Chapter 5
Elastic Constitutive Equations

Elastic deformation is induced by the reversible deformation of material particles themselves without a mutual slip between them. They therefore exhibit high stiffness. Elastic constitutive equations are classifiable into the three types depending on the exactness in the description of reversibility, i.e. the *hyperelasticity* (or *Green elasticity*) possessing the strain energy function, the *Cauchy elasticity* possessing the one-to-one correspondence between stress and strain and the *hypoelasticity* possessing the linear relation between stress rate and strain rate. As preparation for the study of elastoplasticity in the subsequent chapters, they are explained in this chapter.

5.1 Hyperelasticity

In the hyperelastic material, the one to one correspondence between the stress and the strain exists and further the work done during the loading process from a certain strain to another certain stain is determined uniquely independent of the loading path in that process. Then, the hyperelastic material must possess the *strain (Helmholtz) energy function* which is determined uniquely by a tensor describing deformation of material. For instance, let the deformation gradient tensor $\mathbf{F}$ be adopted for the tensor describing the deformation with the strain energy function, which is the most basic tensor describing the deformation of material. Then, letting the strain energy function per unit volume in the reference configuration be denoted by φ, the work done per the reference unit volume during the change of the deformation gradient from $\mathbf{F}_0$ to $\mathbf{F}$ must be uniquely determined by the values of deformation gradient in the reference and the current states, i.e.

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$$\int_{\mathbf{F}_0}^{\mathbf{F}} dw_0(\mathbf{F}) = \int_{\mathbf{F}_0}^{\mathbf{F}} \frac{\partial \varphi(\mathbf{F})}{\partial \mathbf{F}} : d\mathbf{F} = \varphi(\mathbf{F}) - \varphi(\mathbf{F}_0) \tag{5.1}$$

leading to

$$\dot{w}_0(\mathbf{F}) = \frac{\partial \varphi(\mathbf{F})}{\partial \mathbf{F}} : \dot{\mathbf{F}}$$

Then, the 1st Piola Kirchhoff stress tensor $\mathbf{\Pi}$ is given by

$$\mathbf{\Pi} = \frac{\partial \varphi(\mathbf{F})}{\partial \mathbf{F}} \tag{5.2}$$

noting Eq. $(4.90)_4$.

Substituting Eq. (5.2) into Eqs. (3.19) and (3.23), we obtain various expressions of the hyperelasticity by the deformation gradient as follows:

$$\boxed{\boldsymbol{\sigma} = \frac{1}{\det \mathbf{F}} \frac{\partial \varphi(\mathbf{F})}{\partial \mathbf{F}} \mathbf{F}^T, \quad \boldsymbol{\tau} = \frac{\partial \varphi(\mathbf{F})}{\partial \mathbf{F}} \mathbf{F}^T, \quad \mathbf{S} = \mathbf{F}^{-1} \frac{\partial \varphi(\mathbf{F})}{\partial \mathbf{F}}} \tag{5.3}$$

Furthermore, noting

$$\begin{aligned} \frac{\partial}{\partial F_{iA}} &= \frac{\partial}{\partial C_{PQ}} \frac{\partial C_{PQ}}{\partial F_{iA}} = \frac{\partial}{\partial C_{PQ}} \frac{\partial F_{rP} F_{rQ}}{\partial F_{iA}} = \frac{\partial}{\partial C_{PQ}} (\delta_{ri} \delta_{PA} F_{rQ} + F_{rP} \delta_{ri} \delta_{QA}) \\ &= \frac{\partial}{\partial C_{AQ}} F_{iQ} + \frac{\partial}{\partial C_{PA}} F_{iP} = 2 F_{iP} \frac{\partial}{\partial C_{PA}} \end{aligned} \tag{5.4}$$

and denoting the strain energy function described in terms of the right Cauchy-Green tensor $\mathbf{C}$ or the Green strain $\boldsymbol{E}$ by ψ, one has

$$\left. \begin{aligned} \frac{\partial \varphi}{\partial \mathbf{F}} &= 2\mathbf{F} \frac{\partial \psi}{\partial \mathbf{C}} = \mathbf{F} \frac{\partial \psi}{\partial \boldsymbol{E}} \\ \frac{\partial \psi}{\partial \mathbf{C}} &= \frac{1}{2} \mathbf{F}^{-1} \frac{\partial \varphi}{\partial \mathbf{F}} = \frac{1}{2} \frac{\partial \psi}{\partial \boldsymbol{E}} \\ \frac{\partial \psi}{\partial \boldsymbol{E}} &= \mathbf{F}^{-1} \frac{\partial \varphi}{\partial \mathbf{F}} = 2 \frac{\partial \psi}{\partial \mathbf{C}} \end{aligned} \right\} \tag{5.5}$$

Then, substituting Eq. (5.5) into Eq. (5.3), the hyperelasticity is expressed as follows:

$$\boxed{\begin{aligned}
&\boldsymbol{\sigma} = 2\frac{1}{\det\mathbf{F}}\mathbf{F}\frac{\partial\psi(\mathbf{C})}{\partial\mathbf{C}}\mathbf{F}^T = \frac{1}{\det\mathbf{F}}\mathbf{F}\frac{\partial\psi(\boldsymbol{E})}{\partial\boldsymbol{E}}\mathbf{F}^T\\
&\boldsymbol{\tau} = 2\mathbf{F}\frac{\partial\psi(\mathbf{C})}{\partial\mathbf{C}}\mathbf{F}^T = \mathbf{F}\frac{\partial\psi(\boldsymbol{E})}{\partial\boldsymbol{E}}\mathbf{F}^T\\
&\boldsymbol{\Pi} = 2\mathbf{F}\frac{\partial\psi(\mathbf{C})}{\partial\mathbf{C}} = \mathbf{F}\frac{\partial\psi(\boldsymbol{E})}{\partial\boldsymbol{E}}\\
&\mathbf{S} = 2\frac{\partial\psi(\mathbf{C})}{\partial\mathbf{C}} = \frac{\partial\psi(\boldsymbol{E})}{\partial\boldsymbol{E}}
\end{aligned}} \tag{5.6}$$

As known from Eq. $(5.6)_4$, the constitutive relation for isotropic elastic deformation is described through the elastic potential energy function of the right Cauchy-Green deformation tensor $\mathbf{C}$ in the initial configuration, i.e. the three-dimensional stretching resulting in the volume change and the shape change (pure shear deformation). It is based on the physical background that the elastic deformation is induced by the expansion/contraction of the intervals between material particles connected by the elastic springs. On the other hand, the plastic deformation is induced by the slips between material particles so that it cannot be formulated through the potential energy function but it must be formulated in a rate form.

It holds from Eq. (1.246) for any scalar-valued tensor function $\partial\psi(\boldsymbol{E})$ leading to the isotropic material that

$$\boxed{\frac{\partial\psi(\boldsymbol{E})}{\partial\boldsymbol{E}} = \phi_0^E\mathbf{I} + \phi_1^E\boldsymbol{E} + \phi_2^E\boldsymbol{E}^2} \tag{5.7}$$

where $\phi_0^E, \phi_1^E, \phi_2^E$ are the functions of invariants of $\boldsymbol{E}$. Equation (5.7) reduces to the following equation for the linear elastic material.

$$\frac{\partial\psi(\boldsymbol{E})}{\partial\boldsymbol{E}} = a(\mathrm{tr}\boldsymbol{E})\mathbf{I} + 2b\boldsymbol{E} \tag{5.8}$$

where a, b are the material parameters.

The function $\partial\psi(\mathbf{C})$ is described as

$$\psi(\mathbf{C}) = \psi(\mathrm{I}_C, \mathrm{II}_C, \mathrm{III}_C) \tag{5.9}$$

where

$$\left.\begin{aligned}
&\mathrm{I}_C \equiv \mathrm{tr}\mathbf{C}\\
&\mathrm{II}_C \equiv \frac{1}{2}(\mathrm{tr}^2\mathbf{C} - \mathrm{tr}\mathbf{C}^2)\\
&\mathrm{III}_C \equiv \det\mathbf{C}
\end{aligned}\right\} \tag{5.10}$$

with

$$\left.\begin{aligned} \frac{\partial \mathrm{I}_C}{\partial \mathbf{C}} &= \frac{\partial \mathrm{tr}\mathbf{C}}{\partial \mathbf{C}} = \mathbf{I} \\ \frac{\partial \mathrm{II}_C}{\partial \mathbf{C}} &= \frac{\partial \frac{1}{2}(\mathrm{tr}^2\mathbf{C} - \mathrm{tr}\mathbf{C}^2)}{\partial \mathbf{C}} = \mathrm{I}_C\mathbf{I} - \mathbf{C} \\ \frac{\partial \mathrm{III}_C}{\partial \mathbf{C}} &= \frac{\partial \det \mathbf{C}}{\partial \mathbf{C}} = \mathrm{II}_C\mathbf{I} - \mathrm{I}_C\mathbf{C} + \mathbf{C}^2 = \mathrm{III}_C\mathbf{C}^{-1} \end{aligned}\right\} \tag{5.11}$$

noting Eq. (1.295). Then, substituting Eq. (5.11) into Eq. (5.6), it follows that

$$\boldsymbol{\sigma} = 2\frac{1}{\sqrt{\mathrm{III}_C}}\mathbf{F}\left[\frac{\partial \psi}{\partial \mathrm{I}_C}\mathbf{I} + \frac{\partial \psi}{\partial \mathrm{II}_C}(\mathrm{I}_C\mathbf{I} - \mathbf{C}) + \frac{\partial \psi}{\partial \mathrm{III}_C}(\mathrm{II}_C\mathbf{I} - \mathrm{I}_C\mathbf{C} + \mathbf{C}^2)\right]\mathbf{F}^T \tag{5.12}$$

The strain energy function of the *Mooney-Rivlin model* (Mooney 1940; Rivlin 1948) which is applicable to the elastic deformation of the incompressible rubber is given as

$$\psi = a_1(\mathrm{I}_C - 3) + a_2(\mathrm{II}_C - 3) \quad (\mathrm{III}_C = 1) \tag{5.13}$$

where a_1 and a_2 are material parameters.

Further, the *neo-Hookean model* is given by the simplification setting $a_2 = 0$ as follows:

$$\psi = \frac{1}{2}v(\mathrm{I}_C - 3) \quad (\mathrm{III}_C = 1) \tag{5.14}$$

where v is the material parameter.

Further, the strain energy function of the *Ogden model* (Ogden 1982, 1984) which is applicable to the elastic deformation of the incompressible rubber for a large deformation is given as

$$\psi = \sum_{n=1}^{3}\frac{\beta_n}{\alpha_n}(\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n} - 3) \quad (\lambda_1\lambda_2\lambda_3 = 1) \tag{5.15}$$

where λ_i are the principal values of $\mathbf{U} = \mathbf{C}^{1/2}$, α_n and β_n are material parameters. Equation (5.15) is reduced to Eq. (5.13) for the Mooney-Rivlin model by choosing the material parameters as follows (cf. Hisada 1992):

$$\left.\begin{aligned} \beta_1 &= 2C_1, & \alpha_1 &= 2 \\ \beta_2 &= -2C_2, & \alpha_2 &= -2 \\ \beta_3 &= 0, & \alpha_3 &= 0 \end{aligned}\right\} \tag{5.16}$$

The hyperelastic equation for soils is referred to Sect. 11.10.

The time-differentiation of Eq. (5.6)$_4$ leads to

$$\boxed{\dot{\mathbf{S}} = \frac{\partial^2 \psi(\boldsymbol{E})}{\partial \boldsymbol{E} \otimes \partial \boldsymbol{E}} : \dot{\boldsymbol{E}}} \tag{5.17}$$

Here, the symbol $\otimes$ specifies the fourth-order tensor due to the second-order partial derivative by the second-order tensor, although the expression without this symbol is widely used in a lot of literatures (e.g. Simo and Hughes 1988; Bonet and Wood 1997; Belytschko et al. 2014). Substituting Eqs. (2.128) and (5.17) into Eq. (4.61), the Truesdell rate of Kirchhoff stress $\overset{\circ}{\boldsymbol{\tau}}^{Ol}$ is rewritten as

$$\overset{\Delta}{\boldsymbol{\tau}}^{Ol} = \mathbf{F}\left[\frac{\partial^2 \psi(\boldsymbol{E})}{\partial \boldsymbol{E} \otimes \partial \boldsymbol{E}} : (\mathbf{F}^T \mathbf{d} \mathbf{F})\right]\mathbf{F}^T \left(\overset{\Delta}{\tau}{}_{ij}^{Ol} = F_{iA} F_{jB} F_{kC} F_{lD} \frac{\partial^2 \psi(\boldsymbol{E})}{\partial E_{AB} \partial E_{CD}} d_{kl}\right) \tag{5.18}$$

which is the rate of hyperelastic equation in the current configuration. Here, $\overset{\Delta}{\boldsymbol{\tau}}^{Ol}$ is related to the Zaremba-Jaumann rate of Cauchy stress in Eq. (4.70) as

$$\boxed{\overset{\Delta}{\boldsymbol{\tau}}^{Ol} = J(\overset{\circ}{\boldsymbol{\sigma}}^{w} - \mathbf{d}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{d} + \boldsymbol{\sigma}\mathrm{tr}\mathbf{d})} \tag{5.19}$$

The Zaremba-Jaumann rate of Cauchy stress is related to the strain rate from these equations as

$$\overset{\circ}{\boldsymbol{\sigma}}^{w} = \frac{1}{\det \mathbf{F}} \mathbf{F}\left[\frac{\partial^2 \psi(\boldsymbol{E})}{\partial \boldsymbol{E} \otimes \partial \boldsymbol{E}} (\mathbf{F}^T \mathbf{d} \mathbf{F})\right]\mathbf{F}^T + \mathbf{d}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{d} - \boldsymbol{\sigma}\mathrm{tr}\mathbf{d} \tag{5.20}$$

which is expressed as

$$\overset{\circ}{\boldsymbol{\sigma}}^{w} = \tilde{\mathbf{E}} : \mathbf{d} \tag{5.21}$$

where the *hyperelastic tangent modulus tensor* $\tilde{\mathbf{E}}$ in the current configuration is given by

$$\tilde{E}_{ijkl} \equiv \frac{1}{\det \mathbf{F}} F_{iA} F_{kC} F_{lD} F_{jB} \frac{\partial^2 \psi(\boldsymbol{E})}{\partial E_{AB} \partial E_{CD}} + \Sigma_{ijkl} - \sigma_{ij} \delta_{kl} \tag{5.22}$$

with

$$\Sigma_{ijkl} \equiv \frac{1}{2}(\sigma_{ik}\delta_{jl} + \sigma_{il}\delta_{jk} + \sigma_{jk}\delta_{il} + \sigma_{jl}\delta_{ik}) \quad (\Sigma_{ijkl} = \Sigma_{klij} = \Sigma_{jikl} = \Sigma_{ijlk}) \tag{5.23}$$

5.2 Infinitesimal Elastic Deformation

For the infinitesimal deformation, the hyperelastic constitutive equation can be given as

$$\boldsymbol{\sigma} = \frac{\partial \psi(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}}, \quad \dot{\boldsymbol{\sigma}} = \frac{\partial \psi^2(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} = \mathbf{E} : \dot{\boldsymbol{\varepsilon}}, \quad \mathbf{E} \equiv \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}} = \frac{\partial^2 \psi(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} \tag{5.24}$$

where $\boldsymbol{\varepsilon}$ is the infinitesimal strain in Eq. (2.55).

For the particular Helmholtz free strain energy function (strain energy function)

$$\psi(\boldsymbol{\varepsilon}) = \frac{1}{2} L (\mathrm{tr}\boldsymbol{\varepsilon})^2 + G \mathrm{tr}\boldsymbol{\varepsilon}^2 \tag{5.25}$$

the stress is given by the linear relation to the elastic strain $\boldsymbol{\varepsilon}$ as

$$\boldsymbol{\sigma} = L \varepsilon_v \mathbf{I} + 2G\boldsymbol{\varepsilon} \tag{5.26}$$

i.e.

$$\boldsymbol{\sigma} = \left(L + \frac{2}{3} G \right) \varepsilon_v \mathbf{I} + 2G\boldsymbol{\varepsilon}' \tag{5.27}$$

which is referred to as the *Hooke's law*, where L and G are called the *Lamé constants*. It follows by taking the trace and the deviatoric part of Eq. (5.27) that

$$\varepsilon_v = \frac{3}{3L + 2G} \sigma_m, \quad \boldsymbol{\varepsilon}' = \frac{1}{2G} \boldsymbol{\sigma}' \tag{5.28}$$

where $\sigma_m (\equiv (\mathrm{tr}\boldsymbol{\sigma})/3)$ is the mean stress. Then, the inverse relation of Eq. (5.26) is given by

$$\boldsymbol{\varepsilon} = \frac{1}{3L + 2G} (\mathrm{tr}\boldsymbol{\sigma}) \mathbf{I} + \frac{1}{2G} \boldsymbol{\sigma}' = \frac{4G - 3L}{6G(3L + 2G)} (\mathrm{tr}\boldsymbol{\sigma}) \mathbf{I} + \frac{1}{2G} \boldsymbol{\sigma} \tag{5.29}$$

The inverse relation can be derived by first making the spherical and the deviatoric parts and then combing them as shown above.

Equations (5.26) and (5.29) are rewritten as

$$\boxed{\begin{aligned} \boldsymbol{\sigma} &= K \varepsilon_v \mathbf{I} + 2G\boldsymbol{\varepsilon}' = \left(K - \frac{2}{3} G \right) \varepsilon_v \mathbf{I} + 2G\boldsymbol{\varepsilon} \\ \boldsymbol{\varepsilon} &= \frac{1}{3K} \sigma_m \mathbf{I} + \frac{1}{2G} \boldsymbol{\sigma}' = \left(\frac{1}{3K} - \frac{1}{2G} \right) \sigma_m \mathbf{I} + \frac{1}{2G} \boldsymbol{\sigma} \end{aligned}} \tag{5.30}$$

where

$$K \equiv L + \frac{2}{3}G \tag{5.31}$$

It follows from Eq. (5.30) that

$$\sigma_m = K\varepsilon_v,\ \boldsymbol{\sigma}' = 2G\boldsymbol{\varepsilon}' \tag{5.32}$$

Then, K and G are called the *bulk elastic modulus* and the *shear elastic modulus*, respectively.

Equations (5.26), (5.29) and (5.30) are represented as

$$\boxed{\boldsymbol{\sigma} = \mathbf{E} : \boldsymbol{\varepsilon}} \tag{5.33}$$

using the elastic modulus tensor $\mathbf{E}$ given by

$$\left.\begin{array}{l} \mathbf{E} = \left(L + \frac{2}{3}G\right)\mathcal{T} + 2G\mathcal{I}', \quad E_{ijkl} \equiv \left(L + \frac{2}{3}G\right)\delta_{ij}\delta_{kl} + 2G\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{1}{3}\delta_{ij}\delta_{kl}\right) \\ \mathbf{E}^{-1} = \frac{1}{9L+6G}\mathcal{T} + \frac{1}{2G}\mathcal{I}', \quad (\mathbf{E}^{-1})_{ijkl} = \frac{1}{9L+6G}\delta_{ij}\delta_{kl} + \frac{1}{2G}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{1}{3}\delta_{ij}\delta_{kl}\right) \end{array}\right\} \tag{5.34}$$

and

$$\boxed{\begin{array}{l} \mathbf{E} = K\mathcal{T} + 2G\mathcal{I}', \quad E_{ijkl} \equiv K\delta_{ij}\delta_{kl} + 2G\left[\frac{1}{2}(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}) - \frac{1}{3}\delta_{ij}\,\delta_{kl}\right] \\ \mathbf{E}^{-1} = \frac{1}{9K}\mathcal{T} + \frac{1}{2G}\mathcal{I}', \quad (\mathbf{E}^{-1})_{ijkl} = \frac{1}{9K}\delta_{ij}\,\delta_{kl} + \frac{1}{2G}\left[\frac{1}{2}(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}) - \frac{1}{3}\delta_{ij}\,\delta_{kl}\right] \end{array}} \tag{5.35}$$

where $\mathcal{T}$ is the fourth-order tracing tensor and $\mathcal{I}'$ is the fourth-order deviatoric projection tensor defined in Eq. (1.143) and (1.146), respectively. The inverse relation between the two equation in Eq. (5.35) is confirmed by $(K\mathcal{T} + 2G\mathcal{I}'):[(1/9)K\mathcal{T} + (1/2G)\mathcal{I}'] = (1/3)\mathcal{T} + \mathcal{I}' = \mathcal{I}$, noting Eq. (1.146).

It follows from Eq. $(5.35)_2$ for the uniaxial loading process ($\sigma_{ij} = 0$ for $i = j \neq 1$ and $i \neq j$), noting $\mathcal{T}_{1111} = 1$, $\mathcal{I}'_{1111} = 2/3$, $\mathcal{T}_{2211} = 0$, $\mathcal{I}'_{2211} = -1/3$ that

$$\varepsilon_{11} = \frac{1}{E}\sigma_{11}, \quad \varepsilon_{22} = -\frac{\nu}{E}\sigma_{11} \rightarrow \frac{\varepsilon_{22}}{\varepsilon_{11}} = -\nu \tag{5.36}$$

where

$$E = \frac{9KG}{3K+G}, \quad v = \frac{3K-2G}{2(3K+G)} \tag{5.37}$$

the inverses of which are given as

$$K \equiv \frac{E}{3(1-2v)}, \quad G \equiv \frac{E}{2(1+v)} \tag{5.38}$$

Here, E is the ratio of the axial stress rate to the axial strain rate and is called the *Young's modulus*, and v is the ratio of lateral strain rate to axial strain rate and is called the *Poisson's ratio*. The strain energy function in (5.25) is expressed as

$$\psi(\boldsymbol{\varepsilon}) = \frac{vE}{2(1+v)(1-2v)}(\mathrm{tr}\boldsymbol{\varepsilon})^2 + G\mathrm{tr}\boldsymbol{\varepsilon}^2 \tag{5.39}$$

which must be positive so that the Poisson's ratio is limited in the range

$$-1 < v < 1/2 \tag{5.40}$$

The lower limit and the upper limit correspond to the similar shape and the constant volume, respectively, as known from Eq. (5.38). The former is seen in artificial structures, e.g. honeycomb.

Substituting Eq. (5.38) into Eq. (5.35), the elastic modulus tensor is also described using the Young's modulus and the Poisson's ratio as follows:

$$\boxed{\begin{array}{ll} \mathbf{E} = \dfrac{E}{3(1-2v)}\mathcal{T} + \dfrac{E}{1+v}\mathcal{I}', & E_{ijkl} = \dfrac{E}{3(1-2v)}\delta_{ij}\delta_{kl} + \dfrac{E}{1+v}\left[\dfrac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) - \dfrac{1}{3}\delta_{ij}\delta_{kl}\right] \\ \mathbf{E}^{-1} = \dfrac{1-2v}{3E}\mathcal{T} + \dfrac{1+v}{E}\mathcal{I}', & (\mathbf{E}^{-1})_{ijkl} = \dfrac{1-2v}{3E}\delta_{ij}\,\delta_{kl} + \dfrac{1+v}{E}\left[\dfrac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) - \dfrac{1}{3}\delta_{ij}\delta_{kl}\right] \end{array}} \tag{5.41}$$

or

$$\boxed{\begin{array}{ll} \mathbf{E} = \dfrac{E}{1+v}\left(\dfrac{v}{1-2v}\mathcal{T} + \mathcal{I}\right), & E_{ijkl} = \dfrac{E}{1+v}\left[\dfrac{v}{1-2v}\delta_{ij}\delta_{kl} + \dfrac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\right] \\ \mathbf{E}^{-1} = -\dfrac{1}{E}\left[v\mathcal{T} - (1+v)\mathcal{I}\right], & (\mathbf{E}^{-1})_{ijkl} = -\dfrac{1}{E}\left[v\delta_{ij}\delta_{kl} - \dfrac{1}{2}(1+v)(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\right] \end{array}} \tag{5.42}$$

Table 5.1 Relationships between two independent elastic constants

	$E,\ v$	$G,\ v$	$E,\ G$	$E,\ K$	$G,\ K$	$L,\ G$
E	E	$2(1+v)G$	E	E	$\dfrac{9KG}{3K+G}$	$\dfrac{\mu(3L+2G)}{L+G}$
G	$\dfrac{E}{2(1+v)}$	G	G	$\dfrac{3EK}{9K-E}$	G	G
K	$\dfrac{E}{3(1-2v)}$	$\dfrac{2(1+v)G}{3(1-2v)}$	$\dfrac{EG}{3(3G-E)}$	K	K	$L+\dfrac{2}{3}G$
v	v	v	$\dfrac{E-2G}{2G}$	$\dfrac{3K-E}{6K}$	$\dfrac{3K-2G}{2(3K+G)}$	$\dfrac{L}{2(L+G)}$
L	$\dfrac{vE}{(1+v)(1-2v)}$	$\dfrac{2Gv}{1-v}$	$\dfrac{G(E-2G)}{3G-E}$	$\dfrac{3K(3K-E)}{9K-E}$	$K-\dfrac{2}{3}G$	L

Relationships between two independent elastic constants are listed in Table 5.1.

The Helmholtz free energy function (strain energy function) $\psi(\boldsymbol{\varepsilon})$ and the *Gibbs' free energy function* (complementary energy function) $\phi(\boldsymbol{\sigma})$ are given for the linear elasticity as

$$\psi(\boldsymbol{\varepsilon}) = \frac{1}{2}\boldsymbol{\varepsilon}:\mathbf{E}:\boldsymbol{\varepsilon} = \frac{1}{2}\frac{E}{1+v}\left[\varepsilon_{ij}\varepsilon_{ij} + \frac{E}{1-2v}(\varepsilon_{kk})^2\right] \tag{5.43}$$

$$\phi(\boldsymbol{\sigma}) = \frac{1}{2}\boldsymbol{\sigma}:\mathbf{E}^{-1}:\boldsymbol{\sigma} = \frac{1}{2E}\left[(1+v)\sigma_{ij}\sigma_{ij} - v(\sigma_{kk})^2\right] \tag{5.44}$$

from which it follows that

$$\boldsymbol{\sigma} = \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}} = \mathbf{E}:\boldsymbol{\varepsilon} = \frac{E}{1+v}\left(\frac{v}{1-2v}\varepsilon_v\mathbf{I} + \boldsymbol{\varepsilon}\right) = E\left[\frac{1}{3(1-2v)}\varepsilon_v\mathbf{I} + \frac{1}{1+v}\boldsymbol{\varepsilon}'\right] \tag{5.45}$$

$$\boldsymbol{\varepsilon} = \frac{\partial\phi}{\partial\boldsymbol{\sigma}} = \mathbf{E}^{-1}:\boldsymbol{\sigma} = \frac{1}{E}\left[(1+v)\boldsymbol{\sigma} - 3v\sigma_m\mathbf{I}\right] = \frac{1}{E}\left[(1-2v)\sigma_m\mathbf{I} + (1+v)\boldsymbol{\sigma}'\right] \tag{5.46}$$

5.3 Cauchy Elasticity

The elastic material which does not have a strain energy function but has a one-to-one correspondence between the Cauchy stress and a strain is called the *Cauchy elastic material*. Here, the stress tensor is given by an equation of strain tensor and thus the equation includes six strain components. The equation of six strain components does not fulfill the condition of complete integration leading to the strain energy function so that it does not result in the hyperelasticity in general. Then, the work done by the stress is generally dependent on the deformation path.

For that reason, an energy dissipation/production is induced during the stress or strain cycle.

In the above-mentioned definition, the Cauchy elastic material is described as

$$\boldsymbol{\sigma} = \mathbf{f}(\boldsymbol{e}) \tag{5.47}$$

in terms of the Almansi strain tensor $\boldsymbol{e}$ in Eq. (2.45) or (2.47). Equation (5.47) reduces to the following equation by virtue of Eq. (1.246) for the isotropic material.

$$\boldsymbol{\sigma} = \phi_0^e \mathbf{I} + \phi_1^e \boldsymbol{e} + \phi_2^e \boldsymbol{e}^2 \tag{5.48}$$

where $\phi_0^e, \phi_1^e, \phi_2^e$ are functions of invariants of $\boldsymbol{e}$. Furthermore, for an isotropic linear elastic material, Eq. (5.48) reduces to

$$\boldsymbol{\sigma} = L(\mathrm{tr}\boldsymbol{e})\mathbf{I} + 2G\boldsymbol{e} \tag{5.49}$$

noting Eq. (5.26). Limiting to the infinitesimal strain leading to $\boldsymbol{e} \cong \boldsymbol{\varepsilon}$, Eq. (5.49) results in Eq. (5.26), i.e.

$$\boldsymbol{\sigma} = L(\mathrm{tr}\boldsymbol{\varepsilon})\mathbf{I} + 2G\boldsymbol{\varepsilon} \tag{5.50}$$

Here, substituting Eq. (5.50) with Eq. (2.50) into Eq. (3.31), the *Navier's equation* is obtained by replacing L and G to a and b, respectively, as follows:

$$(a+b)\frac{\partial^2 u_j}{\partial x_j \partial x_i} + b\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \rho b_i = \rho \dot{v}_i$$

$$(a+b)\nabla(\nabla \cdot \mathbf{u}) + b\Delta\mathbf{u} + \rho\mathbf{b} = \rho\,\dot{\mathbf{v}} \tag{5.51}$$

noting Eqs. (1.309), (1.315) and

$$\frac{\partial\left[a\dfrac{\partial u_k}{\partial x_k}\delta_{ij} + 2b\dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right)\right]}{\partial x_j} = a\frac{\partial^2 u_j}{\partial x_j \partial x_i} + b\frac{\partial^2 u_i}{\partial x_j \partial x_j} + b\frac{\partial^2 u_j}{\partial x_j \partial x_i}$$

5.4 Hypoelasticity

The following material, for which the corotational rate of stress is related linearly to the strain rate, is referred to as the *hypoelastic material* by Truesdell (1955).

$$\overset{\circ}{\boldsymbol{\sigma}} = \mathbf{H}(\boldsymbol{\sigma})[\mathbf{d}] \tag{5.52}$$

where the tensor function $\mathbf{H}(\boldsymbol{\sigma})[\mathbf{d}]$ designates the linearity in the strain rate $\mathbf{d}$ and the isotropies in $\boldsymbol{\sigma}$ and $\mathbf{d}$.

In what follows, we adopt the following elastic constitutive relation with the elastic modulus tensor $\mathbf{E}$ incorporated in the infinitesimal elastic deformation in Sect. 5.2, i.e.

$$\boxed{\overset{\circ}{\boldsymbol{\sigma}} = \mathbf{E} : \mathbf{d}} \tag{5.53}$$

which leads to the following relations for the Hooke's law, noting Eqs. (5.30), (5.45) and (5.46).

$$\left.\begin{aligned} \overset{\circ}{\boldsymbol{\sigma}} &= K d_v \mathbf{I} + 2G\mathbf{d}' = \left(K - \frac{2}{3}G\right) d_v \mathbf{I} + 2G\mathbf{d} \\ \mathbf{d} &= \frac{1}{3K}\dot{\sigma}_m \mathbf{I} + \frac{1}{2G}\overset{\circ}{\boldsymbol{\sigma}}' = \left(\frac{1}{3K} - \frac{1}{2G}\right)\dot{\sigma}_m \mathbf{I} + \frac{1}{2G}\overset{\circ}{\boldsymbol{\sigma}} \end{aligned}\right\} \tag{5.54}$$

$$\dot{\sigma}_m = K d_v, \quad \overset{\circ}{\boldsymbol{\sigma}}' = 2G\mathbf{d}' \tag{5.55}$$

$$\left.\begin{aligned} \overset{\circ}{\boldsymbol{\sigma}} &= E\left[\frac{1}{3(1-2v)} d_v \mathbf{I} + \frac{1}{1+v}\mathbf{d}'\right] = \frac{E}{1+v}\left(\frac{v}{1-2v} d_v \mathbf{I} + \mathbf{d}\right) \\ \mathbf{d} &= \frac{1}{E}\left[(1-2v)\dot{\sigma}_m \mathbf{I} + (1+v)\overset{\circ}{\boldsymbol{\sigma}}'\right] = \frac{1}{E}\left[(1+v)\,\overset{\circ}{\boldsymbol{\sigma}} - 3v\dot{\sigma}_m \mathbf{I}\right] \end{aligned}\right\} \tag{5.56}$$

Besides, the following equation in which the Jaumann rate of Cauchy stress is related nonlinearly to the strain rate is called the *hypoplastic material* (Kolymbas and Wu 1993).

$$\overset{\circ}{\boldsymbol{\sigma}} = \mathbf{f}(\mathbf{d}, \boldsymbol{\sigma}), \quad \overset{\circ}{\sigma}_{ij} = f_{ij}(d_{kl}, \sigma_{kl}) \tag{5.57}$$

where f_{ij} is the nonlinear function of d_{kl}, and for rate-independent deformation it is the homogeneous function of d_{kl} in degree-one fulfilling $f_{ij}(|s|d_{kl}) = |s|f_{ij}(d_{kl})$ which implies $(\partial f_{ij}/\partial d_{kl})d_{kl} = f_{ij}$ on account of Euler's theorem for homogeneous function (see **Appendix D**).

While the three popular types of elastic materials are described in this chapter, the other elastic material, called the *Cosserat elastic material*, was advocated by Cosserat and Cosserat (1909). The *couple stress* is related to the *rotational strain* in this material. It has been applied to the prediction of localized deformation (e.g. cf. Mindlin 1963; Muhlhaus and Vardoulaskis 1987).

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