Chapter 14
Damage Model

The elastic deformation due to the deformation of material particles themselves is induced even when the stress is low, the elastoplastic deformation due to the slips between material particles (dislocations of crystal lattices in case of metals and slips between soils particles in soils) is induced when the stress increases up to a certain stress (yield stress) and the damage due to the separations of material particles is induced when the stress further increases. The phenomenological formulation of the deformation up to the failure induced in the damage process within the framework of the continuum mechanics is called the *continuum damage mechanics*.

This chapter addresses the elastoplastic constitutive equation with damage which is extended by incorporating the subloading surface model within the continuum damage mechanics. It is extended further to the description of the elasto-viscoplastic deformation with the damage.

The second law of thermodynamics, i.e. the Clausius-Duhem inequality has been used for the formulation of constitutive relation (e.g. Lemaitre and Chaboche 1990; Lemaitre 1992; Lemaitre and Desmoral 2005; Murakami 2012) but it would be helpless or harmful for the formulation of plastic constitutive relation as will be explained in Chap. 21 and Sect. 21.8 in detail. In addition, the plastic constitutive relation has been formulated originally in terms of the current stress in the actual damaged configuration (e.g. Lemaitre and Chaboche 1990; Lemaitre 1992; Lemaitre and Desmoral 2005; Murakami 2012), although it should be formulated rigorously in terms of the effective stress in the fictitious undamaged configuration. Constitutive equations will be first formulated consistently in terms of the effective stress and thereafter it is transformed to the damaged actual configuration, irrelevantly to the second law of thermodynamics in this chapter.

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14.1 Damage Phenomenon

The conventional elastoplastic model premises on the postulate that the yield surface encloses a purely-elastic domain. Therefore, it describes the abrupt transition from the elastic to the plastic state so that it cannot describe realistically the softening behavior which is observed in the damage phenomenon. Further, it requires the yield-judgment whether or not the stress reaches the yield surface and the operation to pull-back the stress to the yield surface when it goes out from the yield surface in numerical calculation with finite loading increments. The existing damage models (cf. e.g. Kachanov 1958; Rabotnov 1969; Lemaitre and Chaboche 1990; Lemaitre 1992; Lemaitre and Desmoral 2005; de Sauza Neto et al. 2008; Murakami 2012) are based on the conventional elastoplastic model. On the other hand, the subloading surface model (1980, 1989) describes the smooth elastic-plastic transition fulfilling always the smoothness condition in Eq. (7.2). Then, it possesses the distinguished ability as it does not require the yield-judgment and possesses the automatic controlling function to attract the stress to the yield surface in the plastic deformation process so that the stress is automatically pulled-back to the yield surface when it goes out from the yield surface by finite loading increments in numerical calculation.

The extended elastoplastic constitutive equation for the damage phenomenon is formulated by incorporating the concept of subloading surface in this chapter. The formulation is given originally in terms of the effective stress applied in the fictitious undamaged configuration, which is of the identical form to the original subloading surface model formulation without the damage. However, the hypoelasticity cannot be adopted but the hyperelasticity should be adopted for the damage model since we cannot assume that the elastic deformation is far small compared with the plastic deformation in the damage phenomena. Therefore, the formulation is modified to the hyperelastic-based plasticity in terms of the so-called infinitesimal strain. Further, it will be extended to be taken account of the rate-dependent plastic deformation by incorporating the subloading-overstress model described in Chap. 13.

14.2 Damage Variable

Let the relation of the current area da to the area $dA(\leq da)$ in the undamaged initial state be described through the damage variable D in the one-dimensional deformation process as follows:

$$da = \frac{dA}{1-D} \tag{14.1}$$

leading to

$$\sigma = \frac{df}{da} = (1-D)\underset{\sim}{\sigma}, \underset{\sim}{\sigma} = \frac{df}{dA} = \frac{\sigma}{1-D} \tag{14.2}$$

where $1/(1-D)$ $(\geq 1;\ 0 \leq D < 1)$ designates the increase of area caused by the damage and df is the traction. Mechanical quantities in the fictitious undamaged configuration are specified by adding the wave under them, i.e. $(\sim)$. Let Eq. (14.1) be extended to the three-dimensional deformation process through the second-order positive-definite tensor $\mathcal{D}(=\mathcal{D}^T)$ as follows:

$$\mathbf{n}da = (\mathbf{I}-\mathcal{D})^{-1}\mathbf{N}dA, \quad \mathbf{N}dA = (\mathbf{I}-\mathcal{D})\mathbf{n}da \tag{14.3}$$

leading to

$$\boldsymbol{\sigma} = \frac{d\mathbf{f}}{da} = \underset{\sim}{\boldsymbol{\sigma}}\mathbf{N}\frac{dA}{da}, \quad \underset{\sim}{\boldsymbol{\sigma}} = \frac{d\mathbf{f}}{dA} = \boldsymbol{\sigma}\mathbf{n}\frac{da}{dA} \tag{14.4}$$

noting Eq. (3.1), where $\mathbf{N}$ and $\mathbf{n}$ are the unit outward-normal vectors of the initial and the current surface, respectively. $\mathcal{D}$ is referred to as the *damage tensor* and plays the most basic role for the description of constitutive equation of damage.

14.3 Hyperelastic Relation

In what follows, we adopt the *hypothesis of strain equivalence* (Lemaitre 1971) insisting that the strain and its elastic and plastic parts in the fictitious undamaged configuration are equivalent to those in the actual damaged configuration. As described in Sect. 6.9, the infinitesimal strain is additively decomposed as follows:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p \tag{14.5}$$

Now, adopting the linear elasticity in the Hooke's law in Eq. (5.42), i.e.

$$\left.\begin{aligned} E_{0\,ijkl} &= \frac{E_0}{1+\nu}\left[\frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}\right] \\ E^{-1}_{0\,ijkl} &= \frac{1}{E_0}\left[\frac{1}{2}(1+\nu)(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) - \nu\delta_{ij}\delta_{kl}\right] \end{aligned}\right\} \tag{14.6}$$

for which the Helmholtz free energy function $\underset{\sim}{\psi}(\boldsymbol{\varepsilon}^e)$ and the Gibbs' free energy $\underset{\sim}{\phi}(\underset{\sim}{\boldsymbol{\sigma}})$ in the undamaged configuration are given noting Eqs. (5.43) and (5.44) as

$$\underset{\sim}{\psi}(\boldsymbol{\varepsilon}^e) = \frac{1}{2}\boldsymbol{\varepsilon}^e : \mathbf{E}_0 : \boldsymbol{\varepsilon}^e\left(= \frac{1}{2}\underset{\sim}{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}^e\right) = \frac{1}{2}\frac{E_0}{1+\nu}\left[\varepsilon^e_{ij}\varepsilon^e_{ij} + \frac{\nu}{1-2\nu}(\varepsilon^e_{kk})^2\right] \tag{14.7}$$

$$\underset{\sim}{\phi}(\underset{\sim}{\boldsymbol{\sigma}}) = \frac{1}{2}\underset{\sim}{\boldsymbol{\sigma}} : \mathbf{E}_0^{-1} : \underset{\sim}{\boldsymbol{\sigma}} \left(= \frac{1}{2}\underset{\sim}{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}^e \right) = \frac{1}{2E_0}\left[(1+\nu)\underset{\sim}{\sigma}_{ij}\underset{\sim}{\sigma}_{ij} - \nu(\underset{\sim}{\sigma}_{kk})^2 \right] \tag{14.8}$$

where the fourth-order tensor $\mathbf{E}_0$ is assumed to be the constant tensor in the fictitious undamaged configuration, using the Young's modulus E_0 and the Poisson ratio ν in the fictitious undamaged configuration.

The effective stress $\underset{\sim}{\boldsymbol{\sigma}}$ and the elastic strain $\boldsymbol{\varepsilon}^e$ are derived from Eqs. (14.7) and (14.8), noting Eqs. (5.43) and (5.44) as follows:

$$\underset{\sim}{\boldsymbol{\sigma}} = \frac{\partial \underset{\sim}{\psi}(\boldsymbol{\varepsilon}^e)}{\partial \boldsymbol{\varepsilon}^e} = \mathbf{E}_0 : \boldsymbol{\varepsilon}^e, \quad \underset{\sim}{\sigma}_{ij} = \frac{\partial \underset{\sim}{\psi}(\boldsymbol{\varepsilon}^e)}{\partial \varepsilon^e_{ij}} = E_{0\,ijkl}\varepsilon^e_{kl} = \frac{E_0}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij} \right) \tag{14.9}$$

$$\boldsymbol{\varepsilon}^e = \frac{\partial \underset{\sim}{\phi}(\underset{\sim}{\boldsymbol{\sigma}})}{\partial \underset{\sim}{\boldsymbol{\sigma}}} = \mathbf{E}_0^{-1} : \underset{\sim}{\boldsymbol{\sigma}}, \quad \varepsilon^e_{ij} = \frac{\partial \underset{\sim}{\phi}(\underset{\sim}{\boldsymbol{\sigma}})}{\partial \underset{\sim}{\sigma}_{ij}} = \frac{1}{E_0}\left[(1+\nu)\underset{\sim}{\sigma}_{ij} - \nu \underset{\sim}{\sigma}_{kk}\delta_{ij} \right] \tag{14.10}$$

The elastic strain rate is given from Eq. (14.10) as follows:

$$\dot{\boldsymbol{\varepsilon}}^e = \mathbf{E}_0^{-1} : \dot{\underset{\sim}{\boldsymbol{\sigma}}}, \quad \dot{\underset{\sim}{\boldsymbol{\sigma}}} = \mathbf{E}_0 : \dot{\boldsymbol{\varepsilon}}^e \tag{14.11}$$

14.4 Subloading-Damage Model

The plastic strain rate will be formulated based on the concept of subloading surface in this section (cf. Hashiguchi 2015a).

14.4.1 Normal-Yield and Subloading Surfaces

The yield condition is given by

$$f(\hat{\underset{\sim}{\boldsymbol{\sigma}}}) = \underset{\sim}{F}(\underset{\sim}{H}) \tag{14.12}$$

where $\underset{\sim}{H}$ is the isotropic hardening variable in the fictitious undamaged configuration and

$$\hat{\underset{\sim}{\boldsymbol{\sigma}}} \equiv \underset{\sim}{\boldsymbol{\sigma}} - \underset{\sim}{\boldsymbol{\alpha}} \tag{14.13}$$

$\underset{\sim}{\boldsymbol{\alpha}}$ is the kinematic hardening variable in the fictitious undamaged configuration. The rates of these variables are be described as follows:

$$\left.\begin{array}{l} \dot{\underset{\sim}{H}} = \underset{\sim}{f}_{H\varepsilon}(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{H}; \dot{\boldsymbol{\varepsilon}}^p) = \underset{\sim}{f}_{H\varepsilon}(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{H}; \dot{\boldsymbol{\varepsilon}}^p/\|\dot{\boldsymbol{\varepsilon}}^p\|)\|\dot{\boldsymbol{\varepsilon}}^p\| \\ \dot{\underset{\sim}{\boldsymbol{\alpha}}} = \underset{\sim}{\mathbf{f}}_{k\varepsilon}(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{\boldsymbol{\alpha}}, \underset{\sim}{F}; \dot{\boldsymbol{\varepsilon}}^{p\prime}) = \underset{\sim}{\mathbf{f}}_{k\varepsilon}(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{\boldsymbol{\alpha}}, \underset{\sim}{F}; \dot{\boldsymbol{\varepsilon}}^{p\prime}/\|\dot{\boldsymbol{\varepsilon}}^p\|)\|\dot{\boldsymbol{\varepsilon}}^p\| \end{array}\right\} \tag{14.14}$$

which are homogeneous functions of $\dot{\boldsymbol{\varepsilon}}^p$ in degree-one since they are induced only in the plastic loading process $\dot{\boldsymbol{\varepsilon}}^p \neq \mathbf{O}$ and the first-order time-differential quantities. Here, let it be assumed that the function $f(\hat{\underset{\sim}{\boldsymbol{\sigma}}})$ is the homogeneous function of $\hat{\underset{\sim}{\boldsymbol{\sigma}}}$ in degree-one so that the following relation holds.

$$\frac{\partial f(\hat{\underset{\sim}{\boldsymbol{\sigma}}})}{\partial \hat{\underset{\sim}{\boldsymbol{\sigma}}}} : \hat{\underset{\sim}{\boldsymbol{\sigma}}} = f(\hat{\underset{\sim}{\boldsymbol{\sigma}}}) \tag{14.15}$$

Therefore, the yield surface retains the similar shape.

Hereinafter, the yield surface in Eq. (14.12) is called the normal-yield surface in the fictitious undamaged configuration. Further, incorporate the following subloading surface in the fictitious undamaged configuration (see Fig. 14.1).

$$f(\hat{\underset{\sim}{\boldsymbol{\sigma}}}) = R\underset{\sim}{F}(\underset{\sim}{H}) \tag{14.16}$$

where R is the normal-yield ratio.

The rate of the normal-yield ratio is given by

$$\dot{R} = U(R)\|\dot{\boldsymbol{\varepsilon}}^p\| \quad \text{for} \quad \dot{\boldsymbol{\varepsilon}}^p \neq \mathbf{O} \tag{14.17}$$

based on Eq. (7.9).

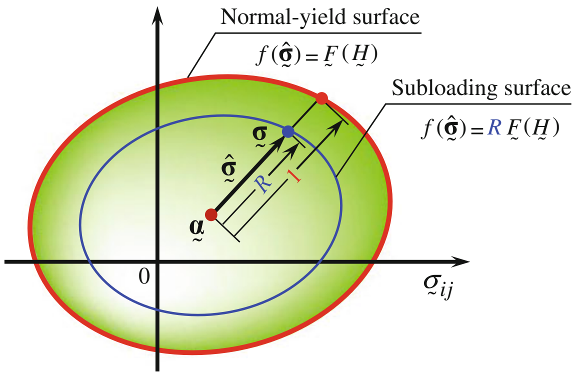

Fig. 14.1 Normal-yield and subloading surfaces in the virtual undamaged configuration

The time-differentiation of Eq. (14.16) leads to

$$\frac{\partial f(\hat{\boldsymbol{\sigma}})}{\partial \hat{\boldsymbol{\sigma}}}:\dot{\boldsymbol{\sigma}} - \frac{\partial f(\hat{\boldsymbol{\sigma}})}{\partial \hat{\boldsymbol{\sigma}}}:\dot{\boldsymbol{\alpha}} - R\dot{F} - \dot{R}F = 0 \tag{14.18}$$

It follows from Eqs. (14.15) and (14.16) that

$$\frac{\partial f(\hat{\boldsymbol{\sigma}})}{\partial \hat{\boldsymbol{\sigma}}}:\hat{\boldsymbol{\sigma}} = RF \tag{14.19}$$

which yields

$$\hat{\mathbf{n}}:\hat{\boldsymbol{\sigma}} = \frac{\partial f(\hat{\boldsymbol{\sigma}})}{\partial \hat{\boldsymbol{\sigma}}}:\hat{\boldsymbol{\sigma}} \Big/ \left\| \frac{\partial f(\hat{\boldsymbol{\sigma}})}{\partial \hat{\boldsymbol{\sigma}}} \right\| = RF \Big/ \left\| \frac{\partial f(\hat{\boldsymbol{\sigma}})}{\partial \hat{\boldsymbol{\sigma}}} \right\| \tag{14.20}$$

leading to

$$1 \Big/ \left\| \frac{\partial f(\hat{\boldsymbol{\sigma}})}{\partial \hat{\boldsymbol{\sigma}}} \right\| = \frac{\hat{\mathbf{n}}:\hat{\boldsymbol{\sigma}}}{RF} \tag{14.21}$$

where

$$\hat{\mathbf{n}} \equiv \frac{\partial f(\hat{\boldsymbol{\sigma}})}{\partial \hat{\boldsymbol{\sigma}}} \Big/ \left\| \frac{\partial f(\hat{\boldsymbol{\sigma}})}{\partial \hat{\boldsymbol{\sigma}}} \right\| \quad (\|\hat{\mathbf{n}}\| = 1) \tag{14.22}$$

The substitution of Eq. (14.21) into Eq. (14.18) leads to

$$\hat{\mathbf{n}}:\left[\dot{\boldsymbol{\sigma}} - \dot{\boldsymbol{\alpha}} - \left(\frac{\dot{F}}{F} + \frac{\dot{R}}{R}\right)\hat{\boldsymbol{\sigma}}\right] = 0 \tag{14.23}$$

Now, adopt the associated flow rule

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\bar{\lambda}}\,\hat{\mathbf{n}} \quad (\dot{\bar{\lambda}} \geq 0) \tag{14.24}$$

where $\dot{\bar{\lambda}}$ is the positive plastic multiplier. Substituting Eqs. (14.14) and (14.17) with Eq. (14.24) into Eq. (14.23), one has

$$\hat{\underset{\sim}{\mathbf{n}}}\mathbf{:}\left[\dot{\underset{\sim}{\boldsymbol{\sigma}}} - \underset{\sim}{\mathbf{f}}_{kn}(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{\boldsymbol{\alpha}}, \underset{\sim}{F}; \hat{\underset{\sim}{\mathbf{n}}}')\,\dot{\bar{\lambda}} - \left(\frac{\underset{\sim}{F}'}{\underset{\sim}{F}}\,\underset{\sim}{f}_{Hn}(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{H}; \hat{\underset{\sim}{\mathbf{n}}})\,\dot{\bar{\lambda}} + \frac{U(R)}{R}\dot{\bar{\lambda}}\right)\hat{\underset{\sim}{\boldsymbol{\sigma}}}\right] \tag{14.25}$$

where

$$\underset{\sim}{f}_{Hn}(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{H}; \hat{\underset{\sim}{\mathbf{n}}}) = \dot{\underset{\sim}{H}}/\dot{\bar{\lambda}}, \quad \underset{\sim}{\mathbf{f}}_{kn}(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{\boldsymbol{\alpha}}, \underset{\sim}{F}; \hat{\underset{\sim}{\mathbf{n}}}') = \dot{\underset{\sim}{\boldsymbol{\alpha}}}/\dot{\bar{\lambda}} \tag{14.26}$$

noting the homogeneities of $\dot{\underset{\sim}{H}}$ and $\dot{\underset{\sim}{\boldsymbol{\alpha}}}$ in degree-one of $\dot{\boldsymbol{\varepsilon}}^p$. It follows from Eqs. (14.24) and (14.25) that

$$\dot{\bar{\lambda}}\frac{\hat{\underset{\sim}{\mathbf{n}}}:\dot{\underset{\sim}{\boldsymbol{\sigma}}}}{\underset{\sim}{\overline{M}}^p}, \dot{\varepsilon}^p \equiv \frac{\hat{\underset{\sim}{\mathbf{n}}}:\dot{\underset{\sim}{\boldsymbol{\sigma}}}}{\underset{\sim}{\overline{M}}^p}\hat{\underset{\sim}{\mathbf{n}}} \tag{14.27}$$

where

$$\underset{\sim}{\overline{M}}^p \equiv \hat{\underset{\sim}{\mathbf{n}}}\mathbf{:}\left[\underset{\sim}{\mathbf{f}}_{kn}(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{\boldsymbol{\alpha}}, \underset{\sim}{F}; \hat{\underset{\sim}{\mathbf{n}}}') + \left(\frac{\underset{\sim}{F}'}{\underset{\sim}{F}}\,\underset{\sim}{f}_{Hn}(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{H}; \hat{\underset{\sim}{\mathbf{n}}}) + \frac{U(R)}{R}\right)\hat{\underset{\sim}{\boldsymbol{\sigma}}}\right] \tag{14.28}$$

14.4.2 Stress Rate Versus Strain Rate Relations

The strain rate is described from Eqs. (14.5), (14.11) and (14.27) as

$$\dot{\boldsymbol{\varepsilon}} = \mathbf{E}_0^{-1}\mathbf{:}\dot{\underset{\sim}{\boldsymbol{\sigma}}} + \frac{\hat{\underset{\sim}{\mathbf{n}}}\mathbf{:}\dot{\underset{\sim}{\boldsymbol{\sigma}}}}{\underset{\sim}{\overline{M}}^p}\hat{\underset{\sim}{\mathbf{n}}} = (\mathbf{E}_0^{-1} + \frac{\hat{\underset{\sim}{\mathbf{n}}}\otimes\hat{\underset{\sim}{\mathbf{n}}}}{\underset{\sim}{\overline{M}}^p})\mathbf{:}\dot{\underset{\sim}{\boldsymbol{\sigma}}} \tag{14.29}$$

from which the plastic multiplier in terms of strain rate is derived as follows:

$$\dot{\bar{\Lambda}} = \frac{\hat{\underset{\sim}{\mathbf{n}}}:\mathbf{E}_0:\dot{\boldsymbol{\varepsilon}}}{\underset{\sim}{\overline{M}}^p + \hat{\underset{\sim}{\mathbf{n}}}:\mathbf{E}_0:\hat{\underset{\sim}{\mathbf{n}}}} \tag{14.30}$$

The stress rate is described from Eqs. (14.11), (14.24) and (14.30) as

$$\dot{\underset{\sim}{\boldsymbol{\sigma}}} = \mathbf{E}_0:\dot{\boldsymbol{\varepsilon}} - \frac{\hat{\underset{\sim}{\mathbf{n}}}:\mathbf{E}_0:\dot{\boldsymbol{\varepsilon}}}{\underset{\sim}{\overline{M}}^p + \hat{\underset{\sim}{\mathbf{n}}}:\mathbf{E}_0:\hat{\underset{\sim}{\mathbf{n}}}}\mathbf{E}_0:\hat{\underset{\sim}{\mathbf{n}}} \tag{14.31}$$

The loading criterion is given by

$$\left.\begin{array}{ll}\dot{\boldsymbol{\varepsilon}}^p \neq \mathbf{O} & \text{for } \dot{\bar{\Lambda}} > 0 \\ \dot{\boldsymbol{\varepsilon}}^p \neq \mathbf{O} & \text{for } \dot{\bar{\Lambda}} \leq 0\end{array}\right\} \tag{14.32}$$

14.5 Hardening Rules

The evolution rules of the isotropic and the kinematic hardening variables are given in this section.

14.5.1 Isotropic Hardening Rule

The isotropic hardening is described by

$$\underset{\sim}{F}(\underset{\sim}{H}) = \underset{\sim}{F}_0[1 + h_1\{1 - \exp(-h_2\underset{\sim}{H})\}], \quad \underset{\sim}{F}' = h_1 h_2 \underset{\sim}{F}_0 \exp(-h_2\underset{\sim}{H}) \tag{14.33}$$

$$\dot{\underset{\sim}{H}} = \sqrt{\frac{2}{3}}\|\dot{\boldsymbol{\varepsilon}}^p\| = \dot{\bar{\lambda}}\, \underset{\sim}{f}_{Hn}(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{H}; \underset{\sim}{\hat{\mathbf{n}}}) \tag{14.34}$$

$$\underset{\sim}{f}_{Hn}(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{H}; \underset{\sim}{\hat{\mathbf{n}}}) = \sqrt{\frac{2}{3}} \tag{14.35}$$

14.5.2 Nonlinear Kinematic Hardening Rule

Internal variables describe state of substructure of material so that tensor-valued internal variables must be given by the hyperelastic-like relation, i.e. the partial differential of the energy storage function of conjugate strain measure.

As described in Sect. 6.9, the plastic strain rate $\boldsymbol{\varepsilon}^p$ be decomposed into the storage part $\boldsymbol{\varepsilon}^p_{ks}$ and the dissipation part $\boldsymbol{\varepsilon}^p_{kd}$, i.e.

$$\boldsymbol{\varepsilon}^p = \boldsymbol{\varepsilon}^p_{ks} + \boldsymbol{\varepsilon}^p_{kd} \tag{14.36}$$

for the kinematic hardening. In addition, incorporate the energy storage function $\psi^k(\boldsymbol{\varepsilon}^p_{ks})$ of the storage part of the plastic strain for the kinematic hardening variable. Then, the kinematic hardening variable is given as follows:

$$\underset{\sim}{\boldsymbol{\alpha}} = \frac{\partial \underset{\sim}{\psi}^{k}(\boldsymbol{\varepsilon}_{ks}^{p\prime})}{\partial \boldsymbol{\varepsilon}_{ks}^{p}} \tag{14.37}$$

Now, adopt the explicit function

$$\underset{\sim}{\psi}^{k}(\boldsymbol{\varepsilon}_{ks}^{p}) = \frac{1}{2} c_k \boldsymbol{\varepsilon}_{ks}^{p\prime} : \boldsymbol{\varepsilon}_{ks}^{p\prime} \tag{14.38}$$

where c_k is the material constant. The kinematic hardening variable is given from Eq. (14.37) with Eq. (14.38) by

$$\underset{\sim}{\boldsymbol{\alpha}} = c_k \boldsymbol{\varepsilon}_{ks}^{p\prime} \tag{14.39}$$

Further, let the dissipative part of plastic strain rate be given, noting Eq. (6.103), as

$$\dot{\boldsymbol{\varepsilon}}_{kd}^{p\prime} = \frac{1}{b_k(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{F})} \underset{\sim}{\boldsymbol{\alpha}} \|\dot{\boldsymbol{\varepsilon}}^{p\prime}\| = \frac{1}{b_k(\underset{\sim}{\boldsymbol{\alpha}}, \underset{\sim}{F})} \underset{\sim}{\boldsymbol{\alpha}} \|\underset{\sim}{\hat{\mathbf{n}}}'\| \,\dot{\bar{\lambda}} \tag{14.40}$$

noting Eq. (14.24), where $b_k(>0)$ is the material function of $\underset{\sim}{\boldsymbol{\sigma}}$ and $\underset{\sim}{F}$ in general. Equation (14.40) satisfies the positivity of the dissipation energy, i.e. $\underset{\sim}{\boldsymbol{\alpha}} : \dot{\boldsymbol{\varepsilon}}_{kd}^{p} = (\underset{\sim}{\boldsymbol{\alpha}} : \underset{\sim}{\boldsymbol{\alpha}}) \|\dot{\boldsymbol{\varepsilon}}^{p\prime}\| / b_k \geq 0$. It follows from Eqs. (14.24), (14.36) and (14.40) that

$$\dot{\boldsymbol{\varepsilon}}_{ks}^{p\prime} = \dot{\boldsymbol{\varepsilon}}^{p\prime} - \dot{\boldsymbol{\varepsilon}}_{kd}^{p\prime} = \dot{\boldsymbol{\varepsilon}}^{p\prime} - \frac{1}{b_k(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{F})} \|\dot{\boldsymbol{\varepsilon}}^{p\prime}\| \underset{\sim}{\boldsymbol{\alpha}} = \dot{\bar{\lambda}} \left(\underset{\sim}{\hat{\mathbf{n}}}' - \frac{1}{b_k(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{F})} \underset{\sim}{\boldsymbol{\alpha}} \|\underset{\sim}{\hat{\mathbf{n}}}'\| \right) \tag{14.41}$$

The following equation is derived from Eqs. (14.39) and (14.41).

$$\left. \begin{aligned} \dot{\underset{\sim}{\boldsymbol{\alpha}}} &= c_k \dot{\boldsymbol{\varepsilon}}_{ks}^{p\prime} = c_k \left(\dot{\boldsymbol{\varepsilon}}^{p\prime} - \frac{1}{b_k(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{F})} \|\dot{\boldsymbol{\varepsilon}}^{p\prime}\| \underset{\sim}{\boldsymbol{\alpha}} \right) = \dot{\bar{\lambda}} \, \underset{\sim}{\mathbf{f}}_{kn} \\ \underset{\sim}{\mathbf{f}}_{kn} &= c_k \, \dot{\bar{\lambda}} \left(\underset{\sim}{\hat{\mathbf{n}}}' - \frac{1}{b_k(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{F})} \underset{\sim}{\boldsymbol{\alpha}} \|\underset{\sim}{\hat{\mathbf{n}}}'\| \right) \end{aligned} \right\} \tag{14.42}$$

which is the modification of the nonlinear kinematic hardening rule by Armstrong and Frederick (1966).

The constitutive relation for the effective stress was described in the previous and this sections. Further, we have to formulate the calculation method of the current stress in the actual configuration from the effective stress, which will be attained through the damage tensor as described in the subsequent sections.

14.6 Damage Tensor

The damage variable is the tensor which transforms the elastic response in the fictitious undamaged configuration to the one in the actual damaged configuration in general. However, assume that the elastic response in the actual damaged configuration is also given by the linear relation between the Cauchy stress and the elastic strain as well as the one in the fictitious undamaged configuration shown in Eq. (14.10), i.e.

$$\boldsymbol{\sigma} = \mathbf{E} : \boldsymbol{\varepsilon}^{e}, \quad \boldsymbol{\varepsilon}^{e} = \mathbf{E}^{-1} : \boldsymbol{\sigma} \tag{14.43}$$

based on the hypothesis of strain equivalence (Lemaitre 1971), where $\mathbf{E}$ is elastic modulus tensor in the actual damaged configuration. It follows from Eqs. (14.10) and (14.43) that

$$\boldsymbol{\sigma} = \mathbf{E} : \mathbf{E}_0^{-1} : \underset{\sim}{\boldsymbol{\sigma}}, \quad \underset{\sim}{\boldsymbol{\sigma}} = \mathbf{E}_0 : \mathbf{E}^{-1} : \boldsymbol{\sigma} \tag{14.44}$$

The relation of $\mathbf{E}$ to $\mathbf{E}_0$ is described using the damage tensor $\mathfrak{D}$ in general as follows:

$$\mathbf{E} = \mathbf{E}(\mathbf{E}_0, \mathfrak{D}) \tag{14.45}$$

where the evolution rule of the damage tensor $\mathfrak{D}$ can be generally given as follows:

$$\dot{\mathfrak{D}} = \mathbf{f}_D(\underset{\sim}{\boldsymbol{\sigma}}, \varepsilon^{eqp}; \dot{\boldsymbol{\varepsilon}}^{p}) \tag{14.46}$$

where $\mathbf{f}_D$ is the homogeneous function of $\dot{\boldsymbol{\varepsilon}}^{p}$ in degree-one.

Now, if we assume the linear relation between the fourth-order tensors $\mathbf{E}_0$ and $\mathbf{E}$, i.e.

$$E_{ijkl} = \mathfrak{D}_{ijklpqrs} E_{0pqrs} \tag{14.47}$$

the damage variable $\mathfrak{D}$ is the eight-order tensor in general.

The following simple transformation rule in terms of the fourth-order tensor $\mathbb{D}$ is adopted by Chaboche (1982).

$$\mathbf{E} = (\mathcal{I} - \mathbb{D}) : \mathbf{E}_0 \tag{14.48}$$

which yields

$$\boldsymbol{\sigma} = (\mathcal{I} - \mathbb{D}) : \underset{\sim}{\boldsymbol{\sigma}}, \quad \underset{\sim}{\boldsymbol{\sigma}} = (\mathcal{I} - \mathbb{D})^{-1} : \boldsymbol{\sigma} \tag{14.49}$$

However, unfortunately the current stress $\boldsymbol{\sigma}$ is generally no longer symmetric even for the effective stress tensor $\underset{\sim}{\boldsymbol{\sigma}}$ which is the symmetric tensor. Besides, it cannot be used as far as the tensor $\mathbb{D}$ is described explicitly by the known physical variable.

Various explicit damage tensors have been proposed as will be described in the following.

14.6.1 Isotropic Damage Tensor

Assume the following isotropic damage tensor with the scalar variable D in Eq. (14.1):

$$\mathbb{D} = D\mathcal{I} \tag{14.50}$$

Substituting Eq. (14.50) into Eq. (14.48), it follows that

$$\mathbf{E} = (\mathcal{I} - \mathbb{D}) : \mathbf{E}_0 = (1 - D)\mathcal{I} : \mathbf{E}_0 = (1 - D)\mathbf{E}_0, \quad \mathbf{E}_0 = \mathbf{E}/(1 - D) \tag{14.51}$$

$$\mathbf{E}^{-1} = \mathbf{E}_0^{-1}/(1 - D), \quad \mathbf{E}_0^{-1} = (1 - D)\mathbf{E}^{-1} \tag{14.52}$$

which is described for the Hooke's law as follows:

$$\left.\begin{aligned} E_{ijkl} &= (1 - D)E_{0ijkl} = (1 - D)\frac{E_0}{1 + v}\left[\frac{v}{1 - 2v}\delta_{ij}\delta_{kl} + \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right] \\ E_{ijkl}^{-1} &= \frac{1}{(1 - D)}E_{0ijkl}^{-1} = \frac{1}{(1 - D)E_0}\left[\frac{1}{2}(1 + v)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - v\delta_{ij}\delta_{kl}\right] \end{aligned}\right\} \tag{14.53}$$

resulting in

$$E = (1 - D)E_0, \quad D = 1 - E/E_0 \tag{14.54}$$

Then, substituting Eq. (14.52) into Eq. (14.44), one has

$$\boldsymbol{\sigma} = (1 - D)\underset{\sim}{\boldsymbol{\sigma}}, \quad \underset{\sim}{\boldsymbol{\sigma}} = \boldsymbol{\sigma}/(1 - D) \tag{14.55}$$

from which it follows that

$$\dot{\boldsymbol{\sigma}} = (1-D)\dot{\underset{\sim}{\boldsymbol{\sigma}}} - \dot{D}\,\underset{\sim}{\boldsymbol{\sigma}}, \quad \dot{\underset{\sim}{\boldsymbol{\sigma}}} = \frac{\dot{\boldsymbol{\sigma}} + \dot{D}\,\underset{\sim}{\boldsymbol{\sigma}}}{1 - D} = \frac{\dot{\boldsymbol{\sigma}} + \dot{D}\dfrac{\boldsymbol{\sigma}}{1-D}}{1 - D} \tag{14.56}$$

The following relations hold from Eqs. (14.51) and (14.52).

$$\boldsymbol{\sigma} = \mathbf{E} : \boldsymbol{\varepsilon}^e = (1-D)\mathbf{E}_0 : \boldsymbol{\varepsilon}^e, \quad \underset{\sim}{\boldsymbol{\sigma}} = \mathbf{E}_0 : \boldsymbol{\varepsilon}^e = \frac{1}{1-D}\mathbf{E} : \boldsymbol{\varepsilon}^e \tag{14.57}$$

$$\boldsymbol{\varepsilon}^e = \mathbf{E}^{-1} : \boldsymbol{\sigma} = \frac{1}{1-D}\mathbf{E}_0^{-1} : \boldsymbol{\sigma} = \mathbf{E}_0^{-1} : \underset{\sim}{\boldsymbol{\sigma}} = (1-D)\mathbf{E}^{-1} : \boldsymbol{\sigma} \tag{14.58}$$

The Helmholtz free energy function $\psi(\boldsymbol{\varepsilon}^e)$ and the Gibbs' free energy $\phi(\boldsymbol{\sigma})$ in the damaged state are given for Eq. (14.53) noting Eqs. (5.40) as

$$\begin{aligned}\psi(\boldsymbol{\varepsilon}^e, D)\left(=\frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\varepsilon}^e\right) &= \frac{1}{2}\boldsymbol{\varepsilon}^e : \mathbf{E} : \boldsymbol{\varepsilon}^e = \frac{1}{2}(1-D)\boldsymbol{\varepsilon}^e : \mathbf{E}_0 : \boldsymbol{\varepsilon}^e \\ &= \frac{1}{2}(1-D)\frac{E_0}{1+v}\left[\varepsilon_{ij}^e\varepsilon_{ij}^e + \frac{v}{1-2v}(\varepsilon_{kk}^e)^2\right]\end{aligned} \tag{14.59}$$

$$\begin{aligned}\phi(\boldsymbol{\sigma}, D)\left(=\frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\varepsilon}^e\right) &= \frac{1}{2}\boldsymbol{\sigma} : \mathbf{E}^{-1} : \boldsymbol{\sigma} = \frac{1}{2(1-D)}\boldsymbol{\sigma} : \mathbf{E}_0^{-1} : \boldsymbol{\sigma} \\ &= \frac{1}{2(1-D)E_0}[(1+v)\sigma_{ij}\sigma_{ij} - v(\sigma_{kk})^2]\end{aligned} \tag{14.60}$$

from which it follows that

$$\sigma_{ij} = \frac{\partial\psi(\boldsymbol{\varepsilon}^e, D)}{\partial\varepsilon_{ij}^e} = (1-D)E_{0ijkl}\varepsilon_{kl}^e = (1-D)\frac{E_0}{1+v}\left(\varepsilon_{ij} + \frac{v}{1-2v}\varepsilon_{kk}\delta_{ij}\right) \tag{14.61}$$

$$\begin{aligned}\varepsilon_{ij}^e &= \frac{\partial\phi(\boldsymbol{\sigma}, D)}{\partial\sigma_{ij}} = \frac{1}{1-D}E_{0ijkl}^{-1}\sigma_{kl} = \frac{1}{(1-D)E_0}[(1+v)\sigma_{ij} - v\sigma_{kk}\delta_{ij}] \\ &= \frac{1}{E_0}[(1+v)\underset{\sim}{\sigma}_{ij} - v\underset{\sim}{\sigma}_{kk}\delta_{ij}]\end{aligned} \tag{14.62}$$

The example of the evolution rule of the isotropic damage variable D is given by Lemaitre and Chaboche (1990) as follows:

$$\dot{D} = \left(\frac{Y}{\zeta}\right)^a \frac{H(\varepsilon^{eqp} - \varepsilon_d^{eqp})}{1-D}\dot{\varepsilon}^{eqp} \tag{14.63}$$

where ζ and a are the material constants, and ε_d^{eqp} is the threshold value of ε^{eqp}, and $H(\,)$ is the Heaviside step function, i.e. $H(s) = 0$ for $s \le 0$ and $H(s) = 1$ for $s > 0$ for a scalar variable s. Y is defined by

$$Y = -\frac{\partial\psi(\boldsymbol{\varepsilon}^e, D)}{\partial D} = \frac{\psi(\boldsymbol{\varepsilon}^e, D)}{1-D} = \frac{1}{2}E_{0\,ijkl}\varepsilon_{ij}^e\varepsilon_{kl}^e = \frac{1}{2}\boldsymbol{\varepsilon}^e : \mathbf{E}_0 : \boldsymbol{\varepsilon}^e = \frac{1}{2}\underset{\sim}{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}^e \tag{14.64}$$

noting Eq. (14.59), which designates the releasing rate of the strain energy due to the damage under the constant strain ($\partial\psi(\boldsymbol{\varepsilon}^e, D)/\partial D < 0$), i.e. the rate of energy dissipated in the crack extension and is called the *strain energy density release rate* in the failure mechanics or the *damage associated variable* (Chaboche 1988). Equation (14.64) is rewritten as follows:

$$Y=\frac{1}{2E_0}[(1+v)\underset{\sim}{\sigma}_{ij}\underset{\sim}{\sigma}_{ij}-v(\underset{\sim}{\sigma}_{rr})^2]=\frac{1+v}{2(1-D)^2E_0}[(1+v)\sigma_{ij}\sigma_{ij}-v(\sigma_{rr})^2] \tag{14.65}$$

or

$$Y=\frac{2}{3}\frac{1+v}{2E_0}\underset{\sim}{\sigma}^{eq\,2}+3\frac{1-2v}{2E_0}\underset{\sim}{\sigma}_m^2 \tag{14.66}$$

noting

$$\begin{aligned}\frac{1}{2}\underset{\sim}{\sigma}_{ij}\varepsilon_{ij}^e&=\frac{1}{2}\underset{\sim}{\sigma}_{ij}\frac{1}{E_0}[(1+v)\underset{\sim}{\sigma}_{ij}-v\underset{\sim}{\sigma}_{kk}\delta_{ij})]=\frac{1}{2E_0}\left[(1+v)\underset{\sim}{\sigma}_{ij}\underset{\sim}{\sigma}_{ij}-v(\underset{\sim}{\sigma}_{kk})^2\right]\\&=\frac{1}{2E_0}\left[(1+v)\left(\underset{\sim}{\sigma}'_{ij}+\frac{1}{3}\underset{\sim}{\sigma}_{kk}\delta_{ij}\right)\left(\underset{\sim}{\sigma}'_{ij}+\frac{1}{3}\underset{\sim}{\sigma}_{kk}\delta_{ij}\right)-v(\underset{\sim}{\sigma}_{kk})^2\right]\\&=\frac{1}{2E_0}\left[(1+v)\underset{\sim}{\sigma}'_{ij}\underset{\sim}{\sigma}'_{ij}+\frac{1}{3}(1-2v)(\underset{\sim}{\sigma}_{kk})^2\right]=\frac{1}{2E_0}\left[(1+v)\frac{2}{3}\underset{\sim}{\sigma}^{eq\,2}+3(1-2v)(\underset{\sim}{\sigma}_m)^2\right]\end{aligned}$$

Equation (14.64) is further rewritten as

$$Y=\frac{\underset{\sim}{\sigma}^{eq\,2}}{2E_0}R_v \tag{14.67}$$

where R_v is defined by the following equation and called the *stress triaxiality function*

$$R_v\equiv\frac{2}{3}(1+v)+3(1-2v)\left(\frac{\underset{\sim}{\sigma}_m}{\underset{\sim}{\sigma}^{eq}}\right)^2 \tag{14.68}$$

$\underset{\sim}{\sigma}_m/\underset{\sim}{\sigma}^{eq}$ is referred to as the *stress triaxiality*.

Analogously to Eqs. (14.55) and (14.56), the relation of the isotropic hardening, the kinematic hardening variables and their rates in the actual damaged and the fictitious undamaged configurations are given by

$$F=(1-D)\underset{\sim}{F},\quad \underset{\sim}{F}=F/(1-D) \tag{14.69}$$

$$\dot{F}=(1-D)\dot{\underset{\sim}{F}}-\dot{D}\,\underset{\sim}{F} \tag{14.70}$$

$$\boldsymbol{\alpha} = (1 - D)\underset{\sim}{\boldsymbol{\alpha}}, \quad \underset{\sim}{\boldsymbol{\alpha}} = \boldsymbol{\alpha}/(1 - D) \tag{14.71}$$

$$\dot{\boldsymbol{\alpha}} = (1 - D)\underset{\sim}{\dot{\boldsymbol{\alpha}}} - \dot{D}\,\underset{\sim}{\boldsymbol{\alpha}} \tag{14.72}$$

Equation (14.50) is widely employed in deformation analyses. However, it would be inapplicable to damage behavior with a strong anisotropy.

14.6.2 On Strain Energy Density Release Rate

Physical interpretation of the strain energy density release rate is given concisely in this section referring to Chaboche (1988).

Consider the elastic deformation with crack extension under the uniaxial loading in Fig. 14.3. It follows in the deformation process from the point a to the point b that

Energy release increment dE_r (from elastic body with crack extension)
= Input energy increment − Strain energy increment
= □a′abb′ − (Δ0bb′ − Δ0aa′)
= □a′abb′ − [(Δ0aa′ + □a′abb′ − Δ0ab) - Δ0aa′]
= Δ0ab

Now, suppose the state that the stress increment is small to be negligible, i.e. $d\sigma \cong 0$ and thus one has

$$\begin{aligned} d\psi &= \Delta 0\mathrm{bb}' - \Delta 0\mathrm{aa}' = (\Delta 0\mathrm{aa}' + \square \mathrm{a}'\mathrm{abb}' - \Delta 0\mathrm{ab}) - \Delta 0\mathrm{aa}' \\ &= \square \mathrm{a}'\mathrm{abb}' - \Delta 0\mathrm{ab} \cong 2\Delta 0\mathrm{ab} - \Delta 0\mathrm{ab} = \Delta 0\mathrm{ab} \\ &= dE_r \end{aligned} \tag{14.73}$$

Therefore, the half of the input energy increment transforms to the energy release increment dE_r and the other half transfers to the strain energy increment $d\psi$.

Now, one has

$$d\varepsilon^e \cong \frac{\varepsilon^e}{1 - D} dD \tag{14.74}$$

noting

$$d\sigma = d(E\varepsilon^e) = d[(1 - D)E_0\varepsilon^e] = -dDE_0\varepsilon^e + (1 - D)E_0 d\varepsilon^e \cong 0$$

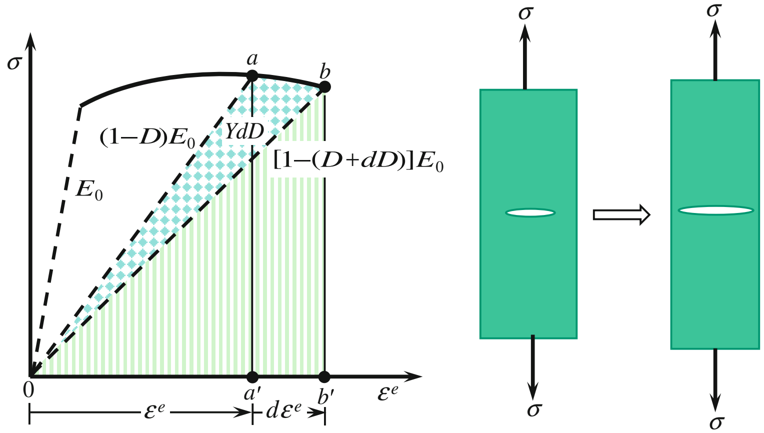

Fig. 14.2 Strain energy release due to crack extension

Then, it follows from Eqs. (14.59) and (14.64) with Eq. (14.74) that

$$d\psi = d\left(\frac{1}{2}\sigma\varepsilon^e\right) \cong \frac{1}{2}\sigma d\varepsilon^e \simeq \frac{1}{2}\frac{\sigma}{1-D}\varepsilon^e dD = \frac{1}{2}\underset{\sim}{\sigma}\varepsilon^e dD = YdD \tag{14.75}$$

Therefore, YdD is shown by Δ0ab under the constant stress state in Fig. 14.2.

14.6.3 Unilateral Damage: Microcrack Closure Effect

The degrees of damage in directions subjected to the tension and the compression stresses are different in some materials, e.g. cast iron, rocks and concretes. It is called the *unilateral damage*, while the identical damage generation in directions subjected to the tension and the compression stresses described in the last section is called the *bilateral damage*. The unilateral damage is formulated by Ladeveze and Lemaitre (1984) as will be described below.

The tensor is described in the spectral representation in Eq. (1.170), i.e.

$$\mathbf{T} = \sum_{P=1}^{3} T_P \mathbf{e}_P \otimes \mathbf{e}_P$$

where T_P are the principal values and $\mathbf{e}_P$ are the principal vectors. The components are described as follows:

$$T_{ij} = \sum_{P=1}^{3} T_P e_{Pi} e_{Pj}, \; e_{Pi} = e_P \cdot e_i \tag{14.76}$$

noting

$$T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j = \mathbf{e}_i \cdot \sum_{P=1}^{3} T_P \mathbf{e}_P \otimes \mathbf{e}_P \mathbf{e}_j \tag{14.77}$$

The components are decomposed into the positive and the negative parts:

$$T_{ij} = \langle \mathbf{T} \rangle_{ij}^{+} + \langle \mathbf{T} \rangle_{ij}^{-} \tag{14.78}$$

where

$$\left. \begin{array}{ll} \langle \mathbf{T} \rangle_{ij}^{+} \equiv \sum_{P=1}^{3} \langle T_P \rangle e_{Pi} e_{Pj} & \text{for} \;\; T_P \geq 0 \\ \langle \mathbf{T} \rangle_{ij}^{-} \equiv \sum_{P=1}^{3} -\langle -T_P \rangle e_{Pi} e_{Pj} & \text{for} \;\; T_P < 0 \end{array} \right\} \tag{14.79}$$

with

$$\langle \mathbf{T} \rangle_{ij}^{+} \langle \mathbf{T} \rangle_{ij}^{-} = 0 \tag{14.80}$$

noting

$$\langle T_P \rangle \langle -T_P \rangle = 0 \tag{14.81}$$

It follows noting Eq. (14.80) from Eq. (14.78) that

$$T_{ij} T_{ij} = \langle \mathbf{T} \rangle_{ij}^{+} \langle \mathbf{T} \rangle_{ij}^{+} + \langle \mathbf{T} \rangle_{ij}^{-} \langle \mathbf{T} \rangle_{ij}^{-} \tag{14.82}$$

$$\langle T_{kk} \rangle^2 = \langle T_{kk} \rangle^2 + \langle -T_{kk} \rangle^2 \tag{14.83}$$

$$\langle T_{kk} \rangle = \langle T_{kk} \rangle - \langle -T_{kk} \rangle \tag{14.84}$$

with

$$\langle \mathbf{T} \rangle_{kl}^{+} \delta_{kl} \neq \langle T_{kk} \rangle, \quad \langle \mathbf{T} \rangle_{kl}^{-} \delta_{kl} \neq -\langle -T_{kk} \rangle \tag{14.85}$$

and the following derivatives hold.

$$\frac{\partial}{\partial T_{ij}}\left(\frac{1}{2}\langle T_{kk}\rangle^2\right) = \langle T_{kk}\rangle\delta_{ij} \tag{14.86}$$

$$\frac{\partial}{\partial T_{ij}}\left(\frac{1}{2}\langle \mathbf{T}\rangle^+_{rs}\langle \mathbf{T}\rangle^+_{rs}\right) = \langle \mathbf{T}\rangle^+_{ij}, \quad \frac{\partial}{\partial T_{ij}}\left(\frac{1}{2}\langle \mathbf{T}\rangle^-_{rs}\langle \mathbf{T}\rangle^-_{rs}\right) = \langle \mathbf{T}\rangle^-_{ij} \tag{14.87}$$

Equations (14.60), (14.62) and (14.65) are described by Eqs. (14.79), (14.82), (14.83) and (14.84) as follows:

$$\phi(\boldsymbol{\sigma}, D) = \frac{1\ +\ v}{2E_0}\frac{\langle\boldsymbol{\sigma}\rangle^+_{ij}\langle\boldsymbol{\sigma}\rangle^+_{ij} + \langle\boldsymbol{\sigma}\rangle^-_{ij}\langle\boldsymbol{\sigma}\rangle^-_{ij}}{1-D} - \frac{v}{2E_0}\frac{\langle\sigma_{kk}\rangle^2 - \langle-\sigma_{kk}\rangle^2}{1-D} \tag{14.88}$$

$$Y = \frac{1+v}{2E_0}\frac{\langle\boldsymbol{\sigma}\rangle^+_{ij}\langle\boldsymbol{\sigma}\rangle^+_{ij} + \langle\boldsymbol{\sigma}\rangle^-_{ij}\langle\boldsymbol{\sigma}\rangle^-_{ij}}{(1-D)^2} - \frac{v}{2E_0}\frac{\langle\sigma_{kk}\rangle^2 + \langle-\sigma_{kk}\rangle^2}{(1-D)^2} \tag{14.89}$$

$$\varepsilon^e_{ij} = \frac{\partial\phi(\boldsymbol{\sigma}, D)}{\partial\boldsymbol{\sigma}} = \frac{1\ +\ v}{E_0}\frac{\langle\boldsymbol{\sigma}\rangle^+_{ij} + \langle\boldsymbol{\sigma}\rangle^-_{ij}}{1-D} - \frac{v}{E_0}\frac{\langle\sigma_{kk}\rangle - \langle-\sigma_{kk}\rangle}{1-D}\delta_{ij} \tag{14.90}$$

Here, noting

$$\begin{aligned}
&(1+v)\langle\boldsymbol{\sigma}\rangle^+_{ij} - v\langle\sigma_{rr}\rangle\delta_{ij} \\
&\quad = (1+v)\langle\boldsymbol{\sigma}\rangle^+_{ij} + \frac{1}{1-2v}[(v+v^2)\langle\boldsymbol{\sigma}\rangle^+_{kl}\delta_{kl} - v(1-2v)\langle\boldsymbol{\sigma}\rangle^+_{kl}\delta_{kl} \\
&\qquad - 3v^2\langle\boldsymbol{\sigma}\rangle^+_{rs}\delta_{rs} - (v+v^2)\langle\sigma_{rr}\rangle + 3v^2\langle\sigma_{rr}\rangle]\delta_{ij} \\
&\quad = (1+v)\langle\boldsymbol{\sigma}\rangle^+_{ij} + \frac{1}{1-2v}[(v+v^2)\langle\boldsymbol{\sigma}\rangle^+_{kl}\delta_{kl} - (v+v^2)\langle\sigma_{rr}\rangle - v(1-2v)\langle\boldsymbol{\sigma}\rangle^+_{kl}\delta_{kl} \\
&\qquad - 3v^2\langle\boldsymbol{\sigma}\rangle^+_{rs}\delta_{rs} + 3v^2\langle\sigma_{rr}\rangle]\delta_{ij} \\
&\quad = (1+v)\left[\langle\boldsymbol{\sigma}\rangle^+_{ij} + \frac{v}{1-2v}(\langle\boldsymbol{\sigma}\rangle^+_{kl}\delta_{kl} - \langle\sigma_{rr}\rangle)\delta_{ij}\right] \\
&\qquad - v\left[\langle\boldsymbol{\sigma}\rangle^+_{kl} + \frac{v}{1-2v}(\langle\boldsymbol{\sigma}\rangle^+_{rs}\delta_{rs} - \langle\sigma_{rr}\rangle)\delta_{kl}\right]\delta_{kl}\delta_{ij}
\end{aligned}$$

Eq. (14.90) is rewritten as

$$\begin{aligned}
\varepsilon^e_{ij} &= \frac{1\ +\ v}{E_0}\left[\frac{\langle\boldsymbol{\sigma}\rangle^+_{ij} + \frac{v}{1-2v}(\langle\boldsymbol{\sigma}\rangle^+_{kl}\delta_{kl} - \langle\sigma_{rr}\rangle)\delta_{ij}}{1-D} - \frac{\langle\boldsymbol{\sigma}\rangle^-_{ij} + \frac{v}{1-2v}(\langle\boldsymbol{\sigma}\rangle^-_{kl}\delta_{kl} - \langle-\sigma_{rr}\rangle)\delta_{ij}}{1-D}\right] \\
&\quad - \frac{v}{E_0}\left[\frac{\langle\boldsymbol{\sigma}\rangle^+_{kl} + \frac{v}{1-2v}(\langle\boldsymbol{\sigma}\rangle^+_{rs}\delta_{rs} - \langle\sigma_{rr}\rangle)\delta_{kl}}{1-D} - \frac{\langle\boldsymbol{\sigma}\rangle^-_{kl} + \frac{v}{1-2v}(\langle\boldsymbol{\sigma}\rangle^+_{rs}\delta_{rs} - \langle-\sigma_{rr}\rangle)\delta_{kl}}{1-D}\right]\delta_{kl}\delta_{ij}
\end{aligned} \tag{14.91}$$

On the other hand, the elastic strain is described in terms of the effective stress by Eq. (5.44) as follows:

$$\varepsilon_{ij}^{e} = \frac{1+v}{E_0}\underset{\sim}{\sigma}_{ij} - \frac{v}{E_0}\underset{\sim}{\sigma}_{kk}\delta_{ij} \tag{14.92}$$

The effective stress is described in terms of the current stress by comparing Eqs. (14.91) and (14.92) as follows:

$$\underset{\sim}{\sigma}_{ij} = \frac{\langle\boldsymbol{\sigma}\rangle_{ij}^{+} + \langle\boldsymbol{\sigma}\rangle_{ij}^{-}}{1-D} + \frac{v}{1-2v}\frac{(\langle\boldsymbol{\sigma}\rangle_{ij}^{+} + \langle\boldsymbol{\sigma}\rangle_{ij}^{-})\delta_{kl} + \langle\sigma_{rr}\rangle - \langle-\sigma_{rr}\rangle}{1-D}\delta_{ij} \tag{14.93}$$

Equations (14.88), (14.89), (14.90) and (14.93) are extended to the unilateral equations as follows:

$$\phi(\boldsymbol{\sigma}, D) = \frac{1 + v}{2E_0}\left(\frac{\langle\boldsymbol{\sigma}\rangle_{ij}^{+}\langle\boldsymbol{\sigma}\rangle_{ij}^{+}}{1-D} + \frac{\langle\boldsymbol{\sigma}\rangle_{ij}^{-}\langle\boldsymbol{\sigma}\rangle_{ij}^{-}}{1-hD}\right) - \frac{v}{2E_0}\left(\frac{\langle\sigma_{kk}\rangle^2}{1-D} - \frac{\langle-\sigma_{kk}\rangle^2}{1-hD}\right) \tag{14.94}$$

$$\begin{aligned} Y &= -\frac{\partial\psi(\boldsymbol{\varepsilon}^e, D)}{\partial D} \\ &= \frac{1+v}{2E_0}\left(\frac{\langle\boldsymbol{\sigma}\rangle_{ij}^{+}\langle\boldsymbol{\sigma}\rangle_{ij}^{+}}{(1-D)^2} + h\frac{\langle\boldsymbol{\sigma}\rangle_{ij}^{-}\langle\boldsymbol{\sigma}\rangle_{ij}^{-}}{(1-hD)^2}\right) - \frac{v}{2E_0}\left(\frac{\langle\sigma_{kk}\rangle^2}{(1-D)^2} + h\frac{\langle-\sigma_{kk}\rangle^2}{(1-hD)^2}\right) \end{aligned} \tag{14.95}$$

$$\varepsilon_{ij}^{e} = \frac{\partial\phi(\boldsymbol{\sigma}, D)}{\partial\boldsymbol{\sigma}} = \frac{1 + v}{E_0}\left(\frac{\langle\boldsymbol{\sigma}\rangle_{ij}^{+}}{1-D} + \frac{\langle\boldsymbol{\sigma}\rangle_{ij}^{-}}{1-hD}\right) - \frac{v}{E_0}\left(\frac{\langle\sigma_{kk}\rangle}{1-D} - \frac{\langle-\sigma_{kk}\rangle}{1-hD}\right)\delta_{ij} \tag{14.96}$$

$$\underset{\sim}{\sigma}_{ij} = \frac{\langle\boldsymbol{\sigma}\rangle_{ij}^{+}}{1-D} + \frac{\langle\boldsymbol{\sigma}\rangle_{ij}^{-}}{1-hD} + \frac{v}{1-2v}\left(\frac{\langle\boldsymbol{\sigma}\rangle_{kl}^{+}\delta_{kl} - \langle-\sigma_{rr}\rangle}{1-D} + \frac{\langle\boldsymbol{\sigma}\rangle_{kl}^{-}\delta_{kl} - \langle-\sigma_{rr}\rangle}{1-hD}\right)\delta_{ij} \tag{14.97}$$

where $h(0 \le h \le 1)$ is the material parameter corresponding to the *bilateral* and the *unilateral crack effect* for h = 1 and h = 0, respectively.

Let the following variables be introduced for more concise description.

$$\sigma_m \equiv \sigma_{kk}/3 \tag{14.98}$$

$$\sigma_m^{+} \equiv \langle\boldsymbol{\sigma}\rangle_{kl}^{+}\delta_{kl}/3, \quad \sigma_m^{-} \equiv \langle\boldsymbol{\sigma}\rangle_{ij}^{-}\delta_{kl}/3 \tag{14.99}$$

$$\sigma_{ij}^{+} \equiv \langle\boldsymbol{\sigma}\rangle_{ij}^{+} + \frac{3v}{1-2v}(\sigma_m^{+} - \langle\sigma_m\rangle)\delta_{ij}, \quad \sigma_{ij}^{-} \equiv \langle\boldsymbol{\sigma}\rangle_{ij}^{-} + \frac{3v}{1-2v}(\sigma_m^{-} - \langle-\sigma_m\rangle)\delta_{ij} \tag{14.100}$$

by which the effective stress in Eq. (14.97) is described as

$$\underset{\sim}{\sigma}_{ij} = \frac{\sigma_{ij}^{+}}{1-D} + \frac{\sigma_{ij}^{-}}{1-hD} \tag{14.101}$$

where the factor $3v/(1-2v)$ is coupling term which accounts for shear effect.

Equations (14.98) to (14.101) are reduced in the one-dimensional state which can be interpreted concisely as follows (Lemaitre 1992):

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{ll}
\text{Tension } \sigma > 0 & \text{Compression } \sigma < 0 \\
\sigma_m = \sigma/3 & \sigma_m = \sigma/3(<0) \\
\left.\begin{array}{l} \sigma_m^{+} = \sigma/3 \\ \sigma_m^{-} = 0 \end{array}\right\} & \left.\begin{array}{l} \sigma_m^{+} = 0 \\ \sigma_m^{-} = -\sigma/3(>0) \end{array}\right\} \\
\left.\begin{array}{l} \sigma_m^{+} - \langle \sigma_m \rangle \\ \sigma_m^{-} - \langle -\sigma_m \rangle \end{array}\right\} = 0 & \left.\begin{array}{l} \sigma_m^{+} - \langle \sigma_m \rangle \\ \sigma_m^{-} - \langle -\sigma_m \rangle \end{array}\right\} = 0 \\
\left.\begin{array}{l} \sigma_{11}^{+} = \sigma \\ \sigma_{11}^{-} = 0 \end{array}\right\} & \left.\begin{array}{l} \sigma_{11}^{+} = 0 \\ \sigma_{11}^{-} = -\sigma \end{array}\right\} \\
\underset{\sim}{\sigma}_{11} = \dfrac{\sigma}{1-D} & \underset{\sim}{\sigma}_{11} = \dfrac{\sigma}{1-hD}(<0)
\end{array} \tag{14.102}$$

Therefore, the difference between the actual stress and the effective stress in the compression state is smaller than that in the tension state if $h < 1$.

14.6.4 Anisotropic (Orthotropic) Damage Tensor

The following asymmetric effective stress was proposed by Murakami and Ohno (1981) and Murakami (1988).

$$\underset{\sim}{\boldsymbol{\sigma}} = (\mathbf{I} - \boldsymbol{\mathcal{D}})^{-1}\boldsymbol{\sigma} \tag{14.103}$$

However, an asymmetric stress tensor makes the mathematical formulation and mechanical analysis very complicated. Then, various symmetrized effective stress tensors have been proposed. For instance,

$$\underset{\sim}{\boldsymbol{\sigma}} = [\boldsymbol{\sigma}(\mathbf{I} - \boldsymbol{\mathcal{D}})^{-1} + (\mathbf{I} - \boldsymbol{\mathcal{D}})^{-1}\boldsymbol{\sigma}]/2 \tag{14.104}$$

and

$$\underset{\sim}{\boldsymbol{\sigma}} = (\mathbf{I} - \boldsymbol{\mathcal{D}})^{-1}\boldsymbol{\sigma}(\mathbf{I} - \boldsymbol{\mathcal{D}})^{-1} \tag{14.105}$$

have been proposed by Murakami and Ohno (1981) and Betton (1986), respectively. However, there would not exist potential function leading to them.

Cordebois and Sidoroff (1982a, b) proposed the effective stress tensor

$$\underset{\sim}{\boldsymbol{\sigma}} = \mathbf{H}\boldsymbol{\sigma}\mathbf{H} \tag{14.106}$$

where

$$\mathbf{H} \equiv (\mathbf{I} - \boldsymbol{\mathcal{D}})^{-1/2}(=\mathbf{H}^T) \tag{14.107}$$

Equation (14.106) is derived from the potential function as follows:

$$\psi = C\mathrm{tr}(\mathbf{H}\boldsymbol{\sigma}\mathbf{H}\boldsymbol{\sigma}) \tag{14.108}$$

$$\boldsymbol{\varepsilon}^e = \frac{\partial \psi}{\partial \boldsymbol{\sigma}} = 2C\mathbf{H}\boldsymbol{\sigma}\mathbf{H} = 2C\underset{\sim}{\boldsymbol{\sigma}} \tag{14.109}$$

However, Eq. (14.109) is physically impertinent in the present form.

Extending Eq. (14.59) to the anisotropic damage, Lemaitre et al. (2000) assumed the following Gibbs energy.

$$\psi = \frac{1+v}{2E}\mathrm{tr}(\mathbf{H}\boldsymbol{\sigma}'\mathbf{H}\boldsymbol{\sigma}') + \frac{3(1-2v)}{2E}\frac{\sigma_m^2}{1-\eta\mathcal{D}_m} \tag{14.110}$$

where

$$\mathcal{D}_m \equiv \frac{1}{3}\mathrm{tr}\boldsymbol{\mathcal{D}} \tag{14.111}$$

η is an hydrostatic sensitivity parameter concerning the Poisson's ratio with damage, while $\eta \cong 3$ is used most often. The particular case chosen as $\boldsymbol{\mathcal{D}} = D\mathbf{I}$ and $\eta = 1$ corresponds to the isotropic damage.

The elastic strain is derived from Eq. (14.110) as follows:

$$\boldsymbol{\varepsilon}^e = \frac{\partial \psi}{\partial \boldsymbol{\sigma}} = \frac{1+v}{E}\underset{\sim}{\boldsymbol{\sigma}} - \frac{3v}{E}\underset{\sim}{\sigma}_m\mathbf{I} \tag{14.112}$$

which is of identical form to the Hooke's law but the effective stress is related to the actual stress as follows (Lemaitre et al., 2000):

$$\underset{\sim}{\boldsymbol{\sigma}} \equiv (\mathbf{H}\boldsymbol{\sigma}'\mathbf{H})' + \frac{\sigma_m}{1-\eta\mathcal{D}_m}\mathbf{I} \tag{14.113}$$

noting

$$\begin{aligned}
\frac{\partial\psi}{\partial\sigma_{ij}} &= \frac{\partial}{\partial\sigma_{ij}}\left[\frac{1+v}{2E}H_{pq}\sigma'_{qr}H_{rs}\sigma'_{sp} + \frac{3(1-2v)}{2E}\frac{\sigma_m^2}{1-\eta\mathcal{D}_m}\right] \\
&= \frac{1+v}{2E}\left[H_{pq}\frac{\partial\sigma'_{qr}}{\partial\sigma_{ij}}H_{rs}\sigma'_{sp} + H_{pq}\sigma'_{qr}H_{rs}\frac{\partial\sigma'_{sp}}{\partial\sigma_{ij}}\right] + \frac{3(1-2v)}{2E}\frac{2\sigma_m}{1-\eta\mathcal{D}_m}\frac{1}{3}\delta_{ij} \\
&= \frac{1+v}{2E}\left[H_{pq}(\delta_{iq}\delta_{jr} - \frac{1}{3}\delta_{qr}\delta_{ij})H_{rs}\sigma'_{sp} + H_{pq}\sigma'_{qr}H_{rs}(\delta_{is}\delta_{jp} - \frac{1}{3}\delta_{sp}\delta_{ij})\right] \\
&\quad + \frac{1-2v}{E}\frac{\sigma_m}{1-\eta\mathcal{D}_m}\delta_{ij} \\
&= \frac{1+v}{2E}\left[(H_{pi}\sigma'_{sp}H_{js} - \frac{1}{3}H_{pq}\sigma'_{sp}H_{qs}\delta_{ij}) + (H_{jq}\sigma'_{qr}H_{ri} - \frac{1}{3}H_{pq}\sigma'_{qr}H_{rp}\delta_{ij})\right] \\
&\quad + \frac{1-2v}{E}\frac{\sigma_m}{1-\eta\mathcal{D}_m}\delta_{ij} \\
&= \frac{1+v}{E}(H_{pi}\sigma'_{sp}H_{js})' + \frac{1-2v}{E}\frac{\sigma_m}{1-\eta\mathcal{D}_m}\delta_{ij} \\
&= \frac{1+v}{E}\left[(H_{pi}\sigma'_{sp}H_{js})' + \frac{\sigma_m}{1-\eta\mathcal{D}_m}\delta_{ij}\right] - \frac{v}{E}\frac{\sigma_m}{1-\eta\mathcal{D}_m}3\delta_{ij} \\
&= \frac{1+v}{E}\left[(H_{pi}\sigma'_{sp}H_{js})' + \frac{\sigma_m}{1-\eta\mathcal{D}_m}\delta_{ij}\right] - \frac{v}{E}[\underbrace{(H_{pi}\sigma'_{sp}H_{js})'_{aa}}_{0} + \frac{\sigma_m}{1-\eta\mathcal{D}_m}\delta_{aa}]\delta_{ij}
\end{aligned}$$

Equation (14.113) is regarded to be the modification of Eq. (14.106) proposed by Cordebois and Sidoroff (1982a, b) so as to conform to the Hooke's elastic behavior.

Equation (14.113) is rewritten as follows:

$$\underset{\sim}{\boldsymbol{\sigma}} = \mathbf{M}(\mathcal{D}) : \boldsymbol{\sigma} \tag{14.114}$$

i.e.

$$\underset{\sim}{\boldsymbol{\sigma}} = \mathbf{M}(\mathcal{D}) : \boldsymbol{\sigma}' + \sigma_m\mathbf{M}(\mathcal{D}) : \mathbf{I} \tag{14.115}$$

where

$$\mathbf{M}(\mathcal{D}) = \mathbf{H}\tilde{\otimes}\mathbf{H} - \frac{1}{3}(\mathbf{H}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{H}^2) + \frac{1}{9}(\mathrm{tr}\mathbf{H}^2)\mathbf{I} \otimes \mathbf{I} + \frac{\mathbf{I} \otimes \mathbf{I}}{3(1-\eta\mathcal{D}_m)} \tag{14.116}$$

with the notation $(\mathbf{A}\tilde{\otimes}\mathbf{B})_{ij} \equiv T_{ik}T_{lj}$ in Eq. (1.151$_4$), noting

$$\begin{aligned}(\mathbf{H}\boldsymbol{\sigma}'\mathbf{H})'_{ij} &= (H_{ik}\sigma'_{kl}H_{lj})' = H_{ik}\sigma'_{kl}H_{lj} - H_{rk}\sigma'_{kl}H_{lr}\delta_{ij}/3 \\ &= H_{ik}(\sigma_{kl} - \sigma_m\delta_{kl})H_{lj} - H_{rk}(\sigma_{kl} - \sigma_m\delta_{kl})H_{lr}\delta_{ij}/3 \\ &= H_{ik}H_{lj}\sigma_{kl} - H_{ik}H_{kj}\sigma_m - H_{lr}H_{rk}\sigma_{kl}\delta_{ij}/3 + H_{rk}H_{kr}\delta_{ij}\sigma_m/3 \\ &= H_{ik}H_{lj}\sigma_{kl} - H_{ir}H_{rj}\sigma_m - H_{lr}H_{rk}\sigma_{kl}\delta_{ij}/3 + H_{rk}H_{kr}\delta_{ij}\sigma_m/3 \\ &= [H_{ik}H_{lj} - (H_{ir}H_{rj}\delta_{kl} + \delta_{ij}H_{lr}H_{rk})/3 + H_{rk}H_{kr}\delta_{ij}\delta_{kl}/9]\sigma_{kl} \\ &= \{[\mathbf{H}\tilde{\otimes}\mathbf{H} - (\mathbf{H}^2\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{H}^2)/3 + (\mathrm{tr}\mathbf{H}^2)\mathbf{I}\otimes\mathbf{I}/9]\boldsymbol{\sigma}\}_{ij}\end{aligned}$$

$$\mathbf{M}(\boldsymbol{\mathcal{D}}):\mathbf{I} = \left[\mathbf{H}\tilde{\otimes}\mathbf{H} - \frac{1}{3}(\mathbf{H}^2\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{H}^2) + \frac{1}{9}(\mathrm{tr}\mathbf{H}^2)\mathbf{I}\otimes\mathbf{I} + \frac{\mathbf{I}\otimes\mathbf{I}}{3(1-\eta\mathcal{D}_m)}\right]:\mathbf{I} = \frac{\mathbf{I}}{1-\eta\mathcal{D}_m}$$

$$\mathbf{H}\tilde{\otimes}\mathbf{H}:\mathbf{I} = \mathbf{H}^2, \mathbf{H}^2\otimes\mathbf{I}:\mathbf{I} = 3\mathbf{H}^2, \mathbf{I}\otimes\mathbf{H}^2:\mathbf{I} = (\mathrm{tr}\mathbf{H}^2)\mathbf{I}$$

Equations (14.114) are inverted as follows (Mengoni and Ponthot 2015):

$$\boldsymbol{\sigma} = \mathbf{H}^{-1}\underset{\sim}{\boldsymbol{\sigma}}'\mathbf{H}^{-1} - \frac{\underset{\sim}{\boldsymbol{\sigma}}':\mathbf{H}^2}{\mathrm{tr}\mathbf{H}^{-2}}\mathbf{H}^{-2} + (1-\eta\mathcal{D}_m)\underset{\sim}{\sigma}_m\mathbf{I} \tag{14.117}$$

$$\boldsymbol{\sigma} = \mathbf{M}^{-1}(\boldsymbol{\mathcal{D}}):\underset{\sim}{\boldsymbol{\sigma}} \tag{14.118}$$

where

$$\mathbf{M}^{-1}(\boldsymbol{D}) = \mathbf{H}^{-1}\underline{\otimes}\mathbf{H}^{-1} - \frac{\mathbf{H}^{-2}\otimes\mathbf{H}^{-2}}{\mathrm{tr}\mathbf{H}^{-2}} + \frac{1}{3}(1-\eta\mathcal{D}_m)\mathbf{I}\otimes\mathbf{I} \tag{14.119}$$

The rate of the damage tensor is given by

$$\dot{\boldsymbol{\mathcal{D}}} = \left(\frac{Y}{S}\right)^m |\mathbf{d}^p| \tag{14.120}$$

where S and m are material parameters and $|\mathbf{d}^p|$ is defined as

$$|\mathbf{d}^P| \equiv \sum_{P=1}^{3}|d_P^p|\mathbf{n}^P\otimes\mathbf{n}^P \tag{14.121}$$

d_P^p and $\mathbf{n}^P$ are the principal values and the normalized principal direction vectors of plastic strain rate. Y is already shown in Eq. (14.67). Then, the principal directions of the damage rate coincide with those of the plastic strain rate.

The plastic strain rate and the fictitious effective stress rate are calculated by the fictitious elastoplastic constitutive relations by inputs of strain rate. Then, the internal variables in the fictitious undamaged configuration and the damage tensor are calculated by the plastic strain rate, and then the current stress is calculated though the damage tensor from the fictitious undamaged stress.

In the uniaxial loading, the damage tensors $\boldsymbol{\mathcal{D}}$ and $\mathbf{H}$ are described in the orthotropic coordinate system the axes of which coincide with the principal directions of the damage tensor as

$$\boldsymbol{\mathcal{D}} = \begin{bmatrix} \mathcal{D}_1 & 0 & 0 \\ 0 & \mathcal{D}_2 & 0 \\ 0 & 0 & \mathcal{D}_2 \end{bmatrix} \tag{14.122}$$

$$\mathbf{H} = \begin{bmatrix} \frac{1}{\sqrt{1-\mathcal{D}_1}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{1-\mathcal{D}_2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{1-\mathcal{D}_3}} \end{bmatrix} \tag{14.123}$$

The effective equivalent stress $\underset{\sim}{\sigma}^{eq}$ is different from the effective stress $\underset{\sim}{\sigma}_1$ as follows:

$$\left.\begin{aligned} \underset{\sim}{\boldsymbol{\sigma}}^{eq} &= \frac{2}{3}\frac{\sigma_1}{1-\mathcal{D}_1} + \frac{1}{3}\frac{\sigma_1}{1-\mathcal{D}_2} \\ \underset{\sim}{\boldsymbol{\sigma}}_1 &= \frac{4}{9}\frac{\sigma_1}{1-\mathcal{D}_1} + \frac{2}{9}\frac{\sigma_1}{1-\mathcal{D}_2} + \frac{1}{3}\frac{\sigma_1}{1-\eta\mathcal{D}_m} \end{aligned}\right\} \tag{14.124}$$

The elastic strain is described by Eq. (14.112) with Eq. (14.113), noting Eqs. (14.122) and (14.123) as follows (Lemaitre et al. 2000):

$$\begin{aligned}
&\begin{bmatrix} \varepsilon_1^e & 0 & 0 \\ 0 & \varepsilon_2^e & 0 \\ 0 & 0 & \varepsilon_3^e \end{bmatrix} \\
&= \frac{1+v}{E}\left(\begin{bmatrix} \frac{1}{\sqrt{1-\mathcal{D}_1}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{1-\mathcal{D}_2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{1-\mathcal{D}_3}} \end{bmatrix}\begin{bmatrix} \frac{2}{3}\sigma_1 & 0 & 0 \\ 0 & -\frac{1}{3}\sigma_1 & 0 \\ 0 & 0 & -\frac{1}{3}\sigma_1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{1-\mathcal{D}_1}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{1-\mathcal{D}_2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{1-\mathcal{D}_3}} \end{bmatrix}\right)' \\
&\quad + \frac{1+v}{E}\frac{\sigma_m}{1-\eta\mathcal{D}_m}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{v}{E}\frac{3\sigma_m}{1-\eta\mathcal{D}_m}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \frac{1+v}{E}\begin{bmatrix} \frac{2}{3}\frac{1}{1-\mathcal{D}_1} & 0 & 0 \\ 0 & -\frac{1}{3}\frac{1}{1-\mathcal{D}_2} & 0 \\ 0 & 0 & -\frac{1}{3}\frac{1}{1-\mathcal{D}_3} \end{bmatrix}'\sigma_1 + \frac{1+v}{E}\frac{\sigma_1/3}{1-\eta\mathcal{D}_m}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{v}{E}\frac{3\sigma_1/3}{1-\eta\mathcal{D}_m}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&\qquad \left(\mathcal{D}_m = \frac{1}{3}\left(\frac{2}{3}\frac{1}{1-\mathcal{D}_1} - \frac{1}{3}\frac{1}{1-\mathcal{D}_2} - \frac{1}{3}\frac{1}{1-\mathcal{D}_3}\right) = \frac{1}{9}\left(\frac{2}{1-\mathcal{D}_1} - \frac{1}{1-\mathcal{D}_2} - \frac{1}{1-\mathcal{D}_3}\right)\right) \\
&= \frac{1+v}{9E}\begin{bmatrix} \frac{4}{1-\mathcal{D}_1} + \frac{1}{1-\mathcal{D}_2} + \frac{1}{1-\mathcal{D}_3} & 0 & 0 \\ 0 & -\frac{2}{1-\mathcal{D}_1} - \frac{2}{1-\mathcal{D}_2} + \frac{1}{1-\mathcal{D}_3} & 0 \\ 0 & 0 & -\frac{2}{1-\mathcal{D}_1} + \frac{1}{1-\mathcal{D}_2} - \frac{2}{1-\mathcal{D}_3} \end{bmatrix}\sigma_1 \\
&\quad + \frac{1-2v}{E}\frac{\sigma_1}{3(1-\eta\mathcal{D}_m)}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned} \tag{14.125}$$

Then, setting

$$\underset{\sim}{E}_1 \equiv \frac{\sigma_1}{\varepsilon_1^e}, \quad \underset{\sim}{v}_{12} \equiv -\frac{\varepsilon_2^e}{\varepsilon_1^e}, \quad \underset{\sim}{v}_{13} \equiv -\frac{\varepsilon_3^e}{\varepsilon_1^e} \tag{14.126}$$

one has

$$\left.\begin{aligned} \frac{E}{\underset{\sim}{E}_1} &= \frac{1+v}{9}\left(\frac{4}{1-\mathcal{D}_1}+\frac{1}{1-\mathcal{D}_2}+\frac{1}{1-\mathcal{D}_3}\right)+\frac{1-2v}{3(1-\eta\mathcal{D}_m)} \\ \underset{\sim}{v}_{12}\frac{E}{\underset{\sim}{E}_1} &= \frac{1+v}{9}\left(\frac{2}{1-\mathcal{D}_1}+\frac{2}{1-\mathcal{D}_2}-\frac{1}{1-\mathcal{D}_3}\right)-\frac{1-2v}{3(1-\eta\mathcal{D}_m)} \\ \underset{\sim}{v}_{13}\frac{E}{\underset{\sim}{E}_1} &= \frac{1+v}{9}\left(\frac{2}{1-\mathcal{D}_1}-\frac{1}{1-\mathcal{D}_2}+\frac{2}{1-\mathcal{D}_3}\right)-\frac{1-2v}{3(1-\eta\mathcal{D}_m)} \end{aligned}\right\} \tag{14.127}$$

The unilateral formulation for the anisotropic damage can be referred to Ladeveze and Lemaitre (1984) Lemaitre and Desmora (2005).

14.7 Subloading-Overstress Damage Model

The subloading-damage model formulated in the preceding sections will be extended to be taken account of the rate-dependent plastic deformation by incorporating the subloading-overstress model (Hashiguchi 2013a) in the following.

Equations (14.29) and (14.31) in the subloading-damage model is extended by incorporating the subloading-overstress model in Eq. (13.29) as follows:

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^{vp} = \mathbf{E}_0^{-1} : \overset{\circ}{\underset{\sim}{\boldsymbol{\sigma}}} + \frac{1}{\bar{\mu}}\frac{\langle R - R_s\rangle^n}{R_m - R}\underset{\sim}{\hat{\mathbf{n}}} \tag{14.128}$$

$$\overset{\circ}{\underset{\sim}{\boldsymbol{\sigma}}} = \mathbf{E}_0 : \dot{\boldsymbol{\varepsilon}} - \frac{1}{\bar{\mu}}\frac{\langle R - R_s\rangle^n}{R_m - R}\mathbf{E}_0 : \underset{\sim}{\hat{\mathbf{n}}} \tag{14.129}$$

where $\dot{\boldsymbol{\varepsilon}}^{vp}$ is the viscoplastic strain rate for which the loading criterion is imposed by incorporating the Macaulay's bracket. $\bar{\mu}$ and n are the material parameters, while $\bar{\mu}$ stands for the viscoplastic coefficient. The surface which passes through the current stress and is similar to the normal-yield surface is called the *dynamic-loading surface* and the ratio of the size of the dynamic-loading surface to the normal-yield surface is described by $R(=f(\underset{\sim}{\hat{\boldsymbol{\sigma}}})/\tilde{F})$ which can be larger than unity and is called the *dynamic-loading ratio*. $R_m(\gg 1)$ is the material constant designating the maximum value of the dynamic-loading ratio, called the *limit dynamic-loading ratio*. The rates of the internal state variables $\underset{\sim}{H}, \underset{\sim}{\boldsymbol{\alpha}}, D, \boldsymbol{\mathcal{D}}$ are given by replacing the plastic strain rate $\dot{\boldsymbol{\varepsilon}}^{p}$ to the viscoplastic strain rate $\dot{\boldsymbol{\varepsilon}}^{vp}$ in Eqs. (14.14), (14.42), (14.63) and (14.120) as follows:

$$\dot{\underset{\sim}{H}} = \underset{\sim}{f}_{H\varepsilon}(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{H}; \dot{\boldsymbol{\varepsilon}}^{vp}) \tag{14.130}$$

$$\dot{\underset{\sim}{\boldsymbol{\alpha}}} = \underset{\sim}{\mathbf{f}}_{k\varepsilon}(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{\boldsymbol{\alpha}}, \underset{\sim}{F}; \dot{\boldsymbol{\varepsilon}}^{vp\prime}) = c_k \left(\dot{\boldsymbol{\varepsilon}}^{vp\prime} - \frac{1}{b_k(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{F})} \underset{\sim}{\boldsymbol{\alpha}} \left\| \dot{\boldsymbol{\varepsilon}}^{vp\prime} \right\| \right) \tag{14.131}$$

$$\dot{D} = \left(\frac{Y}{\zeta} \right)^{a} \frac{\hat{H}[\varepsilon^{eqvp} - \varepsilon_{d}^{eqvp}]}{1 - D} \dot{\boldsymbol{\varepsilon}}^{eqvp} \tag{14.132}$$

$$\dot{\boldsymbol{D}} = \left(\frac{\overline{Y}}{S} \right)^{m} |\mathbf{d}^{vp}| = \left(\frac{\overline{Y}}{S} \right)^{m} \sum_{P=1}^{3} |d_{P}^{vp}| \mathbf{n}^{P} \otimes \mathbf{n}^{P} \tag{14.133}$$

$$\varepsilon^{eqvp} \equiv \sqrt{2/3} \int \left\| \dot{\boldsymbol{\varepsilon}}^{vp} \right\| dt \tag{14.134}$$

The rate of the subloading ratio $R_s (0 \leq R_s \leq 1)$ is given by replacing the normal-yield ratio R to R_s and the plastic strain increment $\dot{\boldsymbol{\varepsilon}}^{p}$ to the viscoplastic strain rate $\dot{\boldsymbol{\varepsilon}}^{vp}$ in Eq. (14.17) for the plastic sliding process and the subloading ratio R_s is identical to the normal sliding-yield ratio R for the elastic sliding process as follows:

$$\dot{R}_s = U(R_s) \left\| \dot{\boldsymbol{\varepsilon}}^{vp} \right\| \quad \text{for} \ \ \dot{\boldsymbol{\varepsilon}}^{vp} \neq \mathbf{O} \tag{14.135}$$

$$R_s = R \quad \text{for} \ \ \dot{\boldsymbol{\varepsilon}}^{vp} = \mathbf{O} \tag{14.136}$$

The smooth transition from the elastic to the viscoplastic state is described by incorporating R_s instead of unity. The response of the subloading-overstress damage model is shown in Fig. 14.3.

14.8 Subloading-Gruson Model

Plastic deformation is induced under hydrostatic stress in porous media even if the base material is the Mises material the plastic deformation behavior of which is independent of hydrostatic stress. The elastoplastic constitutive model taken account of the nucleation and the growth of round voids was proposed first by Gurson (1977) and further studied by Needleman and Rice (1978), Tvergaard and Needleman (1984), Needleman and Tvergaard (1985), etc. The yield surface is

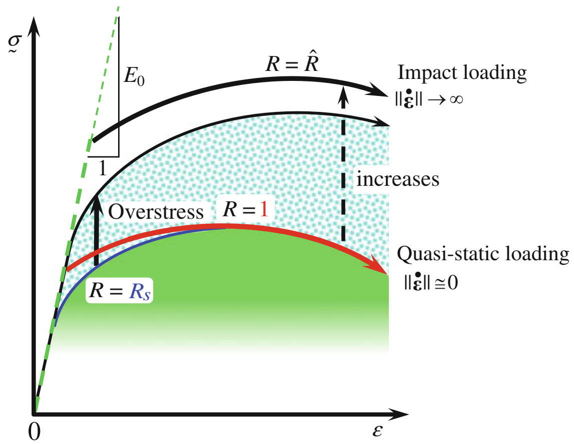

Fig. 14.3 Stress-strain curve predicted by the subloading-overstress damage model

introduced, which is taken account of the void volume fraction and the mean stress with the evolution rule of the void volume fraction. It is often called the *Gurson model* and its elaboration taken account of the void coalescence is called the *GTN (Gurson-Tvergaard-Needleman) model*. The subloading-void(Gurson) model will be described in this section.

The following yield condition is derived by Gurson (1977) by the symmetric deformation analysis of the rigid-plastic Mises material containing a spherical cavity.

$$\psi(\boldsymbol{\sigma}, F, \xi) = \left(\frac{\sigma^{eq}}{F}\right)^2 + 2\xi \cosh\left(\frac{3}{2}\frac{\sigma_m}{F}\right) - \xi^2 - 1 = 0 \tag{14.137}$$

where ξ is the void volume fraction. Equation (14.137) is reduced to the von Mises yield condition, i.e. $\sigma^{eq} = F$ for $\xi = 0$. The dependence of the yield function in Eq. (14.137) is shown in Fig. 14.4.

The rate of the void volume fraction $\dot{\xi}$ is given by sum of the growth rate $\dot{\xi}_{grow}$ and the nucleation rate of new void $\dot{\xi}_{nucl}$ as follows (Needleman and Rice 1978):

$$\dot{\xi} = \dot{\xi}_{grow} + \dot{\xi}_{nucl} \tag{14.138}$$

where

$$\left.\begin{array}{l} \dot{\xi}_{grow} = (1-\xi)\,\mathrm{tr}\,\dot{\boldsymbol{\varepsilon}}_v^p \\ \dot{\xi}_{nucl} = a_1(\dot{F} + \dot{\sigma}_m) + a_2\,\dot{\varepsilon}^{eqp} \end{array}\right\} \tag{14.139}$$

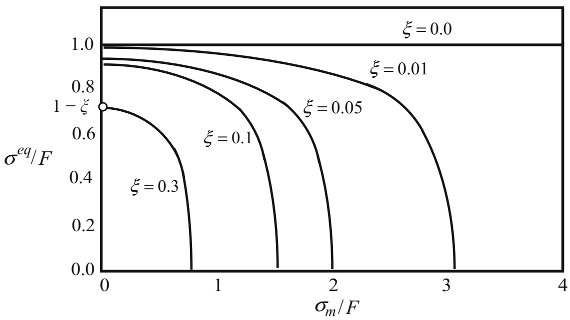

Fig. 14.4 Effect of void volume fraction in Gurson yield surface

The coefficient a_1 and a_2 are given by Chu and Needleman (1980) as follows:

$$\left.\begin{aligned} a_1 &= \frac{f_n}{\sqrt{2\pi}s_n}\exp\left[-\frac{1}{2}\left(\frac{F+\sigma_m-\sigma_n}{s_n}\right)^2\right] \\ a_2 &= \frac{f_n}{\sqrt{2\pi}s_n}\exp\left[-\frac{1}{2}\left(\frac{\varepsilon^{eqp}-\varepsilon_n}{s_n}\right)^2\right] \end{aligned}\right\} \tag{14.140}$$

which is derived postulating that the voids nucleates according to the probability distribution with the stress σ_n and the strain ε_n as their mean values together with s_n as their standard deviation, and f_n is the volume fraction of void nucleating particles.

The subloading surface for the normal-yield surface in Eq. (14.137) is given by replacing F to RF in Eq. (14.137) as follows:

$$\psi(\boldsymbol{\sigma}, F, \xi) = \left(\frac{\sigma^{eq}}{RF}\right)^2 + 2\xi\cosh\left(\frac{3}{2}\frac{\sigma_m}{RF}\right) - \xi^2 - 1 = 0 \tag{14.141}$$

The time-differentiation of Eq. (14.141) is given by

$$\begin{aligned} \dot{\psi}(\boldsymbol{\sigma}, F, \xi) = 2\left(\frac{\sigma^{eq}}{RF}\right)\frac{\dot{\sigma}^{eq}RF - \sigma^{eq}(R\dot{F} + \dot{R}F)}{R^2F^2} \\ + 2\xi\sinh\left(\frac{3}{2}\frac{\sigma_m}{RF}\right)\frac{3}{2}\frac{\dot{\sigma}_m RF - \sigma_m(R\dot{F} + \dot{R}F)}{R^2F^2} - 2\dot{\xi} = 0 \end{aligned} \tag{14.142}$$

from which one has

$$2\left(\frac{\sigma^{eq}}{RF}\right)\left[\dot{\sigma}^{eq} - \sigma^{eq}\left(\frac{\dot{F}}{F} + \frac{\dot{R}}{R}\right)\right] + 3\xi\sinh\left(\frac{3}{2}\frac{\sigma_m}{RF}\right)\left[\dot{\sigma}_m - \sigma_m\left(\frac{\dot{F}}{F} + \frac{\dot{R}}{R}\right)\right] - 2RF\dot{\xi} = 0 \tag{14.143}$$

Assume the associated flow rule

$$\overset{\bullet}{\boldsymbol{\varepsilon}}^{p} = \overset{\bullet}{\bar{\lambda}}\,\mathbf{n}^{\psi} \quad (\overset{\bullet}{\bar{\lambda}} > 0) \tag{14.144}$$

where

$$\mathbf{n}^{\psi} \equiv \frac{\partial\psi}{\partial\boldsymbol{\sigma}} / \left|\left|\frac{\partial\psi}{\partial\boldsymbol{\sigma}}\right|\right| \quad (||\mathbf{n}^{\psi}|| = 1) \tag{14.145}$$

It follows adopting the associated flow rule in Eq. (14.144) with Eqs. (6.42) and (7.9) that

$$2\left(\frac{\sigma^{eq}}{RF}\right)\left[\dot{\sigma}^{eq} - \sigma^{eq}\left(\frac{F'\overset{\bullet}{\bar{\lambda}}h^{\psi}}{F} + \frac{U\overset{\bullet}{\bar{\lambda}}}{R}\right)\right] + 3\xi\sinh\left(\frac{3}{2}\frac{\sigma_m}{RF}\right)\left[\dot{\sigma}_m - \sigma_m\left(\frac{F'\overset{\bullet}{\bar{\lambda}}h^{\psi}}{F} + \frac{U\overset{\bullet}{\bar{\lambda}}}{R}\right)\right]$$
$$- 2RF\left[(1-\xi)\overset{\bullet}{\bar{\lambda}}\,\mathrm{tr}\mathbf{n}^{\psi} + a_1(F'\overset{\bullet}{\bar{\lambda}}h^{\psi} + \dot{\sigma}_m) + a_2\sqrt{\frac{2}{3}}\overset{\bullet}{\bar{\lambda}}\right] = 0$$

resulting in

$$\begin{aligned} &2\left(\frac{\sigma^{eq}}{RF}\right)\dot{\sigma}^{eq} + \left[3\xi\sinh\left(\frac{3}{2}\frac{\sigma_m}{RF}\right) - 2a_1RF\right]\dot{\sigma}_m \\ &\quad - \left\{\left[2\sigma^{eq}\left(\frac{\sigma^{eq}}{RF}\right) + 3\xi\sigma_m\sinh\left(\frac{3}{2}\frac{\sigma_m}{RF}\right)\right]\left(\frac{F'}{F}h^{\psi} + \frac{U}{R}\right)\right] \\ &\quad + 2RF\left[(1-\xi)\mathrm{tr}\mathbf{n}^{\psi} + a_1F'h^{\psi} + \sqrt{\frac{2}{3}}a_2\right]\right\}\overset{\bullet}{\bar{\lambda}} = 0 \end{aligned} \tag{14.146}$$

where

$$h^{\psi} \equiv \dot{H} \,/\, \overset{\bullet}{\bar{\lambda}}\,(=\sqrt{2/3}) \tag{14.147}$$

Noting

$$\dot{\sigma}_m = \frac{1}{3}\mathbf{I}:\dot{\boldsymbol{\sigma}},\ \dot{\sigma}^{eq} = \left(\sqrt{\frac{3}{2}}||\boldsymbol{\sigma}'||\right)^{\bullet} = \sqrt{\frac{3}{2}}\frac{\boldsymbol{\sigma}':\dot{\boldsymbol{\sigma}}}{||\boldsymbol{\sigma}'||} = \frac{3}{2}\frac{\boldsymbol{\sigma}':\dot{\boldsymbol{\sigma}}}{\sigma^{eq}} \tag{14.148}$$

one has

$$2\left(\frac{\sigma^{eq}}{RF}\right)\dot{\sigma}^{eq} + \left[3\xi \sinh\left(\frac{3}{2}\frac{\sigma_m}{RF}\right) - 2a_1 RF\right]\dot{\sigma}_m = \left\{3\left(\frac{\boldsymbol{\sigma}'}{RF}\right) + \left[\xi \sinh\left(\frac{3}{2}\frac{\sigma_m}{RF}\right) - \frac{2}{3}a_1 RF\right]\mathbf{I}\right\} : \dot{\boldsymbol{\sigma}} \tag{14.149}$$

Substituting Eq. (14.149) into Eq. (14.146), the plastic multiplier is derived as follows:

$$\dot{\bar{\lambda}} = \frac{\mathbf{t}^{SG} : \dot{\boldsymbol{\sigma}}}{\bar{M}^{SG}} \tag{14.150}$$

where

$$\mathbf{t}^{SG} \equiv 3\frac{\boldsymbol{\sigma}'}{RF} + \left[\xi \sinh\left(\frac{3}{2}\frac{\sigma_m}{RF}\right) - \frac{2}{3}a_1 RF\right]\mathbf{I} \tag{14.151}$$

$$\bar{M}^{SG} \equiv \left[2\sigma^{eq}\left(\frac{\sigma^{eq}}{RF}\right) + 3\xi\sigma_m \sinh\left(\frac{3}{2}\frac{\sigma_m}{RF}\right)\right]\left(\frac{F'}{F}h + \frac{U}{R}\right) + 2RF\left[(1-\xi)\mathbf{n}^{\psi} + a_1 F' h^{\psi} + \sqrt{\frac{2}{3}}a_2\right] \tag{14.152}$$

The strain rate is given from Eqs. (14.5), (14.144) and (14.150) as

$$\dot{\boldsymbol{\varepsilon}} = \mathbf{E}^{-1} : \dot{\boldsymbol{\sigma}} + \frac{\mathbf{t}^{SG} : \dot{\boldsymbol{\sigma}}}{\bar{M}^{SG}}\mathbf{n}^{\psi} = \left(\mathbf{E}^{-1} + \frac{\mathbf{n}^{\psi} \otimes \mathbf{t}^{SG}}{\bar{M}^{SG}}\right) : \dot{\boldsymbol{\sigma}} \tag{14.153}$$

from which the plastic multiplier in terms of strain rate is derived as follows:

$$\dot{\bar{\Lambda}} = \frac{\mathbf{t}^{SG} : \mathbf{E} : \dot{\boldsymbol{\varepsilon}}}{\bar{M}^{SG} + \mathbf{t}^{SG} : \mathbf{E} : \mathbf{n}^{\psi}} \tag{14.154}$$

The stress rate is described from Eqs. (14.5), (14.144) and (14.154) as

$$\dot{\boldsymbol{\sigma}} = \mathbf{E} : \dot{\boldsymbol{\varepsilon}} - \frac{\mathbf{t}^{SG} : \mathbf{E} : \dot{\boldsymbol{\varepsilon}}}{\bar{M}^{SG} + \mathbf{t}^{SG} : \mathbf{E} : \mathbf{n}^{\psi}}\mathbf{E} : \mathbf{n}^{\psi} = \left[\mathbf{E} - \frac{(\mathbf{E} : \mathbf{n}^{\psi}) \otimes (\mathbf{t}^{SG} : \mathbf{E})}{\bar{M}^{SG} + \mathbf{t}^{SG} : \mathbf{E} : \mathbf{n}^{\psi}}\right] : \dot{\boldsymbol{\varepsilon}} \tag{14.155}$$

The loading criterion is given by the equation same as Eq. (14.32).

The elaboration of the Gurson model was proposed by Tvergaard (1982) (see also Tvergaard and Needleman 1984) by introducing the void coalescence into the yield condition in Eq. (14.137). It is further extended by the concept of the subloading surface as follows:

$$\psi(\boldsymbol{\sigma},\,F,\xi) = \left(\frac{\sigma^{eq}}{RF}\right) + 2\xi^* q_1 \cosh\left(\frac{3}{2}\frac{q_2\sigma_m}{RF}\right) - q_3\xi^{*2} - 1 = 0 \qquad (14.156)$$

where $\xi^*(\xi)$ is the extension of the void volume fraction ξ introduced so as to represent the loss of the load-carrying capacity due to the void coalescence, i.e.

$$\xi^* = \begin{cases} \xi & \text{for } \xi \le \xi_c \\ \xi_c + \left(\dfrac{1}{q_1} - \xi_c\right)\dfrac{\xi - \xi_c}{\xi_f - \xi_c} & \text{for } \xi > \xi_c \end{cases} \qquad (14.157)$$

ξ_c and ξ_f are the critical void volume fraction at the initiation of void coalescence and the void volume fraction at failure (complete loss of load-carrying capacity), respectively. q_1, q_2 and q_3 are the material parameters for the enforcement of the accuracy which are usually chosen as $q_1 = 1.5$, $q_2 = 1.0$ and $q_3 = q_1^2$. The constitutive model with the yield condition in Eq. (14.153) is called the *GTN (Gurson-Tvergaard-Needleman) model*.

The plastic volumetric strain rate is considered in the Gurson model, while it is not considered in the damage model explained in the preceding sections. On the other hand, the decrease of the elastic modulus is not considered in the Gurson model, while it is considered in the damage model. The extended model taken account of both of them would have to be formulated in feature.

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