Quasi-static Evolution, Variational Principles and Implicit Scheme in Gradient Plasticity

Quoc-Son Nguyen

Abstract This paper is devoted to the theory of gradient plasticity. Our attention is focussed on the description of the constitutive equations, on the formulation of the governing equations in terms of the energy potential and the dissipation potential of the solid. The evolution equation is discussed for quasi-static responses. A time-discretization by the implicit scheme of the evolution equation leads to the study of the incremental problem which is different from the rate problem. The incremental problem and associated incremental variational principles are discussed in relation with some existing results of the literature.

1 Introduction

Since two last decades, gradient theories have been much discussed in elasticity, in plasticity as in damage mechanics, see for example [2, 3, 5–7, 9]. This paper is devoted to the study of gradient plasticity. A general and consistent description of the theory of gradient plasticity is considered. Our attention is focussed on the formulation of the constitutive equations and the derivation of the governing equations for the response of a solid under a loading path **in terms of the expression of the energy potential and the dissipation potential of the solid**. Such a synthetic description, still lacking in the literature, appears to be interesting for an overview on the subject. It enables us to include in the same framework all general statements which result from the basic ingredients of the theory such as the evolution equation in quasi-statics and the associated variational principles. In particular, the discretization of the evolution equation by the implicit scheme leads to the formulation of the incremental response which is interesting for the numerical simulation and the stability analysis.

Q.-S. Nguyen (✉)
École Polytechnique-ParisTech, Laboratoire de Mécanique des Solides,
91128 Palaiseau Cedex, France
e-mail: son@lms.polytechnique.fr

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2 Standard Theory of Gradient Plasticity

In an isothermal transformation, the mechanical response of a solid V is described by the fields of displacement $\boldsymbol{u}$, of internal parameter Φ. The internal parameter is a scalar or a tensor and represent physically hidden parameters such as micro-displacements or phase proportions or anelastic strains, etc. The set of state variables $(\nabla u, \phi, \nabla\phi)$ describes the material behaviour and the governing equations can be given in the following way (see [4, 5, 7]).

It is accepted that the rates $(\nabla\dot{u},\ \dot{\phi},\ \nabla\dot{\phi})$ of the state variables are associated with the generalized forces (σ, X, Y) such that a generalized virtual work equation holds

$$\begin{cases} P_i + P_j = P_e \quad \forall\ \delta u,\ \delta\Phi \\ P_i = \int_V (\sigma \cdot \nabla\delta u + X \cdot \delta\phi + Y \cdot \nabla\delta\phi)\, dV, \\ P_j = \int_V \rho\ddot{u} \cdot \delta u\, dV, \\ P_e = \int_V (F_{uv} \cdot \delta u + F_{\phi v} \cdot \delta\phi)\, dV + \int_{\partial V} (F_{us} \cdot \delta u + F_{\phi s} \cdot \delta\phi)\, da \end{cases} \tag{1}$$

where (F_{uv}, F_{us}) and $(F_{\phi v}, F_{\phi s})$ are respectively external body and surface forces associated with the displacement and the internal parameter. It follows that: a

$$\nabla \cdot \sigma + F_{uv} = \rho\ddot{u} = 0 \quad \text{in} \quad V, \qquad \sigma \cdot n = F_{us} \quad \text{on} \quad \partial V_f \tag{2}$$

$$X + \nabla \cdot Y + F_{\phi v} = 0 \qquad \text{in} \quad V, \qquad Y \cdot n = F_{\phi s} \quad \text{on} \quad \partial V \tag{3}$$

Standard gradient models of plasticity also assume that there exists per unit volume an energy potential which is a smooth function $W(\nabla u, \phi, \nabla\phi)$ associated with the energy forces σ, X_e, Y_e:

$$\sigma = W,_{\nabla u}\ , \qquad X_e = W,_{\phi}\ , \qquad Y_e = W,_{\nabla\phi} \tag{4}$$

and a dissipation potential $D(\dot{\phi}, \nabla\dot{\phi})$ which is a convex and positively homogeneous function of degree 1

$$D(a\dot{\phi}, a\nabla\dot{\phi}) = aD(\dot{\phi}, \nabla\dot{\phi}) \qquad \forall \quad a \geq 0 \tag{5}$$

associated with the dissipative forces

$$X_d = \partial_{\dot{\phi}}\ D(\dot{\phi}, \nabla\dot{\phi})\ , \qquad Y_d = \partial_{\nabla\dot{\phi}}\ D(\dot{\phi}, \nabla\dot{\phi}) \tag{6}$$

such that the following equations hold:

$$X = X_e + X_d\ , \qquad Y = Y_e + Y_d. \tag{7}$$

In Eq. (6), the derivative must be understood in the sense of sub-gradients of a convex function, see for example [4, 10]. The dissipation potential can be state-dependent, for exampe via the history of the state variable ϕ.

2.1 Standard Models of Gradient Plasticity

For example, the following model has been discussed by Fleck et al. [2] with $\phi = \epsilon^p$ and

$$\begin{cases} W(\nabla u, \epsilon^p) = \frac{1}{2}(\epsilon - \epsilon^p) : L : (\epsilon - \epsilon^p), \\ D(\dot{\epsilon}^p, \nabla\dot{\epsilon}^p) = R(\gamma)\sqrt{\|\dot{\epsilon}^p\|^2 + \ell^2\|\nabla\dot{\epsilon}^p\|^2}, \\ \gamma = \int_0^t \sqrt{\|\dot{\epsilon}^p\|^2 + \ell^2\|\nabla\dot{\epsilon}^p\|^2)}\, d\tau, \end{cases} \tag{8}$$

with $\epsilon = (\nabla u)_s$ and the notation $\|\dot{\epsilon}^p\| = \sqrt{\dot{\epsilon}^p_{ij}\dot{\epsilon}^p_{ij}}$ and $\|\nabla\dot{\epsilon}^p\| = \sqrt{\dot{\epsilon}^p_{ij,k}\dot{\epsilon}^p_{ij,k}}$.

Here, the dissipation potential is state-dependent via the expression of γ. As in classical plasticity, the model leads to a plastic criterion $f(X^p_d, Y^p_d) \leq 0$ which defines the set of physically admissible forces and to the normality law:

$$\begin{cases} f = (\|X^p_d\|^2 + \frac{1}{\ell^2}\|Y^p_d\|^2)^{1/2} - R(\gamma) \leq 0, \\ \dot{\epsilon}^p = \lambda\frac{\partial f}{\partial X^p_d}\,, \quad \nabla\dot{\epsilon}^p = \lambda\frac{\partial f}{\partial Y^p_d}\,, \quad \lambda \geq 0\,, \qquad f\lambda = 0 \\ \dot{\gamma} = \lambda. \end{cases} \tag{9}$$

The dissipation is

$$d = X^p_d \cdot \dot{\epsilon}^p + Y^p_d \cdot \nabla\dot{\epsilon}^p = R(\gamma)\lambda = \frac{d}{dt}W_d(\gamma) \quad \text{with} \quad R = W'_d(\gamma), \tag{10}$$

$W_d(\gamma)$ being the dissipated energy.

This model can be easily modified to obtain a state-independent dissipation potential. For this, the following model is introduced with $\phi = (\epsilon^p, \gamma)$

$$\begin{cases} W(\nabla u, \epsilon^p, \gamma) = \frac{1}{2}(\epsilon - \epsilon^p) : L : (\epsilon - \epsilon^p) + H(\gamma), \\ D(\dot{\epsilon}^p, \dot{\gamma}, \nabla\dot{\epsilon}^p) = k\sqrt{\|\dot{\epsilon}^p\|^2 + \ell^2\|\nabla\dot{\epsilon}^p\|^2} + \Psi_o(\dot{\gamma} - \sqrt{\|\dot{\epsilon}^p\|^2 + \ell^2\|\nabla\dot{\epsilon}^p\|^2}) \end{cases} \tag{11}$$

where k is a constant and Ψ_o the indicator function

$$\Psi_o(a) = 0 \quad \text{if} \quad a = 0 \quad \text{and} \quad \Psi_o(a) = +\infty \quad \text{if} \quad a \neq 0 \tag{12}$$

which ensures the constraint

$$\dot{\gamma} = \sqrt{\|\dot{\epsilon}^p\|^2 + \ell^2\|\nabla\dot{\epsilon}^p\|^2}.$$

From Eq. (11), the considered model leads to the plastic criterion

$$f(X^p_d, X^\gamma_d, Y^p_d) = (\|X^p_d\|^2 + \frac{1}{\ell^2}\|Y^p_d\|^2)^{1/2} + X^\gamma_d - k \leq 0 \tag{13}$$

and the normality law

$$\begin{cases} \dot{\epsilon}^p = \lambda \frac{\partial f}{\partial X_d^p}\,, \qquad \dot{\gamma} = \lambda \frac{\partial f}{\partial X_d^\gamma} = \lambda\,, \qquad \nabla \dot{\epsilon}^p = \lambda \frac{\partial f}{\partial Y_d^p}\,, \\ f \leq 0\,, \qquad \lambda \geq 0\,, \qquad f\lambda = 0. \end{cases} \tag{14}$$

The dissipation is

$$d = X_d^p \cdot \dot{\epsilon}^p + X_d^\gamma \, \dot{\gamma} + Y_d^p \cdot \nabla \dot{\epsilon}^p = k\lambda = k\dot{\gamma} = k\sqrt{\|\dot{\epsilon}^p\|^2 + \ell^2 \|\nabla \dot{\epsilon}^p\|^2}$$

which gives here the physical interpretation of $k\gamma$ as the dissipated energy.

Since $X_d^\gamma = -X_e^\gamma = H'(\gamma)$, the constitutive equation (8) is recovered with $H'(\gamma) + k = R(\gamma)$. In this model, W_d is the work done by plastic deformation and consists of the dissipated energy $k\gamma$ and the stored energy $H(\gamma)$.

In the same spirit, an interesting model of isotropic hardening is given by

$$\begin{cases} W(\nabla u, \epsilon^p, \gamma, \nabla \gamma) = \frac{1}{2}(\epsilon - \epsilon^p) : L : (\epsilon - \epsilon^p) + H(\gamma) + \frac{g}{2} \nabla \gamma^2, \\ D(\dot{\epsilon}^p, \dot{\gamma}, \nabla \dot{\gamma}) = k\|\dot{\epsilon}^p\| + \kappa \|\nabla \dot{\gamma}\| + \Psi_o(\dot{\gamma} - \|\dot{\epsilon}^p\|). \end{cases} \tag{15}$$

The plastic criterion is given by two inequalities

$$f(X_d^p, X_d^\gamma) = \|X_d^p\| + X_d^\gamma - k \leq 0\,, \qquad \varphi(Y_d^\gamma) = \|Y_d^\gamma\| - \kappa \leq 0 \tag{16}$$

and the normality law is

$$\begin{cases} \dot{\epsilon}^p = \lambda \frac{\partial f}{\partial X_d^p}\,, \quad \dot{\gamma} = \lambda \frac{\partial f}{\partial X_d^\gamma} \quad \text{with} \quad f \leq 0\,, \quad \lambda \geq 0\,, \quad \lambda f = 0, \\ \nabla \dot{\gamma} = \tau \frac{\partial \varphi}{\partial Y_d^\gamma} \quad \text{with} \quad \varphi \leq 0\,, \quad \tau \geq 0\,, \quad \tau\varphi = 0. \end{cases} \tag{17}$$

The reader can also refer to [6, 8, 9] for interesting discussions on a model of energy $W = \frac{1}{2}(\epsilon - \epsilon^p) : L : (\epsilon - \epsilon^p) + \frac{1}{2}\text{curl}(\epsilon^p) : E : \text{curl}(\epsilon^p)$. Here the energy potential depends on the gradient of the plastic strain via the operator 'curl'.

3 Evolution Equation

3.1 Governing Equations for a Solid Under a Loading Path

In the sequel, the assumption of state-independent dissipation is accepted. The conditions $F_{\phi v} = 0$ and $F_{\phi s} = 0$, although not essential, are also admitted. For a solid submitted to a classical loading path, defined by the body forces $F_{uv}(x,t)$, $F_{\phi v}(x,t)$, the surface forces $F_{us}(x,t)$, $F_{\phi,s}(x,t)$ and the imposed displacement $u_g(x,t)$, the response of the solid must satisfy the local equations

$$
\begin{aligned}
&\forall \quad t \in [0,T]: \\
&\qquad \sigma = W,_{\nabla u}\ , \quad X_e = W,_{\phi}\ , \quad Y_e = W,_{\nabla\phi}\ , \\
&\qquad X = X_e + X_d\ , \quad Y = Y_e + Y_d\ , \quad (X_d,\ Y_d) = \partial D(\dot{\phi}, \nabla\dot{\phi})\ , \\
&\qquad \nabla\cdot\sigma + F_{uv} = \rho\ddot{u}\ , \quad X + \nabla\cdot Y = 0 \quad \text{in} \quad V\ , \\
&\qquad \sigma\cdot n = F_{us} \quad \text{on} \quad \partial V_f\ , \quad u = u_g \quad \text{on} \quad \partial V_u\ , \\
&\qquad Y\cdot n = 0 \quad \text{on} \quad \partial V \\
&\text{at} \quad t = 0: \\
&\qquad \boldsymbol{u}(0) = \boldsymbol{u}_o\ , \quad \Phi(0) = \Phi_o, \quad \dot{\mathbf{u}}(0) = \boldsymbol{v}_o\ .
\end{aligned}
\tag{18}
$$

These equations describe the response of the solid from an initial position of state and velocity.

3.2 The Quasi-static Response

It is convenient to introduce as a condensed notation the general displacement $\boldsymbol{U} = (\boldsymbol{u}, \Phi)$ to write simply the energy and dissipation potentials of the solid as :

$$
\boldsymbol{W}(U) = \int_V W(\nabla u, \phi, \nabla\phi)\, dV\ , \qquad \boldsymbol{D}(\dot{U}) = \int_V D(\dot{\phi}, \nabla\dot{\phi})\, dV. \tag{19}
$$

In quasi-static transformation, a variational and condensed form of the evolution equation for the solid can be introduced as in classical plasticity (see [10]).

Evolution Equation. *For all $t \in [0,T]$, the quasi-static response $\boldsymbol{U}(t)$ of the solid submitted to a given loading path $\boldsymbol{F}_g(t)$, $u_g(t)$ satisfies the following variational inequality:*

$$
\boldsymbol{W},_U(U)\cdot(\dot{U}^* - \dot{U}) + D(\dot{U}^*) - D(\dot{U}) - \boldsymbol{F}_g\cdot(\dot{U}^* - \dot{U}) \geq 0 \tag{20}
$$

for all response $\boldsymbol{U}^*(t)$ satisfying the imposed condition $\boldsymbol{u}^*(t) = u_g(t)$ on ∂V_u.

This variational inequality means explicitly that

$$
\begin{cases}
\int_V \sigma : \nabla(\dot{u}^* - \dot{u})\, dV - \int_V F_{vu}\cdot(\dot{u}^* - \dot{u})\, dV - \int_{\partial V_F} F_{gsu}\cdot(\dot{u}^* - \dot{u})\, dS \\
+ \int_V (X_e\cdot(\dot{\phi}^* - \dot{\phi}) + Y_e\cdot(\nabla\dot{\phi}^* - \nabla\dot{\phi}) + D(\dot{\phi}^*, \nabla\dot{\phi}^*) - D(\dot{\phi}, \nabla\dot{\phi}))\, dV \geq 0
\end{cases}
\tag{21}
$$

for all Φ^* and for all $\boldsymbol{u}^*$ admissible.

Thus for all t, it follows from the evolution variational inequality that the equilibrium equation holds

$$\boldsymbol{W},_u(U)\cdot\delta u - F_{gu}\cdot\delta u = 0 \quad \forall \quad \delta u = 0 \quad \text{on} \quad \partial V_u$$

and that $\dot{\boldsymbol{\Phi}}$ must satisfy the following minimum principle:

$$\begin{cases} \boldsymbol{I}(\dot{\boldsymbol{\Phi}}) = \min_{\delta\Phi} \ \boldsymbol{I}(\delta\boldsymbol{\Phi}), \\ \boldsymbol{I}(\delta\boldsymbol{\Phi}) = \int_V (X_e\cdot\delta\phi + Y_e\cdot\nabla\delta\phi + D(\delta\phi,\nabla\delta\phi)\,dV \end{cases} \tag{22}$$

which is the minimum principle I in Fleck & Willis [2].

The force-flux relationships (4), (6), (7) follows from the minimum principle (22). Indeed, the minimum principle holds only if for almost t, $m = \min_{\delta\Phi} \ \boldsymbol{I} = 0$ since **I** is the sum of a linear and a positive homogeneous functionals. Moreover, the rate $\dot{\boldsymbol{\Phi}}$ must be found among the solutions of (22). Such a solution will be denoted as *compatible* rate. The set of compatible rates has the structure of a convex cone since

(i) if $\boldsymbol{\Phi}^* \neq 0$ is compatible then $a\,\boldsymbol{\Phi}^*$ is also compatible for all number $a > 0$;
(ii) if ϕ_1^* and ϕ_2^* are two different compatible rates, then $\alpha\boldsymbol{\Phi}_1^* + (1-\alpha)\boldsymbol{\Phi}_2^*$ is also compatible for $0 \le \alpha \le 1$ since D is a convex function.

If there is no gradient term in the dissipation potential, the proof is very simple. The condition $m = 0$ implies that $\nabla\cdot Y_e - X_e = X_d$ must satisfy the plastic criterion since $m = -\infty$ otherwise. It is also straightforward that an compatible rate $\boldsymbol{\Phi}^*$ has the following expression

$$\phi^* = \lambda^* f,_{X_d} \quad \text{with} \qquad \lambda^* \ge 0\,, \qquad \lambda^* f = 0, \qquad X_d = -W,_\phi + \nabla\cdot W,_{\nabla\phi}\,.$$

When the gradient term does figure in the dissipation potential, the set of compatible rates cannot be easily generated although its definition is mathematically clear.

The question of existence of a solution of (20) has been much discussed in classical plasticity. In gradient plasticity, many discussions have been recently proposed for the existence, regularity and the numerical analysis of a solution (see [1, 3, 6, 9]).

Finally, the evolution equation (20) can be also schematically condensed as

$$\begin{cases} \ W,_u = \ F_u \\ -\boldsymbol{W},_\Phi \in \ \partial\boldsymbol{D}(\dot{\boldsymbol{\Phi}}). \end{cases} \tag{23}$$

This discussion shows in particular that higher gradients can also be included in the same framework. The force-flux relation is still given by Biot equation for the solid and the response of the solid is governed by the evolution equation (20).

4 Time-Discretization by the Implicit Scheme

4.1 Implicit Scheme and Incremental Problem

The numerical analysis of the quasi-static response of a solid to a given loading path is considered in this section. In a time-like discretization, the present value $\boldsymbol{U}$ is assumed at a current step. The incremental problem consists in determining the incremental response $\Delta\boldsymbol{U}$ to an increment of load $(\Delta\boldsymbol{F}_g,\ \Delta\boldsymbol{u}_g)$.

A time discretization of the evolution variational inequality (20) following the implicit scheme consists in replacing $\dot{\mathbf{U}},\ \dot{\mathbf{U}}^*,\ \dot{\mathbf{F}}$ respectively by $\frac{\Delta\boldsymbol{U}}{\Delta t},\ \frac{\Delta\boldsymbol{U}^*}{\Delta t},\ \frac{\Delta\boldsymbol{F}}{\Delta t}$ and $\boldsymbol{U}$ by $\boldsymbol{U}_+ = \boldsymbol{U} + \Delta\boldsymbol{U},\ \ \boldsymbol{F}$ by $\boldsymbol{F}_+ = \boldsymbol{F} + \Delta\boldsymbol{F}$ in the expression (20).

Since the dissipation potential is positively homogeneous of degree 1, it follows that the incremental response $\Delta\boldsymbol{U}$ must be a solution of the incremental problem i.e. satisfy the following variational inequality

$$\begin{aligned}\boldsymbol{W},_U(\boldsymbol{U}+\Delta\boldsymbol{U})\cdot(\Delta\boldsymbol{U}^*-\Delta\boldsymbol{U})+\boldsymbol{D}(\Delta\boldsymbol{U}^*)-\boldsymbol{D}(\Delta\boldsymbol{U})\\ -\boldsymbol{F}_g+\Delta\boldsymbol{F}_g\cdot(\Delta\boldsymbol{U}^*-\Delta\boldsymbol{U})\geq 0 \quad \forall \quad \Delta\boldsymbol{U}^* \quad \text{admissible.}\end{aligned} \tag{24}$$

The implicit scheme ensures that the equilibrium equation and the normality law are satisfied by the increments of the displacement and the internal parameter at the next step.

Conversely, an incremental process $\Delta\boldsymbol{U}(t_n)$, with $t_n = n\Delta t,\ n = 1, 2, ...N$, $N\Delta t = T$ defined by the increment variational inequality (24) and starting from $\boldsymbol{U}_o$ must satisfy at each current increment

$$\begin{cases}\boldsymbol{W},_U(\boldsymbol{U})\cdot(\frac{\Delta\boldsymbol{U}^*}{\Delta t}-\frac{\Delta\boldsymbol{U}}{\Delta t})+\boldsymbol{D}(\frac{\Delta\boldsymbol{U}^*}{\Delta t})-\boldsymbol{D}(\frac{\Delta\boldsymbol{U}}{\Delta t})-\boldsymbol{F}_g\cdot(\boldsymbol{U})\cdot(\frac{\Delta\boldsymbol{U}^*}{\Delta t}-\frac{\Delta\boldsymbol{U}}{\Delta t})\\ +\Delta t\ \{\frac{\Delta\boldsymbol{U}}{\Delta t}\cdot\boldsymbol{W},_{UU}(\boldsymbol{U})-\frac{\Delta\boldsymbol{F}_g}{\Delta t}\}\cdot(\frac{\Delta\boldsymbol{U}^*}{\Delta t}-\frac{\Delta\boldsymbol{U}}{\Delta t})\\ +o(\Delta t)\geq 0 \quad \forall \quad \Delta\boldsymbol{U}^* \quad \text{admissible.}\end{cases} \tag{25}$$

At the limit, when $\Delta t \to 0$, then $\frac{\Delta\boldsymbol{F}}{\Delta t} \to \dot{\mathbf{F}}$, $\frac{\Delta\boldsymbol{U}}{\Delta t} \to \dot{\mathbf{F}}$ and $\frac{\Delta\boldsymbol{U}^*}{\Delta t} \to \dot{\mathbf{U}}^*$ and the evolution equation (20) is recovered since the term of order zero in Δt must be non negative for all $\dot{\mathbf{U}}^*$ admissible.

Moreover, with the choice $\dot{\boldsymbol{\Phi}}^*$ compatible, the term of order zero is zero at the limit. It follows that the term of order Δt in (25) must be non negative at the limit and a variational inequality is obtained for the rate $\dot{\mathbf{U}}$ [10]:

$$\begin{aligned}&\dot{\mathbf{U}}\cdot W,_{UU}\cdot(\dot{U}^*-\dot{U})-\dot{\mathbf{F}}_g\cdot(\dot{U}^*-\dot{U})\geq 0\\ &\forall \quad \dot{\mathbf{U}}^* \quad \text{such that } \dot{\mathbf{u}}^* \text{ is admissible and } \dot{\boldsymbol{\Phi}}^* \text{ compatible.}\end{aligned} \tag{26}$$

4.2 Incremental Minimum Principle

If the energy potential is a convex function (as in the models (11) and (15)), a solution $\Delta \boldsymbol{U}$ of the variational inequality (24) is also a solution of the following minimization problem.

Incremental Minimum Principle. *The increment $\Delta \boldsymbol{U}$ minimizes the functional*

$$\boldsymbol{K}(\Delta \boldsymbol{U}^*) = \boldsymbol{W}(\boldsymbol{U} + \Delta \boldsymbol{U}^*) + \boldsymbol{D}(\Delta \boldsymbol{U}^*) - (\boldsymbol{F}_g + \Delta \boldsymbol{F}_g) \cdot \Delta \boldsymbol{U}^* \tag{27}$$

among the set of admissible increments $\Delta \boldsymbol{U}^*$.

Indeed, the minimum principle (27) results from the variational inequality (24) since the convexity of the energy potential ensures that

$$\boldsymbol{W}(\boldsymbol{U} + \Delta \boldsymbol{U}^*) - \boldsymbol{W}(\boldsymbol{U} + \Delta \boldsymbol{U}) \geq \boldsymbol{W}_{,U}(\boldsymbol{U}_+) \cdot (\Delta \boldsymbol{U}^* - \Delta \boldsymbol{U})\,.$$

The same conclusion also holds if the energy potential is only locally convex. In this case the solution $\Delta \boldsymbol{U}$ of (24) is a local minimum of the functional $\boldsymbol{K}(\Delta \boldsymbol{U}^*)$.

Conversely, a local minimum $\Delta \boldsymbol{U}$ of the functional $\boldsymbol{K}$ is necessarily a local solution of the variational inequality (24) for any smooth energy potential. Indeed, for any $\Delta \boldsymbol{U}^* \in \mathcal{N}$, a neighborhood of $\Delta \boldsymbol{U}$

$$\begin{cases} \boldsymbol{K}(\Delta \boldsymbol{U}) \leq \boldsymbol{K}(1-\alpha)\Delta \boldsymbol{U} + \alpha \Delta \boldsymbol{U}^*) \leq \boldsymbol{W}(\boldsymbol{U}_+ + \alpha(\Delta \boldsymbol{U}^* - \Delta \boldsymbol{U}))+ \\ (1-\alpha)\boldsymbol{D}(\Delta \boldsymbol{U}) + \alpha \boldsymbol{D}(\Delta \boldsymbol{U}^*) - \boldsymbol{F}_{g+} \cdot (\Delta \boldsymbol{U} + \alpha(\Delta \boldsymbol{U}^* - \Delta \boldsymbol{U}) \quad \forall\ \alpha \in [0, 1] \end{cases}$$

since $\boldsymbol{D}$ is a convex function. It follows that

$$\frac{1}{\alpha}(\boldsymbol{W}(\boldsymbol{U}_+ + \alpha(\Delta \boldsymbol{U}^* - \Delta \boldsymbol{U})) - \boldsymbol{W}(\boldsymbol{U}_+) - \boldsymbol{F}_{g+} \cdot (\Delta \boldsymbol{U}^* - \Delta \boldsymbol{U}) + \boldsymbol{D}(\Delta \boldsymbol{U}^*) - \boldsymbol{D}(\Delta \boldsymbol{U}) \geq 0$$

thus (24) results for vanishing α.

The minimum principle (27) deals with stable solutions of the variational inequality (24). The stability is understood here in the sense of a positive external work in any perturbation of the equilibrium $\boldsymbol{U}_+$ (see [10, 11]). In detail, an equilibrium $\boldsymbol{U}_+ = \boldsymbol{U} + \Delta \boldsymbol{U}$ under the applied force $\boldsymbol{F}_+$ and imposed displacement $\boldsymbol{u}_{g+}$ is stable if in any perturbation of this equilibrium, defined by a perturbed path in function of a kinematic time τ

$$\boldsymbol{U}[\tau], \tau \in [0, 1]\,, \qquad \boldsymbol{U}[0] = \boldsymbol{U}_+\,, \qquad \boldsymbol{U}[1] = \boldsymbol{U}_+^* \in \mathcal{N}\,,$$

under the action of some perturbation forces, the work provided by these forces is non-negative.

Indeed, in such a perturbation the energy balance, which results from the constitutive equations (1)–(7) of the solid, shows that the amount of work provided by the perturbed forces is

$$W_{per} = \boldsymbol{W}(\boldsymbol{U}_+^*) - \boldsymbol{W}(\boldsymbol{U}_+) + \int_0^1 \boldsymbol{D}(\frac{d\phi}{d\tau}[\tau])d\tau - \boldsymbol{F}_{g+} \cdot (\boldsymbol{U}_+^* - \boldsymbol{U}_+). \tag{28}$$

From the fact that the dissipation potential is a kind of norm

$$\int_0^1 \boldsymbol{D}(\frac{d\phi}{d\tau}[\tau])d\tau \geq \boldsymbol{D}(\Delta\boldsymbol{U}^* - \Delta\boldsymbol{U}) \geq \boldsymbol{D}(\Delta\boldsymbol{U}^*) - \boldsymbol{D}(\Delta\boldsymbol{U}), \tag{29}$$

it follows that

$$W_{per} \geq \boldsymbol{W}(U_+^*) - W(U_+) + \boldsymbol{D}(\Delta\boldsymbol{U}^*) - \boldsymbol{D}(\Delta\boldsymbol{U}) - \boldsymbol{F}_{g+} \cdot (\Delta\boldsymbol{U}^* - \Delta\boldsymbol{U}) \geq 0. \tag{30}$$

The incremental minimum principle can also written as the following minimum principle concerning the response at the next step $\boldsymbol{U}_+$.

Displacement Minimum Principle. *At time $t + \Delta t$, the generalized displacement $\boldsymbol{U}_+$ minimizes the functional*

$$\bar{\boldsymbol{K}}(U_+^*) = \boldsymbol{W}(U_+^*) + D(U_+^* - U) - \boldsymbol{F}_{g+} \cdot U_+^* \tag{31}$$

among the set of admissible displacements $\boldsymbol{U}_+^*$.

In particular, if the current state is the natural state and if the load increment is the final load, the implicit scheme gives the response of the associated deformation model under the final load.

The reader can refer to [1, 3, 6] for an original mathematical formulation on stable responses. In their approach, the starting point is the displacement minimum principle (31) instead of the evolution equation (20) and the implicit scheme. Their results show in particular that the convergence of the implicit scheme is ensured under the assumption of convexity of the energy potential.

5 Conclusions

Within the framework of standard plasticity, the theory of gradient plasticity is discussed. The governing equations of the response of a solid under a loading path are written in terms of the energy and the dissipation potentials. It is shown that the quasi-static response of the solid is a solution of a variational inequality as in classical plasticity and that higher gradients can also be included in the same spirit. A time-discretization by the implicit scheme of the evolution equation leads to the study of the incremental problem. The increment of the response under a load increment must satisfy a variational inequality and, if the energy potential is convex, an incremental minimum principle. In particular, a local minimum of the incremental minimum principle is a stable solution of the variational inequality.

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