Chapter 4
Objectivity and Objective (Rate) Tensors

Constitutive property of material is independent of observers. Therefore, constitutive equation has to be described by variables obeying the common objective transformation rule described in Sect. 1.3.1. State variables, e.g. stress, strain and back stress tensors in the same configuration obey the common coordinate transformation rule. However, the material-time derivatives of tensors in the current configuration do not obey the objective transformation rule, since they are influenced by the rigid-body rotation. Then, instead of the material-time derivative of tensors, particular time-derivatives of tensors obeying the objective transformation rule have to be adopted in constitutive equations.

The consideration on the fulfillment of objectivity is of great importance for the hypoelastic-based constitutive equation formulated in the current configuration which is influenced directly by the rigid-body rotation, while the hypoelastic-based plasticity is comprehensively explained in this book. Then, the objectivity and the formulation of constitutive relations fulfilling the objectivity will be comprehensively described in this chapter.

4.1 Objectivity

Physical quantities except for scalar ones are observed to be different depending on the state, e.g. position, direction, velocity of observers. On the other hand, mechanical property of material is observed identically independent of the state of observers. In particular, it is observed identically independent of the rigid-body rotation of material. Therefore, a constitutive equation describing material property must be expressed in a common form independent of coordinate systems. Then, it must be described so as not to be influenced by the rigid-body rotation of material. This fact was not so obvious in the olden time and was advocated by Oldroyd (1950) in the middle of the last century. It is referred to as the *principle of material-frame indifference* (Oldroyd 1950) or *principle of objectivity* or simply *objectivity*.

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This would be regarded as the starting point of the modern continuum mechanics which is called sometimes as the *rational mechanics* (Truesdell and Toupin 1960; Truesdell and Noll 1965).

Here, note that components of tensor describing mechanical state of material, e.g. stress, strain and anisotropic internal variables are observed to be changed by the fixed coordinate system if the material rotates, even when the components are observed to be unchanged by the coordinate system rotating concurrently with the material itself. Therefore, the material-time derivative of tensor describing mechanical state is observed to be non-zero, by the fixed coordinate system when the material rotates even when it is observed to be zero by the observer rotating concurrently with the material itself. It is caused by the fact that the material-time derivative of tensor designates the rate of tensor observed by the coordinate system moving in parallel with material but without rotation. Then, the material-time derivative cannot be adopted for the description of constitutive equations in a current rate form.

Machine elements are often subjected not only to deformation but also to rigid-body rotation, as seen in metal forming, gears, wheels, etc. Soils near the side edges of footings, at the bottom ends of piles, etc. undergo a large rigid-body rotation. Therefore, formulations of constitutive equations which is not influenced by the rigid-body rotation are of great importance in practical engineering problems.

4.2 Influence of Rigid-Body Rotation on Various Mechanical Quantities

In order to check whether or not a constitutive equation is formulated so as to satisfy the objectivity principle, it is expedient to examine the influence of rigid-body rotation on the tensor variables used in constitutive equations. Instead, one may examine how the components of these variables are observed by the coordinate systems with the fixed base $\{\mathbf{e}_i\}$ and the rotating base $\{\mathbf{e}_i^*(t)\}$ which are related as

$$\mathbf{e}_i^*(t) = \mathbf{Q}^T(t)\mathbf{e}_i, \quad \mathbf{e}_i^*(0) = \mathbf{e}_i, \quad \dot{\mathbf{e}}_i^*(t) = \dot{\mathbf{Q}}^T(t)\mathbf{e}_i \tag{4.1}$$

provided that the rotating base $\{\mathbf{e}_i^*(t)\}$ coincides with the fixed base $\{\mathbf{e}_i\}$ at the beginning of deforming/rotation $(t = 0)$, where one has

$$\mathbf{Q}(t) = \mathbf{e}_i \otimes \mathbf{e}_i^*(t), \quad \mathbf{Q}(0) = \mathbf{I} \tag{4.2}$$

noting Eq. (1.89) with Eq. (4.1). These bases are illustrated in Fig. 4.1 for the two-dimensional state.

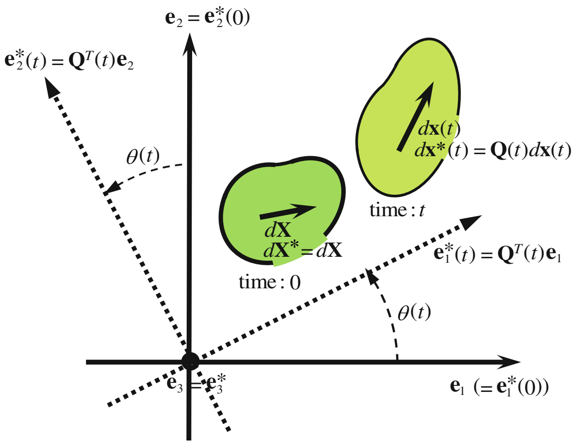

Fig. 4.1 Coordinate systems with the fixed base $\{\mathbf{e}_i\}$ and the rotating base $\{\mathbf{e}_i^*(t)\}$ which coincides with the base $\{\mathbf{e}_i\}$ at the beginning of deformation/rotation (illustrated in two-dimensional state for $\mathbf{e}_3 = \mathbf{e}_3^*$)

The components of the initial infinitesimal line element $d\mathbf{X}$ in the initial state $(t = 0)$ is observed to be identical by these bases, since $\{\mathbf{e}_i^*(t)\}$ coincides with $\{\mathbf{e}_i\}$ in the initial state. On the other hand, the components of the current infinitesimal line-element $d\mathbf{x}(t)$ is observed to be different by these bases as the rotating base $\{\mathbf{e}_i^*(t)\}$ differs from the fixed base $\{\mathbf{e}_i\}$ for $t > 0$. Here, noting $d\mathbf{x}(t) = \mathbf{F}(t)d\mathbf{X}$, we have

$$d\mathbf{X}^* = d\mathbf{X}\,(\{\mathbf{e}_i^*(0)\} = \{\mathbf{e}_i\}),\; d\mathbf{x}^*(t) = \mathbf{Q}(t)d\mathbf{x}(t) = \mathbf{Q}(t)\mathbf{F}(t)d\mathbf{X} \tag{4.3}$$

and

$$d\mathbf{x}^*(t) = \mathbf{F}^*(t)d\mathbf{X}^*(0) = \mathbf{F}^*(t)d\mathbf{X} \tag{4.4}$$

from which it follows that

$$\mathbf{F}^*(t) = \mathbf{Q}(t)\mathbf{F}(t) \tag{4.5}$$

It is known from Eq. (4.5) that the deformation gradient $\mathbf{F}(t)$ is the second-order tensor but it obeys the transformation rule of the first-order tensor. This is based on the fact that the deformation gradient is the two-point tensor as specified in Eq. (2.14).

Substituting Eq. (4.5) into Eqs. (2.20)–(2.22), (2.35), (2.36) and (2.45), the following relations are obtained for various quantities describing a deformation.

$$\mathbf{U}^* = \mathbf{U}, \quad \mathbf{C}^* = \mathbf{C}\ (\mathbf{F}^{*T}\mathbf{F}^* = (\mathbf{QF})^T\mathbf{QF} = \mathbf{F}^T\mathbf{Q}^T\mathbf{QF} = \mathbf{F}^T\mathbf{F}) \tag{4.6}$$

$$\mathbf{V}^* = \mathbf{QVQ}^T, \quad \mathbf{b}^* = \mathbf{QbQ}^T \quad (\mathbf{F}^*\mathbf{F}^{*T} = \mathbf{QF}(\mathbf{QF})^T = \mathbf{QFF}^T\mathbf{Q}^T) \tag{4.7}$$

$$\mathbf{R}^* = \mathbf{QR}\ (\mathbf{F}^* = \mathbf{QF} = \mathbf{QRU} = \mathbf{QRU}^* = \mathbf{R}^*\mathbf{U}^*) \tag{4.8}$$

$$\boldsymbol{E}^* = \boldsymbol{E}\ (= \mathbf{F}^T\boldsymbol{e}\mathbf{F}),\ \boldsymbol{e}^* = \mathbf{Q}\boldsymbol{e}\mathbf{Q}^T \tag{4.9}$$

Noting the relation

$$\dot{\mathbf{F}}^*\mathbf{F}^{*-1} = (\mathbf{QF})^{\boldsymbol{\cdot}}(\mathbf{QF})^{-1} = (\dot{\mathbf{Q}}\mathbf{F} + \mathbf{Q}\dot{\mathbf{F}})\mathbf{F}^{-1}\mathbf{Q}^{-1} = \mathbf{Q}(\dot{\mathbf{F}}\mathbf{F}^{-1} - \dot{\mathbf{Q}}^T\mathbf{Q})\mathbf{Q}^T$$

it holds for the velocity gradient in Eq. (2.75) that

$$\boldsymbol{l}^* = \mathbf{Q}(\boldsymbol{l} - \boldsymbol{\Omega})\mathbf{Q}^T = \mathbf{Q}\boldsymbol{l}\mathbf{Q}^T + \overline{\boldsymbol{\Omega}} \tag{4.10}$$

where $\boldsymbol{\Omega}$ and $\overline{\boldsymbol{\Omega}}$ are defined by

$$\left.\begin{aligned} \boldsymbol{\Omega} &\equiv \dot{\mathbf{Q}}^T\mathbf{Q}, \quad \boldsymbol{\Omega} \equiv \dot{Q}_{ri}Q_{rj}\mathbf{e}_i \otimes \mathbf{e}_j \\ \overline{\boldsymbol{\Omega}} &\equiv \dot{\mathbf{Q}}\mathbf{Q}^T, \quad \overline{\boldsymbol{\Omega}} \equiv \dot{Q}_{ir}Q_{jr}\mathbf{e}_i \otimes \mathbf{e}_j \end{aligned}\right\} \tag{4.11}$$

and they are related by

$$\overline{\boldsymbol{\Omega}} = -\mathbf{Q}\boldsymbol{\Omega}\mathbf{Q}^T,\ \boldsymbol{\Omega} = -\mathbf{Q}^T\overline{\boldsymbol{\Omega}}\mathbf{Q} \tag{4.12}$$

where $\dot{\mathbf{Q}}$ is given by

$$\dot{\mathbf{Q}} = \mathbf{e}_r \otimes \dot{\mathbf{e}}_r^* \tag{4.13}$$

from Eq. (4.2) because of $\dot{\mathbf{e}}_i = \mathbf{0}$. Substituting Eqs. (4.2) and (4.13) into Eq. (4.11), we have

$$\boldsymbol{\Omega} = \dot{\mathbf{e}}_r^* \otimes \mathbf{e}_r^*, \quad \overline{\boldsymbol{\Omega}} = (\dot{\mathbf{e}}_r^* \boldsymbol{\cdot} \mathbf{e}_j^*)\mathbf{e}_i \otimes \mathbf{e}_j \tag{4.14}$$

from which it follows that

$$\dot{\mathbf{e}}_i^* = \boldsymbol{\Omega}\mathbf{e}_i^* \tag{4.15}$$

It is known from Eq. (4.15) that $\boldsymbol{\Omega}$ is the spin of the base $\{\mathbf{e}_i^*\}$.

The substitution of Eq. (4.10) into Eqs. (2.80) and (2.81) yields the following transformation rules.

$$\mathbf{d}^* = \mathbf{Q}\mathbf{d}\mathbf{Q}^T \tag{4.16}$$

$$\mathbf{w}^* = \mathbf{Q}(\mathbf{w} - \boldsymbol{\Omega})\mathbf{Q}^T = \mathbf{Q}\mathbf{w}\mathbf{Q}^T + \overline{\boldsymbol{\Omega}} \tag{4.17}$$

The relative spin in Eq. (2.85), i.e.

$$\boldsymbol{\Omega}^R \equiv \dot{\mathbf{R}}\,\mathbf{R}^T \tag{4.18}$$

obeys the transformation identical to that of **w** as follows:

$$\boldsymbol{\Omega}^{R*} = \mathbf{Q}(\boldsymbol{\Omega}^R - \boldsymbol{\Omega})\mathbf{Q}^T = \mathbf{Q}\boldsymbol{\Omega}^R\mathbf{Q}^T + \overline{\boldsymbol{\Omega}} \tag{4.19}$$

noting

$$\begin{aligned}\dot{\mathbf{R}}^*\mathbf{R}^{*T} &= (\mathbf{QR})^{\bullet}(\mathbf{QR})^T = (\dot{\mathbf{Q}}\mathbf{R} + \mathbf{Q}\dot{\mathbf{R}})\mathbf{R}^T\mathbf{Q}^T \\ &= \mathbf{Q}\dot{\mathbf{R}}\mathbf{R}^T\mathbf{Q}^T + \dot{\mathbf{Q}}\mathbf{Q}^T = \mathbf{Q}\boldsymbol{\Omega}^R\mathbf{Q}^T + \mathbf{Q}\mathbf{Q}^T\dot{\mathbf{Q}}\mathbf{Q}^T = \mathbf{Q}\boldsymbol{\Omega}^R\mathbf{Q}^T - \mathbf{Q}\dot{\mathbf{Q}}^T\mathbf{Q}\mathbf{Q}^T\end{aligned}$$

We obtain the following conclusions for the influence of rigid-body rotation from Eqs. (4.6) to (4.19).

(1) The right Cauchy-Green deformation tensor **C** and the Green strain tensor ***E*** are based in the reference configuration and thus they are observed to be unchangeable, i.e. invariant, obeying the transformation rule of scalar quantities independent of the rigid-body rotation. On the other hand, the left Cauchy-Green deformation tensor **b** and the Almansi strain tensor ***e*** are based in the current configuration and thus obey the transformation rule of second-order tensor.

(2) The strain rate tensor **d** obeys the transformation rule of the second-order tensor. On the other hand, the velocity gradient tensor ***l***, the continuum spin tensor **w** and the relative spin $\boldsymbol{\Omega}^R$ are directly subjected to the influence of rate of rigid-body rotation, lacking the objectivity.

The following transformations hold for stress tensors described in Chap. 3.

$$\boldsymbol{\sigma}^* = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T \quad (\mathbf{t}^* = \mathbf{Q}\mathbf{t} = \mathbf{Q}\boldsymbol{\sigma}\mathbf{n} = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T\mathbf{Q}\mathbf{n} = \boldsymbol{\sigma}^*\mathbf{n}^*) \tag{4.20}$$

$$\boldsymbol{\tau}^* = \mathbf{Q}\boldsymbol{\tau}\mathbf{Q}^T \tag{4.21}$$

$$\begin{gathered}\boldsymbol{\Pi}^* = \mathbf{Q}\boldsymbol{\Pi} \\ (\boldsymbol{\Pi}^* = J\boldsymbol{\sigma}^*\mathbf{F}^{*-T} = J\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T(\mathbf{QF})^{-T} = J\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T\mathbf{Q}^{-T}\mathbf{F}^{-T} = \mathbf{Q}J\boldsymbol{\sigma}\mathbf{F}^{-T} = \mathbf{Q}\boldsymbol{\Pi})\end{gathered} \tag{4.22}$$

$$\mathbf{S}^* = \mathbf{S}$$
$$(\mathbf{S}^* = \mathbf{F}^{*-1}\boldsymbol{\tau}^*\mathbf{F}^{*-T} = (\mathbf{QF})^{-1}(\mathbf{Q}\boldsymbol{\tau}\mathbf{Q}^T)(\mathbf{QF})^{-T} = \mathbf{F}^{-1}\mathbf{Q}^T\mathbf{Q}\boldsymbol{\tau}\mathbf{Q}^T\mathbf{Q}^{-T}\mathbf{F}^{-T} = \mathbf{F}^{-1}\boldsymbol{\tau}\mathbf{F}^{-T} = \mathbf{S}) \tag{4.23}$$

Then, the Cauchy stress tensor $\boldsymbol{\sigma}$ and the Kirchhoff stress tensor $\boldsymbol{\tau}$ obeys the transformation rule of the second-order tensor. On the other hand, the first Piola-Kirchhoff stress tensor $\boldsymbol{\Pi}$ obeys the transformation rule of the first-order tensor so that it is the two-point tensor. The second Piola-Kirchhoff stress tensor $\mathbf{S}$ is the invariant under the superposition of rigid-body rotation.

The consideration of objectivity is of great importance in the formulation of hypoelastic-based plastic constitutive equations since the rates of stress and anisotropic internal state variables in the current configuration are influenced directly by the rigid-body rotation as described above. Then, the time-derivatives of state variables will be further considered in the subsequent sections.

4.3 Material-Time Derivative of Tensor

The material-time derivatives of state variables is the rates of them observed by the coordinate system moving in parallel with material particle as explained in Sect. 2.2. However, it will be mathematically verified in this section that the material-time derivative of tensor does not obey the objective transformation rule and thus it cannot be used in constitutive equations.

Consider the transformation of the material-time derivative of a state variable obeying the objective transformation in Eq. (1.71) and (1.73). The material-time derivative of the tensor $\mathbf{t}$ in the current configuration reads (Hashiguchi 2007a):

$$\begin{aligned}\dot{t}^*_{p_1p_2\cdots p_m} = {} & \dot{Q}_{p_1q_1}Q_{p_2q_2}\cdots Q_{p_mq_m}t_{q_1q_2\cdots q_m} + Q_{p_1q_1}\dot{Q}_{p_2q_2}\cdots Q_{p_mq_m}t_{q_1q_2\cdots q_m} + \cdots \\ & + Q_{p_1q_1}Q_{p_2q_2}\cdots \dot{Q}_{p_mq_m}t_{q_1q_2\cdots q_m} + Q_{p_1q_1}Q_{p_2q_2}\cdots Q_{p_mq_m}\dot{t}_{q_1q_2\cdots q_m}\end{aligned} \tag{4.24}$$

$$\begin{aligned}\dot{t}_{p_1p_2\cdots p_m} = {} & \dot{Q}_{q_1p_1}Q_{q_2p_2}\cdots Q_{q_mp_m}t^*_{q_1q_2\cdots q_m} + Q_{q_1p_1}\dot{Q}_{q_2p_2}\cdots Q_{q_mp_m}t^*_{q_1q_2\cdots q_m} + \cdots \\ & + Q_{q_1p_1}Q_{q_2p_2}\cdots \dot{Q}_{q_mp_m}t^*_{q_1q_2\cdots q_m} + Q_{q_1p_1}Q_{q_2p_2}\cdots Q_{q_mp_m}\dot{t}^*_{q_1q_2\cdots q_m}\end{aligned} \tag{4.25}$$

Noting the relation $\dot{Q}_{p_iq_i} = \delta_{p_is}\dot{Q}_{sq_i} = Q_{p_it}Q_{st}\dot{Q}_{sq_i} = -Q_{p_it}\dot{Q}_{st}Q_{sq_i} = -Q_{p_it}\Omega_{tq_i}$ and replacing $t \rightarrow q_i,\ q_i \rightarrow r_i$, then Eqs. (4.24) and (4.25) can be rewritten as follows:

$$\overset{\bullet}{t}{}^{*}_{p_1p_2\cdots p_m} = Q_{p_1q_1}Q_{p_2q_2}\cdots Q_{p_mq_m}\,(\overset{\bullet}{t}_{q_1q_2\cdots q_m} - \Omega_{q_1r_1}t_{r_1q_2\cdots q_m} - \Omega_{q_2r_2}t_{q_1r_2\cdots q_m} - \cdots - \Omega_{q_1r_m}t_{q_1q_2\cdots r_m}) \tag{4.26}$$

$$\overset{\bullet}{t}_{p_1p_2\cdots p_m} = Q_{q_1p_1}Q_{q_2p_2}\cdots Q_{q_mp_m}\,(\overset{\bullet}{t}{}^{*}_{q_1q_2\cdots q_m} - \overline{\Omega}_{q_1r_1}t^{*}_{r_1q_2\cdots q_m} - \overline{\Omega}_{q_2r_2}t^{*}_{q_1r_2\cdots q_m} - \cdots - \overline{\Omega}_{q_mr_m}t^{*}_{q_1q_2\cdots r_m}) \tag{4.27}$$

It is known from Eqs. (4.26) and (4.27) that the material-time derivative does not obey the objective transformation rule, noting that the components $\overset{\bullet}{t}_{p_1p_2\ldots p_m}$ in the fixed coordinate system is not zero even when the components $\overset{\bullet}{t}{}^{*}_{p_1p_2\ldots p_m}$ in the coordinate system rotating with the material is zero. Equations (4.26) and (4.27) are expressed for the vector **v** and the second-order tensor **t** in symbolic notation as follows:

$$\overset{\bullet}{\mathbf{v}}{}^{*} = \mathbf{Q}(\overset{\bullet}{\mathbf{v}} - \boldsymbol{\Omega}\mathbf{v}),\, \overset{\bullet}{\mathbf{v}} = \mathbf{Q}^{T}(\overset{\bullet}{\mathbf{v}}{}^{*} - \overline{\boldsymbol{\Omega}}\mathbf{v}^{*}) \tag{4.28}$$

$$\overset{\bullet}{\mathbf{t}}{}^{*} = \mathbf{Q}(\overset{\bullet}{\mathbf{t}} - \boldsymbol{\Omega}\mathbf{t} + \mathbf{t}\boldsymbol{\Omega})\mathbf{Q}^{T},\, \overset{\bullet}{\mathbf{t}} = \mathbf{Q}^{T}(\overset{\bullet}{\mathbf{t}}{}^{*} - \overline{\boldsymbol{\Omega}}\mathbf{t}^{*} + \mathbf{t}^{*}\overline{\boldsymbol{\Omega}})\mathbf{Q} \tag{4.29}$$

Consequently, the material-time derivative cannot be adopted in constitutive equations.

In order to see the irrationality for using the material-time derivative of tensor, consider the hypoelastic constitutive equation, which relates the material-time derivative of Cauchy stress tensor linearly to the strain rate tensor and to which the hypoelastic-base plastic constitutive equation described after Chap. 6 also belong, as follows:

$$\overset{\bullet}{\boldsymbol{\sigma}} = \mathbf{H} : \mathbf{d}$$

where the tangent modulus tensor **H** (fourth-order tensor) is the function of stress and anisotropic internal variables in general. It follows from this equation with Eq. (4.29) that

$$\mathbf{d} = \mathbf{H}^{-1} : \mathbf{Q}^{T}(\overset{\bullet}{\boldsymbol{\sigma}}{}^{*} - \overline{\boldsymbol{\Omega}}\boldsymbol{\sigma}^{*} + \boldsymbol{\sigma}^{*}\overline{\boldsymbol{\Omega}})\mathbf{Q}\,(\neq \mathbf{H}^{-1} : \mathbf{Q}^{T}\overset{\bullet}{\boldsymbol{\sigma}}{}^{*}\mathbf{Q} = \mathbf{H}^{-1} : \mathbf{Q}^{T}(\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{T})^{\bullet}\mathbf{Q} = \mathbf{H}^{-1} : \overset{\circ}{\boldsymbol{\sigma}})$$

This leads to the irrational result that the deformation is induced, i.e. $\mathbf{d} \neq \mathbf{O}$ even if the stress observed by the coordinate system rotating ($\overline{\boldsymbol{\Omega}} \neq \mathbf{O}$) with the material itself does not change, i.e. $\overset{\bullet}{\boldsymbol{\sigma}}{}^{*} = \mathbf{O}$. This is caused by the non-objectivity of material-time derivative of tensor, while the strain rate **d** is not a material-time derivative of tensor but is the original tensor defined so as to obey the objective transformation by excluding the continuum spin tensor **w** from the velocity gradient tensor ***l***. In the next section, the objective time-derivative of tensor will be introduced which is based on the rate of tensor observed by the coordinate system deforming/rotating with material itself, satisfying the objectivity.

4.4 Convected Time-Derivative of Tensor

The objective rate of tensor describing the physical quantity must be independent of the spin of rigid-body rotation and thus it has to be given primarily by the *convected rate*, i.e. the rate of tensors observed by the coordinate system deforming and rotating with material itself, i.e. the *convected* (or *convective* or *embedded*) *coordinate system* in which the coordinate axes are etched in material itself. The convected rate, i.e. convected time-derivative is the generalization of the *Lie derivative* (cf. e.g. Truesdell and Toupin; Marsden and Hughes 1983, 1960; Bonet and Wood 1997; Simo 1998; Belytschko et al. 2014; de Souza-Neto et al. 2008). The convected coordinate system turns to the curvilinear coordinate system in general as a deformation proceeds. Therefore, it is required first to study the mathematics on the general curvilinear coordinate system in order to capture the exact physical interpretation of the objective rate tensors. However, it is beyond the level of this book. One can refer to Hashiguchi (2012) for the comprehensive. In what follows, the explanation for the objective rate tensor will be devised so as to be understood without the detailed mathematical formulation in the curvilinear coordinate system.

(1) **Description in convected bases**

Consider the embedded primary base $\{\mathbf{G}_I\}$ in the reference configuration, which becomes $\{\mathbf{g}_i(t)\}$ in the current configuration as the deformation of material is induced. Then, let the reciprocal bases for the primary bases $\{\mathbf{G}_I\}$ and $\{\mathbf{g}_i(t)\}$ be denoted by $\{\mathbf{G}^I\}$ and $\{\mathbf{g}^i(t)\}$, respectively, noting the definition in Eq. (1.45). Here, it should be noted that the reciprocal base $\{\mathbf{g}^i(t)\}$ can be embedded under a pure rotation of material but it cannot be embedded under a deformation of material because it does not keep the reciprocal relation to the primary base $\{\mathbf{g}_i(t)\}$ if deformation is induced. They satisfy

$$\mathbf{G}_I \cdot \mathbf{G}^J = \delta_I^J, \ \mathbf{g}_i(t) \cdot \mathbf{g}^j(t) = \delta_i^j \tag{4.30}$$

by virtue of Eq. (1.46). In addition, the following tensors are the generalized expressions of the identity tensor as can be confirmed by Eq. (1.105), while the identity tensor is called *metric tensor* in the general Euclidian space described in the curvilinear coordinate system.

$$\left.\begin{aligned}
\mathbf{G} &\equiv \mathbf{G}^I \otimes \mathbf{G}_I = \mathbf{G}_I \otimes \mathbf{G}^I \\
&= G^{IJ}\mathbf{G}_I \otimes \mathbf{G}_J = G^I_J \mathbf{G}_I \otimes \mathbf{G}^J \\
&= G^J_I \mathbf{G}^I \otimes \mathbf{G}_J = G_{IJ}\mathbf{G}^I \otimes \mathbf{G}^J \\
\mathbf{g}(t) &\equiv \mathbf{g}^i(t) \otimes \mathbf{g}_i(t) = \mathbf{g}_i(t) \otimes \mathbf{g}^i(t) \\
&= g^{ij}(t)\mathbf{g}_i(t) \otimes \mathbf{g}_j(t) = g^i_j(t)\mathbf{g}_i(t) \otimes \mathbf{g}^j(t) \\
&= g^j_i(t)\mathbf{g}^i(t) \otimes \mathbf{g}_j(t) = g_{ij}(t)\mathbf{g}^i(t) \otimes \mathbf{g}^j(t)
\end{aligned}\right\} \tag{4.31}$$

setting $G^{IJ} \equiv \mathbf{G}^I \cdot \mathbf{G}^J$, $G^I_J \equiv \mathbf{G}^I \cdot \mathbf{G}_J = \delta^I_J$, $G^J_I \equiv \mathbf{G}_I \cdot \mathbf{G}^J = \delta^J_I$, $G_{IJ} \equiv \mathbf{G}_I \cdot \mathbf{G}_J$ and $g^{ij} \equiv \mathbf{g}^i \cdot \mathbf{g}^j$, $g^i_j \equiv \mathbf{g}^i \cdot \mathbf{g}_j = \delta^i_j$, $g^j_i \equiv \mathbf{g}_i \cdot \mathbf{g}^j = \delta^j_i$, $g_{ij} \equiv \mathbf{g}_i \cdot \mathbf{g}_j$. The vector and the tensor based in the reference and the current configurations are called the *Lagrangian vector* and *tensor* and the *Eulerian vector* and *tensor*, respectively. In principle, the Lagrangian and the Eulerian vectors and tensors and their indices are denoted by the uppercase and the lowercase letters, respectively. Further, the tensor based in both of the reference and the current configurations is called the *Lagrangian–Eulerian* or *Eulerian–Lagrangian two-point tensor*, and they are denoted by the uppercase letter, and their indices are denoted by using both of the uppercase and the lowercase letters so as to specify the base vectors in which they are based. The symbol (t) specifying the quantities in the current time is omitted below for the sake of simplicity. The necessity for introducing the primary and the reciprocal bases can be recognized from the typical example that the deformation gradient tensor $\mathbf{F}$, which is the most basic variable for describing the deformation, is specified by the exploiting them as will be shown below.

Regarding the infinitesimal line-element vector $d\mathbf{X}$ in the reference configuration and the infinitesimal line-element $d\mathbf{x}$ in the current configuration to be the primary base vectors $\mathbf{G}_I$ and $\mathbf{g}_i$, respectively, in Eq. (2.15), we obtain the relations between the reference and current base vectors as follows:

$$\mathbf{g}_i = \delta^I_i \mathbf{F}\mathbf{G}_I, \quad \mathbf{G}_I = \delta^i_I \mathbf{F}^{-1}\mathbf{g}_i \tag{4.32}$$

$$\mathbf{F} = \delta^i_I \mathbf{g}_i \otimes \mathbf{G}^I, \quad \mathbf{F}^T = \delta^i_I \mathbf{G}^I \otimes \mathbf{g}_i \tag{4.33}$$

$$\mathbf{F}^{-1} = \delta^I_i \mathbf{G}_I \otimes \mathbf{g}^i, \ \mathbf{F}^{-T} = \delta^I_i \mathbf{g}^i \otimes \mathbf{G}_I \tag{4.34}$$

$$\mathbf{g}^i = \delta^i_I \mathbf{F}^{-T}\mathbf{G}^I, \ \mathbf{G}^I = \delta^I_i \mathbf{F}^T \mathbf{g}^i \tag{4.35}$$

from which we have

$$\left.\begin{array}{l} \dot{\mathbf{g}}_i = \delta^I_i \dot{\mathbf{F}}\mathbf{G}_I = \dot{\mathbf{F}}\mathbf{F}^{-1}\mathbf{g}_i = \boldsymbol{l}\mathbf{g}_i \\ \dot{\mathbf{g}}^i = \delta^I_i \dot{\mathbf{F}}^{-T}\mathbf{G}^I = \dot{\mathbf{F}}^{-T}\mathbf{F}^T\mathbf{g}^i = -\mathbf{F}^{-T}\dot{\mathbf{F}}^T\mathbf{g}^i = -\boldsymbol{l}^T\mathbf{g}^i \end{array}\right\} \tag{4.36}$$

noting Eq. (2.74) and $\dot{\mathbf{I}} = (\mathbf{F}\mathbf{F}^{-1})^{\bullet} = \dot{\mathbf{F}}\mathbf{F}^{-1} + \mathbf{F}\dot{\mathbf{F}}^{-1} = \dot{\mathbf{F}}^{-T}\mathbf{F}^T + \mathbf{F}^{-T}\dot{\mathbf{F}}^T = \mathbf{O}$. While the deformation gradient $\mathbf{F}$ is the two-point tensor based in both the current and the reference configurations, it is further regarded to be the two-point (mixed) identity tensor in the convected bases, i.e. the reference reciprocal base $\mathbf{G}^I$ and the current primary base $\mathbf{g}_i$ from Eq. (4.33).

Vector and tensor in the current base are described by Eqs. (1.44), (1.105) and (1.106) as follows:

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{g}^r)\mathbf{g}_r = (\mathbf{v} \cdot \mathbf{g}_r)\mathbf{g}^r \tag{4.37}$$

$$\mathbf{t} = (\mathbf{g}^r \cdot \mathbf{t}\mathbf{g}^s)\mathbf{g}_r \otimes \mathbf{g}_s = (\mathbf{g}^r \cdot \mathbf{t}\mathbf{g}_s)\mathbf{g}_r \otimes \mathbf{g}^s = (\mathbf{g}_r \cdot \mathbf{t}\mathbf{g}^s)\mathbf{g}^r \otimes \mathbf{g}_s = (\mathbf{g}_r \cdot \mathbf{t}\mathbf{g}_s)\mathbf{g}^r \otimes \mathbf{g}^s \tag{4.38}$$

noting

$$\mathrm{t}_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = (\mathbf{e}_i \cdot \mathbf{t}\mathbf{e}_j)\mathbf{e}_i \otimes \mathbf{e}_j$$

$$= \begin{cases} \{[(\mathbf{e}_i \cdot \mathbf{g}_r)\mathbf{g}^r] \cdot \mathbf{t}[(\mathbf{e}_j \cdot \mathbf{g}_s)\mathbf{g}^s]\}(\mathbf{e}_i \cdot \mathbf{g}^p)\mathbf{g}_p \otimes (\mathbf{e}_j \cdot \mathbf{g}^q)\mathbf{g}_q \\ \quad = (\mathbf{g}^r \cdot \mathbf{t}\mathbf{g}^s)[(\mathbf{e}_i \cdot \mathbf{g}_r)\mathbf{e}_i \cdot \mathbf{g}^p]\mathbf{g}_p \otimes [(\mathbf{e}_j \cdot \mathbf{g}_s)\mathbf{e}_j \cdot \mathbf{g}^q]\mathbf{g}_q \\ \quad = (\mathbf{g}^r \cdot \mathbf{t}\mathbf{g}^s)(\mathbf{g}_r \cdot \mathbf{g}^p)\mathbf{g}_p \otimes (\mathbf{g}_s \cdot \mathbf{g}^q)\mathbf{g}_q \\ \quad = (\mathbf{g}^r \cdot \mathbf{t}\mathbf{g}^s)\mathbf{g}_r \otimes \mathbf{g}_s \\ \{[(\mathbf{e}_i \cdot \mathbf{g}_r)\mathbf{g}^r] \cdot \mathbf{t}[(\mathbf{e}_j \cdot \mathbf{g}^s)\mathbf{g}_s]\}(\mathbf{e}_i \cdot \mathbf{g}^p)\mathbf{g}_p \otimes (\mathbf{e}_j \cdot \mathbf{g}_q)\mathbf{g}^q \\ \quad = (\mathbf{g}^r \cdot \mathbf{t}\mathbf{g}_s)\mathbf{g}_r \otimes \mathbf{g}^s \\ \{[(\mathbf{e}_i \cdot \mathbf{g}^r)\mathbf{g}_r] \cdot \mathbf{t}[(\mathbf{e}_j \cdot \mathbf{g}_s)\mathbf{g}^s]\}(\mathbf{e}_i \cdot \mathbf{g}_p)\mathbf{g}^p \otimes (\mathbf{e}_j \cdot \mathbf{g}^q)\mathbf{g}_q \\ \quad = (\mathbf{g}_r \cdot \mathbf{t}\mathbf{g}^s)\mathbf{g}^r \otimes \mathbf{g}_s \\ \{[(\mathbf{e}_i \cdot \mathbf{g}^r)\mathbf{g}_r] \cdot \mathbf{t}[(\mathbf{e}_j \cdot \mathbf{g}^s)\mathbf{g}_s]\}(\mathbf{e}_i \cdot \mathbf{g}_p)\mathbf{g}^p \otimes (\mathbf{e}_j \cdot \mathbf{g}_q)\mathbf{g}^q \\ \quad = (\mathbf{g}_r \cdot \mathbf{t}\mathbf{g}_s)\mathbf{g}^r \otimes \mathbf{g}^s \end{cases}$$

where the components in the rectangular base are denoted by the roman letter t_{ij} in order to distinguish them from the components denoted by the italic letters t_{ij} in the convected bases. The following expressions in the embedded coordinate system hold from Eqs. (4.37) and (4.38).

$$\mathbf{v} = v^i\mathbf{g}_i = v_i\mathbf{g}^i \tag{4.39}$$

$$\mathbf{t} = t^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = t^i_{\bullet j}\mathbf{g}_i \otimes \mathbf{g}^j = t_i^{\bullet j}\mathbf{g}^i \otimes \mathbf{g}_j = t_{ij}\mathbf{g}^i \otimes \mathbf{g}^j \tag{4.40}$$

where

$$v^i = \mathbf{v} \cdot \mathbf{g}^i, \quad v_i = \mathbf{v} \cdot \mathbf{g}_i \tag{4.41}$$

$$t^{ij} = \mathbf{g}^i \cdot \mathbf{t}\mathbf{g}^j, \quad t^i_{\bullet j} = \mathbf{g}^i \cdot \mathbf{t}\mathbf{g}_j, \quad t_i^{\bullet j} = \mathbf{g}_i \cdot \mathbf{t}\mathbf{g}^j, \quad t_{ij} = \mathbf{g}_i \cdot \mathbf{t}\mathbf{g}_j \tag{4.42}$$

Here, note that the opposite combination of the *contravariant* and the *covariant* (see **Appendix B**) holds between component and base vector of tensor in general, while this fact is obvious for vector by virtue of Eq. (1.47).

(2) **Pull-back and push-forward operations**

The objective time-derivative of tensor would be no more than the rate of variation in tensor observed from the material itself. Then, in order to derive it, we have to incorporate the tensor which changes only when the state of physical quantity is observed to change by the material itself. To this end, we incorporate the tensors defined in the following.

Eulerian tensor: The tensor based in the current configuration, i.e. standing on the current base vectors is called the *Eulerian tensor* (e.g. $\mathbf{b}$, $\mathbf{V}$, $\mathbf{e}$, $\mathbf{R}^E$, $\boldsymbol{l}$, $\boldsymbol{d}$, $\mathbf{w}$, $\boldsymbol{\omega}$, $\boldsymbol{\Omega}^E$, $\boldsymbol{\Omega}^R$, $\boldsymbol{\sigma}$, $\boldsymbol{\tau}$ obeying $\mathbf{t}^* = \mathbf{QtQ}^T$).

Lagrangian tensor: The tensor based in the reference configuration, i.e. standing on the reference base vectors is called the *Lagrangian tensor* (e.g. $\mathbf{C}$, $\boldsymbol{U}$, $\boldsymbol{E}$, $\mathbf{B}$, $\mathbf{R}^L$, $\boldsymbol{\Omega}^L$, $\mathbf{S}$, $\mathfrak{B}$ obeying $\mathbf{T}^* = \mathbf{T}$).

Here, we should notice the following facts.

1) The Eulerian and the Lagrangian tensors possess the same components in the convected coordinate system. Therefore, they are derived by changing the base vectors from each other.
2) The Eulerian tensor is changeable but the Lagrangian tensor remains unaltered when the state observed from the material itself, i.e. components in the convected coordinate system does not change because the current base vectors are changeable but the reference base vectors remain unaltered in the convected coordinate system. In other words, the Eulerian tensor is influenced by the rigid-body rotation but the Lagrangian tensor is independent of that, describing the variation of state observed from the material itself. Here, on the other hand, note that one can represent tensors in any coordinate system. However, the variational rates of physical quantities observed from material itself, which can be adopted in constitutive relation, cannot be described simply in the orthogonal coordinate system.
3) Variation of physical quantity in material itself can be described by the Lagrangian tensor without the influence of superposed rigid-body rotation, while the Eulerian tensor is influenced by the superposed rigid-body rotation. This advantage of the Lagrangian tensor is utilized for the objective time-integration of tensor-valued quantities in numerical calculation as will be described in Sect. 20.10.

Two-point tensor: The tensor based in both the current and the reference configurations, i.e. standing on the current and reference base vectors is called the *two-point tensor* (e.g. $\mathbf{F}$, $\mathbf{R}$, $\boldsymbol{\Pi}(=\mathbf{P}^T)$ obeying $\mathbf{T}^* = \mathbf{QT}$ or $\mathbf{TQ}^T$).

The transformation from the Eulerian tensor to the Lagrangian tensor and its inverse are called the *pull-back* and the *push-forward* operations, respectively, and executed by multiplying the two-point tensor (Lagrangian-Eulerian tensor, e.g. $\mathbf{F}, \mathbf{F}^{-T}, \mathbf{R}$ for the pull-back and Eulerian-Lagrangian tensor, e.g. $\mathbf{F}^T, \mathbf{F}^{-1}, \mathbf{R}^T$ for the push-forward from the left and their inverse ones from the right) describing the deformation and/or rotation in general. In what follows, the concrete examples by the deformation gradient tensor are shown noting Eqs. (4.33), (4.34), (4.39) and (4.40) as follows:

$$\boxed{\begin{array}{l}\overleftarrow{\mathbf{v}}^G = \delta^I_i v^i \mathbf{G}_I = \mathbf{F}^{-1}\mathbf{v},\ \overleftarrow{\mathbf{v}}_G = \delta^i_I v_i \mathbf{G}^I = \mathbf{F}^T\mathbf{v} \\ \overrightarrow{\mathbf{V}}^g = \delta^i_I V^I \mathbf{g}_i = \mathbf{F}\mathbf{V},\ \overrightarrow{\mathbf{V}}_g = \delta^I_i V_I \mathbf{g}^i = \mathbf{F}^{-T}\mathbf{V}\end{array}} \tag{4.43}$$

$$\boxed{\begin{array}{l}\left.\begin{array}{l}\overleftarrow{\mathbf{t}}^{GG}=\delta_i^I\delta_j^J t^{ij}\mathbf{G}_I\otimes\mathbf{G}_J=\mathbf{F}^{-1}\mathbf{t}\mathbf{F}^{-T},\ \overleftarrow{\mathbf{t}}_{\bullet G}^{G}=\delta_i^I\delta_J^j t^i_{\bullet j}\mathbf{G}_I\otimes\mathbf{G}^J=\mathbf{F}^{-1}\mathbf{t}\mathbf{F}\\ \overleftarrow{\mathbf{t}}_{G}^{\bullet G}=\delta_I^i\delta_J^j t_i^{\bullet j}\mathbf{G}_I\otimes\mathbf{G}_J=\mathbf{F}^{T}\mathbf{t}\mathbf{F}^{-T},\ \overleftarrow{\mathbf{t}}_{GG}=\delta_I^i\delta_J^j t^{ij}\mathbf{G}_I\otimes\mathbf{G}^J=\mathbf{F}^{T}\mathbf{t}\mathbf{F}\end{array}\right\}\\ \left.\begin{array}{l}\overrightarrow{\mathbf{T}}^{gg}=\delta_I^i\delta_J^j T^{IJ}\mathbf{g}_i\otimes\mathbf{g}_j=\mathbf{F}\mathbf{T}\mathbf{F}^{T},\ \overrightarrow{\mathbf{T}}_{\bullet g}^{g}=\delta_I^i\delta_j^J T^I_{\bullet J}\mathbf{g}^i\otimes\mathbf{g}^j=\mathbf{F}\mathbf{T}\mathbf{F}^{-1}\\ \overrightarrow{\mathbf{T}}_{g}^{\bullet g}=\delta_i^I\delta_J^j T_I^{\bullet J}\mathbf{g}_i\otimes\mathbf{g}_j=\mathbf{F}^{-T}\mathbf{T}\mathbf{F}^{T},\ \overrightarrow{\mathbf{T}}_{gg}=\delta_i^I\delta_j^J T^{IJ}\mathbf{g}^i\otimes\mathbf{g}^j=\mathbf{F}^{-T}\mathbf{T}\mathbf{F}^{-1}\end{array}\right\}\end{array}} \tag{4.44}$$

noting

$$\mathbf{F}^{-1}\mathbf{v}=\delta_i^I\mathbf{G}_I\otimes\mathbf{g}^i v^r\mathbf{g}_r=\delta_i^I v^i\mathbf{G}_I$$

$$\mathbf{F}^{-1}\mathbf{t}\mathbf{F}^{-T}=\delta_i^I\mathbf{G}_I\otimes\mathbf{g}^i t^{pq}\mathbf{g}_p\otimes\mathbf{g}_q\delta_j^J\mathbf{g}^j\otimes\mathbf{G}_J=\delta_i^I\mathbf{G}_I t^{ij}\delta_j^J\otimes\mathbf{G}_J$$

as the example of Eq. $(4.44)_1$. The over arrow turning left ($\longleftarrow$) and right ($\longrightarrow$) is added for the pull-back and the push-forward operation, respectively. Further, the uppercase letter index G is added in order to specify the replacement of the current base to the reference base in the pull-back operation and the lowercase letter index g is added in order to specify the replacement of the reference base to the current base in the push-forward operation, and they are put in the lower or upper position for the covariant or the contravariant component, respectively, while these symbols were devised by Hashiguchi (2011). Here, note that the pulled-back and push-forward operations of tensors in higher order than two cannot be expressed in the symbolic notations by the multiplications of the deformation gradient tensor but can be represented only by exchanging the current base vectors to the reference base vectors and its inverse as far as quite particular definitions of tensor operations are not adopted.

It is noteworthy that the differences between the contravariant and the covariant forms in the pull-back and the push-forward operations diminish when only rotation is taken account leading to $\mathbf{F}=\mathbf{F}^{-T}=\mathbf{R}$, $\mathbf{F}^{-1}=\mathbf{F}^{T}=\mathbf{R}^{T}$. The tensor ${}^{R}\overleftarrow{\mathbf{t}}=\mathbf{R}^{T}\mathbf{t}\mathbf{R}$ pulled back only by the rotation, regarding $\mathbf{F}=\mathbf{R}$, is called the *rotation-free tensor* or *rotation-insensitive tensor* since the rotation $\mathbf{R}$ is excluded from the Eulerian tensor.

The Lagrangian tensors $\mathbf{C}$, $\boldsymbol{E}$ and $\mathbf{S}$ described in Sect. 4.2 are derived by the pull-back from the Eulerian tensors $\mathbf{g}$, $\boldsymbol{e}$ and $\boldsymbol{\tau}$ as follows:

$$\mathbf{C}=\overleftarrow{\mathbf{g}}_{GG}=\delta_I^i\delta_J^j g_{ij}\mathbf{G}^I\otimes\mathbf{G}^J=\mathbf{F}^{T}\mathbf{g}\mathbf{F}=\mathbf{F}^{T}\mathbf{F}$$
$$\boldsymbol{E}=\overleftarrow{\boldsymbol{e}}_{GG}=\delta_I^i\delta_J^j e_{ij}\mathbf{G}^I\otimes\mathbf{G}^J=\mathbf{F}^{T}\boldsymbol{e}\mathbf{F}$$
$$\mathbf{S}=\overleftarrow{\boldsymbol{\tau}}^{GG}=\delta_i^I\delta_j^J\tau^{ij}\mathbf{G}_I\otimes\mathbf{G}_J=\mathbf{F}^{-1}\boldsymbol{\tau}\mathbf{F}^{-T}$$

and the two-point tensors $\mathbf{F}$ and $\boldsymbol{\Pi}$ are derived by the pull-back from the Eulerian tensors $\mathbf{g}$ and $\boldsymbol{\tau}$ as follows:

$$\mathbf{F} = \left\{ \begin{array}{l} \overleftarrow{\mathbf{g}}^{\hat{g}}_{\bullet G} = \delta^i_I \mathbf{g}_i \otimes \mathbf{G}^I = \mathbf{g}_i \otimes \mathbf{g}^i \delta^r_I \mathbf{g}_r \otimes \mathbf{G}^I \\ \overleftarrow{\mathbf{g}}_{\hat{g}G} = \delta^j_J g_{ij} \mathbf{g}^i \otimes \mathbf{G}^J = g_{ij} \mathbf{g}^i \otimes \mathbf{g}^j \delta^r_J \mathbf{g}_r \otimes \mathbf{G}^J \end{array} \right\} = \mathbf{gF}$$

$$\mathbf{\Pi} = \left\{ \begin{array}{l} \overleftarrow{\boldsymbol{\tau}}^{\hat{g}G} = \delta^J_j \tau^{ij} \mathbf{g}_i \otimes \mathbf{G}_J = \tau^{ij} \mathbf{g}_i \otimes \mathbf{g}_j \delta^J_r \mathbf{g}^r \otimes \mathbf{G}_J \\ \overleftarrow{\boldsymbol{\tau}}_{\hat{g}}^{\bullet G} = \delta^J_j \tau_i^{\bullet j} \mathbf{g}^i \otimes \mathbf{G}_J = \tau_i^{\bullet j} \mathbf{g}^i \otimes \mathbf{g}_j \delta^J_r \mathbf{g}^r \otimes \mathbf{G}_J \end{array} \right\} = \boldsymbol{\tau}\mathbf{F}^{-T}$$

noting Eqs. (4.32), (4.35) and (4.44). The over hat symbol ($^\wedge$) specifies the un-exchange of base vector. $\mathbf{\Pi}$ is the induced two-point tensor from the Kirchhoff stress $\boldsymbol{\tau}$. On the other hand, $\mathbf{F}$ is regarded as the inherent two-point tensors. Here, it can be called the identity tensor in the broad sense, since it possesses the components of the Kronecker's delta in both the current and the reference bases. $\mathbf{F}$ was called the two-point tensor in Eq. (2.14) for the orthogonal coordinate system, where it was described by the components of position vectors in the current and the reference configurations in the orthogonal coordinate systems. Then, the physical meaning of the two-point tensor would be obscure by the expression in the orthogonal coordinate system. On the other hand, the physical meaning of the two-point tensor would be captured clearly by the expression in the convected coordinate system such that it is based extending over the reference and the current bases which are not arbitrary but composed by the definite sets of the embedded base vectors.

The Eulerian tensor changes even when the state of physical quantity observed from the material itself does not change under a material rotation, since the base vectors change even by a rotation of the material. On the other hand, the Lagrangian tensor pulled-back to the ref-erence configuration with the fixed base vectors does not change in the material rotation and thus it called the *rotation-free tensor*, while the Lagrangian tensor inherits the components in the Eulerian tensor. Then, the constitutive relation described by the Lagrangian tensors is used in the deformation analysis. The Eulerian base vectors are calculated by the push-forward op-eration of the Lagrangian (reference) base vectors through the deformation gradient tensor, which is required to capture the Eulerian tensor in the current configuration from the Lagrangian tensor.

The physical and geometrical interpretations for the relations between the above-mentioned Eulerian and Lagrangian tensors can be referred to Hashiguchi and Yamakawa (2012).

(3) **Convected time-derivatives: Objective rate of tensor**

The material-time derivative of the vector $\mathbf{v}$ is described in the current primary base $\{\mathbf{g}_i\}$ and the current reciprocal base $\{\mathbf{g}^i\}$ from Eq. (4.39) by

$$\dot{\mathbf{v}} = \left\{ \begin{array}{l} (v^r \mathbf{g}_r)^\bullet = \dot{v}^r \mathbf{g}_r + v^r \dot{\mathbf{g}}_r \\ (v_r \mathbf{g}^r)^\bullet = \dot{v}_r \mathbf{g}^r + v_r \dot{\mathbf{g}}^r \end{array} \right. \tag{4.45}$$

The first terms in the right-hand sides of Eq. (4.45) represent the rates of the vector $\mathbf{v}$ observed from the embedded coordinate system and thus they are called the *convected time-derivative*. In other words, they mean the rate of physical quantity observed from the embedded coordinate system having the base vectors composed of line-elements etched in a material. Also, they are interpreted as the rates observed from material itself and thus they are independent of rigid-body rotation, possessing the *objectivity*. Here, however, note that the rotation of the embedded base is different from the rotation of the substructure of material in general as known from the fact that the movements of line-elements etched in material coincides with the deformed geometrical appearance of material but it does not necessarily coincide with the movements of material fibers representing the substructure of anisotropic material. The convected time-derivatives of vector are expressed from Eq. (4.45) as

$$\boxed{\begin{aligned}
&\overset{\rightleftarrows}{\dot{\mathbf{v}}}{}^{g} \equiv \dot{v}^{r}\mathbf{g}_{r} = \dot{\mathbf{v}} - v^{r}\dot{\mathbf{g}}_{r} = \dot{\mathbf{v}} - \boldsymbol{l}\mathbf{v} = \dot{\mathbf{v}} + \mathbf{F}\dot{\mathbf{F}}^{-1}\mathbf{v} = \mathbf{F}(\mathbf{F}^{-1}\mathbf{v})^{\bullet} = (\overrightarrow{\overleftarrow{\mathbf{v}}{}^{G}})^{\bullet g}\\
&\overset{\rightleftarrows}{\dot{\mathbf{v}}}{}_{g} \equiv \dot{v}_{r}\mathbf{g}^{r} = \dot{\mathbf{v}} - v_{r}\dot{\mathbf{g}}^{r} = \dot{\mathbf{v}} + \boldsymbol{l}^{T}\mathbf{v} = \dot{\mathbf{v}} + \mathbf{F}^{-T}\dot{\mathbf{F}}^{T}\mathbf{v} = \mathbf{F}^{-T}(\mathbf{F}^{T}\mathbf{v})^{\bullet} = (\overrightarrow{\overleftarrow{\mathbf{v}}{}_{G}})^{\bullet}_{g}
\end{aligned}} \tag{4.46}$$

by using Eq. (4.36), noting $v^{r}\dot{\mathbf{g}}_{r} = v^{r}\boldsymbol{l}\mathbf{g}_{r} = \boldsymbol{l}\mathbf{v}$.

Analogously to the vector described above, the material-time derivative of tensor in the current base is described from Eq. (4.40) by

$$\dot{\mathbf{t}} = \begin{cases}
(t^{ij}\mathbf{g}_{i}\otimes\mathbf{g}_{j})^{\bullet} = \dot{t}^{ij}\mathbf{g}_{i}\otimes\mathbf{g}_{j} + t^{ij}\dot{\mathbf{g}}_{i}\otimes\mathbf{g}_{j} + t^{ij}\mathbf{g}_{i}\otimes\dot{\mathbf{g}}_{j}\\
(t^{i}_{\bullet j}\mathbf{g}_{i}\otimes\mathbf{g}^{j})^{\bullet} = \dot{t}^{i}_{\bullet j}\mathbf{g}_{i}\otimes\mathbf{g}^{j} + t^{i}_{\bullet j}\dot{\mathbf{g}}_{i}\otimes\mathbf{g}^{j} + t^{i}_{\bullet j}\mathbf{g}_{i}\otimes\dot{\mathbf{g}}^{j}\\
(t_{i}^{\bullet j}\mathbf{g}^{i}\otimes\mathbf{g}_{j})^{\bullet} = \dot{t}_{i}^{\bullet j}\mathbf{g}^{i}\otimes\mathbf{g}_{j} + t_{i}^{\bullet j}\dot{\mathbf{g}}^{i}\otimes\mathbf{g}_{j} + t_{i}^{\bullet j}\mathbf{g}^{i}\otimes\dot{\mathbf{g}}_{j}\\
(t_{ij}\mathbf{g}^{i}\otimes\mathbf{g}^{j})^{\bullet} = \dot{t}_{ij}\mathbf{g}^{i}\otimes\mathbf{g}^{j} + t_{ij}\dot{\mathbf{g}}^{i}\otimes\mathbf{g}^{j} + t_{ij}\mathbf{g}^{i}\otimes\dot{\mathbf{g}}^{j}
\end{cases} \tag{4.47}$$

Exploiting Eqs. (1.106) and (4.36) in Eq. (4.47), the following four types of convected time-derivatives are derived.

$$\boxed{\begin{aligned}
&\overset{\rightleftarrows}{\dot{\mathbf{t}}}{}^{gg} \equiv \dot{t}^{ij}\mathbf{g}_{i}\otimes\mathbf{g}_{j} = \dot{\mathbf{t}} - \boldsymbol{l}\mathbf{t} - \mathbf{t}\boldsymbol{l}^{T} = \dot{\mathbf{t}} + \mathbf{F}\dot{\mathbf{F}}^{-1}\mathbf{t} + \mathbf{t}\dot{\mathbf{F}}^{-T}\mathbf{F}^{T} = \mathbf{F}(\mathbf{F}^{-1}\mathbf{t}\mathbf{F}^{-T})^{\bullet}\mathbf{F}^{T} = (\overrightarrow{\overleftarrow{\mathbf{t}}{}^{GG}})^{\bullet gg}\\
&\overset{\rightleftarrows}{\dot{\mathbf{t}}}{}^{g}_{\bullet g} \equiv \dot{t}^{i}_{\bullet j}\mathbf{g}_{i}\otimes\mathbf{g}^{j} = \dot{\mathbf{t}} - \boldsymbol{l}\mathbf{t} + \mathbf{t}\boldsymbol{l} = \dot{\mathbf{t}} + \mathbf{F}\dot{\mathbf{F}}^{-1}\mathbf{t} + \mathbf{t}\dot{\mathbf{F}}\mathbf{F}^{-1} = \mathbf{F}(\mathbf{F}^{-1}\mathbf{t}\mathbf{F})^{\bullet}\mathbf{F}^{-1} = (\overrightarrow{\overleftarrow{\mathbf{t}}{}^{G}_{\bullet G}})^{\bullet g}_{\bullet g}\\
&\overset{\rightleftarrows}{\dot{\mathbf{t}}}{}^{g}_{\bullet g} \equiv \dot{t}_{i}^{\bullet j}\mathbf{g}^{i}\otimes\mathbf{g}_{j} = \dot{\mathbf{t}} + \boldsymbol{l}^{T}\mathbf{t} - \mathbf{t}\boldsymbol{l}^{T} = \dot{\mathbf{t}} + \mathbf{F}^{-T}\dot{\mathbf{F}}^{T}\mathbf{t} + \mathbf{t}\dot{\mathbf{F}}^{-T}\mathbf{F}^{T} = \mathbf{F}^{-T}(\mathbf{F}^{T}\mathbf{t}\mathbf{F}^{-T})^{\bullet}\mathbf{F}^{T} = (\overrightarrow{\overleftarrow{\mathbf{t}}{}_{G}^{\bullet G}})^{\bullet g}_{g}\\
&\overset{\rightleftarrows}{\dot{\mathbf{t}}}{}_{gg} \equiv \dot{t}_{ij}\mathbf{g}^{i}\otimes\mathbf{g}^{j} = \dot{\mathbf{t}} + \boldsymbol{l}^{T}\mathbf{t} + \mathbf{t}\boldsymbol{l} = \dot{\mathbf{t}} + \mathbf{F}^{-T}\dot{\mathbf{F}}^{T}\mathbf{t} + \mathbf{t}\dot{\mathbf{F}}\mathbf{F}^{-1} = \mathbf{F}^{-T}(\mathbf{F}^{T}\mathbf{t}\mathbf{F})^{\bullet}\mathbf{F}^{-1} = (\overrightarrow{\overleftarrow{\mathbf{t}}{}_{GG}})^{\bullet}_{gg}
\end{aligned}} \tag{4.48}$$

The notations $\overset{\rightleftarrows}{\dot{\mathbf{v}}}{}^{g}$, $\overset{\rightleftarrows}{\dot{\mathbf{t}}}{}^{gg}$, $\overset{\rightleftarrows}{\dot{\mathbf{t}}}{}^{g}_{\bullet g}$, etc. were used first by Hashiguchi (2011) to specify the objective time-derivatives and their types of contravariant and covariant, where the indices "g" are added in the upper and lower positions in order to specify the contravariant and the covariant expressions (component positions), respectively, of vector and tensor (**Appendix B**). The objective time-derivative is the rate of tensor observed from material itself but it can be also interpreted from Eqs. (4.46) and (4.48) to be the *current expression of rate of Lagrangian vector* or *tensor* (transformation of vector or tensor physical quantity described in the reference base to the current base), the components of which is independent of a rotation of material. There exist the two and the four types of convected rates of vector and tensor, respectively, as shown in Eqs. (4.46) and (4.48). In particular, $\overset{\rightleftarrows}{\mathbf{t}}{}^{gg}$ and $\overset{\rightleftarrows}{\mathbf{t}}{}_{gg}$ are the general forms of the *Oldroyd rate* (Oldroyd, 1950)) and the *Cotter-Rivlin rate* (Cotter and Rivlin, 1955), respectively, while the former is called the *Lie derivative* based on the hyperelasticity (cf. e.g. Bonet and Wood 1997; Simo 1998; Belytschko et al. 2014; de Souza-Neto et al. 2008).

The convected derivatives in Eqs. (4.46) and (4.48) satisfy the objectivity obviously because they are based on the rates of tensor observed by a material itself. In addition, this fact can be mathematically confirmed as shown below for Eqs. $(4.46)_1$ and $(4.48)_1$ as examples, noting Eq. (4.10).

$$\overset{\rightleftarrows}{\mathbf{v}}{}^{*g} = \mathbf{F}^*(\mathbf{F}^{*-1}\mathbf{v}^*)^{\bullet} = \mathbf{QF}[(\mathbf{QF})^{-1}\mathbf{Qv}]^{\bullet} = \mathbf{Q}[\mathbf{F}(\mathbf{F}^{-1}\mathbf{v})^{\bullet}] \tag{4.49}$$

or

$$\begin{aligned}\overset{\rightleftarrows}{\mathbf{v}}{}^{*g} &= \dot{\mathbf{v}}^* - \boldsymbol{l}^*\mathbf{v}^* = (\mathbf{Qv})^{\bullet} - \mathbf{Q}(\boldsymbol{l} - \boldsymbol{\Omega})\mathbf{Q}^T\mathbf{Qv} = \dot{\mathbf{Q}}\mathbf{v} + \mathbf{Q}\dot{\mathbf{v}} - \mathbf{Q}(\boldsymbol{l} - \dot{\mathbf{Q}}^T\mathbf{Q})\mathbf{v} \\ &= \mathbf{Q}\dot{\mathbf{v}} + \dot{\mathbf{Q}}\mathbf{v} - \mathbf{Q}(\boldsymbol{l} + \mathbf{Q}^T\dot{\mathbf{Q}})\mathbf{v} = \mathbf{Q}(\dot{\mathbf{v}} - \boldsymbol{l}\mathbf{v})\end{aligned} \tag{4.50}$$

and

$$\begin{aligned}\overset{\rightleftarrows}{\mathbf{t}}{}^{*gg} &= \mathbf{F}^*(\mathbf{F}^{*-1}\mathbf{t}^*\mathbf{F}^{*-T})^{\bullet}\mathbf{F}^{*T} = \mathbf{QF}[(\mathbf{QF})^{-1}\mathbf{QtQ}^T(\mathbf{QF}^{-T})]^{\bullet}(\mathbf{QF}^T) \\ &= \mathbf{QF}(\mathbf{F}^{-1}\mathbf{Q}^T\mathbf{QtQ}^T\mathbf{QF}^{-T})^{\bullet}\mathbf{F}^T\mathbf{Q}^T = \mathbf{Q}[\mathbf{F}(\mathbf{F}^{-1}\mathbf{tF}^{-T})^{\bullet}\,\mathbf{F}^T]\mathbf{Q}^T\end{aligned} \tag{4.51}$$

or

$$\begin{aligned}\overset{\rightleftarrows}{\mathbf{t}}{}^{*gg} &= \dot{\mathbf{t}}^* - \boldsymbol{l}^*\mathbf{t}^* - \mathbf{t}^*\boldsymbol{l}^{*T} = (\mathbf{QtQ}^T)^{\bullet} - \mathbf{Q}(\boldsymbol{l}-\boldsymbol{\Omega})\mathbf{Q}^T\mathbf{QtQ}^T - \mathbf{QtQ}^T[\mathbf{Q}(\boldsymbol{l}-\boldsymbol{\Omega})\mathbf{Q}^T]^T \\ &= \dot{\mathbf{Q}}\mathbf{tQ}^T + \mathbf{Q}\dot{\mathbf{t}}\mathbf{Q}^T + \mathbf{Qt}\dot{\mathbf{Q}}^T - \mathbf{Q}(\boldsymbol{l}-\boldsymbol{\Omega})\mathbf{tQ}^T - \mathbf{QtQ}^T\mathbf{Q}(\boldsymbol{l}^T - \boldsymbol{\Omega}^T)\mathbf{Q}^T \\ &= \dot{\mathbf{Q}}\mathbf{tQ}^T + \mathbf{Q}\dot{\mathbf{t}}\mathbf{Q}^T + \mathbf{Qt}\dot{\mathbf{Q}}^T - \mathbf{Q}(\boldsymbol{l}-\dot{\mathbf{Q}}^T\mathbf{Q})\mathbf{tQ}^T - \mathbf{QtQ}^T\mathbf{Q}(\boldsymbol{l}^T - \mathbf{Q}^T\dot{\mathbf{Q}})\mathbf{Q}^T \\ &= \dot{\mathbf{Q}}\mathbf{tQ}^T + \mathbf{Q}\dot{\mathbf{t}}\mathbf{Q}^T + \mathbf{Qt}\dot{\mathbf{Q}}^T - \mathbf{Q}\boldsymbol{l}\mathbf{tQ}^T - \mathbf{QQ}^T\dot{\mathbf{Q}}^T\mathbf{tQ}^T - \mathbf{Qt}\boldsymbol{l}^T\mathbf{Q}^T - \mathbf{Qt}\dot{\mathbf{Q}}^T\mathbf{QQ}^T \\ &= \mathbf{Q}(\dot{\mathbf{t}} - \boldsymbol{l}\mathbf{t} - \mathbf{t}\boldsymbol{l}^T)\mathbf{Q}^T\end{aligned} \tag{4.52}$$

4.5 Corotational Rate Tensors

The convected time-derivatives satisfy the objectivity. However, the objectivity is satisfied even in specialized convected time-derivatives as will be shown below by the particular case in which only the rotation of material is considered.

Let the spin tensors obeying the following coordinate transformation which is seen in Eqs. (4.17), (4.19), etc. be designated by the symbol $\boldsymbol{\omega}\ (=-\boldsymbol{\omega}^T)$ collectively.

$$\boldsymbol{\omega}^* = \mathbf{Q}(\boldsymbol{\omega}-\boldsymbol{\Omega})\mathbf{Q}^T = \mathbf{Q}\boldsymbol{\omega}\mathbf{Q}^T + \overline{\boldsymbol{\Omega}} \tag{4.53}$$

It is readily known from Eqs. (4.46) and (4.48) that the rates of vector and tensor obtained by replacing the velocity gradient $\boldsymbol{l}$ to the spin tensor $\boldsymbol{\omega}$, i.e. by ignoring the rate of deformation are described as follows:

$$\boxed{\overset{\circ}{\mathbf{v}} = \overset{\bullet}{\mathbf{v}} - \boldsymbol{\omega}\mathbf{v}} \tag{4.54}$$

$$\boxed{\overset{\circ}{\mathbf{t}} = \overset{\bullet}{\mathbf{t}} - \boldsymbol{\omega}\mathbf{t} + \mathbf{t}\boldsymbol{\omega}} \tag{4.55}$$

which obey the objective transformation rules and are referred to as the *corotational rate* or *corotational time-derivative*. Their fulfillment of objectivity is obvious from the proof for the convected time-derivatives described in the foregoing, noting that $\boldsymbol{\omega}$ obeys the identical transformation rule to that of $\boldsymbol{l}$. However, Eqs. (4.54) and (4.55) are inapplicable to deformation analysis as far as $\boldsymbol{\omega}$ is not given explicitly as a physical quantity. Needless to say, they must be chosen so as to reflect the rotational rate of material appropriately. In what follows, typical explicit corotational rate vectors and tensors will be shown.

The replacement of $\mathbf{F}=\mathbf{R}$ in Eqs. (4.46) and (4.48) leads Eqs. (4.54) and (4.55) to the corotational rate with the relative spin $\boldsymbol{\omega}=\boldsymbol{\Omega}^R$ tensor in Eq. (2.85) as follows:

$$\boxed{\overset{\circ}{\mathbf{v}}{}^R \equiv \mathbf{R}(\mathbf{R}^T\mathbf{V})^{\bullet} = \overset{\bullet}{\mathbf{v}} - \boldsymbol{\Omega}^R\mathbf{v}} \tag{4.56}$$

$$\boxed{\overset{\circ}{\mathbf{t}}{}^R \equiv \mathbf{R}(\mathbf{R}^T\mathbf{t}\mathbf{R})^{\bullet}\mathbf{R}^T = \overset{\bullet}{\mathbf{t}} - \boldsymbol{\Omega}^R\mathbf{t} + \mathbf{t}\boldsymbol{\Omega}^R} \tag{4.57}$$

$\overset{\circ}{\mathbf{t}}{}^R$ in Eq. (4.57) is called the *Green-Naghdi rate* (Green-Naghdi 1965). The Green-Naghdi rate depends on the initial value of $\mathbf{R}$ describing the rotation of material. In other words, it is influenced by the estimation of initial state of rotation even in isotropic materials. Therefore, it lacks the objectivity in the broad physical sense, which requires the independence of deformation behavior on the rigid-body rotation.

Further, by choosing $\boldsymbol{\omega}=\mathbf{w}(={}_t\overset{\bullet}{\mathbf{R}}(t))$, i.e. the relative spin in Eq. (2.81), Eqs. (4.54) and (4.55) lead to

$$\boxed{\overset{\circ}{\mathbf{v}}^{w} \equiv \dot{\mathbf{v}} - \mathbf{w}\mathbf{v}} \quad (4.58)$$

$$\boxed{\overset{\circ}{\mathbf{t}}^{w} \equiv \dot{\mathbf{t}} - \mathbf{w}\mathbf{t} + \mathbf{t}\mathbf{w}} \quad (4.59)$$

Equation (4.59) is called the *Zaremba-Jaumann rate* (Zaremba 1903; Jaumann 1911).

The accurate numerical time-integration scheme of the corotational rate tensors will be described in Sect. 20.10.

The objectivity is the common requirement for constitutive equations. One can make various objective rates which are given by the convected rates and classified as shown in Eq. (4.46) for vector and in Eq. (4.48) for second-order tensor. However, the other consideration is required for the judgment which one of them is appropriate. In fact, the objective rates described above are determined solely by a geometrical change of outward appearance of material. On the other hand, the spin which reflects the mechanical response is the spin of substructure (microstructure) in material. However, the substructure is invisible from the outward appearance. Generally speaking, the spin of the substructure is not so large as that given by the continuum spin. An explicit form of the spin of substructure in the elastoplastic deformation will be described in Chap. 16.

4.6 Various Stress Rate Tensors

Various rates of the Cauchy stress $\boldsymbol{\sigma}$ and the Kirchhoff stress $\boldsymbol{\tau}\,(=J\boldsymbol{\sigma})$ can be obtained from the aforementioned convected and corotational time-derivatives as will be shown in this section. Corotational time-derivative with a spin tensor is designated by the symbol $(^{\circ})$ as shown in the last section. On the other and, the time-derivatives other than corotational time derivatives are designated by the symbol $(^{\Delta})$.

(a) Contravariant convected rates

Based on Eq. $(4.48)_1$, the contravariant convected rate of the Cauchy stress $\boldsymbol{\sigma}$ is given by

$$\overset{\Delta}{\boldsymbol{\sigma}}^{Ol} \equiv \overset{\overset{\rightleftarrows}{\bullet}}{\boldsymbol{\sigma}}^{gg} = \mathbf{F}(\mathbf{F}^{-1}\boldsymbol{\sigma}\,\mathbf{F}^{-T})^{\bullet}\mathbf{F}^{T} = \mathbf{F}\,\overset{\bullet}{\overline{\mathbf{S}/J}}\,\mathbf{F}^{T} = \dot{\boldsymbol{\sigma}} - \boldsymbol{l}\boldsymbol{\sigma} - \boldsymbol{\sigma}\boldsymbol{l}^{T}(= \overset{\Delta}{\boldsymbol{\sigma}}^{OlT}) \quad (4.60)$$

which is termed the *Oldroyd rate of Cauchy stress* (Oldroyd 1950). Likewise, it holds for the Kirchhoff stress that

$$\overset{\Delta}{\boldsymbol{\tau}}^{Ol} \equiv \overset{\rightleftarrows}{\boldsymbol{\tau}}^{gg} = \mathbf{F}(\mathbf{F}^{-1}\boldsymbol{\tau}\mathbf{F}^{-T})^{\bullet}\mathbf{F}^{T} = \mathbf{F}\dot{\mathbf{S}}\mathbf{F}^{T} = \dot{\boldsymbol{\tau}} - \boldsymbol{l}\boldsymbol{\tau} - \boldsymbol{\tau}\boldsymbol{l}^{T} (= \overset{\Delta}{\boldsymbol{\tau}}^{OlT}) \tag{4.61}$$

which is termed the Oldroyd rate of Kirchhoff stress.

Further,

$$\begin{aligned}\overset{\Delta}{\boldsymbol{\sigma}}^{Tr} &\equiv J^{-1}\overset{\Delta}{\boldsymbol{\tau}}^{Ol} = J^{-1}\overset{\rightleftarrows}{(J\boldsymbol{\sigma})}^{gg} = J^{-1}\mathbf{F}(\mathbf{F}^{-1}(J\boldsymbol{\sigma})\mathbf{F}^{-T})^{\bullet}\mathbf{F}^{T} = J^{-1}\mathbf{F}\dot{\mathbf{S}}\mathbf{F}^{T} = \overset{\Delta}{\boldsymbol{\sigma}}^{Ol} + \boldsymbol{\sigma}\,\mathrm{tr}\,\mathbf{d}\\ &= \dot{\boldsymbol{\sigma}} - \boldsymbol{l}\boldsymbol{\sigma} - \boldsymbol{\sigma}\boldsymbol{l}^{T} + \boldsymbol{\sigma}\,\mathrm{tr}\,\mathbf{d}\ (= \overset{\circ}{\boldsymbol{\sigma}}^{TrT})\end{aligned} \tag{4.62}$$

is termed the *Truesdell rate of Cauchy stress*.

(b) Covariant-contravariant convected rates

The covariant-contravariant convected rate of the Kirchhoff stress $\boldsymbol{\tau}$ is given from Eq. (4.48)$_3$ as

$$\overset{\Delta}{\boldsymbol{\tau}} \equiv \overset{\rightleftarrows}{\boldsymbol{\tau}}_{g}^{\bullet g} = \mathbf{F}^{-T}(\mathbf{F}^{T}\boldsymbol{\tau}\mathbf{F}^{-T})^{\bullet}\mathbf{F}^{T} = \dot{\boldsymbol{\tau}} + \boldsymbol{l}^{T}\boldsymbol{\tau} - \boldsymbol{\tau}\boldsymbol{l}^{T}\ (\neq \overset{\Delta}{\boldsymbol{\tau}}^{T}) \tag{4.63}$$

The particular case of the rate in Eq. (4.63) is given by

$${}_{\Pi}\overset{\Delta}{\boldsymbol{\tau}} \equiv \overset{\rightleftarrows}{\boldsymbol{\tau}}_{\hat{g}}^{\bullet g} = (\boldsymbol{\tau}\mathbf{F}^{-T})^{\bullet}\mathbf{F}^{T} = \dot{\mathbf{\Pi}}\mathbf{F}^{T} = \dot{\boldsymbol{\tau}} - \boldsymbol{\tau}\boldsymbol{l}^{T}\ (\neq {}_{\Pi}\overset{\Delta}{\boldsymbol{\tau}}^{T}) \tag{4.64}$$

which is termed the *relative 1st Piola-Kirchhoff stress rate*. The following stress rate is defined as the nominal stress rate in Eqs. (3.36) and (3.37).

$${}_{\Pi}\overset{\Delta}{\boldsymbol{\sigma}} \equiv \frac{1}{J}\,{}_{\Pi}\overset{\circ}{\boldsymbol{\tau}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{\sigma}\boldsymbol{l}^{T} + \boldsymbol{\sigma}\ \mathrm{tr}\,\mathbf{d}\ (\neq {}_{\Pi}\overset{\Delta}{\boldsymbol{\sigma}}^{T}) \tag{4.65}$$

which is the nominal stress rate and used for the equilibrium equation of rate-form in the current configuration as described in Eq. (3.41).

(c) Covariant convected rates

The *covariant convected rate* of Cauchy stress is given from Eq. (4.48)$_4$ as

$$\overset{\Delta}{\boldsymbol{\sigma}}^{CR} \equiv \overset{\rightleftarrows}{\boldsymbol{\sigma}}_{gg} = \mathbf{F}^{-T}(\mathbf{F}^{T}\boldsymbol{\sigma}\mathbf{F})^{\bullet}\mathbf{F}^{-1} = \dot{\boldsymbol{\sigma}} + \boldsymbol{l}^{T}\boldsymbol{\sigma} + \boldsymbol{\sigma}\boldsymbol{l}\ (= \overset{\Delta}{\boldsymbol{\sigma}}^{CRT}) \tag{4.66}$$

which is termed the *Cotter-Rivlin rate* of Cauchy stress (Cotter and Rivlin 1995). Likewise, the covariant convected rate of Kirchhoff stress is given by

$$\overset{\Delta}{\boldsymbol{\tau}}^{CR} \equiv \overset{\leftrightarrows}{\dot{\boldsymbol{\tau}}}_{gg} = \mathbf{F}^{-T}(\mathbf{F}^T\boldsymbol{\tau}\mathbf{F})^{\bullet}\mathbf{F}^{-1} = \dot{\boldsymbol{\tau}} + \boldsymbol{l}^T\boldsymbol{\tau} + \boldsymbol{\tau}\,\boldsymbol{l}\,(= \overset{\Delta}{\boldsymbol{\tau}}^{CRT}) \tag{4.67}$$

(d) Corotational rates

The following stress rate based on Eq. (4.57) is termed the *Green-Naghdi rate of Cauchy stress* (Green and Naghdi 1965).

$$\overset{\circ}{\boldsymbol{\sigma}}^R \equiv \overset{\leftrightarrows}{\dot{\boldsymbol{\sigma}}}^R = \mathbf{R}(\mathbf{R}^T\boldsymbol{\sigma}\mathbf{R})^{\bullet}\mathbf{R}^T = \dot{\boldsymbol{\sigma}} - \boldsymbol{\Omega}^R\boldsymbol{\sigma} + \boldsymbol{\sigma}\,\boldsymbol{\Omega}^R\,(= \overset{\circ}{\boldsymbol{\sigma}}^{RT}) \tag{4.68}$$

Similarly, the *Green-Naghdi rate of Kirchhoff stress* is given by

$$\overset{\circ}{\boldsymbol{\tau}}^R \equiv \overset{\leftrightarrows}{\dot{\boldsymbol{\tau}}}^R = \mathbf{R}(\mathbf{R}^T\boldsymbol{\tau}\boldsymbol{R})^{\bullet}\mathbf{R}^T = \dot{\boldsymbol{\tau}} - \boldsymbol{\Omega}^R\boldsymbol{\tau} + \boldsymbol{\tau}\boldsymbol{\Omega}^R\,(= \overset{\circ}{\boldsymbol{\tau}}^{RT}) \tag{4.69}$$

The stress rate based on Eq. (4.59) is given by

$$\overset{\circ}{\boldsymbol{\sigma}}^w \equiv \dot{\boldsymbol{\sigma}} - \mathbf{w}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{w}\,(= \overset{\circ}{\boldsymbol{\sigma}}^{wT}) \tag{4.70}$$

which is termed the *Zaremba-Jaumann rate of Cauchy stress* (Zaremba 1903; Jaumann 1911). Likewise, it follows for the Kirchhoff stress that

$$\overset{\circ}{\boldsymbol{\tau}}^w \equiv \dot{\boldsymbol{\tau}} - \mathbf{w}\boldsymbol{\tau} + \boldsymbol{\tau}\mathbf{w}\,(= \overset{\circ}{\boldsymbol{\tau}}^{wT}) \tag{4.71}$$

The stress rate tensors described above are listed in Table 4.1. Here, it should be noted that objective rates must be used also for all internal variables in addition to objective stress rate and strain rate.

The stress rate tensors based on the convected and the corotational time-derivatives satisfy the objectivity. Therefore, the stress rate tensors shown in this section are objective quantities.

Table 4.1 Various stress rate tensors

Oldroyd rate of Cauchy stress: $\overset{\circ}{\boldsymbol{\sigma}}^{Ol} \equiv \dot{\boldsymbol{\sigma}} - \boldsymbol{l}\boldsymbol{\sigma} - \boldsymbol{\sigma}\boldsymbol{l}^T\,(= \overset{\circ}{\boldsymbol{\sigma}}^{OlT})$
Truesdell rate of Cauchy stress: $\overset{\circ}{\boldsymbol{\sigma}}^{Tr} \equiv \overset{\circ}{\boldsymbol{\sigma}}^{Ol} + \boldsymbol{\sigma}\,\mathrm{tr}\,\mathbf{d} = \dot{\boldsymbol{\sigma}} - \boldsymbol{l}\boldsymbol{\sigma} - \boldsymbol{\sigma}\boldsymbol{l}^T + \boldsymbol{\sigma}\,\mathrm{tr}\,\mathbf{d}\,(= \overset{\circ}{\boldsymbol{\sigma}}^{TrT})$
Covariant-contravariant convected rate of the Kirchhoff stress: $\overset{\Delta}{\boldsymbol{\tau}} \equiv \dot{\boldsymbol{\tau}} + \boldsymbol{l}^T\boldsymbol{\tau} - \boldsymbol{\tau}\boldsymbol{l}^T\ (\neq \overset{\Delta}{\boldsymbol{\tau}}^T)$
Nominal stress rate: ${}_{\Pi}\overset{\Delta}{\boldsymbol{\sigma}} \equiv \frac{1}{J}{}_{\Pi}\overset{\Delta}{\boldsymbol{\tau}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{\sigma}\boldsymbol{l}^T + \boldsymbol{\sigma}\ \mathrm{tr}\,\mathbf{d}\,(\neq {}_{\Pi}\overset{\Delta}{\boldsymbol{\sigma}}^T)$
Cotter-Rivlin rate of Cauchy stress: $\overset{\Delta}{\boldsymbol{\sigma}}^{CR} \equiv \dot{\boldsymbol{\sigma}} + \boldsymbol{l}^T\boldsymbol{\sigma} + \boldsymbol{\sigma}\boldsymbol{l}\,(= \overset{\Delta}{\boldsymbol{\sigma}}^{CRT})$
Green-Naghdi rate of Cauchy stress: $\overset{\circ}{\boldsymbol{\sigma}}^R \equiv \dot{\boldsymbol{\sigma}} - \boldsymbol{\Omega}^R\boldsymbol{\sigma} + \boldsymbol{\sigma}\boldsymbol{\Omega}^R\,(= \overset{\circ}{\boldsymbol{\sigma}}^{RT})$
Zaremba-Jaumann rate of Cauchy stress: $\overset{\circ}{\boldsymbol{\sigma}}^w \equiv \dot{\boldsymbol{\sigma}} - \mathbf{w}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{w}\,(= \overset{\circ}{\boldsymbol{\sigma}}^{wT})$

The above-mentioned rate tensors are used for rate-type constitutive equations. In particular, the Oldroyd rate appears in the current rate form of the hyperelastic constitutive equation. The other rates are used for expressions of its variations as will be described in Chap. 5. The corotational time-derivatives are used in the derivation of the consistency condition from the yield condition.

Constitutive equation for irreversible deformation exhibiting the loading-path dependence has to be formulated in a rate-form in terms of objective stress rate and objective strain rate. On the other hand, a hyperelastic constitutive equation must be formulated in terms of objective stress and objective strain. Unfortunately, however, it is difficult to find an objective rate of strain, although various objective rates of stress have been found as explained above in detail. In this situation, the particular spin, called the *logarithmic spin*, by which the corotational rate of the Eulerian-logarithmic (Hencky) strain $\ln \mathbf{V}$ in Eq. $(2.68)_2$ coincides with the strain rate **d**, was proposed and the hyperelastic constitutive equation was derived from the hypoelastic constitutive equation in terms of the logarithmic rate of Cauchy stress and the strain rate **d** by Xiao and his colleagues (Xiao 1995; Xiao et al. 1997, 1999). The formulation of the logarithmic spin is explained in **Appendix C**.

4.7 Time Derivative of Scalar-Valued Tensor Function

Scalar-valued tensor functions of stress and internal variables appear often in continuum mechanics as seen in the strain energy function and the yield function. Then, the time-derivative of scalar-valued tensor function is required in order to derive the rate-type relation of variables, e.g. the consistency condition of yield condition. The time-derivative of scalar function is independent of rigid-body rotation and thus it can be given primarily by its material-time derivative. Here, it should be noticed that the internal variables are formulated by the objective time-derivatives and thus the consistency condition must be transformed to the objective time-derivative. It can be proved that the material-time derivative of scalar-valued tensor function is transformed only to its corotational time-derivative. This fact would seem physically obvious but it must be proved mathematically. To this end, its mathematically exact proof for scalar valued function of general tensor will be given below, referring to the previous studies by Dafalias (1985, 1998; 2011) for vector and second-order tensor and Hashiguchi (2007b) for general tensor.

The corotational rate of general tensor is defined by extending Eq. (4.54) for the vector and Eq. (4.55) for the second-order tensor as follows:

$$\overset{\circ}{\mathbf{t}} = \widehat{\mathbf{R}}\left[\!\left[(\widehat{\mathbf{R}}^T [\![\mathbf{t}]\!]\ )^{\bullet} \right]\!\right] \;\; \overset{\circ}{\mathbf{t}} = \widehat{\mathbf{R}}\left[\!\left[(\widehat{\mathbf{R}}^T [\![\mathbf{t}]\!]\ )^{\bullet} \right]\!\right] \tag{4.72}$$

where use is made of the symbol $[\![\ ]\!]$ for general objective transformation in Eq. (1.84) with the replacement $\mathbf{Q} \rightarrow \widehat{\mathbf{R}}^T$. Here, noting $\dot{f}(\mathbf{t}) = \dot{f}(\widehat{\mathbf{R}}^T[\![\mathbf{t}]\!])$ because of the requirement $f(\mathbf{t}) = f(\widehat{\mathbf{R}}^T[\![\mathbf{t}]\!])$ for scalar variable, one has

$$\begin{aligned}\dot{f}(\mathbf{t}) &= \dot{f}(\widehat{\mathbf{R}}^T[\![\mathbf{t}]\!]) = \frac{\partial f(\widehat{\mathbf{R}}^T[\![\mathbf{t}]\!])}{\partial(\widehat{\mathbf{R}}^T[\![\mathbf{t}]\!])} * (\widehat{\mathbf{R}}^T[\![\mathbf{t}]\!])^{\bullet} = \widehat{\mathbf{R}}^T\left[\!\!\left[\frac{\partial f(\mathbf{t})}{\partial \mathbf{t}}\right]\!\!\right] * (\widehat{\mathbf{R}}^T[\![\mathbf{t}]\!])^{\bullet} \\ &= \frac{\partial f(\mathbf{t})}{\partial \mathbf{t}} * \widehat{\mathbf{R}}\left[\!\!\left[(\widehat{\mathbf{R}}^T[\![\mathbf{t}]\!])^{\bullet}\right]\!\!\right]\end{aligned} \tag{4.73}$$

$$\begin{aligned}\dot{f}(t_{w_1w_2\ldots}) &= \frac{\partial f(\widehat{R}_{u_1w_1}\widehat{R}_{u_2w_2}\ldots t_{u_1u_2\ldots})}{\partial(\widehat{R}_{s_1p_1}\widehat{R}_{s_2p_2}\ldots t_{s_1s_2\ldots})}(\widehat{R}_{v_1p_1}\widehat{R}_{v_2p_2}\ldots t_{v_1v_2\ldots})^{\bullet} \\ &= \frac{\partial f(t_{w_1w_2\ldots})}{\partial t_{s_1s_2\ldots}}\widehat{R}_{s_1p_1}\widehat{R}_{s_2p_2}\ldots(\widehat{R}_{v_1p_1}\widehat{R}_{v_2p_2}\ldots t_{v_1v_2\ldots})^{\bullet}\end{aligned} \tag{4.74}$$

where the symbol * designates the full contraction between derivative components in order between derivative components, i.e. $\mathbf{t} * \mathbf{s} = t_{p_1p_2\cdots p_m}s_{p_1p_2\cdots p_m}$. The derivation of Eq. (4.73) is shown for vector and second-order tensor as follows:

$$\dot{f}(\mathbf{v}) = \dot{f}(\widehat{\mathbf{R}}^T\mathbf{v}) = \frac{\partial f(\widehat{\mathbf{R}}^T\mathbf{v})}{\partial(\widehat{\mathbf{R}}^T\mathbf{v})}\bullet(\widehat{\mathbf{R}}^T\mathbf{v})^{\bullet} = \widehat{\mathbf{R}}^T\frac{\partial f(\mathbf{v})}{\partial \mathbf{v}}\bullet(\widehat{\mathbf{R}}^T\mathbf{v})^{\bullet} = \frac{\partial f(\mathbf{v})}{\partial \mathbf{v}}\bullet\widehat{\mathbf{R}}(\widehat{\mathbf{R}}^T\mathbf{v})^{\bullet} \tag{4.75}$$

$$\begin{aligned}\dot{f}(\mathbf{t}) &= \dot{f}(\widehat{\mathbf{R}}^T\mathbf{t}\widehat{\mathbf{R}}) = \frac{\partial f(\widehat{\mathbf{R}}^T\mathbf{t}\widehat{\mathbf{R}})}{\partial(\widehat{\mathbf{R}}^T\mathbf{t}\widehat{\mathbf{R}})} : (\widehat{\mathbf{R}}^T\mathbf{t}\widehat{\mathbf{R}})^{\bullet} = \widehat{\mathbf{R}}^T\frac{\partial f(\mathbf{t})}{\partial \mathbf{t}}\widehat{\mathbf{R}}^T : (\widehat{\mathbf{R}}^T\mathbf{t}\widehat{\mathbf{R}})^{\bullet} \\ &= \frac{\partial f(\mathbf{t})}{\partial \mathbf{t}} : \widehat{\mathbf{R}}(\widehat{\mathbf{R}}^T\mathbf{t}\widehat{\mathbf{R}})^{\bullet}\widehat{\mathbf{R}}^T\end{aligned} \tag{4.76}$$

Equations (4.73), (4.75) and (4.76) can be satisfied by the corotational rate in Eq. (4.72) amongst objective rates. Then, we have the following relation.

$$\begin{aligned}\dot{f}(\mathbf{t}) &= \frac{\partial f(\mathbf{t})}{\partial \mathbf{t}} * \dot{\mathbf{t}} = \frac{\partial f(\mathbf{t})}{\partial \mathbf{t}} * \overset{\circ}{\mathbf{t}} \\ \dot{f}(t_{q_1q_2\cdots q_m}) &= \frac{\partial f(t_{q_1q_2\cdots q_m})}{\partial t_{p_1p_2\cdots p_m}}\dot{t}_{p_1p_2\cdots p_m} = \frac{\partial f(t_{q_1q_2\cdots q_m})}{\partial t_{p_1p_2\cdots p_m}}\overset{\circ}{t}_{p_1p_2\cdots p_m}\end{aligned} \tag{4.77}$$

which is described for vector and second-order tensor as follows:

$$\overset{\bullet}{f}(\mathbf{v}) = \frac{\partial f(\mathbf{v})}{\partial \mathbf{v}} \cdot \overset{\circ}{\mathbf{v}}, \quad \overset{\bullet}{f}(v_r) = \frac{\partial f(v_r)}{\partial v_i} \overset{\circ}{v}_i \tag{4.78}$$

$$\overset{\bullet}{f}(\mathbf{t}) = \frac{\partial f(\mathbf{t})}{\partial \mathbf{t}} : \overset{\circ}{\mathbf{t}}, \quad \overset{\bullet}{f}(t_{rs}) = \frac{\partial f(t_{rs})}{\partial t_{ij}} \overset{\circ}{t}_{ij} \tag{4.79}$$

It follows from Eq. (4.74) that

$$\begin{aligned} &\frac{\partial f(t_{w_1w_2w_3\ldots})}{\partial t_{s_1s_2s_3\cdots}} \widehat{R}_{s_1p_1}\widehat{R}_{s_2p_2}\widehat{R}_{s_3p_3}\cdots(\overset{\bullet}{\widehat{R}}_{v_1p_1}\widehat{R}_{v_2p_2}\widehat{R}_{v_3p_3}\cdots+\widehat{R}_{p_1v_1}\overset{\bullet}{\widehat{R}}_{p_2v_2}\widehat{R}_{p_3v_3}\cdots+\cdots)t_{v_1v_2v_3\cdots} \\ &= \frac{\partial f(t_{w_1w_2w_3\cdots})}{\partial t_{s_1s_2s_3\cdots}}(\widehat{R}_{s_1p_1}\overset{\bullet}{\widehat{R}}_{v_1p_1}\delta_{s_2v_2}\delta_{s_3v_3}\cdots+\delta_{s_1v_1}\widehat{R}_{s_2p_2}\overset{\bullet}{\widehat{R}}_{v_2p_2}\delta_{s_3v_3}\cdots+\cdots)t_{v_1v_2v_3\cdots} \\ &= \frac{\partial f(t_{w_1w_2w_3\cdots})}{\partial t_{s_1s_2s_3\cdots}}(\omega_{v_1s_1}t_{v_1s_2s_3\cdots}\cdots+\omega_{s_2v_2}t_{s_1v_2s_3\cdots}\cdots+\cdots)=0 \end{aligned} \tag{4.80}$$

which is reduced for vector and second-order tensor as follows:

$$\frac{\partial f(\mathbf{v})}{\partial \mathbf{v}} \cdot \boldsymbol{\omega}\mathbf{v} = \frac{\partial f(v_u)}{\partial v_r}\omega_{ri}v_i = v_i\frac{\partial f(v_u)}{\partial v_r}\omega_{ri} = 0, \text{ i.e.} \quad \mathrm{tr}\left[\left(\mathbf{v}\otimes\frac{\partial f(\mathbf{v})}{\partial \mathbf{v}}\right)\boldsymbol{\omega}\right] = 0 \tag{4.81}$$

$$\mathrm{tr}\left[\left(\frac{\partial f(\mathbf{t})}{\partial \mathbf{t}}\mathbf{t}^T - \mathbf{t}^T\frac{\partial f(\mathbf{t})}{\partial \mathbf{t}}\right)\boldsymbol{\omega}\right] = 0 \tag{4.82}$$

The fulfillments of Eq. (4.81) and (4.82) require for the tensors in the brackets ( ) to be zero or symmetric tensor, while Dafalias (1998) has required for the latter to be zero tensor. The fulfillment of Eq. (4.81) is easily known for $f(\mathbf{v}) = s\mathbf{v}\cdot\mathbf{v}$ because of $\partial f(\mathbf{v})/\partial\mathbf{v} = 2s\mathbf{v}$ with Eq. $(1.130)_3$, and that of Eq. (4.82) for the second-order symmetric tensor $\mathbf{t}$ because of $(\partial f(\mathbf{t})/\partial\mathbf{t})\mathbf{t}^T - \mathbf{t}^T(\partial f(\mathbf{t})/\partial\mathbf{t}) = \mathbf{O}$.

Equation (4.77) is extended for plural variables as follows:

$$\begin{aligned} \overset{\bullet}{f}(\mathbf{t}_1,\mathbf{t}_2,\cdots) &= \frac{\partial f(\mathbf{t}_1,\mathbf{t}_2,\cdots)}{\partial \mathbf{t}_1} * \overset{\bullet}{\mathbf{t}}_1 + \frac{\partial f(\mathbf{t}_1,\mathbf{t}_2,\cdots)}{\partial \mathbf{t}_2} * \overset{\bullet}{\mathbf{t}}_2 + \cdots \\ &= \frac{\partial f(\mathbf{t}_1,\mathbf{t}_2,\cdots)}{\partial \mathbf{t}_1} * \overset{\circ}{\mathbf{t}}_1 + \frac{\partial f(\mathbf{t}_1,\mathbf{t}_2,\cdots)}{\partial \mathbf{t}_2} * \overset{\circ}{\mathbf{t}}_2 + \cdots = \overset{\circ}{f}(\mathbf{t}_1,\mathbf{t}_2,\cdots) \end{aligned} \tag{4.83}$$

which is shown for the function of two tensors in Belytschko et al. (2014). Here, it should be noted that the mathematical property does not hold for each term, i.e.

$$\frac{\partial f(\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_{\mathrm{m}}}{\partial \mathbf{t}_i} * \dot{\mathbf{t}}_i \neq \frac{\partial f(\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_{\mathrm{m}})}{\partial \mathbf{t}_i} * \mathring{\mathbf{t}}_i \text{ (no sum)} \tag{4.84}$$

Scalar-valued functions must be independent of rigid-body rotation so that their material-time derivative possess a unique value which coincides with their corotational time-derivative as can be confirmed by the above-mentioned proof. However, they do not lead to the other convected time-derivatives which depend on the rate of deformation, i.e. velocity gradient. Therefore, corotational time-derivatives can be adopted in the time-derivatives of scalar functions in constitutive relations, e.g. a strain energy function and a yield function of tensors in the current configuration but convected time-derivatives other than corotational rates cannot be adopted in them.

The most popular scalar-valued tensor functions are the principal invariants of tensor. Principal invariants of tensor are described by three independent principal invariants of tensor. Then, the material-time derivatives of the principal invariants are transformed to the corotational time-derivatives merely by replacing all the material time-derivatives of tensor to the corotational time-derivatives of tensor as will be written below.

It follows from Eqs. (4.79) and (4.83) that

$$\left.\begin{aligned} \dot{\bar{\mathrm{I}}} &= (\mathrm{tr}\mathbf{t})^{\bullet} = (\mathrm{tr}\mathbf{t})^{\circ} = \mathbf{I} : \mathring{\mathbf{t}} = \mathrm{tr}\,\mathring{\mathbf{t}} \\ \dot{\bar{\mathrm{II}}} &= (\mathrm{tr}\mathbf{t}^2)^{\bullet} = (\mathrm{tr}\mathbf{t}^2)^{\circ} = 2\mathbf{t}^T : \mathring{\mathbf{t}} = 2(\mathrm{tr}(\mathbf{t}\,\mathring{\mathbf{t}}) \\ \dot{\bar{\mathrm{III}}} &= (\mathrm{tr}\mathbf{t}^3)^{\bullet} = (\mathrm{tr}\mathbf{t}^3)^{\circ} = 3\mathbf{t}^{2T} : \mathring{\mathbf{t}} = 3(\mathrm{tr}(\mathbf{t}^2\,\mathring{\mathbf{t}}) \end{aligned}\right\} \tag{4.85}$$

$$\left.\begin{aligned} \dot{\mathrm{I}} &= (\mathrm{tr}\,\mathbf{t})^{\bullet} = \mathrm{tr}\,\mathring{\mathbf{t}} \\ \dot{\mathrm{II}} &= \frac{1}{2}[(\mathrm{tr}\,\mathbf{t})^2 - \mathrm{tr}\,\mathbf{t}^2]^{\bullet} = (\mathrm{tr}\,\mathbf{t})\mathrm{tr}\,\mathring{\mathbf{t}} - \mathrm{tr}\,(\mathbf{t}\mathring{\mathbf{t}}) \\ \dot{\mathrm{III}} &= (\det\mathbf{t})^{\bullet} = (\det\mathbf{t})\mathbf{t}^{-T} : \mathring{\mathbf{t}} = (\det\mathbf{t})\mathrm{tr}\,(\mathbf{t}^{-1}\mathring{\mathbf{t}}) \end{aligned}\right\} \tag{4.86}$$

for the principal invariants in Eqs. (1.183) and (1.178)–(1.180), noting Eqs. (1.294) and (1.295). Further, it holds that

$$(\mathbf{t}_1 : \mathbf{t}_2)^{\bullet} = \mathring{\mathbf{t}}_1 : \mathbf{t}_2 + \mathbf{t}_1 : \mathring{\mathbf{t}}_2 = \mathrm{tr}(\mathbf{t}_2^T\mathring{\mathbf{t}}_1) + \mathrm{tr}(\mathbf{t}_1^T\mathring{\mathbf{t}}_2) \tag{4.87}$$

for the two tensor variables. If $\mathbf{t}_1$ and $\mathbf{t}_2$ are commutative (possessing same principal directions) leading to $\mathbf{t}_1\mathbf{t}_2 = \mathbf{t}_2\mathbf{t}_1 = \mathbf{t}_1\mathbf{t}_2^T = \mathbf{t}_2^T\mathbf{t}_1$, one has

$$\mathbf{t}_1 : \dot{\mathbf{t}}_2 = \mathbf{t}_1 : \mathring{\mathbf{t}}_2, \dot{\mathbf{t}}_1 : \mathbf{t}_2 = \mathring{\mathbf{t}}_1 : \mathbf{t}_2 \tag{4.88}$$

noting

$$\mathbf{t}_1 : (\mathbf{t}_2\omega - \omega\mathbf{t}_2) = tr\{\mathbf{t}_1(\mathbf{t}_2\boldsymbol{\omega})^T\} - tr\{\mathbf{t}_1(\omega\mathbf{t}_2)^T\} = -\mathrm{tr}(\mathbf{t}_2^T\mathbf{t}_1\boldsymbol{\omega}) + \mathrm{tr}(\mathbf{t}_1\mathbf{t}_2^T\boldsymbol{\omega}) = 0 \tag{4.89}$$

All the equations in Eqs. (4.85)–(4.87) hold for arbitrary corotational tensors as proved here, although they are written explicitly for the Zaremba-Jaumann rate in some literatures (e.g. Prager 1961; Belytschko et al. 2014).

4.8 Work Conjugacy

The work rate done for the unit volume in the current configuration is given by $\boldsymbol{\sigma}:\mathbf{d}(=\sigma_{ij}d_{ij})$. Designating the infinitesimal volumes in a specific region of material in the reference and the current configurations as dV and $dv(=JdV)$, respectively, the work rate $\overset{\bullet}{w}_0$ done per the unit reference volume, i.e. a certain volume element possessing a fixed mass is given from Eqs. (2.35), (2.45), (2.75), (2.80), (2.128), (3.13), (3.19) and (3.23) as follows:

$$\overset{\bullet}{w}_0 = \boldsymbol{\sigma}:\mathbf{d}dv/dV = \mathrm{tr}(\boldsymbol{\sigma}\,\mathbf{d})J = \mathrm{tr}(\boldsymbol{\tau}\boldsymbol{l})$$
$$= \begin{cases} \mathrm{tr}[(\boldsymbol{\tau}\mathbf{F}^{-T})(\boldsymbol{l}\mathbf{F})^T] = \mathrm{tr}(\boldsymbol{\Pi}\overset{\bullet}{\mathbf{F}}^T) \\ \mathrm{tr}[(\mathbf{F}^{-1}\boldsymbol{\tau}\mathbf{F}^{-T})(\mathbf{F}^T\mathbf{d}\mathbf{F})] = \mathrm{tr}(\mathbf{S}\overset{\bullet}{\boldsymbol{E}}) = \mathrm{tr}(\mathbf{S}\overset{\bullet}{\mathbf{C}}/2) \\ \mathrm{tr}(\mathfrak{B}\,\overset{\bullet}{\mathbf{B}}) \end{cases}$$

leading to

$$\boxed{\overset{\bullet}{w}_0 = J\boldsymbol{\sigma}:\mathbf{d} = \boldsymbol{\tau}:\mathbf{d} = \boldsymbol{\Pi}:\overset{\bullet}{\mathbf{F}} = \mathbf{S}:\overset{\bullet}{\boldsymbol{E}} = \mathbf{S}:\overset{\bullet}{\mathbf{C}}/2 = \mathfrak{B}:\overset{\bullet}{\mathbf{B}}} \tag{4.90}$$

where

$$\boxed{\mathfrak{B} \equiv \frac{1}{2}(\mathbf{SU}+\mathbf{US})} \tag{4.91}$$

which is called the *Biot stress tensor*. Equation $(4.90)_7$ is derived also as follows:

$$\begin{aligned} \mathrm{tr}(\mathbf{S}\overset{\bullet}{\boldsymbol{E}}) &= \mathrm{tr}\left[\mathbf{S}\frac{1}{2}(\mathbf{U}\overset{\bullet}{\mathbf{U}}+\overset{\bullet}{\mathbf{U}}\mathbf{U})\right] = \frac{1}{2}\mathrm{tr}(\mathbf{SU}\overset{\bullet}{\mathbf{U}}+\mathbf{US}\overset{\bullet}{\mathbf{U}}) \\ &= \frac{1}{2}\mathrm{tr}(\mathbf{SU}+\mathbf{US})\overset{\bullet}{\mathbf{U}} = \mathrm{tr}(\mathfrak{B}\overset{\bullet}{\mathbf{B}}) \end{aligned} \tag{4.92}$$

noting Eqs. (2.35), $(2.45)_1$ and (2.61).

By taking account of Eq. $(4.90)_4$ into the relation

$$\begin{aligned} \overset{\bullet}{w}_0 &= \sum_{i=1}^{3}\mathbf{f}_i\cdot\overset{\bullet}{\mathbf{n}}_i = \sum_{i=1}^{3}\boldsymbol{\Pi}\mathbf{N}_i\cdot(\mathbf{F}\mathbf{N}_i)^{\bullet} = \sum_{i=1}^{3}\boldsymbol{\Pi}\mathbf{N}_i\cdot\overset{\bullet}{\mathbf{F}}\mathbf{N}_i = \sum_{i=1}^{3}\mathbf{N}_i\cdot\overset{\bullet}{\mathbf{F}}^T\boldsymbol{\Pi}\mathbf{N}_i \\ &= (\overset{\bullet}{\mathbf{F}}^T\boldsymbol{\Pi})_{ii} = \mathrm{tr}\,(\boldsymbol{\Pi}\overset{\bullet}{\mathbf{F}}^T) \end{aligned} \tag{4.93}$$

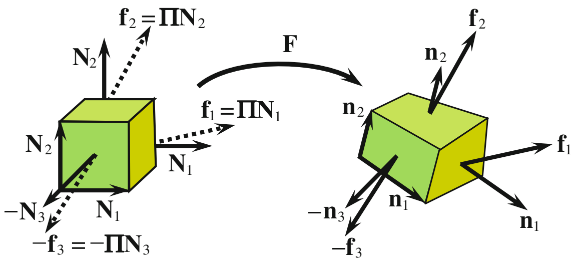

Fig. 4.2 Current cell deformed from reference orthogonal unit cell to which First Piola-Kirchhoff stress applies

we can confirm the fact that $\dot{w}_0$ designates the work rate (power) done in the current cell with the side vectors $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$ which was the orthogonal unit cell with the side vectors $(\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3)$ $(||\mathbf{N}_i||= 1)$ in the reference state, noting Eqs. (1.117), (2.15) and (3.18) with the replacements $dA \rightarrow 1$ and $d\mathbf{f} \rightarrow \mathbf{f}$, as shown in Fig. 4.2. Besides, the first Piola-Kirchhoff stress is calculated supposing that the force $\mathbf{f}_i$ on the current cell formed by the vectors $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$ applies to the reference dell formed by the vectors $(\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3)$.

The work rate reflecting the constitutive property is not concerned with a current unit volume but is concerned with a reference unit volume, noting that the mass in the current unit volume is variable but the mass in the reference unit volume is invariable. The pairs of stresses and strain rates (or rates of deformation gradient) shown in Eq. (4.90) are called the *work-conjugate pair*. Stress and strain rate tensors in the work-conjugacy pair have to be used for the formulation of constitutive equation.

References

Belytschko T, Liu WK, Moran B (2014) Nonlinear finite elements for continua and structures, 2nd edn. Wiley, New York

Bonet J, Wood RD (1997) Nonlinear continuum mechanics for finite element analysis. Cambridge Univ. Press, Cambridge

Cotter BA, Rivlin RS (1955) Tensors associated with time-dependent stresses. Quart Appl Math 13:177–182

Dafalias YF (1985) The plastic spin. J Appl Mech (ASME) 52:865–871

Dafalias YF (1998) Plastic spin: necessity or redundancy ? Int J Plast 14:909–931

Dafalias YF (2011) Finite elastic-plastic deformations: beyond the plastic spin. Theory Appl Mech 38:321–345

de Souza Neto EA, Perić D, Owen DJR (2008) Computational methods for plasticity. Wiley, Chichester

Green AE, Naghdi PM (1965) A general theory of an elastic-plastic continuum. Arch Ration Mech Anal 18:251–281

Hashiguchi K (2007a) General corotational rate tensor and replacement of material-time derivative to corotational derivative of yield function. Comput Model Eng Sci 17:55–62

Hashiguchi K (2007b) Anisotropic constitutive equation of friction with rotational hardening, In: Proceedings of 13th International Symposium Plasticity & its Current Applications, pp 34–36
Hashiguchi K (2011) General interpretations and tensor symbols for pull-back, push-forward and convected derivative, In: Proceedings of JSME 24th Computational Mechanics Conference, pp 669–671
Hashiguchi K (2012) Introduction to finite strain theory for continuum elasto-plasticity. Wiley Series in Computational Mechanics. Wiley, Chichester
Jaumann G (1911) Geschlossenes System physicalisher und chemischer Differentialgesetze. Sitzber Akad Wiss Wien (IIa) 120:385–530
Marsden JE, Hughes TJR (1983) Mathematical foundation of elasticity. Prentice-Hall, Englewood Cliffs
Oldroyd JG (1950) On the formulation of rheological equations of state. Proc R Soc Lond A200:523–541
Prager W (1961) Introduction to mechanics of continua. Ginn & Comp, Boston
Simo JC (1998) Numerical analysis and simulation of plasticity, In: Ciarlet PG, Lions JL (eds) Handbook of numerical analysis, vol 6, Part 3 (Numerical methods for solids). Elsevier, Amsterdam
Truesdell C, Noll W (1965) In: Flugge S (ed) The nonlinear field theories of mechanics, encyclopedia of physics, vol III/3. Springer, Berlin
Truesdell C, Toupin R (1960) In: Flugge S (ed) The classical field theories, encyclopedia of physics, vol III/1. Springer, Berlin
Xiao H (1995) Unified explicit basis-free expressions for time rate and conjugate stress of an arbitrary Hill strain. Int Solids Struct 32:3327–3340
Xiao H, Bruhns OT, Meyers A (1997) Logarithmic strain, logarithmic spin and logarithmic rate. Acta Mech 124:89–105
Xiao H, Bruhns OT, Meyers A (1999) Existence and uniqueness of the integrable-exactly hypoelastic equation $\overset{\circ}{\boldsymbol{\tau}}^{*} = \lambda(\mathrm{tr}\,\mathbf{D})\mathbf{I} + 2\mu\mathbf{D}$ and its significance to finite inelasticity. Acta Mech 138:31–50
Zaremba S (1903) Su une forme perfectionnee de la theorie de la relaxation, In: Bulletin International de l'Academie des Sciences de Cracovie, pp. 594–614 (in French)