Chapter 12
Multiplicative Elastoplasticity: Subloading Finite Strain Theory

The subloading surface model was formulated in the Chaps. 6–11 within the frameworks of the finite hypoelastic-based plasticity in detail and of the infinitesimal hyperelastic-based plasticity (Sect. 6.9) in brief. Finite deformation and rotation cannot be described in the exact sense by these formulations. The multiplicative elastoplastic constitutive equation will be formulated for the subloading surface model with the translation of the elastic-core, although the multiplicative constitutive equation for the initial subloading surface model, in which the elastic-core is fixed in the back stress point, was formulated in an immature form by Hashiguchi and Yamakawa (2012). One must formulate the constitutive equation possessing the generality and the universality to be inherited eternally, while any unconventional model, i.e. cyclic plasticity model other than the subloading surface model has not been extended to the multiplicative finite strain theory. The exact formulation of the multiplicative finite strain theory based on the extended subloading surface model has been attained by Hashiguchi (2016a, b, c, d), which will be explained in detail in this chapter.

12.1 Classification of Elastoplastic Constitutive Equation

The basic frameworks of elastoplasticity are classified as follows:

Infinitesimal elastoplasticity

(1) The infinitesimal strain and its material-time derivative are additively decomposed into the elastic and the plastic parts,
(2) The Cauchy stress is used as the stress measure,
(3) The elastic deformation is formulated in the hyperelastic relation,
(4) The initial and the current configurations are not distinguished,
(5) The infinitesimal elastic and plastic deformation is described, ignoring a rotation,

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Finite elastoplasticity

There are the following two frameworks for the description of finite deformation/rotation.

Hypoelastic-based plasticity

(1) The symmetric and anti-symmetric parts of the velocity gradient are defined to be the strain rate and the spin, respectively and further they are is additively decomposed into the elastic and the plastic parts,
(2) The Cauchy stress and its corotational rate (also for internal variables) are used,
(3) The elastic strain rate is formulated in the hypoelastic relation,
(4) The pertinent time-integration of stress rate is required,
(5) The formulation is executed in the current configuration which is influenced by the material rotation,
(6) The finite plastic deformation and the finite rotation are described under the restriction of the infinitesimal elastic deformation.

Multiplicative hyperelastic-based plasticity

(1) The multiplicative decomposition of the deformation gradient tensor is used consistently,
(2) The additive decomposition of the strain rate and the spin tensors in the intermediate configuration into the elastic and the plastic parts are used, which are decomposed definitely into these parts,
(3) The Mandel stress in the intermediate configuration is used as the stress measure,
(4) The elastic deformation is formulated in the hyperelastic relation,
(5) The formulation is executed in the intermediate configuration which is not influenced by the material rotation,
(6) The finite elastic and the plastic deformation and rotation are described exactly.

Then, it realizes the exact description of the finite deformation/rotation.

The formulation of the subloading surface model in the multiplicative hyperelastic-based plasticity was given by Hashiguchi and Yamakawa (2012) in the immature form for the initial subloading surface model in which the elastic-core is fixed so that it is limited to the description of the monotonic loading behavior. The subloading surface model with the translation of the elastic-core will be formulated in this section within the framework of the multiplicative hyperelastic-based plasticity. It is to be the first cyclic (unconventional) elastoplasticity model in the multiplicative hyperelastic-based plasticity for the exact description of finite elastoplastic deformation/rotation.

12.2 Further Multiplicative Decomposition of Plastic Deformation Gradient

The deformation gradient $\mathbf{F}$ is multiplicatively decomposed into the elastic deformation gradient $\mathbf{F}^e$ and the plastic deformation gradient $\mathbf{F}^e$ as described in Sect. 6.1. Further, decompose $\mathbf{F}^p$ into the plastic storage part $\mathbf{F}^p_{ks}$ causing the kinematic hardening and its plastic dissipative part $\mathbf{F}^p_{kd}$ multiplicatively (Lion 2000). Analogously, decompose $\mathbf{F}^p$ into the plastic storage part $\mathbf{F}^p_{cs}$ causing the translation of elastic-core and its plastic dissipative part $\mathbf{F}^p_{cd}$ multiplicatively as follows:

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p, \quad \mathbf{F}^p = \mathbf{F}^p_{ks}\mathbf{F}^p_{kd}, \quad \mathbf{F}^p = \mathbf{F}^p_{cs}\mathbf{F}^p_{cd} \tag{12.1}$$

The configurations based on these decompositions are illustrated in Fig. 12.1.

Based on the right Cauchy-Green deformation tensor

$$\mathbf{C} \equiv \mathbf{F}^T\mathbf{F} \tag{12.2}$$

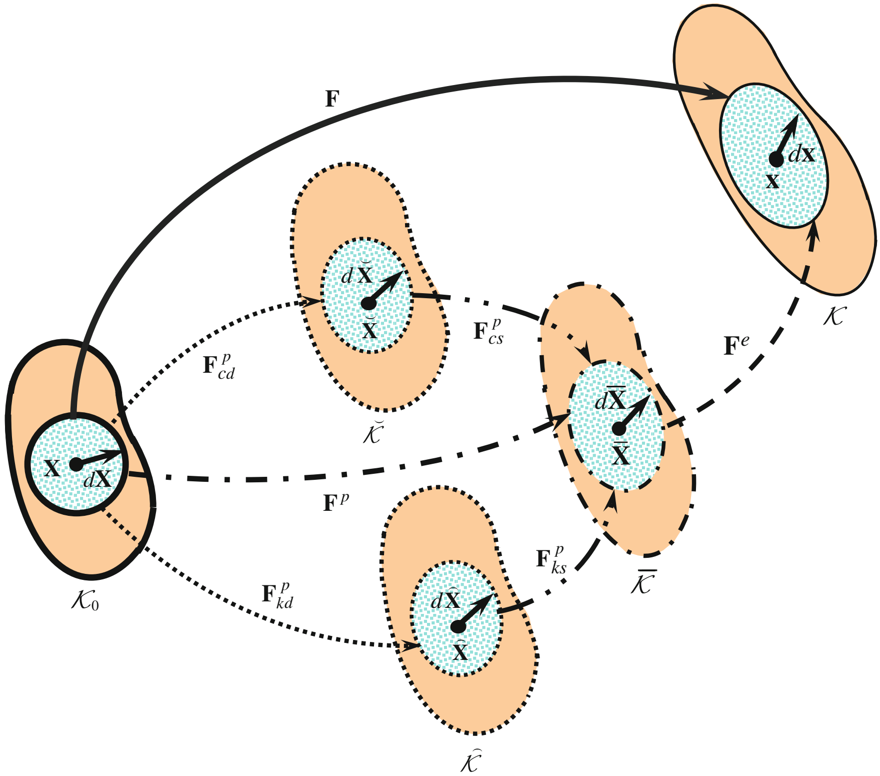

Fig. 12.1 Multiplicative decompositions of deformation gradient for elastoplastic material with translations of kinematic hardening and elastic-core

the following tensors of the storage parts $\overline{\mathbf{C}}^e, \widehat{\mathbf{C}}^p_{ks}, \breve{\mathbf{C}}^p_{cs}$ and the dissipative parts $\mathbf{C}^p, \widehat{\mathbf{C}}^p_{kd}, \breve{\mathbf{C}}^p_{cd}$ are defined.

$$\left.\begin{array}{l}\overline{\mathbf{C}}^e \equiv \mathbf{F}^{eT}\mathbf{F}^e = (\mathbf{R}^e\overline{\mathbf{U}}^e)^T\mathbf{R}^e\overline{\mathbf{U}}^e = \overline{\mathbf{U}}^{e2},\ \mathbf{C}^p \equiv \mathbf{F}^{pT}\mathbf{F}^p, \\ \widehat{\mathbf{C}}^p_{ks} \equiv \mathbf{F}^{pT}_{ks}\mathbf{F}^p_{ks} = \widehat{\mathbf{U}}^{p\,2}_{ks}, \mathbf{C}^p_{kd} \equiv \mathbf{F}^{pT}_{kd}\mathbf{F}^p_{kd}, \\ \breve{\mathbf{C}}^p_{cs} \equiv \mathbf{F}^{pT}_{cs}\mathbf{F}^p_{cs} = \widehat{\mathbf{U}}^{p\,2}_{cs}, \mathbf{C}^p_{cd} \equiv \mathbf{F}^{pT}_{cd}\mathbf{F}^p_{cd}\end{array}\right\} \tag{12.3}$$

where one has

$$\left.\begin{array}{l}\overline{\mathbf{C}}^p_{ks} \equiv {}^p_{ks}\widehat{\mathbf{C}}^p_{ks\bar{G}\bar{G}} = \mathbf{F}^{p-T}_{ks}\widehat{\mathbf{C}}^p_{ks}\mathbf{F}^{p-1}_{ks} \\ \overline{\mathbf{C}}^p_{cs} \equiv {}^p_{cs}\breve{\mathbf{C}}^p_{cs\bar{G}\bar{G}} = \mathbf{F}^{p-T}_{cs}\breve{\mathbf{C}}^p_{cs}\mathbf{F}^{p-1}_{cs}\end{array}\right\} = \overline{\mathbf{G}} \tag{12.4}$$

$\overline{\mathbf{G}}$ is the metric tensors defined in Eq. (4.31) in the intermediate configuration. The hat symbols $(^-)$, $(^\frown)$ and $(^\smile)$ are added to the variables based in the intermediate configuration $\overline{\mathcal{K}}$, the kinematic hardening intermediate configuration $\widehat{\mathcal{K}}$ and the elastic-core intermediate configuration $\breve{\mathcal{K}}$, respectively. The superscript and/or subscript is (are) added in the left side in order to specify the pull-back or push-forward due to the elastic (or plastic) deformation gradient.

In order to explain clearly, the equations described already in Eqs. (6.11) to (6.18) in Sect. 6.1 are again written in the following Eqs. (12.5)–(12.11).

The velocity gradient $\boldsymbol{l}$ in the current configuration $\mathcal{K}$ is additively decomposed into the elastic and the plastic parts:

$$\boldsymbol{l} = \boldsymbol{l}^e + \boldsymbol{l}^p \tag{12.5}$$

where

$$\left.\begin{array}{l}\boldsymbol{l} \equiv \dot{\mathbf{F}}\mathbf{F}^{-1}, \\ \boldsymbol{l}^e \equiv \dot{\mathbf{F}}^e\mathbf{F}^{e-1}, \quad \boldsymbol{l}^p \equiv \mathbf{F}^e\dot{\mathbf{F}}^p\mathbf{F}^{p-1}\mathbf{F}^{e-1} = {}^e\overrightarrow{\overline{\mathbf{L}}}{}^{p\,g}_{\bullet g} = \mathbf{F}^e\overline{\mathbf{L}}^p\mathbf{F}^{e-1} \\ \overline{\mathbf{L}}^p \equiv \dot{\mathbf{F}}^p\mathbf{F}^{p-1}\end{array}\right\} \tag{12.6}$$

Further, the velocity gradient $\overline{\mathbf{L}}$ defined as the *contravariant-covariant* pull-back (Eq. (4.44)) of the velocity gradient tensor $\boldsymbol{l}$ in the current configuration to the intermediate configuration $\overline{\mathcal{K}}$ can be additively decomposed into the purely elastic and the purely plastic parts as follows:

$$\overline{\mathbf{L}} = \overline{\mathbf{L}}^e + \overline{\mathbf{L}}^p \tag{12.7}$$

where

$$\left.\begin{array}{l}\overline{\mathbf{L}} \equiv {}^e\overleftarrow{\boldsymbol{l}}{}^{\overline{G}}_{\bullet\overline{G}} = \mathbf{F}^{e-1}\boldsymbol{l}\mathbf{F}^e \\ \overline{\mathbf{L}}^e \equiv {}^e\overleftarrow{\boldsymbol{l}}{}^{\overline{G}}_{\bullet\overline{G}} = \mathbf{F}^{e-1}\boldsymbol{l}^e\mathbf{F}^e = \mathbf{F}^{e-1}\dot{\mathbf{F}}^e, \quad \overline{\mathbf{L}}^p \equiv {}^e\overleftarrow{\boldsymbol{l}^p}{}^{\overline{G}}_{\bullet\overline{G}} = \mathbf{F}^{e-1}\boldsymbol{l}^p\mathbf{F}^e = \dot{\mathbf{F}}^p\mathbf{F}^{p-1}\end{array}\right\} \tag{12.8}$$

Therefore, $\overline{\mathbf{L}}$ and $\overline{\mathbf{L}}^e$, $\overline{\mathbf{L}}^p$ can be pertinently adopted in the formulation of elasto-plastic constitutive equation. It follows from Eq. (12.8) that

$$\left.\begin{array}{l}\overline{\mathbf{L}} = \overline{\mathbf{D}} + \overline{\mathbf{W}} \\ \overline{\mathbf{L}}^e = \overline{\mathbf{D}}^e + \overline{\mathbf{W}}^e, \quad \overline{\mathbf{L}}^p = \overline{\mathbf{D}}^p + \overline{\mathbf{W}}^p\end{array}\right\} \tag{12.9}$$

$$\overline{\mathbf{D}} = \overline{\mathbf{D}}^e + \overline{\mathbf{D}}^p, \quad \overline{\mathbf{W}} = \overline{\mathbf{W}}^e + \overline{\mathbf{W}}^p \tag{12.10}$$

where

$$\left.\begin{array}{ll}\overline{\mathbf{D}} = \mathrm{sym}[\overline{\mathbf{L}}], & \overline{\mathbf{W}} = \mathrm{ant}[\overline{\mathbf{L}}] \\ \overline{\mathbf{D}}^e = \mathrm{sym}[\overline{\mathbf{L}}^e], & \overline{\mathbf{W}}^e = \mathrm{ant}[\overline{\mathbf{L}}^e] \\ \overline{\mathbf{D}}^p = \mathrm{sym}[\overline{\mathbf{L}}^p], & \overline{\mathbf{W}}^p = \mathrm{ant}[\overline{\mathbf{L}}^p]\end{array}\right\} \tag{12.11}$$

The rate of $\overline{\mathbf{C}}^e$ is given from Eqs. (12.3)$_1$ and (12.7) as

$$\dot{\overline{\mathbf{C}}}^e = 2\mathrm{sym}[\overline{\mathbf{C}}^e\overline{\mathbf{L}}^e] = 2\mathrm{sym}[\overline{\mathbf{C}}^e(\overline{\mathbf{L}} - \overline{\mathbf{L}}^p)] \tag{12.12}$$

noting

$$\begin{aligned}\dot{\overline{\mathbf{C}}}^e &= (\mathbf{F}^{eT}\mathbf{F}^e)^{\bullet} = \mathbf{F}^{eT}\dot{\mathbf{F}}^e + \dot{\mathbf{F}}^{eT}\mathbf{F}^e = \mathbf{F}^{eT}\mathbf{F}^e(\mathbf{F}^{e-1}\dot{\mathbf{F}}^e) + (\dot{\mathbf{F}}^{eT}\mathbf{F}^{e-T})\mathbf{F}^{eT}\mathbf{F}^e \\ &= \overline{\mathbf{C}}^e\overline{\mathbf{L}}^e + \overline{\mathbf{L}}^{eT}\overline{\mathbf{C}}^e\end{aligned}$$

Further, the plastic velocity gradient $\overline{\mathbf{L}}^p$ is additively decomposed for the kinematic hardening as follows:

$$\overline{\mathbf{L}}^p = \overline{\mathbf{L}}^p_{ks} + \overline{\mathbf{L}}^p_{kd} \tag{12.13}$$

where

$$\overline{\mathbf{L}}^p_{ks} \equiv \dot{\mathbf{F}}^p_{ks}\mathbf{F}^{p-1}_{ks}, \overline{\mathbf{L}}^p_{kd} \equiv {}^{p}_{ks}\overset{\rightarrow}{\widehat{\mathbf{L}}}{}^{p\,\bar{G}}_{kd\bullet\bar{G}} = \mathbf{F}^p_{ks}\widehat{\mathbf{L}}^p_{kd}\mathbf{F}^{p-1}_{ks} \tag{12.14}$$

$$\left.\begin{array}{l}\widehat{\mathbf{L}}^p_{kd} = \dot{\mathbf{F}}^p_{kd}\mathbf{F}^{p-1}_{kd} \equiv {}^{p}_{ks}\overset{\leftarrow}{\overline{\mathbf{L}}}{}^{p\,\widehat{G}}_{kd\bullet\widehat{G}} = \mathbf{F}^{p-1}_{ks}\overline{\mathbf{L}}^p_{kd}\mathbf{F}^p_{ks} = \widehat{\mathbf{D}}^p_{kd} + \widehat{\mathbf{W}}^p_{kd} \\ \widehat{\mathbf{D}}^p_{kd} = \mathrm{sym}[\widehat{\mathbf{L}}^p_{kd}], \quad \widehat{\mathbf{W}}^p_{kd} = \mathrm{ant}[\widehat{\mathbf{L}}^p_{kd}]\end{array}\right\} \tag{12.15}$$

noting

$$\begin{aligned}\overline{\mathbf{L}}^p &= (\mathbf{F}^p_{ks}\mathbf{F}^p_{kd})^{\bullet}(\mathbf{F}^p_{ks}\mathbf{F}^p_{kd})^{-1} = (\dot{\mathbf{F}}^p_{ks}\mathbf{F}^p_{kd} + \mathbf{F}^p_{ks}\dot{\mathbf{F}}^p_{kd})\mathbf{F}^{p-1}_{kd}\mathbf{F}^{p-1}_{ks} \\ &= \dot{\mathbf{F}}^p_{ks}\mathbf{F}^{p-1}_{ks} + \mathbf{F}^p_{ks}\dot{\mathbf{F}}^p_{kd}\mathbf{F}^{p-1}_{kd}\mathbf{F}^{p-1}_{ks}\end{aligned}$$

Analogously, the following additive decomposition of the velocity gradient holds for the elastic-core.

$$\overline{\mathbf{L}}^{p} = \overline{\mathbf{L}}^{p}_{cs} + \overline{\mathbf{L}}^{p}_{cd} \tag{12.16}$$

where

$$\overline{\mathbf{L}}^{p}_{cs} \equiv \dot{\mathbf{F}}^{p}_{cs}\mathbf{F}^{p-1}_{cs}, \quad \overline{\mathbf{L}}^{p}_{cd} \equiv {}^{p}_{cs}\overleftarrow{\breve{\mathbf{L}}}{}^{p}{}_{cd\bullet\bar{G}}^{\bar{G}} = \mathbf{F}^{p}_{cs}\breve{\mathbf{L}}^{p}_{cd}\mathbf{F}^{p-1}_{cs} \tag{12.17}$$

$$\left.\begin{array}{l} \breve{\mathbf{L}}^{p}_{cd} = \dot{\mathbf{F}}^{p}_{cd}\mathbf{F}^{p-1}_{cd} = {}^{p}_{cs}\overleftarrow{\overline{\mathbf{L}}}{}^{p}{}_{cd\bullet\bar{G}}^{\bar{G}} = \mathbf{F}^{p-1}_{cs}\overline{\mathbf{L}}^{p}_{cd}\mathbf{F}^{p}_{cs} = \widehat{\mathbf{D}}^{p}_{cd} + \breve{\mathbf{W}}^{p}_{cd} \\ \breve{\mathbf{D}}^{p}_{cd} = \mathrm{sym}\left[\breve{\mathbf{L}}^{p}_{cd}\right], \quad \breve{\mathbf{W}}^{p}_{cd} = \mathrm{ant}\left[\breve{\mathbf{L}}^{p}_{cd}\right] \end{array}\right\} \tag{12.18}$$

The time-derivative of $\widehat{\mathbf{C}}^{p}_{ks}$ in Eq. (12.4) is given by

$$\dot{\widehat{\mathbf{C}}}{}^{p}_{ks} = 2{}^{p}_{ks}\overleftarrow{\overline{\mathbf{D}}}{}^{p}_{ks\widehat{G}\widehat{G}} = 2\mathbf{F}^{pT}_{ks}\overline{\mathbf{D}}^{p}_{ks}\mathbf{F}^{p}_{ks} = 2\mathbf{F}^{pT}_{ks}\left(\overline{\mathbf{D}}^{p} - \overline{\mathbf{D}}^{p}_{kd}\right)\mathbf{F}^{p}_{ks} \tag{12.19}$$

noting

$$\begin{aligned} \dot{\widehat{\mathbf{C}}}{}^{p}_{ks} &= (\mathbf{F}^{pT}_{ks}\mathbf{F}^{p}_{ks})^{\bullet} = \mathbf{F}^{pT}_{ks}\dot{\mathbf{F}}^{p}_{ks} + \dot{\mathbf{F}}^{pT}_{ks}\mathbf{F}^{p}_{ks} = \mathbf{F}^{pT}_{ks}\dot{\mathbf{F}}^{p}_{ks}\mathbf{F}^{p-1}_{ks}\mathbf{F}^{p}_{ks} + \mathbf{F}^{pT}_{ks}\mathbf{F}^{p-T}_{ks}\dot{\mathbf{F}}^{pT}_{ks}\mathbf{F}^{p}_{ks} \\ &= \mathbf{F}^{pT}_{ks}\dot{\mathbf{F}}^{p}_{ks}\mathbf{F}^{p-1}_{ks}\mathbf{F}^{p}_{ks} + \mathbf{F}^{pT}_{ks}(\dot{\mathbf{F}}^{p}_{ks}\mathbf{F}^{p-1}_{ks})^{T}\mathbf{F}^{p}_{ks} = \mathbf{F}^{pT}_{ks}\overline{\mathbf{L}}^{p}_{ks}\mathbf{F}^{p}_{ks} + \mathbf{F}^{pT}_{ks}\overline{\mathbf{L}}^{pT}_{ks}\mathbf{F}^{p}_{ks} = 2\mathbf{F}^{pT}_{ks}\overline{\mathbf{D}}^{p}_{ks}\mathbf{F}^{p}_{ks} \end{aligned}$$

Analogously, one has

$$\dot{\widehat{\mathbf{C}}}{}^{p}_{cs} = 2{}^{p}_{cs}\overleftarrow{\overline{\mathbf{D}}}{}^{p}_{cs\widehat{G}\widehat{G}} = 2\mathbf{F}^{pT}_{cs}\overline{\mathbf{D}}^{p}_{cs}\mathbf{F}^{p}_{cs} = 2\mathbf{F}^{pT}_{cs}(\overline{\mathbf{D}}^{p} - \overline{\mathbf{D}}^{p}_{cd})\mathbf{F}^{p}_{cs} \tag{12.20}$$

Further, it follows for the dissipative parts that

$$\left.\begin{array}{l} \dot{\mathbf{C}}^{p} = 2\,{}^{p}\overleftarrow{\overline{\mathbf{D}}}{}^{p}{}_{GG} = 2\mathbf{F}^{pT}\overline{\mathbf{D}}^{p}\mathbf{F}^{p}, \\ \dot{\mathbf{C}}^{p}_{kd} = 2{}^{p}_{kd}\overleftarrow{\overline{\mathbf{D}}}{}^{p}_{kdGG} = 2\mathbf{F}^{pT}_{kd}\overline{\mathbf{D}}^{p}_{kd}\mathbf{F}^{p}_{kd},\ \dot{\mathbf{C}}^{p}_{cd} = 2{}^{p}_{cd}\overleftarrow{\overline{\mathbf{D}}}{}^{p}_{cdGG} = 2\mathbf{F}^{pT}_{cd}\overline{\mathbf{D}}^{p}_{cd}\mathbf{F}^{p}_{cd} \end{array}\right\} \tag{12.21}$$

noting

$$\begin{aligned} \dot{\mathbf{C}}^{p} &= (\mathbf{F}^{pT}\mathbf{F}^{p})^{\bullet} = \mathbf{F}^{pT}\dot{\mathbf{F}}^{p} + \dot{\mathbf{F}}^{pT}\mathbf{F}^{p} = \mathbf{F}^{pT}(\dot{\mathbf{F}}^{p}\mathbf{F}^{p-1} + \mathbf{F}^{p-T}\dot{\mathbf{F}}^{pT})\mathbf{F}^{p} \\ &= \mathbf{F}^{pT}[\dot{\mathbf{F}}^{p}\mathbf{F}^{p-1} + (\dot{\mathbf{F}}^{p}\mathbf{F}^{p-1})^{T}]\mathbf{F}^{p} \end{aligned}$$

12.3 Stress Measures

Introduce the second Piola-Kirchhoff stress tensor in the intermediate configuration, which is the contravariant pulled-back (Eq. (4.44)) of the Kirchhoff stress tensor, i.e.

$$\overline{\mathbf{S}}\left(=\overline{\mathbf{S}}^{T}\right)\equiv{}^{p}\overrightarrow{\mathbf{S}}^{\bar{G}\bar{G}}=\mathbf{F}^{p}\mathbf{S}\mathbf{F}^{pT}=\mathbf{F}^{e-1}\left(\mathbf{F}\mathbf{S}\mathbf{F}^{T}\right)\mathbf{F}^{e-T}\equiv{}^{e}\overleftarrow{\boldsymbol{\tau}}^{\overline{G}\overline{G}}=\mathbf{F}^{e-1}\boldsymbol{\tau}\mathbf{F}^{e-T} \tag{12.22}$$

and the *Mandel stress*

$$\overline{\mathbf{M}}\equiv\overline{\mathbf{C}}^{e}\overline{\mathbf{S}}=\mathbf{F}^{eT}\boldsymbol{\tau}\mathbf{F}^{e-T}(\neq\overline{\mathbf{M}}^{T}) \tag{12.23}$$

noting

$$\overline{\mathbf{C}}^{e}\overline{\mathbf{S}}=\left(\mathbf{F}^{eT}\mathbf{F}^{e}\right)\left(\mathbf{F}^{e-1}\boldsymbol{\tau}\mathbf{F}^{e-T}\right)=\mathbf{F}^{eT}\boldsymbol{\tau}\mathbf{F}^{e-T} \tag{12.24}$$

Here, note that the work-conjugate stress measure with the strain rare $\overline{\mathbf{L}}$ in the intermediate configuration is the Mandel stress $\overline{\mathbf{M}}$ as known from

$$\begin{aligned}\boldsymbol{\tau}:\boldsymbol{l}&=\mathrm{tr}\left[\left(\mathbf{F}^{e}\overline{\mathbf{S}}\mathbf{F}^{eT}\right)\left(\mathbf{F}^{e}\overline{\mathbf{L}}\mathbf{F}^{e-1}\right)^{T}\right]=\mathrm{tr}(\mathbf{F}^{e}\overline{\mathbf{S}}\mathbf{F}^{eT}\mathbf{F}^{e-T}\overline{\mathbf{L}}^{T}\mathbf{F}^{eT})\\&=\mathrm{tr}(\mathbf{F}^{eT}\mathbf{F}^{e}\overline{\mathbf{S}}\overline{\mathbf{L}}^{T})=\mathrm{tr}(\bar{\mathbf{C}}^{e}\bar{\mathbf{S}}\bar{\mathbf{L}}^{T})=\overline{\mathbf{C}}^{e}\overline{\mathbf{S}}:\overline{\mathbf{L}}=\overline{\mathbf{M}}:\overline{\mathbf{L}}\end{aligned}$$

Further, the *contravariant* push-forward (Eq. (4.44)) of the kinematic hardening variable $\widehat{\mathbf{S}}_{k}$ and the elastic-core $\widehat{\mathbf{S}}_{c}$ to the intermediate configuration $\overline{\mathcal{K}}$ is given by

$$\left.\begin{array}{ll}\overline{\mathbf{S}}_{k}\equiv{}^{p}_{ks}\overrightarrow{\widehat{\mathbf{S}}}_{k}^{\overline{G}\overline{G}}=\mathbf{F}^{p}_{ks}\widehat{\mathbf{S}}_{k}\mathbf{F}^{pT}_{ks}(=\overline{\mathbf{S}}_{k}^{T}), & \widehat{\mathbf{S}}_{k}\equiv{}^{p}_{cs}\overleftarrow{\overline{\mathbf{S}}}_{k}^{\widehat{G}\widehat{G}}=\mathbf{F}^{p-1}_{ks}\overline{\mathbf{S}}_{k}\mathbf{F}^{p-T}_{ks}(=\widehat{\mathbf{S}}_{k}^{T})\\ \overline{\mathbf{S}}_{c}\equiv{}^{p}_{cs}\overrightarrow{\breve{\mathbf{S}}}_{c}^{\overline{G}\overline{G}}=\mathbf{F}^{p}_{cs}\widehat{\mathbf{S}}_{c}\mathbf{F}^{pT}_{cs}\left(=\overline{\mathbf{S}}_{c}^{T}\right), & \widehat{\mathbf{S}}_{c}\equiv{}^{p}_{cs}\overleftarrow{\overline{\mathbf{S}}}_{c}^{\breve{G}\breve{G}}=\mathbf{F}^{p-1}_{cs}\overline{\mathbf{S}}_{c}\mathbf{F}^{p-T}_{cs}(=\breve{\mathbf{S}}_{c}^{T})\end{array}\right\} \tag{12.25}$$

Further, the Mandel-like variables $\overline{\mathbf{M}}_{k}$ and $\overline{\mathbf{M}}_{c}$ for the kinematic hardening variable and the elastic-core, respectively, are defined as

$$\left.\begin{array}{c}\overline{\mathbf{M}}_{k}=\overline{\mathbf{C}}^{p}_{ks}\overline{\mathbf{S}}_{k}=\overline{\mathbf{G}\mathbf{S}}_{k}=\overline{\mathbf{S}}_{k}=\mathbf{F}^{p}_{ks}\widehat{\mathbf{S}}_{k}\mathbf{F}^{pT}_{ks}={}^{p}_{ks}\overrightarrow{\overline{\mathbf{M}}}_{k}{}^{\bullet\bar{G}}_{\bar{G}}=\mathbf{F}^{p-T}_{ks}\widehat{\mathbf{M}}_{k}\mathbf{F}^{pT}_{ks}(\neq\overline{\mathbf{M}}_{k}^{T})\\ \widehat{\mathbf{M}}_{k}=\widehat{\mathbf{C}}^{p}_{ks}\widehat{\mathbf{S}}_{k}={}^{p}_{ks}\overrightarrow{\mathbf{M}}_{k}{}^{\bullet\bar{G}}_{\bar{G}}=\mathbf{F}^{pT}_{ks}\overline{\mathbf{M}}_{k}\mathbf{F}^{p-T}_{ks}(\neq\widehat{\mathbf{M}}_{k}^{T})\end{array}\right\} \tag{12.26}$$

$$\left.\begin{array}{l}\overline{\mathbf{M}}_{c}=\overline{\mathbf{C}}^{p}_{cs}\overline{\mathbf{S}}_{c}=\bar{\mathbf{G}}\bar{\mathbf{S}}_{c}=\overline{\mathbf{S}}_{c}={}^{p}_{cs}\overrightarrow{\breve{\mathbf{M}}}_{c}{}^{\bullet\bar{G}}_{\bar{G}}=\mathbf{F}^{p-T}_{cs}\breve{\mathbf{M}}_{c}\mathbf{F}^{pT}_{cs}\left(\neq\overline{\mathbf{M}}_{c}^{T}\right)\\ \breve{\mathbf{M}}_{c}=\breve{\mathbf{C}}^{p}_{cs}\breve{\mathbf{S}}_{c}={}^{p}_{cs}\overleftarrow{\overline{\mathbf{M}}}_{c}{}^{\bullet\breve{G}}_{\breve{G}}=\mathbf{F}^{pT}_{cs}\overline{\mathbf{M}}_{c}\mathbf{F}^{p-T}_{cs}\left(\neq\breve{\mathbf{M}}_{c}^{T}\right)\end{array}\right\} \tag{12.27}$$

noting Eq. (12.4), (12.25) and $\overline{\mathbf{S}}_{k}=\mathbf{F}^{p}_{ks}\widehat{\mathbf{S}}_{k}\mathbf{F}^{pT}_{ks}=\mathbf{F}^{p}_{ks}\mathbf{C}^{p-1}_{ks}\widehat{\mathbf{M}}_{k}\mathbf{F}^{pT}_{ks}=\mathbf{F}^{p-T}_{ks}\widehat{\mathbf{M}}_{k}\mathbf{F}^{pT}_{ks}$.

The material-time derivative of the kinematic hardening variable $\overline{\mathbf{S}}_{k}$ in the intermediate configuration is given by

$$\dot{\overline{\mathbf{S}}}_k = \mathbf{F}^p_{ks}\overset{\circ}{\hat{\mathbf{S}}}_k\mathbf{F}^{pT}_{ks} + 2\mathrm{sym}\left[\overline{\mathbf{L}}^p_{ks}\overline{\mathbf{S}}_k\right] \tag{12.28}$$

from Eqs. (12.14) and (12.25), noting

$$\begin{aligned}
\dot{\overline{\mathbf{S}}}_k &= \mathbf{F}^p_{ks}\overset{\circ}{\hat{\mathbf{S}}}_k\mathbf{F}^{pT}_{ks} + \dot{\mathbf{F}}^p_{ks}\hat{\mathbf{S}}_k\mathbf{F}^{pT}_{ks} + \mathbf{F}^p_{ks}\hat{\mathbf{S}}_k\dot{\mathbf{F}}^{pT}_{ks} \\
&= \mathbf{F}^p_{ks}\overset{\circ}{\hat{\mathbf{S}}}_k\mathbf{F}^{pT}_{ks} + \dot{\mathbf{F}}^p_{ks}\mathbf{F}^{p-1}_{ks}\overline{\mathbf{S}}_k\mathbf{F}^{p-T}_{ks}\mathbf{F}^{pT}_{ks} + \mathbf{F}^p_{ks}\mathbf{F}^{p-1}_{ks}\overline{\mathbf{S}}_k\mathbf{F}^{p-T}_{ks}\dot{\mathbf{F}}^{pT}_{ks} \\
&= \mathbf{F}^p_{ks}\overset{\circ}{\hat{\mathbf{S}}}_k\mathbf{F}^{pT}_{ks} + \dot{\mathbf{F}}^p_{ks}\mathbf{F}^{p-1}_{ks}\overline{\mathbf{S}}_k + \overline{\mathbf{S}}_k\mathbf{F}^{p-T}_{ks}\dot{\mathbf{F}}^{pT}_{ks} \\
&= \mathbf{F}^p_{ks}\overset{\circ}{\hat{\mathbf{S}}}_k\mathbf{F}^{pT}_{ks} + \overline{\mathbf{L}}^p_{ks}\overline{\mathbf{S}}_k + (\overline{\mathbf{L}}^p_{ks}\overline{\mathbf{S}}_k)^T
\end{aligned}$$

Further, the material-time derivative of $\overline{\mathbf{M}}_k$ is given from Eq. (12.28) with Eqs. (12.13) and (12.26) by

$$\dot{\overline{\mathbf{M}}}_k = \dot{\overline{\mathbf{S}}}_k = \mathbf{F}^p_{ks}\overset{\circ}{\hat{\mathbf{S}}}_k\mathbf{F}^{pT}_{ks} + 2\mathrm{sym}\left[\overline{\mathbf{L}}^p_{ks}\overline{\mathbf{M}}_k\right] = \mathbf{F}^p_{ks}\overset{\circ}{\hat{\mathbf{S}}}_k\mathbf{F}^{pT}_{ks} + 2\mathrm{sym}\left[\left(\overline{\mathbf{L}} - \overline{\mathbf{L}}^p_{kd}\right)\overline{\mathbf{M}}_k\right] \tag{12.29}$$

Analogously, the following relation holds for Mandel-like elastic-core stress.

$$\dot{\overline{\mathbf{M}}}_c = \dot{\overline{\mathbf{S}}}_c = \mathbf{F}^p_{cs}\overset{\circ}{\check{\mathbf{S}}}_c\mathbf{F}^{pT}_{cs} + 2\mathrm{sym}\left[\overline{\mathbf{L}}^p_{cs}\overline{\mathbf{M}}_c\right] = \mathbf{F}^p_{cs}\overset{\circ}{\check{\mathbf{S}}}_c\mathbf{F}^{pT}_{cs} + 2\mathrm{sym}\left[\left(\overline{\mathbf{L}} - \overline{\mathbf{L}}^p_{cd}\right)\overline{\mathbf{M}}_c\right] \tag{12.30}$$

12.4 Hyperelastic Constitutive Equations

Now, the 2nd Piola-Kirchhoff stress push-forwarded to the intermediate configuration, $\overline{\mathbf{S}}$, is given by the following equation with the strain energy function $\psi(\overline{\mathbf{C}}^e)$, noting Eq. $(5.6)_4$, where $\overline{\mathbf{C}}^e$ stands for the purely elastic deformation because of $\overline{\mathbf{C}}^e = \overline{\mathbf{U}}^{e2}$ as shown in Eq. (12.3).

$$\overline{\mathbf{S}} = 2\frac{\partial\psi^e(\overline{\mathbf{C}}^e)}{\partial\overline{\mathbf{C}}^e} \tag{12.31}$$

and the Mandel stress is given by

$$\overline{\mathbf{M}} \equiv \overline{\mathbf{C}}^e\overline{\mathbf{S}} = 2\overline{\mathbf{C}}^e\frac{\partial\psi^e(\overline{\mathbf{C}}^e)}{\partial\overline{\mathbf{C}}^e}(\neq \overline{\mathbf{M}}^T) \tag{12.32}$$

The rate of the Mandel stress is given noting Eq. (12.12) as

$$\dot{\overline{\mathbf{M}}} = (\overline{\mathbf{C}}^e\overline{\mathbf{S}})^{\bullet} = \overline{\mathbb{L}}^e : \dot{\overline{\mathbf{C}}}^e = \overline{\mathbb{L}}^e : \mathrm{sym}[\overline{\mathbf{C}}^e(\overline{\mathbf{L}} - \overline{\mathbf{L}}^p)] \tag{12.33}$$

where $\overline{\mathbb{L}}^e$ is the fourth-order hyperelastic tangent modulus tensor given by

$$\overline{\mathbb{L}}^e \equiv \frac{\partial \overline{\mathbf{M}}}{\partial \overline{\mathbf{C}}^e} = \overline{\mathbf{S}} + \frac{1}{2}\overline{\mathbf{C}}^e : \overline{\mathbb{C}}^e \tag{12.34}$$

with

$$\overline{\mathbb{C}}^e \equiv 2\frac{\partial \overline{\mathbf{S}}}{\partial \overline{\mathbf{C}}^e} = 4\frac{\partial^2 \psi^e(\overline{\mathbf{C}}^e)}{\partial \overline{\mathbf{C}}^e \otimes \partial \overline{\mathbf{C}}^e} \tag{12.35}$$

Further, let the 2nd Piola-Kirchhoff stress-like variables for the kinematic hardening variable $\widehat{\mathbf{L}}_k$ based in $\widehat{\mathcal{K}}$ and for the elastic-core $\breve{\mathbf{S}}_c$ based in $\breve{\mathcal{K}}$ be formulated by the potential energy functions $\psi^k(\widehat{\mathbf{C}}^p_{ks})$ and $\psi^c(\breve{\mathbf{C}}^p_{cs})$, noting Eq. (12.25) with Eq. (12.4), as follows:

$$\widehat{\mathbf{S}}_k = 2\frac{\partial \psi^k(\widehat{\mathbf{C}}^p_{ks})}{\partial \widehat{\mathbf{C}}^p_{ks}}, \quad \breve{\mathbf{S}}_c = 2\frac{\partial \psi^c(\breve{\mathbf{C}}^p_{cs})}{\partial \breve{\mathbf{C}}^p_{cs}} \tag{12.36}$$

$$\overline{\mathbf{S}}_k = {}^{p}_{ks}\vec{\widehat{\mathbf{S}}}_k^{\overline{G}\,\overline{G}} = 2\mathbf{F}^p_{ks}\frac{\partial \psi^k(\widehat{\mathbf{C}}^p_{ks})}{\partial \widehat{\mathbf{S}}^p_{ks}}\mathbf{F}^{pT}_{ks}, \quad \overline{\mathbf{S}}_c = {}^{p}_{cs}\vec{\breve{\mathbf{S}}}_c^{\overline{G}\,\overline{G}} = 2\mathbf{F}^p_{cs}\frac{\partial \psi^c(\breve{\mathbf{C}}^p_{cs})}{\partial \breve{\mathbf{C}}^p_{cs}}\mathbf{F}^{pT}_{cs} \tag{12.37}$$

$$\widehat{\mathbf{M}}_k = \widehat{\mathbf{C}}^p_{ks}\widehat{\mathbf{S}}_k = 2\widehat{\mathbf{C}}^p_{ks}\frac{\partial \psi^k(\widehat{\mathbf{C}}^p_{ks})}{\partial \widehat{\mathbf{C}}^p_{ks}}, \quad \breve{\mathbf{M}}_c = \breve{\mathbf{C}}^p_{cs}\breve{\mathbf{S}}_c = 2\breve{\mathbf{C}}^p_{cs}\frac{\partial \psi^c(\breve{\mathbf{C}}^p_{cs})}{\partial \breve{\mathbf{C}}^p_{cs}} \tag{12.38}$$

$$\overline{\mathbf{M}}_k = \overline{\mathbf{C}}^p_{ks}\overline{\mathbf{S}}_k = \overline{\mathbf{S}}_k = 2\mathbf{F}^p_{ks}\frac{\partial \psi^k(\widehat{\mathbf{C}}^p_{ks})}{\partial \widehat{\mathbf{C}}^p_{ks}}\mathbf{F}^{pT}_{ks}, \quad \overline{\mathbf{M}}_c = \overline{\mathbf{C}}^p_{cs}\overline{\mathbf{S}}_c = \overline{\mathbf{S}}_c = 2\mathbf{F}^p_{cs}\frac{\partial \psi^c(\breve{\mathbf{C}}^p_{cs})}{\partial \breve{\mathbf{C}}^p_{cs}}\mathbf{F}^{pT}_{cs} \tag{12.39}$$

The tensors $\overline{\mathbf{M}}, \overline{\mathbf{M}}_k, \overline{\mathbf{M}}_c$ satisfy the symmetries, i.e. $\overline{\mathbf{M}} = \overline{\mathbf{M}}^T$, $\overline{\mathbf{M}}_k = \overline{\mathbf{M}}_k^T$, $\overline{\mathbf{M}}_c = \overline{\mathbf{M}}_c^T$ for particular cases of the strain energy functions ψ^e, ψ^k, ψ^c only of the tensors $\overline{\mathbf{C}}^e$, $\widehat{\mathbf{C}}^p_{ks}$, $\breve{\mathbf{C}}^p_{cs}$, respectively, while the elastic-isotropy is caused only from the first one.

The rates of $\widehat{\mathbf{S}}_k$ and $\breve{\mathbf{S}}_c$ are given from Eq. (12.36) as

$$\overset{\bullet}{\hat{\mathbf{S}}}_k = \widehat{\mathbb{C}}^k : \frac{1}{2}\overset{\bullet}{\hat{\mathbf{C}}}{}^p_{ks}, \; \overset{\bullet}{\breve{\mathbf{S}}}_c = \breve{\mathbb{C}}^c : \frac{1}{2}\overset{\bullet}{\breve{\mathbf{C}}}{}^p_{cs} \tag{12.40}$$

where

$$\widehat{\mathbb{C}}^k \equiv 2\frac{\partial \widehat{\mathbf{S}}^p_{ks}}{\partial \widehat{\mathbf{C}}^p_{ks}} = 4\frac{\partial^2 \psi^k(\widehat{\mathbf{C}}^p_{ks})}{\partial \widehat{\mathbf{C}}^p_{ks} \otimes \partial \widehat{\mathbf{S}}^p_{ks}}, \quad \breve{\mathbb{C}}^c \equiv 2\frac{\partial \breve{\mathbf{S}}^p_{ks}}{\partial \breve{\mathbf{C}}^p_{cs}} = 4\frac{\partial \psi^c(\breve{\mathbf{C}}^p_{cs})}{\partial \breve{\mathbf{C}}^p_{cs} \otimes \partial \breve{\mathbf{C}}^p_{cs}} \tag{12.41}$$

Substituting Eq. (12.40) with Eq. (12.19) into Eq. (12.29), $\overset{\bullet}{\overline{\mathbf{M}}}_k$ is given as follows:

$$\overset{\bullet}{\overline{\mathbf{M}}}_k = \mathbf{F}^p_{ks}\widehat{\mathbb{C}}^k : \mathbf{F}^{pT}_{ks}(\overline{\mathbf{D}}^p - \overline{\mathbf{D}}^p_{kd})\mathbf{F}^p_{ks}\mathbf{F}^{pT}_{ks} \; + \; 2\mathrm{sym}[(\overline{\mathbf{L}}^p - \overline{\mathbf{L}}^p_{kd})\overline{\mathbf{M}}_k] \tag{12.42}$$

Analogously, it follows for the elastic-core that

$$\overset{\bullet}{\overline{\mathbf{M}}}_c = \mathbf{F}^p_{cs}\breve{\mathbb{C}}^c : \mathbf{F}^{pT}_{cs}(\overline{\mathbf{D}}^p - \overline{\mathbf{D}}^p_{cd})\mathbf{F}^p_{cs}\mathbf{F}^{pT}_{cs} + 2\mathrm{sym}[(\overline{\mathbf{L}}^p - \overline{\mathbf{L}}^p_{cd})\overline{\mathbf{M}}_c] \tag{12.43}$$

12.5 Normal-Yield and Subloading Surfaces

The *normal-yield surface* with the isotropic and the kinematic-hardenings is described following Eq. (9.1) in the intermediate configuration by

$$f(\hat{\overline{\mathbf{M}}}) = F(H) \tag{12.44}$$

and the *subloading surface* following Eq. (9.2) by

$$f(\underline{\overline{\mathbf{M}}}) = RF(H) \tag{12.45}$$

in the intermediate configuration, which are depicted in Fig. 12.2, where

$$\underline{\overline{\mathbf{M}}} \equiv \overline{\mathbf{M}} - \underline{\overline{\mathbf{M}}}_k (\neq \underline{\overline{\mathbf{M}}}^T) \tag{12.46}$$

$$\left.\begin{array}{c} \hat{\overline{\mathbf{M}}} \equiv \overline{\mathbf{M}} - \overline{\mathbf{M}}_k (\neq \hat{\overline{\mathbf{M}}}^T), \quad \tilde{\overline{\mathbf{M}}} \equiv \overline{\mathbf{M}} - \overline{\mathbf{M}}_c (\neq \tilde{\overline{\mathbf{M}}}^T), \\ \hat{\overline{\mathbf{M}}}_c \equiv \overline{\mathbf{M}}_c - \overline{\mathbf{M}}_k (\neq \overline{\mathbf{M}}_c^T) \end{array}\right\} \tag{12.47}$$

$\underline{\overline{\mathbf{M}}}_k$ is the conjugate point in the subloading surface to $\overline{\mathbf{M}}_k$ in the normal-yield surface. The following relations hold.

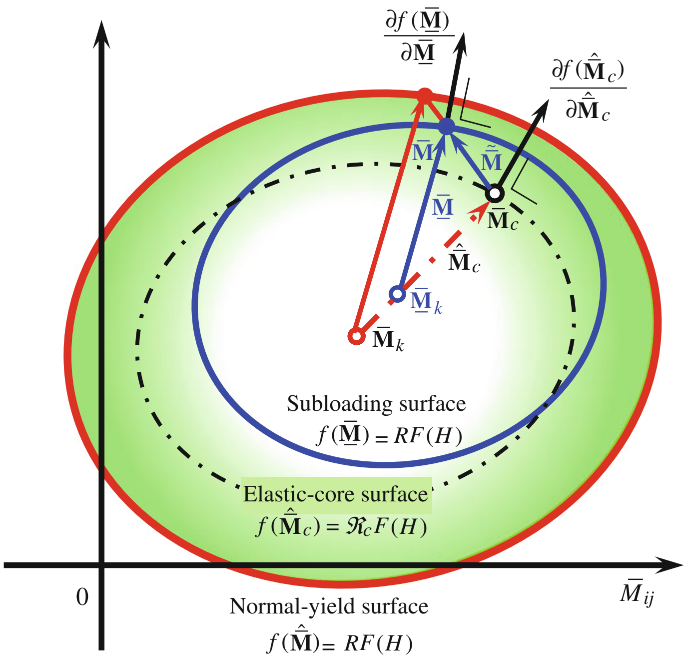

Fig. 12.2 Normal-yield, subloading and elastic-core surfaces in the intermediate configuration in multiplicative elastoplasticity theory

$$\underline{\bar{\mathbf{M}}}_k = \bar{\mathbf{M}}_c - R\hat{\bar{\mathbf{M}}}_c \; (\underline{\bar{\mathbf{M}}}_k - \bar{\mathbf{M}}_c = R(\bar{\mathbf{M}}_c - \bar{\mathbf{M}}_k)) \tag{12.48}$$

the rate of which is given by

$$\dot{\underline{\bar{\mathbf{M}}}}_k = R\dot{\bar{\mathbf{M}}}_k + (1 - R)\dot{\bar{\mathbf{M}}}_c - \dot{R}\hat{\bar{\mathbf{M}}}_c \tag{12.49}$$

leading to

$$\dot{\underline{\bar{\mathbf{M}}}} \equiv \dot{\bar{\mathbf{M}}} - R\dot{\bar{\mathbf{M}}}_k - (1 - R)\dot{\bar{\mathbf{M}}}_c + \dot{R}\hat{\bar{\mathbf{M}}}_c \tag{12.50}$$

The variables in the hypoelastic-based plasticity described in the previous sections correspond to the following variables in the intermediate configuration for the multiplicative hyperelastic-based plasticity.

$$\left.\begin{array}{l} \boldsymbol{\sigma} \rightarrow \bar{\mathbf{M}}, \; \bar{\boldsymbol{\sigma}} \rightarrow \underline{\bar{\mathbf{M}}} = \bar{\mathbf{M}} - \underline{\bar{\mathbf{M}}}_k \\ \boldsymbol{\alpha} \rightarrow \bar{\mathbf{M}}_k, \; \bar{\boldsymbol{\alpha}} \rightarrow \underline{\bar{\mathbf{M}}}_k \\ \mathbf{c} \rightarrow \bar{\mathbf{M}}_c, \; \hat{\mathbf{c}} \rightarrow \hat{\bar{\mathbf{M}}}_c = \bar{\mathbf{M}}_c - \bar{\mathbf{M}}_k \\ \tilde{\boldsymbol{\sigma}} \rightarrow \tilde{\bar{\mathbf{M}}} = \bar{\mathbf{M}} - \bar{\mathbf{M}}_c \end{array}\right\} \tag{12.51}$$

The *elastic-core surface* which passes through the elastic-core $\overline{\mathbf{M}}_c$ and is similar to the normal-yield surface with respect to the back-stress $\overline{\mathbf{M}}_k$ in the hyperelastic-based-plasticity is given following Eq. (9.17) as follows:

$$f(\hat{\overline{\mathbf{M}}}_c) = \mathfrak{R}_c F(H), \quad \text{i.e. } \mathfrak{R}_c = f(\hat{\overline{\mathbf{M}}}_c)/F(H) \tag{12.52}$$

12.6 Plastic Flow Rules

The plastic strain rate is given in the following associated flow rule proposed by Hashiguchi (2016a, b, c, d).

$$\boxed{\overline{\mathbf{D}}^p = \dot{\bar{\lambda}}\,\underline{\overline{\mathsf{N}}}\ (\dot{\bar{\lambda}} \geq 0)} \tag{12.53}$$

where $\dot{\bar{\lambda}}$ is the plastic multiplier and

$$\underline{\overline{\mathsf{N}}} \equiv \mathrm{sym}\left[\frac{\partial f(\underline{\overline{\mathbf{M}}})}{\partial \overline{\mathbf{M}}}\right] \Big/ \left\| \mathrm{sym}\left[\frac{\partial f(\underline{\overline{\mathbf{M}}})}{\partial \overline{\mathbf{M}}}\right] \right\| \quad (\|\underline{\overline{\mathsf{N}}}\| = 1) \tag{12.54}$$

which is the normalized and symmetrized tensor. If the symmetries of the Mandel stress and Mandel-like kinematic hardening variable, i.e. $\overline{M} = \overline{M}^T$ and $\overline{M}_k = \overline{M}_k^T$ hold, which are provided by the strain energy functions ψ^e and ψ^k of only $\overline{\mathbf{C}}^e$ and $\widehat{\mathbf{C}}^p_{ks}$, respectively, one obtains the symmetry $\partial f(\underline{\overline{\mathbf{M}}})/\partial \overline{\mathbf{M}} = (\partial f(\underline{\overline{\mathbf{M}}})/\partial \overline{\mathbf{M}})^T$.

The symmetric plastic dissipative parts of the plastic velocity gradients for the kinematic hardening variable and the elastic-core in Eqs. (12.15) and (12.18) are assumed following Eqs. (9.11) and (9.15) as follows:

$$\boxed{\overline{\mathbf{D}}^p_{kd} = \frac{1}{b_k}\|\overline{\mathbf{D}}^{p\prime}\|\mathrm{sym}[\overline{\mathbf{M}}_k]} = \frac{1}{b_k}\dot{\bar{\lambda}}\|\underline{\overline{\mathsf{N}}}'\|\mathrm{sym}[\overline{\mathbf{M}}_k] \tag{12.55}$$

$$\boxed{\overline{\mathbf{D}}^p_{cd} = \frac{\mathfrak{R}_c}{\xi}\|\overline{\mathbf{D}}^p\|\hat{\overline{\mathsf{N}}}_c} = \frac{\mathfrak{R}_c}{\xi}\dot{\bar{\lambda}}\hat{\overline{\mathsf{N}}}_c \tag{12.56}$$

where

$$\hat{\overline{\mathsf{N}}}_c \equiv \mathrm{sym}\left[\frac{\partial f(\hat{\overline{\mathbf{M}}}_c)}{\partial \hat{\overline{\mathbf{M}}}_c}\right] \Big/ \left\| \mathrm{sym}\left[\frac{\partial f(\hat{\overline{\mathbf{M}}}_c)}{\partial \hat{\overline{\mathbf{M}}}_c}\right] \right\| \quad (\|\hat{\overline{\mathsf{N}}}_c\| = 1) \tag{12.57}$$

$$\hat{\overline{\mathbf{M}}}_c \equiv \overline{\mathbf{M}}_c - \overline{\mathbf{M}}_k \tag{12.58}$$

In the material parameter $u = \bar{u}\exp(u_c \mathfrak{R}_c C_\sigma)$ in Eq. (9.47) for the Masing effect, $\mathfrak{R}_c$ is given by Eq. (12.51) and C_σ is given by

$$C_\sigma \equiv \hat{\underline{\mathrm{N}}}_c : \underline{\mathrm{N}} (-1 \le C_\sigma \le 1) \tag{12.59}$$

Let the spins $\overline{\mathbf{W}}^p$, $\widehat{\mathbf{W}}^p_{kd}$ and $\breve{\mathbf{W}}^p_{cd}$ in Eqs. (12.11), (12.15) and (12.18), which are induced by the plastic and the dissipative parts, be given by extending Eq. (9.48) as follows:.

$$\boxed{\begin{aligned}
\overline{\mathbf{W}}^p &= \eta^p(\overline{\mathbf{M}}\,\overline{\mathbf{D}}^p - \overline{\mathbf{D}}^p\,\overline{\mathbf{M}}) = \eta^p\,\dot{\bar{\lambda}}(\overline{\mathbf{M}}\,\underline{\mathrm{N}} - \underline{\mathrm{N}}\,\overline{\mathbf{M}}) \\
\overline{\mathbf{W}}^p_{kd} &= \eta^p_k(\overline{\mathbf{M}}\,\overline{\mathbf{D}}^p_{kd} - \overline{\mathbf{D}}^p_{kd}\overline{\mathbf{M}}) = (\eta^p_k/b_k)\,\dot{\bar{\lambda}}\,||\underline{\mathrm{N}}'||(\overline{\mathbf{M}}\,\mathrm{sym}[\overline{\mathbf{M}}_k] - \mathrm{sym}[\overline{\mathbf{M}}_k]\overline{\mathbf{M}}) \\
\overline{\mathbf{W}}^p_{cd} &= \eta^p_c(\overline{\mathbf{M}}\,\overline{\mathbf{D}}^p_{cd} - \overline{\mathbf{D}}^p_{cd}\overline{\mathbf{M}}) = \eta^p_c(\mathfrak{R}_c/\xi)\,\dot{\bar{\lambda}}(\overline{\mathbf{M}}\,\hat{\underline{\mathrm{N}}}_c - \hat{\underline{\mathrm{N}}}_c\overline{\mathbf{M}})
\end{aligned}} \tag{12.60}$$

where η^p_k and η^p_c are the material parameters, while the flow rules in Eqs. (12.53), (12.55) and (12.56) are exploited. The plastic spin tensor $\overline{\mathbf{W}}^p$ diminishes if the symmetry of the Mandel stress, i.e. $\overline{\mathbf{M}} = \overline{\mathbf{M}}^T$ due to the elastic isotropy and the plastic isotropy due to $\overline{\mathbf{M}}_k = \overline{\mathbf{M}}_c = \mathbf{O}$ hold. Further, the spin tensors $\overline{\mathbf{W}}^p_{kd}$ and $\overline{\mathbf{W}}^p_{cd}$ diminish for the plastic-isotropy due to $\overline{\mathbf{M}}_k = \overline{\mathbf{M}}_c = \mathbf{O}$.

The velocity gradients are given by substituting Eqs. (12.53), (12.55), (12.56) and (12.60) into Eqs. (12.9), (12.15) and (12.18) as follows:

$$\left.\begin{aligned}
\overline{\mathbf{L}}^p &= \dot{\bar{\lambda}}[\underline{\mathrm{N}} + \eta^p(\overline{\mathbf{M}}\,\underline{\mathrm{N}} - \underline{\mathrm{N}}\,\overline{\mathbf{M}})] \\
\overline{\mathbf{L}}^p_{kd} &= (1/b_k)\dot{\bar{\lambda}}||\underline{\mathrm{N}}'||\{\mathrm{sym}[\overline{\mathbf{M}}_{\mathrm{k}}] + \eta^{\mathrm{p}}_{\mathrm{k}}(\overline{\mathbf{M}}\,\mathrm{sym}[\overline{\mathbf{M}}_{\mathrm{k}}] - \mathrm{sym}[\overline{\mathbf{M}}_{\mathrm{k}}]\overline{\mathbf{M}})\} \\
\overline{\mathbf{L}}^p_{cd} &= (\mathfrak{R}_c/\xi)\,\dot{\bar{\lambda}}[\hat{\underline{\mathrm{N}}}_c + \eta^p_c(\overline{\mathbf{M}}\,\hat{\underline{\mathrm{N}}}_c - \hat{\underline{\mathrm{N}}}_c\overline{\mathbf{M}})]
\end{aligned}\right\} \tag{12.61}$$

The substitutions of Eq. (12.61) into Eqs. (12.33), (12.42) and (12.43) yield:

$$\dot{\overline{\mathbf{M}}} = \overline{\mathbb{L}}^e : \mathrm{sym}[\overline{\mathbf{C}}^e\{\mathbf{L} - \dot{\bar{\lambda}}\,[\underline{\mathrm{N}} + \eta^p(\overline{\mathbf{M}}\,\underline{\mathrm{N}} - \underline{\mathrm{N}}\,\overline{\mathbf{M}})]\}] \tag{12.62}$$

$$\begin{aligned}
\dot{\overline{\mathbf{M}}}_k = \dot{\bar{\lambda}}\{&\mathbf{F}^p_{ks}\widehat{\mathbb{C}}^k : \mathbf{F}^{pT}_{ks}(\underline{\mathrm{N}} - (1/b_k)||\underline{\mathrm{N}}'||\mathrm{sym}[\overline{\mathbf{M}}_k])\mathbf{F}^p_{ks}\mathbf{F}^{pT}_{ks} + 2\mathrm{sym}[(\underline{\mathrm{N}} + \eta^p(\overline{\mathbf{M}}\,\underline{\mathrm{N}} - \underline{\mathrm{N}}\,\overline{\mathbf{M}}) \\
&- (1/b_k)||\underline{\mathrm{N}}'||\{\mathrm{sym}[\overline{\mathbf{M}}_{\mathrm{k}}] + \eta^{\mathrm{p}}_{\mathrm{k}}(\overline{\mathbf{M}}\,\mathrm{sym}[\overline{\mathbf{M}}_{\mathrm{k}}] - \mathrm{sym}[\overline{\mathbf{M}}_{\mathrm{k}}]\overline{\mathbf{M}})\})\overline{\mathbf{M}}_{\mathrm{k}}]\}
\end{aligned} \tag{12.63}$$

$$\begin{aligned}
\dot{\overline{\mathbf{M}}}_c = \dot{\bar{\lambda}}\{&\mathbf{F}^p_{cs}\widehat{\mathbb{C}}^c : \mathbf{F}^{pT}_{cs}(\underline{\mathrm{N}} - (\mathfrak{R}_c/\xi)\hat{\underline{\mathrm{N}}}_c)\mathbf{F}^p_{cs}\mathbf{F}^{pT}_{cs} \\
&+ 2\mathrm{sym}[(\underline{\mathrm{N}} + \eta^p(\overline{\mathbf{M}}\,\underline{\mathrm{N}} - \underline{\mathrm{N}}\,\overline{\mathbf{M}}) - (\mathfrak{R}_c/\xi)[\hat{\underline{\mathrm{N}}}_c + \eta^p_c(\overline{\mathbf{M}}\,\hat{\underline{\mathrm{N}}}_c - \hat{\underline{\mathrm{N}}}_c\overline{\mathbf{M}})])\overline{\mathbf{M}}_c]\}
\end{aligned} \tag{12.64}$$

12.7 Plastic Strain Rate

The elastic constitutive equation is given by Eqs. (12.31)–(12.33). The plastic strain rate formulated by Hashiguchi (2016a, b, c, d) will be shown in this section. The formulations given in this section is not necessary in the numerical calculation by the return-mapping based on the closet-point projection described in Chap. 20.

The time-differentiation of Eq. (12.45) leads to the consistency condition of the subloading surface as follows:

$$\frac{\partial f(\underline{\overline{\mathbf{M}}})}{\partial \underline{\overline{\mathbf{M}}}} : \dot{\underline{\overline{\mathbf{M}}}} - \dot{R}F - R\dot{F} = 0 \tag{12.65}$$

It holds from Eq. (12.45) that

$$\frac{\partial f(\underline{\overline{\mathbf{M}}})}{\partial \underline{\overline{\mathbf{M}}}} : \underline{\overline{\mathbf{M}}}(= f(\underline{\overline{\mathbf{M}}})) = RF \tag{12.66}$$

by the Euler's theorem for the homogenous function $f(\underline{\overline{\mathbf{M}}})$ of $\underline{\overline{\mathbf{M}}}$ in degree-one, and then it follows that

$$\underline{\overline{\mathbf{N}}} : \underline{\overline{\mathbf{M}}} = \frac{\partial f(\underline{\overline{\mathbf{M}}})}{\partial \underline{\overline{\mathbf{M}}}} : \underline{\overline{\mathbf{M}}} / \left|\left| \frac{\partial f(\underline{\overline{\mathbf{M}}})}{\partial \underline{\overline{\mathbf{M}}}} \right|\right| = f(\underline{\overline{\mathbf{M}}}) / \left|\left| \frac{\partial f(\underline{\overline{\mathbf{M}}})}{\partial \underline{\overline{\mathbf{M}}}} \right|\right| = RF / \left|\left| \frac{\partial f(\underline{\overline{\mathbf{M}}})}{\partial \underline{\overline{\mathbf{M}}}} \right|\right|$$

which leads to

$$1 / \left|\left| \frac{\partial f(\underline{\overline{\mathbf{M}}})}{\partial \underline{\overline{\mathbf{M}}}} \right|\right| = \frac{\underline{\overline{\mathbf{N}}} : \underline{\overline{\mathbf{M}}}}{RF} \tag{12.67}$$

where

$$\underline{\overline{\mathbf{N}}} \equiv \frac{\partial f(\underline{\overline{\mathbf{M}}})}{\partial \underline{\overline{\mathbf{M}}}} / \left|\left| \frac{\partial f(\underline{\overline{\mathbf{M}}})}{\partial \underline{\overline{\mathbf{M}}}} \right|\right| (\neq \underline{\overline{\mathbf{N}}}^T, ||\underline{\overline{\mathbf{N}}}|| = 1) \tag{12.68}$$

The substitution of Eq. (12.67) into Eq. (12.65) leads to

$$\underline{\overline{\mathbf{N}}} : \dot{\underline{\overline{\mathbf{M}}}} - \left(\frac{\dot{F}}{F} + \frac{\dot{R}}{R} \right) \underline{\overline{\mathbf{N}}} : \underline{\overline{\mathbf{M}}} = 0 \tag{12.69}$$

The substitution of Eq. (12.50) into Eq. (12.69) leads to

$$\underline{\overline{\mathbf{N}}} : \dot{\underline{\overline{\mathbf{M}}}} - \underline{\overline{\mathbf{N}}} : \left[\frac{\dot{F}}{F} \underline{\overline{\mathbf{M}}} + \frac{\dot{R}}{R} (\underline{\overline{\mathbf{M}}} - R \hat{\underline{\overline{\mathbf{M}}}}_c) + R \dot{\underline{\overline{\mathbf{M}}}}_k + (1 - R) \dot{\underline{\overline{\mathbf{M}}}}_c \right] = 0 \tag{12.70}$$

Further, substituting the relation

$$\underline{\overline{\mathbf{M}}} - R\hat{\overline{\mathbf{M}}}_c = \overline{\mathbf{M}} - \underline{\overline{\mathbf{M}}}_k - (\overline{\mathbf{M}}_c - \underline{\overline{\mathbf{M}}}_k) = \tilde{\overline{\mathbf{M}}} \tag{12.71}$$

Eq. (12.70) is rewritten as

$$\underline{\overline{\mathbf{N}}}:\overset{\bullet}{\overline{\mathbf{M}}} - \underline{\overline{\mathbf{N}}}:\left[\frac{F'\overset{\bullet}{H}}{F}\underline{\overline{\mathbf{M}}} + \frac{\overset{\bullet}{R}}{R}\tilde{\overline{\mathbf{M}}} + R\overset{\bullet}{\overline{\mathbf{M}}}_k + (1-R)\overset{\bullet}{\overline{\mathbf{M}}}_c\right] = 0 \tag{12.72}$$

where

$$\overset{\bullet}{H} = f_{Hd}(\overline{\mathbf{M}}, H, \overline{\mathbf{D}}^p/||\overline{\mathbf{D}}^p||)||\overline{\mathbf{D}}^p|| = f_{Hn}(\overline{\mathbf{M}}, H, \underline{\overline{\mathbf{N}}})\,\overset{\bullet}{\overline{\lambda}} \tag{12.73}$$

$$\overset{\bullet}{R} = U(R)||\overline{\mathbf{D}}^p|| = U(R)\,\overset{\bullet}{\overline{\lambda}} \quad \text{for} \quad \overline{\mathbf{D}}^p \neq \mathbf{O} \tag{12.74}$$

based on Eqs. (6.37) and (7.9) with Eq. (12.53). The normal-yield ratio is calculated in general from Eq. (9.45) as follows:

$$f(\tilde{\overline{\mathbf{M}}} + R\hat{\overline{\mathbf{M}}}_c) = RF(H) \tag{12.75}$$

which is explicitly described for the Mises metals from Eq. (10.32) as

$$R = \frac{\tilde{\overline{\mathbf{M}}}':\hat{\overline{\mathbf{M}}}'_c + \sqrt{(\tilde{\overline{\mathbf{M}}}':\hat{\overline{\mathbf{M}}}'_c)^2 + \left(\frac{2}{3}F^2 - ||\hat{\overline{\mathbf{M}}}'_c||^2\right)||\tilde{\overline{\mathbf{M}}}'||^2}}{\frac{2}{3}F^2 - ||\hat{\overline{\mathbf{M}}}'_c||^2} \tag{12.76}$$

The substitutions of Eqs. (12.63), (12.64), (12.73) and (12.74) into Eq. (12.72) lead to the consistency condition:

$$\underline{\overline{\mathbf{N}}}:\overset{\bullet}{\overline{\mathbf{M}}} - \overline{M}^p\,\overset{\bullet}{\overline{\lambda}} = 0 \tag{12.77}$$

from which it follows that

$$\overset{\bullet}{\overline{\lambda}} = \frac{\underline{\overline{\mathbf{N}}}:\overset{\bullet}{\overline{\mathbf{M}}}}{\overline{M}^p}, \quad \overline{\mathbf{D}}^p = \frac{\underline{\overline{\mathbf{N}}}:\overset{\bullet}{\overline{\mathbf{M}}}}{\overline{M}^p}\underline{\overline{\mathbf{N}}} \tag{12.78}$$

where

$$
\begin{aligned}
\overline{M}^{p} \equiv \underline{\overline{\mathbf{N}}} : \Bigg[& \frac{F' f_{Hn}(\overline{\mathbf{M}}, F, \underline{\overline{\mathbf{N}}})}{F} \underline{\overline{\mathbf{M}}} + \frac{U(R)}{R} \tilde{\overline{\mathbf{M}}} \\
& + R\{\mathbf{F}^{p}_{ks} \widehat{\mathbb{C}}^{k} : \mathbf{F}^{pT}_{ks} (\underline{\overline{\mathbf{N}}} - (1/b_k)\|\underline{\overline{\mathbf{N}}}'\| \mathrm{sym}[\overline{\mathbf{M}}_k]) \mathbf{F}^{p}_{ks} \mathbf{F}^{pT}_{ks} + 2\mathrm{sym}[(\underline{\overline{\mathbf{N}}} + \eta^{p}(\overline{\mathbf{M}}\,\underline{\overline{\mathbf{N}}} - \underline{\overline{\mathbf{N}}}\,\overline{\mathbf{M}}) \\
& - (1/b_k)\|\underline{\overline{\mathbf{N}}}'\| \{\mathrm{sym}[\overline{\mathbf{M}}_k] + \eta^{p}_{k}(\overline{\mathbf{M}}\mathrm{sym}[\overline{\mathbf{M}}_k] - \mathrm{sym}[\overline{\mathbf{M}}_k]\overline{\mathbf{M}})\}) \overline{\mathbf{M}}_k]\} \\
& + (1-R)\mathbf{F}^{p}_{cs} \breve{\mathbb{C}}^{c} : \mathbf{F}^{pT}_{cs} (\underline{\overline{\mathbf{N}}} - (\mathfrak{R}_c/\xi)\hat{\overline{\mathrm{N}}}_c) \mathbf{F}^{p}_{cs} \mathbf{F}^{pT}_{cs} \\
& + 2\mathrm{sym}[(\underline{\overline{\mathbf{N}}} + \eta^{p}(\overline{\mathbf{M}}\,\underline{\overline{\mathbf{N}}} - \underline{\overline{\mathbf{N}}}\,\overline{\mathbf{M}}) - (\mathfrak{R}_c/\xi)[\hat{\overline{\mathrm{N}}}_c + \eta^{p}_{c}(\overline{\mathbf{M}}\,\hat{\overline{\mathrm{N}}}_c - \hat{\overline{\mathrm{N}}}_c \overline{\mathbf{M}})]) \overline{\mathbf{M}}_c]\} \Bigg]
\end{aligned}
\tag{12.79}
$$

The substitutions of Eq. (12.62) into Eq. (12.77) lead to the consistency condition:

$$
\underline{\overline{\mathbf{N}}} : 2\overline{\mathbb{L}}^{e} : \mathrm{sym}[\overline{\mathbf{C}}^{e}\overline{\mathbf{L}}] - \{\underline{\overline{\mathbf{N}}} : \overline{\mathbb{L}}^{e} : \mathrm{sym}[\overline{\mathbf{C}}^{e}\{\underline{\overline{\mathbf{N}}} + \eta^{p}(\overline{\mathbf{M}}\,\underline{\overline{\mathbf{N}}} - \underline{\overline{\mathbf{N}}}\,\overline{\mathbf{M}})\}] + \overline{M}^{p}\} \dot{\overline{\Lambda}} = 0 \tag{12.80}
$$

using the symbol $\dot{\overline{\Lambda}}$ for the plastic multiplier in terms of the strain rate instead of $\dot{\overline{\lambda}}$ in terms of the stress rate. The plastic multiplier is given from Eq. (12.80) as follows:

$$
\dot{\overline{\Lambda}} = \frac{2\underline{\overline{\mathbf{N}}} : \overline{\mathbb{L}}^{e} : \mathrm{sym}[\overline{\mathbf{C}}^{e}\overline{\mathbf{L}}]}{\overline{M}^{p} + \underline{\overline{\mathbf{N}}} : \overline{\mathbb{L}}^{e} : \mathrm{sym}[\overline{\mathbf{C}}^{e}\{\underline{\overline{\mathbf{N}}} + \eta^{p}(\overline{\mathbf{M}}\,\underline{\overline{\mathbf{N}}} - \underline{\overline{\mathbf{N}}}\,\overline{\mathbf{M}})\}]} \tag{12.81}
$$

The loading criterion is given by

$$
\left.
\begin{array}{ll}
\overline{\mathbf{D}}^{p} \neq \mathbf{O} & \text{for } \dot{\overline{\Lambda}} > 0 \\
\overline{\mathbf{D}}^{p} = \mathbf{O} & \text{for others}
\end{array}
\right\} \tag{12.82}
$$

which can be given actually as

$$
\left.
\begin{array}{ll}
\overline{\mathbf{D}}^{p} \neq \mathbf{O} & \text{for } \underline{\overline{\mathbf{N}}} : \overline{\mathbb{L}}^{e} : \mathrm{sym}[\overline{\mathbf{C}}^{e}\overline{\mathbf{L}}] > 0 \\
\overline{\mathbf{D}}^{p} = \mathbf{O} & \text{for others}
\end{array}
\right\} \tag{12.83}
$$

12.8 Calculation Procedures

The calculation procedure by the above-mentioned formulations is described in this section.

First, the plastic multiplier $\dot{\bar{\Lambda}}$ is calculated by the input of the velocity gradient $\bar{\mathbf{L}}$ into Eq. (12.81). The forward-Euler method or the return-mapping projection can be adopted to this calculation. Then, substituting it into Eq. (12.61), the plastic and the dissipative parts $\bar{\mathbf{L}}^p$, $\bar{\mathbf{L}}^p_{kd}$ and $\bar{\mathbf{L}}^p_{cd}$ are calculated. Thereafter, the stress and the tensor-valued internal variables are calculated by the method described below.

The rates of the plastic gradient and its dissipative parts are given from Eqs. $(12.6)_3$, $(12.15)_1$ and $(12.18)_1$ as follows:

$$\left.\begin{aligned} \dot{\mathbf{F}}^p &= \bar{\mathbf{L}}^p \mathbf{F}^p \\ \dot{\mathbf{F}}^p_{kd} &= \widehat{\mathbf{L}}^p_{kd} \mathbf{F}^p_{kd} = \mathbf{F}^{p-1}_{ks} \bar{\mathbf{L}}^p_{kd} \mathbf{F}^p_{ks} \mathbf{F}^p_{kd} \\ \dot{\mathbf{F}}^p_{cd} &= \breve{\mathbf{L}}^p_{cd} \mathbf{F}^p_{cd} = \mathbf{F}^{p-1}_{cs} \bar{\mathbf{L}}^p_{cd} \mathbf{F}^p_{cs} \mathbf{F}^p_{cd} \end{aligned}\right\} \tag{12.84}$$

where $\bar{\mathbf{L}}^p$, $\bar{\mathbf{L}}^p_{kd}$ and $\bar{\mathbf{L}}^p_{cd}$ are given by Eq. (12.61). The storage parts $\mathbf{F}^e$, $\mathbf{F}^p_{ks}$ and $\mathbf{F}^p_{cs}$ of the deformation gradient are given by substituting the time-integrations of Eq. (12.84) into

$$\mathbf{F}^e = \mathbf{F}\mathbf{F}^{p-1}, \quad \mathbf{F}^p_{ks} = \mathbf{F}^p \mathbf{F}^{p-1}_{kd}, \quad \mathbf{F}^p_{cs} = \mathbf{F}^p \mathbf{F}^{p-1}_{cd} \tag{12.85}$$

Further, $\bar{\mathbf{C}}^e, \widehat{\mathbf{C}}^p_{ks}$ and $\breve{\mathbf{C}}^p_{cs}$ are calculated by substituting Eq. (12.85) into Eq. (12.3). Further, the stress $\bar{\mathbf{S}}$, the kinematic hardening variable $\widehat{\mathbf{S}}_k$ and the elastic-core $\breve{\mathbf{S}}_c$ are calculated by substituting $\bar{\mathbf{C}}^e, \widehat{\mathbf{C}}^p_{ks}$ and $\breve{\mathbf{C}}^p_{cs}$ into Eqs. (12.31) and (12.36). The isotropic hardening variable and the normal-yield ratio are calculated by the time-integration of Eqs. (12.73), (12.74) and (12.75).

The plastic constitutive equation with the plastic modulus in Eq. (12.78) is not necessary to be used in the numerical calculation by the return-mapping in which the plastic strain rate is calculated by use of only the plastic flow rule in Eq. (12.53) and then the stress and internal variables are calculated.

The time-integrations of Eq. (12.84) for the deformation gradient tensors $\mathbf{F}^p$, $\mathbf{F}^p_{kd}$ and $\mathbf{F}^p_{cd}$ can be executed in high efficiency by the tensor exponential method (Miehe 1996; Weber and Anand 1990; Hashiguchi and Yamakawa 2012).

The numerical calculation scheme for the multiplicative elastoplasticity adopting the initial subloading surface model ($\bar{\mathbf{M}}_c = \mathbf{O}$) without the plastic spins ($\bar{\mathbf{W}}^p = \bar{\mathbf{W}}^p_{kd} = \bar{\mathbf{W}}^p_{cd} = \mathbf{O}$) can be referred to Hashiguchi and Yamakawa (2012).

12.9 Cyclic Stagnation of Isotropic Hardening of Metals

The stagnation of the isotropic hardening for a while after the reverse of loading was described in Sect. 10.2. It will be extended to the framework of the multiplicative finite strain theory in this section.

The normal-isotropic hardening surface in the intermediate configuration is given by

$$g(\tilde{\bar{\mathbf{M}}}_k) = \tilde{K} \tag{12.86}$$

where

$$\tilde{\bar{\mathbf{M}}}_k \equiv \bar{\mathbf{M}}_k - \bar{\boldsymbol{\Theta}} (\neq \tilde{\bar{\mathbf{M}}}_k^T) \tag{12.87}$$

The scalar variable K and the second-order tensor variable $\bar{\boldsymbol{\Theta}}(\neq \bar{\boldsymbol{\Theta}}^T, \mathrm{tr}\bar{\boldsymbol{\Theta}} = 0)$ designate the size and the center, respectively, of the normal-isotropic hardening surface, the evolution rules of which will be formulated later. Further, the sub-isotropic hardening surface, which always passes through the back stress $\bar{\mathbf{M}}_k$ in the intermediate configuration and has a similar shape and a same orientation to the normal-isotropic hardening surface is expressed by the following equation (see Fig. 12.3).

$$g(\tilde{\bar{\mathbf{M}}}_k) = \tilde{R}\tilde{K} \tag{12.88}$$

The normal-isotropic hardening ratio $\tilde{R}$ is calculable from the equation $\tilde{R} = g(\tilde{\bar{\mathbf{M}}}_k)/\tilde{K}$ in terms of the known values $\bar{\mathbf{M}}_k$, $\bar{\boldsymbol{\Theta}}$ and $\tilde{K}$.

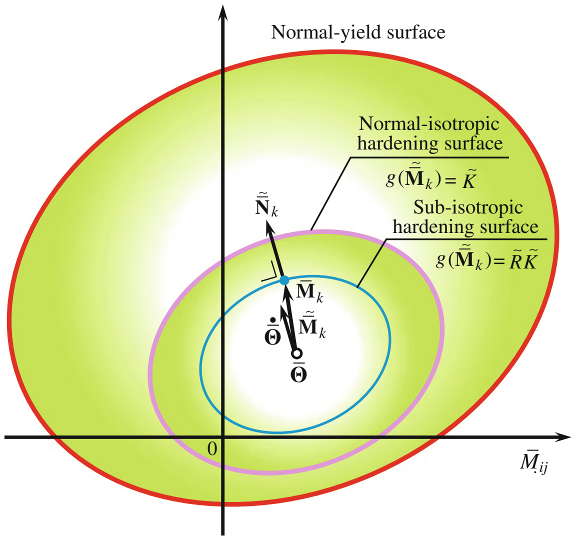

Fig. 12.3 Normal- and sub-isotropic hardening surfaces in multiplicative elastoplasticity

The consistency condition of the sub-isotropic hardening surface is given by

$$\frac{\partial g(\tilde{\overline{\mathbf{M}}}_k)}{\partial \tilde{\overline{\mathbf{M}}}_k} : \dot{\overline{\mathbf{M}}}_k - \frac{\partial g(\tilde{\overline{\mathbf{M}}}_k)}{\partial \tilde{\overline{\mathbf{M}}}_k} : \dot{\overline{\boldsymbol{\Theta}}} = \tilde{R}\dot{\tilde{K}} + \dot{\tilde{R}}\tilde{K} \tag{12.89}$$

The rates of $\tilde{K}$ and $\overline{\boldsymbol{\Theta}}$ are given by the following equations based on Eqs. (10.21) and (10.22).

$$\dot{\tilde{K}} = C\tilde{R}^{\varsigma}\langle \partial f \tilde{\overline{\mathbf{N}}}_k : \dot{\overline{\mathbf{M}}}_k \rangle \left\| \frac{\partial g(\tilde{\overline{\mathbf{M}}}_k)}{\partial \tilde{\overline{\mathbf{M}}}_k} \right\| \tag{12.90}$$

$$\dot{\overline{\boldsymbol{\Theta}}} = (1-C)\tilde{R}^{\varsigma}\langle \tilde{\overline{\mathbf{N}}}_k : \dot{\overline{\mathbf{M}}}_k \rangle \tilde{\overline{\mathbf{N}}}_k \tag{12.91}$$

where are the material constants and

$$\tilde{\overline{\mathbf{N}}}_k \equiv \frac{\partial g(\tilde{\overline{\mathbf{M}}}_k)}{\partial \tilde{\overline{\mathbf{M}}}_k} / \left|\left| \frac{\partial g(\tilde{\overline{\mathbf{M}}}_k)}{\partial \tilde{\overline{\mathbf{M}}}_k} \right|\right| \quad (\neq \tilde{\overline{\mathbf{N}}}_k^T) \tag{12.92}$$

Substituting Eqs. (12.91) and (12.92) for the evolution rules of K and $\boldsymbol{\Theta}$ into Eq. (12.90), the rate of the normal-isotropic hardening ratio is given by

$$\begin{aligned} \dot{\tilde{R}} &= \frac{1}{\tilde{K}}[\langle \frac{\partial g(\tilde{\overline{\mathbf{M}}}_k)}{\partial \tilde{\overline{\mathbf{M}}}_k} : \dot{\overline{\mathbf{M}}}_k \rangle - \frac{\partial g(\tilde{\overline{\mathbf{M}}}_k)}{\partial \tilde{\overline{\mathbf{M}}}_k} : (1-C)\tilde{R}^{\varsigma}\langle \overline{\mathbf{N}}_k : \dot{\overline{\mathbf{M}}}_k \rangle \overline{\mathbf{N}}_k - \tilde{R}C\tilde{R}^{\varsigma}\langle \frac{\partial g(\tilde{\overline{\mathbf{M}}}_k)}{\partial \tilde{\overline{\mathbf{M}}}_k} : \dot{\overline{\mathbf{M}}}_k \rangle] \\ &= \frac{1}{\tilde{K}}\langle \frac{\partial g(\tilde{\overline{\mathbf{M}}}_k)}{\partial \tilde{\overline{\mathbf{M}}}_k} : \dot{\overline{\mathbf{M}}}_k \rangle \{1 - [1 - C(1-\tilde{R})]\tilde{R}^{\zeta}\} \end{aligned} \tag{12.93}$$

which is the monotonically-decreasing function of $\tilde{R}$ fulfilling

$$\dot{\tilde{R}} \begin{cases} = \dfrac{1}{\tilde{K}}\langle \dfrac{\partial f(\tilde{\overline{\mathbf{M}}}_k)}{\partial \tilde{\overline{\mathbf{M}}}_k} : \dot{\overline{\mathbf{M}}}_k \rangle \ (>0) \ \text{ for } \ \tilde{R} = 0 \\ < \dfrac{1}{\tilde{K}}\langle \dfrac{\partial f(\tilde{\overline{\mathbf{M}}}_k)}{\partial \tilde{\overline{\mathbf{M}}}_k} : \dot{\overline{\mathbf{M}}}_k \rangle \ (>0) \ \text{ for } \ \tilde{R} < 1 \\ = 0 \ \text{ for } \ \tilde{R} = 1 \\ < 0 \ \text{ for } \ \tilde{R} > 1 \end{cases} \tag{12.94}$$

Therefore, $\overline{\mathbf{M}}_k$ is attracted automatically to the normal-isotropic hardening surface even if it goes out from that surface by virtue of the inequality $\dot{\tilde{R}} < 0$ for $\tilde{R} > 1$ as shown in Eq. (12.96). Furthermore, the judgment of whether $\overline{\mathbf{M}}_k$ lies on the normal-isotropic hardening surface is not required in the present formulation.

The evolution rule of isotropic hardening is given analogously to Eqs. (10.27) and (10.28), noting Eq. (12.63) as follows:

$$\boxed{\dot{H} = \tilde{R}^{\upsilon}\langle \tilde{\overline{\mathbf{N}}}_k : \overline{\mathbf{A}}/\|\overline{\mathbf{A}}\|\rangle f_{Hn}\,\dot{\bar{\lambda}}} = f_{Hsn}\,\dot{\bar{\lambda}} \tag{12.95}$$

where υ is the material constant and

$$f_{Hsn} \equiv \tilde{R}^{\upsilon}\langle \tilde{\overline{\mathbf{N}}}_k : \overline{\mathbf{A}}/\|\overline{\mathbf{A}}\|\rangle f_{Hn} \tag{12.96}$$

$$\begin{aligned}\overline{\mathbf{A}} \equiv\; & \mathbf{F}^{p}_{ks}\widehat{\mathbb{C}}^{k} : \mathbf{F}^{pT}_{ks}(\underline{\overline{\mathbf{N}}} - (1/b_k)\|\underline{\overline{\mathbf{N}}}\|\mathrm{sym}[\overline{\mathbf{M}}_k])\mathbf{F}^{p}_{ks}\mathbf{F}^{pT}_{ks} + 2\mathrm{sym}[(\underline{\overline{\mathbf{N}}} + \eta^{p}(\overline{\mathbf{M}}\,\underline{\overline{\mathbf{N}}} - \underline{\overline{\mathbf{N}}}\,\overline{\mathbf{M}})\\ & - (1/b_k)\|\underline{\overline{\mathbf{N}}}'\|\{\mathrm{sym}[\overline{\mathbf{M}}_k] + \eta^{p}_{k}(\overline{\mathbf{M}}_k\,\mathrm{sym}[\overline{\mathbf{M}}_k] - \mathrm{sym}[\overline{\mathbf{M}}_k]\overline{\mathbf{M}}_k)\})\overline{\mathbf{M}}_k]\end{aligned} \tag{12.97}$$

The plastic modulus is given by replacing f_{Hn} to f_{Hsn} in Eq. (12.79).

The function $g(\tilde{\overline{\mathbf{M}}}_k)$ is given in the simplest form as follows:

$$g(\tilde{\overline{\mathbf{M}}}_k) = \|\tilde{\overline{\mathbf{M}}}_k\| \tag{12.98}$$

which will be used in the subsequent sections for the comparisons with test data. It follows from Eqs. (12.92) and (12.98) that

$$\tilde{\overline{\mathbf{N}}}_k \equiv \frac{\partial g(\tilde{\overline{\mathbf{M}}}_k)}{\partial \tilde{\overline{\mathbf{M}}}_k} = \frac{\tilde{\overline{\mathbf{M}}}_k}{\|\tilde{\overline{\mathbf{M}}}_k\|} \tag{12.99}$$

The incorporation of the tangential-inelastic strain rate described in Sect. 9.10 into the multiplicative elastoplasticity requires a further study.

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