leq j \leq M$) are constant real numbers with $M \leq N$. With zero initial conditions, the fractional order system transfer function is given as:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\sum_{j=0}^{M} b_j (s^\alpha)^j}{\sum_{i=0}^{N} a_i (s^\alpha)^i} \tag{2}$$

This fractional transfer function can be decomposed into several elementary fundamental functions corresponding to different types of poles, in s^α, as:

$$G(s) = \sum_{k=1}^{K} H_k(s) \tag{3}$$

where the functions $H_k(s)$ are given, according to the poles of the fractional system, as:

- For a simple real pole:

$$H_k(s) = \frac{1}{(s^\alpha + p)} \tag{4}$$

- For a pair of complex poles with negative real part:

$$H_k(s) = \frac{\omega_n^2}{[s^{2\alpha} + 2\zeta\omega_n s^\alpha + \omega_n^2]} \tag{5}$$

$$H_k(s) = \frac{s^\alpha + \omega_n \zeta}{[s^{2\alpha} + 2\zeta\omega_n s^\alpha + \omega_n^2]} \tag{6}$$

- For a pair of complex poles with null real part:

$$H_k(s) = \frac{\omega_n^2}{[s^{2\alpha} + \omega_n^2]} \tag{7}$$

$$H_k(s) = \frac{\omega_n s^\alpha}{[s^{2\alpha} + \omega_n^2]} \tag{8}$$

In previous works (Charef 2006a) and (Charef et al. 2011), (Nezzari et al. 2011), (Boucherma et al. 2011), the elementary fundamental functions defined in (4), (5), (6) and (7) have been approximated by rational ones in order to represent them by linear time-invariant system models so as to derive their closed form impulse and step responses as well as their performance characteristics. Using these approximations, simple analog circuits have been also derived to represent the above irrational functions of the fractional order system. This paper gives a rational function approximation of the fundamental function represented by the irrational transfer function of (8) which corresponds to pure complex conjugate poles or eigenvalues, in s^{α}. In (Boucherma et al. 2011), the approximation of (8) has been done for $0.5 < \alpha < 1$. In this work the approximation of (8) will be done for $0 < \alpha \le 0.5$. First, the basic ideas and the derived formulations of the approximation technique are presented. Then, the impulse and step responses of this type of fractional system are derived. Finally, illustrative examples are presented to show the exactitude and the usefulness of the approximation method.

2 Rational Function Approximation

For $(\tau_0)^{\alpha} = \frac{1}{\omega_n}$, (8) can be rewritten as:

$$H(s) = \frac{X(s)}{E(s)} = \frac{(\tau_0 s)^{\alpha}}{[1 + (\tau_0 s)^{2\alpha}]}, \quad 0 < \alpha \le 0.5 \tag{9}$$

The above irrational function is the transfer function of the linear fractional order system represented by the following fundamental linear fractional order differential equation:

$$(\tau_0)^{2\alpha} \frac{d^{2\alpha} x(t)}{dt^{2\alpha}} + x(t) = (\tau_0)^{\alpha} \frac{d^{\alpha} e(t)}{dt^{\alpha}} \tag{10}$$

In this context, the transfer function of (9) has two pure complex conjugate poles, in s^{α}. To represent the linear fractional order system of (10) by a linear time-invariant system model so as to derive their closed form impulse and step responses, its irrational transfer function of (9) will be approximated by a rational function. To do so, we will consider two cases based on the fractional derivative α.

2.1 Case 1: $0 < \alpha < 0.5$

For this case, the function of (9) can be decomposed in two functions as follows:

$$H(s) = H_1(s) \times H_2(s) = (\tau_0 s)^{\alpha} \times \frac{1}{1 + (\tau_0 s)^{2\alpha}} \tag{11}$$

where $H_1(s) = (\tau_0 s)^{\alpha}$ and $H_2(s) = \frac{1}{1 + (\tau_0 s)^{2\alpha}}$.

In a given frequency band of interest $[\omega_L, \omega_H]$, around the frequency $\omega_0 = (1/\tau_0)$, the fractional order differentiator $H_1(s) = (\tau_0 s)^{\alpha}$ can be approximated by a rational function as follows (b, 2006):

$$H_1(s) = (\tau_0 s)^{\alpha} \cong \tau_0^{\alpha}\left[K_D \frac{\prod_{i=0}^{N_1}\left(1+\frac{s}{z_i}\right)}{\prod_{i=0}^{N_1}\left(1+\frac{s}{p_i}\right)}\right] \tag{12}$$

where the poles pi and the zeros z_i ($0 \leq i \leq N1$), the constant K_D and the number N1 of the approximation are given by:

$$\begin{aligned} p_i &= p_0\ (ab)^i,\ z_i = z_0\ (ab)^i, K_D = (\omega_c)^{\alpha}, \\ N_1 &= \left\{Integer\left[\frac{\log(\omega_{\max}/z_0)}{\log(ab)}\right]+1\right\} \end{aligned} \tag{13}$$

For some given real values y (dB), δ and β, the approximation parameters a, b, p_0, z_0, ω_c and ω_{max} can be calculated as:

$$\begin{aligned} a &= 10^{\left[\frac{y}{10(1-\alpha)}\right]}, b = 10^{\left[\frac{y}{10\alpha}\right]}, \omega_c = \delta\omega_L, \\ \omega_{\max} &= \beta\omega_H,\ z_0 = \omega_c\sqrt{b}\ and\ p_0 = az_0 \end{aligned} \tag{14}$$

By the decomposition of the rational function of (12), we will get:

$$H_1(s) = (\tau_0 s)^{\alpha} \cong \tau_0^{\alpha}\left(K_D + \sum_{i=0}^{N_1}\frac{k_i s}{\left(1+\frac{s}{p_i}\right)}\right) \tag{15}$$

$$k_i = -\frac{K_D}{p_0(ab)^i}\frac{\prod_{j=0}^{N_1}\left(1-a(ab)^{(i-j)}\right)}{\prod_{j=0,i\neq j}^{N_1}\left(1-(ab)^{(i-j)}\right)}, i = 0, 1, \ldots, N1 \tag{16}$$

Because $0 < \alpha < 0.5$ the number 2α is then $0 < 2\alpha < 1$; hence, in a given frequency band [0, ω_H], the fractional system $H_2(s) = \frac{1}{1+(\tau_0 s)^{2\alpha}}$ can be approximated by a rational function as follows (Charef 2006a):

$$H_2(s) = \frac{1}{1+(\tau_0 s)^{2\alpha}} \cong \sum_{j=1}^{2N_2-1}\frac{kk_j}{\left(1+\frac{s}{pp_j}\right)} \tag{17}$$

where the poles pp_j and the residues kk_j (for $1 \leq j \leq 2N_2-1$), and the number N_2 of the approximation are given, for some given real values λ and β, by:

$$pp_j = \frac{(\lambda)^{(j-N)}}{\tau_0}$$
$$kk_j = \frac{1}{2\pi}\left[\frac{\sin[(1-\alpha)\pi]}{\cosh[\alpha\,\log(\frac{1}{\tau_0 pp_j})] - \cos[(1-\alpha)\pi]}\right] \tag{18}$$
$$N_2 = Integer\left[\frac{\log[\tau_0\beta\omega_H]}{\log(\lambda)}\right] + 1$$

Therefore, the function of (11) is approximated by a rational function as follows:

$$H(s) \cong \tau_0^{\alpha}\left(K_D + \sum_{i=0}^{N_1}\frac{k_i s}{\left(1+\frac{s}{p_i}\right)}\right)\left(\sum_{j=1}^{2N_2-1}\frac{kk_j}{\left(1+\frac{s}{pp_j}\right)}\right) \tag{19}$$

$$\begin{aligned} H(s) \cong & \left(\sum_{j=1}^{2N_2-1}\frac{\left(\tau_0^{\alpha}K_D\right)\,\left(pp_j kk_j\right)}{\left(s+pp_j\right)}\right) \\ & + \left(\sum_{i=0}^{N_1}\sum_{j=1}^{2N_2-1}\frac{\left(\tau_0^{\alpha}\right)(p_i k_i)\left(pp_j kk_j\right)s}{(s+p_i)\left(s+pp_j\right)}\right) \end{aligned} \tag{20}$$

By the decomposition of the rational function of (20), we will get:

$$\begin{aligned} H(s) \cong & \left(\sum_{j=1}^{2N_2-1}\frac{\left(\tau_0^{\alpha}K_D\right)\left(pp_j kk_j\right)}{\left(s+pp_j\right)}\right) \\ & + \left(\sum_{i=0}^{N_1}\sum_{j=1}^{2N_2-1}\frac{A_{ij}}{(s+p_i)} + \frac{B_{ij}}{\left(s+pp_j\right)}\right) \end{aligned} \tag{21}$$

where the residues A_{ij} and B_{ij} (for $0 \le i \le N_1$ and $1 \le j \le 2N_2 - 1$) are given by:

$$A_{ij} = \frac{\left(\tau_0^{\alpha}\right)\left(p_i^2 k_i\right)\left(pp_j kk_j\right)}{p_i - pp_j}, \quad B_{ij} = \frac{\left(\tau_0^{\alpha}\right)(p_i k_i)\left(pp_j^2 kk_j\right)}{pp_j - p_i} \tag{22}$$

Hence, we can write that:

$$\begin{aligned} H(s) \cong & \left(\sum_{j=1}^{2N_2-1}\frac{\left(\tau_0^{\alpha}K_D\right)\left(pp_j kk_j\right)}{\left(s+pp_j\right)}\right) \\ & + \left(\sum_{j=1}^{2N_2-1}\frac{\sum_{i=0}^{N_1}B_{ij}}{\left(s+pp_j\right)}\right) + \left(\sum_{i=0}^{N_1}\frac{\sum_{j=1}^{2N_2-1}A_{ij}}{(s+p_i)}\right) \end{aligned} \tag{23}$$

$$H(s) \cong \left(\sum_{i=0}^{N_1} \frac{\overline{A}_i}{(s+p_i)}\right) + \left(\sum_{j=1}^{2N_2-1} \frac{\overline{B}_j}{(s+pp_j)}\right) \tag{24}$$

where the residues $\overline{A}_i$ ($0 \leq i \leq N_1$), and $\overline{B}_j$ ($1 \leq j \leq 2N_2-1$), are given by:

$$\overline{A}_i = \sum_{j=1}^{2N_2-1} A_{ij}, \quad \overline{B}_j = \left[\left(\tau_0^{\alpha} K_D\right)\left(pp_j kk_j\right)\right] + \sum_{i=0}^{N_1} B_{ij} \tag{25}$$

2.2 Case 2: α = 0.5

For this case, the function of (9) can be rewritten as:

$$H(s) = \frac{(\tau_0 s)^{0.5}}{1 + (\tau_0 s)} \tag{26}$$

From Eq. (15), the above function is approximated by a rational function as follows:

$$H(s) \cong \tau_0^{0.5} \left(K_D + \sum_{i=0}^{N_1} \frac{k_i s}{\left(1 + \frac{s}{p_i}\right)}\right)\left(\frac{1}{1 + (\tau_0 s)}\right) \tag{27}$$

$$H(s) \cong \left(\frac{\tau_0^{(-0.5)} K_D}{(s + 1/\tau_0)}\right) + \left(\sum_{i=0}^{N_1} \frac{\left(\tau_0^{(-0.5)}\right)(p_i k_i) s}{(s+p_i)(s+1/\tau_0)}\right) \tag{28}$$

By the decomposition of the rational function of (28), we will get:

$$H(s) \cong \left(\frac{\tau_0^{(-0.5)} K_D}{(s + 1/\tau_0)}\right) + \left(\sum_{i=0}^{N_1} \left(\frac{C_i}{(s+p_i)} + \frac{D_i}{(s+1/\tau_0)}\right)\right) \tag{29}$$

where the residues C_i and D_i (for $0 \leq i \leq N_1$) are given by:

$$C_i = \frac{\left(\tau_0^{(-0.5)}\right)\left(p_i^2 k_i\right)}{p_i - 1/\tau_0}, \quad D_i = \frac{\left(\tau_0^{(-1.5)}\right)(p_i k_i)}{1/\tau_0 - p_i} \tag{30}$$

Hence, we can write that:

$$H(s) \cong \left(\sum_{i=0}^{N_1} \frac{\overline{C}_i}{(s+p_i)}\right) + \left(\frac{\overline{D}}{(s+1/\tau_0)}\right) \tag{31}$$

with the residues $\overline{C}_i = C_i$ (for $0 \le i \le N_1$) and $\overline{D} = \left[\tau_0^{(-0.5)} K_D\right] + \left[\sum_{i=0}^{N_1} D_i\right]$.

3 Time Responses

3.1 Case 1: $0 < \alpha < 0.5$

From Eq. (24), we have that:

$$H(s) = \frac{X(s)}{E(s)} = \left(\sum_{i=0}^{N_1} \frac{\overline{A}_i}{(s + p_i)}\right) + \left(\sum_{j=1}^{2N_2-1} \frac{\overline{B}_j}{(s + pp_j)}\right) \tag{32}$$

then,

$$X(s) = \left[\left(\sum_{i=0}^{N_1} \frac{\overline{A}_i}{(s + p_i)}\right) + \left(\sum_{j=1}^{2N_2-1} \frac{\overline{B}_j}{(s + pp_j)}\right)\right] E(s) \tag{33}$$

For e(t) = δ(t) the unit impulse E(s) = 1, we will have:

$$X(s) = \left[\left(\sum_{i=0}^{N_1} \frac{\overline{A}_i}{(s + p_i)}\right) + \left(\sum_{j=1}^{2N_2-1} \frac{\overline{B}_j}{(s + pp_j)}\right)\right] \tag{34}$$

Hence, the impulse response of (10) is:

$$\begin{aligned} x(t) = L^{-1}\{X(s)\} = &\left(\sum_{i=0}^{N_1} \overline{A}_i \exp(-p_i t)\right) \\ &+ \left(\sum_{j=1}^{2N_2-1} \overline{B}_j \exp(-pp_j t)\right) \end{aligned} \tag{35}$$

For e(t) = u(t) the unit step E(s) = 1/s, (33) will be:

$$X(s) = \left[\left(\sum_{i=0}^{N_1} \frac{\overline{A}_i}{(s + p_i)}\right) + \left(\sum_{j=1}^{2N_2-1} \frac{\overline{B}_j}{(s + pp_j)}\right)\right]\left(\frac{1}{s}\right) \tag{36}$$

$$X(s) = \left(\sum_{i=0}^{N_1} \frac{\overline{A}_i}{(s + p_i)}\left(\frac{1}{s}\right)\right) + \left(\sum_{j=1}^{2N_2-1} \frac{\overline{B}_j}{(s + pp_j)}\left(\frac{1}{s}\right)\right) \tag{37}$$

$$X(s) = \left(\sum_{i=0}^{N_1} \left(\frac{\overline{A}_i}{p_i}\right)\left(\frac{1}{s} - \frac{1}{(s+p_i)}\right)\right) + \left(\sum_{j=1}^{2N_2-1} \left(\frac{\overline{B}_i}{pp_j}\right)\left(\frac{1}{s} - \frac{1}{(s+pp_j)}\right)\right) \quad (38)$$

Hence, the step response of (10) is:

$$x(t) = L^{-1}\{X(s)\} = \left(\sum_{i=0}^{N_1} \left(\frac{\overline{A}_i}{p_i}\right)[1 - \exp(-p_i t)]\right) + \left(\sum_{j=1}^{2N_2-1} \left(\frac{\overline{B}_j}{pp_j}\right)[1 - \exp(-pp_j t)]\right) \quad (39)$$

3.2 Case 2: α = 0.5

From Eq. (31), we have that:

$$H(s) = \frac{X(s)}{E(s)} = \left(\sum_{i=0}^{N_1} \frac{\overline{C}_i}{(s+p_i)}\right) + \left(\frac{\overline{D}}{(s+1/\tau_0)}\right) \quad (40)$$

then,

$$X(s) = \left[\left(\sum_{i=0}^{N_1} \frac{\overline{C}_i}{(s+p_i)}\right) + \left(\frac{\overline{D}}{(s+1/\tau_0)}\right)\right] E(s) \quad (41)$$

For e(t) = δ(t) the unit impulse E(s) = 1, we will have:

$$X(s) = \left[\left(\sum_{i=0}^{N_1} \frac{\overline{C}_i}{(s+p_i)}\right) + \left(\frac{\overline{D}}{(s+1/\tau_0)}\right)\right] \quad (42)$$

Hence, the impulse response of (10), for α = 0.5, is:

$$x(t) = L^{-1}\{X(s)\} = \left(\sum_{i=0}^{N_1} \overline{C}_i \exp(-p_i t)\right) + \left(\overline{D} \exp(-t/\tau_0)\right) \quad (43)$$

For e(t) = u(t) the unit step E(s) = 1/s, (41) will be:

$$X(s) = \left[\left(\sum_{i=0}^{N_1} \frac{\overline{C}_i}{(s+p_i)}\right) + \left(\frac{\overline{D}}{(s+1/\tau_0)}\right)\right]\left(\frac{1}{s}\right) \tag{44}$$

$$X(s) = \left(\sum_{i=0}^{N_1} \frac{\overline{C}_i}{(s+p_i)}\left(\frac{1}{s}\right)\right) + \left(\left(\frac{\overline{D}}{(s+1/\tau_0)}\right)\left(\frac{1}{s}\right)\right) \tag{45}$$

$$\begin{aligned} X(s) = &\left(\sum_{i=0}^{N_1}\left(\frac{\overline{C}_i}{p_i}\right)\left(\frac{1}{s} - \frac{1}{(s+p_i)}\right)\right) \\ &+ \left(\overline{D}\tau_0\left(\frac{1}{s} - \frac{1}{(s+1/\tau_0)}\right)\right) \end{aligned} \tag{46}$$

Hence, the step response of f (10), for α = 0.5, is:

$$\begin{aligned} x(t) = L^{-1}\{X(s)\} = &\left(\sum_{i=0}^{N_1}\left(\frac{\overline{C}_i}{p_i}\right)[1 - \exp(-p_i t)]\right) \\ &+ \left(\overline{D}\,\tau_0[1 - \exp(-t/\tau_0)]\right) \end{aligned} \tag{47}$$

4 Illustrative Example

Let us first consider the fractional system represented by the following fundamental linear fractional order differential equation with α = 0.35 and τ_0 = 2 as:

$$(2)^{0.7}\frac{d^{0.7}x(t)}{dt^{0.7}} + x(t) = (2)^{0.35}\frac{d^{0.35}e(t)}{dt^{0.35}} \tag{48}$$

its transfer function is given by:

$$H(s) = \frac{(2s)^{0.35}}{1 + (2s)^{0.7}} \tag{49}$$

Its rational function approximation, in a given frequency band, is given as:

$$H(s) = \frac{(2s)^{0.35}}{1 + (2s)^{0.7}} = \left(\sum_{i=0}^{N_1} \frac{\overline{A}_i}{(s+p_i)}\right) + \left(\sum_{j=1}^{2N_2-1} \frac{\overline{B}_j}{(s+pp_j)}\right) \tag{50}$$

For the fractional order differentiator $(2s)^{0.35}$, the frequency band of approximation is $[\omega_L, \ \omega_H]$ = [10^{-4} rad/s, 10^4 rad/s], around ω_0 = ($1/\tau_0$) = 0.5 rad/s, y = 1 dB, δ = 0.1, and β = 100. For the fractional system $\frac{1}{1+(2s)^{0.7}}$, the frequency band of approximation [0, ω_H] = [0, 10^4 rad/s], λ = 4 and β = 100. Then, the approximation parameters of H(s) are:

$$a = 1.4251,\ b = 1.9307,\ \omega_c = 10^{-5},\ \omega_{max} = 10^6,$$
$$p_0 = 1.9802{*}10^{-5},\ K_D = 0.0178,\ N_1 = 25,\ and\ N_2 = 11$$

Hence, poles p_i, the residues $\overline{A}_i$ ($0 \le i \le 25$), the poles pp_j and the residues $\overline{B}_j$ ($1 \le j \le 20$), are given by:

$$p_i = \left(1.9802 * 10^{-5}\right)(2.7514)^i$$
$$\overline{A}_i = \sum_{j=1}^{21} \frac{\left[\left(1.1230 * 10^{-7}\right)\left((2.7514)^i(4)^{(j-11)}\right) * \frac{\prod_{j=0}^{25}\left[1-\left((1.4251)(2.7514)^{(i-j)}\right)\right]}{\prod_{j=0,i\neq j}^{25}\left(1-(2.7514)^{(i-j)}\right)}\right]}{\left[\pi\left[(0.5)(4)^{(j-11)} - (1.9802 * 10^{-5})(2.7514)^i\right]\right]} * \left[\frac{\sin[(0.3)\pi]}{\cosh[(0.35)\log((4)^{(11-j)})] - \cos[(0.3)\pi]}\right] \tag{51}$$

$$pp_j = (0.5)(4)^{(j-11)}$$
$$\overline{B}_j = \frac{(0.0057)}{\pi}(4)^{(j-11)}\left[\frac{\sin[(0.3)\pi]}{\cosh[(0.35)\log((4)^{(11-j)})] - \cos[(0.3)\pi]}\right] + \sum_{i=0}^{25}\left\{\frac{-(0.0029)(4)^{2(j-11)}\frac{\prod_{j=0}^{25}\left[1-(1.4251)(2.7514)^{(i-j)}\right]}{\prod_{j=0,i\neq j}^{25}\left(1-(2.7514)^{(i-j)}\right)}}{\pi\left[(0.5)(4)^{(j-11)} - (1.9802 * 10^{-5})(2.7514)^i\right]} \times \left[\frac{\sin[(0.3)\pi]}{\cosh[(0.35)\log((4)^{(11-j)})] - \cos[(0.3)\pi]}\right]\right\} \tag{52}$$

Figures 1 and 2 show the bode plots of the fundamental linear fractional order system transfer function of (49) and of its proposed rational function approximation of (50). We can easily see that they are all quite overlapping in the frequency band of interest.

Figures 3 and 4 show, respectively, the impulse and the step responses of the fractional order system of (48).

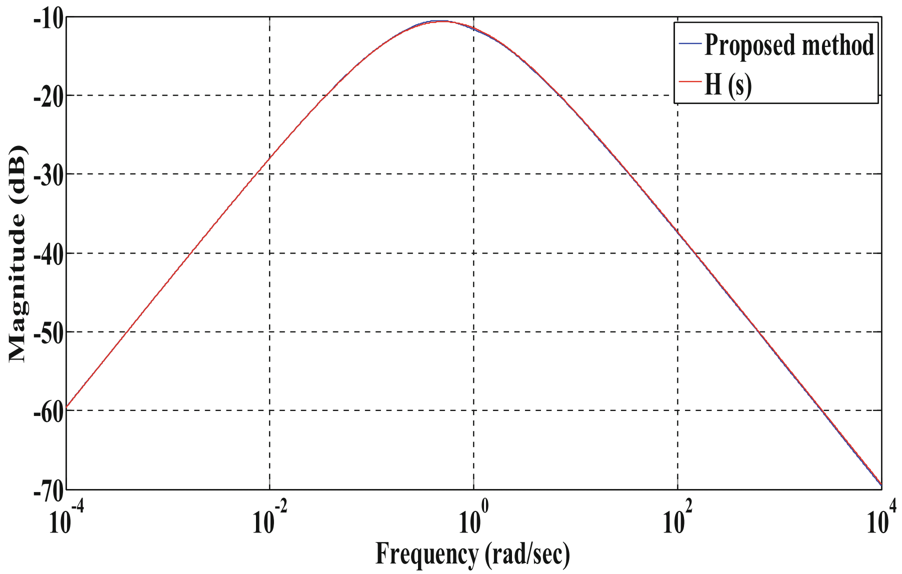

Fig. 1. Magnitude bode plots of (49) and of its proposed approximation.

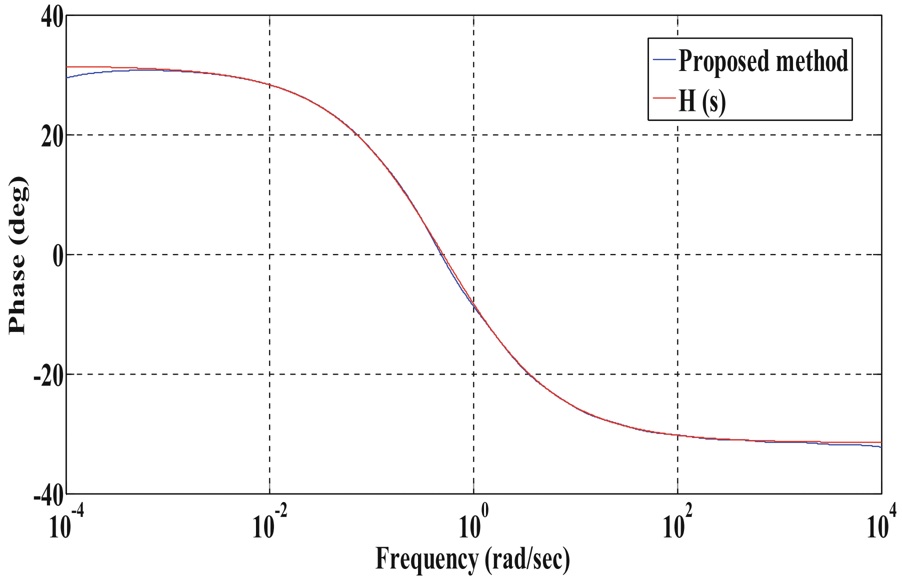

Fig. 2. Phase bode plots of (49) and of its proposed approximation

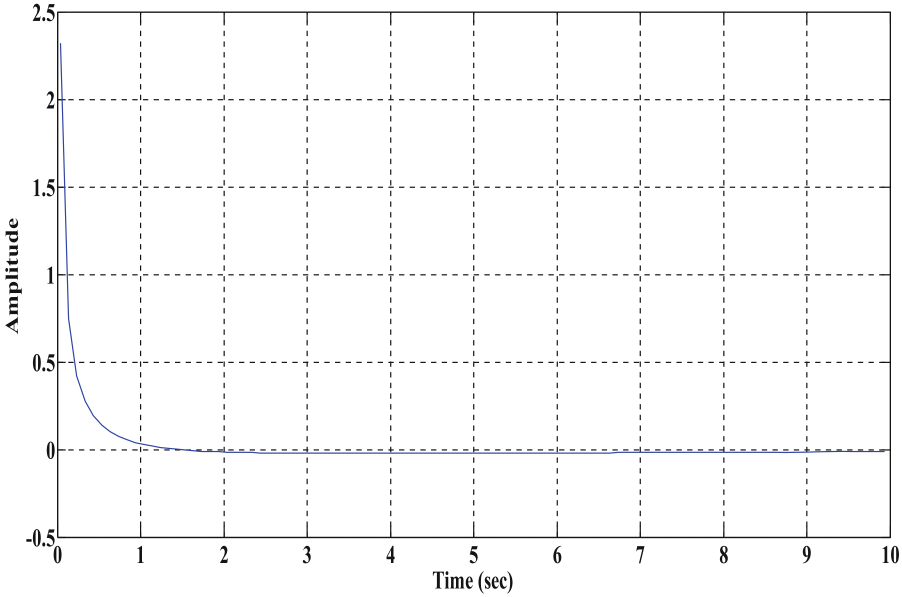

Fig. 3. Impulse response of (48) from its proposed approximation.

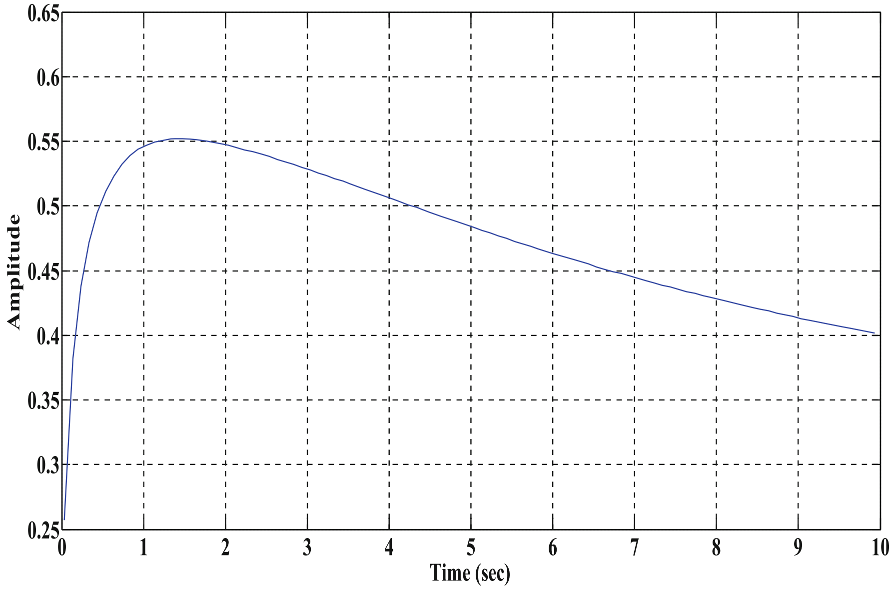

Fig. 4. Step response of (48) from its proposed approximation.

As a second example, we will consider the fractional order system represented by the following fundamental linear fractional order differential equation with $\alpha = 0.5$ and $\tau_0 = 0.16$ as:

$$(0.16)\frac{dx(t)}{dt} + x(t) = (0.16)^{0.5}\frac{d^{0.5}e(t)}{dt^{0.5}} \tag{53}$$

its transfer function is given by:

$$H(s) = \frac{(0.16\,s)^{0.5}}{1 + (0.16\,s)} \tag{54}$$

Its rational function approximation, in a given frequency band, is given as:

$$H(s) = \frac{(0.16\,s)^{0.5}}{1 + (0.16\,s)} = \left(\sum_{i=0}^{N_1} \frac{\overline{C}_i}{(s + p_i)}\right) + \left(\frac{\overline{D}}{(s + 1/\tau_0)}\right) \tag{55}$$

For the fractional differentiator $(0.16\ s)^{0.5}$, the frequency band of approximation is $[\omega_L,\ \omega_H] = [10^{-4}$ rad/s, 10^4 rad/s], around $\omega_0 = (1/\tau_0) = 6.25$ rad/s, $y = 1$ dB, $\delta = 0.1$, and $\beta = 100$. Then, the approximation parameters of H(s) are:

$$a = 1.5849,\ b = 1.5849,\ \omega_c = 10^{-5},\ \omega_{max} = 10^6,$$
$$p_0 = 1.9953 * 10^{-5},\ K_D = 0.0032\ \mathit{and}\ N_1 = 28$$

Hence, poles p_i, the residues $\overline{C}_i$ ($0 \le i \le 25$), and the residue $\overline{D}$, are given by:

$$\begin{aligned} p_i &= \left(1.9953 * 10^{-5}\right)(2.5119)^i \\ \overline{C}_i &= \left(\frac{-(1.5963 * 10^{-7})(2.5119)^i}{\left[(1.9953 * 10^{-5})(2.5119)^i - 6.25\right]}\right) \\ &\quad * \left(\frac{\prod_{j=0}^{28}\left(1 - (1.5849)(2.5119)^{(i-j)}\right)}{\prod_{j=0,\,i\neq j}^{28}\left(1 - (2.5119)^{(i-j)}\right)}\right) \end{aligned} \tag{56}$$

$$\begin{aligned} \overline{D} &= (0.008) \\ &\quad + \sum_{i=0}^{28}\left[\left(\frac{-(0.05)}{\left[6.25 - (1.9953 * 10^{-5})(2.5119)^i\right]}\right) * \left(\frac{\prod_{j=0}^{28}\left(1 - (1.5849)(2.5119)^{(i-j)}\right)}{\prod_{j=0,\,i\neq j}^{28}\left(1 - (2.5119)^{(i-j)}\right)}\right)\right] \end{aligned} \tag{57}$$

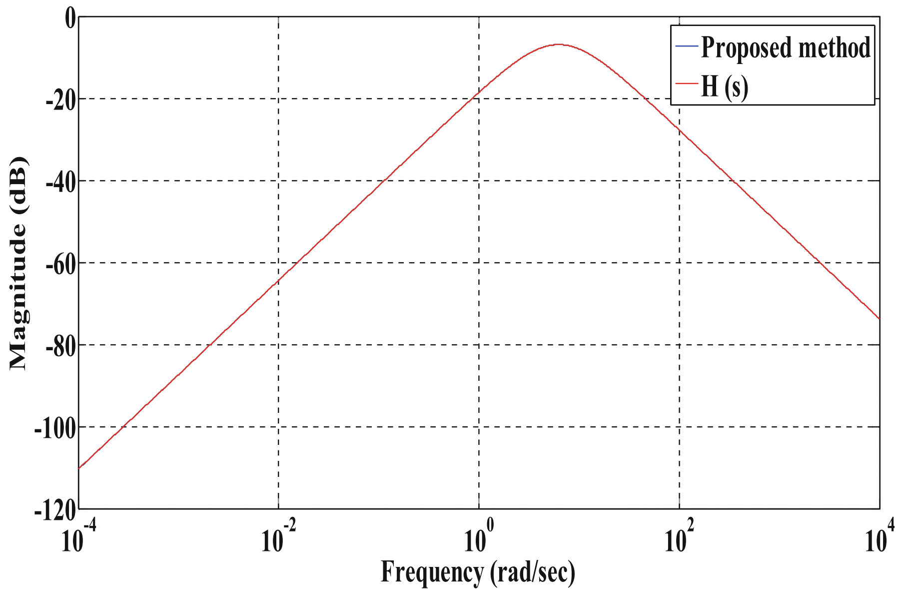

Fig. 5. Magnitude bode plot of (54) and of its proposed approximation.

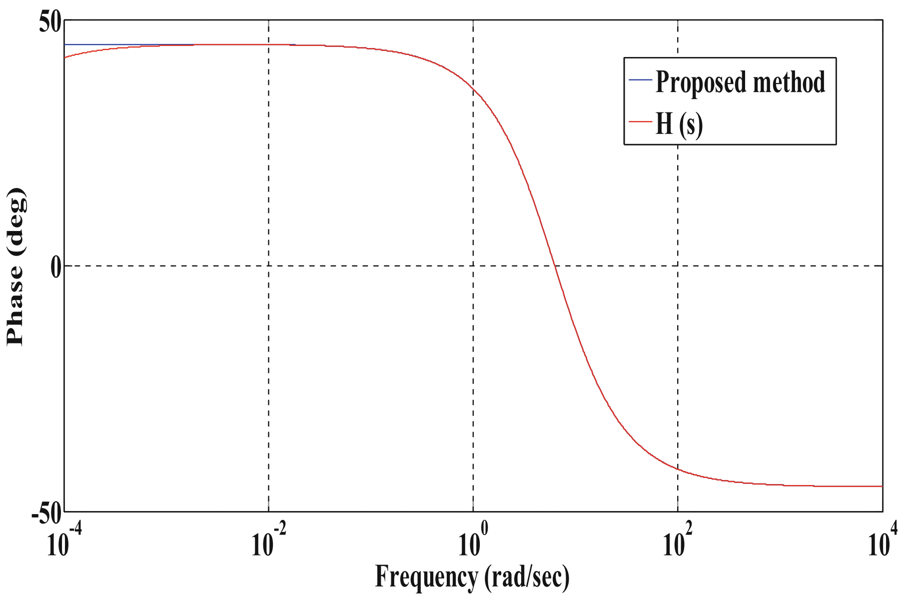

Fig. 6. Phase bode plot of (54) and of its proposed approximation.

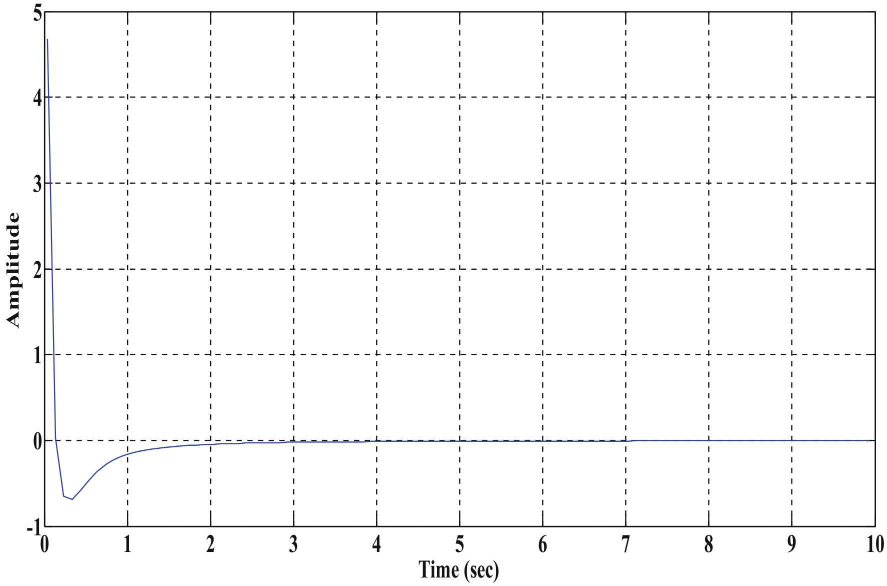

Fig. 7. Impulse response of (53) from its proposed approximation.

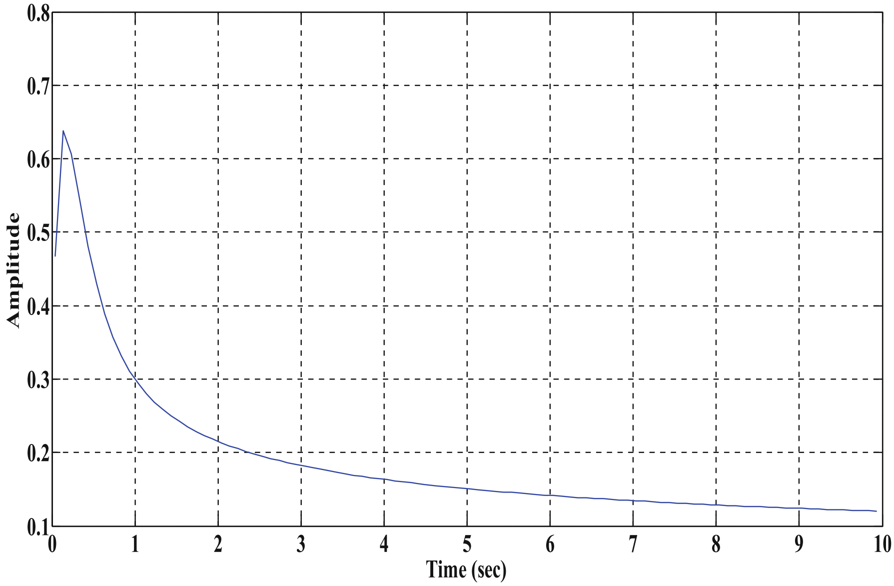

Fig. 8. Step response of (53) from its proposed approximation.

Figures 5 and 6 show the bode plots of the fundamental linear fractional system transfer function of (54) and of its proposed rational function approximation of (55). We note that they are overlapping in the frequency band of interest.

Figures 7 and 8 show, respectively, the impulse and the step responses of the fractional order system of (53).

5 Conclusion

In this paper, we have presented a rational function approximation of the fractional order transfer function $H(s) = \frac{(\tau_0 s)^{\alpha}}{[1 + (\tau_0 s)^{2\alpha}]}, \quad for\, 0 < \alpha \le 0.5$. This fractional order transfer function is one of the fundamental functions of the linear fractional system of commensurate order, represented by the linear fractional state-space $D^{\alpha}x(t) = A\,x(t)$. H(s) corresponds to pure complex conjugate eigenvalues of the A matrix. First, closed form of the approximation technique has been derived. Then, the impulse and step responses of this type of fractional system have been obtained. Finally, illustrative examples are presented to show the exactitude and the usefulness of the approximation method.

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