Trends in Mathematics
Research Perspectives

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New Trends in Analysis and Interdisciplinary Applications

Selected Contributions of the 10th ISAAC Congress, Macau 2015

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Preface

The International Society for Analysis, its Applications and Computation (ISAAC) is organizing biennial international congresses since 1997 in different places all over the world. Besides highlighting the newest developments in analysis, its applications, and computation through plenary and session talks, these congresses became well-liked social events where young scientists are honored with ISAAC awards, highly advancing and active mathematicians are elected as honorary ISAAC members, and the society is electing its officers for the next 2-years period. After the congresses the ISAAC members elect a new board, the group in charge of electing, supporting, and controlling the officers, for the following 2 years.

The 10th International ISAAC Congress was organized at the University of Macau between the 3rd and the 8th of August 2015. The proceedings of this 10th congress are published in two volumes, one containing the plenary lectures and the other one including the session talks and some personal citations for meritorious ISAAC members. Some sessions publish their own proceeding issues separately. The plenary talks appear as "Mathematical Analysis, Probability and Applications—Plenary Lectures: ISAAC 2015, Macau, China, T. Qian and L. Rodino., eds. Springer Proceedings in Mathematics and Statistics, 177, 2016."

At the Macau congress Professor Lo Yang from the Chinese Academy of Sciences was elected as honorary member (see the citation by Gary Gundersen in this volume) and Dr. Jinsong Liu, also from the Chinese Academy of Sciences in Beijing, received the ISAAC award for young scientists. As newly introduced, five analysts were chosen for special mention of young scientists: Dr. Marcello D'Abbicco from Universidade de Sao Paulo, Brazil; Dr. Matteo Dalla Riva from Universidade de Aveiro, Portugal; Dr. Pei Dang from Macau University of Science and Technology, Macau, China; Dr. Humberto Gil Silva Rafeiro from Pontificia Universidad Javeriana, Bogota, Colombia; Dr. Yan Yang from Zhongshan (Sun Yat-Sen) University, Guangzhou, China.

Two more citations were included in this volume, one for Professor Wei Lin from Zhongshan University in Guangzhou by Yongzhi Xu, Dao-Qing Dai, and Yu-Qiu Zhao, and one for Professor Rudolf Gorenflo from FU Berlin by Francesco Mainardi. Unfortunately two colleagues from the participants of the Macau congress have passed away recently. Alan McIntosh, a plenary speaker at the congress, passed away in August 2016, and Juri Rappoport, one of the very active special interest group and session organizers and long-time ISAAC member, passed away in March 2016. Some short obituaries will appear in the next ISAAC newsletter to be sent out in January, 2017. The Macau ISAAC congress proved to be also very efficient in recruiting a large number of new ISAAC life members. This has happened mainly because of the activities of some special interest groups. ISAAC has now over 200 life members and two institutional members, i.e., the Scientific Centre CEAF in Lisbon, Portugal, and the Springer International Publishing AG in Basel, Switzerland.

The **76** articles collected in the present volume are selected from session talks. They are the outgrowth and further development of the talks presented at the conference by participants from different countries all over the world, including United States, UK, Australia, Canada, Russia, Kazakhstan, China, India, Hong Kong, Japan, Korea, Macau, and members of the European Union. Most of them contain new results. All the papers were strictly refereed. This volume reflects the latest developments in the area of analysis, its applications, and computation. As in the previous years, some of the sessions or interest groups decided to publish their own volumes of proceedings and are therefore excluded from the present collection. This volume contains eight different chapters.

In Part I, we include eleven articles on complex-analytic methods for applied sciences, complex geometry, and generalized functions. L.A. Alexeyeva and G.K. Zakir'yanova use the method of generalized functions to solve nonstationary boundary value problems (BVP) for strictly hyperbolic systems and construct the generalized solutions of BVP subject to shock waves. Olaf Bar uses a fast algorithm to determine the flux around closely spaced nonoverlapping disks in a conductive plane. Roman Czapla and Vladimir V. Mityushev construct a conformal mapping of the square with disjoint circular holes onto the square with disjoint slits. Piotr Drygas and Vladimir Mityushev study two-dimensional elastic composites with nonoverlapping inclusions by means of boundary value problems for analytic functions, following Muskhelishvili's approach. Victoria Hoskins gives an example of a linear action of the additive group on an affine algebraic variety arising in the construction of an algebraic symplectic reduction, with nonfinitely generated ring of invariants. Priska Jahnke and Ivo Radloff generalize a theorem of Van de Ven to the case when the ambient space is a homogeneous manifold different from a projective space. Irina V. Melnikova considers different types of solutions to abstract stochastic Cauchy problems, especially generalized solutions, as regularized in a broad sense. Mateusz Muchacki discusses image processing algorithms in different areas of science. E. Pesetskaya and N. Rylko extend the method of functional equations to boundary value problems for a half-plane, strips, and rectangles with circular inclusions. Daniel S. Sage illustrates the theory of meromorphic connections on curves whose leading term is nilpotent in the case of rank-2 flat vector bundles, where much of the Lie-theoretic complexity is absent. Finally, Anna Stolińska and Magdalena Andrzejewska present a result of a qualitative investigation where a case study is treated using an eye tracking technology.

In Part II, we include eleven articles on complex and functional-analytic methods for differential equations and applications. H. Begehr constructs harmonic Green and Neumann functions using the parqueting-reflection principle for strips and hyperbolic strips in the complex plane. Zaiqiang Ku and Li Cheng study the SIR model and obtain the relationship between the number of affected individuals within a special period of time with applications to Ebata virus propagation. M.B. Muratbekov treats such issues as existence of the resolvent and discreteness of the spectrum for the Schrödinger operator with a parameter changing sign. S.A. Avdonin, G.Y. Murzabekova, and K.B. Nurtazina consider source identification problems for the heat equation with memory on an interval and on graphs without cycles and propose a stable efficient identification algorithm. N. Rajabov suggests new methods for investigating the model Volterra type integral equation with logarithmic singularity and the kernel of which consists of a composition of polynomial functions with logarithmic singularity and functions with singular points. K.N. Ospanov studies a three-term second-order differential equation with unbounded intermediate coefficient and gives solvability results and some conditions for compactness of the resolvent of the corresponding operator. Gian Rossodivita and Judith Vanegas find all linear first-order partial differential operators with elliptic complex numbers-valued coefficients associated with an elliptic generalized-analytic operator. Simon Serovajsky and collaborators consider a bacteria population under the action of bactericidal antibiotics. A. Tungatarov solves the Cauchy problem for a system of n-th order nonlinear ordinary differential equations. Yufeng Wang and Yanjin Wang study the Hilbert-type boundary-value problem for rotation-invariant polyanalytic functions on the unit disc. Finally, Shouguo Zhong, Ying Wang, and Pei Dang study a Riemann boundary value problem with square roots on the real axis for a sectionally holomorphic unknown function having zeros in the upper and lower half-planes, and obtain a solution and the explicit solvability condition.

Part III includes twelve articles on functions theory of one and several complex variables. Zhixue Liu and Tingbin Cao make use of Nevanlinna theory and the Zalcman-Pang lemma to obtain interesting results of normalness criteria for a certain type of differential polynomials studied. Guantie Deng, Haichou Li, and Tao Qian present some results on rational approximation, Laplace integral representation, and Fourier spectrum characterization of functions in Hardy H^p spaces on tubes for the whole range of p in $[1, +\infty]$ in the several complex variables setting. Robert Xin Dong obtains lower bounds for the Arakelov metrics for certain compact Riemann surfaces. Min Ku and Fuli He construct the Bergman kernel on the unit ball of $\mathbb{R}^{2n}$ in the setting of Hermitian Clifford analysis and then derive the Plemelj formula for the Bergman integral on the unit ball. Further, Xiao-Min Li, Cui Liu, and Hong-Xun Yi study the uniqueness question for transcendental meromorphic functions that share four distinct finite real values. Fanning Meng, Jianming Lin, Wenjun Yuan, and Zhigang Wang study a complete intersection surface singularity of Brieskorn type and propose a sufficient condition for coincidence of the fundamental cycle and the minimal cycle on a minimal resolution space. Ming-Sheng Liu investigates properties and characteristics of a subclass of starlike functions on the unit disk and proves a covering theorem. Yongmin Liu and Yanyan

Yu obtain the boundedness and compactness of the generalized integration operator from the $Q_K(p;q)$ space to the little Zygmund-type space. Katsuhiko Matsuzaki studies the hyperbolic metric a domain in the plane obtained by removing its integer lattice points. Wenjun Yuan, Fanning Meng, and Shengjiang Chen study a normality criterion related to the famous Hayman conjecture and obtain four criteria. Jingshi Xu and Xiaodi Yang introduce variable exponent Besov and Triebel-Lizorkin spaces associated with a non-negative self-adjoint operator and give equivalent norms and atomic decompositions of these new spaces. Hongfen Yuan, Tieguo Ji, and Hongyan Ji derive a decomposition theorem for the kernel of the polynomial slice Dirac operator using the generalized Euler operator in $\mathbb{R}^{m+1}$, generalizing the well-known Almansi decomposition theorem. Shengjiang Chen, Weichuan Lin, and Wenjun Yuan consider properties of meromorphic functions, which share a set with their first derivatives. Finally, Cuiping Zeng derives a normality criterion for differential polynomials, which improves an earlier result by Fang and Hong.

Part IV is devoted to harmonic analysis and nonlinear PDEs and includes seven articles. Neal Bez et al. propose a conjecture concerning the shape of initial data that make the classical Strichartz estimates extremal for the wave propagator with initial data of Sobolev regularity $\frac{d-1}{4}$ in all spatial dimensions $d > 3$, complementing an earlier conjecture of Foschi in the critical case of $\frac{1}{2}$ regularity. Jishan Fan and Tohru Ozawa study global weak solutions to the 3D time-dependent Ginzburg-Landau-Maxwell equations with the Coulomb gauge and obtain uniform bounds of solutions with respect to the dielectric constant. Tokio Matsuyama and collaborators prove L_p-boundedness of functions of Schrödinger operators on an open set of $\mathbb{R}^d$. Tokio Matsuyama and Michael Ruzhansky consider the Cauchy problem for the Kirchhoff equation and establish the almost global existence of Gevrey space solutions. Kiyoshi Mochizuk and Igor Trooshin treat an inverse scattering problem on a graph with infinite rays and a loop joined at different points, and reconstruct a potential on the basis of the scattering data of the operator. Lukasz T. Stępień presents some exact (functionally invariant) solutions of self-dual Yang-Mills equations in the $SU(2)$ case. Finally, Vladimir B. Vasilyev discusses basic principles for constructing a theory of boundary value problems on manifolds with nonsmooth boundaries.

Part V contains ten articles on integral transforms and reproducing kernels. Miki Aoyagi considers the Vandermonde matrix-type singularity learning coefficients in statistical learning theory. Dong Hyun Cho, Suk Bong Park, and Min Hee Park introduce several scale formulas on $C[0;t]$ for the generalized analytic conditional Wiener integrals of cylinder functions and of functions in a Banach algebra, which corresponds to the Cameron-Storvick Banach algebra. B.I. Golubov and S.S. Volosivets generalize the results of C.W. Onneweer on membership of multiplicative Fourier transforms to Besov-Lipschitz or Herz spaces. Byoung Soo Kim reviews results on shifting for the Fourier-Feynman transform. T. Matsuura and S. Saitoh develop some general integral transform theory, based on the recent general concept of generalized reproducing kernels. Juri Rappoport considers the application of the Kontorovich-Lebedev integral transforms and dual integral equations to the solution of certain mixed boundary value problems and reduces the diffusion and elasticity problems to the solution of a proper mixed boundary value problem for the Helmholtz equation. S. Saitoh and Y. Sawano introduce a general concept of a generalized delta function as a generalized reproducing kernel and consider all separable Hilbert spaces. Yoshihiro Sawano proves that the mapping $R : H_K(E) \ni f \mapsto (f|E_1, f|E_2) \in H_{K|E_1 \times E_2}(E_1) \oplus H_{K|E_2 \times E_2}(E_2)$ is isomorphic if and only if $K|E_1 \times E_2 = 0$, where K is a positive definite function on $E = E_1 + E_2$. G.K. Zakir'yanova studies a system of hyperbolic equations of second order, and by using the Fourier transform of generalized functions constructs the fundamental and generalized solutions. Finally, Haizhang Zhang and Jun Zhang justify substituting inner products with semi-inner-products in Banach spaces, discuss the notion of reproducing kernel Banach spaces, and develop regularized learning schemes in the spaces.

Part VI deals with recent advances in sequence spaces and includes five articles. Awad A. Bakery gives sufficient conditions on a sequence space under which the finite-rank operators are dense in the space of all the operators considered. Binod Chandra Tripathy addresses developments on rates of convergence of sequences. Further, Paritosh Chandra Das studies a bounded difference sequence space with a statistical metric. Shyamal Debnath and Debjani Rakshit introduce the notion of rough convergence in general metric spaces and the set of rough limit points and prove several results concerning this set. Finally, Amar Jyoti Dutta introduces a class of sequences of interval numbers and establishes some properties like completeness, linearity, symmetry, as well as some inclusion relations.

Part VII focuses on recent progress in evolution equations and includes twelve articles. Marcello D'Abbicco and his collaborators find a critical exponent for the global existence of small-data solutions to the semilinear fractional wave equation in low space dimensions. Andrei V. Faminskii applies a classical Strichartz-type argument for an abstract one-parameter set of linear continuous operators and rigorously justifies a Strichartz-type estimate in a non-endpoint case. M.R. Ebert, L. Fitriana, and F. Hirosawa prove that the elastic energy satisfies a better estimate than the kinetic energy if the propagation speed is in $L_1(\mathbb{R}_+)$. Anahit Galstian gives estimates for the lifespan of solutions of a semilinear wave equation in the de Sitter spacetime with flat and hyperbolic spatial part under some conditions on the order of the nonlinearity. Christian P. Jäh discusses the connection of the backward uniqueness property with the regularity of the principal part coefficients measured by moduli of continuity and obtains a new backward uniqueness result for higher-order equations. Xiaojun Lu and Xiaofen Lv study the strong unique continuation property for the electromagnetic Schrödinger operator with complex-valued coefficients and its applications. Makoto Nakamura studies the Cauchy problem for nonlinear complex Ginzburg-Landau type equations in Sobolev spaces under the variance of the space and remarks some properties of the spatial variance on the problem. Belkacem Aksas and Salah-Eddine Rebiai consider boundary and internal stabilization problems for the fourth-order Schrödinger equation in a smooth bounded domain of $\mathbb{R}^n$ and establish the decay of the solutions. Pham Trieu Duong and Michael Reissig give a survey of an external damping problem and present a research approach to show essential conditions for the differential operators that influence the decay estimates and the global existence of solutions to initial value problems. Yuta Wakasugi deals with the critical exponent for the Cauchy problem of the system of semilinear damped wave and wave equations and proves some blow-up results. Jens Wirth's work is devoted to Cauchy problems for t-dependent hyperbolic systems with lower order terms allowed to become singular at a final time and describes the associated loss of Sobolev regularity in terms of the full symbol of the operator.

Part VIII is devoted to wavelet theory and image processing and includes six articles. Kensuke Fujinoki considers two-dimensional average interpolating wavelets and describes properties of the bi-orthogonal bases and associated filters, such as order of zeros, regularity, and decay. Keiko Fujita studies the Gabor transformation for a square integrable function on the two-dimensional sphere and its inverse transformation, and by using an integral over $\mathbb{R}^3$ gives the inverse Gabor transformation in explicit form. Nobuko Ikawa and his collaborators investigate the relation between the slow component of auditory brainstem response and the number of averagings using discrete stationary wavelet analysis and present a new model to analyze the phase shifts of the spontaneous electroencephalogram. Further, Kiyoshi Mizohata shows how to deal with big data written in Japanese and explains several interesting results obtained by wavelet analysis. Akira Morimoto and collaborators propose an image source separation method using N-tree discrete wavelet transforms and present the results of numerical experiments to show the validity of the proposed method. Finally, Marcin Piekarczyk and Marek R. Ogiela undertake a study of the approach to personal authentication based on analyzing biomechanical characteristics related to palm movements and propose a matching scheme.

The editors wish to sincerely thank Macau government and the University of Macau, whose support made possible the success of the conference. We wish to thank the postgraduates in the Department of Mathematics who contributed kind and generous support as volunteers during the congress. We are grateful to the organizers of the 21 sessions of the congress for their work. They spent a key large amount of time inviting participants, arranging and chairing their sessions, and creating a familiar and workshop-like atmosphere within their sessions. Finally, we sincerely thank the session leaders, including V. Mityushev, H. Begehr, A. Schmitt, Wenjun Yuan, Ming-Sheng Liu, Michael Oberguggenberger, J. Wirth, Juri Rappoport, V. Georgiev, B.C. Tripathy, Marcello D'Abbicco, Qiuhui Chen, and Keiko Fujita, who participated in collecting contributions to this proceedings volume and in the refereeing process of the submissions.

Macau, China — Pei Dang
Aveiro, Portugal — Min Ku
Macau, China — Tao Qian
Torino, Italy — Luigi G. Rodino

Citation

Professor Lo Yang: Honorary Member of ISAAC

Professor Lo Yang is a well-known outstanding mathematician whose research achievements and contributions to the mathematical community are invaluable. His extensive collection of high-quality research contains profound results on a variety of topics in complex analysis. Most of his research is in the value distribution theory of meromorphic functions, which is the study of the growth and distribution of values of entire and meromorphic functions. Modern value distribution theory began with the fundamental work of Rolf Nevanlinna.

Particular areas of Lo Yang's research include Nevanlinna deficiencies of meromorphic functions and their derivatives and primitives, deficient functions, angular distribution theory, Borel directions and other singular directions, normal families, and other areas. His research has had a far-reaching impact on value distribution theory. He likes to work on big problems and has had outstanding success.

His many striking results include the following: (a) Lo Yang and G. H. Zhang showed that the number of deficient values of a meromorphic function f of finite positive order cannot exceed the number of Borel directions of f (with a corresponding result for entire functions). (b) Lo Yang showed that the sum of the deficiencies of the kth derivative of a meromorphic function cannot exceed a specific constant that depends on k, which improved results of E. Mues and W.K. Hayman. (c) Lo Yang and Y. Wang proved an inequality for the sums of deficiencies of a meromorphic function and its kth derivative and found all cases of equality.

(d) Lo Yang showed that a meromorphic function of finite lower order p has at most a countable number of deficient functions, and the sum of the corresponding deficiencies cannot exceed a specific constant that depends on p. (e) For every transcendental meromorphic function, Lo Yang proved the existence of a Hayman direction of Picard type, and with Q.D. Zhang, proved the existence of a Hayman direction of Borel type. As an expert in the field, Lo Yang wrote the very important reference book, *Value Distribution Theory*, which is an invaluable resource for both established researchers and mathematicians entering the field.

His publications are very well written and they exhibit his love of mathematics. His style of writing is a pleasure to read. Lo Yang has achieved numerous honors and awards and he has held many important positions. He was elected Member of the Chinese Academy of Sciences in 1980, where he was the youngest Academician from 1980 to 1990. Lo Yang was the Director of the Institute of Mathematics (1987–1995) and the founding President of the Academy of Mathematics and Systems Science (1998–2002) in the Chinese Academy of Sciences. He was the Secretary General (1983–1987) and President (1992–1995) of the Chinese Mathematical Society.

In addition to the enormous contributions of his research and his excellent work in the above important positions, Lo Yang has contributed to the mathematical community in many other significant ways. These additional contributions include his several expository surveys of important results in complex analysis, his numerous lectures at universities throughout the world, his extensive work at developing Chinese mathematics and arranging international scientific exchanges, his service on editorial boards of several mathematical publications, and the countless times he has helped others with their research.

Gary G. Gundersen

Citation

Professor Wei Lin: Citation for His 80th Birthday

Professor Wei Lin is a professor in the Department of Mathematics at Zhongshan (Sun Yat-sen) University in Guangzhou, China. Lin is perhaps most well known for his involvement in the development of a new classification system for partial differential equations, a project completed in collaboration with Ci-Quian Wu, under the direction of Professor Loo-Keng Hua, and published in their 1979 work, Systems of Second-Order Linear Partial Differential Equations with Constant Coefficients, Two Independent Variables and Two Functions. Professor Lin's research continues to expand upon his earlier work, though since the 1990s he has developed an increased focus on wavelet theory, publishing papers on the nonuniform sampling problems in shift-invariant spaces and the Galerkin Method and the subdivision algorithm, among many others.

Lin was born on September 7, 1934, in Shantou, Guangdong Province, China. In 1952, he was admitted to Zhongshan University, and he completed his undergraduate work there by 1956. Following his graduation, Lin remained at Zhongshan, where he began his postgraduate studies, majoring in geometric theory of complex functions and completing his thesis in 1960 (no degree was conferred, as prior to 1977 no degree system existed in China). Following completion of his thesis, Lin received an appointment as a lecturer at Zhongshan, a position he held until 1978, this duration due largely to the ossification of positions in Chinese universities during the period. Lin was promoted to associate professor in 1978 and achieved full professor status in 1983. The Academic Degrees Committee of the State Council approved Professor Lin as an advisor for PhD students in 1986.

In addition to his work at Zhongshan University, Lin has also been actively involved in a variety of other roles in the mathematical community. Between 1990 and 1995, he was the Chairman of the Mathematical Society of Guangdong; he was later named Honor Chairman, a position that he continues to hold. Lin was a member of the board of directors of the International Society for Analysis, its Applications and Computation (ISAAC) between 1997 and 2002, and from 2006 to 2010. Professor Lin has served on the editorial committee of several well-respected journals, including the *Journal of Applicable Analysis*, *Advances in Mathematics*, and *Control Theory and Application*.

Among the most important of Professor Lin's myriad publications is the aforementioned Systems of Second-Order Linear Partial Differential Equations. Lin and his coauthors completed the classification of the system with two variables and two unknown functions, underwent theoretical analysis of the canonic systems, and found applications in mathematical physics, such as elasticity. Their initial 1979 Chinese-language publication was later enhanced by a 1985 English companion, which carried the work further and included revisions and research reports by Professors Robert Gilbert, Y.K. Cheung, C.Q. Wu, and, of course, Lin himself.

On top of his impressive theoretical work, the issues of applying mathematics to practical systems have been one of Professor Lin's concerns since the 1970s. During this period, Lin turned to a brand new mathematical-related field, automatic control theory and applications. On the theoretical level, Lin and his colleagues developed the distributed control systems, while simultaneously applying their work to the creation of a digitally controlled steel-cutting machine for Wen Chong Shipyard in Guangzhou.

Throughout his career, Professor Lin has worked not only to advance his field but also to create bridges within the international community. In 1982, Professor Lin visited Professor Gilbert at the University of Delaware. During this 1 year visit, Lin's warm and friendly personality helped to build a bridge for long-lasting international collaborations among Gilbert, Lin, Heinrich Begehr, and other mathematicians in analysis, application, and computation. Lin has also received visiting appointments at a number of institutions across the globe, including the University of Delaware, the Free University of Berlin, the University of Kazan, and the National University of Singapore.

Yongzhi Xu
Dao-Qing Dai
Yu-Qiu Zhao

Citation

Professor Rudolf Gorenflo: Citation for His 85th Birthday

As a long standing collaborator of Professor Rudolf Gorenflo (since 1994), I am pleased and honored to edit an overview of his life story and professional career on the occasion of his 85th birthday for ISAAC readers. He is a well-known mathematician, an expert in the fields of Differential and Integral Equations, Numerical Mathematics, Fractional Calculus, Applied Analysis, Special Functions, and Mathematical Modeling. His list of publications yields a vivid sight of his ability for collaboration with other researchers and on his wide spectrum of research activities; see his profile in GOOGLE SCHOLAR and the cited paper of FCAA in 2011.

Footnote: For readers interested in Fractional Calculus and in other details of Gorenflo's scientific life may I refer to the survey paper published in the journal *Fractional Calculus and Applied Analysis*, Vol. 14 No. 1 (2011), pp. 3–18, entitled Professor Rudolf Gorenflo and his contribution to fractional calculus by Yuri Luchko, Francesco Mainardi, and Sergei Rogosin.

Rudolf Gorenflo is a descendant of Huguenots who in the seventeenth century came to Germany as religious refugees. He was born on July 31, 1930, as first son of a farmer in Friedrichstal near Karlsruhe. As a child he learnt and practiced all kinds of agricultural work, thereby during the war also learning how to hide in holes against low-flying fighter planes hunting peasants in the fields. As a remarkable fact it may be noted that between ages 12 and 16 he played the organ in the village church at religious services, replacing the regular organist who served in the army during the war. His mathematical talents showed early. He mastered the number system before he learnt to read alphabetical texts; aged 5 years he well knew the multiplication table, and a little later he liked to play with decimal expansions of fractions, for example by pencil and paper calculation he found the length of the period of the number 1/49 to be 42, contrary to the naive expectation that it should be 48. When he was 12 his parents (recognizing his inclination) arranged for him school education in a gymnasium, and he could get a kind of higher instruction (languages including Latin, mathematics and science, history, religion). He has in living grateful memory his teachers of mathematics. He was, e.g., fascinated learning that the power, the imaginary unit raised to itself, turns out to be a real number.

After Matura examination he became a student of mathematics at the Technical University of Karlsruhe. Under the guidance of Professor Hans Wittich he choose theory of analytic functions as his field of research for his diploma (1956) and degree of Doctor rerum naturalium (1960). His diploma thesis "Meromorphic Periodic Functions of Finite Order" was a critical detailed exposition of papers by Pham Tinh-Quat, a Vietnamese researcher working in Paris with the famous Georges Valiron. His doctoral thesis treated questions of deducing asymptotic properties of an entire function from the infinitary behavior of the sequence of its Taylor coefficients. It has the horribly long title "On the Wiman-Valiron comparison method for power series and its application to the theory of entire transcendental functions." In it appears as an example the Mittag-Leffler function which now, 55 years later, still is a companion of Rudolf Gorenflo. From 1955 until 1961 R. Gorenflo was a teaching assistant in Prof. Wittich's institute. Feeling that there are no promising career prospects for pure mathematicians at the beginning of the 1960s R. Gorenflo looked around for and found a position in industry. Now, leaving the warm environment of university research and teaching made him scientifically homeless. Instead of developing a clear mathematical profile by working for the rest of his life deeper and deeper in the specialization he had entered through his two theses he became a random walker through mathematics and its applications. Entering a commercial-industrial enterprise was not his death as a scientist. After having received his degree of Dr. rer. nat, he worked as a research mathematician first in a telecommunication company in Stuttgart (Standard Electric Lorenz, then in the German branch of US International Telephone and Telegraph Company ITT), then 8 years (1962–1970) in the Max Planck Institute for Plasma Physics in Garching near Munich, then as professor of Mathematics 3 years at the Technical University in Aachen, and since 1973 at the Free University of Berlin. He has collaborated with very different kinds of coauthors in several countries, in varying mathematical and physical disciplines and has supervised many students. His fields of research comprise complex analysis, random numbers and Monte-Carlo simulation, integral transforms and special functions, fractional calculus, diffusion processes (analysis, discretization, stochastics, and simulation), and recent specialization distributed order fractional diffusion processes. A few applications may be mentioned: simulation of queuing systems, simulation of particle flights in rarefied gases, evaluation of spectroscopic measurements, calculation of magnetic fields for toroidal devices of plasma containment in controlled nuclear fusion. At SEL and in the Institute for Plasma-Physics large part of his work consisted in close collaboration with engineers and physicists by which he developed his outstanding ability to work fruitfully with non-mathematicians.

Thus, a researcher, he has worked in complex analysis. He contributed to the theory of entire and meromorphic functions and complex differential equations. Later he devoted to its application to modeling plane electric and magnetic fields and flows, in analysis, simulation, and numerical treatment of diffusion processes (of one- and of multicomponent type). All the applications were achieved by systems of parabolic differential equations in theory and numerical treatment of integral equations occurring in evaluation of physical measurements. In this latter activity he found his way into the field of inverse and ill-posed problems, first to integral equations of Abel type where he made acquaintance with differential and integral operators of non-integer order and to whose theory and practice he later worked intensively with several coauthors. Noteworthy to mention here are his joint books with Sergio Vessella on Abel Integral Equations: Analysis and Applications (Springer1991) and with Dang Dinh Ang et al. on Moment Theory and Some Inverse Problems in Potential Theory and Heat Conduction (Springer 2002). In the 1980s R. Gorenflo jointly with the geophysicist Prof. Andreas Vogel was a chief organizer of several conferences on applications of mathematics in geophysics with main emphasis on inverse problems. Gorenflo's research results attained international peer recognition. Then he were invited to have a research visit the universities in Delaware (Newark, Del.), Kingston Ontario, Florence (Italy), Beijing, Tokyo, Hanoi, Ho Chi Minh City, Manila, Kraków. In recent years he developed close collaboration with the Centre for Mathematical and Statistical Sciences (Director Prof. A.M. Mathai) in Pala in the southern Indian state of Kerala, India, starting in 2009 with a course he gave there on power laws and their applications.

Around 1990 R. Gorenflo started his final career as a specialist in fractional calculus which was highly stimulated through friendship and collaboration with myself (Francesco Mainardi). This started with the conference hold in Summer of 1994 in Bordeaux. As their field of highest common interest they choose fractional relaxation and diffusion with related stochastic processes. Jointly with several coauthors, including Mainardi's students, they contributed substantially to the development and applications of fractional calculus and propagated their results in many international conferences. Their joint work culminated in their joint book with the late Prof. Anatoly Kilbas and Prof. Sergei Rogosin from Minsk on *Mittag-Leffler Functions: Related Topics and Applications. Theory and Applications*. Springer-Verlag 2014.

Other aspects should not be forgotten. R. Gorenflo was one of West German mathematicians who during the Cold War cultivated close personal and scientific contacts with colleagues in the other separated part of Germany and with colleagues from socialist countries. He participated in conferences in eastern parts of the world and had visitors and research fellows from there in Berlin, contacts that were strengthened and continued after the great change of 1989. He acted as one of the founding editors of the journal *Fractional Calculus and Applied Analysis* and is member of editorial committees of several other journals. In Aachen and Berlin he has guided many students in their works for diploma, doctor's degree, and teacher's examination, among them several from outside of Germany.

Rudolf Gorenflo: a short outline of his life.

Born on 31 July 1930 in Friedrichstal near Karlsruhe.
1950–1956: Student of Mathematics and Physics at Technical University in Karlsruhe.
1956: Diploma in mathematics.
1960: Promotion to Dr. rer. nat. (doctor rerum naturalium).
1957–1961: Scientific assistant at Technical University in Karlsruhe.
1961–1962: Mathematician at Standard Electric Lorenz Company in Stuttgart.
1962–1970: Research mathematician at Max-Planck Institute for Plasma Physics in Garching near Munich.
1970: Habilitation in Mathematics at Technical University in Aachen.
1971–1973: Professor at Technical University in Aachen.
1972: Guest professor at the University of Heidelberg.
since October 1973: Full professor at Free University of Berlin.
1976–1982: Deputy leader of Free University Research Project Optimization and Approximation (Leader K.-H. Hoffmann).
1982–1989: Director of Third Mathematical Institute of Free University of Berlin.
1980–1984: President of Berlin Mathematical Society.
1983–1988: Head of Research Project Modelling and Discretization (Free University of Berlin).
1989–1994: Head of Research Project Regularization (Free University of Berlin).
1995–2003: Head of Research Project Convolutions (Free University of Berlin).
1994–1997: Leading member of NATO Collaborative Research Project Fractional Order Systems, with R. Rutman, University of Massachusetts Dartmouth.
1995: Guest professor at the University of Tokyo.
1996–2013: Visiting professor of the Department of Physics, University of Bologna for periods of 2–4 weeks (almost) every year.
Since October 1998: Professor emeritus at Free University of Berlin.
After emeritation: continued activity in teaching, scientific research and international collaboration, refereeing and reviewing, and editorial boards of journals.
Rudolf Gorenflo is member of several scientific associations.
Family status: married since August 1959, two sons, one daughter.

Francesco Mainardi

ISAAC: How It Became What It Is

H. Begehr, R.P. Gilbert, L. Rodino, M. Ruzhansky, and M.W. Wong

Abstract A history of ISAAC from its founding in 1996 until its 10th biennial International Congress in 2015.

Keywords History of ISAAC

Mathematics Subject Classification (2010) 01A74

1 Foundation

"Let us found an international society for analysis!" announced R.P. Gilbert upon entering the office of H. Begehr, at the Free University of Berlin (FU), late in the morning as usual after a long night working on mathematics. This idea came to him in the beginning of the 1990s, almost 25 years ago. "What for?" replied Begehr, "Do you want to become the president of something?" But it was not just a mood. There

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was already the idea to call the society after Newton "ISAAC," the International Society for Analysis, its Applications and Computation.

In the 1970s R.P. Gilbert had been honored by an Alexander von Humboldt Award, at that time still restricted to US citizens, which had enabled him to extend his stay in Berlin for an additional semester at FU Berlin in 1974/1975. About 15 years later he used the AvH-reinvitation program for another visit at FU. Begehr and Gilbert were involved in writing a monograph [1] and the idea of an ISAAC was put aside for a while. During January 15–17, 1993, there was the conference on Complex Analysis and its Applications at the newly founded Hong Kong University of Science and Technology, organized by Chung Chun Yang. Here Gilbert brought up the idea about ISAAC at the conference and the response from colleagues was indeed supportive. In particular, C.C. Yang was enthusiastic. But his aim was different; in his view the main purpose behind the founding of the ISAAC was to promote and restore the prestige of complex analysis as an area in mathematical research. In this sense he was quite active at that time, e.g., with his book series in complex analysis with Gordon and Breach and with Kluwer. However, Gilbert had in mind a less focussed society which would "promote analysis, its applications, and its interaction with computation." This idea was adopted by the colleagues at the conference and was imbedded in ISAAC's constitution. The goal of ISAAC was to act as a society in the broad field of analysis and should never be dominated just by some particular subfield.

Three more years passed by before Gilbert had prepared to incorporate the society as a nonprofit organization in the state of Delaware. This registration had to be renewed every year for about US$30 and was terminated in 2005. The registration was important because otherwise a tax rate of 2% would have applied even to membership fees paid in the USA.

In 1996 activities of ISAAC began in ernest with Gilbert opening a bank account in Delaware and Begehr one in Berlin. C.C. Yang, the founding vice president, was asked to act as a treasurer for Asia. In April Gilbert had set up registration forms for memberships. They were sent out internationally and the founding members became registered. Indeed, this enabled Begehr sending US$1000 to Gilbert in that June for paying the cost of registration. Shangyou Zhang, a young colleague of Gilbert, set up a home page for ISAAC at the University of Delaware and consequently became the ISAAC corresponding secretary. A second website was established at the Freie Universität Berlin with a link to the main site in Delaware. And Yuri Hohlov opened a site in Moscow. He also managed to create the ISAAC logo, a drawing of the head of ISAAC Newton. Hohlov had put a list of ISAAC fellows on his site, which contained many mathematicians to whom Begehr had sent an invitation to join ISAAC. This publication was risky as none of the persons on the list were ever asked if they agree. By the way, only some were recruited at that time and had paid their dues. Nevertheless, the list survived somehow on the main website at Delaware now in the "List (all ISAAC related people)," containing more than 500 names; however their email addresses are not visible for the public but can be used to send messages. Probably it was never used again and has not been updated. The Moscow page disappeared, however, as soon as Hohlov joined the Lenin Library in Moscow as an employee.

2 ISAAC Publications, Proceedings

Gilbert prepared the 1st International ISAAC Congress at the University of Delaware (UD) early in 1996. This Congress took place from June 3 to 7, 1997. It was necessary to reduce registration fee for the congress, as well as the society membership fee, for attendees from countries with weak economies. Also it was decided that the Congress fee should contain some part of the ISAAC membership fee for the following 2 years. The congress was organized in parallel sessions, altogether more than 25, and plenary talks; see the ISAAC home page “mathissac.org” under “congresses”. The congress fee of US$100 was moderate for the developed countries. Several publishers were present, not just for book presentations but with editors who were interested in obtaining book contracts with ISAAC and its members. In particular John Martindale from Kluwer was enthusiastic about the new society and was expressing his wish that ISAAC will soon compete with SIAM in magnitude and importance. He signed a contract with Gilbert about a book series “International Society for Analysis, Applications and Computation.” Ten volumes have appeared between 1998 and 2003. In 1997 the market for scientific literature was still okay. Kluwer had guaranteed for US$500 each volume published in this series and for the first two volumes this amount was actually paid. Obviously Kluwer did not make money with the series and stopped paying. When Kluwer then merged with Springer after 2003 and the name Kluwer disappeared, the book market for proceedings had collapsed because of the restricted book budgets of university libraries. Springer had decided not to publish ISAAC’s proceedings anymore and as a consequence had terminated the Kluwer ISAAC series. John Martindale continued to work for Springer. He managed, e.g., in 2006 to get the English translation of the first volume of the *Selected Works of S.L. Sobolev* published by Springer. The Russian original had appeared in 2003. But Martindale was not happy with the new company and had quit. A translation of the second Sobolev volume, published in Russian language in the same year 2006, has not appeared. However, Springer was not interested in further collected works, for examples, the translation of the 2009 in Russian language published Selected Works of S.M. Nikol’skii. Therefore some years later this publisher has changed its policy. Already in 2003 Birkhäuser, since 1985 part of Springer, has started its series “Trends in Mathematics” and in 2011 Springer itself has started its “Springer Proceedings in Mathematics,” now “Springer Proceedings in Mathematics & Statistics,” where proceedings volumes are published. In 2013 Birkhäuser has started the subseries “Research Perspectives” of the “Trends in Mathematics.” Since the 2013 congress in Krakow the proceedings of ISAAC are published in these series, the plenary talks volumes in the Springer series, and the general proceedings volumes with Birkhäuser. The general Krakow proceedings volume is the second book in this newly opened subseries. But the general proceedings volume gets smaller and smaller as more sections are publishing their own volumes in order to avoid length restrictions for manuscripts.

As a consequence to the situation with publishers H. Begehr had started to tight ISAAC's strings with World Scientific in Singapore. This company had published the main proceedings from 2001 in Berlin in a very reliable way with convincing results. The problem was the requirement of bulk orders. Joji Kajiwara had succeeded in finding financial resources for paying the printing costs for the two proceedings volumes from 1999 of together more than 1,600 pages.

By the way he did a marvelous job although he had to look after his sick spouse daily for years. On the occasion of his 70th birthday ISAAC has honored him with a "Distinguished ISAAC Service Award".

Joji Kajiwara Fukuoka 2015

It was presented to him at the 12th International Conference of Finite and Infinite Dimensional Complex Analysis and Applications in Tokyo in July 2004, one of a yearly held series of conferences, organized by mathematicians at first from the countries China, Japan, and Korea, to which later Thailand and Vietnam and recently India have joined in. This series was once initiated by Joji Kajiwara.

The Fukuoka proceedings volumes were the only ones of all ISAAC congresses where almost all contributions are published in total. Maybe it was just an accident that some contributions from the session "Analytic Extension Formulas and their Applications" were published together with the ones from a conference from 11 to 13 January, 2000, on "Applications of Analytic Extensions" at the Research Institute for Mathematical Sciences of the Kyoto University in an extra volume in the ISAAC Kluwer series 2001 under the title "Analytic Extension Formulas and their Applications" edited by Saburou Saitoh, Nakao Hayashi, and Masahiro Yamamoto.

From the Delaware 1st ISAAC congress, four sections have published their papers in extra volumes outside the ISAAC Kluwer series; see the ISAAC home

23rd International Conference on Finite or Infinite Dimensional Complex Analysis and Applications Kyushu Sangyo University, Fukuoka, Japan, 24.–28. 08., 2015

page mathisaac.org. For the 3rd congress in Berlin an extra volume with the plenary lectures and some main talks from sessions had appeared in the ISAAC Kluwer series. All the other contributions made up two volumes of about 1500 pages published with World Scientific in Singapore. At the 4th congress in Toronto, Man Wah Wong had founded his special interest group under the ISAAC umbrella on pseudo-differential operators and a book series "Pseudo-Differential Operators, Theory and Applications" published with Springer-Birkhäuser. The contributions to the ISAAC congresses within the sessions on pseudo-differential operators are since 2003 published in the Birkhäuser series "Operator Theory: Advances and Applications." Until the 7th congress in London World Scientific has published the main parts of the contributed talks. At the Krakow congress in 2013 representatives from Springer were present and new relations to this publisher were tied. Since then the proceedings are split into two parts. While the plenary talks are published in the series "Springer Proceedings in Mathematics & Statistics," the contributed talks of most of the sessions are appearing in a volume from the Birkhäuser series "Trends in Mathematics, Research Perspectives." After the 10th congress in Macao, ISAAC and Springer signed a contract on publishing monographs and proceedings volumes under the logo of ISAAC (see Appendix). This is the second contract about ISAAC publications as the SAAC book series with World Scientific, see http://www.worldscientific.com/series/saac, is also effective since 2003. Only the main proceedings of the 8th ISAAC congress 2011 in Moscow were published without World Scientific and Springer for monetary reasons. They have appeared in three volumes published by the Peoples' Friendship University of Russia in Moscow.

A list of ISAAC publications is available at the home page mathisaac.org; see also the links there to the SIGs' home pages.

3 Members

The first ISAAC members were according to an email from Begehr to Gilbert from May 9, 1997: H. Begehr, M. Essen, R. Gorenflo, M. Kracht, S. Louhivaara, H. Malonek, W. Watzlawek, E. Wegert as life members (they had paid DM (German Marks) 250) and as annual members (the fee was DM 40) in 1995 K. Guerlebeck, P. Kravanja, M. Reissig, St. Ruscheweyh, J. Schnitzer and in 1996 Yu. Hohlov and M. Reissig. Not yet listed here are R.P. Gilbert and C.C. Yang. They both were supposed to recruit members in America and in Asia, respectively. At the end of 2015 ISAAC had 204 life members registered from 49 counties. The largest groups are from Japan (22), Russia (19), Italy (16), USA (13), China (12), Germany (10), and Kazakhstan (10). Seven members are from each Georgia and Serbia, 6 from Austria, also 6 from India, and 6 from Uzbekistan, 5 from Canada, 5 from UK, and 5 from Turkey, 4 as well from Belgium, from France and from Romania. Four countries have 3 ISAAC members: Armenia, Portugal, Sweden, and Tajikistan. The countries Algeria, Belarus, Cuba, Finland, Malaysia, Poland, Switzerland, Ukraine, and Vietnam are represented each by 2 members, and Argentina, Australia, Brazil, Egypt, Guadeloupe (France), Iran, Israel, Korea, Kyrgystan, Macedonia, Mexico, Moldova, Saudi Arabia, South Africa, Spain, Taiwan, United Arab Emirates, and Venezuela have one citizen in ISAAC. Among the "paid" members no continuation of membership fee is yet cultivated. Among the few members, however, is one institutional member from Portugal, the Center for Functional Analysis, Linear Structures and Applications in Lisbon. It has joined ISAAC in 2009 and is since then an ISAAC member. And recently in spring 2016 Springer International Publishing AG has become the second institutional ISAAC member.

Four mathematicians have been decorated as Honorary ISAAC Members. Victor Burenkov had initiated to honor Sergei Mikhailovich Nikol'skii, see [2], Academician of the Russian Academy of Sciences; at the 3rd congress in Berlin, Nikol'skii was at the age of 96, one of the plenary speakers there. At the York University in Toronto, Lee Lorch (see [3]) was—also suggested by Victor Burenkov—elected as Honorary Member. Oleg Vladimirovich Besov [4], former student and coauthor of S.M. Nikol'skii and also Academician at the Russian Academy of Sciences, became Honorary ISAAC Member at the Catania ISAAC congress in 2005. He was the first plenary speaker there and again V. Burenkov had suggested him. Lo Yang, Academician at the Chinese Academy of Sciences, was elected Honorary ISAAC Member at the 10th ISAAC Congress at the University of Macau in Macao 2015, see [5].

4 ISAAC Budget

Because international money transfer was and still is expensive, the idea at the beginning was to have local treasurers, collecting fees and only from time to time transfer bigger amounts. Due to the economical situation C.C. Yang did not collect any membership fees. Gilbert collected money in North America; Begehr had opened an ISAAC bank account in Berlin. The only financial resources of ISAAC are the membership and the congress participation fees. Already between 1995 and 1997 Begehr had collected DM (German Marks) 285 for paid membership fees and DM 2000 for life membership fees. For the registration of ISAAC as nonprofit scientific society in Delaware DM 1556.30 was transferred to Gilbert in 1997. Further in the first years ISAAC had in 1998 collected DM 1390 for conference fees and between the 2001 and 2003 congresses about Euro 610 and US$140 from so-called paid members and about Euro 1960 and US$1400 from life members.

But because of the monthly due bank charges, DM 109.50 until autumn of 1997, and only rare use, the account was terminated in October 1997. Since then Begehr uses his private bank account also for ISAAC. The account in Newark, Delaware, with Gilbert was dissolved in December 2006. The collected amount was used from Man Wah Wong to finance his ISAAC workshop on pseudo-differential operators from 11 to 16 December 2006 at the Fields Institute in Toronto. But Zhang still held an account for paying fees through credit cards. He had arranged for this possibility in connection with the 3rd congress in Berlin after this was demanded mainly from USA participants. In 2011 he did transfer the entire amount to Begehr and closed the account in the US on 11.11. 2011. This was just after Gilbert had retired from the University of Delaware. The main part of this amount was collected from congress fees paid via credit cards since 2001. During 2005 and 2009 Wong has collected some membership fees. This money was spent together with contributions from the Berlin account to finance the conference on “Homogenization, Inverse Problems and Applied Analysis” at the University of Central Florida in Orlando, Florida, from January 13 to 15, 2007, organized by Miao-jung Ou from the University of Delaware in honor of Gilbert on the occasion of his 75th birthday.

At its 2015 meeting in Macao, the board decided to open a bank account again for ISAAC. But this became difficult in the meantime. Banks are very careful nowadays with opening new bank accounts because of problems with money washing. Before ISAAC is not registered again as a (nonprofit) scientific society, there is no chance for an own society account. In recent years, however ISAAC’s financial situation has improved. The 3rd (Berlin), the 7th (London), and the 10th (Macao) congresses were the only ones ISAAC was able to collect funds from. Recently many non-ISAAC members from SIGs are joining ISAAC as life members and thus increasing the ISAAC property.

The basis for the property of ISAAC was laid at the Berlin congress. This congress was financially supported mainly by the German Research Foundation (DFG), the Berlin Government, and the Free University of Berlin. Unfortunately there was an economical crisis before the congress which had forced the local

75th Birthday Party of R.P. Gilbert Orlando, Florida, 2007

government to stop spending from the budget. As a consequence, when finally, 1 week before the congress, the amount of over DM 100,000 (about Euro 50,000) was transferred, this was too late to guarantee financial support to participants from the former Soviet Union as they had not enough time to apply for visa. Most of the support from the city had to be returned but some part could be saved for the society for paying bulk orders of the proceedings etc.

In 2005 ISAAC had besides the cash in Begehr's FU account Euro 22,240 furnished by the Berlin congress. Part of this amount was spent for bulk orders in 2009 for the Berlin (about US$9000) and the Ankara (about US$5000) proceedings to World Scientific. In 2009 from the London congress about Euro 11,000 remained after having paid the World Scientific bulk order. From the Macao congress about Euro 4000 was transferred early in 2016 by Tao Qian to the society. He has been able to financially support many more participants of the congress than it was possible in the events before 2015.

But all the congresses were organized independently of the ISAAC budget. Only for the Toronto and the Catania congresses in 2003 and 2005 some financial engagement of the society for plenary speakers and some young participants was required. Up to the 2nd congress in Fukuoka and the 10th in Macao all ISAAC awards for young researchers were financed by ISAAC. C.C.Yang had suggested to start the ISAAC Awards for young scientists of age below 40. He had managed to finance 10 awardees at the Fukuoka congress. In Berlin again 10 young mathematicians were decorated. The price was equipped with a certificate, DM 800 supported by Berlin Mathematical Society, Daimler Chrysler, Motorola, and Siemens, and additionally some books provided by Elsevier, Kluwer, Springer, and

World Scientific. Afterwards, the number of awards was reduced to one at most two. At the Macao congress besides one award some Special Mention of Young Scientist, just a certificate for outstanding research work, were given to five candidates. The other congress very successful also from a financial point of view was the London congress. It got financial support from the London Mathematical Society, the IMU, the Engineering and Physics Research Council, the Oxford Centre in Collaborational and Applied Mathematics, the Oxford Centre for Nonlinear Partial Differential Equations, the Bath Institute for Computer Systems, and the Imperial College London (Strategic Fund and Department of Mathematics).

Remarkable is also the funding of the Fukuoka congress from the Commemorative Association for the Japan World Exposition (1970). It served to equip any proceedings contributor who had registered at the congress with a whole set of two proceedings volumes of more than 1600 pages. From the Berlin congress on it became usual that participants had the choice to order the proceedings volume during the registration process. When this habit was given up without notice at the Krakow congress, this caused some irritation.

5 Constitution

For getting ISAAC registered in Delaware, Gilbert had to create a constitution. It has served for several years, but was not very much observed in the first years. Already in the constitution the idea of special interest groups (SIGs) within the ISAAC society was established. In 2003 Man Wah Wong has founded the first SIG on pseudo-differential operators. Because of a discussion in the ISAAC board about representatives of several, in the meantime built, active special interest groups (SIGs) without elections in the board, before and during the 2009 London ISAAC congress, Michael Ruzhansky and Begehr, on request of the board, have adjusted the constitution somehow to what was practiced. The ISAAC community has accepted the new constitution by an electronic voting and it was amended in 2013 just before the 9th congress in Krakow. One regulation in the new constitution proved to be very effective in the sequel. Some of the SIGs had quite some large numbers of members, but only a few of whom were registered ISAAC members. Large enough SIGs are allowed to delegate a representative in the board. But non-ISAAC members in a SIG are only counted once as a SIG member. Afterwards they have to join ISAAC or are not counted again. This rule was applied for the first time in 2015. And because several SIGs wanted to keep their representative in the board, ISAAC enjoyed more new life members than ever within a short period between the 10th International ISAAC Congress in Macao and the following board election at the beginning of 2016.

The constitution is available at the ISAAC home page mathisaac.org.

6 Officers and Board, ISAAC Congresses

Gilbert had served as the founding president until the 3rd International ISAAC Congress in 2001 in Berlin. At the 2nd congress 1999 in Fukuoka organized by Joji Kajiwara the first ISAAC board meeting took place (see the Appendix). According to the minutes of this meeting, besides Gilbert and Begehr, Erwin Brüning, Louis Fishman, Ismael Herrera, Ilpo Laine, Saburo Saitoh, Boris Vainberg, Man Wah Wong, and Yongshi Xu took part. Already at the first Congress in Newark, Delaware not only the site of the next but also the next two congresses were determined as Fukuoka Institute of Technology, Japan, and FU Berlin, Germany, with Kajiwara and Begehr, respectively, as local organizer.

At the 3rd congress the democratic structure, manifested in the constitution, was finally practiced. The open board meeting was preceded by a Member Meeting attended by many congress participants who even were not ISAAC members (see the Attachment). However they could not participate in the voting. Gilbert had stepped down as president and had nominated Begehr as first elected president for the period of 2 years. After his election Begehr suggested Gilbert as an honorary president. The board was in favor. But neither a vice president nor a new secretary was nominated. C.C. Yang continued as the vice president and Begehr served also as secretary. For the board meeting in Berlin Gilbert and Begehr just had appointed some colleagues independently of their membership to ISAAC but they were appealed to become ISAAC members. Attendances on this meeting can be viewed in the respective list of the minutes of the board meeting of the 3rd ISAAC congress in Berlin (see Attachment).

Board members were Grigor Barsegian, Carlos Berenstein, Alain Bourgeat, Erwin Brüning, Victor Burenkov, William Cherry, Christian Constanda, George Csordas, Julii Dubinski, Abduhamid Dzhuraev, Maths Essen, Antonio Fasano, Louis Fishman, Klaus Hackl, Ismael Herrera, Adi Ben Israel, Joji Kajiwara, Ilpo Laine, Irina Lasiecka, Wei Lin, Fon-Che Liu, Rolando Magnanini, Takafumi Murai, Ivan Netuka, John Ryan, Saburou Saitoh, Promarz Tamrazov, Domingo Tarzia, Boris Vainberg, Armand Wirgin, Man Wah Wong, and Yongzhi Xu.

At this meeting it was decided to split the proceedings into two parts, one with just the plenary and some main talks from the sessions published by Kluwer in the ISAAC series and others published by World Scientific.

For the election of a board Zhang had set up an electronic voting system on the ISAAC website. This lead to the first democratically elected ISAAC Board consisting of Heinrich Begehr (President), Alain Bourgeat, Victor Burenkov, Julii Dubinskii, Robert Gilbert (Honorary President), Joji Kajiwara, Ilpo Laine, Michael Reissig, John Ryan, Saburou Saitoh, and Man Wah Wong.

ISAAC still had very few members. The main task in the following years was to recruit members. Only with a certain number of members the society would start to develop itself. Life members, just paying a certain amount only once at the beginning of membership was attractive as well for the members as for the society. At first the fee was US$200 later after beginning of 2010 Euro 300. Such an amount

Congress Photo by A. Begehr Berlin 2001

is not very much for people in countries with a strong economy; but the society could collect a bigger amount at a moment while the budget still was very low. For many candidates however, this amount was not affordable because of the economy in their countries. Many of those colleagues were interested in ISAAC and their fees were waived or reduced. After a few years ISAAC counted over 100 life members while the regular annual, as "paid" members listed, ones never were more than 20 at the same time.

In 2003 the 4th congress was held at York University in Toronto, organized by Man Wah Wong. This 4th ISAAC congress was hit by SARS. Because of this disease many scientists refused to participate. While in the preceding Berlin congress there almost 40 sessions, in Toronto there were just 15 sessions. In the minutes of the Toronto board meeting ISAAC is reported to have, besides 1 honorary member, 54 life members, 44 of which had joined after 2001, and 16 paid members. Begehr was reelected as president. At the board meeting were present Begehr, Victor Burenkov, Gilbert, Ilpo Laine, Michael Reissig, Saburou Saitoh, M.W. Wong, and C.C. Yang.

The board election after the 2003 congress took place in two separate ballots. As well three vice presidents were elected, Erwin Brüning for Africa, Man Wah Wong for America, and C.C. Yang for Asia, and Victor Burenkov, Massimo Lanza de Cristoforis, Ilpo Laine, Michael Reissig, John Ryan, Saburo Saitoh, and Masahiro Yamamoto as board members. Additionally, Gilbert as the Honorary President belonged to the board. This board elected the next president at the congress in Catania in 2005. By the regulation of the constitution the president may only once be reelected. ISAAC should develop as an international society dominated neither by some person nor by a group. Man Wah Wong was elected and served for the next 4 years. The reelection of presidents became a habit in the following years.

New vice presidents were determined by electronic election, namely Victor Burenkov replacing M.W. Wong, Saburo Saitoh, and Erwin Brüning. Begehr and Zhang were confirmed as secretary and treasurer and as webmaster and secretary, respectively. Besides the officers members of the board were for the board 2006

Lee Lorch, decoration as honorary ISAAC member V. Burenkov, H. Begehr photo by R.P Gilbert Toronto 2003

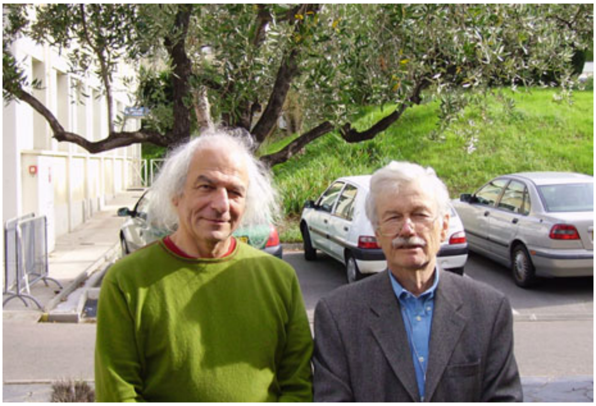
to 2008: Okay Celebi, Anatoly Kilbas, Massimo Lanza de Cristoforis, Michael Reissig, Luigi Rodino, Bert-Wolfgang Schulze, Joachim Toft.

The decision for the site of the following congress had been taken already in the year before the Toronto congress at the "International Conference to Celebrate Robert Pertsch Gilbert's 70th Birthday" at CAES du CNRS in Fréjus, France, on "Acoustics, Mechanics, and the Related Topics of Mathematical Analysis" organized by Armand Wirgin from June 18 to 22, 2002.

Franco Nicolosi had been participating and was enthusiastic to organize the ISAAC congress after the Toronto one already in 2004. But ISAAC did not give in and did not change the congress period. Probably Franco had to postpone his retirement at the University of Catania in order to run the ISAAC congress.

There were no changes of officers at the Ankara congress in 2007 which was organized by Okay Celebi. As representatives in the board of the SIG pseudo-differential operators were Luigi Rodino and Bert-Wolfgang Schulze, for the SIG

M.W. Wong, opening speech of president Ankara 2007


A. Wirgin, R.P. Gilbert CNRS-Marseille, Fréjus 2002

Franco Nicolosi, 70th birthday conference Catania 2009

special functions and reproducing kernels Alain Berlinet and Anatoly Kilbas were nominated. Seven Board members were elected electronically: Bogdan Bojarski, Okay Celebi, Massimo Lanza de Cristoforis, Michael Reissig, John Ryan, Joachim Toft, Masahiro Yamamoto.

O.A. Celebi, heading towards the opening speech Ankara 2007

H.T. Kaptanoglu, R.P. Gilbert, T. Shaposhnikova, V. Maz'ya Ankara 2007

M. Ruzhansky, ISAAC awardee 2007, session talk Ankara 2007

In London 2009 a successor of M.W. Wong had to be chosen. There was an exciting board meeting. Two groups supporting different candidates were labeled. While one group wanted to elect Luigi Rodino, a member from Wong's SIG, the other just aimed for a change to another subgroup. After all Michael Ruzhansky, also from the pseudo-differential operators group, was elected.

He was the local organizer of the 9th ISAAC congress in London, which was very well planned and held. Vice presidents were elected for the next 2 years by electronic voting: Victor Burenkow for Europe and Africa, Yongzhi Steve Xu for America, Masahiro Yamamoto for Asia. As the former president Man Wah Wong became board member by invitation. Moreover, eight representatives of six SIGs were delegated so that only five members could be elected by the general members. Begehr and Zhang were confirmed in their positions. But Zhang just wanted to be the webmaster without also being secretary. The new regulation for filling the board seats said that larger SIGs may delegate two members, smaller ones just one. Larger SIGs were in 2009 the pseudo-differential operators and the partial differential equations groups. The total number of board members including the officers is 20. The honorary president, Gilbert, is counted as an officer. The five elected board members were Bogdan Bojarski, Okay Celebi, Massimo Lanza de Cristoforis, Michael Reissig, and Joachim Toft. The six SIG representatives were Luigi Rodino and Bert-Wolfgang Schulze for pseudo-differential operators, Mitsuro Sugimoto and Daniele Del Santo for partial differential equations, Michael Oberguggenberger for generalized functions, Sergei Rogosin for complex analysis, Juri Rappoport for integral transforms and applications, and Saburo Saitoh for reproducing kernels.

Congress Dinner, M. Ruzhansky, main organizer, with wife, H. Begehr, next generation London 2009

As Begehr had pointed out, ISAAC was not following the rules from the constitution. Therefore at the board meeting of the Moscow congress it was agreed upon a reform of the constitution to be worked out by the president and the secretary before the next congress.

Opening Ceremony, V. Burenkov, main organizer Moscow 2011

Group on stairs to Main Lecture Hall Moscow 2011

Excursion on Moscow River V. Burenkov, P.D. Lamberti, M. Lanza de Cristoforis, A. Mohammed, S. Bernstein, unknown, unknown, F. Sommen Moscow 2011

This was done before the 2013 Krakow congress organized by Vladimir Mityushev. The new constitution was accepted by electronically voting. Fifty-one members have voted, 47 in favor, 1 opposed, 3 abstained. The constitution was put on the home page and is effective since August 5, 2013. Vladimir Mityushev explained that by Polish law it will not be possible to transfer to ISAAC money from collected congress fees. He offered to use the amount to build up a new ISAAC home page at the Pedagogical University in Krakow. This was accepted by the board and in Macao 2015 confirmed where V. Mityushev had explained the progress in building up this site, isaacmath.org.

According to the new constitution a new president and just one vice president were elected in Krakow. Luigi Rodino became president and Michael Reissig vice president. Begehr was acclaimed as secretary and treasurer. Zhang is still webmaster until the new home page will work.

Michael Reissig had suggested to start an ISAAC newsletter to increase the contact within the society. Four have appeared until the following congress in 2015 at the University of Macau in Macao. This was ISAAC congress no. 10 organized by Tao Qian.

Michael Ruzhansky, opening speech of president Krakow 2013

Vladimir Mityushev, session talk of local organizer Krakow 2013

The board, according to the new regulation, consists besides the honorary president, the president, the vice president, the past president, secretary and treasurer, webmaster, local organizer of the next congress, just one representative of each active SIG and of elected members up to altogether 21. For the period 2014–2016 the SIG representatives are Man Wah Wong for pseudo-differential operators, Mitsuro Sugimoto for pdes, Stevan Pilipovic for generalized functions, Sergei Rogosin for complex analysis, Yuri Rappoport for integral transforms and reproducing kernels, Irene Sabadini for Clifford and quaternionic analysis, Anatoly Golberg for complex variables and potential theory, and Tynysbek Kalmenov for spectral analysis and bvps. The seven elected members are Okay Celebi, Massimo Lanza de Cristoforis, Michael Oberguggenberger, Joachim Toft, Ville Turunen, Masahiro Yamamoto, and Jens Wirth.

Board Meeting, Krakow 2013

Luigi Rodino, speech as newly elected President Closing Ceremony, Krakow 2013

This board elected the new officers in Macao. Luigi Rodino was reelected; Joachim Toft became the vice president after Michael Reissig had stepped down. Begehr was elected as secretary and treasurer. The next local congress organizer is Joachim Toft. His Linnäus University in Växjö will host the 11th International ISAAC Congress in 2017. Because he is also the vice president this time eight board members will be elected at the beginning of 2016.

Tao Qian getting up to present Macao as next congress site Krakow 2013

Tao Qian Opening Speech, Alan McIntosh First Plenary Lecture Macao 2015

Unfortunately the home pages of the congresses disappear after a while as the universities close them down. For the Berlin, Toronto, and Catania congresses however are the abstracts still available, as the AMCA service from Toronto was used for these three congresses (see http://at.yorku.ca/cgi-bin/amca/submit/cahk-01).

Abstracts for the 3rd congress are http://at.yorku.ca/c/a/h/k/01.htm, for the 4th congress http://at.yorku.ca/c/a/k/u/01.htm, and for the 5th one http://at.yorku.ca/c/

Audience During Plenary Talk of L. Cohen Macao 2015

ISAAC Special Mention Award Decoration by L. Rodino and M. Reissig Macao 2015

J. Toft Photo by M. Ruzhansky in Matsumoto, Japan, February 2016 discussing the 2017 ISAAC Congress

a/p/g/01.htm. Later congresses saved the fee for this service, but the abstracts are not available any more on the Internet. Some rudimental home page for the 4th and the 5th congress are still available via link from the AMCA page http://www.math.yorku.ca/isaac03/, and the AMCA page http://mathisaac.org/c/cs/isaac/c/05/a, respectively. But on the ISAAC home pages, mathisaac.org, the websites of most of the congresses are still available. Zhang has reconstructed the one for Delaware. No home pages are there any more from Fukuoka, Berlin, Ankara, and Moscow.

The ISAAC congresses are held in sessions. Between 20 and 30 sessions are organized on varying subjects. The Delaware congress had 25 sessions. At the Fukuoka congress one session with the title "Special Session" organized by Michael Reissig presided a meeting on the occasion of Begehr's 60th birthday. Six talks were presented: R.P. Gilbert (About the person H. Begehr), Ju. Duinskij (About Begehr's results in the theory of boundary value problems in $\mathbb{C}^n$), W. Sprößig (On hypercomplex boundary value problems in H. Begehr's work), Xing Li (Complex analytic methods in mechanics), A. Cialdea (Contributions of H. Begehr to the theory of Hele-Shaw flows), M. Reissig (Contribution of H. Begehr to the abstract Cauchy-Kovalevsky theory).

The number of plenary speakers of the ISAAC congresses is 10 to 12. The first plenary talk was delivered by H. Begehr on June 3, 1997, 8:45–9:45 on Riemann Hilbert Boundary Value Problems in $\mathbb{C}^n$.

Only one of the plenary speakers was invited twice: Saburou Saitoh was one at the Berlin and at the Macao congress. The plenary talks at the Berlin congress were video-recorded. The diskettes are still in Begehr's hands and can occasionally be shown at some later ISAAC congress.

H. Begehr's 60th Birthday Party, Photo E. Wegert Fukuoka 1999

As an international scientific society ISAAC is well equipped with analysts from a large group of countries representing the major fields in analysis, its applications, and computation. But some geographic areas such as South America and Australia are not adequately present and further scientific spreading would be favorable. Also ISAAC has not yet managed to organize congresses there. ISAAC is overly dominated by Europeans and mainly has its meetings there. ISAAC needs to move around and to widen its membership more internationally.

7 Special Interest Groups (SIGs)

As was pointed out earlier already, from the very beginning of ISAAC, Gilbert had fixed the idea of special interest groups within the society for particular activities such as workshops, local conferences etc. in the constitution. There he had listed as examples Approximation Theory, Function Theoretic Methods in Partial Differential Equations, Asymptotics and Homogenization, Free and Moving Boundary Problems, Inverse Problems and Control Theory, and Symbolic Computation and Analysis, but none was really organized until Man Wah Wang has efficiently created the SIG for pseudo-differential operators on 15. 09. 2003. Before only some people in Function Theoretic Methods in Partial Differential Equations had developed some activities in organizing small conferences and workshops mainly the ones in 1998 and 2004 at METU in Ankara organized by Okay Celebi, Gilbert, Begehr, and Wolfgang Tutschke and one in 2002 in connection with a Nato Advanced Research Workshop organized by Grigor Barsegian, and Begehr, and the ISAAC Conference on Complex Analysis, Partial Differential Equations, and Mechanics of Continua dedicated to the Centenary of Ilia Vekua at Tbilisi State University in April 2007 organized by George Jaiani. But the very first ISAAC activity was the Graz conference on generalized analytic functions in honor of Gilbert's 65th birthday organized mainly by Wolfgang Tutschke a few months before the Delaware ISAAC congress.

It was then Man Wah Wong who had motivated many colleagues in joining this SIG and also ISAAC in connection with the Toronto congress. Man Wah Wong became ISAAC president, Michael Ruzhansky, successor of M.W. Wong as president from 2009 to 2013, and Luigi Rodino, president since 2013, and one is vice presidents right now since 2015, Joachim Toft. Moreover, Wong had founded a book series "Pseudo-Differential Operators, Theory and Applications" published with Springer-Birkhäuser. The contributions to the ISAAC congresses from the sessions on pseudo-differential operators are since 2003 published in Gohberg's Birkhäuser series "Operator Theory: Advances and Applications" where also proceedings of other conferences of this SIG are published; see the site of the group http://www.math.yorku.ca/IGPDO/, e.g., from the ISAAC home page mathisaac.org.

Several SIGs followed and at the end of 2015 there are altogether nine SIGs. Very active ones are also the SIG on partial differential equations, and the SIG on generalized functions, the SIG on Clifford and quaternionic analysis. The two SIGs founded by Saburou Saitoh on reproducing kernels and on integral transforms and applications are even older than the pseudo-differential operator group. They had organized a conference "Applications of analytic extensions" in January 2000 at the Research Institute for Mathematical Sciences at the Kyoto University. Later both groups merged to the SIG integral transforms and reproducing kernels. Other SIGs are on inverse problems and industrial mathematics, on complex analysis, on complex variables and potential theory, and finally the youngest one on spectral analysis and boundary value problems.

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M.E. Muldoon. M.W. Wong (World Scientific, Singapore, 2005), pp. vii-viii
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5. G.G. Gundersen, Professor Yang Lo: Honorary Member of ISAAC, *New Trends in Analysis and Interdisciplinary Applications*, ed. by P. Dang, M. Ku, T. Qian, L. Rodino (Birkhäuser, Basel, 2017, xi–xii)

Appendix

1. List of ISAAC Presidents

Robert Pertsch Gilbert, Founding President, 1996–2001,
Heinrich Begehr, 2001–2005,
Man Wah Wong, 2005–2009,
Michael Ruzhansky, 2009–2013,
Luigi Rodino, 2013–2017.

2. List of International ISAAC Congresses

Newark, Delaware, USA, 03.–07. 06. 1997 (R.P. Gilbert; S. Zhang)
Fukuoka, Japan, 16.–21. 08. 1999 (J. Kajiwara)
Berlin, Germany, 20.–25. 08. 2001 (H. Begehr)
Toronto, Canada, 11.–16. 08. 2003 (M.W. Wong)
Catania, Italy, 25.–30. 07. 2005 (F. Nicolosi)
Ankara, Turkey, 13.–18. 08. 2007 (A.O. Çelebi)
London, GB, 13.–18. 08. 2007 (M. Ruzhansky; J. Wirth)
Moscow, Russia, 22.–27. 08. 2011 (V. Burenkov)
Krakow, Poland, 05.–09. 08. 2013 (V.V. Mityushev)
Macao, China, 03.–08. 08. 2015 (T. Qian)
Växjö, Sweden, 14.-18. 08. 2017 (J. Toft)

3. List of ISAAC Supported Conferences

Generalized Analytic Functions—Theory and Applications, on the occasion of the 65th anniversary of R.P. Gilbert, 06.–10. 01. 1997, Technical University of Graz, Austria,
Workshop on Recent Trends in Complex Methods for Partial Differential Equations, 06.–10. 06. 1998, Middle East Technical University, Ankara, Turkey,
Applications of Analytic Extensions, 11.–13. 01. 2000, Research Institute for Mathematical Sciences, Kyoto University, Kyoto, Japan,
Complex Analysis, Differential Equations and Related Topics, devoted to the memory of Professor Artashes Shahinyan, 17.–21. 09. 2002, in union with the NATO Advanced Research Workshop Topics in Analysis and its Applications, 22.–25. 09. 2002, Yerevan, Armenia,
Workshop on Recent Trends in Applied Complex Analysis, 01.–05. 06. 2004, Middle East Technical University, Ankara, Turkey,

Function Spaces, Approximation Theory, Nonlinear Analysis, dedicated to the centennial of Sergei Mikhailovich Nikol'skii, 23.–29. 05. 2005, Lomonosov Moscow State University and V.A. Steklov Mathematical Institute of RAS, Moscow, Russia,
ISAAC Workshop on Pseudo-Differential Operators: PDE and Time-Frequency Analysis, 11.–15. 12. 2006, Fields Institute, Toronto, Canada, see (http://www.fields.utoronto.ca/programs/scientific/06-07/ISAAC/),
Homogenization, Inverse Problems and Applied Analysis, dedicated to the 75th birthday of R.P. Gilbert, 13.–15. 01. 2007, University of Central Florida, Orlando, USA,
Complex Analysis, Partial Differential Equations, and Mechanics of Continua, dedicated to the Centenary of Ilia Vekua, 23.–27. 04. 2007, I. Vekua Institute of Applied Mathematics, Iv. Javakhishvili Tbilisi State University, Tbilisi, Georgia,
Analysis, PDEs and Applications on the occasion of the 70th birthday of Vladimir Maz'ya, 30. 06.–03. 07. 2008, University of Rome, Italy,
International Symposium on Nonlinear Partial Differential Equations and Applications, in honor of Franco Nicolosi on the occasion of his 70th birthday, 22.–25. 06. 2009, Catania, Italy,
Chinese-German Workshop on Analysis of Partial Differential Equations and Applications, 14.–18. 02. 2011, Freiberg, Germany,
Analytical Methods of Analysis and Differential Equations (AMADE) 2011 in honor of A. Kilbas, 12.–19. 09. 2011, Minsk, Belarus,
Italian-German Workshop on Modern Aspects of Phase Space Analysis, 13.–17. 02. 2012, Freiberg, Germany,
Satellite conference of ECM Fourier Analysis and Pseudo-differential Operators, 25.–30. 06. 2012, Helsinki, Finland,
Conference on Applied Analysis and Mathematical Biology in honor of R.P. Gilbert's 80th birthday, 08.–09. 08. 2012, University of Delaware, Newark, USA,
Topics in PDE, Microlocal and Time Frequency Analysis, 03.–08. 09. 2012, Novi Sad, Serbia,
Complex Analysis & Dynamical Systems VI, 19.–24. 05. 2013, Nahariya, Israel,
Conference on Fourier Analysis and Approximation Theory, 04.–08. 11. 2013, Centre de Ricerca Matemàtica, Barcelona, Spain,
Complex and Harmonic Analysis, 11.–13. 06. 2014, Holon, Israel,
Days of Analysis, 03.–07. 07. 2014, Novi Sad, Serbia,
Recent Trends in Mathematical Analysis and its Applications, 21.–23. 12. 2014, Indian Institute of Technology, Roorkee, India,
2nd International Conference on Mathematical Computation (ICMC 2015), 05.–10. 01. 2015, Haldia Institute of Technology, Haldia, India,
Complex Analysis & Dynamical Systems VII, 10.–15. 05. 2015, Nahariya, Israel,
Analytical Methods of Analysis and Differential Equations (AMADE) 2015, 14.–19. 09. 2015, Minsk, Belarus,
Boundary Value Problems, Functional Equations, Applications, 3rd Meeting, 20.–23. 04. 2016, University of Rzeszow, Poland,

Modern Methods, Problems and Applications of Operator Theory and Harmonic Analysis dedicated to Stefan Samko, 24.–29. 04. 2016, Rostov-on-Don, Russia,
Harmonic Analysis and Approximation Theory, 06.–10. 06. 2016, Centre de Recerca Matemàtica, Catalunya, Spain,
Actual Problems in Theory of Partial Differential Equations dedicated to the centenary of Andrey V. Bitsadze, 15.–19. 06. 2016, Lomonosov Moscow State University and V.A. Steklov Mathematical Institute of RAS, Moscow, Russia,
14th International Conference on Integral Methods in Science and Engineering, 25.–29. 07. 2016, University of Padua, Italy,
24'ICFIDCAA-2016 24th International conference on finite or infinite dimensional complex analysis and applications, 22.–26. 08. 2016, Anand International College of Engineering, Jaipur, India,
International conference on generalized functions GF2016, 05.–09. 09. 2016, Centre for Advanced Academic Studies, Dubrovnik, Croatia,
VI Russian-Armenian Conference on Mathematical Analysis, Mathematical Physics and Analytical Mechanics, 11.–16. 09. 2016, Rostov-on-Don, Russia,
International Conference Nonlinear Analysis and its Applications, 18.–24. 09. 2016 (postponed to 2017), Samarkand State University of Uzbekistan,
International Conference on Mathematical Analysis and its Applications (ICMAA 2016), 28. 11.–02. 12. 2016, Indian Institute of Technology, Roorkee, India,
3rd International Conference on Mathematical Computation (ICMC 2017), 17.–21. 01. 2017, Haldia Institute of Technology, Haldia, India.

4. Minutes of the ISAAC Board Meeting on August 19, 1999, 17:15–17:45 at Fukuoka Institute of Technology in Fukuoka, Japan

Present: R.P. Gilbert, President, H. Begehr, E. Bruening, L. Fishman, I. Herrera, I. Laine, Man Wah Wong, S. Saitoh, B. Vainberg, Y. Xu

1. Proceedings problems with the present publisher Kluwer are discussed. ISAAC might switch to publish its proceedings by itself in case the publisher will ask for subsidies. In future the costs of the proceedings will have to be collected from the participants together with congress fees. This could be about US$100 together with some US$40–50 for membership fee for the term until the next congress and probably additional $100 for local organization. In the future the proceedings might become about 1000 pages per congress. For the proceedings of this congress Prof. Kajiwara has found some funds for paying the printing costs. The suggested restriction to 6 pages per contribution is discussed. Many board members would prefer to enlarge this to about 10 pages. This has to be discussed with Prof. Kajiwara who only knows the side conditions for the publication of these proceedings. As many of the preregistered colleagues did not arrive at the congress, it seems likely that the contributions could be increased slightly. The session organizers are responsible to collect the manuscripts in time and to contact Prof. Kajiwara about their length. They are also responsible to get the manuscripts properly refereed. The manuscripts together with copies of the reviews should be sent to the editors in time.

2. Next congress in Berlin. It seems desirable that the local organization and the president/board work closely together in preparing the congresses. For Berlin a careful selection of sessions and session organizers should be ensured. People will be invited to organize a session and they then should invite colleagues to participate in their session. The board will decide about the plenary talks. One afternoon during the congress should be reserved for an excursion. The congress should be properly advertised in particular not just on the website of ISAAC. As in Berlin the president, the vice presidents, and the board will be elected it is desirable to have registered members. Any participant of the congress having paid the fee which will include a membership fee (see 1.) is an ISAAC member for the period until the next congress. He/she is thus able to take part in the voting both actively and passively.
3. Further congresses. The following possibilities are offered: 2003: Mexico (I. Herrera), Toronto (Man Wah Wong), Durbam, South Africa (E. Bruening), 2005: Joensuu (I.Laine). After Berlin ISAAC should have its next congress on another continent. The different possibilities are discussed and also when during the year it should take place.
4. Workshops and regional meetings. ISAAC intends to have local workshops in between its biannual congresses. The local workshops should find own financial support. The perspective proceedings should be prepared under the same rules as the ones for the congresses. They are edited by the local organizers. Eventually these proceedings could be electronically published. Until now we have had the following local meetings: 1997: Generalized analytic functions—theory and applications, Graz, Austria, 1998: Recent trends in complex methods for partial differential equations, Ankara, Turkey.
5. Memberships and purpose of ISAAC. The aim of ISAAC was roughly fixed in the constitution and an accompanying letter from the president, after ISAAC had got registered in Delaware in 1996. ISAAC does not want to be just another AMS. It is also not inclined to cover the whole area of analysis but is devoted to complex and functional analysis, its applications, and computation. The ISAAC website is suggested to be improved. In particular the congresses and workshops as well as the proceedings appeared have to be advertised. Until now three volumes have appeared with Kluwer, three more are in press, three volumes from the first congress have appeared elsewhere, additionally one will appear as a special issue in a journal.

Proceedings appeared in our Kluwer series: Generalized analytic functions, theory and applications to mechanics. Eds. H. Florian, K. Hackl, F.J. Schnitzer, W. Tutschke. ISAAC 1, Kluwer, Dordrecht, 1998.

Partial differential and integral equations. Eds. H. Begehr, R.P. Gilbert, G.-C. Wen. ISAAC 2, Kluwer, Dordrecht, 1999.

Reproducing kernels and their applications. Eds. S. Saitoh, D. Alpay, J.A. Ball, T. Ohsawa. ISAAC 3, Kluwer, Dordrecht, 1999.

Proceedings appeared elsewhere: Inverse problems, tomography, and image processing. Proc. of sessions from 1. int. congress of ISAAC. Ed. A.G. Ramm. Plenum Press, New York, 1998.

Spectral and scattering theory. Proc. of sessions from 1. int. congress of ISAAC. Ed. A.G. Ramm. Plenum Press, New York, 1998.

Dirac operators in analysis. Eds. J. Ryan, D. Struppa. Pitman Research Notes No. 394. Addison Wesley Longman, Harlow, 1998.

Proceedings in print: Recent developments in complex analysis and computer algebra. Eds. R.P. Gilbert, J. Kajiwara, Y.S. Xu. Kluwer, Dordrecht.

Direct and inverse problems of mathematical physics. Eds. R.P. Gilbert, J. Kajiwara, Y.S. Xu. Kluwer, Dordrecht.

Complex methods for partial differential equations. Eds. H. Begehr, A.O. Celebi, W. Tutschke. Kluwer, Dordrecht.

Orthogonal polynomials and computer algebra. Eds. R.A. Askey, W. Koepf, T.H. Koornwinder. Special issue of Journal of Symbolic Computation.

R.P. Gilbert, President, H. Begehr, Secretary

5. Minutes of the ISAAC Member and Board Meeting on Tuesday, August 21, 2001, 7:15–9pm in Berlin During the 3rd ISAAC Congress

Participants: (*: board member)

T. Aliyev (Azerbaijan), G.E.G. Almeida (United Kingdom), C. Andreian Cazacu (Romania), G. Barsegian (Armenia), H. Begehr* (Germany), A. Bourgeat* (France), V. Burenkov (United Kingdom), V.V. Dmitrieva (Russia), A.D. Dzabrailov (Azerbaijan), L. Fishman* (USA), V.A. Gaiko (Belarus), R.P. Gilbert* (USA), V.S. Guliev (Azerbaijan), H. Guliev (Azerbaijan), G. Jaiani (Georgia), G. Khimshiashvili (Georgia, Poland), W. Lin* (China), I.S. Louhivaara (Germany, Finland), N. Manjavidze (Georgia), G.F. Roach* (United Kingdom), J. Ryan* (USA), S. Saitoh* (Japan), R.S. Saks (Russia), V. Stanciu (Romania), P. Tamrazov* (Ukraine), B. Vainberg* (USA), S. Voldop'yanov (Russia), A. Wirgin* (France), M.W. Wong* (Canada), S. Xu* (USA), C.C. Yang* (Hong Kong), M. Yamamoto (Japan)

The main problem of ISAAC is still the lack of registered members. For this reason there are no elections for a new board. It is decided that the old board will remain but C.H. FitzGerald (USA), St. Krantz (USA), A.G. Ramm (USA) will be replaced. The board members are expected to pay membership fee (US$20 per year, lifetime membership fee US$200). Moreover, they are expected to participate in the organization and/or in the congresses itself at least every second time. Members from countries with weak economy may apply for exemption from paying the membership fee. The board will decide about exemptions. New board members are G. Barsegian (Armenia), C. Berenstein (USA), V. Burenkov (UK), G. Csordas (USA), A. Dzhuraev (Tajikistan), and R. Magnanini (Italy).

The location for the 4th International ISAAC Congress is not yet fixed. At the 2nd congress Mexico, South Africa, and Canada were discussed and for 2005 Joensuu (Finland) was offered. ISAAC's policy is to have its congresses on different continents; the American continent is favorable for 2003. The strong offer from Minsk (Belarus) for organizing the next congress is gratefully recognized. It should be accepted for an intermediate conference or workshop in 2002 as there the facilities are optimal for meetings with up to 200 participants. Other options for locations of the next congresses are Taiwan, Maryland (USA), Rome, and Armenia. Joensuu has withdrawn for the 2005 congress as there will be another conference on complex analysis in 2005. They offer to organize the 2007 congress. Other locations for smaller ISAAC conferences or workshops are Armenia and the new branch of the Banach center in Bedlevo near Posnan. ISAAC members are welcome to organize such workshops in even numbered years.

The proceedings of ISAAC congresses, conferences, and workshops will not any more be published automatically in the Kluwer ISAAC series. All publishers have changed their policy in accepting proceedings only under bulk orders of the editors. This economical reason forces ISAAC to split the proceedings into two parts. The plenary and some elected main talks will be published in the ISAAC series with Kluwer, and the main proceedings volumes with World Scientific. Here a section of open problems is suggested. The ISAAC series will change its character in the sense that also monographs might be published there. World Scientific has applied to become an ISAAC member. The board is in favor of this idea and fixes the fee for a lifetime membership for a company at US$2000.

After the Member meeting is closed, the Board meeting takes place. During this meeting R.P Gilbert steps down as the interim president. He nominates H. Begehr as the new president. The board approves this motion. The new president suggests to elect R.P. Gilbert as honorary president. This is accepted without objections. There are no proposal to elect a new vice president and a new secretary. Hopefully at the next congress there will be enough ISAAC members to elect all officers and board members. It is discussed if ISAAC should have more than one vice president, e.g., one for each continent. At last the board approves the motion of the president to accept S.M. Nikol'skii, Academician of the Russian Academy of Sciences, one of the plenary speakers of this congress, 96 years old and one of the dominant Russian analysts of the last century, well known, e.g., for his research results about O.V. Besov's function spaces, the Nikol'skii-Besov spaces.

H. Begehr

6. Minutes of the ISAAC Board Meeting on 12. August, 2003, 7–9pm at York University in Toronto

Attendant: H. Begehr, V.I. Burenkov, R.P. Gilbert, I. Laine, M. Reissig, S. Saitoh, M.W. Wong, C.C. Yang

0. The agenda proposed is approved.
1. Report of the president.
   a. At this moment the society has 54 life members, 1 honorary member, and 16 paid members. Forty-four life members have joined since 2001. Everyone is invited to RECRUIT further members.
   b. There are two different ISAAC home pages, one at the Univ. of Delaware and the other at FU Berlin. These were briefly described. It would be appreciated

if members would visit them regularly and use the services offered, e.g., the open problem section or the job search and the advertisements section.

c. The financial situation is reported on: R.P. Gilbert has an ISAAC account in Newark, Delaware, of about US$3500. H. Begehr has US$1540 and about Euro 24,000. The exact figure in Euro will be available as soon as the bill for the World Scientific proceedings volumes from the 3rd congress (US$9000) will be paid. From 1995 to 1997, DM 285/DM 2000 has been collected for paid/life membership fees, and in 1998 DM 1390 has been collected for conference fees. From this amount DM 109.50/DM 1556.20 was paid for account costs/transferred to R.P. Gilbert for ISAAC registration costs. The account was canceled in October 1997. The remaining amount was transferred into Euro 1027.34. From 2001 DM 1026 = Euro 524.59 and Euro 82.50 as well as US$140 were collected for paid membership fees and Euro 1962.35 and US$1400 for life membership fees. Thus H. Begehr has Euro 3596.78 and US$1540 in his hands. Moreover, at the ISAAC account at the FU Berlin there are Euro 28,774.32. From this amount US$9000 have to be paid to World Scientific and US$400 for the ISAAC Award to Prof. Umezu in Toronto.
d. Special interest group. M.W. Wong has formed a special interest group on pseudo-differential operators. It has four directors and several other members. Special interest groups are supposed to organize workshops and coordinate scientific activities in their area. Members of these groups are supposed to be ISAAC members. (It is discussed if these special interest groups are a good way to expand and develop ISAAC.)

2. Elections. The board agrees that the elections by the ISAAC members for the officers and the board members should be done electronically. With the first ballot the officers will be elected. A second ballot will be used to elect the board. Officers and board members will serve for a 2 years period. The board had a split vote concerning the period for the president: 6 votes for a 2-years period with the possibility for a second 2-years term, 2 votes for a 4-years period. It was agreed upon that as the president also the vice presidents are only re-eligible for a second time in sequence. Moreover, it was agreed to elect two vice presidents this time. President and the vice presidents should represent each of the global areas, the Americas, Europe, and Asia. The secretary may be reelected any time.

   H. Begehr is nominated for president, M.W. Wong and C.C. Yang for vice presidents, and S. Zhang as secretary and webmaster. The present board members are wiling to serve again at the board. The four absent members will be asked if they want to serve as board members again. Further candidates may be nominated by ISAAC members. The board agrees that the Founding and Honorary President is a lifetime board member. Also former presidents will become lifetime board members. An enlargement of the board up to 20 members is discussed but as long as the number of members is still small this is not considered as suitable.
3. Further activities. The next ISAAC congress will be organized in Catania. This decision was made by the board last year after the ISAAC conference in Frejus. F. Nicolosi has agreed to act as local organizer. There are no information yet

how far the preparation, if any already, has gone. This will be discussed with Prof. Nicolosi soon. There are two more offers for this 2005 congress: one from the University and the Academy of Sciences from Minsk and one from the Middle East Technical University in Ankara. Both these possibilities were discussed. In case Catania will not work out the board will decide again. In that case both candidates should work out a detailed offer including information about the congress fee, financial supports, accommodations, flight connections, and proceedings. An essential condition will be that the membership fee for the society can be collected together with the congress fee and then handed over to ISAAC. As another possible place for a congress Moscow is suggested.

Ankara has also suggested an ISAAC workshop at the Middle East Technical University to be organized by O. Celebi, H. Begehr, and R.P. Gilbert.

4. Organization of the congresses. I. Laine has suggested to carefully select the sessions for future congresses. The stochastic way it was done so far is not satisfactory. Related sessions should neither overlap in the schedule nor should there be too many. Fewer and broader sessions are preferable. I. Laine will provide the board with some particular suggestions. An ISAAC congress opens a window on analysis, its applications, and computation. Different congresses open different windows. A sequence of congresses should more or less cover the whole area. In the future topics and organizers of session will be decided by the board as is done already with the plenary speakers. It is discussed if session organizers should be asked to become ISAAC members first.
5. In the ISAAC series with Kluwer since 1998 altogether 10 volumes—all proceedings—have appeared. As proceedings are not easily sold any more Kluwer does not pay any money anymore to ISAAC for volumes published. This was done only for the first few volumes. Now ISAAC is asked to combine publication of proceedings with a bulk order. The series are open for publishing monographs also. Everybody is asked to think about using the ISAAC series for publishing books. The lectures from the annually organized "minicorsi" at the university of Padua will appear in the ISAAC series.
6. Miscellaneous. The high congress fees are criticized. It is discussed if the proceedings can be published electronically. The congress fee should be negotiated with perspective local organizers for congresses. The proceedings of this 4th Intern. ISAAC Congress will appear with Kluwer in our series. M.W. Wong has negotiated with Kluwer for an 800 pages volume and a bulk order of 100 copies. The contributions from the session on pseudo-differential operators will appear in Gohberg's Birkhaeuser series on Operator Theory, Advances and Applications. Registered participants will receive a copy. V. Burenkov suggests to make Prof. Lee Lorch an Honorary Member of ISAAC at the closing ceremony. Prof. Lorch is a distinguished mathematician at York University and a fellow of the Royal Society of Canada. He is a participant of this congress and about 87 years old. The board agreed to this suggestion with 6:0:2.

   Prof. J. Kajiwara is nominated for a "Distinguished ISAAC Service Award". He has served ISAAC extremely well in organizing the 2nd Intern. ISAAC Congress in Fukuoka, in preparing the proceedings volumes of this congress almost all by himself, and as a board member. This is accepted by the board. The award will be presented to him on the occasion of his 70th birthday at the celebration at the 12th International Conference on Finite or Infinite Dimensional Complex Analysis and Applications to be held next July in Tokyo.

H. Begehr

7. Minutes of ISAAC Board Meeting in Catania During the 5th International ISAAC Congress, 26 July, 2005, 17.00–20:45

Attendants: H. Begehr, E. Bruning, V. Burenkov, M. Lanza De Cristoforis, F. Nicolosi (guest as local organizer), M. Reissig, S. Saitoh, M.W. Wong, M. Yamamoto, C.C. Yang

Excused: J. Ryan, R.P. Gilbert (not at the congress), I. Laine (not at the congress)

Further guests for point 1 of the agenda: O. Celebi, A. Kilbas, A. Jerbashian

The board agrees to the agenda suggested.

Site of the 6th International ISAAC Congress.

O. Celebi (METU Ankara), A. Kilbas (Univ. Minsk), and A. Jerbashian (Univ. Yerevan) are reporting about the conditions in their institutions, city/country for organizing the congress and answering questions of board members. Of particular interests are as follows:

International traffic connection of the city (airplane and train connections) and local transportation,

Possibilities of accommodation for 300 to 400 participants as well in higher quality hotels as in student dormitories/youth hotels close to the congress site,

Lecture halls for about 15 parallel sessions,

Chances to get financial support from local, national, and international sources (societies, institutions, companies),

Experience in organizing conferences.

After the three reporters have left, the board agrees in demanding more detailed proposals in written form from the three universities. A questionnaire will be worked out which should be answered within the proposals until beginning of September 2005. The board members will be informed electronically about the proposals and should decide via e-mail contact.

Remark Prof. Kilbas has handed over two letters from the President of the Belarus Academy of Sciences and from the Rector of the Univ. Minsk inviting ISAAC to organize the ISAAC congress in Minsk. The letters were sent once in 2001 to ISAAC. After the board meeting J. Toft has offered to organize the next ISAAC congress in Sweden. Other universities have expressed their interest to host ISAAC congresses in the near future. They are Univ. of New Delhi and Univ. of Wuhan; also colleagues from Moscow are willing to organize an ISAAC congress in Moscow.

Money Report.

This is a continuation of the money report from the last board meeting in Toronto.

Since the Toronto congress ISAAC has the following income:

Euro 2905, US$320, Yen 7700 : with H. Begehr,
Euro 220, US$40. : with M.W. Wong,
US$220 (?) : with R.P. Gilbert,
Euro 640 (?) : with F. Nicolosi.

Remark the amount with R.P. Gilbert is not confirmed; the amount with F. Nicolosi is estimated (expected membership fees collected in Catania).

The amount at FU Berlin from the 2001 congress:

Euro 22,240

Remark This amount differs by Euro 1020 from the figure given at the meeting. As was mentioned there some part of the amount given there is NATO money for the Yerevan NATO workshop in 2002.

Since the Toronto congress ISAAC has spent:

US$165 by M.W. Wong for the AMCA service,
Euro 7934 by H. Begehr for plenary speakers in Catania,
Euro 980 by H. Begehr for participants support in Catania,
Euro 400 by H. Begehr for one ISAAC award in Catania,

A final financial report on the total amount of ISAAC money will be given later.

Award Procedure.

The board agrees that strict regulations for the procedure as well for nominations/applications as criteria for the selection process have to be formulated and be published on the ISAAC home page. This has to be done early enough before the next congress. The board is expecting some proposal which then can be modified before deciding upon.

President Election.

The board agrees in the principle to re-elect the president in sequel only once. ISAAC should not become the society of just one person. In accordance with R.P. Gilbert who is not able to attend the congress for health reasons M.W. Wong is nominated as a candidate. Also V. Burenkov and C.C. Yang are nominated. The board agrees that F. Nicolosi may vote in the election process. The result of the voting is:

V. Burenkov 3 votes, M.W. Wong 5 votes, C.C. Yang 2 votes.
M.W. Wong is elected and accepts: He is the new ISAAC president for the next 2 years.

The board agrees to H. Begehr being the secretary and treasurer for this period. Also S. Zhang is welcome to continue his work as webmaster and secretary for ISAAC.

Other Elections.

The vice presidents and the board will be elected electronically in October/November this year by all life, honorary, and paid members. The board nominates E. Bruning, V. Burenkov, F. Nicolosi, S. Saitoh, and C.C. Yang for the positions of three vice presidents.

For the board election ISAAC members may nominate candidates in September. They will be informed via e-mail. All present board members will be candidates, additionally F. Nicolosi.

The suggestion to enlarge the board, as proposed already in Toronto, is met by the regulation that all officers (honorary president, president, vice presidents, secretary and treasurer, webmaster and secretary) are board members and that moreover each special interest group will delegate one representative (director) to the board.

The board election will be combined with the election of the vice presidents.

Various.

S. Saitoh has established two new special interest groups on

Integral Transforms (jointly with A. Kilbas),

Reproducing Kernels (jointly with A. Berlinet).

V. Burenkov is nominating O. Besov as honorary ISAAC member. The board approves this.

Remark O. Besov was decorated as honorary member on the final lunch during the excursion on 30th July by F. Nicolosi. A certificate could not be produced in Catania and will be mailed later. V. Burenkov will write a citation for O. Besov which will be put on the respective place on the Berlin ISAAC home page.

Heinrich Begehr

8. Frame Agreement Between Springer and ISAAC from 2015

The International Society for Analysis, its Applications and Computation (ISAAC) and the Springer International Publishing AG signed a joint Frame Agreement.

The aim of the Agreement is to promote the publication of proceedings of workshops and conferences, monographs, and textbooks, edited, or written, by ISAAC members, members of the ISAAC special interest groups, or any scientist willing to take part in such ISAAC activities. The volumes will appear with the ISAAC logo and a short sentence stating the belonging to the Frame Agreement.

The procedure is as follows: Colleagues interested in these ISAAC activities may send a text or a proposal to luigi.rodino@unito.it who will forward the material, with a short presentation, to the competent Springer editor. At this moment, the potential editor/author will interact directly with Springer. In particular, Springer will take care of the refereeing process and will choose the appropriate Springer series where the book will be published. The publication will then be governed by a separate contract between editor/author and Springer.

Other editors/authors, producing independently (outside the aforesaid procedure) Springer books related to Analysis, Applications and Computation, should be possibly invited by ISAAC and Springer to add, before publication, the ISAAC logo in the cover. They are also welcome to enter in this way the Frame Agreement.

We should also draw the attention of colleagues to the ISAAC Series on Analysis, Applications and Computation, co-edited by Heinrich Begehr of Freie Universität Berlin in Germany, Robert Pertsch Gilbert of the University of Delaware in the USA, and M.W. Wong of York University in Canada and published by World Scientific. To date, six volumes have been published and another one is scheduled to be published in 2016.

Contents

Part I
Complex-Analytic Methods for Applied Sciences, Complex Geometry and Generalized Functions

Vladimir Mityushev, Alexander Schmitt, and Oberguggenberger Michael

Generalized Functions Method for Solving Nonstationary Boundary Value Problems for Strictly Hyperbolic Systems with Second-Order Derivatives

L.A. Alexeyeva and G.K. Zakir'yanova

Abstract The method of generalized functions has been elaborated for solving nonstationary boundary value problems (BVPs) for strictly hyperbolic systems. Considered solutions may belong to the class of regular functions with discontinuous derivatives on moving surfaces, that is, wave fronts (shock waves). Generalized solutions of BVP subject to shock waves have been constructed. Singular boundary integral equations have been obtained that allow for the solution of BVP.

Keywords Boundary value problem • Generalized function method • Generalized solution • Hyperbolic system • Shock wave

Mathematics Subject Classification (2010) Primary 76L05; Secondary 30G99

The goal of this paper is to elaborate the method of boundary integral equations (BIEs) for solving nonstationary boundary value problems (BVPs) for strictly hyperbolic systems, which can be used to describe the dynamics of anisotropic continuum media. It is based on the generalized functions method (GFM), which gives the possibility of constructing the solutions of BVP in the space of distributions [1, 2]. The solutions considered may belong to the class of regular functions with discontinuous derivatives on moving surfaces. This case is typical of dynamic processes with shock waves in media. Here, on the basis of GFM in the space of distributions, the dynamic analogues of Kirchhoff's, Gauss's, and Green's formulas have been constructed using the fundamental Green's matrix of the system [3] and the new fundamental matrices generated by it [4]. For construction of their regular integral representations, the original method of regularization of the integral kernel with strong singularities on the fronts of fundamental solutions has been elaborated

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[1, 2, 4]. The resolving singular BIEs are constructed, and integral representations of classical solutions are given.

1 Statement of Nonstationary BVPs

We consider a system of hyperbolic equations of the form

$$L_{ij}(\partial_x, \partial_t)u_j(x,t) + G_i(x,t) = 0,\ (x,t) \in R^{N+1} \tag{1}$$

$$L_{ij}\,(\partial_x, \partial_t) = C_{ij}^{ml}\partial_m\partial_l - \delta_{ij}\partial_t^2, \quad i,j,m,l = \overline{1,N},$$

$$C_{ij}^{ml} = C_{ij}^{lm} = C_{ji}^{ml} = C_{ml}^{ij}, \tag{2}$$

where $x = (x_1,\ldots,x_N)$, $\partial_j = \partial/\partial x_j$, $\partial_t = \partial/\partial t$, $\delta_{ij} = \delta_j^i$ is the Kronecker symbol, and there is everywhere tensor convolution over repeated indices.

The matrix C_{ij}^{ml} satisfies the condition of strong hyperbolicity:

$$W(n,v) = C_{ij}^{ml}n_m n_l v^i v^j > 0 \quad \forall n \neq 0, \quad v \neq 0. \tag{3}$$

We construct a solution of (1) in the bounded region $x \in S^- \subset R^N$, $t > 0$; $D^- = S^- \times (0,\infty)$, $D_t^- = S^- \times (0,t)$; $D = S \times (0,\infty)$, $D_t = S \times (0,t)$, and $n(x)$ is a unit normal to S.

The vector function u is continuous and twice differentiable everywhere on D^-, except perhaps on some characteristic surface F in R^{N+1}, which in R^N describes wave fronts F_t with gaps where the derivatives are discontinuous.

We have the following conditions on the wave fronts [5]:

$$[u_i(x,t)]_{F_t} = 0 \;\Rightarrow m_l\,[u_{i,t}]_{F_t} + c[u_{i,l}]_{F_t} = 0, \tag{4}$$

$$m_l\left[\sigma_i^l\right]_{F_t} = -c\,[u_{i,t}]_{F_t}\,. \tag{5}$$

Here $\sigma_i^l = C_{ij}^{ml}u_{j,m}$, $u_{i,m} = \partial_m u_i$, $u_{i,t} = \partial_t u_i$. It is a *shock wave*. The speed of motion c of the wave front is defined by the characteristic equations of system (1):

$$\det\{C_{ij}^{ml}\nu_m\nu_l - \nu_t^2\delta_{ij}\} = 0, \tag{6}$$

where $(\nu,\nu_t) = (\nu_1,\ldots,\nu_N,\nu_t)$ is a normal to the characteristic surface F:

$$c = -\frac{\nu_t}{\|\nu\|^2}, \quad \|\nu\|^2 = \nu_j\nu_j. \tag{7}$$

By virtue of (3), Eq. (6) has $2N$ real roots, $c = \pm c_k(\nu)$, $0 < c_k \leq c_{k+1}$, $k = \overline{1, N}$, which in general depend on the direction of motion of F_t.

We must construct the solutions of Eq. (1) that satisfy (4) and (5) as well as the *initial conditions*

$$u_i(x, 0) = u_i^0(x), \quad x \in S^- + S, \quad u_{i},_t(x, 0) = u_i^1(x), \quad x \in S^- \tag{8}$$

and *boundary conditions*

$$\sigma_i^l(x, t) n_l(x) = g_i(x, t), \quad x \in S, \quad t \geq 0, \quad i = \overline{1, N}. \tag{9}$$

We call such solutions *classical*.

Remark 1.1 Wave fronts arise if the following condition of coordination of initial and boundary data is not satisfied:

$$w_i(x, +0) = u_i^0(x), \quad u_i^S,_t(x, +0) = u_i^1(x), \quad x \in S. \tag{10}$$

This is a typical typical situation in problems in mechanics when the boundary vector $g(x, t)$, which describes boundary stresses, has the character of a shock.

2 BVP Statement in the Space of Distributions $D'_N(R^{N+1})$

To consider Eq. (1) in the space of distributions, we consider the generalized functions

$$\widehat{u}(x, t) = u H_D^-(x, t), \ \widehat{G}_k(x, t) = G_k H_D^-(x, t), \tag{11}$$

where

$$H_D^-(x, t) = H_S^-(x) H(t),$$

is the characteristic function of the space-time cylinder D^-, $H(t)$ is the Heaviside function, and $H_S^-(x)$ is the characteristic function of $S^- \in R^N$.

Their generalized derivatives are equal to

$$\partial_j H_D^- = -n_j(x)\delta_S(x)H(t), \quad \partial_t H_D^- = H_D^-(x)\delta(t) \quad \Rightarrow$$
$$\partial_j \hat{u} = u,_j H_D^- - u n_j(x)\delta_S(x)H(t), \quad \partial_t \hat{u} = u,_t H_D^-(x, t) + u_0(x) H_D^-(x)\delta(t).$$

Here δ_S is a singular generalized function representing a *simple layer on the surface* S [1], while $g_k(x, t)\delta_S(x)H(t)$ denotes a *simple layer on D*. If we substitute these formulas and their second derivatives into Eq. (1), then [by virtue of the conditions on the gaps (4) and (5)], the densities of the layers of the independent singular functions on the wave fronts are equal to zero. As a result, we get the following theorem.

Theorem 2.1 *If u is a classical solution of BVP, then $\hat{u}$ is its generalized solution for the singular function*

$$\widehat{G}_i = H_D^- \widehat{G}_i + u_i^0(x) H_S^-(x) \delta'(t) + u_i^1(x) H_S^-(x) \delta(t) \\ + g_i(x,t) \delta_S(x) H(t) - (C_{ij}^{ml} u_j(x,t) n_m(x) \delta_S(x) H(t))_{,l} \,.$$

From this, the simple method of construction of wave front conditions for hyperbolic systems follows: *one has only to write the equations in the space of distributions and set the densities of the layers of the independent singular functions to zero.*

3 Generalized Solution of Nonstationary BVP

We denote by $\widehat{U}_i^k(x,t)$ the matrix of fundamental solutions of Eq. (1) (*Green tensor*). It satisfies (1) with $F_i = \delta_i^k \delta(x) \delta(t)$ and the radiation conditions (see construction U in [3]). The following theorem has been proved.

Theorem 3.1 *If $u(x,t)$ is the classical solution of a BVP, the generalized solution $\widehat{u}$ is representable in the form of a sum of convolutions ($*$)*

$$\widehat{u}_i = U_i^k * \widehat{G}_k + \partial_t U_i^k \underset{x}{*} u_k^0(x) H_S^-(x) + U_i^k \underset{x}{*} u_k^1(x) H_S^-(x) \\ + U_i^k * g_k(x,t) \delta_S(x) H(t) - C_{kj}^{ml} \partial_l U_i^k * u_j(x,t) n_m(x) \delta_S(x) H(t). \tag{12}$$

Here the symbol $*$ denotes the full convolution on (x,t); the variable under $*$ corresponds to an incomplete convolution only on x or t as the case may be.

In mechanics problems, this formula defines the solution of a BVP: media displacements through boundary tensions and displacements, initial values of displacements, and their speeds. At zero initial data, this formula is the *dynamic analogue of Green's formula* for elliptic equations [6]. When there is no boundary (summands with layers disappear), it gives the solution of the Cauchy problem and generalizes the formulas of Poisson ($N = 2$) and Kirchhoff ($N = 3$) for the classical wave equations [1]. But to write this formula in integral form, we regularize one of the convolutions with hypersingularities on wave fronts. For this, we use the antiderivative of the Green tensor for t:

$$\widehat{V}_i^k(x,t) = \widehat{U}_i^k(x,t) \underset{t}{*} H(t) \quad \Rightarrow \quad \partial_t \widehat{V}_i^k = \widehat{U}_i^k, \tag{13}$$

which is the solution of Eq. (1) for $G_i = \delta_i^k \delta(x) H(t)$. Applying the rules of differentiation of convolutions and generalized functions, we obtain

$$\begin{aligned} C_{kj}^{ml} \partial_t \widehat{V}_i^k * (u_j n_m(x) \delta_S(x) H(t))_{,l} &= C_{kj}^{ml} \partial_l \widehat{V}_i^k * (u_j n_m(x) \delta_S(x) H(t))_{,t} \\ &= C_{kj}^{ml} \partial_l \widehat{V}_i^k * u_{j,t}\, n_m(x) \delta_S(x) H(t) \\ &\quad + C_{kj}^{ml} \partial_l \widehat{V}_i^k \underset{x}{*} u_j^0(x) n_m(x) \delta_S(x), \end{aligned}$$

which together with Theorem 3.1 yields the following theorem.

Theorem 3.2 *If $u(x,t)$ is the classical solution of BVP, then the generalized solution $\widehat{u}$ is representable in the form*

$$\begin{aligned} \widehat{u}_i = {} & U_i^k * \widehat{G}_k + U_i^k \underset{x}{*} u_k^1(x) H_S^-(x) \\ & + \partial_t U_i^k \underset{x}{*} u_k^0(x) H_S^-(x) + U_i^k * g_k(x,t) \delta_S(x) H(t) \\ & - C_{kj}^{ml} \partial_l V_i^k * u_{j,t}(x,t) n_m(x) \delta_S(x) H(t) - C_{kj}^{ml} \partial_l V_i^k \underset{x}{*} u_j^0(x) n_m(x) \delta_S(x). \end{aligned} \tag{14}$$

To put this solution in integral form and construct BIEs for the BVP's solution, we require some new tensors.

4 Fundamental Tensors $\widehat{V}$, $\widehat{T}$, $\widehat{W}$, $\widehat{U}^{(s)}$, $\widehat{T}^{(s)}$ and Their Properties

$$\widehat{S}_{ik}^m(x,t) = C_{ij}^{ml} \partial_l \widehat{U}_j^k, \qquad \Gamma_i^k(x,t,n) = \widehat{S}_{ik}^m n_m, \tag{15}$$

$$\widehat{T}_k^i(x,t,n) = -\Gamma_i^k(x,t,n) = -C_{ij}^{ml} n_m \partial_l \widehat{U}_j^k \tag{16}$$

Properties of symmetry:

$$\begin{aligned} & \widehat{U}_i^k(x,t) = \widehat{U}_i^k(-x,t), \quad \widehat{U}_i^k(x,t) = \widehat{U}_k^i(x,t), \quad \widehat{S}_{ik}^m(x,t) = -\widehat{S}_{ik}^m(-x,t), \\ & \widehat{T}_i^k(x,t,n) = -\widehat{T}_i^k(-x,t,n) = -\widehat{T}_i^k(x,t,-n). \end{aligned} \tag{17}$$

The following theorems were proved in [4].

Theorem 4.1 *$\widehat{T}_i^k(x,t,n)$ (for fixed k and n) is the generalized solution of Eq. (1) corresponding to the multipole $G_i = C_{ik}^{ml} n_m \delta_{,l}(x) \delta(t)$.*

The antiderivative of the multipolar tensor is given by

$$\widehat{W}_j^k(x,t,n) = \widehat{T}_j^k(x,t,n) \underset{t}{*} H(t), \tag{18}$$

$$\widehat{W}_i^k(x,t,n) = -\widehat{W}_i^k(-x,t,n) = -\widehat{W}_i^k(x,t,-n). \tag{19}$$

In addition, the *static Green's tensor* of the static equations is $\widehat{U}_i^{k(s)}(x)$:

$$L_{ij}(\partial_x, 0)\widehat{U}_j^{k(s)}(x) + \delta_i^k \delta(x) = 0, \quad \widehat{U}_i^{k(s)}(x) \to 0, \quad \|x\| \to \infty. \tag{20}$$

Finally, the *static multipolar tensor* is given by

$$\widehat{T}_i^{k(s)}(x,n) = -C_{kj}^{ml} n_m \partial_l \widehat{U}_j^{i(s)}, \tag{21}$$

$$\widehat{T}_i^{k(s)}(x,n) = -\widehat{T}_i^{k(s)}(-x,n) = -\widehat{T}_i^{k(s)}(x,-n).$$

Theorem 4.2 $\widehat{T}_i^{k(s)}$ *is the generalized solution of the static equations*

$$L_{ij}(\partial_x, 0)T_j^{k(s)} - n_m C_{ki}^{ml} \delta_{,l}(x) = 0.$$

Theorem 4.3 (Analogue of Gauss's Formula) *If S is any closed Lyapunov surface in R^N, then*

$$P.V. \int_S T_k^{i(s)}(y-x, n(y)) dS(y) = \delta_k^i H_S^-(x), \tag{22}$$

where this integral is singular only for $x \in S$. It is calculated in the sense of principal values, and $H_S^-(x) = 0.5$.

Theorem 4.4 *The following representations are valid:*

$$\widehat{V}_i^k(x,t) = U_i^{k(s)}(x)H(t) + V_i^{k(d)}(x,t), \tag{23}$$

$$\widehat{W}_i^k(x,t) = T_i^{k(s)}(x)H(t) + W_i^{k(d)}(x,t). \tag{24}$$

Here $U_i^{k(s)}(x)H(t)$, $T_i^{k(s)}(x)H(t)$ are regular functions for $x \neq 0$. For $\|x\| \to 0$, we have

$$U_i^{k(s)}(x) \sim \ln\|x\| A_{ik}^N(e_x), \quad T_i^{k(s)}(x) \sim \|x\|^{-1} B_{ik}^N(e_x), \quad N = 2,$$

$$U_i^{k(s)}(x) \sim \|x\|^{-N+2} A_{ik}^N(e_x), \quad T_i^{k(s)}(x) \sim \|x\|^{-N+1} B_{ik}^N(e_x), \quad N > 2.$$

Here $e_x = x/\|x\|$, $A_{ik}^N(e_x)$, and $B_{ik}^N(e_x)$ are continuous functions, bounded on the sphere $\|e_x\| = 1$; $V_i^{k(d)}$, $W_i^{k(d)}$ are regular functions, continuous for $x = 0$, $t > 0$. For every N, we have

$$V_i^{k(d)}(x,t) = 0 \qquad W_i^{k(d)}(x,t) = 0 \quad \text{for } \|x\| > \max_{k=\overline{1,M}} \max_{\|e\|=1} c_k(e)t,$$

and for odd N, these equalities hold, and $\|x\| < \min_{k=\overline{1,M}} \min_{\|e\|=1} c_k(e)t$. These properties were proved in [4] using some theorems from [7].

5 Integral Form of the BVP Solution: Dynamic Analogue of the Kirchhoff–Green Formulas

If we write $\hat{u}$ (14) in integral form, it becomes (for zero initial conditions)

$$\widehat{u}_k(x,t) = \int_D \{T_k^i(x-y, n(y), t-\tau)u_i(y,t) + U_k^i(x-y, t-\tau)g_i(y,\tau)\}dD(y,\tau),$$

similar to Green's formulas for the solutions of elliptic systems or Somigliana's formula for static BVP of elasticity. But the first summand with T_k^i is not integrable because of existence of hypersingularities on waves fronts of fundamental solutions. The integral representation of the BVP solution gives the regularized formula of Theorem 3.2, obtained using Theorem 4.4. For problems with zero initial conditions, we get the following theorem.

Theorem 5.1 *If $x \in D^-$, then the classical solution of the BVP is representable in the form*

$$u_k = U_k^i(x,t) * G_i(x,t) + U_k^i(x,t) * g_i(x,t)\delta_s(x)H(t)$$
$$- \int_S T_k^{i(s)}(x-y)u_i(y,t)dS(y) - \int_S dS(y) \int_0^t W_k^{i(d)}(x-y, n(y), t-\tau)u_{i,t}(y,\tau)d\tau.$$

6 Singular BIEs

Using the properties of the static multipolar tensor (22), we construct the singular BIEs that resolve the nonstationary BVP [4].

Theorem 6.1 *If S is a Lyapunov surface, then the classical solution of the nonstationary BVP satisfies the following singular BIEs: for $x \in S$, $t > 0$ $(k = \overline{1,N})$,*

$$
\begin{aligned}
0.5u_k(x,t) = {} & U_k^i(x,t) * G_i(x,t) + U_k^i(x,t) * g_i(x,t)\delta_s(x)H(t) \\
& - V.P.\int_S T_k^{i(s)}(x-y)u_i(y,t)dS(y) \\
& - \int_S dS(y) \int_0^t W_k^{i(d)}(x-y,n(y),t-\tau)u_{i,t}(y,\tau)d\tau \\
& - \int_S W_k^{i(d)}(x-y,n(y),t)u_i^0(y)dS(y) \\
& + \left(U_k^i(x,t) \underset{x}{*} u_i^0(y)H_S^-(x)\right)_{,t} + U_i^k \underset{x}{*} u_k^1(x)H_S^-(x).
\end{aligned}
$$

These equations allow us to define unknown boundary functions and construct the generalized solution, which for regular initial and boundary conditions gives the classical solution by virtue of the du Bois–Reymond lemma of the theory of generalized functions [6].

7 Conclusion

The question of the unsolvability of singular BIEs for a certain class of functions represents an independent task of functional analysis. Numerical solution of these equations can be obtained using the method of boundary elements. Special cases of solutions of nonstationary problems in the theory of elasticity are considered in [8–10].

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A Fast Algorithm to Determine the Flux Around Closely Spaced Non-overlapping Disks

Olaf Bar

Abstract This paper is devoted to application of the fast algorithm to determine the flux around closely spaced non-overlapping disks on the conductive plane. This method is based on successive approximations applied to functional equations. When the distances between the disks are sufficiently small, convergence of the classical method of images fails numerically. In this talk, the limitations on geometric parameters are described.

Keywords Multiply connected domain • Non-overlapping disks • Poincaré series

1 Introduction

Fibrous composites can be modeled by unidirectional circular cylinders embedded in the matrix [1, 2]. The electrical or thermal conductivity in the plane perpendicular to fibers can be described by Laplace's equation. The standard numerical methods (like FEM) refer to calculation of the local fields. In special cases (like a periodic layout) we can use the Weierstrass function to solve this problem. In this paper we focus on the case without symmetry. The method used in this work gives an approximate analytical solution of the discussed problem [3]. This method is based on inversions with respect to the circles which transform the harmonic functions from inside to outside of the disks.

2 Description of the Fast Poincaré Series Method

Denote :

$$\mathbb{D}_k = \{z \in \mathbb{C} : |z - a_k| < r_k (k = 1, 2, \ldots, \mathcal{N}) \tag{1}$$

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Introduce the inversion with respect to the k-circle:

$$z^*_{(k)} = \frac{r_k^2}{\overline{z - a_k}} + a_k \tag{2}$$

The goal of this method is to find a function $u(z)$ harmonic in $\mathbb{D}$ and continuously differentiable in the closure of $\mathbb{D}$ with the following boundary conditions [4]

$$u(t) = u_k, \quad |t - a_k| = r_k, k = 1, 2, \dots, \mathcal{N}. \tag{3}$$

This problem can be reduced to the Riemann–Hilbert problem [5]

$$\text{Re } \varphi(t) = u_k, \quad |t - a_k| = r_k,\ k = 1, 2, \dots, \mathcal{N}, \tag{4}$$

on the function $\varphi(t)$ analytic in $\mathbb{D}$.

The exact solution for the flux between two circles is known [6]

$$\Psi(z) = \frac{1}{z - z_{12}} - \frac{1}{z - z_{21}} \tag{5}$$

where $\Psi(z) = \varphi'(z)$. The function $\Psi(z)$ describes the flux for the known difference $u_1 - u_2$ and the z_{12}, z_{21} satisfy the quadratic equation $z^*_{(1)} = z^*_{(2)}$. This function can be used as the zero-th approximation for the fast algorithm.

Define the analytic function [2]:

$$f_{km}(z) := \begin{cases} 0, & k = m, \\ \sum_{\ell \in J_m; \ell \neq k} \Psi(z; m, \ell)(z), & k \in J_m, \\ \sum_{\ell \in J_m} \Psi(z; m, \ell)(z), & k \in J_m^*, \end{cases}$$

where J_m^* is the complement of $J_m \cup \{m\}$ to $\{1, 2, \dots, n\}$.

The following algorithm can be applied. First, we compute auxiliary functions $\psi_k(z)$ by the following iterations:

$$\psi_k^{(0)}(z) = f_{km}(z), \tag{6}$$

$$\psi_k^{(p)}(z) = \sum_{m \neq k} \left(\frac{r_m}{z - a_m} \right)^2 \overline{\psi_m^{(p-1)} \left(z^*_{(m)} \right)} + f_{km}(z),\ p = 1, 2, \dots. \tag{7}$$

The p-th approximation of the complex flux $\psi(z) = \varphi'(z)$ is calculated by formula

$$\psi^{(p)}(z) = \sum_{m=1}^{n} \left(\frac{r_m}{z - a_m} \right)^2 \overline{\psi_m^{(p)}\left(z_{(m)}^*\right)} + \psi_\delta(z), \quad z \in \mathbb{D}, \tag{8}$$

where $\psi_\delta(z) = \sum_{\ell \in J_m} \Psi(z; m, \ell)$.

The potential $\varphi(z)$ is obtained by integration of $\psi(z)$.

3 Results

Calculations were made for four circles.

Introduce the vector of the circle centers: [−1,−i, 1.1, 1.2 i] (see Fig. 1) and the radius $r = \frac{\sqrt{2}}{2}\delta$. The limit case $\delta = 1$ yields tangent circles $\mathbb{D}_1$ and $\mathbb{D}_2$.

The accuracy of calculation is associated with the accuracy of the boundary conditions. The procedure to obtain the potential from the flux requires integration of the long algebraic expressions. The analytical integration procedure requires a lot of time and it is possible at most for the 8–9 iterations. Thus instead of checking the boundary conditions we check the equivalent conditions [2] (Fig. 2):

$$\operatorname{Im} \Psi(t) := \operatorname{Im} \left[\frac{t - a_k}{r_k} \psi(t) \right] = 0, \quad |t - a_k| = r_k, \ k = 1, \dots, 4, \tag{9}$$

The function $\psi(z)$ describes all the fluxes in $\mathbb{D}$ with constant real potentials on the boundary. It follows from general theory [5] that the problem (9) has $(n-1)\mathbb{R}$-linear independent solutions (in our case $n - 1 = 3$). In order to determine $\psi(z)$ one can construct basis functions and their $\mathbb{R}$-linear combination. In order to determine the

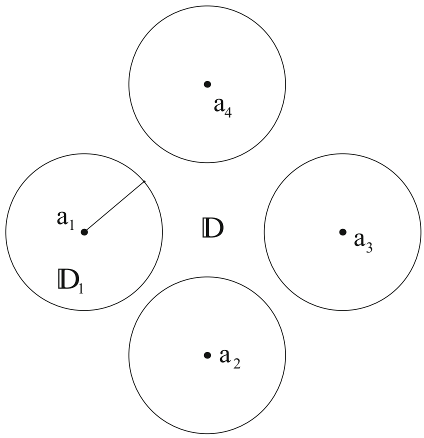

Fig. 1 Four disks and the multiply connected region $\mathbb{D}$

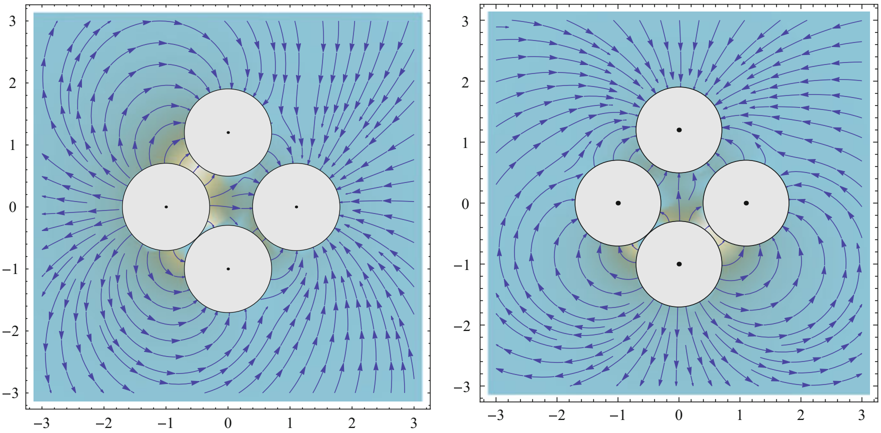

Fig. 2 Example of stream basis functions $\Psi_1(z)$ and $\Psi_2(z)$

Table 1 Calculation for $\delta = 0.98$

Iterations	U_{err}	$\operatorname{Im}\Psi_{\text{err}}$	$\frac{\operatorname{Im}\Psi_{\text{err}}}{U_{\text{err}}}$
1	0.38	1.40	3.7
2	0.18	1.10	6.1
3	0.16	0.80	5.0
4	0.089	0.61	6.8
5	0.074	0.42	5.7
6	0.048	0.31	6.5
7	0.037	0.22	4.6
8	0.025	0.15	6.0
9	0.0193	0.12	6.2

flux $\psi(z)$ corresponding to the potential Re $\varphi(z)$ with prescribed boundary values u_j one has to find the basis solutions $\Psi_j(z)$ $(j = 1, 2, 3)$ of the problem (9). Index j denotes that the corresponding functions $u_{j,k} = \delta_{j,k}$ is a potential on the boundary of k-disk (δ_{kj} denotes the Kronecker symbol) (Table 1).

Introduce: $\operatorname{Im}\Psi_{\text{err}} = \max(|\operatorname{Im}\Psi(t)|)$; $U_{\text{err}} = \frac{\max(\operatorname{Re}\varphi(t)) - \min(\operatorname{Re}\varphi(t))}{\max(\operatorname{Re}\varphi(t))}$ where $t \in \partial\mathbb{D}$.

During the calculation it turned out that the maximum of the boundary condition error occurred between the closest disks $\mathbb{D}_1$ and $\mathbb{D}_2$. Additionally the maximum of the error turns into places between one disk and the other. So as the boundary condition error it was adopted the maximum variation of U on the all disks.

When the number of iterations is less than 9, it is possible to compare the boundary values comparing the $\operatorname{Im}\Psi(t)$ and $U(t)$. The dependence of $\ln(U_{\text{err}})$ on the number of iteration is linear. Figure 3 shows this relation for the eight iterations depending on δ. Although it is impossible to obtain U_{err} for higher number of iterations,we can use the linear approximation to estimate this error. This prediction for U_{err} is shown in Table 2.

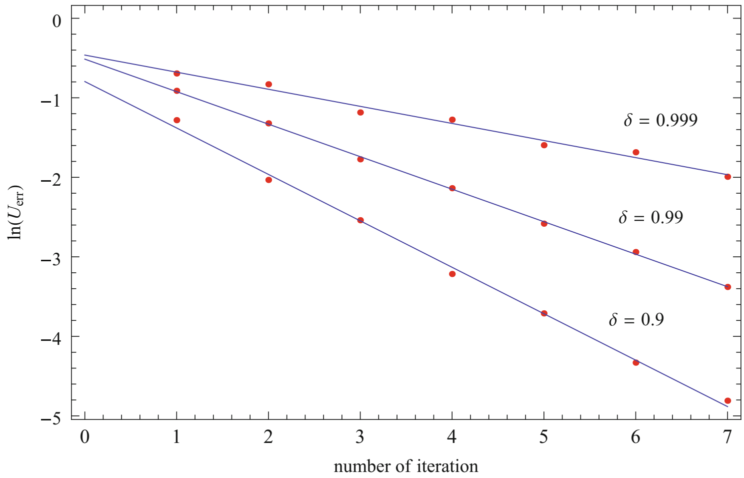

Fig. 3 Dependence of logarithm of the error on number of iterations

Table 2 Number of iterations on acceptable U_{err}

Accept. U_{err}	$\delta = 0.98$	$\delta = 0.99$	$\delta = 0.999$
0.05	6	8	13
0.02	9	12	17
0.01	11	14	20

Table 3 Calculation for Im Ψ_{err}

	Im Ψ_{err}		
Iterations	$\delta = 0.9$	$\delta = 0.99$	$\delta = 0.999$
8	0.026	0.246	1.76
9	0.015	0.160	1.20
10	0.0082	0.111	1.17
11	0.0049	0.072	0.791
12	0.0027	0.050	0.779
13	0.0016	0.032	0.525
14	0.0009	0.022	0.523
15	0.0005	0.015	0.351

Table 3 presents the calculation errors for greater number of iterations. The parameter δ was changed between 0.9 and 0.999.

The results presented in this paper show that at the present stage of research, we can perform at most 15 iterations, thus it is the limitation of the method. For four nonoverlapping disks the minimum distance between inclusions is equal to $\delta = 0.99$. The corresponding relative error holds less than 5%.

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Conformal Mapping of Circular Multiply Connected Domains Onto Domains with Slits

Roman Czapla and Vladimir V. Mityushev

Abstract The conformal mapping of the square with circular disjoint holes onto the square with disjoint slits is constructed. This conformal mapping is considered as a solution of the Riemann–Hilbert problem for a multiply connected domain in a class of double periodic functions. The problem is solved by reduction to a system of functional equations.

Keywords Circular multiply connected domain • Conformal mapping • Multiply connected domain with slits • Riemann–Hilbert problem

1 Introduction

Analytical formulae for conformal mapping of multiply connected domains with slits onto circular domains are the canonical formulae of complex analysis. Such a formula can be referred to the Schwarz–Christoffel formula. Analytical formulae for conformal mapping between various canonical slit domains with sufficiently well-separated boundary components were given in [3] and the works cited therein. The geometrical restrictions were eliminated in [6] and a formula for an arbitrary circular multiply connected domain was constructed by means of the Poincaré series. It is worth noting that the Poincaré series is the second derivative of the Schottky–Klein prime function [3] that makes a possibility to use various form of analytical formulae. The uniform convergence of the Poincaré series for an arbitrary circular multiply connected domain can extend the validity of the constructions by DeLillo et al [3] by modifications explained in [6].

Analogous study was performed in [7] where the Schwarz–Christoffel formula for conformal mapping of multiply connected domains bounded by polygons onto circular domains was constructed. The formula given in [7] does not contain any geometrical restriction on boundary components. However, the general Schwarz–Christoffel formula contains the accessory parameters contrary to formulae for

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conformal mappings onto slit domains. This is the reason why the special formula [6] for slit domains is preferable than the general one.

Besides the canonical slit domains, conformal mapping of multiply connected domains with slits of various inclinations onto circular domains is applied to boundary value problems of fracture mechanics. The results of [6] were developed to domains bounded by mutually disjoint arbitrarily oriented slits in [2].

The above presented results concern the canonical multiply connected domains bounded by disjoint circles. However, a straight line can be treated a circle on the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \infty$. This fact was used in [4] to extend the method of functional equations [10] to strips and rectangles with circular holes.

In this paper, we follow the method outlined in [4] to construct the conformal mapping of the square with circular holes onto the square with slits of given inclinations. The conformal mapping is constructed as a solution of the Riemann–Hilbert problem for a doubly periodic multiply connected domain. The latter problem is reduced to a system of functional equations.

2 Basic Problem

Let $z = x + iy$ denote a complex variable on the complex plane $\mathbb{C}$ and $\mathcal{G} = \Big\{z \in \mathbb{C}\colon \big|\mathrm{Re}[z]\big| \le \frac{1}{2}\ \&\ \big|\mathrm{Im}[z]\big| \le \frac{1}{2}\Big\}$ stands for the unit square. Consider non-overlapping disks $D_k = \{z \in \mathcal{G}\colon |z - a_k| < r_k\}$ $(k = 1, 2, \ldots, N)$. Let $\mathcal{D}$ denote the complement of the closed disks $|z - a_k| \le r_k$ to the square $\mathcal{G}$. Consider the second complex variable $\zeta = u + iv$ on the complex plane with non-overlapping slits Γ_k lying in the square $\mathcal{G}$. Here, each slit Γ_k has two sides, hence, it is considered as a closed curve. Let $\mathcal{D}'$ stand for the complement of all the slits Γ_k to $\mathcal{G}$. Find a conformal mapping φ of the circular multiply connected domain $\mathcal{D}$ onto $\mathcal{D}'$. Function φ has to satisfy the following boundary conditions:

$$\mathrm{Im}\left[e^{-i\alpha_k}\varphi(t)\right] = c_k, \quad |t - a_k| = r_k,\ k = 1, 2, \ldots, N \tag{1}$$

$$\mathrm{Re}\left[\varphi\left(\pm\frac{1}{2} + iy\right)\right] = \pm\frac{1}{2}, \quad -\frac{1}{2} \le y \le \frac{1}{2}, \tag{2}$$

$$\mathrm{Im}\left[\varphi\left(x \pm \frac{i}{2}\right)\right] = \pm\frac{1}{2}, \quad -\frac{1}{2} \le y \le \frac{1}{2}, \tag{3}$$

where α_k is the given inclination angle of Γ_k, a_k and r_k are center and radius of the k-th disk, c_k are undetermined real constants. The condition (1) means that each slit Γ_k lies on the line $-\sin\alpha_k u + \cos\alpha_k v = c_k$, i.e., the circle $|z - a_k| = r_k$ maps onto the slit Γ_k. The conditions (2) and (3) show that the boundary of the square $\mathcal{G}$ is mapped onto itself and four corner points are fixed points of the conformal mapping. The fixed point condition should not be imposed in general case. But we shall consider only symmetric domains $\mathcal{D}$ and $\mathcal{D}'$ when this condition is automatically fulfilled.

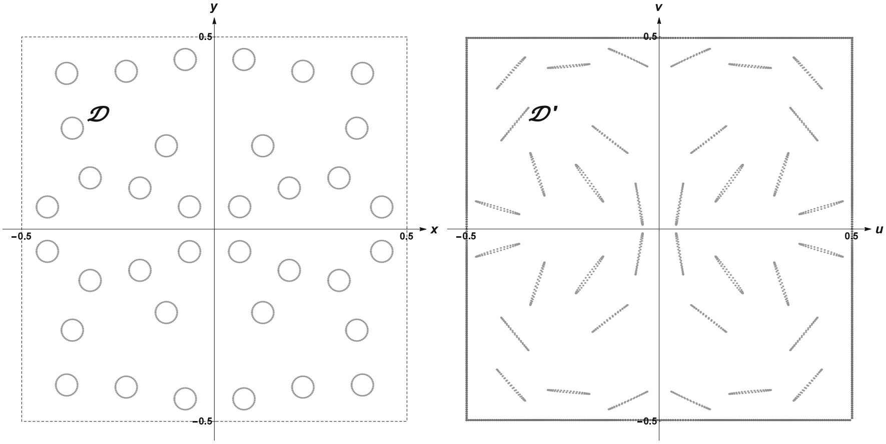

Fig. 1 Domains $\mathcal{D}$ and $\mathcal{D}'$

More precisely, we assume that the holes of $\mathcal{D}$ and the slits of $\mathcal{D}'$ are symmetric with respect to the axes as shown in Fig. 1 that yields the symmetry of the conformal mapping including the symmetry of the constants c_k and $\varphi(0) = 0$. Instead of the symmetric square one can consider the one fourth small square without any symmetry condition.

The conditions (1)–(3) can be considered as the Riemann–Hilbert problem which has a unique solution if one of the undetermined constants c_k is fixed [10]. Therefore, the conformal mapping and the unique solution of the Riemann–Hilbert problem (1)–(3) coincide. This implies the uniqueness of the discussed conformal mapping. In the same time, this implies that the unique solution of (1)–(3) is a univalent function.

We now proceed to solve the problem (1)–(3). Introduce an auxiliary function $\widetilde{\varphi}(z) = \varphi(z) - z$. Then (1)–(3) become

$$\operatorname{Im}\left[e^{-i\alpha_k}\widetilde{\varphi}(t)\right] = c_k - \operatorname{Im}\left[e^{-i\alpha_k}t\right], \quad |t - a_k| = r_k,\ k = 1, 2, \ldots, N, \tag{4}$$

$$\operatorname{Re}\left[\widetilde{\varphi}\left(\pm\frac{1}{2} + iy\right)\right] = 0, \quad -\frac{1}{2} \le y \le \frac{1}{2}, \tag{5}$$

$$\operatorname{Im}\left[\widetilde{\varphi}\left(x \pm \frac{i}{2}\right)\right] = 0, \quad -\frac{1}{2} \le x \le \frac{1}{2}. \tag{6}$$

The problem (4) can be reduced to $\mathbb{R}$-linear problem [10]

$$\widetilde{\varphi}(t) = \varphi_k(t) + e^{2i\alpha_k}\overline{\varphi_k(t)} + ie^{i\alpha_k}c_k - t, \qquad |t - a_k| = r_k, \tag{7}$$

where φ_k are analytic in $|z - a_k| < r_k$ and continuously differentiable in $|z - a_k| \le r_k$, $k = 1, 2, \ldots, N$. The equivalence of the boundary conditions (4) and (7) was justified in [2] and [6]. Multiply Eq. (7) by $e^{-i\alpha_k}$:

$$e^{-i\alpha_k}\widetilde{\varphi}(t) = e^{-i\alpha_k}\varphi_k(t) + e^{i\alpha_k}\overline{\varphi_k(t)} + ic_k - e^{-i\alpha_k}t$$

that is equivalent to the relation

$$e^{-i\alpha_k}\widetilde{\varphi}(t) = 2\mathrm{Re}\left[e^{-i\alpha_k}\varphi_k(t)\right] + ic_k - e^{-i\alpha_k}t. \tag{8}$$

One can see that the imaginary part of (8) yields (4). The inverse way from (4) to (8) is based on the solution to the Dirichlet problem for the disk D_k

$$2\mathrm{Re}\left[e^{-i\alpha_k}\varphi_k(t)\right] = \mathrm{Re}\left[e^{-i\alpha_k}\widetilde{\varphi}(t) + e^{-i\alpha_k}t\right]. \tag{9}$$

with respect to $e^{-i\alpha_k}\varphi_k(z)$ with given $\widetilde{\varphi}$ (for details see [10]).

We now demonstrate that $\widetilde{\varphi}$ is a double periodic function:

$$\widetilde{\varphi}(z+i) = \widetilde{\varphi}(z) = \widetilde{\varphi}(z+1). \tag{10}$$

The symmetry of the conditions (4)–(6) with respect to the x-axis and y-axis implies that

$$\overline{\widetilde{\varphi}(\overline{z})} = \widetilde{\varphi}(z), \quad z \in \mathcal{D} \tag{11}$$

$$\overline{-\widetilde{\varphi}(\overline{-z})} = \widetilde{\varphi}(z). \quad z \in \mathcal{D}; \tag{12}$$

It follows from (11) that $\overline{\widetilde{\varphi}\left(x+\frac{i}{2}\right)} = \widetilde{\varphi}\left(x-\frac{i}{2}\right)$ for $\frac{1}{2} \leq x \leq \frac{1}{2}$, hence $\mathrm{Re}\ \widetilde{\varphi}\left(x+\frac{i}{2}\right) = \mathrm{Re}\ \widetilde{\varphi}\left(x-\frac{i}{2}\right)$. Using (6) we get $\widetilde{\varphi}\left(x+\frac{i}{2}\right) = \widetilde{\varphi}\left(x-\frac{i}{2}\right)$. In the same way, the condition (12) gives (5).

The $\mathbb{R}$-linear problem (7) can be reduced to functional equations. We introduce the function [10]:

$$\Phi(z) = \begin{cases} \varphi_k(z) - \sum\limits_{m=1}^{N} e^{2i\alpha_m} \sum\limits_{m_1,m_2}^{*} \left[\overline{\varphi_m\left(\overline{\frac{r_m^2}{z-a_m-m_1-m_2 i}} + a_m\right)} - \overline{\varphi_m(a_m)}\right] + f_k(z), \\ \qquad |z-a_k| \leq r_k, \qquad k = 1, 2, \dots N, \\ \widetilde{\varphi}(z) - \sum\limits_{m=1}^{N} e^{2i\alpha_m} \sum\limits_{m_1,m_2} \left[\overline{\varphi_m\left(\overline{\frac{r_m^2}{z-a_m-m_1-m_2 i}} + a_m\right)} - \overline{\varphi_m(a_m)}\right], \ z \in \mathcal{D}, \end{cases} \tag{13}$$

where $f_k(z) = ie^{i\alpha_k}c_k - z + e^{2i\alpha_k}\overline{\varphi_k(a_k)}$ and

$$\sum_{m=1}^{N} \sum_{m_1,m2}^{*} V_{(m_1,m_2)m} := \sum_{m \neq k} \sum_{m_1,m_2} V_{(m_1,m_2)m} + \sum_{m_1,m_2}{}' V_{(m_1,m_2)k}.$$

In the sum $\sum'_{m_1,m_2}$ the integer numbers m_1, m_2 range from $-\infty$ to $+\infty$ except the case when $m_1^2 + m_2^2 = 0$.

We calculate the jump across the circle $|t - a_k| = r_k$

$$\Delta_k := \Phi^+(t) - \Phi^-(t), \quad |t - a_k| = r_k,$$

where $\Phi^+(t) := \lim_{z \to t\, x \in \mathcal{D}} \Phi(z)$ and $\Phi^-(t) := \lim_{z \to t\, x \in D_k} \Phi(z)$. Using (7) we get $\Delta_k = 0$.

The definition of Φ in $|z - a_k| \le r_k$ $(k = 1, 2, \ldots, N)$ yields the following system of functional equations

$$\varphi_k(z) - \sum_{m=1}^{N} e^{2i\alpha_m} \sum\nolimits^*_{m_1,m_2} \left[\overline{\varphi_m\left(\frac{r_m^2}{\overline{z - a_m - m_1 - m_2 i}} + a_m \right)} - \overline{\varphi_m(a_m)} \right] + f_k(z) = C. \tag{14}$$

Let φ_k $(k = 1, 2, \ldots, N)$ be a solution of (14). Then the function $\widetilde{\varphi}$ can be found from the definition of Φ in $\mathcal{D}$

$$\widetilde{\varphi}(z) = \sum_{m=1}^{N} e^{2i\alpha_m} \sum_{m_1,m_2} \left[\overline{\varphi_m\left(\frac{r_m^2}{\overline{z - a_m - m_1 - m_2 i}} + a_m \right)} - \overline{\varphi_m(a_m)} \right] + C. \tag{15}$$

3 Numerical Example

Solution to the functional equations (14) can be found by the method of approximations [1]. The zero order approximation is

$$\varphi_k^{(0)}(z) = C - i e^{i\alpha_k} c_k + z.$$

Then (15) implies that

$$\widetilde{\varphi}^{(0)}(z) = \sum_{m-1}^{N} e^{2i\alpha_m} \left[r_m^2 \sum_{m_1,m_2} \left(\frac{1}{z - a_m - m_1 - m_2 i} \right) \right], \tag{16}$$

where C is taken to be zero. The zero order approximation of function φ has the form

$$\varphi^{(0)}(z) = \sum_{m=1}^{N} e^{2i\alpha_m} r_m^2 E_1(z - a_m) + z, \tag{17}$$

where E_1 denotes the Eisenstein function which can be expressed in terms of the Weierstrass ζ-function [11]

$$E_1(z) = \sum_{m_1, m_2} \frac{1}{z - m_1 - i m_2} = \zeta(z) - \pi z. \tag{18}$$

One can expect that the zero approximation (17) gives sufficiently good results for not high density of slits. Let $\mathcal{G}$ be the unit square. Consider 36 non-overlapping circular disks D_k of radius $r = 0.028$ symmetric with respect to the x-axis and y-axis distributed in $\mathcal{G}$. Conformal mapping of the considered domain $\mathcal{D}$ onto the square with slits of the inclinations randomly chosen on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is presented in Fig. 1. Higher order approximations will be constructed in a separate paper.

4 Discussion

The successive approximations applied to (14) yield an infinite series after substitution into (15). This is the generalized Poincaré series [8] constructed for doubly periodic generators (inversions with respect to circles composed with translations on the square lattice). The present result demonstrates that the method of functional equations to solve the Riemann–Hilbert problem for complicated multiply connected domains obtained from circular ones by symmetries with respect to straight lines yields generalized Poincaré series, i.e., solves the considered problem exactly. In his plenary ISAAC 2015 talk, Darren Crowdy discussed particular cases of the above problems by use of the Schottky–Klein prime function $S(z)$ constructed for a class of domains restricted by the separation condition and the Fourier–Mellin transforms. One can see that the derivative of $\ln S(z)$ is the Poincaré θ_1-series. This simple observation shows that the most general constructions of solutions to such a type of boundary value problems can be found in [4–10] and the works cited therein.

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Functional Equations for Analytic Functions and Their Application to Elastic Composites

Piotr Drygaś and Vladimir Mityushev

Abstract Two-dimensional elastic composites with non-overlapping inclusions is studied by means of the boundary value problems for analytic functions following Muskhelishvili's approach. We develop a method of functional equations to reduce this problem for a circular multiply connected domain to functional-differential equations. Analytical formulae for the effective constants are deduced.

Keywords Eisenstein series • Functional equation • Natanzon series • Two-dimensional elastic composite

Mathematics Subject Classification (2010) Primary 30E25; Secondary 74Q15

1 Introduction

Two-dimensional elastic composites with non-overlapping inclusions can be discussed through boundary value problems for analytic functions following Muskhelishvili's approach [16]. A method of functional equations was proposed to solve the Riemann–Hilbert and $\mathbb{R}$-linear problems for multiply connected domains [13]. These results were applied to description of the local fields and the effective conductivity tensor for 2D composites [1, 4, 8, 9, 12, 14, 15]. In the present note, we develop this method of functional equations to elastic problems modelled by the biharmonic equation. We reduce the problem for a circular multiply connected domain to a system functional-differential equations and propose a constructive method for their solution in terms of the generalized Eisenstein and Natanzon functions [2, 6, 7, 17]).

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2 Statement of the Elastic Problem

Consider n disks $D_k = \{z \in \mathbb{C} : |z - a_k| < r\}$, $(k = 1, .., n)$ in the complex plane $\mathbb{C}$. Let $\Gamma_k = \partial D_k$, $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $\dot{D} = \mathbb{C} \setminus \bigcup_{j=1}^{n} \left(D_j \cup \Gamma_j\right)$, $D = \dot{D} \cup \{\infty\}$ where the circle Γ_k is orientated in counter-clockwise sense. Further, the limit case as $n \to \infty$ will be considered following the method described in [5]. This means that we formally introduce an infinite sequence of non-overlapping disks D_k $(k = 1, 2, \ldots)$. After, we fix a number n and consider only first n disks. Since the number n is arbitrary in our study, hence we can take the limit $n \to \infty$ in the final formulae.

The component of the stress tensor can be determined by the Kolosov–Muskhelishvili formulas [16]

$$\sigma_{xx} + \sigma_{yy} = \begin{cases} 4\mathrm{Re}\varphi_k'(z), \; z \in D_k, \\ 4\mathrm{Re}\varphi_0'(z), \;\; z \in D, \end{cases} \tag{1}$$

$$\sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} = \begin{cases} -2\left[\overline{z\varphi_k''(z)} + \overline{\psi_k'(z)}\right], \; z \in D_k, \\ -2\left[\overline{z\varphi_0''(z)} + \overline{\psi_0'(z)}\right], \;\; z \in D, \end{cases}$$

Introduce constants $B_0 = \frac{\sigma_{xx}^{\infty} + \sigma_{yy}^{\infty}}{4}$, $\Gamma_0 = \frac{\sigma_{yy}^{\infty} - \sigma_{xx}^{\infty} + 2i\sigma_{xy}^{\infty}}{2}$, where σ_{xx}^{∞}, σ_{xy}^{∞}, σ_{yy}^{∞} are the given stresses at infinity. Introduce the functions $\varphi_0(z) = B_0 z + \varphi(z)$, $\psi_0(z) = \Gamma_0 z + \psi(z)$ where $\varphi(z)$ and $\psi(z)$ are analytical in D and bounded at infinity, $\varphi_k(z)$ and $\psi_k(z)$ are analytical in D_k and all one twice differentiable in the closures of the considered domains. The ideal contact between different materials is expressed by means of the following boundary conditions [16]

$$\varphi_k(t) + \overline{t\varphi_k'(t)} + \overline{\psi_k(t)} = \varphi_0(t) + \overline{t\varphi_0'(t)} + \overline{\psi_0(t)}, \tag{2}$$

$$\mu\left(\kappa_1\varphi_k(t) - \overline{t\varphi_k'(t)} - \overline{\psi_k(t)}\right) = \mu_1\left(\kappa\varphi_0(t) - \overline{t\varphi_0'(t)} - \overline{\psi_0(t)}\right), \tag{3}$$

$t \in \partial D_k$. In these equations μ is a shear modulus, κ is Kolosov constant and $\kappa = 3 - 4\nu$ in plane strain; $\kappa = (3 - \nu)/(1 + \nu)$ in plane stress; ν is Poisson's ratio. Index 1 denotes physical constants for inclusions.

3 Functional-Differential Equations

Let $z_{(k)}^{*} = \frac{r^2}{\overline{z - a_k}} + a_k$ denote the inversion with respect to the circle Γ_k. Introduce the functions $\Phi_k(z) = \overline{z_{(k)}^{*}}\varphi_k'(z) + \psi_k(z)$, $|z - a_k| \le r$, analytic in D_k except point a_k, where its principal part has the form $r^2\,(z - a_k)^{-1}\,\varphi_k'(a_k)$. The problem (2), (3) can be reduced to the system of functional equations following [11, 13]

$$\left(\frac{\mu_1}{\mu}+\kappa_1\right)\varphi_k(z)=\left(\frac{\mu_1}{\mu}-1\right)\sum_{m\neq k}\left[\overline{\Phi_m(z^*_{(m)})}-(z-a_m)\,\overline{\varphi_m'(a_m)}\right]$$
$$-\left(\frac{\mu_1}{\mu}-1\right)\overline{\varphi_k'(a_k)}\,(z-a_k)$$
$$+\frac{\mu_1}{\mu}(1+\kappa)B_0z+p_0,\ |z-a_k|\leq r, \tag{4}$$

and

$$\left(\kappa\frac{\mu_1}{\mu}+1\right)\Phi_k(z)=\left(\kappa\frac{\mu_1}{\mu}-\kappa_1\right)\sum_{m\neq k}\overline{\varphi_m(z^*_{(m)})}$$
$$+\left(\frac{\mu_1}{\mu}-1\right)\sum_{m\neq k}\left(\frac{r^2}{z-a_k}+\overline{a_k}-\frac{r^2}{z-a_m}+\overline{a_m}\right)$$
$$\left[\left(\overline{\Phi_m(z^*_{(m)})}\right)'-\overline{\varphi_m'(a_m)}\right]$$
$$+\frac{\mu_1}{\mu}(1+\kappa)B_0\left(\frac{r^2}{z-a_k}+\overline{a_k}\right)$$
$$+\frac{\mu_1}{\mu}(1+\kappa)\Gamma_0z+\omega(z),\ |z-a_k|\leq r,\ k=1,2,\ldots,n, \tag{5}$$

where

$$\omega(z)=\sum_{k=1}^{n}\frac{r^2q_k}{z-a_k}+q_0, \tag{6}$$

q_0 is a constant and

$$q_k=\varphi_k'(a_k)\left((\kappa-1)\frac{\mu_1}{\mu}-(\kappa_1-1)\right)-\overline{\varphi_k'(a_k)}\left(\frac{\mu_1}{\mu}-1\right),\ k=1,2,\ldots,n. \tag{7}$$

The unknown functions $\varphi_k(z)$ and $\Phi_k(z)$ $(k=1,2,\ldots,n)$ are related by $2n$ Eqs. (4) and (5).

Introduce the Banach space $\mathcal{H}^{(2,2)}\left(\bigcup_{j=1}^{n}D_k\right)$ as the space of functions f of the form $f(z)=f_k(z)$, $z\in D_k$, analytic in $\bigcup_{j=1}^{n}D_k$, endowed with the norm

$$\|f\|^2_{\mathcal{H}^{(2,2)}}:=\sum_{j=1}^{n}\left(\sup_{0<r<r_k}\int_0^{2\pi}|f_j\left(re^{i\theta}+a_j\right)|^2d\theta+\right.$$
$$\left.\sup_{0<r<r_k}\int_0^{2\pi}|f_j'\left(re^{i\theta}+a_j\right)|^2d\theta+\sup_{0<r<r_k}\int_0^{2\pi}|f_j''\left(re^{i\theta}+a_j\right)|^2d\theta\right).$$

The functional equations contain compositions of $\varphi_k(z)$ and $\Phi_k(z)$ with inversions which define compact operators in the Banach space $\mathcal{H}^{(2,2)}\left(\bigcup_{j=1}^n D_k\right)$. Hence, the functional equations (4) and (5) can be effectively solved by use of the symbolic computations. After their solution $\varphi(z)$ and $\psi(z)$ can be found

$$\frac{\mu_1}{\mu}(1+\kappa)\,\varphi(z)=\left(\frac{\mu_1}{\mu}-1\right)\sum_{m=1}^{n}\left[\overline{\Phi_m(z^*_{(m)})}-(z-a_m)\,\overline{\varphi'_k(a_k)}\right]+p_0,\ z\in D, \tag{8}$$

$$\begin{aligned}\frac{\mu_1}{\mu}(1+\kappa)\,\psi(z)&=\omega(z)-\left(\frac{\mu_1}{\mu}-1\right)\sum_{m=1}^{n}\left(\frac{r^2}{z-a_m}+\overline{a_m}\right)\left[\left(\overline{\Phi_m(z^*_{(m)})}\right)'-\overline{\varphi'_m(a_m)}\right]\\&\quad+\left(\kappa\frac{\mu_1}{\mu}-\kappa_1\right)\sum_{m=1}^{n}\overline{\varphi_m(z^*_{(m)})},\ z\in D.\end{aligned} \tag{9}$$

Theorem 3.1 ([3]) *For sufficiently small coefficients μ, μ_k, κ and κ_k ($k=1,\ldots,n$) the method of successive approximations applied to* (4) *and* (5) *converges in* $\mathcal{H}^{(2,2)}\left(\bigcup_{j=1}^n D_k\right)\times\mathcal{H}^{(2,2)}\left(\bigcup_{j=1}^n D_k\right)$.

For instance, the zero approximation has the form

$$\varphi_k^{(0)}(z)=-\frac{1-\frac{\mu}{\mu_k}}{1+\frac{\mu}{\mu_k}\kappa_k}\frac{(\kappa+1)}{2-\frac{\mu}{\mu_k}+\frac{\mu}{\mu_k}\kappa_k}B_0\,(z-a_k)+(\kappa+1)B_0z+p_0, \tag{10}$$

$$\psi_k^{(0)}(z)=(1+\kappa)\Gamma_0z+q_0,\ |z-a_k|\le r,\ k=1,2,\ldots,n. \tag{11}$$

Let the approximation of the order $(p-1)$ be known. Then the p-th approximation for $\varphi_k^{(p)}(z)$ has the form

$$\begin{aligned}\varphi_k^{(p)}(z)&=\left(1+\frac{\mu}{\mu_k}\kappa_k\right)^{-1}\\&\quad\times\left(\sum_{m\neq k}\left(1-\frac{\mu}{\mu_m}\right)\left[\overline{\Phi_m^{(p-1)}(z^*_{(m)})}-(z-a_m)\,\overline{\left(\varphi_m^{(p-1)}\right)'(a_m)}\right]\right.\\&\quad\left.-\frac{\left(1-\frac{\mu}{\mu_k}\right)(\kappa+1)B_0}{2-\frac{\mu}{\mu_k}+\frac{\mu}{\mu_k}\kappa_k}(z-a_k)+(\kappa+1)B_0z+p_0\right).\end{aligned} \tag{12}$$

Analogous formulae can be written for $\psi_k^{(p)}(z)$.

An alternative method is to solve Eqs. (4) and (5) by series with undetermined coefficients proposed in [11]. We are looking for the analytic potentials φ_k and ψ_k

in the form of the series in r^2

$$\varphi_k(z) = \sum_{j=0}^{\infty}\sum_{s=0}^{\infty} \alpha_{k,j}^{(s)} r^{2s} (z-a_k)^j, \;\; \psi_k(z) = \sum_{j=0}^{\infty}\sum_{s=0}^{\infty} \beta_{k,j}^{(s)} r^{2s} (z-a_k)^j. \tag{13}$$

Selecting the terms with the same powers $(z-a_k)^j$ and r^{2s} we arrive at an iterative method to find $\alpha_{k,j}^{(p)}$ and $\beta_{k,j}^{(p)}$. When the coefficients are determined, the functions (13), hence (8) and (9), can be approximately constructed. The stress and deformation tensors can be calculated by the Kolosov–Muskhelishvili formulae [16]. Then, the effective elastic moduli of macroscopically isotropic fibrous composites can be calculated as follows:

$$\mu_{\text{eff}} = \frac{\langle \sigma_{xx} - \sigma_{yy} \rangle}{2\langle \varepsilon_{xx} - \varepsilon_{yy} \rangle}, \;\; k_{\text{eff}} = \frac{\langle \sigma_{xx} + \sigma_{yy} \rangle}{2\langle \varepsilon_{xx} + \varepsilon_{yy} \rangle}. \tag{14}$$

Here, the limit average over the plane is introduced

$$\langle A \rangle = \lim_{Q_n \to \infty} \frac{1}{|Q_n|} \iint\limits_{Q_n} A \; dxdy.$$

In the latter limit, it is assumed that the infinitely many points a_k are distributed in the plane. After long symbolic computations we get

$$\begin{aligned}
\mu_{\text{eff}} = \mu &- \frac{(\kappa+1)\mu\,(\mu-\mu_1)}{\kappa\mu_1+\mu} f \\
&+ \Bigg(-\frac{2B_0(\kappa+1)\mu(\mu-\mu_1)(-\kappa\mu_1+(\kappa_1-1)\mu+\mu_1)}{\pi\Gamma_0(\kappa\mu_1+\mu)((\kappa_1-1)\mu+2\mu_1)} \\
&+ \frac{2e_3^{(1)}(\kappa+1)\mu\overline{\Gamma_0}(\mu-\mu_1)^2}{\pi\Gamma_0(\kappa\mu_1+\mu)^2} + \frac{\kappa(\kappa+1)\mu(\mu-\mu_1)^2}{(\kappa\mu_1+\mu)^2} \Bigg) f^2 + O(f^3)
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
k_{\text{eff}} = 7k &+ \frac{k(\kappa+1)\mu\,(-\kappa_1\mu+(\kappa-1)\mu_1+\mu)}{(\kappa-1)\,((\kappa_1-1)\,\mu+2\mu_1)} f \\
&+ \Bigg(\frac{e_2^{(0)}k(\kappa+1)\,(\mu-\mu_1)\,\overline{\Gamma_0}\,(\kappa_1\mu+\mu_1)\,(-\kappa_1\mu+\kappa\mu_1+\mu-\mu_1)}{\pi B_0(\kappa-1)\,(\kappa_1+1)\,(\kappa_1\mu-\mu+2\mu_1)\,(\kappa\mu_1+\mu)} \\
&+ \frac{\overline{e_2^{(0)}}k\Gamma_0(\kappa+1)\,(\mu-\mu_1)^2\,(-\kappa_1\mu+(\kappa-1)\mu_1+\mu)}{\pi B_0(\kappa-1)\,(\kappa_1+1)\,((\kappa_1-1)\,\mu+2\mu_1)\,(\kappa\mu_1+\mu)}
\end{aligned}$$

$$- \left. \frac{2k(\kappa+1)\mu\left(-\kappa_1\mu+(\kappa-1)\mu_1+\mu\right)\left((\kappa_1-1)\mu-\kappa\mu_1+\mu_1\right)}{(\kappa-1)^2\left((\kappa_1-1)\mu+2\mu_1\right)^2}\right) f^2$$

$$+O(f^3), \tag{16}$$

where

$$e_2^{(0)} = \lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}\sum_{m\neq k}\frac{1}{(a_k-a_m)^2},\ e_3^{(1)} = \lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}\sum_{m\neq k}\frac{\overline{a_k-a_m}}{(a_k-a_m)^3} \tag{17}$$

are the generalized Eisenstein and Natanzon series [2, 3, 10, 17] which depend on the values of B_0 and Γ_0. The Eisenstein summation must be applied for the conditionally convergent sums (17). Take $B_0 = 0$, $\Gamma_0 = i$ in (15) and $B_0 = 1$, $\Gamma_0 = 0$ in (16). Then, we get $e_2^{(0)} = \pi$ (see discussion in [10, 15]). It is a generalization of the Rayleigh formula [18] for a regular array. Numerical simulations suggest that $e_3^{(1)} = \frac{\pi}{2}$. A rigorous proof of the latter formula has been unknown.

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An Example of Algebraic Symplectic Reduction for the Additive Group

Victoria Hoskins

Abstract We give an example of a linear action of the additive group on an affine algebraic variety arising in the construction of an algebraic symplectic reduction, with non-finitely generated invariant ring.

Keywords Algebraic symplectic reduction • Geometric invariant theory • Non-reductive group actions

Mathematics Subject Classification (2010) Primary 14L24, 53D20; Secondary 13A50

1 Introduction

Geometric invariant theory (GIT) gives a method for constructing quotients of reductive group actions in algebraic geometry [5]; the reductivity of the group ensures rings of invariants are finitely generated. The invariant theory of the simplest non-reductive group, the additive group $\mathbb{G}_a = \mathbb{C}^+$, has been extensively studied. Weitzenböck [8] showed any linear $\mathbb{G}_a$-action on $\mathbb{A}^n$ has finitely generated ring of invariants. Zariski proved any $\mathbb{G}_a$-action on a normal affine variety of dimension at most three has finitely generated invariant ring [9]. In this note, we give a linear $\mathbb{G}_a$-action on a non-normal reducible affine variety of dimension 3, whose ring of invariants is non-finitely generated.

The example arises in the construction of an algebraic symplectic reduction for $\mathbb{G}_a$. Symplectic reduction is a method for constructing quotients in symplectic geometry and is performed by taking a quotient of a level set of a moment map. For reductive groups, one uses GIT to obtain an algebraic symplectic quotient; for example, hypertoric varieties and Nakajima quiver varieties arise as algebraic symplectic reductions of reductive group actions. In [2], the groundwork is laid

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for combining non-reductive GIT with algebraic symplectic reduction. In this note, we study a $\mathbb{G}_a$-action on an algebraic symplectic variety for which the reduction at all regular values of the moment map is an algebraic symplectic variety and, at the only non-regular value 0, we show the ring of $\mathbb{G}_a$-invariant functions on the moment map zero level set Z is non-finitely generated. For every non-reductive group G, there exists a rational G-representation on a finitely generated algebra for which the ring of invariants is non-finitely generated by a Theorem of Popov. Our example provides such a representation for $\mathbb{G}_a$. As far as the author is aware, this is the first example of a $\mathbb{G}_a$-action on a three-dimensional affine variety Z for which the ring of invariants is non-finitely generated. By the result of Zariski, every $\mathbb{G}_a$-action on a normal affine variety of dimension at most 3 has a finitely generated ring of invariants. In particular, we will see that the moment map zero level set Z is reducible and non-normal.

The outline of this paper is as follows: Section 2 is an overview of algebraic symplectic reduction and Sect. 3 contains all the results and calculations for an action of the additive group.

We use the following conventions: we always work over the complex numbers and by variety, we mean a finite type separated reduced scheme.

2 An Overview of Algebraic Symplectic Reduction

Let (M, ω) be an algebraic symplectic manifold; that is, M is a smooth quasi-projective complex variety and ω is a closed non-degenerate algebraic 2-form on M. Suppose a linear algebraic group G acts symplectically on (M, ω). Then, for $A \in \mathfrak{g} = \text{Lie } G$ and $m \in M$, we let A_m denote the infinitesimal action of A on m:

$$A_m := \frac{d}{dt} \exp(tA) \cdot m|_{t=0}.$$

By varying $m \in M$, we get a vector field on M denoted M_A, called the infinitesimal action of A.

The goal of algebraic symplectic reduction is to construct a quotient of this action which is also an algebraic symplectic manifold by using a moment map for the action.

Definition 2.1 A moment map for a symplectic action of a linear algebraic group G on an algebraic symplectic manifold (M, ω) is a G-equivariant algebraic morphism $\mu : M \to \mathfrak{g}^*$ which lifts the infinitesimal action in the sense that for each $A \in \mathfrak{g}^*$, we have an agreement of 1-forms

$$d\mu_A = \omega(M_A, -)$$

where $\mu_A : M \to \mathbb{C}$ is given by pairing the moment map μ with $A \in \mathfrak{g}$ and M_A is the vector field on M given by the infinitesimal action of A. We say the action of G is Hamiltonian if a moment map exists.

We note that a moment map may not exist and may not be unique. If a moment map exists, then for each $\eta \in \mathfrak{g}^*$, the level set $\mu^{-1}(\eta)$ is an algebraic subvariety of M, which is smooth if and only if η is a regular value of the moment map. Moreover, $\mu^{-1}(\eta)$ is invariant under the action of the stabiliser subgroup $G_\eta \subset G$. For central elements η, we have that $G_\eta = G$ acts on $\mu^{-1}(\eta)$. The idea of algebraic symplectic reduction is to take an algebraic quotient of this action and show that ω descends to a form on this quotient.

GIT enables the construction of algebraic quotients of reductive group actions [5]. Given an action of a reductive group G on a quasi-projective variety X, GIT gives a categorical quotient of the action on an open subset of X using a linearisation $L = (\mathcal{L}, \sigma)$ of the action (that is, a lift σ of the action to a line bundle $\mathcal{L}$ over X). More precisely, the linearisation L determines an open subset $X^{ss}(L) \subset X$ of semistable points and the GIT quotient $X^{ss}(L) \to X//_L G$ is constructed using invariant sections of powers of L. The GIT is a good quotient of the G-action on $X^{ss}(L)$ and restricts to a geometric quotient on the stable locus (for the definitions of the (semi)stable sets and good and geometric quotients, see [6]).

One can always find a linearisation $L = (\mathcal{L}, \sigma)$ of our G-action on M, as M is smooth (for example, see [1, Theorem 7.3]); then this restricts to a linearisation of the G_η-action on $\mu^{-1}(\eta)$, which we also denote by L.

Definition 2.2 For a Hamiltonian action of a reductive group G on an algebraic symplectic manifold (M, ω) with moment map $\mu : M \to \mathfrak{g}^*$ and a linearisation L of the G-action on M, we define the algebraic symplectic reduction at a central value $\eta \in \mathfrak{g}$ to be the GIT quotient $\mu^{-1}(\eta)//_L G$.

By the infinitesimal lifting property of μ, a central value η is a regular value of μ if and only if G acts on $\mu^{-1}(\eta)$ with finite stabiliser groups. In this case, the action is closed and so the GIT quotient of this action is a geometric quotient that has at worst orbifold singularities. If G acts on $\mu^{-1}(\eta)^{ss}(L)$ freely, then it follows from Luna's étale slice Theorem that $\mu^{-1}(\eta)//_L G$ is smooth and the form descends to a symplectic form on this GIT quotient by the Marsden–Weinstein Theorem [4].

For a non-reductive group G, one can use techniques of non-reductive geometric invariant theory to construct a quotient of the action [3]. The main difference of the non-reductive theory is that invariant rings may not be finitely generated.

3 The Example

Let us consider the simplest non-trivial representation of the additive group $\mathbb{G}_a$: we let $\mathbb{G}_a$ act linearly on $V = \mathbb{A}^2$ by

$$s \cdot (z_1, z_2) = (z_1 + s z_2, z_2)$$

where $s \in \mathbb{G}_a$ and $(z_1, z_2) \in \mathbb{A}^2$. If we identify $T^*V \cong V \times V^*$ and take dual coordinates (α_1, α_2) on V^*, then the cotangent lift of the action is the linear action given by

$$s \cdot (z_1, z_2, \alpha_1, \alpha_2) = (z_1 + sz_2, z_2, \alpha_1, \alpha_2 - s\alpha_1).$$

The cotangent bundle T^*V has a natural algebraic symplectic Liouville form

$$\omega = dz_1 \wedge d\alpha_1 + dz_2 \wedge d\alpha_2.$$

By Weitzenböck's Theorem [8], the ring of $\mathbb{G}_a$-invariant functions on the cotangent bundle is finitely generated. To compute the ring of $\mathbb{G}_a$-invariants, we use an algorithm of van den Essen [7]. Given a rational $\mathbb{G}_a$-action on a finitely generated algebra A, we let D denote the derivation of the $\mathbb{G}_a$-action; then the ring of invariants is the kernel of D. The algorithm works by picking $a \in A$ such that $b := D(a) \neq 0$ and $D(b) = 0$, then in the localisation A_b, we see that $D(a/b) = 1$ and so the ring of $\mathbb{G}_a$-invariants in A_b is finitely generated by van den Essen [7, Proposition 2.1]. In fact, if $x_1, \ldots, x_n$ are generators of A, then $(-a/b) \cdot x_1, \ldots, (-a/b) \cdot x_n, y^{\pm 1}$ are generators of $(A_b)^{\mathbb{G}_a}$. Then

$$A^{\mathbb{G}_a} = A \cap (A_b)^{\mathbb{G}_a}$$

and a sequence of generators for $A^{\mathbb{G}_a}$ can be obtained by following the algorithm described in loc. cit §3–5.

We claim that by applying this algorithm, we have

$$\mathbb{C}[T^*V]^{\mathbb{G}_a} = \mathbb{C}[z_2, \alpha_1, z_1\alpha_1 + z_2\alpha_2].$$

Indeed, we can take $a = z_1$ and $b = z_2$; then

$$(\mathbb{C}[T^*V]_{z_2})^{\mathbb{G}_a} = \mathbb{C}[z_2^{\pm 1}, \alpha_1, \alpha_2 + \alpha_1 z_1/z_2]$$

and so the result follows from [7, Lemma 3.1].

Proposition 3.1 *This $\mathbb{G}_a$-action on the cotangent bundle T^*V with its Liouville symplectic form is Hamiltonian with moment map $\mu : T^*V \to \mathbb{C}$ given by*

$$\mu(z_1, z_2, \alpha_1, \alpha_2) = z_2\alpha_1.$$

Furthermore, $\eta \in \mathbb{C}$ is a regular value of the moment map if and only if $\eta \neq 0$.

Proof The infinitesimal action of $A \in \mathrm{Lie}\,(\mathbb{G}_a) = \mathbb{C}$ on $(z, \alpha) \in T^*V$ is given by

$$(A_z, A_\alpha) = (Az_2, 0, 0, -A\alpha_1).$$

For $p = (z_1, z_2, \alpha_1, \alpha_2) \in T^*V$ and $p' = (z_1', z_2', \alpha_1', \alpha_2') \in T_p(T^*V) \cong T^*V$, we have

$$d_p\mu(p') = (z_2\alpha_1' + z_2'\alpha_1) = \omega(1_p, p').$$

Moreover μ is $\mathbb{G}_a$-equivariant, as the coadjoint action of $\mathbb{G}_a$ is trivial and μ is clearly $\mathbb{G}_a$-invariant. Hence, μ is a moment map.

Furthermore, $d_p\mu$ is surjective if and only if $z_2 \neq 0$ or $\alpha \neq 0$. Therefore, $\eta \in \mathbb{C}$ is a regular value of μ if and only if $\eta \neq 0$. □

To construct an algebraic symplectic reduction, we can use a linearisation of the $\mathbb{G}_a$-action on T^*V. Since T^*V is an affine space, the only line bundle on T^*V is the trivial line bundle. In fact, as the character group of $\mathbb{G}_a$ is trivial, the only linearisation of the action is the trivial one (see [1, Theorem 7.2]). The trivial linearisation on T^*V corresponds to the non-reductive GIT quotient

$$T^*V//\mathbb{G}_a := \operatorname{Spec} \mathbb{C}[T^*V]^{\mathbb{G}_a} \cong \mathbb{A}^3.$$

We first construct the algebraic symplectic reduction at a regular value of the moment map $\eta \in \mathbb{C}^*$. The following result shows that there is a categorical quotient of the $\mathbb{G}_a$-action on $\mu^{-1}(\eta)$; in fact, we show the quotient is a trivial $\mathbb{G}_a$-torsor, which, in particular, is a geometric quotient.

Proposition 3.2 *For $\eta \in \mathbb{C}^*$, the level set $\mu^{-1}(\eta) \subset T^*V$ is a smooth quadric hypersurface and the $\mathbb{G}_a$-action on $\mu^{-1}(\eta)$ is free. The ring of $\mathbb{G}_a$-invariant functions on $\mu^{-1}(\eta)$ is finitely generated and, moreover,*

$$\pi : \mu^{-1}(\eta) \to \mu^{-1}(\eta)//\mathbb{G}_a := \operatorname{Spec} \mathbb{C}[\mu^{-1}(\eta)]^{\mathbb{G}_a}$$

is a trivial $\mathbb{G}_a$-torsor. Hence, the quotient $\mu^{-1}(\eta) \to \mu^{-1}(\eta)//\mathbb{G}_a$ is a geometric quotient and $\mu^{-1}(\eta)//\mathbb{G}_a$ is a smooth affine variety which is naturally algebraic symplectic.

Proof The first statement is a simple computation. For the second statement, we observe that $D(z_1\alpha_1/\eta) = 1$ on $\mu^{-1}(\eta)$, where D denotes the derivation of the $\mathbb{G}_a$-action. Hence, by van den Essen [7, Proposition 2.1], the ring of $\mathbb{G}_a$-invariants on $\mu^{-1}(\eta)$ is finitely generated and we have

$$\mathbb{C}[\mu^{-1}(\eta)]^{\mathbb{G}_a} = \mathbb{C}[z_2, \alpha_1, z_1\alpha_1 + z_2\alpha_2]/(z_2\alpha_1 - \eta) = \mathbb{C}[z_2^{\pm 1}, \eta z_1/z_2 + z_2\alpha_2].$$

There is a $\mathbb{G}_a$-equivariant morphism from $\mu^{-1}(\eta)$ to the trivial $\mathbb{G}_a$-torsor over $\mu^{-1}(\eta)//\mathbb{G}_a$ given by

$$f : \mu^{-1}(\eta) \to (\mu^{-1}(\eta)//\mathbb{G}_a) \times \mathbb{A}^1, \quad p = (z_1, z_2, \alpha_1, \alpha_2) \mapsto (\pi(p), z_1\alpha_1/\eta),$$

which we claim is an isomorphism. First f is injective: if

$$f(z_1, z_2, \alpha_1, \alpha_2) = f(z_1', z_2', \alpha_1', \alpha_2'),$$

then $z_2 = z_2'$ and $\alpha_1 = \alpha_1'$ and, as $z_2\alpha_1 = \eta \neq 0$, we have $z_1 = z_1'$ and $\alpha_2 = \alpha_2'$. Let (q_1, q_2) be coordinates on $\mu^{-1}(\eta)//\mathbb{G}_a \cong \mathbb{A}^1 - \{0\} \times \mathbb{A}^1$ and r be the coordinate on $\mathbb{A}^1$. For surjectivity, if $(q_1, q_2, r) \in \mu^{-1}(\eta)//\mathbb{G}_a \times \mathbb{A}^1$, then $f(rq_1, q_1, \eta/q_1, (q_2 - \eta r)/q_1) = (q_1, q_2, r)$. Therefore, by Zariski's Main Theorem, f is an isomorphism.

Since the trivial $\mathbb{G}_a$-torsor is a geometric quotient, so is π. Furthermore, as π is a trivial $\mathbb{G}_a$-torsor and $\mu^{-1}(\eta)$ is smooth, so is $\mu^{-1}(\eta)//\mathbb{G}_a$, and there is a short exact sequence

$$0 \to T_p(G \cdot p) \to T_p(\mu^{-1}(\eta) \to T_{\pi(p)}(\mu^{-1}(\eta)//\mathbb{G}_a) \to 0,$$

for each $p \in \mu^{-1}(\eta)$. One can then follow the arguments of the Marsden–Weinstein Theorem [4] to prove that $T_p(\mu^{-1}(\eta)) = T_p(G \cdot p)^\omega$, i.e. $T_p(G \cdot p)$ is isotropic, and so there is an induced symplectic form on the quotient. □

A straight forward calculation gives a description of the singular locus of the zero level set as the $\mathbb{G}_a$-fixed locus:

$$\operatorname{Sing} \mu^{-1}(0) = \{(z_1, z_2, \alpha_1, \alpha_2) : z_2 = \alpha_1 = 0\} = \mu^{-1}(0)^{\mathbb{G}_a}.$$

Since the singular locus has codimension 1, the zero level set $\mu^{-1}(0)$ is not normal. The zero level set has two irreducible components: $\{z_2 = 0\}$ and $\{\alpha_1 = 0\}$, which are normal, and so the normalisation of $\mu^{-1}(0)$ is the disjoint union of these two irreducible components.

Proposition 3.3 *The ring of $\mathbb{G}_a$-invariant functions on $\mu^{-1}(0)$ is non-finitely generated.*

Proof We claim that for $n \geq 0$, the functions

$$z_1^n\alpha_1 \quad \text{and} \quad z_2\alpha_2^n$$

are $\mathbb{G}_a$-invariant functions on $\mu^{-1}(0)$. Indeed, for $s \in \mathbb{G}_a$,

$$\begin{aligned} s \cdot z_1^n\alpha_1 &= (z_1 + sz_2)^n\alpha_1 = (z_1^n + sz_1^{n-1}z_2 + \cdots z_2^n)\alpha_1 \\ &= z_1^n\alpha_1 + (z_2\alpha_1)(sz_1^{n-2} + \cdots + z_2^{n-1}) \end{aligned}$$

and similarly $z_2\alpha_2^n$ is $\mathbb{G}_a$-invariant on $\mu^{-1}(0)$. However, the functions z_1 and α_2 are not $\mathbb{G}_a$-invariant. In particular, for each n, the invariant functions $z_1^n\alpha_1$ and $z_2\alpha_2^n$ are not polynomials in the invariant functions $z_1^m\alpha_1$ and $z_2\alpha_2^m$ for $m < n$. In other words, each of these invariants is a new generator of $\mathbb{C}[\mu^{-1}(0)]^{\mathbb{G}_a}$ and so this ring is non-finitely generated. □

In this example, we also see that taking $\mathbb{G}_a$-invariants is not right exact, as the homomorphism of invariant rings

$$\mathbb{C}[T^*V]^{\mathbb{G}_a} \to \mathbb{C}[\mu^{-1}(0)]^{\mathbb{G}_a}$$

is not surjective: the invariant functions $z_1^n\alpha_1$ and $z_2\alpha_2^n$ on $\mu^{-1}(0)$ do not extend to invariant functions on T^*V.

Furthermore, one cannot construct a quotient by taking the spectrum of the ring of invariants, as this is not finitely generated. Instead, one should use the techniques of non-reductive GIT developed by Doran and Kirwan [3]. This idea is pursued in a joint work with Doran [2], where in a general set-up, we use notions of stability and a Hilbert–Mumford criterion to determine an open subset of the zero level set of the moment map which admits an algebraic symplectic quotient. More precisely, the open subset is the completely stable locus in the sense of [3, Definition 5.2.11], which can be determined by extending the linear $\mathbb{G}_a$-action to SL_2 and using the Hilbert–Mumford criterion.

Let us outline how this works in our example. For the $\mathbb{G}_a$-action on $\mu^{-1}(0)$, the completely stable locus $\mu^{-1}(0)^{\bar{s}}$ is the largest open set possible, namely the complement to the $\mathbb{G}_a$-fixed locus:

$$\mu^{-1}(0)^{\bar{s}} := \{(z_1, z_2, \alpha_1, \alpha_2) \in \mu^{-1}(0) : z_2 \neq 0 \text{ or } \alpha_1 \neq 0\},$$

which coincides with the smooth locus of $\mu^{-1}(0)$. In general, the completely stable locus is a proper subset of the complement to the $\mathbb{G}_a$-fixed locus. It follows from [3, Theorem 5.3.1] that the completely stable locus admits a Zariski locally trivial geometric quotient. In our example, the completely stable locus is the union of two connected components, each of which is isomorphic to $\mathbb{A}^2 \times (\mathbb{A}^1 - \{0\})$ and has a geometric quotient which is isomorphic to $\mathbb{A}^1 \times (\mathbb{A}^1 - \{0\})$ (in fact, these quotients are both trivial $\mathbb{G}_a$-torsors). Furthermore, there is an induced algebraic symplectic form on the geometric quotient $\mu^{-1}(0)^{\bar{s}}/\mathbb{G}_a$ which is constructed as in the proof of Proposition 3.2.

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Submanifolds with Splitting Tangent Sequence

Priska Jahnke and Ivo Radloff

Abstract A theorem of Van de Ven states that a projective submanifold of complex projective space whose holomorphic tangent bundle sequence splits holomorphically is necessarily a linear subspace. Note that the sequence always splits differentiably but in general not holomorphically. We are interested in generalizations to the case, when the ambient space is a homogeneous manifold different from projective space: quadrics, Grassmannians or abelian manifolds, for example. Split submanifolds are closely related to totally geodesic submanifolds.

Keywords Complex homogeneous spaces • Totally geodesic submanifolds

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1 Introduction

Let M be a complex projective manifold and $X \subset M$ a complex compact submanifold. Denote the holomorphic tangent bundle of M by T_M and the normal bundle of X by $N_{X/M}$.

Definition 1.1 The submanifold $X \subset M$ has *splitting tangent sequence*, iff

$$0 \longrightarrow T_X \longrightarrow T_M|_X \xrightarrow{\alpha} N_{X/M} \longrightarrow 0 \tag{1}$$

splits holomorphically, i.e. $\exists \beta : N_{X/M} \to T_M|_X$ such that $\alpha \circ \beta = \mathrm{id}_{N_{X/M}}$.

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More generally, assume $f : X \to M$ be a holomorphic immersion from some compact complex manifold X to M. Then we call f *split*, if the natural sequence

$$0 \longrightarrow T_X \longrightarrow f^* T_M \longrightarrow N_f \longrightarrow 0$$

admits a holomorphic splitting.

Remark 1.2 Let

$$0 \longrightarrow S \longrightarrow E \longrightarrow Q \longrightarrow 0$$

be any short exact sequence of vector bundles on a complex manifold X. The (holomorphic) splitting obstruction is the image

$$[\delta(\mathrm{id}_Q)] \in \mathrm{Ext}^1(Q, S),$$

where $\delta : \mathrm{Hom}(Q, Q) \longrightarrow \mathrm{Ext}^1(Q, S)$ is the first connecting morphism. In particular, (1) splits if $h^1(X, T_X \otimes N^*_{X/M}) = 0$ (h^1-criterion).

Now fix a hermitian metric on E. Then the corresponding connection gives rise to the *second fundamental form* $B \in \mathcal{A}^{0,1}(\mathrm{Hom(Q, S)})$, whose class

$$[B] = [\delta(\mathrm{id}_Q)] \in \mathrm{Ext}^1(Q, S)$$

coincides with the splitting obstruction. Hence the vanishing of the second fundamental form implies the splitting of the sequence.

2 Projective Space and Abelian Manifolds

Example Consider first M a projective space. That a submanifold X has splitting tangent sequence depends not only on the geometry of X, but also on the embedding:

1. Let $\mathbb{P}_m \subset \mathbb{P}_n$ be a linear subspace. Then

$$H^1(\mathbb{P}_m, T_{\mathbb{P}_m} \otimes N_{\mathbb{P}_m/\mathbb{P}_n}) = \bigoplus_{n-m} H^1(\mathbb{P}_m, T_{\mathbb{P}_m}(-1)) = 0.$$

   Hence the tangent sequence splits by the h^1-criterion.
2. Let $C \subset \mathbb{P}_2$ be a conic. Then $T_C = \mathcal{O}(2)$ and $N_{C/\mathbb{P}_2} = \mathcal{O}(4)$, where $\mathcal{O}(1)$ denotes the tautological hyperplane bundle on $C \simeq \mathbb{P}_1$. The tangent bundle of $\mathbb{P}_2$ splits on C as a direct sum $T_{\mathbb{P}_2}|_C = \mathcal{O}(a) \oplus \mathcal{O}(b)$ for some integers a, b. Counting degrees in the tangent sequence we find $a + b = 6$. On the other hand, the dual Euler sequence of $\mathbb{P}_2$ restricted to C remains surjective on H^0-level, implying

$H^0(C, \Omega^1_{\mathbb{P}_2}(1)|_C) = 0$. This means $a, b \leq 3$, hence $a = b = 3$ and the tangent sequence does not split.

In general, the case M a projective space was first studied by A. Van de Ven in 1958:

Theorem 2.1 (Van de Ven [14]) *A submanifold $X \subset \mathbb{P}_n$ has splitting tangent sequence if and only if $X \simeq \mathbb{P}_m$ is a linear subspace.*

There can be found several different proofs of this result in the literature, compare, for example, [6, 11, 13] or [3]. One idea is the following:

Sketch of Proof: For the splitting of linear subspaces of $\mathbb{P}_n$ see the above example. Assume, on the other hand, $X \subset \mathbb{P}_n$ is split, $\dim(X) = m \geq 1$. Then the natural surjective map

$$T_M(-1) \longrightarrow T_X(-1) \longrightarrow 0$$

implies $T_X(-1)$ is generated by global sections, where $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}_n}(1)|_X$ is the restriction of the hyperplane bundle. Then by Wahl's Theorem (or use Mori's Theorem or projective connections)

$$(X, \mathcal{O}_X(1)) \simeq (\mathbb{P}_m, \mathcal{O}_{\mathbb{P}_m}(1)) \text{ or } (\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}(2)).$$

Now we are done since the tangent sequence of a conic in $\mathbb{P}_2$ does not split as seen above. □

For several reasons, another interesting class for M are abelian varieties. Here we can show:

Theorem 2.2 (Jahnke [3]) *Let M be an abelian variety. Then a submanifold $X \subset M$ has splitting tangent sequence if and only if X is abelian.*

The proof relies on the fact that X is abelian if and only if T_X is trivial, and on a general result on vector bundles: any globally generated vector bundle with trivial determinant must be trivial.

The vanishing of the second fundamental form was studied in a more general situation by N. Mok. In general this should be a stronger condition than just the holomorphic splitting, but it turns out that at least for projective spaces and abelian varieties both notions coincide. Adopted to our situation here, Mok proves in his paper:

Theorem 2.3 (Mok [12]) *Let M be a projective space, abelian or a ball quotient and $f : X \to M$ a holomorphic immersion, where X is a compact complex manifold. Then f is split, if and only if the second fundamental form vanishes, or equivalently, if $f : X \to M$ is totally geodesic.*

We may ask for which further types of M the splitting of a submanifold already implies the vanishing of the second fundamental form.

3 Homogeneous Manifolds

Besides projective spaces and abelian manifolds we find homogeneous manifolds a suitable category to consider for the following reasons: if $X \subset M$ is split, the splitting morphism induces a surjective map

$$T_M \longrightarrow T_X.$$

This means X inherits several positivity properties of M:

- if M is homogeneous, then also X is,
- if T_M is ample, 1-ample etc., then also T_X is,
- if M admits a projective connection, then also X does [5], etc.

Moreover, by a theorem of Borel and Remmert, any homogeneous projective manifold M splits into a product

$$M \simeq A \times G/P,$$

where A is abelian and G/P a rational homogeneous space.

We now first collect some general facts on splitting submanifolds (see [3]):

Lemma 3.1 *1.) Let $X \subset Y \subset M$ be projective manifolds. Then*

1. *if $X \subset M$ is split, then $X \subset Y$ is split,*
2. *if $X \subset Y$ and $Y \subset M$ are split, then $X \subset M$ is split.*

2.) If $M \simeq Y \times Z$, then all fibers of the projections are split in M.

Proposition 3.2 *Let $M \simeq Y \times Z$ and $f : X \to Y$ a split holomorphic immersion (resp. embedding). Let $g : X \to Z$ be any map. Then $(f \times g) : X \to M$ is a split immersion (resp. embedding).*

Proof Denote the projections of M by π_Y and π_Z, respectively. Then we have $T_M \simeq \pi_Y^* T_Y \oplus \pi^* T_Z$. Hence

$$(f \times g)^* T_M \simeq f^* T_Y \oplus g^* T_Z.$$

Since f is split by assumption, there exists a holomorphic splitting morphism $\beta : f^* T_Y \to T_X$, such that $\beta \circ T_f = \mathrm{id}_{T_X}$, where $T_f : T_X \to T_Y$ denotes the tangent map of f. Then

$$\gamma : \begin{cases} (f \times g)^* T_M \simeq f^* T_Y \oplus g^* T_Z \longrightarrow T_X \\ \qquad\qquad\qquad (v, w) \qquad \mapsto \ \beta(v) \end{cases}$$

is a splitting morphism for $f \times g$. $\square$

Example Using this result we can construct some interesting examples:

1. Assume $M \simeq A \times G/P$ homogeneous. Take some split submanifold $X \subset G/P$ and define $g \equiv 0$. Then $X \hookrightarrow M$ is split.
2. Again for $M \simeq A \times G/P$ now assume there exists an elliptic curve $E \subset A$. Then $f : E \hookrightarrow A$ is split. Pick some rational curve $C \subset G/P$ and take $g : E \xrightarrow{2:1} \mathbb{P}_1 \simeq C$ a double cover. Then $(f \times g)(E) \subset M$ is split.
3. Consider $M \simeq \mathbb{P}_1 \times \mathbb{P}_1, f = \mathrm{id}_{\mathbb{P}_1}$ on the first factor and for some $d \geq 1$

$$g : \begin{cases} \mathbb{P}_1 & \longrightarrow & \mathbb{P}_1 \\ [s:t] & \mapsto & [s^d : t^d] \end{cases}$$

   on the second factor. Then $C = (f \times g)(\mathbb{P}_1) \subset M$ is a rational curve of bidegree $(1, d)$, which is split for any d by Proposition 3.2.

We conclude that we should first understand splitting submanifolds of the rational homogeneous part G/P. The projective space is well known, for some further cases there is a classification in [3], we sum up the results:

Theorem 3.3 (Jahnke [3])

1.) Assume X is a submanifold of the n-dimensional quadric $M = Q_n$, $n \geq 2$. Then

(i) *If* $\dim(X) \geq 2$, *then X is split if and only if X is a complete intersection subquadric or a linear subspace.*
(ii) *If* $\dim(X) = 1$ *and X is split, then $X \simeq \mathbb{P}_1$ is a rational curve.*

2.) If M is a Grassmannian, then any split submanifold is again rational homogeneous.

Remark 3.4 Concerning rational curves $C \subset Q_n$:

1. Assume $C \subset Q_2 \subset Q_n$ is a curve of bidegree $(1, d)$ in some 2-dimensional complete intersection subquadric of Q_n. Then C is split in Q_2, hence also split in Q_n.
2. Assume $n \geq 6$ and $L \simeq \mathbb{P}_3 \subset Q : n$ a linear subspace. Then L is split in Q_n. Take C a rational curve of degree $d \geq 2$ in L. Then C is non-split in L, hence also non-split in Q_n.

We now compare this result with the list of all submanifolds with vanishing second fundamental form:

Theorem 3.5 (Chen/Nagano, Klein [1, 7]) *Let $X \subset Q_n$ a submanifold with vanishing second fundamental form. Then X is a complete intersection subquadric or a linear subspace.*

This means for $\dim(X) \geq 2$ split submanifolds of Q_n are exactly those with vanishing second fundamental form. But for curves this is not the case anymore: rational curves in Q_n with vanishing second fundamental form are lines or conics. The curves of degree $d \geq 3$ lying in some complete intersection subquadric constructed above have splitting tangent sequence, but non-vanishing second fundamental form.

4 Manifolds with Holomorphic Projective Connection

Another interesting case are manifolds admitting a holomorphic projective connection for the following reason:

Proposition 4.1 (Jahnke/Radloff [5]) *Let M be compact Kähler carrying a holomorphic projective connection. Let $\iota : N \hookrightarrow M$ be a compact submanifold that splits in M. Then N carries a holomorphic projective connection.*

Here, a manifold M is said to carry a *holomorphic normal projective connection* if the (normalized) Atiyah class of the holomorphic cotangent bundle has the form

$$a(\Omega^1_M) = \frac{c_1(K_M)}{m+1} \otimes id_{\Omega^1_M} + id_{\Omega^1_M} \otimes \frac{c_1(K_M)}{m+1} \in H^1(M, \Omega^1_M \otimes T_M \otimes \Omega^1_M),$$

where we use $\Omega^1_M \otimes T_M \otimes \Omega^1_M \simeq \Omega^1_M \otimes \mathcal{E}nd(\Omega^1_M) \simeq \mathcal{E}nd(\Omega^1_M) \otimes \Omega^1_M$.

Compact Kähler Einstein manifolds with holomorphic projective connection were studied by Kobayashi and Ochiai in [8], the only such varieties are $\mathbb{P}_n(\mathbb{C})$, finite étale quotients of tori and ball quotients. Here, the connection is always flat, i.e. M admits a projective structure. In the non Kähler Einstein case we found in dimension three exactly one new example, that are families of fake elliptic curves $M \longrightarrow C$ [4]. Again, the connection is flat. This classification generalizes to higher dimensions (now assuming flatness):

Theorem 4.2 (Jahnke/Radloff [6]) *On a projective manifold M_m which is* not *Kähler Einstein, the following conditions are equivalent:*

1. *M carries a flat holomorphic normal projective connection.*
2. *Up to a finite étale covering, M admits an abelian group scheme structure $f : M \longrightarrow C$ over a compact Shimura curve C such that the Arakelov inequality $2 \deg f_* \Omega^1_{M/C} \leq (m-1) \deg K_C = (m-1)(2g_C - 2)$ [2] is an equality.*
3. *Up to a finite étale covering, $M \simeq Z \times_C Z \times_C \cdots \times_C Z$, where $Z \longrightarrow C$ is a Kuga fiber space constructed from the rational corestriction $Cor_{F/\mathbb{Q}}(A)$ of a division quaternion algebra A defined over a totally real number field F such that*

$$A \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_2(\mathbb{R}) \oplus \mathbb{H} \oplus \cdots \oplus \mathbb{H}.$$

Here $\mathbb{H}$ denotes the Hamiltonian quaternions.

That 2. implies 3. is due to Viehweg and Zuo [15], and by construction, a manifold as in 3. admits a projective structure (see [6] for details). For the converse, first show that the existence of a rational curve implies M is the projective space.

Hence, by the cone theorem, we may assume K_M is nef, and also K_F is nef for any smooth submanifold. By results of Aubin and Yau we conclude that if M is also big, then M is a ball quotient, and if $K_M \equiv 0$, then M is a finite étale quotient of a torus. In the remaining case, we now show that M must have *(generically) large fundamental group* and use results of Kollár [9] and Lai [10] to prove that there exists a finite étale cover $M' \longrightarrow M$ which is a good minimal model and, again up to a finte étale cover, M is an abelian group scheme over a base N of general type.

We already studied splitting submanifolds of the projective space and abelian manifolds. The case M a ball quotient is treated by Yeung in [16]. In the remaining case as described in Theorem 4.2 we have

Proposition 4.3 (Jahnke/Radloff [5]) *Let* $\dim(M) = 3$ *and* $\pi : M \longrightarrow C$ *be a modular family of fake elliptic curves. A compact submanifold* N *of* M *of dimension* $0 < \dim N < 3$ *splits in* M *if and only if*

1. *N is an étale multisection of π or*
2. *N is an elliptic curve in a fiber of π.*

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Generalized Solutions to Stochastic Problems as Regularized in a Broad Sense

Irina V. Melnikova

Abstract We consider different types of solutions to abstract stochastic Cauchy problems, especially generalized solutions, as regularized in a broad sense. By the regularization of a problem in a broad sense, we mean construction of proper well-posed problems related to the original one, whose solutions, in distinct from the regularization in the ill-posed problem theory, are not necessarily approximations to a solution of the original problem.

Keywords Distribution • Gelfand–Shilov spaces • Generalized solution • Laplace and Fourier transforms • Semigroup • White noise • Wiener process

Mathematics Subject Classification (2010) 47D06, 46F12, 46F25, 60G20

Models of various evolution processes considered with regard to random perturbations lead to the Cauchy problem for equations with an inhomogeneity in the form of white noise in infinite dimensional spaces. Among them, important for applications is the first-order Cauchy problem

$$X'(t) = AX(t) + F(t, X) + B(t, X)\mathbb{W}(t),\ t \geq 0 \text{ (or } t \in [0, T]), \quad X(0) = \zeta, \qquad (1)$$

where A is the generator of a regularized semigroup in a Hilbert space H, F is a nonlinear map in H, $\mathbb{W} = \{\mathbb{W}(t), t \geq 0\}$ is a white noise process in another Hilbert space $\mathbb{H}$, and $B(t, X) \in L(\mathbb{H}, H)$.

The problem (1) is ill-posed due to several reasons: the generation by A just a regularized semigroup, properties of F and B, and the well-known irregularity property of white noise processes. Because of the irregularity of $\mathbb{W}$, which is not a process in the usual sense, the problem (1), similarly to the finite dimensional case,

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is considered in the integral form :

$$X(t) = \zeta + \int_0^t AX(s)ds + \int_0^t F(s,X)ds + \int_0^t BdW(s), \quad t \geq 0, \tag{2}$$

where $\int_0^t BdW(s)$ is the Itô integral w.r.t. (with respect to) a $\mathbb{H}$-valued Wiener process $\{W(t), t \geq 0\}$, a "primitive" of $\mathbb{W}$, and $W(t) = W(t,\omega), \omega \in (\Omega, \mathcal{F}, P)$ (see, e.g., [2, 3, 7]). The problem (2) is usually written in the short form :

$$dX(t) = AX(t)dt + Fdt + BdW(t), \ \ t \geq 0, \quad X(0) = \zeta. \tag{3}$$

To overcome the issues related to the ill-posedness of the problem, along with weak and mild solutions to (2) we consider

- solutions generalized w.r.t. the spatial variable $x \in \mathbb{R}^n$ for the problem with differential operators $A = A\,(i\partial/\partial x)$ in Gelfand–Shilov spaces.
  For the problem in the differential form (1) we consider
- solutions generalized w.r.t. the time variable $t \in \mathbb{R}$ in spaces of abstract distributions and ultra-distributions, where the spaces are closely related to the type of the semigroup generated by A;
- solutions generalized w.r.t. the random variable $\omega \in \Omega$ in spaces of abstract stochastic distributions.

Each of these solutions allows us to overcome a part of the ill-posedness issues. Further we consider different types of regularization in the broad sense related to each type of solutions. We do not consider here the regularization using methods of ill-posed problem theory; such regularization of stochastic problems needs special investigations (for (1) with $F = B = 0$, see, e.g, [6]). We study the problem using known techniques of abstract (ultra)distribution and Gelfand–Shilov generalized functions, as well as new ones extending the white noise theory to the infinite dimensional case. The solutions mentioned can be found, e.g., in [1, 4, 8].

1 Regularization in the Case of the Problem (2)

If A is the generator of C_0-class solution operators $U(t), t \geq 0$, to the corresponding homogeneous Cauchy problem, the linear problem (with $F = 0$) has a weak solution

$$X(t) = U(t)\zeta + \int_0^t U(t-s)B\,dW(s), \ \ t \geq 0\,, \tag{4}$$

that is a process pathwise satisfying the equation

$$\langle X(t), y\rangle \ = \ \langle \zeta, y\rangle \ + \int_0^t \langle X(s), A^*y\rangle\, ds \ + \int_0^t \langle BdW(s), y\rangle \ \ P_{a.s.}, \ \ t \ \geq \ 0, \tag{5}$$

$y \in \operatorname{dom} A^*$, and in the case $F \neq 0$, a mild solution

$$X(t) = U(t)\zeta + \int_0^t U(t-s)F(s,X)\,ds + \int_0^t U(t-s)B\,dW(s), \qquad t \geq 0. \tag{6}$$

The solutions (4) and (6) can be considered as (weakly) regularized by "test" functions $y \in \operatorname{dom} A^*$ and, as we will see below, *any generalized solution can be considered as regularized by test functions corresponding to the problem.*

Now consider the regularization of solutions, in particular solution operators related to A, the generator of a regularized semigroup: integrated, convoluted, or R-semigroup. A family of bounded operators $\{S(t), t \in [0,\tau)\}, \tau \leq \infty$, satisfying the equations with smooth bounded operators $R(t)$ in H:

$$A\int_0^t S(s)f\,ds = S(t)f - R(t)f,\ f \in H, \qquad S(t)Af = AS(t)f,\ f \in \operatorname{dom} A,$$

is a K-convoluted semigroup S_K in H with the generator A if $R(t) = K(t)I$, in particular an n-times integrated semigroup S_n if $K(t) = \frac{t^n}{n!}$, and $S = S_R$ is an R-semigroup if $R(t) \equiv R$. The semigroup is local if $\tau < \infty$. These semigroups are the following regularizations of solution operators $U(t)$:

$$S_K(t) = (U * K)(t), \quad S_n(t) = \int_0^t U(s)\frac{(t-s)^{n-1}}{(n-1)!}ds, \quad S_R(t) = U(t)R.$$

For A, the generator of a regularized semigroup S, instead of (4) and (6) we have (weak) regularized solutions with S instead of U for corresponding regularized problems. Further we will consider generalized solutions obtained via these regularized ones and additionally regularized by corresponding test functions.

Now mention regularization of the white noise term in the case of (5). Here W is a Q-Wiener process: $W = W_Q(t) = \sum_{i=1}^{\infty}\sigma_i\beta_i(t)e_i, t \geq 0$, where β_i are independent Brownian motions and $\{e_i\}$ is a basis in $\mathbb{H}$ such that $Qe_i = \sigma_i^2 e_i$, $\sum_{i=1}^{\infty}\sigma_i^2 < \infty$. The regularization is due to "integration" of $\mathbb{W}$ and due to σ_i, which distinguish a Q-Wiener process W_Q from a weak Wiener process $W(t) = \sum_{i=1}^{\infty}\beta_i(t)e_i, t \geq 0$, converging just weakly in $\mathbb{H}$. Usage of a Q-Wiener process or a weak Wiener process in a stochastic problem depends on a model that results in the problem. In Sect. 3 we show regularization that allows us to use a weak Wiener process W and corresponding $\mathbb{W} = W'$.

2 Regularization in the Case of Generalized w.r.t. *t* Solutions

Let $\mathcal{D}'(H)$ be the space of abstract (H-valued) distributions, $\mathcal{D}'_{\{M_q\}}(H)$ the space of abstract ultra-distributions, and $\{W_Q(t),\ t \geq 0\}$ be a Q-Wiener process with values in a Hilbert space $\mathbb{H}$: $W_Q(t) = W_Q(t,\omega), \omega \in \Omega;\ W_Q(t,\omega) \in \mathbb{H}, t \geq 0, P_{\text{a.s.}};\ W_Q(t,\cdot) \in L^2(\Omega;\mathbb{H})$.

A Q-white noise $\mathbb{W}$ is defined as the generalized t-derivative of W:

$$\langle\varphi, \mathbb{W}\rangle := -\langle\varphi', W_Q\rangle = -\int_0^\infty W_Q(t)\varphi'(t)\,dt, \qquad \varphi \in \mathcal{D}\ (\varphi \in \mathcal{D}_{\{M_q\}}), \tag{7}$$

where W_Q continued by zero for $t < 0$ is regarded as a (regular) element of $\mathcal{D}_0'(\mathbb{H})$ $(\mathcal{D}_{\{M_q\},0}'(\mathbb{H}))$ $P_{a.s.}$ and of $\mathcal{D}_0'\big(L^2(\Omega;\mathbb{H})\big)$ $\big(\mathcal{D}_{\{M_q\},0}'\big(L^2(\Omega;\mathbb{H})\big)\big)$. The space is chosen in dependence on the type of a semigroup generated by A, namely an integrated or convoluted one.

Using the general idea of reducing boundary-value differential problems to equations with δ-functions and their derivatives multiplied by boundary (initial) data in spaces of distributions, we consider the generalized w.r.t. t linear stochastic Cauchy problem (1) with $\mathbb{W} = W_Q{}'$ and A generating an integrated (convoluted) semigroup as follows:

$$\langle\varphi, X'\rangle = A\langle\varphi, X\rangle + \langle\varphi, \delta\rangle\zeta + \langle\varphi, B\mathbb{W}\rangle, \quad \varphi \in \mathcal{D}\ (\varphi \in \mathcal{D}_{\{M_q\}}). \tag{8}$$

It is well known that if A is the generator of a C_0-semigroup, then its resolvent $\mathcal{R}(\lambda), \lambda > \varpi$, is the Laplace transform of solution operators $U(t), t \geq 0$, satisfies the MFPHY-condition, and solution operators can be obtained via the inverse Laplace transform of $\mathcal{R}$ (see e.g., [6]). Generally, the resolvent satisfies the following conditions:

(R1) $\Lambda_\varpi = \{\lambda \in \mathbb{C} :\ \mathrm{Re}\lambda > \varpi,\ \varpi \in \mathbb{R}\} \subseteq \rho(A)$ and

$$\left\| \frac{d^k}{d\lambda^k}\left(\frac{\mathcal{R}(\lambda)}{\lambda^n}\right) \right\| \leq \frac{Ck!}{(\mathrm{Re}\lambda - \varpi)^{k+1}}, \qquad \lambda \in \Lambda_\varpi, \quad k \in \mathbb{N}_0;$$

(R2) $\Lambda^{\ln}_{n,\nu,\varpi} = \{\lambda \in \mathbb{C} :\ \mathrm{Re}\lambda > n\nu \ln|\lambda| + \varpi\} \subseteq \rho(A)$ and

$$\|\mathcal{R}(\lambda)\| \leq C|\lambda|^n, \qquad \lambda \in \Lambda^{\ln}_{n,\nu,\varpi};$$

(R3) $\Lambda^{M}_{\alpha,\gamma,\varpi} = \{\lambda \in \mathbb{C} :\ \mathrm{Re}\lambda > \alpha M(\gamma|\lambda|) + \varpi\} \subseteq \rho(A)$ and

$$\|\mathcal{R}(\lambda)\| \leq Ce^{\varpi M(\gamma|\lambda|)}, \qquad \lambda \in \Lambda^{M}_{\alpha,\gamma,\varpi},$$

where $M(x)$ is a positive function increasing not faster than $x^h,\ h < 1$;

(R4) The set $\rho(A)$ does not fill any interval of the type $\lambda > \varpi$.

Note that (R1) coincides with the MFPHY-condition as $n = 0$. Change of the resolvent behavior when we pass from (R1) to (R4) reflects strengthening of the ill-posedness of (1), which is connected with the character of peculiarities of solution operators and corresponds, respectively, to an exponentially bounded n-times integrated semigroup, local n-times integrated semigroup, convoluted semigroup, and an R-semigroup.

In the case of integrated and convoluted semigroups the techniques of the Laplace transform, now the generalized Laplace transform, can be used for regularization of solution operators $U(t), t \geq 0$, and X, generalized w.r.t. t solution to (8): $\langle \varphi, X\rangle = \langle \varphi, U\zeta\rangle + \langle \varphi, U * B\mathbb{W}\rangle$. In this case the regularization (by φ and K) is the following:

$$\langle \varphi, U\rangle = \langle \varphi, \frac{d^n}{dt^n} S_n\rangle, \quad \langle \varphi, U\rangle = \langle \varphi, P_{\text{ult}} S_K\rangle,$$
$$S_K(t) = \mathcal{L}^{-1}[\widetilde{K}\mathcal{R}](t) = (U * K)(t),$$

where $\varphi \in \mathcal{D}$ in the case of S_n and $\varphi \in \mathcal{D}_{\{M_q\}}$ in the case of S_K, $\widetilde{K} = \mathcal{L}[K]$ is the Laplace transform of K (its order of decreasing at infinity is defined by smoothness of K), and P_{ult} is an ultra-differential operator defined by the equality $P_{\text{ult}} K = \delta$.

As for regularization of solutions to the nonlinear Cauchy problem, we obtained a generalized solution to (1) with the generator of a C_0-semigroup in spaces of abstract stochastic algebras [4, 5], which were constructed on the basis of Colombeau algebras (see, e.g., [9]). Here, in addition to the regularization of solutions, the regularization by convolution with δ-shaped functions in the very construction of elements of the algebra holds.

In the case of R-semigroups the regularization of solution operators is performed by smoothing the operator $(\lambda I - A)^{-1}$ (which is not the resolvent in this case) by the operator R. In the next section we consider the regularization in the case of differential operators A generating R-semigroups.

3 Regularization in the Case of Generalized w.r.t. x and ω Solutions

First, consider generalized w.r.t. x solutions to the linear Cauchy problem (2) with $A = A\,(i\partial/\partial x)$, $(m \times m)$-operator-matrix of differential operators.

In contrast to the case of an abstract operator A, where the technique of the (generalized) Laplace transform of solution operators representing the resolvent of A is used, here main results are based on the estimation of the Fourier transformed solution operators $e^{tA(s)}, s = \sigma + i\tau$. It is proved (see, e.g., [4]) that the family of convolution operators with the regularized Green function $G_{\mathrm{R}}(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\sigma x}\mathrm{R}(\sigma)e^{tA(\sigma)}\,d\sigma, x \in \mathbb{R}$:

$$[S_R(t)f](x) := G_{\mathrm{R}}(t,x) * f(x), \quad f \in L^2_m(\mathbb{R}), \tag{9}$$

forms an R-semigroup $\{S_R(t), t \in [0, \tau)\}$ in $L^2_m(\mathbb{R})$ with

$$Rf(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\sigma x}\mathrm{R}(\sigma)\widetilde{f}(\sigma)\,d\sigma, \quad x \in \mathbb{R},$$

where $\mathtt{R}$ is taken in such a way that it neutralizes the growth of $\|e^{tA(\sigma)}\|_m$ and $\widetilde{f}$ is the Fourier transform of $f \in L^2_m(\mathbb{R}) = L^2(\mathbb{R}) \times \cdots \times L^2(\mathbb{R})$.

Since A generates only a regularized semigroup, we cannot construct a solution in the form (4) as we did while constructing the solution to the Itô integrated Cauchy problem (5) in the case of a C_0-semigroup $\{U(t), t \geq 0\}$. In the case of R-semigroup defined by (9) we construct X, regularized (by ψ and R) solution[1]

$$\langle \psi(x), X(t,x)\rangle = \langle \psi(x), R^{-1}S_R(t,x)\zeta(x) + R^{-1}S_R(t-h,x)BW(h,x)\,dh\rangle,$$

$\psi \in \Psi$. Here test function spaces Ψ are determined by estimates on $\|e^{tA(s)}\|_m$, which in turn are determined by estimates for eigen-values of $A(s)$ given in the Gelfand–Shiov classification (see, e.g., [4]).

For the generalized w.r.t. x linear Cauchy problem (2) *with* $\zeta \in L^2_m(\mathbb{R})$ *and* $A\,(i\partial/\partial x)$ *generating a Petrovsky correct system, there exists a unique* (*regularized by* ψ *and* R) *solution* $X(t,\cdot) \in \mathcal{S}'_m$; *for a conditionally correct system,* $X(t,\cdot) \in (\mathcal{S}^{\alpha,\mathtt{A}})'_m$, *where parameters* $\alpha, \mathtt{A}$ *are determined by estimates of* $\|e^{tA(s)}\|_m$ ($\alpha = \frac{1}{h}$, $\frac{1}{h\,e\,\mathtt{A}^h} > a_0$); *for an incorrect system,* $X(t,\cdot) \in (\mathcal{S}^{\alpha,\mathtt{A}})'_m$, *where* $\alpha = \frac{1}{p_0}$ *and* $\frac{1}{p_0\,e\,\mathtt{A}^{p_0}} > b_1$.

Note the important fact that here, due to test functions $\psi \in \Psi \subset L^2_m(\mathbb{R})$ we can consider weak Wiener processes W.

In conclusion, without going into details, introduce a generalized w.r.t. ω solution to the linear Cauchy problem with multiplicative singular white noise $\mathbb{W}$ defined in the space $(\mathcal{S})_{-\rho}(\mathbb{H})$, $\rho \in [0,1)$, an infinite dimensional extension of the white noise space. Replacing the Itô integral in (2) with the Hitsuda–Skorohod integral and differentiating w.r.t. t, we come to the Cauchy problem

$$\frac{dX(t)}{dt} = AX(t) + B\big(t, X(t)\big) \diamond \mathbb{W}(t), \quad t \geq 0, \qquad X(0) = \zeta. \tag{10}$$

Applying the S-transform to (10), we obtain the following problem:

$$\frac{d}{dt}\hat{X}(t,\theta) = A\hat{X}(t,\theta) + B\big(\hat{X}(t,\theta)\big)\hat{\mathbb{W}}(t,\theta), \;\; t \geq 0, \;\; \theta \in \mathcal{S}, \quad \hat{X}(0,\theta) = \hat{\zeta}(\theta),$$

where $\hat{X}(t,\theta) = \mathrm{S}[X(t)](\theta)$, $\hat{\mathbb{W}}(t,\theta) = \mathrm{S}[\mathbb{W}(t)](\theta)$, and $\hat{\zeta}(\theta) = \mathrm{S}\zeta(\theta)$.

Under certain, not restrictive conditions on operators A and B, we obtain $\hat{X}$ via the approximation based on the contraction operators theorem and then $X = \mathrm{S}^{-1}[\hat{X}]$, a (regularized by test functions from $(\mathcal{S})_\rho$) solution to (10) with $X(t) \in (\mathcal{S})_{-\rho}(H)$, $t \geq 0$.

[1] Recall that if we write an argument of a generalized function, we mean the function is applied to test functions of this argument.

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Image Processing Algorithms in Different Areas of Science

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Abstract The main objective of this paper is to present some of the key issues in the field of computer processing of digital images. They are used in a large amount of fields of science and education. Using modern controllers (MS Kinect, PS Move) can recognize and interpret the gestures of the user. It can be used in the development of interesting applications for teaching tools.

Keywords Image processing algorithms • Kinect

Motion sensors, created for home entertainment systems, which at the end of the first decade of the twenty-first century changed the approach to the issues of cooperation between man and machine provided the opportunity of natural human interaction with operated device. There are many systems based on so-called natural user interface, which key element is the user himself, whose gestures, body movements, and speech are interpreted by appropriate mechanisms without direct contact of the user with the hardware. One of the most important systems of this type are: Perceptive Pixel, Microsoft Pixel Sense, 3D Immersive Touch, or the popular Microsoft Kinect. To a large extent they are based on creating a scene using different types of user tracking sensors (cameras, IR cameras, motion sensors, and microphones) and allow to determine his location in a controlled space. With this type of solutions direct and intuitive contact has become possible, on which Natural User Interface systems (NUI, NI) are based.

From the perspective of educational applications (but actually in virtually any application area) of NUI based systems, the solutions based on the last of these solutions can be distinguished, namely, Microsoft Kinect (Xbox Kinect), or rather its implementation in the MS Windows environment. The device itself is equipped with a number of sensors, such as RGB camera, infrared wave emitter IR camera, and four microphones. This package allows you to interact with your computer without using any additional physical controller. Thanks to Kinect SDK user library provided by Microsoft, it is possible to track user movements using skeletal

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representation of the silhouette of a person registered by the device. This gives a fairly wide range of possibilities of application of the described environment, both in applications for the youngest computer users (for whom traditional controller can be a barrier to contact with the unit), and in activities related to the therapeutic function of the NUI system applications associated with, e.g., rehabilitation of older people. In the traditional sense of a computer system, the user in order to be able to interact with the machine uses a peripheral input device—to enter instructions, then waits for the result on the output device—usually a computer monitor. The set of this type I/O devices in conjunction with the operating system, can be defined very simply as the user interface. Similarly, any system—also in purely software terms—mediating in man-machine communication can be named user interface either it is an operating system, application running locally, Web application or manageable Web page.

The evolution of user interfaces—from simple text to advanced, often intuitive graphics solutions, as well as recently popular touch solutions—indicates a desire to maximize the simplification of the interaction between man and machine (towards direct and intuitive). The last of the steps (for the moment) is to use interfaces that eliminate any peripheral device in favor of gestures or interpretation of the movement of human body. Speaking of systems acquiring data from observations of the real world, we enter into another area of issues which is computer vision (CV). It covers three main issues. The first one deals with the problems of obtaining a digital image of sufficiently high quality. This area consists of image acquisition process and image pre-processing process, which eliminates noise and distortion. In the second stage specific features are extracted from the picture—essential for the further stages of the work. This stage is called image analysis. The third and final stage of the process of computer vision is usually associated with classification or recognition of recorded images. One of the overriding motives of CV development was to create a system capable of perceiving and understanding the image in a manner similar to human. In this case, the analysis is done with geometric, physical, or static models. Many tasks included in CV schemes can be analyzed as mathematic problems—in view of the fact that a significant part of the required image processing is based on statistics, geometry operations, or processes optimization.

Among many issues that are raised within CV applications we can distinguish three basic ones: image recognition, motion analysis, and reconstruction of three-dimensional scenes. The very process of image recognition is the task of finding a well-known object in the image or scene. This task easy for a human is still a major challenge for vision system. It requires appropriate strategies—and the performance of the plotted sequence of actions: (1) detection—detection of a plurality of segments in the picture, (2) classifications—determination of the best fit of the plurality of segments with a set of elements of the representation object model, (3) location—determination of the position of the detected object in three-dimensional space or on a plane. For a rigid object with known dimensions there are 3–6 degrees of freedom: the three-dimensional object requires the calculation of six parameters (three translations and three rotation angles), and the flat object—three parameters (two translations and one angle).

In order to classify the strategies of image recognition, 3D object detection problem due to the method of searching a set of models and data segments can be divided into: optimal strategies and heuristic strategies [3]. In the optimal strategies there are two approaches: a systematic search and Bayes for object instances. Both the first and the second strategy can be difficult due to the computational complexity resulting from the large number of possible instances. In the heuristic strategies there are three main methods: data-driven fit, model-driven fit, and algorithm for generation and verification of hypotheses (a combination of the two previous strategies). They are often used because of their characteristics, namely finding a "good enough" solution as soon as possible. Practical research show that the most effective algorithm is just the latest example of the heuristic strategy (algorithm generation and test hypotheses), and its most popular implementation is RANSAC algorithm [2]. The key objective in the process of image recognition is to determine whether the information about a specific object, the related characteristics and important properties is included in the analyzed data. Performing this type of action sequences is not much of a problem for the computer, but in general situations (applications) it is very hard to get a satisfactory solution. Much more effective are in this case systems that focus on specific cases—e.g. facial recognition, letters, signs recognition, etc.

In the processes of motion analysis the sequence of images to predict further action—change of the parameters or object position on the stage is tested (usually by tracking a set of points or objects). The most common problems with the reproduction of three-dimensional scenes—are the problems associated with the process of image acquisition (3D to 2D mapping, scene illumination, etc.). The above analysis may be cover simple sets of points embedded in 3D space and complex spatial models. The ideal and most importantly universally accessible controller implementing basic functions and objectives of natural user interface (NUI) is Microsoft Kinect. It allows a fully natural communication, which does not require additional controllers or markers to facilitate data acquisition. The controller performs image analysis through the main points located on the head, arms, body, and legs of a simplified human model (20 points). The emitter and an infrared camera can effectively and quite accurately determine the location of an object in the visual field sensors. The algorithm calculating the distance to each point of analyzed scene is based on relationships that arise from used imaging technique—stereoscopy—where on the basis of two angles, one can calculate the actual distance from the observed objects. In case of the described controller, the simplification exhausting the fact that one of the observers (in this case the IR emitter that produces a beam of pseudo-random patterns) of the scene is also the reference point for the image viewed by a second observer (camera R, reading automatically the location of points generated by the emitter to the objects in the scene) were used. This allows to determine the distance of objects in the scene. The very process of detection of objects is based on generating a point cloud (containing more than 300,000 of them).

Based on this simple solution, a number of projects supporting various fields of science were created. From educational games to robot control systems. For example, in the Department of Computer Science of Cracow Pedagogical University

Virtual Reality Educational system based on developed there Gesture Description Language (GDL) [1] was created, basing on an open environment of well-known and popular among the youth game allows to transfer into the virtual world with which player interacts through gestures interpreted by the Kinect controller.

Examples of uses of the systems described above could be listed indefinitely. From simple, uncomplicated entertainment applications, through education to complex, advanced models used in areas such as artificial intelligence, robotics, and neuroscience. Imperfection of existing systems poses researchers new challenges, which, however, as demonstrated the achievements in this field should be resolved in the coming years.

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Boundary Value Problems for Half-Plane, Strip and Rectangle with Circular Inclusions

E. Pesetskaya and N. Rylko

Abstract The aim of this work is to extend the method of functional equations to boundary value problems for half-plane, strips, and rectangles with circular inclusions.

Keywords Boundary value problem • Functional equations • Multiply connected domain • Poincaré series • R-linear problem • Riemann–Hilbert problem • Schottky double

Mathematics Subject Classification (2010) 30E25

1 Introduction

Exact formulae for solutions of the boundary value problem are of great interest of the theory of analytic functions. The famous Poisson integral solves the Dirichlet problem for a disk. The corresponding Poisson kernel is expressed in terms of the elementary rational function $\frac{\zeta+z}{\zeta-z}\frac{1}{\zeta}$ and yields the Schwarz operator for the disk (see formula (2.7.1) in [8]). Many attempts were applied to extend the Poisson formula to multiply connected domains with circular holes. The main line of this study was based on the construction of the Poisson kernel in the form of the absolutely convergent Poincaré θ_2-series beginning from [2]. An equivalent approach was based on the Schottky–Klein prime function [1]. In both the approaches holes of the domains were located far away each from other. Such a geometrical restriction is called the separation condition.

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The generalized almost uniformly convergent Poincaré series and the generalized Schottky–Klein prime functions were constructed in [6–8] for an arbitrary multiply connected domain that yields the exact Poisson formula, the Schwarz operator, Green's functions, Bergman kernel, the Schwarz–Christoffel formula and solves the scalar Riemann–Hilbert and $\mathbb{R}$-linear problems (see references in [7]). It is worth noting that the Poincaré series and other corresponding objects in general are not expressed in terms of the elementary functions that effective computational methods require. The structure of the objects is based on the functions z^{-2}, z^{-1}, and $\ln z$ related by differentiation. The corresponding functions constructed for doubly connected domains are expressed in terms of the Weierstrass elliptic functions, for instance $\wp(z)$, $\zeta(z)$, and $\ln\sigma(z)$. The multi-(quasi-)periodic functions corresponding to the classic Schottky groups are the Poincaré θ_2-, θ_1- series and the Schottky–Klein prime function. Therefore, all the fundamental functions can be obtained from the Poincaré θ_2-series by integration term by term of the uniformly convergent series. Though the Poincaré series is in general conditionally, hence slowly, convergent, a fast algorithm was proposed in [9]. The above scheme is the most general to solve the Riemann–Hilbert and $\mathbb{R}$-linear problems.

Dareen Crowdy gave talks (for instance, a plenary talk at the 10th ISAAC Congress) where he proposed possible extensions to boundary value problems for a strip and for a rectangle with circular holes (see Fig. 1). However, these problems can be easily treated as the above problems considered by Mityushev. For instance, the boundary of the strip (two straight lines) can be considered as two circles on the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \infty$. It is worth noting that the previous exact formulae by Dareen Crowdy [3] had geometrical restrictions, namely the separation condition. This excludes the case of tangent circles. Following Mityushev's scheme the holes can be tangent that yields solution to various problems even more general than Dareen Crowdy presented. This scheme was discussed by Mityushev at the hard-to-reach paper [5].

In the present paper, we develop Mityushev's method of functional equations to the domains displayed in Fig. 1. A boundary value problem for rectangle is discussed in the separate paper [4] of the present volume.

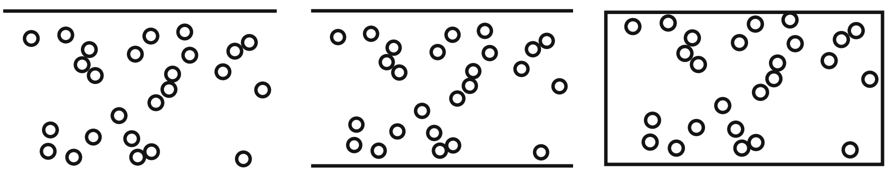

Fig. 1 Half-Plane, strip, and rectangle with circular inclusions

2 Half-Plane

Let $z = x_1 + ix_2$ denote a complex variable on the complex plane $\mathbb{C}$ and $i = \sqrt{-1}$. Consider a multiply connected domain $D \subset \mathbb{C}$ consisting of the half-plane Im $z < 0$ with n circular non-overlapping inclusions $D_k = \{z \in \mathbb{C} : |z - a_k| < r\}$ $(k = 1, 2, \ldots, n)$. The real axis is denoted by $L_0 = \{z \in \mathbb{C} : \text{Im } z = 0\}$ and the circles by $L_k = \{z \in \mathbb{C} : |z - a_k| = r\}$. Let each circle L_k be positively oriented, i.e., when one is following along L_k the domain D_k is on left. Let the domain D be occupied by a conducting material with the thermal conductivity λ normalized to unity and the inclusions D_k by a perfect conductor. Let a temperature distribution $f(x_1)$ be given on L_0. For simplicity, it is assumed that $f(x_1)$ decays at infinity at least as $|x_1|^{-1}$. Then, the temperature distribution $T(z) \equiv T(x_1, x_2)$ in D satisfies the boundary value problem

$$\begin{array}{ll} \nabla^2 T = 0 & \text{in } D, \\ T = f & \text{on } L_0, \\ T = T_k & \text{on } L_k \\ \int_{L_k} \frac{\partial T}{\partial n} ds = 0 & (k = 1, 2, \ldots, n), \end{array} \tag{1}$$

where $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ stands for the Laplace operator and T_k are undetermined constants.

Following [8] introduce the complex potential $\varphi(z)$ in such a way that

$$T(z) = \text{Re } \varphi(z), \tag{2}$$

where the function $\varphi(z)$ is single-valued and analytic in the domain D. Following [8] we can reduce the boundary value problem (1) to the $\mathbb{R}$-linear problem [5]

$$\varphi(z) = \varphi_k(z) - \overline{\varphi_k(z)} + T_k, \quad |z - a_k| = r \ (k = 1, 2, \ldots, n), \tag{3}$$

$$\varphi(z) = \varphi_0(z) - \overline{\varphi_0(z)} + f_0(z), \quad \text{Im } z = 0, \tag{4}$$

where the auxiliary functions $\varphi_k(z)$ are analytic in the domains $|z - a_k| < r$ $(k = 1, 2, \ldots, n)$ and Im $z > 0$, respectively. Moreover these functions are continuously differentiable in the closures of the considered domains. The function $f_0(z)$ is analytic in Im $z > 0$ and satisfies the boundary value problem

$$\text{Re } f_0(x_1) = f(x_1), \quad -\infty < x_1 < \infty. \tag{5}$$

It can be exactly written by Poisson's formula for a half-plane [8].

Differentiation of (3) and (4) on the tangent parameters of L_0 yields the $\mathbb{R}$-linear problem [8]

$$\psi(z) = \psi_k(z) + \left(\frac{r}{z - a_k}\right)^2 \overline{\psi_k(z)}, \quad |z - a_k| = r \ (k = 1, 2, \ldots, n), \tag{6}$$

$$\psi(z) = \psi_0(z) - \overline{\psi_0(z)} + f_0'(z), \quad \text{Im } z = 0. \tag{7}$$

The problem (6) and (7) is reduced to the system of functional equations

$$\psi_k(z) = \sum_{m \neq k} \left(\frac{r}{z - a_m} \right)^2 \overline{\psi_m(z^*_{(m)})} - \overline{\psi_0(\overline{z})}, \quad |z - a_k| \leq r \; (k = 1, 2, \ldots, n), \tag{8}$$

$$\psi_0(z) = \sum_{m=1}^{n} \left(\frac{r}{z - a_m} \right)^2 \overline{\psi_m(z^*_{(m)})} - f_0'(z), \quad \operatorname{Im} z \geq 0. \tag{9}$$

After its solution the complex flux in the domain D can be found by formula following from the definition of $\Phi(z)$ in D

$$\psi(z) = \sum_{m=1}^{n} \left(\frac{r}{z - a_m} \right)^2 \overline{\psi_m(z^*_{(m)})} - \overline{\psi_0(\overline{z})}, \quad z \in D. \tag{10}$$

Theorem 2.1 ([8, 10]) *The functional equations (8)–(9) can be solved by the method of successive approximations.*

Using Theorem 2.1 we can write the function $\psi(z)$ by (10) in the form of the uniformly convergent Poincaré θ_2-series (cf. [6, 8]). The function $\varphi(z)$ is obtained by integration of the latter series term by term. This yields $\varphi(z)$ in the form of the Poincaré θ_1-series. Its further integration yields the Schottky–Klein prime function. Ultimately, the temperature distribution is defined in (2). Numerical examples are presented in [10].

3 Strip

Let us consider in the complex plane $\mathbb{C}$ of the complex variable z a strip domain D_2 containing a hole with the boundary $G_1 \cup G_2 \cup \partial D_1$, the center $a \in \mathbb{R}$, and the radius r (see Fig. 2). A width of the strip is $2b$. Let the domain D_2 be occupied by a conducting material with a conductivity λ. Consider a problem when the potential $T(z) \equiv T(x, y)$ satisfies the Laplace equation $\Delta T = 0$ in D_2 with the Dirichlet boundary conditions

$$T(t) = h_1(t), \; t \in G_1, \; T(t) = h_2(t), \quad t \in G_2, \quad T(t) = h_3(t), \; t \in \partial D_1. \tag{11}$$

For simplicity, let given functions h_1 and h_2 form functions Hölder continuous on $G_1 \cup G_2$, then $T(z)$ is continuous at infinity, h_3 is a Hölder continuous function on ∂D_1.

Using a conformal mapping $w = \frac{1}{z}$, we arrive at the Dirichlet problem in the complex plane of the complex variable w (see Fig. 2) for $T(w)$ harmonic in a domain D_2' and Hölder continuous in its closure. In particular, $T(w)$ is continuous at

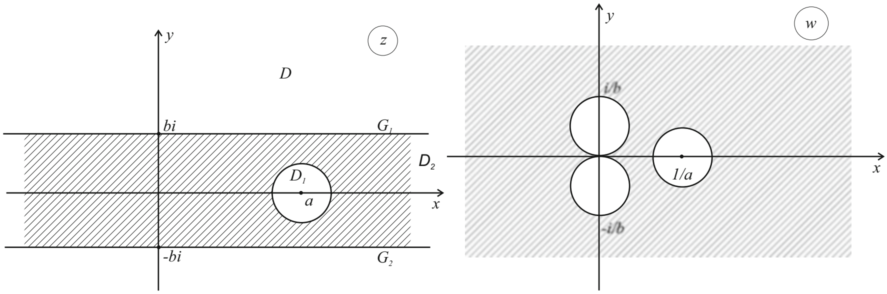

Fig. 2 Strip with a hole and conformally equivalent triply connected circular domain

$w = 0$ and at $w = \infty$. According to the Logarithmic Conjugation Theorem [8], the harmonic function $T(w)$ is related to an analytic function $\varphi(w)$ by formula

$$T(w) = \operatorname{Re}(\varphi(w)) + A_1 \ln(w - a_1) + A_2 \ln(w - a_2) + A_3 \ln(w - a_3), \tag{12}$$

where A_j are undetermined real constants, a_1, a_2, a_3 are centers of circles $\partial D_1'$, G_1', G_2' obtained by conformal mapping $w = \frac{1}{z}$. In our case, $a_1 = \frac{1}{a}$, $a_2 = -\frac{\imath b}{2}$, $a_3 = \frac{\imath b}{2}$ are centers of the circles $\partial D_1'$, G_1', G_2', respectively. Let us denote $w_k^* := \frac{r_k^2}{\overline{w - a_k}} + a_k$ the inversion with respect to a circle with the center a_k and radius r_k. Using Decomposition Theorem, we find the function $\varphi(w)$ in the form

$$\varphi(w) = \overline{\varphi_1(w_1^*)} + \overline{\varphi_2(w_2^*)} + \overline{\varphi_3(w_3^*)}. \tag{13}$$

Here, a function φ_k is analytic in $|w - a_k| < r_k$ and Hölder continuous on $|w - a_k| \leq r_k$ except $w = 0$ where φ_k is almost bounded. Substituting (13) into (12) and using (11) we get the following boundary conditions for $k = 1, 2, 3$

$$\operatorname{Re}(\overline{\varphi_1(w_1^*)}) + \operatorname{Re}(\overline{\varphi_2(w_2^*)}) + \operatorname{Re}(\overline{\varphi_3(w_3^*)}) + \sum_{i=1}^{3} A_i \ln(w - a_i) = h_k(w). \tag{14}$$

Following [8], we arrive at

$$\operatorname{Re}(\overline{\varphi_1(w_1^*)}) + \operatorname{Re}(\overline{\varphi_2(w_2^*)}) + \operatorname{Re}(\overline{\varphi_3(w_3^*)}) = \operatorname{Re} g_k(w) - \sum_{i=1}^{3} A_i \ln(w - a_i), \tag{15}$$

where

$$g_k(w) = -\frac{1}{\pi \imath} \int\limits_{T_k} \frac{h_k(\tau)}{\tau - w}\, d\tau + \frac{1}{2\pi \imath} \int\limits_{T_k} \frac{h_k(\tau)}{\tau - a_k}\, d\tau + \imath \gamma_k$$

is a solution of the Schwarz problem $\mathrm{Re} g_k(w) = h_k(w)$ with respect to Hölder continuous function g_k. On a boundary of any circle, we have

$$\mathrm{Re}\overline{\varphi_k(w_k^*)} = \mathrm{Re}\varphi_k(w), \quad A_k \ln(w - a_k) = A_k \ln r_k.$$

One can consider the problem (15) as the Schwarz problem with respect to the function $\overline{\varphi_1(w_1^*)} + \overline{\varphi_2(w_2^*)} + \overline{\varphi_3(w_3^*)}$ analytic in the disc $|w - a_k| < r_k$ with the prescribed real part of boundary values $\mathrm{Re}(g_k(w) - \sum_{i=1}^{3} A_i \ln(w - a_i))$. Using these facts and (13), we get the system of functional equations which after differentiation becomes

$$\psi_k(w) = \sum_{m \neq k}^{3} \left(\frac{r_m}{w - a_m} \right)^2 \overline{\psi_m(w_m^*)} - \sum_{m \neq k}^{3} \frac{A_m}{w - a_m} + g_k'(w), \qquad k = 1, 2, 3, \tag{16}$$

on $\psi_k(z) := \varphi_k'(w)$. Let us represent the system (16) in the form of an operator equation $\Psi = A\Psi + G$. This non-homogeneous equation has a unique solution in the class of analytical in $\bigcup_{k=1}^{3} \{|w - a_k| < r_k\}$ functions Hölder continuous on $\bigcup_{k=1}^{3} \{|w - a_k| \leq r_k\}$ for each G from the same class. A solution can be explicitly found by the method of successive approximations [8].

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Regular Strata and Moduli Spaces of Irregular Singular Connections

Daniel S. Sage

Abstract In joint work with C. Bremer, the author has developed a geometric theory of fundamental strata which provides a new approach to the study of meromorphic G-connections on curves (for complex reductive G). In this theory, a fundamental stratum associated to a connection at a singular point plays the role of the local leading term of the connection. In this paper, we illustrate this theory for $G = \mathfrak{gl}_2(\mathbb{C})$ (i.e. for connections on rank two vector bundles). In particular, we show how this approach can be used to construct explicit moduli spaces of irregular singular connections on the projective line with specified singularities and formal types.

Keywords Formal connections • Fundamental strata • Irregular singularities • Meromorphic connections • Moduli spaces • Regular strata

In recent years, there has been extensive interest in meromorphic connections on curves due to their role as Langlands parameters in the geometric Langlands correspondence. In particular, connections with irregular singularities are the geometric analogue of Galois representations with wild ramification.

The classical approach to studying the local behavior of irregular singular meromorphic connections on curves depends on the leading term of the connection matrix being well-behaved. Let V be a trivializable vector bundle over $\mathbb{P}^1$ endowed with a meromorphic (automatically flat) connection ∇. Upon fixing a local parameter t at a singular point y and a local trivialization, one can express the connection near y as

$$d + (M_{-r}t^{-r} + M_{1-r}t^{1-r} + \cdots)\frac{dt}{t}, \tag{1}$$

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with $M_i \in \mathfrak{gl}_n(\mathbb{C})$, $M_{-r} \neq 0$ and $r \geq 0$. From a more geometric point of view, setting $F = \mathbb{C}((t))$, this formula defines the induced connection $\hat{\nabla}_y$ on the formal punctured disk $\operatorname{Spec}(F)$.

When M_{-r} is well-behaved, this leading term contains important information about the connection. As a first example, if M_{-r} is nonnilpotent, then the expansion of ∇ at y with respect to any local trivialization must begin in degree $-r$ or below. Moreover, if $\hat{\nabla}_y$ is irregular, r is the slope of the connection at y. (The slope is an invariant introduced by Katz [6] that gives one measure of how singular a connection is at a given point.)

Much more can be said in the irregular singular *nonresonant* case when $r > 0$ and M_{-r} is regular semisimple. We assume that $r > 0$ so that we are in the irregular singular case. In this case, asymptotic analysis [9] guarantees that ∇ can be diagonalized at y by an appropriate gauge change so that $\nabla = d + (D_{-r}t^{-r} + D_{1-r}t^{1-r} + \cdots D_0)\frac{dt}{t}$, with each D_i diagonal. The diagonal 1-form here is called a *formal type* of $\hat{\nabla}_y$. When all of the irregular singularities on a meromorphic connection on $\mathbb{P}^1$ are of this form, Boalch has shown how to construct well-behaved moduli spaces of such connections; he has further realized the isomonodromy equations as an integrable system on an appropriate moduli space [1].

However, many interesting connections do not have regular semisimple leading terms. Consider, for example, the generalized Airy connections:

$$d + \begin{pmatrix} 0 & t^{-(s+1)} \\ t^{-s} & 0 \end{pmatrix} \frac{dt}{t} = d + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t^{-(s+1)} \frac{dt}{t} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} t^{-s} \frac{dt}{t}, \tag{2}$$

for $s \geq 0$. Note that when $s = 1$, this is the usual Airy connection with the irregular singular point at 0 instead of ∞. Also, when $s = 0$, this is the GL_2 version of the Frenkel–Gross rigid flat G-bundle on $\mathbb{P}^1$ [7]. For the generalized Airy connections, the leading term is nilpotent, and it is no longer the case that one can read off the slope directly from the leading term. Indeed, the slope is $s + \frac{1}{2}$, not $s + 1$.

In a recent series of papers joint with Bremer [2–5], we have generalized these classical results to meromorphic connections on curves (or even flat G-bundles for reductive G) whose leading term is nilpotent. We have introduced a new notion of the "leading term" of a formal connection through a systematic analysis of its behavior in terms of suitable filtrations on the loop algebra. This theory has already proved useful in applications to the geometric Langlands program [8].

In this paper, we will illustrate our theory in the case of rank 2 flat vector bundles, where much of the Lie-theoretic complexity is absent. In this case, up to $\mathrm{GL}_2(F)$-conjugacy, one need only consider two filtrations on $\mathfrak{gl}_2(F)$, the degree filtration and the (standard) *Iwahori filtration*.

Let $\mathfrak{o} = \mathbb{C}[[t]]$ be the ring of formal power series, and let $\omega = \left(\begin{smallmatrix} 0 & 1 \\ t & 0 \end{smallmatrix}\right)$. Then the Iwahori filtration is defined by

$$\mathfrak{i}^r = \begin{pmatrix} t^{\lceil r/2 \rceil}\mathfrak{o} & t^{\lfloor r/2 \rfloor}\mathfrak{o} \\ t^{\lfloor r/2 \rfloor + 1}\mathfrak{o} & t^{\lceil r/2 \rceil}\mathfrak{o} \end{pmatrix}. \tag{3}$$

Recalling that the standard Iwahori subgroup $I \subset \mathrm{GL}_2(\mathfrak{o})$ consisting of the invertible matrices which are upper triangular modulo t, one sees that $\mathfrak{i} := \mathrm{Lie}(I)$ is just $\mathfrak{i}^0$; moreover, $\mathfrak{i}^r = \mathfrak{i}\omega^r = \omega^r\mathfrak{i}$. A matrix is homogeneous of degree $2s$ (resp. $2s+1$) with respect to the Iwahori filtration if it is in $\left(\begin{smallmatrix} \mathbb{C}t^s & 0 \\ 0 & \mathbb{C}t^s \end{smallmatrix}\right)$ (resp. $\left(\begin{smallmatrix} 0 & \mathbb{C}t^s \\ \mathbb{C}t^{s+1} & 0 \end{smallmatrix}\right)$). In particular, the matrix of the generalized Airy connection is Iwahori-homogeneous of degree $-(2s+1)$.

The groups $\mathrm{GL}_2(\mathfrak{o})$ and I are examples of "parahoric subgroups." For any parahoric P, there is an associated filtration $\mathfrak{p}^j$ of $\mathfrak{gl}_2(F)$; this filtration satisfies $\mathfrak{p}^{j+e_P} = t\mathfrak{p}^j$ for $e_P \in \{1,2\}$. For $\mathrm{GL}_2(\mathfrak{o})$, the filtered subspaces are $t^j\mathfrak{gl}_2(\mathfrak{o})$. For simplicity, we will take P to be I or $\mathrm{GL}_2(\mathfrak{o})$ in this paper. Note $e_{\mathrm{GL}_2(\mathfrak{o})} = 1$ and $e_I = 2$.

It will be convenient to view any one-form $\nu \in \Omega^1(\mathfrak{gl}_2(F))$ as a continuous $\mathbb{C}$-linear functional on (subspaces of) $\mathfrak{gl}_2(F)$ via $Y \mapsto \mathrm{Res}\,\mathrm{Tr}\,Y\nu$. Any such functional on $\mathfrak{p}^r$ can be represented as $X\frac{dt}{t}$ for some $X \in \mathfrak{p}^{-r}$. For our standard examples, a functional $\beta \in (\mathfrak{p}^r/\mathfrak{p}^{r+1})^\vee$ can be written uniquely as $\beta^\flat\frac{dt}{t}$ for $\beta^\flat$ homogeneous.

A GL_2-*stratum* is a triple (P, r, β) with $P \subset \mathrm{GL}_2(F)$ a parahoric subgroup, r a nonnegative integer, and $\beta \in (\mathfrak{p}^r/\mathfrak{p}^{r+1})^\vee$. The stratum is called *fundamental* if $\beta^\flat$ is nonnilpotent. A formal connection $\hat{\nabla}$ contains (P, r, β) if $\hat{\nabla} = d + X\frac{dt}{t}$ with $X \in \mathfrak{p}^{-r}$ and β is induced by $X\frac{dt}{t}$. The following theorem shows that fundamental strata provide the correct notion of the leading term of a connection.

Theorem 1 *Any formal connection $\hat{\nabla}$ contains a fundamental stratum (P, r, β) with $r/e_P = \mathrm{slope}(\hat{\nabla})$; in particular, the connection is irregular singular if and only if $r > 0$. Moreover, if (P', r', β') is any other stratum contained in $\hat{\nabla}$, then $r'/e_{P'} \geq r/e_P$ with equality if (P', r', β') is fundamental. The converse hold if $\hat{\nabla}$ is irregular singular.*

The theorem shows that the classical slope of a connection can also be defined in terms of the fundamental strata contained in it. For flat G-bundles, the analogous result serves to define the slope [4].

Example 1 The connection in (1) (with the M_i's in $\mathfrak{gl}_2(\mathbb{C})$) contains the stratum $(\mathrm{GL}_2(\mathfrak{o}), r, M_{-r}t^{-r}\frac{dt}{t})$; it is fundamental if and only if M_{-r} is nonnilpotent, in which case the slope is r. If M_{-r} is upper triangular with a nonzero diagonal entry, then $\hat{\nabla}$ contains a fundamental stratum of the form $(I, 2r, \beta)$, where β is induced by the diagonal component of $M_{-r}t^{-r}$. Again, one sees that the slope is $2r/2 = r$. On the other hand, if M_{-r} has a nonzero entry below the diagonal, then $\hat{\nabla}$ contains a nonfundamental stratum of the form $(I, 2r+1, \beta')$.

Example 2 The generalized Airy connection with parameter s contains the nonfundamental stratum $(\mathrm{GL}_2(\mathfrak{o}), s+1, \left(\begin{smallmatrix} 0 & t^{-(s+1)} \\ 0 & 0 \end{smallmatrix}\right)\frac{dt}{t})$. It also contains the fundamental stratum $(I, 2s+1, \omega^{-(2s+1)}\frac{dt}{t})$, whence its slope is $s + \frac{1}{2}$.

In order to construct well-behaved moduli spaces, we need a condition on strata that is analogous to the nonresonance condition for diagonalizable connections. This is accomplished through the notion of a *regular stratum*. Let $S \subset \mathrm{GL}_2(F)$ be a (not necessarily split) maximal torus. Up to $\mathrm{GL}_2(F)$-conjugacy, there are two distinct maximal tori: $T(F)$ and $\mathbb{C}((\omega))^\times$ (nonzero Laurent series in ω). For our standard examples, we say that (P, r, β) is S-regular if S is the centralizer of $\beta^\flat$. (See [3, 5] for the general definition.)

Example 3 If M_{-r} is regular semisimple, then the stratum $(\mathrm{GL}_2(\mathfrak{o}), r, M_{-r}t^{-r}\frac{dt}{t})$ is $Z(M_{-r})(F)$-regular.

Example 4 The stratum $(I, 2s+1, \omega^{-(2s+1)}\frac{dt}{t})$ contained in the generalized Airy connection is $\mathbb{C}((\omega))^\times$-regular. On the other hand, if (P, r, β) is $\mathbb{C}((\omega))^\times$-regular, then $r/e_P \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$.

From now on, we assume that S is $T(F)$ or $\mathbb{C}((\omega))^\times$. Note that $\mathfrak{s} = \mathrm{Lie}(S)$ is $\mathfrak{t}(F)$ and $\mathbb{C}((\omega))$ in these two cases, and both are endowed with an obvious filtration by powers of t or ω. We call a connection containing an S-regular stratum *S-toral*. An S-toral connection can be "diagonalized" into $\mathfrak{s} = \mathrm{Lie}(S)$. Again, for simplicity, we will only describe what this means for S equal to $T(F)$ and $\mathbb{C}((\omega))^\times$. For any $r > 0$ such that $\mathfrak{s}^r$ contains a regular semisimple element of homogeneous degree r, one can define a quasiaffine variety $\mathcal{A}(S, r) \subset \mathfrak{s}^r\frac{dt}{t}$ of *S-formal types* of depth r: $\mathcal{A}(T, r) = \{D_{-r}t^{-r} + \cdots + D_0 \mid D_i \in \mathfrak{t}, D_{-r} \text{ regular}\}\frac{dt}{t}$ and $\mathcal{A}(\mathbb{C}((\omega))^\times, 2s+1) = \{p(\omega^{-1})\frac{dt}{t} \mid p \in \mathbb{C}[\omega^{-1}], \deg(p) = 2s+1\}$. We remark that if we set $P_{T(F)} = \mathrm{GL}_n(\mathfrak{o})$ and $P_{\mathbb{C}((\omega))^\times} = I$, then an S-formal type $A = X\frac{dt}{t}$ of depth r gives rise to the S-regular stratum $(P_S, r, X\frac{dt}{t})$.

Theorem 2 *If $\hat{\nabla}$ contains the S-regular stratum (P, r, β), then $\hat{\nabla}$ is $P^1 := 1 + \mathfrak{p}^1$-gauge equivalent to a unique connection of the form $d + A$ for $A \in \mathcal{A}(S, r)$ with leading term $\beta^\flat\frac{dt}{t}$.*

Before discussing moduli spaces, we need to define the notion of a *framable* connection. Suppose that ∇ is a flat G-bundle on $\mathbb{P}^1$. Upon fixing a global trivialization ϕ, we can write $\nabla = d + [\nabla]$, where $[\nabla]$ is the matrix of the connection. Assume that the formal connection $\hat{\nabla}_y$ at y has formal type A_y. We say that $g \in \mathrm{GL}_2(\mathbb{C})$ is a *compatible framing* for ∇ at y if $g \cdot \hat{\nabla}_y$ contains the regular stratum determined by A_y. For example, if $A_y = (D_{-r}t^{-r} + \cdots + D_0)\frac{dt}{t}$, then g is a global gauge change such that $g \cdot \hat{\nabla}_y = d + (D_{-r}t^{-r} + Xt^{-r+1})\frac{dt}{t}$ with $X \in \mathfrak{gl}_2(\mathfrak{o})$. The connection ∇ is framable at y if there exists a compatible framing.

We now explain how moduli spaces of connections can be defined for meromorphic connections ∇ on $\mathbb{P}^1$ such that $\hat{\nabla}_y$ is toral at each irregular singularity. We also want to allow for regular singular points. If the residue of a regular singular connection is "nonresonant," in the sense that the eigenvalues do not differ by a nonzero integer, then its formal isomorphism class is determined by the adjoint orbit of the residue. Accordingly, our starting data will consist of:

- A nonempty set $\{x_i\} \subset \mathbb{P}^1$ of irregular singular points;
- $\mathbf{A} = (A_i)$, a set of S_i-formal types with positive depths r_i at the x_i's;
- A set $\{y_j\} \subset \mathbb{P}^1$ of regular singular points disjoint from $\{x_i\}$;
- A corresponding collection $\mathbf{C} = (C_j)$ of nonresonant adjoint orbits.

Let $\mathcal{M}(\mathbf{A}, \mathbf{C})$ be the moduli space classifying meromorphic rank 2 connections (V, ∇) on $\mathbb{P}^1$ with V trivializable such that:

- ∇ has irregular singular points at the x_i's, regular singular points at the y_j's, and no other singular points;
- ∇ is framable and has formal type A_i at x_i;
- ∇ has residue at y_j in C_j.

We will construct this moduli space as the Hamiltonian reduction of a product over the singular points of symplectic manifolds, each of which is endowed with a Hamiltonian action of $\mathrm{GL}_2(\mathbb{C})$. At a regular singular point with adjoint orbit C, the symplectic manifold is C viewed as the coadjoint orbit $C\frac{dt}{t}$. The symplectic manifold at an irregular singular point with formal type A will be denoted $\mathcal{M}_A$; it is called an extended orbit. To define it, let $\mathcal{O}_A$ be the P_S-coadjoint orbit of $A|_{\mathfrak{p}_S} \in \mathfrak{p}_S^\vee$. If A is a $T(F)$-formal type, then $\mathcal{M}_A = \mathcal{O}_A$. The $\mathrm{GL}_2(\mathbb{C})$-action is the usual coadjoint action, and the moment map μ_A is just restriction of the functional α to $\mathfrak{gl}_2(\mathbb{C})$. The definition is more complicated when A is a $\mathbb{C}((\omega))^\times$-formal type. In this case, let $B \subset \mathrm{GL}_2(\mathbb{C})$ be the upper triangular subgroup. Then, $\mathcal{M}_A = \{(Bg, \alpha) \mid (\mathrm{Ad}^*(g)(\alpha))|_{\mathfrak{i}} \in \mathcal{O}_A)\} \subset (B\backslash \mathrm{GL}_2(\mathbb{C})) \times \mathfrak{gl}_2(\mathfrak{o})^\vee$. The group $\mathrm{GL}_2(\mathbb{C})$ acts on $\mathcal{M}_A$ via $h(Bg, \alpha) = (Bgh^{-1}, Ad^*(h)\alpha)$ with moment map $\mu_A : (Bg, \alpha) \mapsto \alpha|_{\mathfrak{gl}_2(\mathbb{C})}$.

We can now describe the structure of $\mathcal{M}(\mathbf{A}, \mathbf{C})$.

Theorem 3 *The moduli space $\mathcal{M}(\mathbf{A}, \mathbf{C})$ is obtained as a symplectic reduction of the product of local data:*

$$\mathcal{M}(\mathbf{A}, \mathbf{C}) \cong \left[\left(\prod_i \mathcal{M}_{A_i}\right) \times \left(\prod_j C_j\right)\right] /\!\!/_0 \mathrm{GL}_2(\mathbb{C}).$$

Remark 4 For other variants and a realization of the isomonodromy equations as an integrable system, see [2, 3].

Here, $\mathrm{GL}_2(\mathbb{C})$ acts diagonally on the product manifold, so that the moment map μ for the product is the sum of the moment maps of the factors. Since each factor involves a functional on $\mathfrak{gl}_2(\mathfrak{o})$ or $\mathfrak{gl}_2(\mathbb{C})$, the definition of the local moment maps shows that $\mu^{-1}(0)$ is the set of tuples for which the restrictions of these functionals to $\mathfrak{gl}_2(\mathbb{C})$ sum to 0. Writing each functional as a 1-form, this is just the condition that the sum of the residues vanish.

We conclude this paper with two illustrations of the theorem, each with one irregular singular and one regular singular point, say at 0 and ∞. Take $A^s = \mathrm{diag}(a, b)t^{-1}\frac{dt}{t} \in \mathcal{A}(T(F), 1)$ (so $a \neq b$) and $A^e = \omega^{-1}\frac{dt}{t} \in \mathcal{A}(\mathbb{C}((\omega))^\times, 1)$. Also, let C be an arbitrary nonresonant adjoint orbit. Below, we use the identifications $\mathfrak{gl}_2(\mathfrak{o})^\vee = \mathfrak{gl}_2(\mathbb{C})[t^{-1}]\frac{dt}{t}$ and $\mathfrak{i}^\vee = \mathfrak{t}[\omega^{-1}]\frac{dt}{t}$. Under these identifications, the restriction map $\mathfrak{gl}_2(\mathfrak{o})^\vee \to \mathfrak{i}^\vee$ has fiber $\mathbb{C}e_{12}\frac{dt}{t}$.

Example 5 ($\mathcal{M}(A_0^s, C_\infty)$) We first observe that $\mathrm{Ad}^*(1+t\,\mathfrak{gl}_2(\mathfrak{o}))(A^s) = A^s + \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}\frac{dt}{t}$ with $u, v \in \mathbb{C}$ arbitrary. Indeed, if $X, Y \in \mathfrak{gl}_2(\mathbb{C})$, then $(1 + tX)Y(1 + tX)^{-1} \equiv$

$Y + t[X, Y] \pmod{t^2}$, and the claim follows since $\operatorname{ad}(\operatorname{diag}(a, b))(\mathfrak{gl}_2(\mathbb{C}))$ consists of the off-diagonal matrices. Since $\mathrm{GL}_2(\mathfrak{o}) = \mathrm{GL}_2(\mathbb{C}) \ltimes (1 + t\,\mathfrak{gl}_2(\mathfrak{o}))$, we get

$$\operatorname{Ad}^*(\mathrm{GL}_2(\mathfrak{o}))(A^s) = \operatorname{Ad}^*(\mathrm{GL}_2(\mathbb{C})) \left\{ \begin{pmatrix} at^{-1} & u \\ v & bt^{-1} \end{pmatrix} \frac{dt}{t} \mid u, v \in \mathbb{C} \right\}. \tag{4}$$

The moduli space is the space of $\mathrm{GL}_2(\mathbb{C})$-orbits of pairs (α, Y) with $Y \in C$, and $\operatorname{Res}(\alpha) + Y = 0$. One sees from (4) that every orbit has a representative with α of the form $\left(\begin{smallmatrix} at^{-1} & u \\ v & bt^{-1} \end{smallmatrix}\right) \frac{dt}{t}$ for some $u, v \in \mathbb{C}$. Since T is the stabilizer of the leading term, it follows that the moduli space is the same as the set of T-orbits of pairs (α, Y) with α in this standard form. We claim that

$$|\mathcal{M}(A_0^s, C_\infty)| = \begin{cases} 2, & \text{if } C \text{ is regular nilpotent} \\ 1, & \text{if } C = 0 \text{ or } C \text{ is regular semisimple with trace } 0 \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

To see this, note that there are unique representatives for the T-orbits of standard α's by taking $(u, 1)$ with $u \in \mathbb{C}$, $(1, 0)$, and $(0, 0)$. Each $(u, 1)$ with $u \neq 0$ gives rise to Y regular semisimple with trace 0 and determinant u. The pairs $(1, 0)$ and $(0, 1)$ both lead to regular nilpotent Y's while $(0, 0)$ just gives $Y = 0$.

Example 6 ($\mathcal{M}(A_0^e, C_\infty)$) Here, the moduli space is the space of $\mathrm{GL}_2(\mathbb{C})$-orbits of triples (Bg, α, Y), where $(Bg, \alpha) \in \mathcal{M}_{A^e}$, $Y \in C$, and $\operatorname{Res}(\alpha) + Y = 0$. This is the same as the space of B-orbits of triples (B, α, Y). Using $I = T \ltimes I^1$, an argument similar to the one in the previous example shows that

$$\operatorname{Ad}^*(I)(A^e) = \operatorname{Ad}^*(T) \left\{ \begin{pmatrix} z & t^{-1} \\ 1 & -z \end{pmatrix} \frac{dt}{t} \mid z \in \mathbb{C} \right\}. \tag{6}$$

It follows easily that

$$\alpha = \begin{pmatrix} z & vt^{-1} + w \\ v^{-1} & -z \end{pmatrix} \frac{dt}{t} \tag{7}$$

for some $z, v, w \in \mathbb{C}$ with $v \neq 0$. In fact, each B-orbit has a unique representative with $v = 1$ and $z = 0$. This means that the only adjoint orbits C that give nonempty moduli space are the orbits of $\left(\begin{smallmatrix} 0 & -w \\ -1 & 0 \end{smallmatrix}\right)$. Thus, $\mathcal{M}(A_0^e, C_\infty)$ is a singleton if C is regular nilpotent or regular semisimple with trace zero; otherwise, it is empty. We remark that in the regular nilpotent case, the unique such connection is the GL_2 version of the Frenkel–Gross rigid connection, and this argument shows that this connection is indeed uniquely determined by its local behavior.

Remark 5 By setting $C = 0$ in these examples, we obtain the corresponding one singularity moduli spaces: $|\mathcal{M}(A_0^s)| = 1$ and $\mathcal{M}(A_0^e) = \emptyset$.

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Analysis of the Strategy Used to Solve Algorithmic Problem: A Case Study Based on Eye Tracking Research

Anna Stolińska and Magdalena Andrzejewska

Abstract In this article is presented the result of qualitative investigation in which a case study method and eye tracking technology were used. The participant has analyzed the algorithm shown in the flowchart. The path of saccades and fixations were recorded and the researchers followed the process of solving linear equations. The results confirmed the hypothesis that eye tracking technology can be used to optimize the educational process of learning programming.

Keywords Attention • Eye movements • Eye tracking • Flowchart of algorithms • Mathematical literacy

Mathematics Subject Classification (2010) Primary 97C30, 97C70; Secondary 97P99

1 Introduction

Mathematical reasoning and an ability to solve problems in an ordered and algorithmic way play an increasing role in the modern society. Mathematics, algorithmic and programming skills closely connected with them become more and more important, turning into a key competency for almost all workers. Lots of students find learning to program hard—they have difficulties in analyzing algorithms—a fundamental skill that involves decoding, understanding and predicting results. Failures of students connected with solving algorithmic problems result repeatedly in a decrease in motivation to learn programming which is thought to be a complicated skill, because of its complexity or a need to master a syntax of a programming language [1]. Those problems became a challenge for scientists who try to find effective methods of teaching mathematics, algorithms and programming

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[2]. More friendly environment and programming tools are being developed [3], studies on learning strategies are being conducted, innovative methods of teaching, such as gamification [4], techniques of visualization and simulation are being implemented [5]. However, it seems that the reasons for these problems should be looked for at in the first stages of the education in a scope of mathematical reasoning, during which students acquire basics to construct more advanced skills. One of the most fundamental elements of educating in a scope of programming is the ability to make and analyze algorithms which present a solution to a problem. A question can be asked what is more difficult for students: finding this solution or presenting it in such a way so that it takes the form of a correct algorithm, returning good results for any input data. It is also interesting in what way students read the ready algorithms, what causes that they cannot follow a course in their execution and cannot find mistakes in them. From the studies performed it results [6] that a graphic presentation of the algorithms seems to be more effective (in the meaning of its correct reading and determining a result of its operation), because it is easier to follow a control course (an order of executing instructions). Observing of which elements in a flow diagram focus attention of the students and finding the optimum (leading to solving a problem) information processing patterns can result in discovering strategies undertaken by the students and subsequently it can contribute to improving the education process in the scope of algorithms. In solving algorithmic problems, the ability to select and order information is very important. These tasks are aided by attention, an elementary cognitive process which is determined largely by the visual system. The goal of finding an objective method of analysis of the attention management process, including guidance and concentration, while solving an algorithmic problem, led researchers to study the techniques used in the eye tracking studies. The main goal of researchers was an answer to the question, **if the eye tracking technique allows to discover strategies of solving the algorithmic problems**, what subsequently could help a teacher select the optimum information processing patterns, which could be implemented in the process of teaching mathematics and algorithms.

2 Eye Tracking Technique

Eye tracking studies provide information on motor, optical and visual functions of eyeballs and provide a base for making an analysis of a psychological character [7]. In the oculomotor investigations it is assumed that the movement of the eyeballs is directed at those elements of a visual scene (namely the image presented) you think about, which are important for a viewer, therefore eye movement parameters are interpreted as indicators of cognitive processes [8]. The fundamental cognitive function is attention which is a certain kind of concentration, owing to which you can focus on the selected stimuli. An important indicator which enables to analyze the process of directing the attention is time devoted to processing data and a sequence in which the elements of the visual scene are observed. Eye tracking which enables, among other things, to measure those indicators turns out to be a useful technique in the education studies [9].

Saccades and fixations of which values can be presented as figures or visualized data owing to processing made by different types of programming are the main parameters measured during the investigations by an eye tracker. **Fixations** are motor functions of an eye which are interpreted as dwelling a gaze on a specific object, lasting on average 200–300 ms. In reality they consist of so-called internal fixation movements. They are: micro-jitter—alternating rhythmical movements of usually low amplitude, a micro-saccade (an eye movement lasting on average from 30 to 80 ms) and a micro-drift—a slow eye movement which prevents disappearance of a still image on the retina, connected with adaptation of receptors [10]. **Saccades** are jerky movements shifting a gaze to find an element of the visual scene. During a saccade an eye moves with the speed of 30–700°/s. Between the movement amplitude and its speed there is a close relationship—the higher amplitude the quicker the eye movement is. Latency from the moment of stimulus appearance to releasing the saccade is about 150–250 ms [11]. It is assumed that during the saccade it is not possible to see or to change the trajectory planned earlier and that they can be released in an intentional or involuntary way.

One of the very popular and frequently present forms of presenting investigation results is a *heat map*, which shows the areas to which a tested person directed his/her gaze differentiating them by colors. Warm colors (red, orange) indicate a great interest of a user in the specific area, whereas cold colors mean a lower concentration of focus on the specific region. The elements to which the gaze was not directed at all are not marked by any color (Fig. 1).

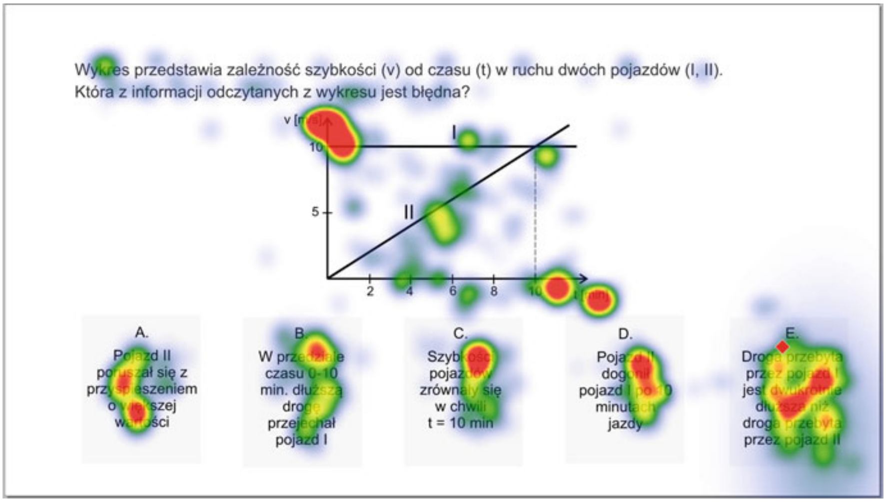

Fig. 1 An example of a picture presenting the heat map—involvement of attention of a student solving a test task

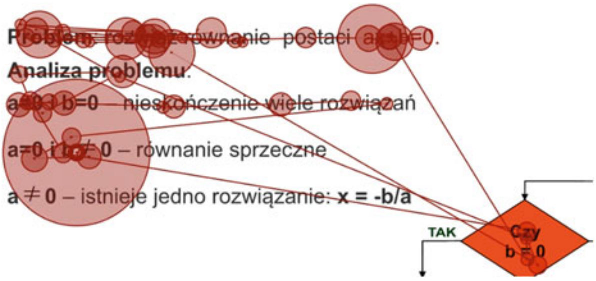

Fig. 2 An example of a scan path presenting the fixations while reading the task (a fragment of an image registered during the tests)

Scan paths by gaze (gaze plot) presenting a saccade movement after the visual scene are used to make an analysis of the sequence of the areas viewed. The results represented in such a way depict the areas of dwelling visual attention in a form of circles of which the size is proportional to the fixation time (Fig. 2).

An important function of majority of applications for managing the data obtained as a result of the eye tracking measurements is also a possibility to generate diagrams, a matrix, a percentage distribution of data (some software provides even over 100 statistically variables) and to single out areas of interest (AOI) with the key results for them. They include, among other things: *dwell time*, being the total time of duration of the fixations and saccades in the selected area of interest, the number of *visits* and *revisits*, *a sequence*—order of gaze hits into the AOIs based on *entry time* (average duration that elapsed before the first fixation).

Operation of the majority of eye trackers available on the market consists of locating a place on which the tested person focuses his/her gaze by directing infrared light (harmless for an eye and invisible for a human being) towards an eyeball and making a measurement of relative positions of a pupil and so-called a corneal reflection—a light reflection on the eye cornea. Those reflections in the form of reflexes are well visible and can be recorded by a camera. The eye tracking measurement requires equipment calibration to be made, which consists of displaying to the tested participant sequentially highlighting dots on the screen. The task of the tested person is to follow those highlights with his/her eyes. Glasses or contact lenses can interfere with good calibration because they disturb the correct course of the infrared light beam. After the correct calibration of the tested person, tests can be performed. Technological characteristics of the eye tracking equipment available on the market are diverse, although for majority of tests in a scope of the human being–computer interactions of a frequency measurement of 60 Hz is already sufficient. For investigations on a text read much more higher frequency is required, and it is deemed that it should be about 500 Hz or more [12].

3 Methodology and Methods

3.1 Eye Tracking Apparatus

In the experiment there was an eye tracker used of the firm SensoMotoric Instruments iView™Hi-Speed500/1250 recording a stream of data with 500 Hz time resolution such as for example coordinates x and y of a gaze position (in pixels or millimeters), a pupil width, parameters of the fixations and saccades or blinks. A comfortable construction of the interface used in this system allows for stable holding of a motionless head, without limiting the field of vision of the examined. This equipment is characterized by high tolerance to glasses and contact lenses. During the test visual images were presented on an LCD screen of 23″ screen diagonal, with resolution of Full HD 1920 × 1080. Before each test 9-point calibration with validation and other operations were made and their aim was to prepare the tested person in such a way that the obtained results could be deemed as reliable and non-distorted. A position of the chin support was corrected, among others, so that the tested person was in the most comfortable position, with simultaneous maintenance of the eye position centered in relation to the midpoint of the screen. Additionally, while testing all persons were provided the same environment conditions such as a temperature, lighting and acoustic insulation. Software Experiment Center™of the SensoMotoric Instruments firm was used to design the tests. It allows selecting stimuli in the form of a text, an image, a video, a www website, a pdf file or others, depending on the experiment conditions. The results were processed on basis of software SMI BeGaze™2.4.

3.2 Participants

Fifty two students of third forms of a junior high school at the age of 16, including 25 girls and 27 boys took part in the experiment. Measurement data of 7 persons were rejected due to technical reasons (improper calibration) and 45 cases were qualified for further analysis. Among the tested students a group of award winners in the subject completion in physics was singled out—this group consisted of 16 persons. The eyesight of all the tested students was normal or corrected to normal. All the students dealt with solving the algorithmic tasks in their school education, which was confirmed by a diagnostic survey before the test.

3.3 Procedure

The algorithmic tasks discussed in the article and solved by the tested students selected were presented in the form of a flowchart. The solution correctness

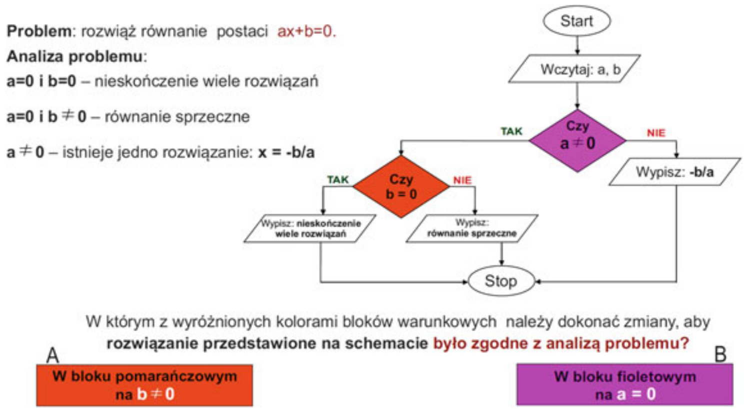

Fig. 3 The task which the students were solving

indicator was 66.7%, the correct answer was indicated by 30 students, among which 13 students were the finalists of the subject completion in physics. The students were solving the task presented in Fig. 3.

The task required to present correctly a solution of a linear equation in the form of $ax + b = 0$ by a flowchart. In the instruction an analysis of the problem was included, therefore, the task boiled down to verifying correctness of the record of the conditions entered in the blocks marked in orange or violet. Indicating the correct answer (marked by letter B), that is determining that in the first conditional block a condition should be written: "if $a = 0$", it could be done by adopting two strategies—one in which it was enough to check the content of the output block placed in the right arm of the algorithm (for the "No" alternative) or an alternative one, resulting from the assumption that there is a condition that a takes a value different from 0. In this second case instead of the conditional block "if $b = 0$", should be present the output block "Write out: $-b/a$".

4 Results

The eye tracking investigations provide quantitative data for which statistical interference can be performed. However, in the event of the problem considered in this article an application of qualitative methodology, which is used in the situation when a researcher is interested in deepened knowledge on a certain subject and getting to the root of the problem when representativeness of the results is not necessary, but rather knowing the essence of the phenomenon, seems to be more appropriate. Qualitative data can include any record of information in a written, audio or video form. Therefore, the material, obtained as a result of measurements—dynamic gaze paths with fixation duration time and films showing points in which there is a gaze of the tested person—a *bee swarm*, was analyzed. The main aim of the studies was to verify if the eye tracking technique enables us to identify strategies of solving algorithmic problems, that is why it was decided to use a qualitative analysis which is characteristic for a case study and to provide an answer on basis of some selected cases. However, the results, which were obtained on basis of an analysis of the film materials, were supplemented with numerical data connected with an eye activity of the cases in question in the determined areas of interest. The preliminary selection of the research material allowed to specify the people who solved the task correctly (30 persons). The survey made after the examination enabled subsequently to reject those students who declared that they could not establish the correct answer and had made a random choice (11 persons). Among other 19 cases, after watching all the films 2 students were selected (in the further part of the article marked by identifiers P1 and P2)—those who obtained the highest results out of all the tasks solved while being tested (at 80% level) and solved the task at different time (P1 about 75 s, P2 about 108 s.) And they analyzed them differently (Fig. 4).

On the board with the task there were singled out areas of interest (AOI) for which the program BeGaze generated information on the gaze activity of the tested. The following areas were analyzed and interpreted:

- AOI 001—the problem and the mathematical analysis of its solution,
- AOI 002—the formula describing the linear equation,
- AOI 006—the area of the flowchart,
- AOI 008—the main conditional block, for coefficient a,
- AOI 010—the output block area when there is one solution,
- AOI 009—the chart area, with the conditional block for coefficient b and appropriate output blocks,
- AOI 011—the conditional block for coefficient b,
- AOI 014—the area with the instruction to the task.

The parameters which were the subject of interest were following:

- Dwell Time—expressed by percentage of time percent which the tested dwelt on observing the area,
- Fixation Count—a number of fixations in the given area,

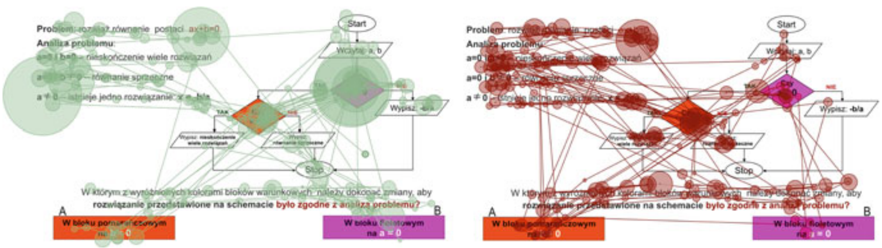

Fig. 4 Fixations and saccade paths—a record from the screen made while the tasks were being solved by the tested P1 and P2

Table 1 Presentation of the selected AOI parameters for cases P1 and P2

Subject	Area of interest	Dwell time [%]	Fixation count	Revisits	Average fixation [ms]
P1	AOI 001	32.9	77	5	286.5
P2		36.8	160	30	219.9
P1	AOI 002	4.2	7	2	437.0
P2		3.1	9	5	367.5
P1	AOI 006	38.6	80	13	324.0
P2		33.1	136	36	235.5
P1	AOI 008	15.5	26	11	427.1
P2		11.1	42	17	265.5
P1	AOI 009	13.5	31	17	308.8
P2		13.5	60	26	226.0
P1	AOI 010	2.6	6	0	304.8
P2		3.3	13	8	270.0
P1	AOI 011	9.5	24	14	283.2
P2		6.9	30	25	247.8
P1	AOI 014	13.5	45	11	195.6
P2		14.2	63	13	210.0

- Revisits—a number of re-fixations (re-visits) in the given area,
- Average Fixation—the average time of the fixation duration in this area.

The obtained results (values of the parameters above) are presented in Table 1.

By analyzing the data in Table 1 it can be noticed that the tested P1 devoted less time in percent to a mathematical analysis of solving the problem than P2 (area A001, Dwell Time P1 32.9%, P2 36.8%), and at the same time he gazed longer at the formula describing the linear equation (area A002, Dwell Time P1 4.2%, P2 3.1%). Independently of the time devoted to the analysis of those two areas the P1 behavior is also characterized by a lower number of fixations—in the case of area A001 it is almost twice lower (Fixation Count P1 77, P2 160), and further, what seems to be very important a much lower number of revisits particularly for area A001 (Revisits P1 5, P2 30). It is worth noticing that such a relation took place in each AOI in question—regardless of the time percentage which the tested P1 devoted to its observation, made lower number of fixations comparing to P2. This regularity also concerns a number of revisits to the particular areas of interest. The tested P1 observed the area of the flowchart longer in percent (area A006, Dwell Time P1 38.6%, P2 33.1%), he gazed longer at both conditional blocks (area A008, Dwell Time P1 15.5%, P2 1.1%; area A011, Dwell Time P1 9.5%, P2 6.9%), although the analysis of the flowchart area, with the conditional block for coefficient b and appropriate output blocks, took him the same time percentage as the tested P2 (area A009, Dwell Time P1 and P2 13.5%). However, on the output block in the case when there is one solution, the tested P1 gazed for a shorter time (area A010, Dwell Time P1 2.6% i P2 3.3%). Participant P1 also analyzed the text of the instruction itself for a shorter time (area A014, Dwell Time P1 13.5% and P2 14.2%). It should be noticed that only in the case of this area, P1 showed shorter average time of fixation duration fixation (area A014, Average Fixation P1 195.6 ms and P2 210.0%), and the longer average time of the fixation duration is usually interpreted as a sign of focusing—more careful (deeper) data processing.

5 Conclusions

Both observations of eyesight paths and figures from AOI allowed to notice that case P1 looking for the solution referred less often to its analysis presented in the task content. It seems that already at the stage of reading the task content and its mathematical analysis he identified the problem as the linear equation solution known to him, after recognizing the mathematical problem (being a standard school task) he analyzed the flowchart regarding its conformity with the task solution, which was well known to him. Case P2 concentrated (being guided by the instruction given in the content) on conformity of the solution presented in the flowchart with the problem analysis, that is why during following the flow chart he moved his gaze many times to particular cases of the equation solution presented in its mathematical analysis making their verification on the algorithm flow chart. Analysis of the video films lets you also observe that the P1 student made the decision as the result of adopting the strategy which can be recognized as optimal—first he paid attention to the content of the output block located on the high branch of the algorithm. Such a solution made determining quickly the correct answer possible. However, it should be noticed that the qualitative analysis enabled to determine that in the P1 case additional time spent on perception was not used to examine the secondary elements, but to reexamine the adopted solution. To conclude it should be stated that on the basis of the eyesight analysis and the values of the selected oculomotor parameters in the areas of interest singled out, it was possible to recognize and indicate difference in the strategy of solving an algorithmic task by two selected students who obtained the correct answers. This result confirms that techniques of visual activity recording can be used to identify cognitive processes and it is in compliance with the opinion of other researchers [8] who have stated that eye movements reflect the human thought processes. Use of the techniques of the visual activity recording to study learning process understood widely widened our knowledge in the scope of cognitive processes. Eye tracking can be useful to identify behaviors and reactions which experiment participants cannot or do not want to describe. As the experiment results show it can be helpful in the studies on methods of teaching of information technology, in particular in the scope of algorithmic and programming because it allows observing and differentiating information processing patterns.

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Part II
Complex and Functional-Analytic Methods for Differential Equations and Applications

Heinrich Begehr

Green and Neumann Functions for Strips

H. Begehr

Abstract Harmonic Green and Neumann functions are constructed using the parqueting-reflection principle for strips and hyperbolic strips in the complex plane.

Keywords Green and Neumann functions • Hyperbolic strip • Parqueting-reflection principle • Strip

Mathematics Subject Classification: 31A25, 35J25

1 Introduction

A set of plane domains $\{D_i : i \in I\}, I$ some index set, is said to provide a parqueting of the complex plane $\mathbb{C}$ if $\cup_{i \in I} \overline{D_i} = \mathbb{C}$ while $D_i \cap D_k = \emptyset$ for $i, k \in I, i \neq k$. Occasionally a domain D is providing a parqueting of $\mathbb{C}$ through continued reflections of D at its parts of the boundary ∂D. For this it is necessary that ∂D is composed by arcs from circles and straight lines. Such domains can be, e.g., convex polygons, disc sectors, infinite sectors, circular ring domains, circular concave or convex lenses, straight strips, half planes, hyperbolic strips, hyperbolic half planes. In case a parqueting of $\mathbb{C}$ is provided the parqueting-reflection principle helps to construct the harmonic Green and Neumann functions as well for the original domain as for all the other domains from the parqueting.

Definition A domain D of the complex plane $\mathbb{C}$ with piecewise smooth boundary ∂D is called admissible for the parqueting-reflection principle, if continued reflections at the boundary parts achieve a parqueting of $\mathbb{C}$.

The Parqueting-Reflection Principle The original domain D is called a pole-domain. Any direct reflection of D at some part of ∂D is called a zero-domain. A reflection of a zero-domain at some part of its boundary is a pole-domain.

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Choosing an arbitrary point $z \in D$ it will become a pole. Any reflection of z into a zero-domain will become a zero. Any reflection of a zero in a zero-domain will become a pole in a pole-domain and vice versa.

Let $\{z\} \cup \{z_k\}$ and $\{\widetilde{z}_0\} \cup \{\widetilde{z}_k\}$ be the sets of poles and zeros respectively, from this construction. Then the harmonic Green function of D is

$$G_1(z,\zeta) = \log \Big|\frac{\zeta - \widetilde{z}_0}{\zeta - z} \prod_k \frac{\zeta - \widetilde{z}_k}{\zeta - z_k}\Big|^2.$$

A candidate for the harmonic Neumann function for D is

$$N_1(z,\zeta) = -\log \Big|(\zeta - z)(\zeta - \widetilde{z}_0) \prod_k (\zeta - z_k)(\zeta - \widetilde{z}_k)\Big|^2.$$

Here in case of infinite products for the Neumann functions certain factors achieving convergence have to be inserted.

The principle was applied for a circular ring in [10, 13], for a triangle, see [5–7, 14], for some concave and convex lenses [8, 9], for disc sectors [15], for half ring [4], for half hexagon[11, 12], for quarter ring[11, 12], for hyperbolic strip [1, 2]. The results for the plane strip and a hyperbolic strip are reported on here. For details see [3]. For the half strip see also [3].

2 Dirichlet and Neumann Problems in a Strip

Let for $0 < \alpha < \pi, a \in \mathbb{R}^+$ the set S_1 be the strip

$$S_1 = \{z \in \mathbb{C} : z = e^{2i\alpha}\overline{z} + 2iate^{i\alpha}, 0 < t < 1\}.$$

The boundary parts

$$\partial^- S_1 = \{z \in \mathbb{C} : z = e^{2i\alpha}\overline{z}\}, \partial^+ S_1 = \{z \in \mathbb{C} : z = e^{2i\alpha}\overline{z} + 2iae^{i\alpha}\}$$

are parallel lines with angle α against the positive real axis, the first one passing through the origin, the second one above the first in distance a to the former.

Continued reflection of S_1 at the boundaries provides a parqueting of $\mathbb{C}$ through the strips

$$S_{k+1} = \{z \in \mathbb{C} : z = e^{2i\alpha}\overline{z} + 2ia(k+t)e^{i\alpha}, 0 < t < 1\}, k \in \mathbb{Z},$$

$$\mathbb{C} = \bigcup_{k \in \mathbb{Z}} \overline{S_k}, \quad S_k \cap S_l = \emptyset \quad for \quad k \neq l.$$

An arbitrary point $z \in S_1$ has the representation $z = e^{2i\alpha}\overline{z} + 2iate^{i\alpha}$ for some $t, 0 < t < 1$. Reflecting this point at the line $z = e^{2i\alpha}\overline{z} + 2iae^{i\alpha}$ gives the image

$$z_2 = e^{2i\alpha}\overline{z} + 2iae^{i\alpha} \in S_2.$$

Reflecting this point at the line $z = e^{2i\alpha}\overline{z} + 4iae^{i\alpha}$ leads to

$$z_3 = e^{2i\alpha}\overline{z_2} + 4iae^{i\alpha} = z + 2iae^{i\alpha} \in S_3.$$

Inductively $z_{2k+1} = z + 2iake^{i\alpha} \in S_{2k+1}, \quad z_{2k} = e^{2i\alpha}\overline{z} + 2iake^{i\alpha} \in S_{2k}, \quad k \in \mathbb{Z}$, follow. Choosing the z_{2k} as zeros and the z_{2k+1} as poles leads to the meromorphic function

$$\begin{aligned} P(z,\zeta) &= \frac{\zeta - e^{2i\alpha}\overline{z}}{\zeta - z} \prod_{k=1}^{\infty} \frac{\zeta - e^{2i\alpha}\overline{z} - 2iake^{i\alpha}}{\zeta - z - 2iake^{i\alpha}} \frac{\zeta - e^{2i\alpha}\overline{z} + 2iake^{i\alpha}}{\zeta - z + 2iake^{i\alpha}} \\ &= \frac{\zeta - e^{2i\alpha}\overline{z}}{\zeta - z} \prod_{k=1}^{\infty} \frac{(\zeta - e^{2i\alpha}\overline{z})^2 - (2iake^{i\alpha})^2}{(\zeta - z)^2 - (2iake^{i\alpha})^2} = \frac{\sin \pi \frac{\zeta - e^{2i\alpha}\overline{z}}{2iae^{i\alpha}}}{\sin \pi \frac{\zeta - z}{2iae^{i\alpha}}}. \end{aligned}$$

$G_1(z,\zeta) = \log |P(z,\zeta)|^2$ is the Green function for S_1. For $\alpha = 0, a = 1$ this is a classical result, see, e.g., [6]. $N_1(z,\zeta) = -\log\left|\sin \pi \frac{\zeta - e^{2i\alpha}\overline{z}}{2iae^{i\alpha}} \sin \pi \frac{\zeta - z}{2iae^{i\alpha}}\right|^2$ is the Neumann function for S_1. By the way, the symmetry of both the Green and the Neumann functions are obvious.

Theorem 1 *The Dirichlet problem*

$$w_{z\overline{z}} = f, \quad in \quad S_1, \quad w = \gamma \quad on \quad \partial S_1,$$

$$f \in L_{p,2}(S_1;\mathbb{C}), 2 < p, \ \gamma \in C(\partial S_1;\mathbb{C}), \lim_{x\to\infty} x^{1+\epsilon}\gamma(x+iy) = 0, x+iy \in S_1, 0 < \epsilon,$$

is uniquely solvable by

$$\begin{aligned} w(z) = &-\frac{1}{2a}\int_{\partial^- S_1} \gamma(\zeta)\mathrm{Re}\cot \pi \frac{\zeta - z}{2iae^{i\alpha}} ds_\zeta - \frac{1}{2a}\int_{\partial^+ S_1} \gamma(\zeta)\mathrm{Re}\cot \pi \frac{\zeta - z}{2iae^{i\alpha}} ds_\zeta \\ &+\frac{1}{\pi}\int_{S_1} f(\zeta) \log\left|\frac{\sin \pi \frac{\zeta - e^{2i\alpha}\overline{z}}{2iae^{i\alpha}}}{\sin \pi \frac{\zeta - z}{2iae^{i\alpha}}}\right|^2 d\xi d\eta. \end{aligned}$$

Theorem 2 *The Neumann problem*

$$w_{z\overline{z}} = f, \quad in \quad S_1, \quad \partial_\nu w = \gamma \quad on \quad \partial S_1,$$

$$f \in L_{p,2}(S_1;\mathbb{C}), 2 < p, \ \gamma \in C(\partial S_1;\mathbb{C}), \lim_{x\to\infty} x^{1+\epsilon}\gamma(x+iy) = 0, x+iy \in S_1, 0 < \epsilon,$$

is for any $c \in \mathbb{C}$ solvable by

$$w(z) = c - \frac{1}{2\pi}\int_{\partial - S_1} \gamma(\zeta) \log\left|\sin \pi \frac{\zeta - z}{2iae^{i\alpha}}\right|^2 ds_\zeta$$
$$-\frac{1}{2\pi}\int_{\partial + S_1} \gamma(\zeta) \log\left|\sin \pi \frac{\zeta - z}{2iae^{i\alpha}}\right|^2 ds_\zeta$$
$$+\frac{1}{\pi}\int_{S_1} f(\zeta) \log\left|\sin \pi \frac{\zeta - e^{2i\alpha}\overline{z}}{2iae^{i\alpha}} \sin \pi \frac{\zeta - z}{2iae^{i\alpha}}\right|^2 d\xi d\eta.$$

Remark The Neumann problem is unconditionally solvable for the strip due to the circumstance that the normal derivative of the Neumann function vanishes at the boundary of the strip as long as the other variable lies inside the strip. The proofs of both results are straightforward by verification.

3 Dirichlet and Neumann Problems for the Hyperbolic Strip

A hyperbolic strip is given by two circles not intersecting one another and orthogonal to the unit circle. There are two main different cases, either one of the two discs is enclosed in the other or they are disjoint. While the latter case can be transformed into the first one by reflection the converse is not always possible. Here the following case is studied. For four real numbers m_1, m_2 greater than 1 and positive r_1, r_2 given such that $1 + r_1^2 = m_1^2, 1 + r_2^2 = m_2^2$ the circles $\partial D_{-m_1}(r_1), \partial D_{m_2}(r_2)$ where $D_m(r) = \{|z - m| < r\}$ for $0 < r, 1 < m, 1 + r^2 = m^2$ are orthogonal to the unit circle $\partial\mathbb{D}$. For $r_1 + r_2 < m_1 + m_2$ the relations $-1 < r_1 - m_1 < 0 < m_2 - r_2 < 1$ hold. Both circles are disjoint and $D = \mathbb{D} \setminus \overline{\{D_{-m_1}(r_1) \bigcup D_{m_2}(r_2)\}}$ is a hyperbolic strip, see [1, 2].

The disc $D_{m_2}(r_2)$ is reflected at $\partial D_{-m_1}(r_1)$ onto $D_{-m_3}(r_3), m_3 = \frac{2\alpha\beta - m_2(\alpha^2+\beta^2)}{(\alpha^2+\beta^2)-2\alpha\beta m_2}$, $m_3^2 = r_3^2 + 1, \alpha = m_1 m_2 + 1, \beta = m_1 + m_2$. The circle $\partial D_{-m_3}(r_3)$ is orthogonal to $\partial\mathbb{D}$ and $D_{-m_3}(r_3)$ is compactly included in $D_{-m_1}(r_1)$. Reflecting now $D_{-m_1}(r_1)$ at $\partial D_{-m_3}(r_3)$, etc. gives a set of nested discs $D_{-m_{2k+1}}(r_{2k+1}), \alpha_{2k-1} = m_{2k-1}m_{2k+1} - 1,\ \beta_{2k-1} = m_{2k-1} - m_{2k+1}, k \in \mathbb{N}$,

$$m_{2k+3}^2 = r_{2k+3}^2 + 1,\ m_{2k+3} = \frac{2\alpha_{2k-1}\beta_{2k-1} + m_{2k-1}(\alpha_{2k-1}^2 + \beta_{2k-1}^2)}{\alpha_{2k-1}^2 + \beta_{2k-1}^2 + 2\alpha_{2k-1}\beta_{2k-1}m_{2k-1}},$$

shrinking monotonically towards the point -1. Interchanging the roles of m_1 and m_2 produces a sequence of discs $D_{m_{2k}}(r_{2k})$, shrinking monotonically towards the point 1, with $\alpha_{2k} = m_{2k}m_{2k+2} - 1, \beta_{2k} = m_{2k} - m_{2k+2}, k \in \mathbb{N}$, and

$$r_{2k+4}^2 + 1 = m_{2k+4}^2,\ m_{2k+4} = \frac{2\alpha_{2k}\beta_{2k} + (\alpha_{2k}^2 + \beta_{2k}^2)m_{2k}}{\alpha_{2k}^2 + \beta_{2k}^2 + 2\alpha_{2k}\beta_{2k}m_{2k}}.$$

Reflecting an arbitrary point $z \in D = \mathbb{D} \setminus \{\overline{D_{-m_1}(r_1) \cup D_{m_2}(r_2)}\}$ at $\partial D_{-m_1}(r_1)$ gives

$$z_1 = -\frac{m_1\overline{z} + 1}{\overline{z} + m_1} \in \mathbb{D} \cap \{D_{-m_1}(r_1) \setminus \overline{D_{-m_3}(r_3)}\}.$$

The reflection of z_1 at $\partial D_{-m_3}(r_3)$ is

$$z_3 = -\frac{m_3\overline{z_1} + 1}{\overline{z_1} + m_3} = \frac{\alpha_1 z - \beta_1}{-\beta_1 z + \alpha_1},\ \alpha_1 = m_1 m_3 - 1, \beta_1 = m_1 - m_3.$$

Continuing, then $z_{2k-1} \in \mathbb{D} \cap \{D_{-m_{2k-1}}(r_{2k-1}) \setminus \overline{D_{-m_{2k+1}}(r_{2k+1})}\}$ is reflected at $\partial D_{-m_{2k+1}}(r_{2k+1})$ onto

$$z_{2k+1} = -\frac{m_{2k+1}\overline{z_{2k-1}} + 1}{\overline{z_{2k-1}} + m_{2k+1}} = \frac{\alpha_{2k-1} z_{2k-3} - \beta_{2k-1}}{-\beta_{2k-1} z_{2k-3} + \alpha_{2k-1}}.$$

The continued reflections of an arbitrary point $z \in D = \mathbb{D} \setminus \overline{\{D_{-m_1}(r_1) \cup D_{m_2}(r_2)\}}$ at $\partial D_{m_2}(r_2)$ leads to the points $z_{2k} \in D_{m_{2k}}(r_{2k}) \setminus \overline{D_{m_{2k+2}}(r_{2k+2})}, k \in \mathbb{N}$. They are

$$z_2 = \frac{m_2\overline{z} - 1}{\overline{z} - m_2}, z_{2k} = \frac{m_{2k}\overline{z_{2k-2}} - 1}{\overline{z_{2k-2}} - m_{2k}} = \frac{\alpha_{2k-2} z_{2k-4} + \beta_{2k-2}}{\beta_{2k-2} z_{2k-4} + \alpha_{2k-2}}, 1 < k.$$

The two families of hyperbolic strips $D_{-m_{2k-1}}(r_{2k-1}) \setminus \overline{D_{-m_{2k+1}}(r_{2k+1})}, D_{m_{2k}}(r_{2k}) \setminus \overline{D_{m_{2k+2}}(r_{2k+2})}, k \in \mathbb{N}$, together with the strip D itself compose a parqueting of the unit disc $\mathbb{D}$. Reflecting $\mathbb{D}$ at its boundary completes the parqueting of the complex plane $\mathbb{C}$. The last reflection transforms the points $z_k \in \mathbb{D}$ onto $\overline{z_k}^{\,-1}$.

Remark 1 The reflection at $\partial\mathbb{D}$ maps each of the discs $D_{-m_1}(r_1), D_{m_2}(r_2)$ onto itself and the domain D onto $\mathbb{C} \setminus \overline{\{\mathbb{D} \cup D_{-m_1}(r_1) \cup D_{m_2}(r_2\}}$, [2].

The parqueting-reflection principle [4–6, 8, 9, 11–15] leads to the meromorphic function in $\mathbb{C}$

$$P(z, \zeta) = \frac{1 - \overline{z}\zeta}{\zeta - z} \frac{\zeta - z_1}{1 - \overline{z_1}\zeta} \frac{\zeta - z_2}{1 - \overline{z_2}\zeta} \prod_{k=1}^{\infty} \frac{1 - \overline{z_{4k-1}}\zeta}{\zeta - z_{4k-1}} \frac{1 - \overline{z_{4k}}\zeta}{\zeta - z_{4k}} \frac{\zeta - z_{4k+1}}{1 - \overline{z_{4k+1}}\zeta} \frac{\zeta - z_{4k+2}}{1 - \overline{z_{4k+2}}\zeta},$$

converging for z and ζ in D. The harmonic Green function for the hyperbolic strip D is $G_1(z, \zeta) = \log |P(z, \zeta)|^2$, see [2]. The respective Neumann function is, see [1],

$$N_1(z, \zeta) = -\log |Q(z, \zeta)|^2, z \in D, \zeta \in D,$$

with

$$Q(z,\zeta) = (\zeta - z)(1 - \overline{z}\zeta) \prod_{k=1}^{\infty} \frac{\zeta - z_{2k-1}}{\zeta + 1} \frac{1 - \overline{z_{2k-1}}\zeta}{\overline{z_{2k-1}}(1+\zeta)} \frac{\zeta - z_{2k}}{\zeta - 1} \frac{1 - \overline{z_{2k}}\zeta}{\overline{z_{2k}}(1-\zeta)}.$$

The respective boundary value problems for the hyperbolic strip are treated, e.g., in [3].

Theorem 3 *The Dirichlet problem*

$$w_{z\overline{z}} = f \quad in \quad D, \quad f \in L_p(D;\mathbb{C}), \quad 2 < p,$$
$$w = \gamma \quad on \quad \partial D, \gamma \in C(\partial D;\mathbb{C}), \gamma\left(-\frac{1}{2} \pm i\frac{m_1}{r_1}\right) = \gamma\left(\frac{1}{2} \pm i\frac{m_2}{r_2}\right) = 0,$$

is uniquely solvable by

$$w(z) = -\frac{1}{4\pi}\int_{\partial D} \gamma(\zeta)\partial_{\nu_\zeta} G_1(z,\zeta) ds_\zeta - \frac{1}{\pi}\int_D f(\zeta) G_1(z,\zeta) d\xi d\eta.$$

Theorem 4 *The Neumann problem for the Poisson equation*

$$w_{z\overline{z}} = f \quad in \quad D, \quad \partial_\nu w = \gamma \quad on \quad \partial D,$$

for $f \in L_p(D;\mathbb{C}), 2 < p, \gamma \in C(\partial D;\mathbb{C}), \gamma(-\frac{1}{2} \pm i\frac{m_1}{r_1}) = \gamma(\frac{1}{2} \pm i\frac{m_2}{r_2}) = 0$, *is solvable if and only if*

$$\frac{1}{2\pi}\int_{\partial D} \gamma(\zeta) ds_\zeta = \frac{2}{\pi}\int_D f(\zeta) d\xi d\eta.$$

The solution then is with some arbitrary $c \in \mathbb{C}$

$$w(z) = c + \frac{1}{4\pi}\int_{\partial D} \gamma(\zeta) N_1(z,\zeta) ds_\zeta - \frac{1}{\pi}\int_D f(\zeta) N_1(z,\zeta) d\xi d\eta.$$

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The Forecast of Ebola Virus Propagation

Zaiqiang Ku and Li Cheng

Abstract To control and alleviate the outbreak of Ebola hemorrhagic fever in West Africa, it is quite important to understand the transmission of Ebola virus and establish a proper medicine transportation system. The population in the regions suffering from Ebola virus disease (EVD) is divided into three groups including susceptible (S), infected (I), and recovered (R) ones. According to the transmitting speed of Ebola virus is changed, the SIR model is established to obtain the relationship between the number of affected individuals within a specific period of time. The outbreak of EVD in Sierra Leone would reach State Emergency Level 3 announced by WHO within 30 weeks starting July 2nd, 2014, and the number of affected individuals would decrease gradually.

Keywords Ebola virus disease • Medicine delivery • SIR model

Mathematics Subject Classification (2010) 92C50, 34A30

1 Introduction

1.1 Problem Background

Ebola hemorrhagic fever (EHF) is caused by the Ebola virus (EV), an acute hemorrhagic disease. It has a very high infectivity, and the fatality rate is as high as 50–88%. People are mainly infected through the contact with infectious patients or animals via fluid, feces, secretion, etc. The quantity of the medicine needed, speed of manufacturing of the vaccine or drug, locations of delivery and any other decisive factors should be considered to develop a feasible vaccine or drug delivery system such that the pressure on the current fight against Ebola could be relieved.

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1.2 Previous Research

Studying on mathematical model for Ebola virus, as early as in 1996, Fauci [1] used the SIR and SEIR model to simulate Zaire twice Ebola outbreak process: the Yambuku crisis in 1976 and the Kikwit epidemic in 1995 [2]. The final conclusion as follows: when the basic reproductive rate was satisfying $1.72 < R_0 < 8.60$, then the Ebola virus infection was not so serious as before and the potential death of population could be reduced.

1.3 Our Works

The remainder of the paper is organized as follows. In Sect. 2.1, we establish a Ebola virus propagation model. We first simulate the parameters of the SIR model, and then spread the process of the Ebola virus and eliminate the analysis.

2 Ebola Virus Propagation Model

2.1 SIR Model Based on the Transmission of Ebola Virus

SIR model divides the region of disease into three types of status: susceptible state (S), infection state (I), and recovery mode (R) [3–6]. This model is considering the effect of eliminating the Ebola virus.

2.2 Terms, Definitions, and Symbols

Before constructing a mathematical model for the problem, let us introduce the following terms, definitions, and symbols used in this paper.

- S: denotes the susceptible state, and its number is recorded as $S(t)$.
- I: denotes the infection state, and its number is recorded as $I(t)$.
- R: denotes the recovery mode, and its number is recorded as $R(t)$.
- N: the total number of population in the disease area (city or country).
- Patients' daily contact rate (infection) is λ. Daily cure rate is μ, and the number of people from infection to restore state per unit time is μI.
- The contact coefficient during infectious period is $\delta = \frac{\mu}{\lambda}$.

2.3 Assumptions

We make the following three assumptions.

- Over a period of time, the research object is ideal, without thinking of moving and other causes of death, so the total population maintains a constant level N.
- Only the patients who have been cured after going through the treatment have long-term immunity, anyone else has no immunity.
- If people who have no immunity contact with an infected person, then they become contagious.

2.4 The Foundation of Model

Obviously, the ideal situation is

$$S(t) + I(t) + R(t) = N. \tag{1}$$

The model described by differential equations is

$$\begin{cases} \dfrac{dS}{dt} = -\lambda SI, S(0) = S_0 \\ \dfrac{dI}{dt} = \lambda SI - \mu I, I(0) = I_0 \\ \dfrac{dR}{dt} = \mu I. \end{cases} \tag{2}$$

2.5 Solution and Result

On the basis of Eqs. (1) and (2), the solution can be obtained as follows:

$$S(t) = S_0 e^{-\frac{R}{S}}. \tag{3}$$

From the first three items of its Taylor expansion, by eliminating the change rate of the number of people, we have the approximate solution

$$\frac{dR}{dt} = \mu\left[N - R - S_0\left(1 - \frac{R}{\delta} + \frac{1}{2}\left(\frac{R}{S}\right)^2\right)\right]. \tag{4}$$

The initial value of the solution is obtained under the cumulative removal number:

$$R(t) = \frac{\delta^2}{S_0}\left[\frac{S_0}{\delta} - 1 + \alpha \tanh\left(\frac{1}{2}\alpha\mu t - \varphi\right)\right], \tag{5}$$

where $\alpha = \frac{\delta^2}{S_0}\left[\left(\frac{S_0}{\delta} - 1\right)^2 + \frac{2S_0 I_0}{\delta^2}\right]^{\frac{1}{2}}$, and $\tanh\varphi = \frac{S_0 - \delta}{\alpha\delta}$.

Therefore, Eq. (4) can be reduced to

$$\frac{dR}{dt} = \frac{\mu\alpha^2\delta^2}{2S_0}\frac{1}{\mathrm{ch}^2(\frac{\mu\alpha t}{2} - \varphi)}. \tag{6}$$

The final infection incidence area calculation is as follows:

$$\begin{cases} I(t) = I_0 + S_0 - S + \delta \ln\frac{S}{S_0}, I(0) = I_0 \\ S(t) = S_0 e^{-\frac{R}{\delta}}, S(0) = S_0 \\ R(t) = \frac{\delta^2}{S_0}\left[\frac{S_0}{\delta} - 1 + \alpha \tanh\left(\frac{1}{2}\alpha\mu t - \varphi\right)\right] \\ \delta = \frac{\mu}{\lambda} \\ \alpha = \frac{\delta^2}{S_0}\left[\left(\frac{S_0}{\delta} - 1\right)^2 + \frac{2S_0 I_0}{\delta^2}\right]^{\frac{1}{2}} \\ \tanh\varphi = \frac{S_0 - \delta}{\alpha\delta} \\ S(t) + I(t) + R(t) = N. \end{cases} \tag{7}$$

2.6 Analysis of the Result

In this subsection, we analyze the effect of $R(t), S(t), I(t)$.

(1) Analysis of the changes of $R(t)$.

We can get $\frac{1}{\mathrm{ch}^2(\frac{\mu\alpha t}{2} - \varphi)} \leq 1$, because of $\mathrm{ch}^2(\frac{\mu\alpha t}{2} - \varphi) \geq 1$.

In the item (6) if and only if $\frac{\mu\alpha t}{2} - \varphi = 0$, that is, $t = \frac{2\varphi}{\mu\alpha}$. $\frac{dR}{dt}$ gets the maximum value, also the number of the recovery mode is the largest.

(2) Analysis of the changes of $S(t), I(t)$.

The first two equations of the model are independent of $R(t)$, so the two equations of the relationship between $I(t)$ and $S(t)$ can be used.

First we consider

$$\begin{cases} \dfrac{dS}{dt} = -\lambda SI, S(0) = S_0 \\ \dfrac{dI}{dt} = \lambda SI - \mu I, I(0) = I_0. \end{cases} \tag{8}$$

Eliminating dt, we thus get the first order equation $\frac{dI}{dS} = -1 + \frac{\delta}{S}$. By solving the equation, we get $I(s) = I_0 + S_0 - S + \delta \ln \frac{S}{S_0}$. Denote $\lim_{t\to\infty} S(t) = S_\infty$, $\lim_{t\to\infty} I(t) = I_\infty$, and $\lim_{t\to\infty} R(t) = R_\infty$.

(i) The final uninfected healthy proportion is S_∞. It is the inner root of the equation $I_0 + S_0 - S + \delta \ln \frac{S}{S_0} = 0$.

(ii) If $S_0 > \delta$, then $I(t)$ first increases. When $S_0 = \delta$, $I(t)$ reaches the maximum value: $I_m = S_0 + I_0 - \delta(1 + \ln \frac{S_0}{\delta})$ and then $I(t)$ decreases and tends to 0. $S(t)$ is monotonically decreasing to S_∞ only if the number of infected persons $I(t)$ has a growing period, and then infectious diseases are spreading. The value δ is a threshold. When $S_0 > \delta$, the infectious diseases will spread.

(iii) If $S_0 \le \delta$, $I(t)$ is monotonically decreasing to 0, and $S(t)$ is monotonically decreasing to S_∞. Therefore reducing the infectious period, where the contact number is δ, making $S_0 \le \delta$. Hence, the infection does not spread.

Therefore, the higher the level of medical treatment is, the higher is the daily cure rate. Thus, improving the health and medical level is an effective way to control the spread of infection.

3 Empirical Analysis

On the basis of the SIR model established in Sect. 2, we test the model parameters to verify whether it is consistent with reality or not. If it is consistent, the model may be thought initially reasonable, and it can be applied in practice according to the scope of application and make further verification. If it is not consistent, the parameters need more carefully checking until we find a satisfying result.

Step 1: Analysis of spreading of Ebola

The data of Ebola virus's epidemic changing situation occurred in Guinea from July 2, 2014 to February 4, 2015 published by WHO, are shown in Table 1.

Step 2: The revised SIR model

Taking July 2, 2014 as the base point and forecasting it to the following 100 weeks. We can draw an image of the number of those infected, those recovered, those infectious over the changes of time with MATLAB.

Table 1 2014 Guinea Ebola virus caused cumulative cases and death numbers

Date	7.2	7.7	7.8	7.14	7.16	7.21	7.24	7.28	7.31	8.3	8.4
Total cases	413	412	408	409	406	410	415	427	460	472	485
Total deaths	303	305	207	309	304	310	314	319	339	346	358
Date	8.8	8.12	8.13	8.15	8.19	8.21	8.22	8.28	9.6	9.8	9.12
Total cases	495	506	510	519	543	579	607	648	812	862	861
Total deaths	367	373	377	380	394	396	406	430	517	555	557
Date	9.16	9.18	9.22	9.24	9.18	9.22	9.24	9.26	10.1	10.3	10.8
Total cases	936	942	1008	1022	942	1008	1022	1074	1157	1199	1298
Total deaths	595	601	632	635	601	632	635	648	710	739	768
Date	10.11	10.15	10.17	10.22	10.25	10.29	10.31	11.5	11.7	11.12	11.14
Total cases	1350	1472	1519	1540	1553	1906	1667	1731	1760	1878	1919
Total deaths	778	843	862	904	926	997	1018	1041	1054	1142	1166
Date	11.19	11.26	12.3	12.10	12.17	12.24	1.7	1.14	1.21	1.28	2.4
Total cases	1971	2134	2164	2292	2416	2597	2775	2806	2871	2917	2975
Total deaths	1192	1260	1327	1428	1525	1607	1781	1814	1876	1910	1994

Step 3: To verify the similarity

We find that the number of infected persons reached the peak in the thirtieth week. Therefore, the authors of this paper believe that with the continuous development of effective drugs and more effective measures of prevention and cure taken, the Ebola outbreak will be eased on February 1, 2015.

4 Conclusions

In this paper, the problem was to build a realistic, sensible, and useful model that considers the spread of the disease, the quantity of the medicine needed, possible feasible delivery systems, locations of delivery, speed of manufacturing of the vaccine or drug. We draw the conclusion that the communication model could effectively simulate the Ebola virus, and could also predict the trend of development of the next stage of Ebola.

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On Existence of the Resolvent and Discreteness of the Spectrum of the Schrödinger Operator with a Parameter Changing Sign

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Abstract In this work such issues as existence of the resolvent and discreteness of the spectrum for the Schrödinger operator with a parameter changing sign are studied.

Keywords Resolvent • Schrödinger operator • Spectrum

Mathematics Subject Classification (2010) Primary 47A10; Secondary 35J10

In this research we study the Schrödinger operator with a parameter changing sign

$$-\Delta + (-k(x)t^2 + ita(x) + c(x)) \tag{1}$$

in the space $L_2(\mathbb{R}^n)$, where $-\infty < t < \infty,\ i^2 = -1$, $k(x)$ is a piecewise continuous and bounded function in $\mathbb{R}^n$.

As well-known, when $t = 0$ the spectral properties of the Schrödinger operator $\Delta + c(x)$ are highly dependent on the behavior of $c(x)$ at infinity. In this case the spectral characteristics of the Schrödinger operator are well studied by A.M. Molchanov, T. Kato, T. Carleman, M. Rid and B. Saymon, M.Sh. Birman, V.G. Maz'ya, M. Otelbaev, T. Carleman, B.S. Pavlov, Yu.M. Berezanskii, R.S. Ismagilov, Ya.T. Sultanaev, K.H. Boymatov, and others.

Issues about the discreteness of the spectrum and the estimates of approximate numbers (s-numbers) of the Schrödinger operator

$$-\Delta + q_1(x) + iq_2(x),\ (q_1(x) \geq 0,\ q_2(x) \geq 0)$$

are studied by V.B. Lidsky, M. Otelbaev, T. Kato, and others.

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Here we will study for the operator (1) such issues as:
(1) the existence of the resolvent; (2) the discreteness of the spectrum.
Further, we assume that the coefficients $a(x)$ and $c(x)$ satisfy the condition
(i) $|a(x)| \geq \delta_0 > 0$, $c(x) \geq \delta > 0$ are continuous function in $\mathbb{R}^n$.

Theorem 1 *Let the condition i) be fulfilled. Then the operator* $L + \lambda E$ *is continuously invertible for* $\lambda \geq 0$.

It is known that the question of the discreteness of the spectrum is closely related to the concept of capacity in the case $2 \leq n$. Avoiding complicated formulations and definitions, we do not consider this case.

Consider the operator

$$Lu = (-\Delta)^l u + (-k(x)t^2 + ita(x) + c(x))u \tag{2}$$

in $L_2(\mathbb{R}^n)$, $l > 0$ is an integer, $u \in D(L)$.

Theorem 2 *Let the condition i) be fulfilled and let* $2l > n$. *Then the spectrum of (2) is discrete if and only if for each cube* Q_d *the quantity* $\int\limits_{Q_d} c(t)dt$ *tends to* $+\infty$, *when the cube* Q_d *goes to infinity keeping the length of the edge.*

Remark The theorem holds also for $2l \leq n$, but here one needs to use the concept of capacity.

In particular, in the one-dimensional case, the operator (1) has the form

$$l_t u = -u'' + (-k(x)t^2 + ita(x) + c(x))u, \quad u \in D(l).$$

Theorem 3 *Let the condition i) be fulfilled. Then the operator* $l_t + \lambda E$ *is continuously invertible for* $\lambda \geq 0$.

Theorem 4 *Let the condition i) be fulfilled. Then the resolvent of the operator* l_t *is compact if and only if for any* $w > 0$

$$\lim_{|x|\to\infty} \int_x^{x+w} c(t)dt = \infty.$$

Suppose that in addition to the conditions i) the following condition: ii) $\mu_0 = \sup\limits_{|x-t|\leq 1} \frac{c(x)}{c(t)} < \infty$, $\mu_1 = \sup\limits_{|x-t|\leq 1} \frac{a(x)}{a(t)} < \infty$; be satisfied. Then the following theorem holds:

Theorem 5 *Let the conditions i)–ii) be fulfilled. Then the resolvent of the operator* l_t *is compact if and only if*

$$\lim_{|x|\to\infty} c(x) = \infty.$$

Let A be a completely continuous linear operator and let $|A| = \sqrt{A^*A}$.

The eigenvalues of the operator $|A|$ are called s-values of the operator A (Schmidt eigenvalues of the operator A).

The nonzero s-values are numbered according to decreasing magnitude and observing their multiplicities and so

$$s_j(A) = \lambda_j(|A|), \ \ j = 1, 2, \ldots .$$

The nonzero s-values of the operator l_t^{-1} are also numbered according to decreasing magnitude and observing their multiplicities and so

$$s_k(l_t^{-1}) = \lambda_k(|l_t^{-1}|), \ \ j = 1, 2, \ldots .$$

Theorem 6 *Let the conditions i)–ii) be fulfilled. Then the estimate*

$$c^{-1}\lambda^{-\frac{1}{2}} mes(x \in \mathbb{R} : Q_t(x) \le c^{-1}\lambda^{-1}) \le N(\lambda)$$
$$\le c\lambda^{-1} mes(x \in \mathbb{R} : K_t^{\frac{1}{2}}(x) \le c\lambda^{-1}),$$

holds, where $Q_t(x) = |t^2 + ita(x) + c(x)|$, $K_t(x) = |ta(x)| + c(x)$*, the constant* $c > 0$ *does not depend on* $Q_t(x), K_t(x)$ *and* λ*.*

Example Let $a(x) = |x| + 1, c(x) = |x| + 1$ The singular numbers of the operator l_t^{-1} are denoted by $s_{k,t}$ $(k = 1, 2, \ldots)$. It is easy to verify that all the conditions of Theorem 6 are satisfied. Hence, according to Theorem 6 we have

$$c^{-1}\frac{1}{(|t| + 1)^{\frac{4}{3}} k^{\frac{2}{3}}} \le s_{k,t} \le c\frac{1}{(|t| + 1)^{\frac{1}{3}} k^{\frac{1}{3}}}, k = 1, 2, \ldots .$$

The results obtained in this study can be used to study as well the existence of the resolvent, the discreteness of the spectrum, as the coercive estimates of differential operators of hyperbolic type in the case of degeneration or the case of unbounded domains.

Proof of Theorems 3–6 In space $L_2(\mathbb{R})$ we consider the operator

$$(l_t + \mu E)u = -u'' + (-k(x)t^2 + ita(x) + c(x) + \mu)u$$

with the domain $D(l_t)$ of compactly supported functions, which together with their derivatives up to second order belong to $L_2(\mathbb{R})$.

The operator $(l_t + \mu E)$ admits a closure, which is also denoted by $(l_t + \mu E)$.

Remark We note that previously the following cases:

1) $lu = -u'' + c(x)u, \quad u \in D(l)$ (A.M. Molchanov, T. Carleman, T. Kato, M.Sh. Birman, V.G. Maz'ya, M. Otelbaev, B.S. Pavlov, Yu.M. Berezansky, R.S. Ismagilov, K.H. Baymatov and others);

2) $\bar{l}u = -u'' + (ia(x) + c(x))u, \quad u \in D(\bar{l})$ (V.B. Lidsky, M. Otelbaev, T. Kato, K.H. Baymatov and others)

have been well studied.

Lemma 1 *Let the condition i) be fulfilled and* $\mu \geq 0$. *Then the estimate*

$$\|(l_t + \mu E)u\|_2 \geq c\|u\|_2$$

holds for $u \in D(l)$, *where* $c = c(\delta_0, \delta)$.

Lemma 2 *Let the condition i) be fulfilled and* $\mu \geq 0$. *Then the following estimates:*

a) $\| \sqrt{c(x)}(l_t + \mu E)^{-1}\|_{2\to 2} < \infty$;
b) $\| \sqrt{|ta(x)|}(l_t + \mu E)^{-1}\|_{2\to 2} < \infty$;
c) $\|\frac{d}{dx}(l_t + \mu E)^{-1}\|_{2\to 2} < \infty$

hold.

The proofs of Lemmas 1–2 follow from Lemmas 2.9–2.11 of the paper [6].

Proofs of Theorems 3–4 Let $t = 0$. Then the operator $l_0 + \mu E$ is a Sturm-Liouville operator with the potential $c(y)$, i.e.

$$(l_0 + \mu E)u = -u'' + (c(x) + \mu)u, \quad u \in D(l_0)$$

In this case, we obtain the proofs of the theorems by reproducing all the computations and arguments used in the papers [2, 7].

In the case $t \neq 0$, we take Lemmas 1–2 into account and use methods of the papers [3, 6] and the book [1].

Let us introduce the sets

$$M = \{u \in L_2(\mathbb{R}) : \|(l_t u + \mu E)u\|_2 + \|u\|_2 \leq 1\},$$

$$\tilde{M}_c = \{u \in L_2(\mathbb{R}) : \|u'\|_2 + \| \sqrt{|ta(x)|}u\|_2^2 + \| \sqrt{c(x) + \mu}u\|_2^2 \leq c\},$$

$$\tilde{\tilde{M}}_{c^{-1}} = \{u \in L_2(\mathbb{R}) : \| - u''\|_2 + \|t^2 u\|_2^2 + \|ita(x)u\|_2^2 + \|(c(x) + \mu)u\|_2^2 \leq c^{-1}\}.$$

Lemma 3 *Let the condition i) be fulfilled. Then the inclusions*

$$\tilde{\tilde{M}}_{c^{-1}} \subseteq M \subseteq \tilde{M}_c,$$

are valid, where $c > 0$ *is a constant.*

Lemma 4 *Let the condition i) be fulfilled. Then the estimate*

$$c^{-1}\tilde{\tilde{d}}_k \leq s_{k+1} \leq c\tilde{d}_k, \quad k = 1, 2, \ldots,$$

holds, where $c > 0$ is a constant, s_k are s-values of the operator l_t^{-1}, $d_k, \tilde{d}_k, \tilde{\tilde{d}}_k$ are the Kolmogorov widths related to the sets $M, \tilde{M}, \tilde{\tilde{M}}$.

Definition 1 The magnitude

$$d_k = \inf_{\{y_k\}} \sup_{u \in M} \inf_{v \in y_k} \|u - v\|_2,$$

is called Kolmogorov k-widths of the set, where $\{y_k\}$ is the set of all k-dimensional subspaces of $L_2(\mathbb{R})$.

Lemma 5 *Let the condition i) be fulfilled. Then the estimate*

$$\tilde{\tilde{N}}(c\lambda) \le N(\lambda) \le \tilde{N}(c^{-1}\lambda),$$

holds, where $N(\lambda) = \sum_{s_{k+1} > \lambda} 1$ is the number of s_k's greater than $\lambda > 0$, $\tilde{N}(\lambda) = \sum_{\tilde{d}_k > \lambda} 1$ is the number of $\tilde{d}_k$'s greater than $\lambda > 0$, $\tilde{\tilde{N}}(\lambda) = \sum_{\tilde{\tilde{d}}_k > \lambda} 1$ is the number of $\tilde{\tilde{d}}_k$'s greater than $\lambda > 0$.

Lemmas 3–5 are proved in the same way as in papers [4, 5]. Hence, using the properties of weighted spaces obtained in [7, 8], we prove Theorems 3–6.

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Source Identification for the Differential Equation with Memory

S.A. Avdonin, G.Y. Murzabekova, and K.B. Nurtazina

Abstract We consider source identification problems for the heat equation with memory on an interval and on graphs without cycles (trees). We propose a stable efficient identification algorithm which reduces to the solving of linear integral Volterra equations of the second kind.

Keywords Heat equation with memory • Identification algorithm • Metric graphs • Source identification • Volterra equation

Mathematics Subject Classification (2010) Primary 35Q93; Secondary 45K05, 93B30

1 Introduction

In this paper, we consider source identification problems for the heat equation with memory on intervals and graphs. On an interval, the problem is described by the equation

$$u_t(x,t) - \int_0^t Q(t-s)u_{xx}(x,s)\,ds = f(t)g(x),\ 0 < x < l,\ 0 < t < T; \tag{1}$$

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with the boundary and initial conditions

$$u(0,t) = u(l,t) = 0, \;\; u(x,0) = 0. \tag{2}$$

The functions $Q, f \in H^1(0,T)$ are known, and we assume that $f(0) \neq 0$ and $Q(0) > 0$. The function $g \in L^2(0,l)$ is unknown and has to be recovered from the observation $\mu(t) := u_x(0,t), \; t \in [0,T]$. We will prove that this problem is solvable for $T \geq l/\sqrt{Q(0)}$.

The heat equation with memory was proposed by Cattaneo [1] and, in a more general form, by Gurtin and Pipkin [2]. After differentiation, Eq. (1) with the right-hand side *fg* set to zero takes the form of the viscoelasticity equation:

$$u_{tt}(x,t) - Q(0)\, u_{xx}(x,t) - \int_0^t Q'(t-s)\, u_{xx}(x,s)\, ds = 0. \tag{3}$$

Both Eqs. (1) and (3) possess a finite speed of the wave propagation equal to $\sqrt{Q(0)}$. Heat and wave equations with memory arise in many problems of physics and engineering. Controllability problems for these equations were actively studied in recent time (see, e.g., [3–5] and the references therein), source identification problems for these equations on an interval were considered in [6, 7]. Source identification problems for the wave and beam equations with constant coefficients on trees were studied, correspondingly, in [8, 9]. The papers [6–9] are based on the identification algorithm developed in [10, 11] for the hyperbolic type equations.

We propose a quite different approach to source identification problems for the heat and wave equations with memory. It is similar to the method described in [12, 13] for the wave equation without memory. The main advantage of our approach is its locality: to recover an unknown coefficient on a part of the interval we use an information relevant only to this subinterval. This allows us to develop a very efficient identification algorithm which is much simpler than algorithms proposed in [6–9]. More of that, starting with identification problems on an interval we extend our approach to equations on star graphs and arbitrary trees.

2 The Case of One Interval

Without loss of generality, we assume that $Q(0) = 1$ (it can be achieved by a simple change of variable $x \mapsto \sqrt{Q(0)}x$). Solution of the initial boundary value problem (1), (2) can be presented in the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x), \;\; \phi_n(x) = \sqrt{\frac{2}{l}} \sin \frac{\pi n x}{l}.$$

Substituting to (1), (2) leads to the following equations for $a_n(t)$:

$$a_n'(t) + \omega_n^2 \int_0^t Q(t-s)\, a_n(s)\, ds = f(t) \int_0^l g(x)\phi_n(x)\, dx,\ a_n(0) = 0,\ \ n \in \mathbb{N}.$$

It follows that the coefficients a_n can be written as

$$a_n(t) = \kappa_n \int_0^t f(\tau)\, s_n(t-\tau)\, d\tau, \ \text{ where } \kappa_n = \int_0^l g(x)\phi_n(x)\, dx,\ \omega_n = \frac{\pi n}{l},$$

and $s_n(t)$ satisfies the initial value problem

$$s_n'(t) + \omega_n^2 \int_0^t Q(t-\tau)\, s_n(\tau)\, d\tau = 0,\ s_n(0) = 1.$$

In [3, 4] it was proved that the functions $s_n(t)$ are asymptotically close to $\cos \omega_n t$ and form a Riesz sequence in $L^2(0,T)$ for $T \geq l$. Therefore we can justify the following formal presentations of the observation:

$$\begin{aligned}\mu(t) &= u_x(0,t) \\ &= \sum_{n=1}^{\infty} \phi_n'(0) \int_0^t f(\tau)\, s_n(t-\tau)\, d\tau \int_0^l g(x)\, \phi_n(x)\, dx = \int_0^l g(x)\, w(x,t)\, dx\,, \\ &\qquad \text{where } w(x,t) = \sum_{n=1}^{\infty} \left[\phi_n'(0) \int_0^t f(\tau)\, s_n(t-\tau)\, d\tau \right] \phi_n(x). \qquad (4)\end{aligned}$$

One can check (similar calculations are performed in [5]) that the function w is a solution to the initial boundary value problem

$$w_{tt}(x,t) = w_{xx}(x,t) + \int_0^t Q'(t-s)\, w_{xx}(x,s)\, ds\ = 0;\ \ 0 < x < l,\ t > 0, \qquad (5)$$

$$w(0,t) = h(t),\ w(l,t) = 0;\ w(x,0) = 0,\ w_t(x,0) = 0, \qquad (6)$$

where h is uniquely determined by f : $h(t) + \int_0^t Q(t-s)\, h(s)\, ds = f(t)$.

It is not difficult to prove (see [14, Sect. 1.2] for details) that for $0 < t < l$ the function w can be presented in the form

$$\begin{cases} w(x,t) = h(t-x) + (Bh)(x,t), & x < t, \\ 0, & x > t, \end{cases} \qquad (7)$$

where Bh is a more regular term compared with h, and $(Bh)(x,t) = 0$ for $x \geq t$. In particular, w satisfies the equality

$$w(t-0,t) = h(0) = f(0). \qquad (8)$$

Therefore, formula (4) can be written in the form

$$\mu(t) = \int_0^t g(x)w(x,t)\,dx = f(0)\,G(t) - \int_0^t w_x(x,t)\,G(x)\,dx, \quad 0 \le t \le l, \tag{9}$$

where $G(x) = \int_0^x g(\xi)d\xi$. This is a second kind Volterra equation for $G(x)$. Solving it, we find $G(x)$ and then, $g(x)$. We can summarize our results as follows.

Theorem 2.1 *For any $f \in H^1(0,T)$, the observation $u_x(0,t), t \in [0,T]$, belongs to $H^1(0,T)$. The function g is recovered by solving the Volterra equation of second kind (9) on the interval $[0,T]$, and $T = l$ is generally the minimal identification time. The identification is stable, more exactly, for every $T \le l$, the following estimates are valid:*

$$c||u_x(0,\cdot)||_{H^1(0,T)} \le ||g||_{L^2(0,T)} \le C||u_x(0,\cdot)||_{H^1(0,T)}\,, \tag{10}$$

with positive constants c, C independent of g.

3 A Star Graph

Our method can be extended to source identification problems for the heat equations with memory on graphs. First we consider a star graph Γ consisting of N edges e_j identified with intervals $[0, l_j]$, $j = 1, \ldots, N$, connected at an internal vertex γ_0 which we identify with the set of the left end points of the intervals. The boundary vertices γ_j are identified with the right end points of the corresponding intervals (Fig. 1). The following initial boundary value problem is considered on each interval:

$$\begin{cases} \partial_t u_j(x,t) - \int_0^t Q(t-s)\,\partial_x^2 u_j(x,s)\,ds = f_j(t)g_j(x), & 0 < x < l_j, 0 < t < T, \\ u_j(l_j,t) = 0, & 0 < t < T, \\ u_j(x,0) = 0, & 0 < x < l_j. \end{cases} \tag{11}$$

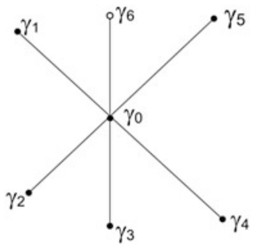


Fig. 1 A star graph

At the internal vertex we impose the standard Kirchhoff–Neumann matching conditions

$$\begin{cases} u_1(0,t) = \cdots = u_N(0,t),\ 0 < t < T, \\ \sum_{j=1}^{N} \partial_x u_j(0,t) = 0, \qquad 0 < t < T. \end{cases} \tag{12}$$

In mechanical systems, the first condition expresses the continuity of the solution, the second is the balance of forces (Newton's second law).

The problem is to recover the unknown functions $g_j,\ j = 1,\dots,N,$ from the observations

$$\mu_j(t) := \partial_x u_j(l_j,t),\ j = 1,\dots,N-1,\ t \in [0,T].$$

Note that we use the observations at all but one boundary vertices.

On the first step we recover the functions $g_j,\ j = 1,\dots,N-1,$ using corresponding observations $\mu_j(t),\ j = 1,\dots,N-1,$ with the help of a similar algorithm to that described in Sect. 2. It can be done in the time interval of length $\max_{j=1,\dots,N-1}\{l_j\}$. Indeed, to recover g_j we use the solution w of the wave equation with memory on the graph with zero right-hand side and the boundary conditions $w(l_j,t) = f_j(t),\ w(l_i,t) = 0$ for $i \neq j$. (See the definition of a similar function in (17) below.) This function is certainly different from the function used for one interval in (5), (6), but our identification procedure requires only the observation at the point l_j in the time interval $[0,l_j]$. Boundary observation in this time interval "does not feel" the other edges of the graph, therefore, the identification algorithm is the same as for one interval.

Then, since the $g_j,\ j = 1,\dots,N-1$ are known, we can consider the initial boundary value problem on the star graph where instead of the first line of (11) we have

$$\begin{cases} \partial_t v_j(x,t) - \int_0^t Q(t-s)\,\partial_x^2 v_j(x,s)\,ds = f_j(t)g_j(x),\ \ j \neq N, \\ \partial_t v_N(x,t) - \int_0^t Q(t-s)\,\partial_x^2 v_N(x,s)\,ds = 0. \end{cases}$$

Subtracting the solution of this problem from the solution of (11), we reduce our identification problem to the case where all g_j except g_N are equal to zero:

$$\begin{cases} \partial_t u_j - \int_0^t Q(t-s)\,\partial_x^2 u_j(x,s)\,ds = 0,\ j \neq N, & 0 < x < l_j,\ 0 < t < T, \\ \partial_t u_N - \int_0^t Q(t-s)\,\partial_x^2 u_N(x,s)\,ds = f_N(t)g_N(x), & 0 < x < l_N,\ 0 < t < T, \\ u_j(l_j,t) = 0, & 0 < t < T, \\ u_j(x,0) = 0, & 0 < x < l_j \end{cases} \tag{13}$$

with the Kirchhoff–Neumann conditions at $x = 0$.

Our next step is to obtain the spectral representation of the solution of (13). Let $\Phi_n = (\phi_{n,1}, \ldots, \phi_{n,N})$ and ω_n^2, $n \in \mathbb{N}$, be eigenfunctions and eigenvalues of the following eigenvalue problem on the graph Γ :

$$\begin{cases} -\phi_j'' = \omega^2 \phi_j, \;\; 0 < x < l_j, \; j = 1, \ldots, N, \\ \phi_j(l_j) = 0, \; \phi_1(0) = \cdots = \phi_N(0), \; \sum_{j=1}^N \phi_j'(0) = 0. \end{cases} \tag{14}$$

Applying the Fourier method, one can obtain the formula

$$u_j(x,t) = \sum_{n=1}^{\infty} \phi_{n,j}(x) \left[\int_0^{l_N} g_N(x)\, \phi_{n,N}(x)\, dx \right] \int_0^t f_N(\tau)\, s_n(t-\tau)\, d\tau, \tag{15}$$

Now we demonstrate how to recover g_N using any of the boundary observations μ_j, $j = 1, \ldots, N-1$, say, μ_1. Changing formally the order of summation and integration in (15) (it can be justified similarly to the case of one interval considered in the previous section) we get

$$\partial_x u_j(l_j,t) = \int_0^{l_N} g_N(x) \left[\sum_{n=1}^{\infty} \phi_{n,N}(x)\, \phi_n'(l_j) \int_0^t f_N(\tau)\, s_n(t-\tau)\, d\tau \right] dx. \tag{16}$$

In this formula, the expression in the brackets (it will be denoted by $w_N(x,t)$) is the restriction to the edge e_N of the solution, $w(x,t)$, of the homogeneous wave equation with memory on Γ with the Dirichlet boundary control applied to the boundary vertex γ_1:

$$\begin{cases} \partial_t^2 w_j - \partial_x^2 w_j(x,t) - \int_0^t Q'(t-s)\, \partial_x^2 w_j(x,s)\, ds = 0, & j = 1, \ldots, N, \\ w_1(l_1,t) = h_N(t), \; w_j(l_j,t) = 0, \; j \neq 1, & 0 < t < T, \\ w_j(x,0) = \partial_t w_j(x,0) = 0, & 0 < x < l_j, j = 1, \ldots, N, \end{cases}$$
$$h_N(t) := f_N(t) + \int_0^t Q(t-\tau) f_N(\tau)\, d\tau. \tag{17}$$

This solution satisfies also the Kirchhoff–Neumann matching conditions at $x = 0$.

Taking into account that the speed of the wave propagation in our system is one, we present the formula (16) as [compare with (9)]

$$\mu_1(t) = \int_0^{t-l_1} g_N(x) w_N(x,t)\, dx, \;\;\; l_1 \leq t \leq l_1 + l_N. \tag{18}$$

From this formula one can recover g_N in the time interval $l_1 \leq t \leq l_1 + l_N$. Indeed, for $0 < t < l_1$, $w_1(x,t) = h_N(t+x-l_1) + (B_1 h_N)(x,t)$ [see (7)] and $w_i(x,t) = 0$, $i \neq 1$.

Using the Kirchhoff–Neumann matching conditions at $x = 0$, one can find the values of w_N at the front points:

$$w_N(t - l_1, t) = \frac{2}{N} f_N(0)$$

for $l_1 < t \leq l_N$ [compare with (8)]. Substituting into (18) and putting $t = \tau + l_1$, $G_N(x) = \int_0^x g_N(\xi)\, d\xi$, we obtain a second kind Volterra equation for $G_N(x)$:

$$\mu_1(\tau + l_1) = \frac{2}{N} f_N(0)\, G_N(\tau) - \int_0^\tau G_N(x)\, \partial_x w_N(x, \tau + l_1)\, dx, \quad 0 < \tau < l_N. \tag{19}$$

From this equation one can now find $G_N(x)$ and, so, $g_N(x)$ for $0 \leq x \leq l_N$.

Similarly, we can consider the case of the observation on the whole boundary. The sharp identification time is generally smaller in this situation. We summarize our results in the form of

Theorem 3.1 *For a star graph described by Eqs. (11), (12), a stable reconstruction of sources g_j is possible using N or $N - 1$ boundary observations. In the first case the sharp identification time is* $1/2 \max_{i,j=1,\dots,N,\, i\neq j} \{l_i + l_j\}$*, in the second* $\max_{j=1,\dots,N-1} \{l_j + l_N\}$ *(assuming no observation at l_N).*

Remark 3.2 It can be proved that generally a stable source reconstruction for a star graph is impossible if we use less than $N - 1$ boundary observations. However, a uniqueness result for this inverse problem may take place for smaller number of boundary observation. This number must be greater than or equal to the multiplicity of the spectrum of the operator $-d^2/dx^2$ on the graph Γ with Dirichlet boundary conditions and Kirchhoff–Neumann matching conditions.

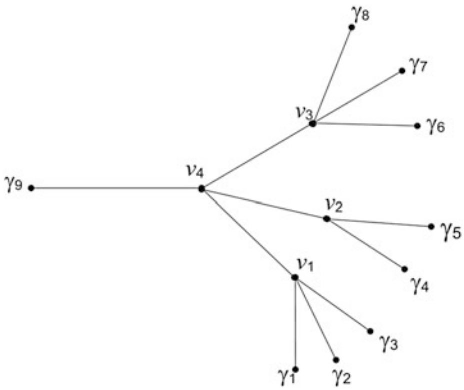
4 A Tree

Let Γ be a finite compact metric graph without cycles (tree), $E = \{e_j\}_{j=1}^N$ is the set of its edges and $V = \{v_j\}_{j=1}^{N+1}$ is the set of vertices. We denote the set of exterior vertices by $\{\gamma_1, \dots, \gamma_m\} = \partial\Gamma \subset V$. This set plays the role of the graph boundary.

On such a graph, we consider the heat equation with memory with Dirichlet boundary conditions at boundary vertices and Kirchhoff–Neumann matching conditions at internal vertices. The observations will be the set of normal derivatives $\{\partial u_j\}$ evaluated at all $\gamma_j \in \partial\Gamma$, or at all but one boundary vertices $\gamma_j \in \partial\Gamma_m := \partial\Gamma \setminus \{\gamma_m\}$ during the time interval $[0, T]$. The problem is to find the functions $g_j(x)$, $j = 1, \dots, N$, from the observation and determine the sharp T.

We consider a subgraph of Γ which is a star graph consisting of all edges incident to an internal vertex v. This star graph is called a *sheaf* if all but one of its edges are the boundary edges adjacent to the boundary vertices of $\partial\Gamma_m$. It is known that any tree contains at least one sheaf.

Fig. 2 A metric tree



Using the results of Sect. 3 we are able to solve the identification problem on an arbitrary tree. First, applying the techniques of the previous section we find the functions g_i on the sheaves, and we can further consider these functions to be zero there. Then we repeat the procedure to move further, on each step we use the observations on the original boundary to recover the functions g_i on the sheaves of the reduced graph (i.e. the graph Γ without the sheaves). On the final step we come to the edge incident to the root $\{\gamma_m\}$.

For the tree presented on Fig. 2 we use the boundary observations at the vertices $\gamma_1, \gamma_2, \gamma_3$ to recover the functions g_j on the sheaf with vertices $\gamma_1, \gamma_2, \gamma_3, v_4$ and v_1. Similarly, we recover the functions g_j on the sheaf with vertices γ_4, γ_5, v_4 and v_2 from the boundary observations at the vertices γ_4, γ_5 and on the sheaf with vertices $\gamma_6, \gamma_7, \gamma_8, v_4$ and v_3 from the boundary observations at the vertices $\gamma_6, \gamma_7, \gamma_8$. Then we can assume that all functions g_j are zero on these sheaves and find g_j on the edge incident to the root γ_9 from the observation at any of the edges $\gamma_1, \ldots, \gamma_8$.

The identification problem can be solved using this algorithm at a finite number of steps. Our results can be formulated as follows.

Theorem 4.1 *For the heat equation with memory on a tree, a stable reconstruction of sources g_j is possible from the derivatives of the solution evaluated at all or all but one boundary vertices. In the first case the sharp identification time is one half of the tree diameter, in the second it is* $\max\limits_{j=1,\ldots,m-1} dist\{\gamma_m, \gamma_j\}$ *(assuming no observation at γ_m).*

Analog of Remark 3.2 is valid for trees: for a stable reconstruction we need not less than $m-1$ boundary observations, for uniqueness of the identification the number of observations has to be not less than the multiplicity of the spectrum of the Laplacian on the graph.

5 Conclusions

In this paper, we propose a recursive algorithm which allows the recovery of the sources in the heat equation with memory on the original tree starting from the leaves and reducing the problem to smaller and smaller subtrees up to the rooted edge. In this sense, it can be considered as an analog of the leaf-peeling method proposed in [15] and developed further in [16–18] for boundary inverse problems on trees with unknown coefficients of differential equations.

Our approach works also for arbitrary graphs (with cycles). If the graph has cycles, boundary observations do not guarantee unique solvability of the source identification problem. We need also additional observations at the internal vertices. It will be a topic of a forthcoming paper.

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New Methods for Volterra Type Integral Equation with Boundary Singular Point

Rajabov Nusrat

Abstract In this work we suggest new methods for investigating the model Volterra type integral equation with singularity and logarithmic singularity, the kernel of which consists of a composition of polynomial functions with logarithmic singularity and functions with singular points. The problem of investigating this type of integral equation for $n = 2m$ reduces to the problem to investigate the Volterra type integral equation (4) for $n = 2$ and for $n = 2m + 1$, it is reduced to m Volterra integral equation (4) and one integral equation (5) for $n = 1$.

Keywords Boundary singularity • Logarithmic singularity • Singular kernel • Volterra type integral equation

Mathematics Subject Classification (2010) Primary 44A15; Secondary 35C10, 45E10

Let $\Gamma = \{x : \ a < x < b\}$ be an interval at the real axis and consider an integral equation

$$\varphi(x) + \int_a^x \left[\sum_{j=1}^{n} A_j \ln^{j-1}\left(\frac{x-a}{t-a}\right) \right] \frac{\varphi(t)}{t-a} dt = f(x), \tag{1}$$

where $A_j (1 \leq j \leq n)$ are given constants, $f(x)$ is a given function on $\overline{\Gamma}$, and $\varphi(x)$ to be found.

The solution of the integral equation (1) is sought in a class on functions $\varphi(x) \in C(\overline{\Gamma})$ vanishing at the singular point $x = a$, i.e.,

$$\varphi(x) = o[(x-a)^{\varepsilon}], \ \varepsilon > 0 \quad \text{at} \quad x \to a.$$

Assume that the solution of Eq. (1) is a function $\varphi(x) \in C^{(n)}(\Gamma)$. Besides let in Eq. (1), the function $f(x) \in C^{(n)}(\Gamma)$. Then differentiating integral equation (1) n

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times we obtain the following nth order degenerate ordinary differential equation

$$(D_x)^n\varphi(x) + A_1(D_x)^{n-1}\varphi(x) + A_2(D_x)^{n-2}\varphi(x) \\ +2!A_3(D_x)^{n-3}\varphi(x) + \cdots + (n-1)!A_n\varphi(x) = (D_x)^n f(x), \tag{2}$$

where $D_x = (x-a)\frac{d}{dx}$.

The homogeneous differential equation (2) is corresponding to the following characteristic equation

$$\lambda^n + A_1\lambda^{n-1} + A_2\lambda^{n-2} + 2!A_3\lambda^{n-3} + 3!A_4\lambda^{n-4} + \cdots + (n-1)!A_n = 0. \tag{3}$$

The papers [1–5] are devoted to the integral equation (1) for the different cases when $n = 1, 2, 3$. The case of Eq. (1) when the parameters $A_j (1 \le j \le n)$ are such that the roots of the characteristic equation (3) are real, different and positive is investigated in [3]. But in the investigation of the other cases for the roots of the characteristics equation (3) difficulties of analytical character arise for the representation of the manifold of solutions to Eq. (1).

In this connection we offer here a method for representing the manifold of solutions of integral equation (1) for $n = 1$ and $n = 2$. This theory was constructed in [1] for $n = 1$ and in [2] for $n = 2$. In [2] other possible cases are investigated for $n = 2$. Depending on the cases $n = 2m$ and $n = 2m + 1$ this gives the possibility to give the solution of Eq. (1) in explicit form connected with m second order algebraic equations.

Here a new method is offered. When $n = 2m$ the general solution of the integral equation (1) is represented by the solutions of m integral equations of the type

$$T_{p_j,q_j}(\varphi) \equiv \varphi(x) + \int_a^x \left[p_j + q_j \ln\left(\frac{x-a}{t-a}\right)\right]\frac{\varphi(t)}{t-a}dt = f(x). \tag{4}$$

and when $n = 2m + 1$ the general solution of (1) is representable by the solutions to m integral equations of type (4) and one solution to the integral equation

$$\Pi_\lambda(\varphi) \equiv \varphi(x) + \lambda \int_a^x \frac{\varphi(t)}{t-a}dt = g(x). \tag{5}$$

The respective theory is constructed in [1, 2].

Let in integral equation (1) $n = 2m$. Then we represent the integral equation in the form

$$\prod_{j=1}^{m} T_{p_j,q_j}(\varphi) = f(x), \tag{6}$$

where p_j, q_j $(1 \le j \le m)$ are constants, which are the coefficients of the following characteristic equations

$$(\lambda^{(j)})^2 + p_j\lambda^{(j)} + q_j = 0 \ (1 \le j \le m). \tag{7}$$

Later on we denote the roots of the characteristic equation (7) by $\lambda_k^{(j)}$ $(k = 1, 2,\ 1 \leq j \leq m)$.

We can represent the integral equation (1) in the form (6) when the roots of the characteristic equation (7) are connected with the parameters $A_j (1 \leq j \leq n)$ of Eq. (1) by

$$A_1 = -\sum_{j=1}^{m}(\lambda_1^{(j)} + \lambda_2^{(j)}),\ A_2 = \sum_{\substack{j,k=1\\ j\neq k}}^{m}(\lambda_1^{(k)}\lambda_2^{(j)}) + \sum_{\substack{j,k=1\\ j\neq k}}^{m}(\lambda_1^{(k)}\lambda_1^{(j)}) + \sum_{\substack{j,k=1\\ j\neq k}}^{m}(\lambda_2^{(k)}\lambda_2^{(j)}),$$

$$2!A_3 = \sum_{\substack{j,k,s=1\\ j\neq k\neq s}}^{m}(\lambda_1^{(k)}\lambda_2^{(j)}\lambda_1^{(s)}) + \sum_{\substack{j,k,s=1\\ j\neq k\neq s}}^{m}(\lambda_2^{(k)}\lambda_1^{(j)}\lambda_2^{(s)}), \ldots, (n-1)!A_n = \prod_{j=1}^{m}\lambda_1^{(j)}\lambda_2^{(j)}. \tag{8}$$

Equation (6) is written as the system

$$\psi_m(x) = T_{p_m,q_m}(\varphi),\ \psi_{m-1}(x) = T_{p_{m-1},q_{m-1}}(\psi_m),$$
$$\psi_{m-2}(x) = T_{p_{m-2},q_{m-2}}(\psi_{m-1}), \ldots, \psi_2(x) = T_{p_1,q_1}(\psi_1), T_{p_1,q_1}(\psi_1) = f(x). \tag{9}$$

So, in this case the problem of finding the general solution of the integral equation (1) is reduced to the problem of finding the solution of the system (9) of Volterra integral equations.

In particular if the roots of the characteristic equation (7) are real, equal, and negative and the constants $p_j (1 \leq j \leq m)$ satisfy the following inequalities

$$|p_m| > |p_{m-1}| > |p_{m-2}| > \cdots > |p_1|, \tag{10}$$

and the function $f(x) \in C(\overline{\Gamma}), f(a) = 0$ with the asymptotic behavior

$$f(x) = o[(x-a)^{\delta_m}],\ \delta_m > |p_m|, \quad \text{at} \quad x \to a, \tag{11}$$

then the solution of the integral equation (1) is given by the formula

$$\varphi(x) = \prod_{j=1}^{m}\left(T^{1C_1^{m-j+1},C_2^{m-j+1}}_{p_{m-j+1},q_{m-j+1}}\right)^{-1}(f), \tag{12}$$

where $C_1^{m-j+1}, C_2^{m-j+1} (1 \leq j \leq m)$ are arbitrary constants

$$\left(T^{1C_1^{m-j+1},C_2^{m-j+1}}_{p_{m-j+1},q_{m-j+1}}\right)^{-1}(f) = (x-a)^{\frac{|p_{m-j+1}|}{2}}[C_1^{m-j+1} + \ln(x-a)C_2^{m-j+1}]$$
$$+f(x) - \int_a^x \left(\frac{x-a}{t-a}\right)^{\frac{|p_{m-j+1}|}{2}}\left[p_{m-j+1} - q_{m-j+1}\ln\left(\frac{x-a}{t-a}\right)\right]\frac{f(t)}{t-a}dt.$$

So, we have proved the following confirmation.

Theorem 1 (Main Theorem) *Let in integral equation (1) $n = 2m$, the parameters $A_j (1 \le j \le n)$ be connected with the coefficients of the algebraic equation (7) given by formula (8). Moreover, let the function $f(x) \in C(\overline{\Gamma})$, $f(a) = 0$ with asymptotic behavior (11) and in (7) the parameters $p_j (1 \le j \le m)$ satisfy conditions (10). Then the integral equations (1) in the class of functions $\varphi(x) \in C(\overline{\Gamma})$ vanishing in the point $x = a$ are solvable, and its general solution contains $2m$ arbitrary constants and is given by formula (12), where $C_k^{m-j+1} (k = 1, 2, 1 \le j \le m)$ are arbitrary constants.*

Remark 1 The representation of the manifold of solutions of the integral equation (1) in form (6) for the case $n = 2m$ gives the possibility to write the general solution in dependence of the roots of the characteristic equation (7).

When $n = 2m + 1$, we represent Eq. (1) in the form

$$\prod_{j=1}^{m} T_{p_j, q_j}(\psi) = f(x), \tag{13}$$

where

$$\psi(x) = \varphi(x) + p_{m+1} \int_a^x \frac{\varphi(t)}{t - a} dt \equiv \Pi_{m+1}(\varphi), \tag{14}$$

$p_j, q_j (1 \le j \le m)$ are the coefficients of the algebraic equations

$$(\mu^{(j)})^2 + p_j \mu^{(j)} + q_j = 0 \quad (1 \le j \le m). \tag{15}$$

In this case Eq. (13) is represented in the form

$$\psi(x) + \int_a^x \left[\sum_{j=1}^{2m} B_j \ln^{j-1} \left(\frac{x - a}{t - a} \right) \right] \frac{\psi(t)}{t - a} dt = f(x), \tag{16}$$

where the parameters $B_j (1 \le j \le 2m)$ are connected with the roots of the algebraic equations (15) defined by the formulas

$$B_1 = -\sum_{j=1}^{m} (\mu_1^{(j)} + \mu_2^{(j)}),\ B_2 = \sum_{\substack{j,k=1 \\ j \ne k}}^{m} (\mu_1^{(k)} \mu_2^{(j)}) + \sum_{\substack{j,k=1 \\ j \ne k}}^{m} (\mu_1^{(k)} \mu_1^{(j)}) + \sum_{\substack{j,k=1 \\ j \ne k}}^{m} (\mu_2^{(k)} \mu_2^{(j)}),$$

$$2! B_3 = \sum_{\substack{j,k,s=1 \\ j \ne k \ne s}}^{m} (\mu_1^{(k)} \mu_2^{(j)} \mu_1^{(s)}) + \sum_{\substack{j,k,s=1 \\ j \ne k \ne s}}^{m} (\mu_2^{(k)} \mu_1^{(j)} \mu_2^{(s)}), \dots, (n-1)! B_n = \prod_{j=1}^{m} \mu_1^{(j)} \mu_2^{(j)}, \tag{17}$$

where $\mu_1^{(j)}, \mu_2^{(j)} (1 \le j \le m)$ are the roots of the algebraic equations (15).

Substituting $\psi(x)$ from (14) into formula (16) and taking the equation

$$\int_a^x \ln^{j-1}\left(\frac{x-a}{t-a}\right)\left[\int_a^t \frac{\varphi(\tau)}{\tau-a}d\tau\right]\frac{dt}{t-a} = \frac{1}{j}\int_a^x \ln^j\left(\frac{x-a}{t-a}\right)\frac{\varphi(t)}{t-a}dt,$$

into account we obtain

$$\varphi(x) + \int_a^x \left[\sum_{j=1}^{2m+1} A_j \ln^{j-1}\left(\frac{x-a}{t-a}\right)\right]\frac{\varphi(t)}{t-a}dt = f(x), \tag{18}$$

where $A_1 = p_{m+1} + B_1, A_2 = B_2 + B_1 p_{m+1}, A_3 = B_3 + \frac{B_2 p_{m+1}}{2}$,

$$A_4 = B_4 + \frac{B_3 p_{m+1}}{3}, \ldots, A_{2m} = B_{2m} + \frac{B_{2m-1}p_{m+1}}{2m-1}, A_{2m+1} = \frac{B_{2m}p_{m+1}}{2m}.$$

Substituting into these equations the $B_j (1 \le j \le 2m)$ from formula (17), we have

$$A_1 = p_{m+1} - \sum_{j=1}^m (\mu_1^{(j)} + \mu_2^{(j)}),\ A_2 = \sum_{\substack{j,k=1\\ j\neq k}}^m (\mu_1^{(k)}\mu_2^{(j)}) + \sum_{\substack{j,k=1\\ j\neq k}}^m (\mu_1^{(k)}\mu_1^{(j)})$$

$$+ \sum_{\substack{j,k=1\\ j\neq k}}^m (\mu_2^{(k)}\mu_2^{(j)}) - \sum_{j=1}^m (\mu_1^{(j)} + \mu_2^{(j)})p_{m+1},$$

$$2!A_3 = \sum_{\substack{j,k,s=1\\ j\neq k\neq s}}^m (\mu_1^{(k)}\mu_2^{(j)}\mu_1^{(s)}) + \sum_{\substack{j,k,s=1\\ j\neq k\neq s}}^m (\mu_2^{(k)}\mu_1^{(j)}\mu_2^{(s)})$$

$$+p_{m+1}\left[\sum_{\substack{j,k=1\\ j\neq k}}^m (\mu_1^{(k)}\mu_2^{(j)}) + \sum_{\substack{j,k=1\\ j\neq k}}^m (\mu_1^{(k)}\mu_1^{(j)}) + \sum_{\substack{j,k=1\\ j\neq k}}^m (\mu_2^{(k)}\mu_2^{(j)})\right], \ldots \tag{19}$$

where $\mu_1^{(j)}, \mu_2^{(j)}$ are the roots of the algebraic equations (15).

So we have proved the following confirmation.

Theorem 2 (Main Theorem) *Let the parameters $A_j (1 \le j \le n)$ in integral equations* (1) *be connected with the roots of the characteristic equations* (15) *and the number p_{m+1} given by formula* (19)*. Then the problem of finding the solution of the integral equation* (1) *for $n = 2m + 1$, of the integral equation* (18)*, reduces to* *the problem of finding the solution of the integral equation*

$$\prod_{j=1}^{m} T_{p_j,q_j}(\Pi_{m+1}(\varphi)) = f(x). \tag{20}$$

Introducing in the equality (20) the new functions

$$\psi_{m+1}(x) = \prod_{m+1}(\varphi), \psi_m(x) = T_{p_m,q_m}(\psi_{m+1}), \ \ \psi_{m-1}(x) = T_{p_{m-1},q_{m-1}}(\psi_m),$$

$$\psi_{m-2}(x) = T_{p_{m-2},q_{m-2}}(\psi_{m-1}), \ldots, \psi_2(x) = T_{p_1,q_1}(\psi_1), T_{p_1,q_1}(\psi_1) = f(x).$$

this equation is reduced to the system of m integral equations of type (4) and one integral equation of type (5).

In particular case, if all the roots of the characteristic equations (15) are real, equal, negative and

$$|p_{m+1}| > |p_m| > |p_{m-1}| > |p_{m-2}| > \cdots > |p_1|, \tag{21}$$

and the function $f(x) \in C(\overline{\Gamma}), f(a) = 0$ with asymptotic behavior

$$f(x) = o[(x-a)^{\delta_{m+1}}], \ \ \delta_{m+1} > |p_{m+1}|, \quad \text{as} \quad x \to a, \tag{22}$$

then the solution of the integral equation (1) for $n = 2m + 1$ is given by the formula

$$\varphi(x) = \left(\Pi_{p_{m+1}}^{C_{m+1}}\right)^{-1} \left[\prod_{j=1}^{m} \left(T_{p_{m-j+1},q_{m-j+1}}^{1C_1^{m-j+1},C_2^{m-j+1}}\right)^{-1}(f)\right], \tag{23}$$

where $C_1^{m-j+1}, C_2^{m-j+1} (1 \le j \le m)$, C_{m+1} are arbitrary constants,

$$\left(\Pi_{p_{m+1}}^{C_{m+1}}\right)^{-1}(\omega) \equiv (x-a)^{|p_{m+1}|} C_{m+1} + \omega(x) - p_{m+1} \int_a^x \left(\frac{x-a}{t-a}\right)^{|p_{m+1}|} \frac{\omega(t)}{t-a} dt.$$

So, we have proved the following confirmation.

Theorem 3 (Main Theorem) *Let in integral equation* (1) $n = 2m + 1$*, the parameters* $A_j (1 \le j \le n)$ *be connected with coefficients of the algebraic equation* (15) *by formula* (19)*. Moreover, let the roots of the characteristic equation* (15) *be real, equal, and positive, and the function* $f(x) \in C(\overline{\Gamma})$*,* $f(a) = 0$ *with asymptotic behavior* (22) *and in* (15) *parameters* $p_j (1 \le j \le m)$*,* p_{m+1} *satisfy conditions* (21)*. Then integral equations* (1) *in the class of function* $\varphi(x) \in C(\overline{\Gamma})$*, vanishing at the point* $x = a$ *are solvable, and its general solution contains* $2m + 1$ *arbitrary constants and is given by formula* (23)*, where* $C_k^{m-j+1} (k = 1, 2, 1 \le j \le m)$*,* C_{m+1} *are arbitrary constants.*

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Approximate Properties of Solutions of Second Order Differential Equations with Unbounded Drift Coefficient

K.N. Ospanov

Abstract In this work we consider the three-member second order differential equation with unbounded intermediate coefficient. We give the solvability results and some conditions for compactness of the resolvent of corresponding operator. Furthermore, we discuss the estimates for the k-diameters of the set of solutions.

Keywords Compactness • Maximal regularity • Schatten class • Solvability

Mathematics Subject Classification (2010) Primary 34A30; Secondary 34L05

1 Introduction

Let $1 < p < +\infty$. Consider the following differential equation

$$L_0 y := -y'' + r(x)y' + q(x)y = f(x) \ (x \in \mathbf{R}), \tag{1}$$

where r is a continuously differentiable function, q is continuous, and $f \in L_p = L_p(\mathbf{R})$.

Definition 1.1 The function $y \in L_p$ is called a solution of Eq. (1), if there is a sequence $\{y_n\}_{n=1}^{+\infty} \subset C_0^{(2)}(\mathbf{R})$ such that $\|y_n - y\|_p \to 0$ and $\|L_0 y_n - f\|_p \to 0$ as $n \to +\infty$. Here $C_0^{(2)}(\mathbf{R})$ is the set of twice differentiable functions with compact support, and $\|\cdot\|_p$ is the norm of L_p.

Equation (1) has a number of applications. In the theory of stochastic processes associated with the dynamics of the Brownian motion, the Ornstein–Uhlenbeck differential equation (OU equation) is used, which in the one dimensional case is Eq. (1) with some unbounded intermediate coefficient r. The OU equation first was

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studied in [1]. Its study is important for investigation of the known Fokker–Planck and Kramer equations. In the third decade of the twentieth century OU equation was studied also by M. Smoluchowski, A. Fokker, M. Plank, H.C. Burger, R. Furth, L. Zernike, S. Goudsmitt, M.C. Wang and others.

Furthermore, Eq. (1) describes the propagation of small oscillations in the viscoelastic compressible medium [2], the dynamics of a stratified compressible fluid [3], the vibrational motion in media with resistance, which depends on velocity [4].

Recently Eq. (1) and its multidimensional generalizations were studied by J. Pruss, R. Shnaubelt, A. Rhandi, G. Da Prato, V. Vespri, P. Clement, G. Metafune, D. Pallara, M. Hieber, L. Lorenzi (see, for example, [5], where there are some references). However, in the above works it is assumed that in (1) the intermediate term $r(x)y'$ is a small perturbation of $-y'' + q(x)y$.

If we denote $p(x) := \exp \int\limits_a^x [-r(t)]dt$, then $Ly = p^{-1}\left[-(py')' + (qp)y\right]$, and then the problem is reduced to the Sturm–Liouville equation

$$-(py')' + (qp)y = f.$$

But this equation was less studied, when $p \to 0(|x| \to +\infty)$ (the degenerate case).

Thus, for (1) the growth of the function $|r|$ at infinity is important. If the growth of $|r|$ is weaker than the growth of q, and $q > 0$, then (1) becomes the well-known cases of the Sturm–Liouville equation. The case where $|r|$ increased faster than q was studied relatively less [6, 7].

We discuss the conditions under which

(a) Eq. (1) is uniquely solvable in the space L_p,
(b) the inverse L^{-1} to the operator L corresponding to Eq. (1) is compact,
(c) some upper estimates of the Kolmogorov k-diameters of the set $M = \{y \in L_p : \|Ly\|_p \le 1\}$ hold.

2 Results

Let $1 < p < +\infty$, $1/p + 1/(p') = 1$. For given g and h, we denote that

$$\alpha_{g,h}(t) := \|g\|_{L_p(0,\,t)} \left\|h^{-1}\right\|_{L_{p'}(t,\,+\infty)} \quad (t > 0),$$

$$\beta_{g,h}(\tau) := \|g\|_{L_p(\tau,\,0)} \left\|h^{-1}\right\|_{L_{p'}(-\infty,\,\tau)} \,(\tau < 0),$$

$$\gamma_{g,h} := \max\left(\sup_{t>0} \alpha_{g,h}(t),\ \sup_{\tau<0} \beta_{g,h}(\tau)\right).$$

In the future, we shall always assume that r is a continuously differentiable function and q is continuous.

Theorem 2.1 *Let* $1 < p < +\infty, f \in L_p$ *and*

$$|r(x)| \geq 1, \ \gamma_{1,\sqrt[p]{|r|}} < +\infty, \tag{2}$$

$$\sup_{x,\eta\in\mathbb{R},\,|x-\eta|\leq 1} |r(x)|/|r(\eta)| < +\infty. \tag{3}$$

Assume that $\gamma_{q,r} < +\infty$. *Then Eq.* (1) *has a unique solution y and for y the following inequality holds:*

$$\| -y''\|_p + \|ry'\|_p + \|qy\|_p \leq C_1\|Ly\|_p, \tag{4}$$

where C_1 *does not depend on y.*

We recall that if the estimate (4) holds, then the operator L corresponding to Eq. (1) is called separable in the space L_p by Everitt and Giertz [8].

Theorem 2.2 *Assume that the conditions of Theorem 2.1 are fulfilled. Then the operator* L^{-1} *is completely continuous in* L_p, *if and only if the condition*

$$\overline{\lim}_{x\to+\infty} \ x\left[\left(\int_x^{+\infty} |r|^{-p'}dt\right)^{p/p'} + \left(\int_{-\infty}^{-x} |r|^{-p'}dt\right)^{p/p'}\right] = 0 \tag{5}$$

holds.

Note that (3) is a condition on the oscillation of the function r. For example, $y = e^{|x|}$ satisfies (3). But $y = 1 + e^{|x|}\sin^2 x$ does not satisfy (3). In the following theorem the expression (3) is replaced by other conditions.

Theorem 2.3 *Assume that* $1 < p < +\infty$, $f \in L_p$ *and r does not increase in the interval* $(-\infty, 0)$, *does not decrease in* $(0, +\infty)$, *satisfies* (2) *and for some* $\alpha \in (0, 1/p)$ *the condition*

$$\max\left(\sup_{\tau<0}\left[\int_{-\infty}^{\tau} \tilde{r}_\alpha(s)ds\right]^{1/(p')}, \sup_{t>0}\left[\int_t^{+\infty} \tilde{r}_\alpha(s)ds\right]^{1/(p')}\right) < +\infty,$$

holds, where $\tilde{r}_\alpha(s) = |s|^{1/(p-1)}|r(s)|^{-p(1-\alpha)}$.

Let $\gamma_{q,\sqrt[p]{r}} < +\infty$. *Then*

i) Eq. (1) *is uniquely solvable in* L_p;
ii) for the solution y the inequality

$$\| -y''\|_p + \|\sqrt[p]{r}y'\|_p + \|\left(|q| + |r|^{\alpha/p}\right)y\|_p \leq C_2\|f\|_p \tag{6}$$

holds, where C_2 *does not depend on y;*
iii) the inverse operator L^{-1} *is compact in* L_p.

Assume that the inverse L^{-1} to operator L corresponding to Eq. (1) is compact in L_p. Let M be the set of such solutions y of (1) that $\|Ly\|_p \equiv \|f\|_p \le 1$. We recall that the Kolmogorov k-diameters of M are

$$d_k(M) = \inf_{Q\in T_k} \sup_{v\in M} \inf_{u\in Q} \|u - v\|_p,\ k = 1, 2, \dots ,$$

where T_k is the class of all subsets Q of L_p with $\dim Q \le k$.

We can see that the k-diameters d_k describe the order of the approximation of solutions of Eq. (1) by elements of finite-dimensional subspaces of L_p. The bounds for d_k are important for estimating the quality of the approximate methods for solving equation (1).

Theorem 2.4 *Assume that the conditions of Theorem 2.1 and (5) hold. Then*

$$d_k(M) \le C_4 \inf\left\{\lambda > 0 :\ \sqrt{\lambda}\mu\left\{x \in \mathbf{R} : |q(x)| \le C_5\lambda^{-p}\right\} \le k\right\}, \tag{7}$$

where μ is the Lebesgue measure.

Consider the particular case $p = 2$ (the Hilbert case). Assume that L^{-1} is the inverse to the operator L corresponding to the equation (1). We denote by $s_k(L^{-1})$ $(k = 1,\ 2,\ \dots)$ the s-numbers of L^{-1} (eigenvalues of $\sqrt{L^{-1}\,(L^*)^{-1}}$). Recall that σ_θ $(0 < \theta < +\infty)$ is the set of the compact operators A, which satisfy $\sum_{k=1}^{+\infty} [s_k(A)]^\theta < \infty$. Note that σ_1 is the class of nuclear operators, and σ_2 is the class of Hilbert–Schmidt operators. Then Theorem 2.4 gives the following result.

Corollary 2.5 *Let $p = 2, \theta > 1/2$. Assume that the conditions of Theorem 2.2 are fulfilled, and*

$$\int_{\mathbf{R}} |q(x)|^{(1-2\theta)/2} dx < +\infty.$$

Then $L^{-1} \in \sigma_\theta$.

The following theorem gives the estimate for the k-diameters $d_k(M)$ of the set of solutions of Eq. (1) when r is an oscillating function.

Theorem 2.6 *Assume that the conditions of Theorem 2.3 hold. Then*

$$d_k(M) \le C_4 \inf\left\{\lambda > 0 :\ \sqrt{\lambda}\ \mu\left\{x \in \mathbf{R} : |q(x)| \le C_5\lambda^{-1}\right\} \le k\right\}, \tag{8}$$

where μ is the Lebesgue measure.

3 Examples

Example 3.1 Consider the equation

$$Ly := -y'' + (1+x^2)^{\omega/2}y' + |x|^m y = f, \tag{9}$$

where $\omega > 0$, $m > 0$. Then:

A) It is easy to see that

$$\sup_{|x-z|\le 1} [(1+x^2)^{\omega/2}]/[(1+z^2)^{\omega/2}] \le 3^{\omega/2}.$$

Hence if

$$\omega \ge (m+2/p)(p-1), \tag{10}$$

then the conditions of Theorems 2.1 and 2.2 hold. So the equation (9) for any $f \in L_p$ has a unique solution y and for y the following inequality holds: $\|y''\|_p + \|(1+x^2)^{\omega/2}y'\|_p + \||x|^m y\|_p \le C_2\|f\|_p, 0 < m \le \omega/(p-1)-2/p$. Furthermore, the inverse operator L^{-1} is compact in L_p.

B) By Theorem 2.4

$$d_k \le C_8 k^{-(2m)/m+4p},\ 0 < m \le \omega/(p-1) - 2/p. \tag{11}$$

Example 3.2 Let

$$Ly := -y'' + (1+x^2)^{\rho/2}s_\epsilon(x)y' + |x|^\tau y = f, \tag{12}$$

where $\rho > 0$ and $\tau > 0$, $s_\epsilon(x) \in C^{(1)}_{loc}(\mathbb{R})$ given as

$$s_\epsilon(x) = \begin{cases} (1+x^2)^{1/2}[5+\sin^2 x], \text{if } x \in [k\pi, (k+1/2-\epsilon)\pi) \\ 6(1+(k+1/2)^2\pi^2)^{1/2}, \text{if } x \in [(k+1/2)\pi, (k+1)\pi - \epsilon), k \in \mathbb{Z}, \end{cases}$$

where $0 < \epsilon < \pi/4$. Then the conditions of Theorems 2.1 and 2.2 are not fulfilled. However, it is easy to see that the conditions of Theorem 2.3 are fulfilled, if

$$\rho \ge p(\tau + 2/p)(p-1). \tag{13}$$

By Theorem 2.3 Eq. (12) has a unique solution, and by Theorem 2.6

$$d_k \le C_9 k^{-2\tau/(\tau+4p)},\ 0 < \tau \le \rho/[p(p-1)] - 2/p. \tag{14}$$

The comparison between (10) and (13), (11) and (14) shows the following. Under the condition (3), both Theorems 2.1 and 2.3 are applicable to Eq. (1), however, Theorem 2.1 covers a wider class of equations (1). For example, Theorem 2.3 is not applicable to the equation

$$-y'' + (1+x^2)^2 y' + |x|y = f, \tag{15}$$

when $p = 3$. At the same time, by Theorems 2.1 and 2.2 Eq. (15) is uniquely solvable, and for the k-diameters d_k the upper estimate

$$d_k \leq C_{10} k^{-2/13}$$

holds. Note that Theorems 2.1 and 2.2 are not applicable in the case of a strongly oscillating function $r(x)$.

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Associated Operators to the Space of Elliptic Generalized-Analytic Functions

Gian Rossodivita and Judith Vanegas

Abstract We find all linear first order partial differential operators with elliptic complex numbers-valued coefficients that are associated to an elliptic generalized-analytic operator. As an application, the solvability of initial value problems involving these operators is shown.

Keywords Associated spaces • Elliptic complex numbers • Elliptic generalized-analytic functions • Initial value problems

Mathematics Subject Classification (2010) Primary 35F10; Secondary 35A10, 15A66

1 Introduction

Let $\mathcal{F}$ be a given differential operator. Then a function space $\mathcal{X}$ is called an associated space to $\mathcal{F}$ if $\mathcal{F}$ transforms $\mathcal{X}$ into itself. An example is the space of holomorphic functions which is associated to the complex differentiation because the complex derivative of a holomorphic function is again holomorphic. The concept of associated spaces leads to conditions under which an initial value problem of Cauchy-Kovalevsky type has a solution. For example, consider the problem

$$\partial_t \omega = \mathcal{F}(t, z, \omega, \partial_z \omega), \quad u(0, z) = \varphi(z), \tag{1}$$

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where φ satisfies a partial differential equation $\mathcal{G}(\omega) = 0$. Then this problem is solvable provided that the initial function φ belongs to the associated space $\mathcal{X}$ of $\mathcal{F}$ containing all the solutions for $\mathcal{G}(\omega) = 0$, and that the elements of $\mathcal{X}$ satisfy an interior estimate, i.e., an estimate for the first order derivatives of the solutions (see [5]).

Generalized Complex Numbers are defined as complex numbers of the form $z = x + iy$ where the product of two complex numbers is induced by the relation $i^2 = -\beta i - \alpha$, for α and β real numbers subject to the ellipticity condition $4\alpha - \beta^2 > 0$ (see [6]).

In this work we show necessary and sufficient conditions on the coefficients of the operator $\mathcal{F}$ defined by

$$\mathcal{F}\omega = C(z)\partial_z\omega + A(z)\omega + B(z)\bar{\omega} + G(z), \tag{2}$$

where ω, $C(z)$, $A(z)$, $B(z)$ and $G(z)$ are elliptic complex-valued functions and $z = x + iy$, so that $\mathcal{F}$ and the generalized-analytic operator

$$\mathcal{G}\omega = \partial_{\bar{z}}\omega - a(z)\omega - b(z)\bar{\omega}, \tag{3}$$

where $a(z), b(z)$ are given functions, be an associated pair of operators.

Necessary and sufficient conditions for evolution operators transforming holomorphic functions into themselves are given in [3], and [2] in the framework of complex analysis and elliptic complex analysis, respectively. Sufficient conditions for evolution operators transforming generalized-analytic functions into themselves are given in [4]. The results in our paper provide a generalization of all these results. As an application, we show the solvability of initial value problems involving the operators $\mathcal{F}$ and $\mathcal{G}$.

2 The Elliptic Complex Numbers

Let $\alpha, \beta \in \mathbb{R}$ be parameters such that $4\alpha - \beta^2 > 0$. If $x, y \in \mathbb{R}$, the set

$$\mathbb{C}(\alpha, \beta) = \{\, x + iy \;:\; i^2 = -\alpha - \beta i\}$$

is the set of the elliptic (or generalized) complex numbers.

In this algebra, the product is defined for two numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ as $z_1z_2 = (x_1x_2 - \alpha y_1y_2) + i(x_1y_2 + y_1x_2 - \beta y_1y_2)$. Taking this into account it is easy to prove that

- $x^2 - \beta xy + \alpha y^2$ is positive, if $x^2 + y^2 > 0$
- $(\frac{\beta+2i}{\sqrt{4\alpha-\beta^2}})^2 = -1.$

It can be also shown that the $z = x + iy$ has a well defined inverse that is given by $z^{-1} = \frac{x-\beta y-iy}{x^2-\beta xy+\alpha y^2}$ for every $z \neq 0$.

In general for $z = x + yi$, we define

$$\bar{z} = (x - \beta y) - yi\ ,\ |z|^2 = x^2 - \beta xy + \alpha y^2\ .$$

The norm also satisfies the property

$$|zw| = |z||w| \text{ and } \left|\frac{1}{z}\right| = \frac{1}{|z|}\ .$$

2.1 *Differentiability in the Elliptic Complex Numbers*

Let Ω be a bounded domain with sufficiently smooth boundary $\partial\Omega$. Consider a continuous function $f : \Omega \subset \mathbb{C}(\alpha, \beta) \to \mathbb{C}(\alpha, \beta)$ such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous in Ω, then starting from

$$\tilde{f}(z) := f(z_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0),$$

where $z_0 = x_0 + y_0 i \in \Omega$, and after a straightforward calculation we have

$$\tilde{f}(z) = f(z_0) + \frac{1}{\beta + 2i}\Big[\big((\beta + i)\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\big)(z - z_0) + \Big(i\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\Big)\overline{(z - z_0)}\Big].$$

Comparing these two representations, it is natural to describe the coefficients of $(z - z_0)$ and $\overline{(z - z_0)}$ as partial elliptic complex derivatives of f with respect to z and $\bar{z}$, respectively, at z_0, i.e., we define

$$\partial_z f := \frac{(\beta + i)\partial_x f + \partial_y f}{\beta + 2i} \text{ and } \partial_{\bar{z}} f := \frac{i\partial_x f - \partial_y f}{\beta + 2i}.$$

Therefore, for $w = u + vi$, we get

$$\partial_{\bar{z}} w = \frac{1}{\beta + 2i}[-(\partial_y u + \alpha\partial_x v) + i(\partial_x u - \beta\partial_x v - \partial_y v)].$$

and the expression $\partial_{\bar{z}} w = 0$ is equivalent to the following real system of equations

$$\partial_y u + \alpha\partial_x v = 0, \quad \partial_x u - \beta\partial_x v - \partial_y v = 0,$$

which is a generalization of the ordinary Cauchy-Riemann equations. In this case we say the function w is holomorphic in $\mathbb{C}(\alpha, \beta)$. Since there exist differentiable functions that are not holomorphic in the ordinary sense, but they are holomorphic for some suitable choice of real numbers α and β, a more encompassing concept of holomorphicity is obtained in the framework of elliptic complex numbers.

The product rule holds for the Cauchy-Riemann operator: Direct computation shows that for every pair of differentiable functions f_1 and f_2 it holds

$$\partial_{\bar{z}}(f_1 \cdot f_2) = \partial_{\bar{z}} f_1 \cdot f_2 + f_1 \cdot \partial_{\bar{z}} f_2,$$

where the product is understood to be in the generalized complex algebra.

3 Associated Spaces

Definition 3.1 ([4, 5]) Let $\mathcal{F}$ be a first order differential operator depending on t, x, u and $\partial_i u$ for $i = 0, 1, \ldots, n$, while $\mathcal{G}$ is a differential operator with respect to the spacelike variables x_i with coefficients not depending on time t. $\mathcal{F}$ is said to be associated with $\mathcal{G}$ if $\mathcal{F}$ maps solutions for the differential equation $\mathcal{G}u = 0$ into solutions of the same equation for a fixedly chosen t, i.e.,

$$\mathcal{G}u = 0 \Rightarrow \mathcal{G}(\mathcal{F}u) = 0.$$

The function space $\mathcal{X}$ containing all the solutions for the differential equation $\mathcal{G}u = 0$ is called an associated space of $\mathcal{F}$.

Next we will determine necessary and sufficient conditions such that the operator $\mathcal{F}$ defined by (2) be associated to the elliptic generalized-analytic operator (3).

3.1 Necessary and Sufficient Conditions on the Coefficients of $\mathcal{F}$

We consider the operator $\mathcal{F}$ defined by (2)

$$\mathcal{F}\omega = C(z)\partial_z \omega + A(z)\omega + B(z)\bar{\omega} + G(z),$$

where $C(z)$, $A(z)$, $B(z)$ and $G(z)$ are continuously differentiable functions and with values in $\mathbb{C}(\alpha, \beta)$. We will determine conditions over $C(z), A(z), B(z)$ and $G(z)$ such that

$$\mathcal{G}\omega = \partial_{\bar{z}}\omega - a(z)\omega - b(z)\bar{\omega} = 0 \Rightarrow \mathcal{G}(\mathcal{F}\omega) = 0, \tag{4}$$

where $a(z), b(z)$ are given continuous functions with values in $\mathbb{C}(\alpha, \beta)$.

Applying the operator $\mathcal{G}$ to $\mathcal{F}(\omega)$ and assuming that ω is a generalized-analytic function, we get

$$\begin{aligned}\mathcal{G}(\mathcal{F}\omega) &= \partial_{\bar{z}}\mathcal{F}\omega - a(z)\mathcal{F}\omega - b(z)\,\overline{\mathcal{F}\omega}\\ &= (\partial_{\bar{z}}C(z))\,\partial_z\omega + (C(z)\partial_z a(z))\,\omega\\ &\quad + (C(z)\partial_z b(z))\,\bar{\omega} + (C(z)b(z))\,\partial_z\bar{\omega} + \partial_{\bar{z}}A(z)\omega\\ &\quad + (A(z)b(z))\,\bar{\omega} + \partial_{\bar{z}}B(z)\bar{\omega} + B(z)\overline{\partial_z\omega}\\ &\quad - (a(z)B(z))\,\bar{\omega} - \left(b(z)\overline{C(z)}\right)\overline{\partial_z\omega}\\ &\quad - \left(b(z)\,\overline{A(z)}\right)\bar{\omega} - \left(b(z)\,\overline{B(z)}\right)\omega\\ &\quad +\partial_{\bar{z}}G(z) - a(z)G(z) - b(z)G(z).\end{aligned}$$

Using $\partial_z\bar{\omega} = \overline{\partial_{\bar{z}}\omega}$, we obtain

$$\begin{aligned}\mathcal{G}(\mathcal{F}\omega) &= (\partial_{\bar{z}}C(z))\,\partial_z\omega + \left(B(z) - b(z)\overline{C(z)}\right)\overline{\partial_z\omega}\\ &\quad + \left(C(z)\partial_z a(z) + C(z)b(z)\overline{b(z)} + \partial_{\bar{z}}A(z) - b(z)\,\overline{B(z)}\right)\omega\\ &\quad + \left(C\partial_z b + Cb\bar{a} + Ab + \partial_{\bar{z}}B - aB - b\bar{A}\right)\bar{\omega}\\ &\quad +\partial_{\bar{z}}G(z) - a(z)G(z) - b(z)G(z).\end{aligned}$$

Now we denote $G_1 = B(z) - b(z)\overline{C(z)}$, $G_2 = C(z)\partial_z a(z) + C(z)b(z)\overline{b(z)} + \partial_{\bar{z}}A(z) - b(z)\,\overline{B(z)}$, $G_3 = C(z)\partial_z b(z) + C(z)b(z)\overline{a(z)} + A(z)b(z) + \partial_{\bar{z}}B(z) - a(z)B(z) - b(z)\overline{A(z)}$ and $G_4 = \partial_{\bar{z}}G(z) - a(z)G(z) - b(z)G(z)$. Then $\mathcal{G}(\mathcal{F}(\omega))$ can be rewritten as

$$\mathcal{G}(\mathcal{F}(\omega)) = (\partial_{\bar{z}}C(z))\,\partial_z\omega + G_1\overline{\partial_z\omega} + G_2\omega + G_3\bar{\omega} + G_4. \tag{5}$$

Therefore we observe that if

$$\partial_{\bar{z}}C(z) = G_1 = G_2 = G_3 = G_4 = 0$$

are met, then $\mathcal{G}(\mathcal{F}\omega) = 0$ if $\mathcal{G}(\omega) = 0$. It means that the following conditions

$$\partial_{\bar{z}}C(z) = 0 \tag{6}$$

$$B(z) = b(z)\overline{C(z)} \tag{7}$$

$$C(z)\partial_z a(z) + \partial_{\bar{z}}A(z) = 0 \tag{8}$$

$$C\partial_z b + \partial_{\bar{z}}b\overline{C} + b\partial_{\bar{z}}\overline{C} + b\left(C\bar{a} - a\bar{C}\right) + b\left(A - \overline{A}\right) = 0 \tag{9}$$

$$\partial_{\bar{z}}G(z) - a(z)G(z) - b(z)G(z) = 0. \tag{10}$$

are sufficient conditions.

Now we assume that $(\mathcal{F}, \mathcal{G})$ is an associated pair, i.e., $\mathcal{G}(\mathcal{F}(\omega) = 0$ if only ω is a generalized-analytic function. In order to obtain the conditions on the coefficients of the operator $\mathcal{F}$ we will start by choosing special functions of the associated space, in this case generalized-analytic functions, and we will write out the relations assuming that $\mathcal{F}$ is generalized analytic for those functions. Then choosing the generalized-analytic function $w = 0$ we have $\partial_{\bar{z}} G(z) - a(z)G(z) - b(z)G(z) = 0$ and so G is generalized analytic and the term G_4 can be omitted from (5). We now look for a function satisfying $\partial_z \omega = 0$ and $\partial_{\bar{z}} \omega = a(z)\,\omega + b(z)\,\bar{\omega}$, which is equivalent to look for a function satisfying $\partial_y \omega + (\beta + 2i)\partial_x \omega = 0$. Then we choose the functions $\omega = 1$ and $\omega = i$ and observe that they are generalized-analytic functions if $a(z) = -b(z)$ and if $a(z) = -\frac{(\beta+i)^2}{\alpha} b(z)$, respectively. So we can consider two cases for Eq. (5): If $\omega = 1$ then $G_2\,\omega + G_3\,\bar{\omega} = 0$ and if $\omega = i$ then $i\,G_2 - (\beta + i)\,G_3 = 0$. This implies $(\beta + 2i)\,G_2 = 0$. Hence $G_2 = 0$ and $G_3 = 0$, and Eq. (5) becomes

$$\mathcal{G}(\mathcal{F}(\omega)) = (\partial_{\bar{z}} C(z))\,\partial_z \omega + G_1\,\overline{\partial_z \omega} = 0. \tag{11}$$

Now we look for a special generalized-analytic function ω satisfying $\partial_z \omega = 1$, it means satisfying the relation $\partial_y \omega + (\beta + i)\partial_x \omega = (\beta + 2i)$. We observe that the function $\omega(x, y) = x + iy$ satisfies this relation and it is generalized analytic if $a(z) = z^{-1}(1 - b(z)\bar{z})$. Next we look for another special generalized-analytic function ω satisfying $\partial_z \omega = i$, i.e., satisfying $\partial_y \omega + (\beta + i)\partial_x \omega = i\,(\beta + 2i)$. In this case we find the function $\omega(x, y) = -\alpha y + i(x - \beta y)$ which is generalized analytic if $a(z) = -\frac{(\beta+i)^2}{\alpha} z^{-1} b(z)\bar{z}$. Therefore using the functions $\omega(x, y) = x + iy$ and $\omega(x, y) = -\alpha y + i(x - \beta y)$ in (11), we get $\partial_{\bar{z}} C(z) + G_1 = 0$ and $i\,\partial_{\bar{z}} C(z) - (\beta + i)G_1 = 0$, which implies $-(\beta + 2i)\,G_1 = 0$ and so $\partial_{\bar{z}} C(z) = 0$ and $G_1 = 0$.

Therefore the following statement is true:

Theorem 3.2 *The operator $\mathcal{F}$ is associated to the generalized-analytic operator $\mathcal{G}$ if and only if the conditions (6)–(10) are satisfied.*

4 Solvability of Initial Value Problems

We consider the initial value problem

$$\partial_t \omega(t, z) = \mathcal{F}(t, z, \omega, \partial_z \omega) \tag{12}$$

$$\omega(0, z) = \varphi(z), \tag{13}$$

where $t \in [0, T]$ is the variable time, $z = x + iy$ and φ is a generalized-analytic function. z, $\omega = \omega(t, z)$ and φ are $\mathbb{C}(\alpha, \beta)$-valued functions.

It is well known that this problem can be rewritten as

$$\omega(t,z) = \varphi(z) + \int_0^t \mathcal{F}(\tau, z, \omega(\tau, z), \partial_z \omega(\tau, z))d\tau. \tag{14}$$

Consequently, the solution of the initial value problem (12), (13) is a fixed point of the operator

$$T\omega(t,z) = \varphi(z) + \int_0^t \mathcal{F}(\tau, z, \omega(\tau, z), \partial_z \omega(\tau, z))d\tau. \tag{15}$$

and vice versa.

To apply a fixed point theorem as the Contraction Mapping Principle, the operator (15) should map a certain Banach space B of generalized-analytic functions into itself. Since the operator $\mathcal{F}$ also depends on the derivative $\partial_z\omega$, that map exists in case the derivatives $\partial_z(T\omega(t,z))$ do exist and can be estimated accordingly. Therefore, one has to restrict the operator to a space of generalized-analytic functions for which the derivatives $\partial_z\omega$ of a generalized-analytic function ω can be estimated by ω itself. This space is the so-called associated space and the estimates for the derivatives $\partial_z\omega$ can be attained by using the so-called interior estimate.

First order interior estimates can be obtained via a generalized version of the Cauchy Pompeiu operator for elliptic numbers [1] and thus the method of associated operators [5] is applied to solve initial value problems with initial functions that are generalized analytic in elliptic complex numbers.

In consequence, we have the following theorem:

Theorem 4.1 *Let $\mathcal{F}$ be the operator defined by (2). Suppose $\mathcal{F}$ and the operator $\mathcal{G}$ defined by (3) form an associated pair of operators, for each fixed $t \in [0,T]$, and the solutions of the corresponding equation $\mathcal{G}u = 0$, satisfy an interior estimate of first order. Then the initial value problem (12) and (13) is solvable provided that the initial function is an elliptic generalized-analytic function.*

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Identification of Nonlinear Differential Systems for Bacteria Population Under Antibiotics Influence

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Abstract A bacteria population under bactericidal antibiotics influence is considered. A part of the bacteria is resistant to the antibiotic. The system is described by nonlinear differential equations.

Keywords Identification • Mathematical model • Nonlinear differential equations • Population dynamics

Mathematics Subject Classification (2010) Primary 34A55; Secondary 92C50

1 Introduction

The problem antimicrobial resistance (AMR) has achieved alarming dimensions. A laboratory monitoring in India showed that proportion of strains *Escherichia coli* resistance to carbapenem increased from 7% in 2008 to 12% in 2014, *Klebsiella* sp. was 22% in 2008 and increased to 60% in 2014 [1]. In the USA, about two million men contaminated of resistant bacteria at least for one antibiotic in the USA [2]. The same problem we observed in the EC [3]. However, the arsenal of antibiotics is

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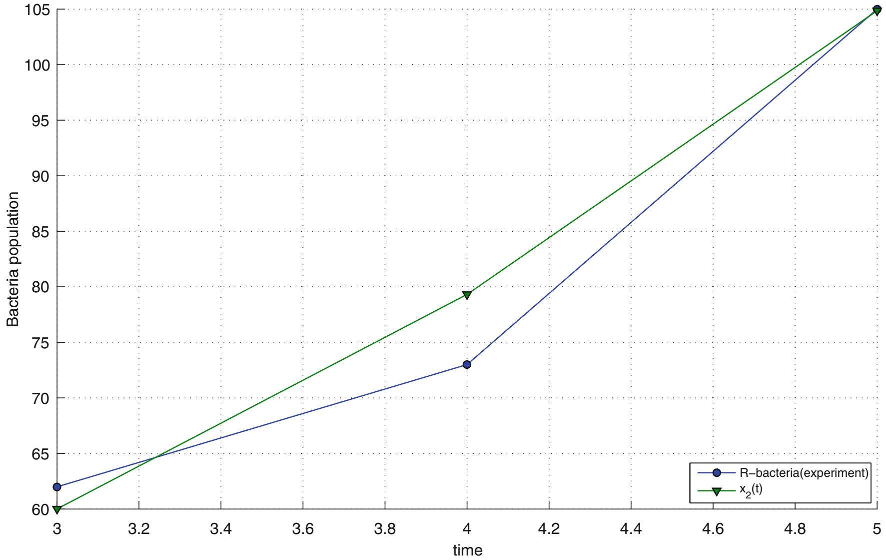
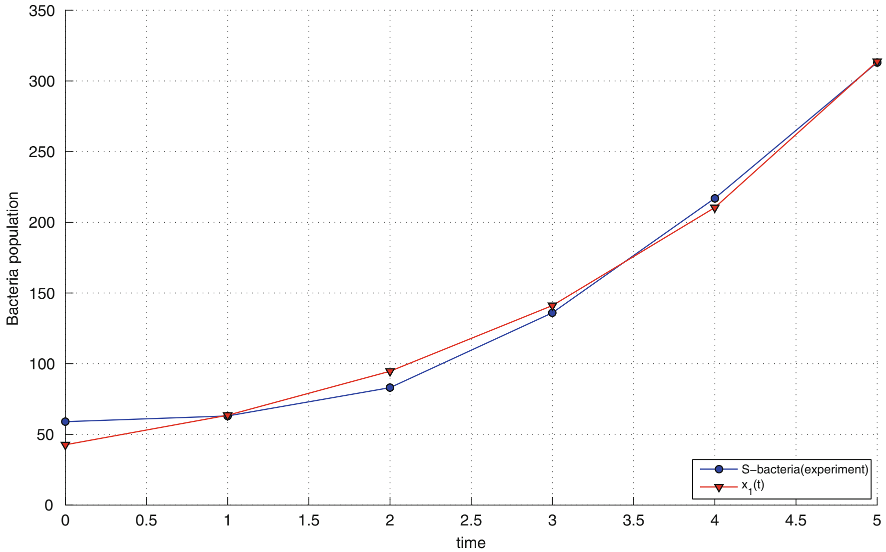
practically exhausted, and those under clinical trials do not pass because of the high toxicity [4]. In coming years, it can lead to a post-antibiotic era or a second pre-antibiotic era, when death can occur because of infection with usual scratches [5].





For solution this global challenge offers a variety of approaches, including mathematical modeling of AMR [6]. There are many works on modeling the spread of antibiotic resistance in the population and nosocomial epidemiology [7]. Some models do not incorporate the history of antibiotic usage in patients [8]. Others on the contrary involve the initial data about the history of the patient's treatment with antibiotics [9]. All this is aimed at the development and management of antibiotic therapy to prevent the development and spread of antibiotic resistance [10]. There are other mathematical models describing the growth and behavior of bacteria [11]. The best known model describing the dynamics of population growth is the Logistic function of Verhulst which takes into account carrying capacity, as well as others, including various modifications [12]. Usually, the characteristic of the growth curve of bacteria is used in the pharmacodynamic models to optimize therapy [13]. Furthermore, using the parameters of several growth curves of the bacteria minimum inhibitory concentration (MIC) may be determined with specified accuracy, whereas now widely used for this is the disk-diffusion method or the more laborious tube dilution test [14]. The parameters of bacterial growth curve may be used in different models, for example in the study of the development of antibiotic resistance and the possibility of reversion [15].

2 Mathematical Model

For determining the considered phenomenon, we use the following suppositions.

1. The bacteria population is under bactericidal antibiotic influence.
2. The population is nonhomogeneous; it consists of antibiotic sensitive bacteria and antibiotic resistant bacteria.
3. The sensitive bacteria are more viable in the absence of the antibiotic.
4. The habitat is limited.
5. Transitions from one to the other type of bacteria by mutation and transmission of plasmids carrying the gene for resistance to the antibiotic are conceded.

We propose the following systems of nonlinear differential equations.

$$\begin{aligned}\dot{x}_1 &= a_1 x_1 - b_1(x_1 + x_2)x_1 + a_{12}x_2 - c\theta(t)x_1^d,\\ \dot{x}_2 &= a_2 x_2 - b_2(x_1 + x_2)x_2 + a_{21}x_1\end{aligned}$$

with initial conditions

$$x_1(0) = x_{10},\ x_2(0) = x_{20}.$$

The functions x_1 and x_2 describe the number of sensitive and resistant bacteria (S-bacteria and R-bacteria) here; x_{10} and x_{20} are its initial values. The parameters a_1 and a_2 are increase in the number of both types of bacteria; b_1 and b_2 describe the influence of the habitat limitation; and a_{12} and a_{21} characterize the transitions from one to the other type of bacteria by mutation and/or transmission of plasmids carrying the gene for resistance to the antibiotic. The parameters c and d are characteristics of the antibiotic. The function θ is equal to one during the action of antibiotics, and zero in the rest of the time.

We get the following results.

Theorem 2.1 *If the parameters x_{10}, x_{20}, a_{12}, a_{21} are small enough, and $\theta = 0$, then the functions x_1 and x_2 increase exponentially at the initial stage of the process.*

Indeed, when the initial bacteria numbers are small enough, transitions from one type of bacteria to the other are rare, and no antibiotic is applied by the suppositions of the theorem. Then the last term on the right-hand side of the first equation is absent, and we can neglect the second and third terms on the right-hand side of both equations. As a result, for each type of bacteria we obtain the Malthus equation with exponential solution.

Theorem 2.2 *If x_{20} is much less than x_{10}, and $\theta = 0$, then the function x_1 increases up to a certain value.*

Indeed, suppose the absence of an antibiotic and a negligible number of sensitive bacteria. Then the second term on the right-hand side of the first equation will be most important after the increase of the function x_1. Therefore, the system gradually tends to the equilibrium typical for the Verhulst model.

Theorem 2.3 *If the coefficients a_{12}, a_{21} are small enough, and $\theta = 0$, then the function x_2 tends to zero.*

Indeed, the vitality of sensitive bacteria is greater compared to the resistant because of the inequality condition of the theorem. Then, in the absence of antibiotics and the fairly rare intrapopulation transitions weaker resistant bacteria are dying out, which corresponds to the model of Volterra.

Theorem 2.4 *Suppose the conditions of Theorem 2.3 except the smallness of the parameters a_{12} and a_{21}. Then the function x_2 decreases to a positive value, and the equilibrium position of the function x_1 is less than the value that was obtained under the conditions of Theorem 2.2.*

Indeed, in this case the presence of the third term on the right-hand side of the second equation ensures a positive limit value of x_2. The number of resistant bacteria thereby supported by transitions, and reducing the value of the equilibrium position of sensitive bacteria is due to a non-zero value of the maximum number of resistant bacteria.

Theorem 2.5 *If the function θ is positive starting from a certain point in time, then the function x_1 decreases to a sufficiently small positive value, and the function x_2 increases to an equilibrium position.*

Indeed, the positive value of the function θ indicates the presence of the antibiotic. Then the last negative term is present on the right-hand side of the first equation. Therefore the function x_1 decreases. However, its velocity decreases gradually to zero with decreasing values of this function. On the other hand, the antibiotic does not affect the resistant bacteria. Thus, the number of resistant bacteria x_2 increases. This growth is constrained by the limited living space characterized by the second term on the right-hand side of the second equation.

Theorem 2.6 *If starting from a certain point in time the function takes a positive value, and then zero, then after some time the system tends to the same equilibrium position, which was observed under the conditions of Theorem 2.4.*

Indeed, if the function θ is equal to zero (i.e., after the finish of the treatment), the system is in the state described by Theorem 2.6. Thus, the stronger antibiotic-sensitive bacteria supplant the resistant bacteria.

Computer calculations have confirmed the results of the qualitative analysis.

3 Identification of the Mathematical Model

The next step of the analysis is the identification of the systems. We determine parameters of the equations by inverse problems with using test data. The necessary experiments were conducted at the Scientific Center for Anti-infectious Drugs, Almaty.

The bacterial growth rate was measured for susceptible (S) and resistant (R) microorganisms. The tested microorganism was *Escherichia coli* (ATCC 8739). Resistant *E. coli* was selected from a susceptible one by cultivation on medium with increasing concentrations of antibiotic (Ampicillin trihydrate, drug substance, Sigma Aldrich). The microorganisms were cultivated between the wells of microtiter plates at a temperature of 37 °C. For cultivating minimal media were containing salts, amino acids, glycerol, glucose, and fermentative peptone (Himedia, India). The bacterial growth was estimated by changing the optical density (OD) measured and recorded a microwell plate reader (Multiscan Ascent, Agilent Technologies, USA). OD was determined at a wavelength of 540 nm. The plate was shaken for 10 s just before each reading. The reading interval was 1 h. For the first hour after adding antibiotic OD was measured every 15 min. Then measurements were done every hour until growth curve shows steady trend for increasing OD. Stock solution of antibiotic were added into the experimental wells at the end of the exponential phase of growth—beginning of stationary phase of growth. The equal volume was added to all wells (16 μl). The final volume of suspension in the wells was 286 μl. The final concentration of ampicillin in the wells was 3400 μg/ml. This corresponds to 4.0 minimal inhibitory concentrations (MIC) of ampicillin for resistant *E. coli* in MIC assays with initial concentration of bacterial suspension 107 CFU/ml (colony forming unit per milliliter). MIC assays of ampicillin were performed by broth microdilution method [16]. For the test quality control positive and negative controls of growth were used. We consider two stages of the phenomenon. This is the evolution of the bacteria population without antibiotic, and the system under the antibiotic influence. We neglect the third term on the right-hand side of the equations describing intrapopulation transitions for both cases. Thus, we get two inverse problems.

We determine the parameters a_1, a_2, b_1, and b_2 with using measuring of the functions x_1 and x_2 for the first inverse problem. The results of computing are its following values $a_1 = 0.399$, $a_2 = 0.274$, $b_1 = 4.77 \cdot 10^{-9}$, and $b_2 = 3.92 \cdot 10^{-10}$.

We chose the value $d = 3$ for the second inverse problem. This is explained by the following considerations. The linear terms of the equations characterize the natural exponential increase of the population by the Malthus model. The square terms of the system describe the bounded increase of the population because of the habitat limitedness by the Verhulst model. Therefore decreasing of the population by antibiotic must be described by higher-order term. Thus, we find the coefficient c by the second inverse problem. We get $c = 2.71 \cdot 10^{-5}$.

These results will be used for further study of antibiotic resistance, in particular, the phenomenon of reversion, i.e. restore the sensitivity of bacteria to antibiotics.

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The Cauchy Problem for Some System of n-th Order Nonlinear Ordinary Differential Equations

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Abstract In this paper the Cauchy problem for some system of n-th order nonlinear ordinary differential equations is solved.

Keywords Cauchy problem • General solution • n-th order nonlinear ODE

Mathematics Subject Classification (2010) Primary 34B05; Secondary 34L30

1 Introduction

Let $t_1 > 0$ and $n \geq 1$ be a natural number. We consider the system

$$\begin{aligned} \frac{d^n u}{dt^n} &= f(t)u - g(t)v + h(t, u, v) \\ \frac{d^n v}{dt^n} &= g(t)u - f(t)v + q(t, u, v) \end{aligned} \tag{1}$$

in the interval $[0, t_1]$, where $f(t), g(t) \in C[0, t_1]$ and the functions $h(t, u, v), q(t, u, v)$ are continuous in the set of variables in the domain

$$G = \{(t, u, v) : 0 \leq t < \delta, |u - \alpha_1| < \sigma_1, |v - \beta_1| < \sigma_2\}.$$

Here $\delta, \alpha_1, \beta_1, \sigma_1, \sigma_2$ are real numbers so that $\sigma_1 > 0, \sigma_2 > 0, 0 < \delta < t_1$.

In the particular case $n = 1$ the general solution of system (1) and the solution of the Cauchy problem for it are given in [1]. Solutions of system (1) will be sought from the class

$$C^n[0,\ t_1]. \tag{2}$$

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Let us consider the Cauchy problem for system (1).

Cauchy Problem Find the solution of system (1) from the class (2) satisfying the conditions

$$u(0) = \alpha_1, v(0) = \beta_1, u'(0) = \alpha_2, v'(0) = \beta_2, \dots, u^{(n-1)}(0) = \alpha_n, v^{(n-1)}(0) = \beta_n, \tag{3}$$

where $\alpha_k, \beta_k, (k = 1, 2, \dots, n)$ are given real numbers.

2 Construction of the General Solution of the Corresponding Linear System

First we consider the system

$$\begin{aligned} \tfrac{d^n u}{dt^n} &= f(t)u - g(t)v + h(t), \\ \tfrac{d^n v}{dt^n} &= g(t)u - f(t)v + q(t) \end{aligned} \tag{4}$$

on $[0, t_1]$, where $f(t), g(t), h(t), q(t) \in C[0, t_1]$.

In this work using the method employed in [1] we are obtaining an explicit form of the general solution of the system (4) and are solving the Cauchy problem for it. For simplicity we will study the system (4) in $[0, t_1]$, the coefficients are taken from the class of continuous functions in $[0, t_1]$. Using the proposed method we can obtain the general solution of system (4) in an arbitrary bounded domain of the real axis, the coefficients can be taken from the class of measurable and essentially bounded functions. But in this case the derivative of n-th order of the solutions belongs to the class of measurable and essentially bounded functions. Solutions of system (4) will be sought from the class (2). To construct the general solution we use a method developed in [1]. To accomplish this multiplying the second equation of system (4) by the imaginary unit $i = \sqrt{-1}$ and then adding it to the first equation we obtain

$$\frac{d^n w}{dt^n} - p(t)w = s(t), \tag{5}$$

where $p(t) = f(t) + ig(t), s(t) = h(t) + iq(t), w = u + iv$.

Obviously, $p(t), s(t) \in C[0, t_1], w \in C^n[0, t_1]$. From (3) follows

$$w(0) = \alpha_1 + i\beta_1, w'(0) = \alpha_2 + i\beta_2, \dots, w^{(n-1)}(0) = \alpha_n + i\beta_n. \tag{6}$$

Integrating equation (5) n times gives

$$w(t) = (Bw)(t) + s_0(t) + \sum_{k=1}^{n} c_k t^{n-1}, \tag{7}$$

where c_k, $(k = 0, 1, 2, \ldots, n-1)$ are arbitrary complex numbers,

$$(Bw)(t) = \int_0^t \int_0^{y_1} \int_0^{y_2} \ldots \int_0^{y_{n-1}} p(\tau)w(\tau)d\tau dy_{n-1}dy_{n-2}\ldots dy_1,$$

$$s_0(t) = (Bs)(t) = \int_0^t \int_0^{y_1} \int_0^{y_2} \ldots \int_0^{y_{n-1}} p(\tau)s(\tau)d\tau dy_{n-1}dy_{n-2}\ldots dy_1.$$

Applying the operator $(B\cdot)(t)$ to both sides of Eq. (7) we have

$$(Bw)(t) = (B^2w)(t) + s_1(t) + \sum_{k=1}^{n} c_k a_{k,1}(t), \tag{8}$$

where

$$(B^2w)(t) = (B(Bw)(t))(t), s_1(t) = (Bs_0)(t)$$
$$= \int_0^t \int_0^{y_1} \int_0^{y_2} \ldots \int_0^{y_{n-1}} p(\tau)s_0(\tau)d\tau dy_{n-1}dy_{n-2}\ldots dy_1,$$

$$a_{k,1}(t) = \int_0^t \int_0^{y_1} \int_0^{y_2} \ldots \int_0^{y_{n-1}} \tau^{k-1}p(\tau)d\tau dy_{n-1}dy_{n-2}\ldots dy_1 \quad (k = 1, 2, \ldots, n).$$

From (7) and (8) follows

$$w(t) = (B^2w)(t) + s_0(t) + s_1(t) + \sum_{k=1}^{n} c_k(t^{k-1} + a_{k,1}(t)). \tag{9}$$

Again applying the operator $(B\cdot)(t)$ to both sides of Eq. (9) we obtain

$$(Bw)(t) = (B^3w)(t) + s_1(t) + s_2(t) + \sum_{k=1}^{n} c_k(a_{k,1}(t) + a_{k,2}(t)). \tag{10}$$

where

$$(B^3w)(t) = (B(B^2w)(t))(t),$$

$$a_{k,2}(t) = (Ba_{k,1})(t) = \int_0^t \int_0^{y_1} \int_0^{y_2} \ldots \int_0^{y_{n-1}} p(\tau)a_{k,1}(\tau)d\tau dy_{n-1}dy_{n-2}\ldots dy_1.$$

From (10) and (7) follows

$$w(t) = (B^3w)(t) + s_0(t) + s_1(t) + s_2(t) + \sum_{k=1}^{n} c_k(t^{k-1} + a_{k,1}(t) + a_{k,2}(t)).$$

Continuing this procedure m times we obtain the following integral representation for the solutions of Eq. (5)

$$w(t) = (B^m w)(t) + \sum_{j=0}^{m-1} s_j(t) + \sum_{k=1}^{n} c_k \left(t^{k-1} + \sum_{j=1}^{m-1} a_{k,j}(t) \right), \tag{11}$$

where $(B^m w)(t) = (B(B^{m-1}w)(t))(t)$,

$$s_j(t) = (Bs_{j-1})(t) = \int_0^t \int_0^{y_1} \int_0^{y_2} \dots \int_0^{y_{n-1}} p(\tau)s_{j-1}(\tau)d\tau dy_{n-1}dy_{n-2}\dots dy_1,$$

$$(j = 1, 2, \dots),$$

$$a_{k,j}(t) = (Ba_{k,j-1})(t) = \int_0^t \int_0^{y_1} \int_0^{y_2} \dots \int_0^{y_{n-1}} p(\tau)a_{k,j-1}(\tau)d\tau dy_{n-1}dy_{n-2}\dots dy_1,$$

$$(k = 1, 2, \dots, n-1), (j = 2, 3, \dots).$$

Taking the definition of the integrated operators $(B^k f)(t)$ and the integrated functions $a_{k,j}(t)$ into consideration the following estimates are obtained:

$$\begin{aligned} &|(B^m w)(t)| \le |w|_0 \cdot \tfrac{(\sqrt[n]{|p|_0 t})^{nm}}{(nm)!}, |s_j(t)| \le |s_0|_0 \cdot \tfrac{(\sqrt[n]{|p|_0 t})^{jn}}{(jn)!}, \\ &|a_{k,j}| \le \tfrac{|p|_0^j}{(k+jn-1)!} \cdot t^{k+jn-1}, (m = 1, 2, \dots), (k = 1, 2, \dots, n-1), \end{aligned} \tag{12}$$

where $|f|_0 = \max_{t\in[0,t_1]}|f(t)|$.

If we pass to the limit as $m \longrightarrow \infty$ in (11) taking into account (12) we obtain the solution of Eq. (5):

$$w(t) = \sum_{m=1}^{\infty} c_k I_k + F(t), \tag{13}$$

where c_k, $(k = 1, 2, \dots, n)$ are arbitrary complex numbers,

$$I_k(t) = t^{k-1} + \sum_{m=1}^{\infty} a_{k,m}(t), \quad F(t) = \sum_{j=0}^{\infty} s_j(t).$$

From the form of the functions $I_k(t)$, $(k = 1, 2, \dots, n)$ and $F(t)$ follow

$$I_1(0) = 1, \;\; I_k(0) = 0, (k \ne 1), \;\; F^k(0) = 0, (k = 1, 2, \dots, n-1), \tag{14}$$

$$I_1(0) = I_2'(0) = I_3''(0) = \dots = I_n^{(n-1)}(0) = 1.$$

By virtue of (12) we obtain the estimates

$$|I_k(t)| \leq t^k \exp(|p|_0 t^n), \quad |F(t)| \leq |s_0|_0 \exp(|p|_0 t^n), (k = 1, 2, \dots, n-1).$$

From (14) follows that the Wronskian of the functions $I_1(t), I_2(t), \dots, I_n(t)$ in the point $t = 0$ is not equal to zero. Therefore, the functions $I_1(t), I_2(t), \dots, I_n(t)$ are linearly independent of $[0, t_1]$. From the definition of $I_k(t), (k = 1, 2, \dots, n)$ and $F(t)$ follow

$$\frac{d^n I_k}{dt^n} - p(t) I_k = 0, (k = 1, 2, \dots, n), \qquad \frac{d^n F}{dt^n} - p(t) F = s(t). \tag{15}$$

Hence, the formula (13) defines the general solution of Eq. (5). Separating the real and imaginary parts of the (13) we obtain the general solution of the system (4):

$$\begin{aligned} u(t) &= \textstyle\sum_{k=1}^n (c_{k1} \mathrm{Re} I_k(t) - c_{k2} \mathrm{Im} I_k(t)) + \mathrm{Re} F(t), \\ v(t) &= \textstyle\sum_{k=1}^n (c_{k2} \mathrm{Re} I_k(t) + c_{k1} \mathrm{Im} I_k(t)) + \mathrm{Re} F(t), \end{aligned} \tag{16}$$

where $c_{k1}, c_{k2}, (k = 1, 2, \dots, n)$ are arbitrary real numbers. Thus, the following theorem holds.

Theorem 2.1 *The general solution of system (1) from the class (2) is given by the formula (16).*

3 The Solution of the Cauchy Problem for the Corresponding Linear System

First we will solve the Cauchy problem for Eq. (5). To solve the problem we use the formula (13). Substituting the function $w(t)$, given by (13) in the initial conditions (6) and taking (14) into account we obtain

$$c_1 = \alpha_1 + i\beta_1, c_2 = \alpha_2 + i\beta_2, c_3 = \alpha_3 + i\beta_3, \dots, c_n = \alpha_n + i\beta_n$$

and hence the function

$$w(t) = \sum_{k=1}^{n} (\alpha_k + i\beta_k) I_k(t) + F(t) \tag{17}$$

gives a solution of the Cauchy problem for the system (4). Highlighting the real and imaginary parts of (17) we obtain the solution of the Cauchy problem for the system (4):

$$\begin{aligned} u(t) &= \textstyle\sum_{k=1}^{n} (\alpha_k \mathrm{Re} I_k(t) - \beta_k \mathrm{Im} I_k(t)) + \mathrm{Re} F(t), \\ v(t) &= \textstyle\sum_{k=1}^{n} (\beta_k \mathrm{Re} I_k(t) + \alpha_k \mathrm{Im} I_k(t)) + \mathrm{Im} F(t). \end{aligned} \tag{18}$$

Thus, the following theorem holds.

Theorem 3.1 *The Cauchy problem for the system (4) has a solution, which is given by the formula (18).*

The obtained results remain in force in the case:

$$f(t), g(t), h(t), q(t) \in S[t_1, t_2], w(t) \in C^{n-1}[t_1, t_2] \cap S_\infty^n [t_1, t_2] .$$

Here $S_\infty^n [t_1, t_2]$ is the class of functions $f(t)$, for which $f^n(t) \in S[t_1, t_2]$, $S[t_1, t_2]$ is the class of measurable and essentially bounded functions in $[t_1, t_2]$ and $-\infty < t_1 < t_2 < \infty$.

4 Cauchy Problem for the System of *n*-th Order Nonlinear Ordinary Differential Equations

Let $t_1 > 0$. We consider the system (1) in the interval $[0, t_1]$, where $f(t), g(t) \in C[0, t_1]$ and the functions $h(t, u, v), q(t, u, v)$ are continuous in the set of variables in the domain

$$G = \{(t, u, v) : 0 \le t < \delta, |u - \alpha_1| < \sigma_1, |v - \beta_1| < \sigma_2\}.$$

Here $\delta, \alpha_1, \beta_1, \sigma_1, \sigma_2$ are real numbers so that $\sigma_1 > 0, \sigma_2 > 0, 0 < \delta < t_1, u(0) = \alpha_1,\ v(0) = \beta_1$. The connection between the numbers δ and σ_1, σ_2 will be defined later.

Multiplying the second equation of the system (1) by $i = \sqrt{-1}$ then adding it to the first equation of (1) we get

$$\frac{d^n w}{dt^n} - p(t)w = s(t, w), \tag{19}$$

where $w = u(t) + iv(t),\ p(t) = f(t) + ig(t),\ s(t, w) = h(t, u, v) + iq(t, u, v)$.

Obviously, $p(t) \in C[0, t_1]$ and the function $s(t, w)$ is continuous in the set of variables in the domain

$$G = \{(t, w) : 0 \le t < \delta, |w - \gamma_1| < \sigma\},$$

where $\gamma_1 = \alpha_1 + i\beta_1, \sigma = \sigma_1 + \sigma_2$.

We find the solution to Eq. (19) from the class (2) satisfying the following conditions

$$w(0) = \gamma_1, w'(0) = \gamma_2, \ldots, w^{n-1}(0) = \gamma_n, \tag{20}$$

where $\gamma_k = \alpha_k + i\beta_k, (k = 1, 2, \ldots, n)$. The equalities are given by (6).

Using formula (13) we have

$$w(t) = \sum_{k=1}^{n} c_k I_k + F(t, w), \tag{21}$$

where

$$F(t, w) = \sum_{j=0}^{\infty} s_j(t, w),$$

$$s_0(t, w) = \int_0^t \int_0^{y_1} \int_0^{y_2} \ldots \int_0^{y_{n-1}} p(\tau) s(\tau, w) d\tau dy_{n-1} dy_{n-2} \ldots dy_1,$$

$$s_j(t, w) = \int_0^t \int_0^{y_1} \int_0^{y_2} \ldots \int_0^{y_{n-1}} p(\tau) s_{j-1}(\tau, w) d\tau dy_{n-1} dy_{n-2} \ldots dy_1, (j = 1, 2, \ldots).$$

From (12) follows

$$|I_1(t) - 1| \le \exp(|p|_0 t^n) - 1, |I_k(t)| \le t^{k-1} \exp(|p|_0 t^n), (k = 2, 3, \ldots, n), \tag{22}$$

$$|(I_1(t_3) - I_k(t_2))| \le t_1^{k+n-2}(|p|_0 + (k-1)!(\exp(|p|_0 t_1) - 1)) \cdot (t_3 - t_2),$$
$$(k = 1, 2, \ldots, n), \tag{23}$$

where $0 \le t_2 < t_3 \le t_1$. Let a_1 be the maximum of the function $|s(t, w)|$ in the domain G. From the form of the function $F(t, w)$ follows

$$|F(t, w)| \le a_1 |p|_0 t^n \exp(|p|_0 t^n), \tag{24}$$

$$|F(t_3, w) - F(t_2, w) \le a_1 |p|_0 \delta^{n-1} \exp(|p|_0 t_1^n)(t_3 - t_2). \tag{25}$$

From the form of the functions $I_k(t)$, $(k = 1, 2, \ldots, n)$, $F(t, w)$ and (15) follows that the right-hand side of the equality (21) belongs to the class $C^n[0, t_1]$. If we take the derivative n times of both sides of the equality (21), then we obtain (19). Therefore, the following theorem holds.

Theorem 4.1 *Any solution from the class* $C[0, t_1]$ *of Eq. (21) is a solution to Eq. (19) from the class (2).*

We consider the solution to Eq. (19) from the class (2) satisfying initial conditions (20). We obtain $c_k = \gamma_k, (k = 1, 2, \ldots, n)$ from (21) by taking the equalities (14) into account. Thus, any solution from the class $C[0, t_1]$ of the equation

$$w(t) = (Dw)(t), \tag{26}$$

where $(Dw)(t) = \sum_{k=1}^{n} \gamma_k I_k + F(t, w)$,, is a solution of the Cauchy problem for Eq. (19).

Let

$$\gamma_1(\exp(|p|_0\delta^n) - 1) + \exp(|p|_0\delta^n) \cdot \sum_{k=2}^{\infty} \gamma_k \delta^{k-1} + a_1|p|_0\delta^n \cdot \exp(|p|_0\delta^n) < \sigma, \tag{27}$$

Inequality (27) always might be obtained for small values of the number δ. Let us prove the existence of continuous solutions to the system (1) in some neighborhood of the point $t = 0$.

Theorem 4.2 *Let $f(t), g(t) \in C[0, t_1]$ and the functions $h(t, u, v), q(t, u, v)$ be continuous in the set of variables in the domain*

$$G = (t, u, v) : 0 \le t < \delta, |u - \alpha_1| < \sigma_1, |v - \beta_1| < \sigma_2.$$

Then on the interval $[0, \delta]$, where the number δ satisfies the condition (27) there exists at least one solution to the system (1) from the class (2) satisfying the conditions (3).

Proof If there exists a solution of Eq. (26) from the class C[0, t1], then by virtue of Theorem 4.1 by highlighting real and imaginary parts of it we obtain a solution of the system (1) from the class $C^n[0, t_1]$. Therefore, by Theorem 4.2 it is sufficient to prove the existence of solutions from the class $C[0, t_1]$ of Eq. (26).

Let $||w|| = \max_{0 \le t < \delta} |w(t)|$. We consider the operator D which is defined by the equality

$$(Dw)(t) = \sum_{k=1}^{n} \gamma_k I_k(t) + F(t, w)$$

on the sphere $||w - \gamma_1|| \le \delta$. For any element $w(t)$ of the sphere $||w - \gamma_1|| < \delta$ in force of the inequalities (22), (24) we get

$$|(Dw)(t)| \le \gamma_1(\exp(|p|_0\delta^n) - 1) + \exp(|p|_0\delta^n) \cdot \sum_{k=2}^{n} |\gamma_k|\delta^{k-1} \tag{28}$$

If $t_2, t_3, (t_2 < t_3)$, are two arbitrary points of the interval $[0, \delta]$, then by inequalities (4), (25) we have

$$\begin{aligned} |(Dw)(t_3) - (Dw)(t_2)| \le (\delta^{k+n-2}(|p|_0 + (k-1)!(\exp(|p|_0\delta - 1)) \\ +a_1|p|_0\delta^{n-1}(\exp(|p|_0\delta^n))(t_2 - t_3). \end{aligned} \tag{29}$$

By the Arzela-Ascoli theorem from (28), (29) follows that the operator D transforms the sphere $||w - \gamma_1|| \le \delta$ into a compact set. We show that operator D transforms this into itself. Indeed, inequalities (22), (24) give us

$$|(Dw)(t) - \gamma_1| \le \gamma_1(\exp(|p|_0\delta^n) - 1) \\ + \exp(|p|_0\delta^n) \cdot \sum_{k=2}^{\infty} \gamma_k \delta^{k-1} + a_1|p|_0\delta^n \exp(|p|_0\delta^n). \qquad (30)$$

Using the inequalities (27), (30) we have $|(Dw)(t) - \gamma_1| < \sigma$. Therefore, the operator D satisfies all the conditions of Schauder's theorem. Hence, there exists a fixed point of this operator, i.e. such a function $w(t)$, so that

$$w(t) = \sum_{k=1}^{n} \gamma_k I_k(t) + F(t, w).$$

Therefore, by Theorem 4.1 there exists a solution to the Cauchy problem for the system (1). □

The theorem is proved.

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Hilbert Boundary-Value Problem for Rotation-Invariant Polyanalytic Functions

Yufeng Wang and Yanjin Wang

Abstract In this article, we will consider the Hilbert-type boundary-value problem of rotation-invariant polyanalytic functions on the unit disc. By use of the decomposition of rotation-invariant polyanalytic functions, the Hilbert problem of rotation-invariant polyanalytic functions is reduced to n corresponding problems of rotation-invariant analytic functions. Then the expression of solution and the condition of solvability are obtained.

Keywords Hilbert boundary-value problem • Rotation-invariant polyanalytic function

Mathematics Subject Classification (2010) Primary 30G30; Secondary 45E05

1 Introduction and Preliminary Results

In recent years, the theory of boundary-value problems (BVPs) for analytic functions has been generalized to those of different classes of functions, including polyanalytic functions, polyharmonic functions, metaanalytic functions, and other function classes derived from complex partial differential equations. Polyanalytic function is a natural generalization of analytic function [1].

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As in [2], we introduce some symbols as follows. Firstly, a rotation group with respect to the origin is defined as follows:

$$G = \left\{\rho_k : \ \rho_k(z) = e^{\frac{2k\pi i}{m}} z, \ k = 0, 1, 2, \ldots, m-1\right\} \tag{1}$$

with $|G| = m \in \mathbb{Z}$.

Definition 1.1 (See [2]) Suppose f be a polyanalytic function [1] of order n on Ω, where Ω is a rotation-invariant open set under the group G. If

$$f(\rho_1(z)) = f(z), \quad \forall z \in \Omega, \tag{2}$$

then we say that f is a rotation-invariant polyanalytic function of order n under the group G, or simply automorphic polyanalytic function. Here G and ρ_1 are given by *(1)*. The collection of all the automorphic polyanalytic functions with respect to G on Ω is denoted as $H_n^G(\Omega)$.

The symbol $H_1^G(\Omega)$ represents the family of automorphic analytic functions on Ω. As in [1], $H_n(\Omega) = \{f : \partial_{\bar{z}}^n f(z) = 0, z \in \Omega\}$ is denoted as the class of polyanalytic functions of order n on Ω, where $\partial_{\bar{z}} = 1/2[\partial/\partial x + i(\partial/\partial y)]$ is the Cauchy-Riemann operator.

Next, a circular ring domain with center at the origin is denoted as

$$D(0; r, R) = \{z \in \mathbb{C} : \ r < |z| < R\} \tag{3}$$

with $0 \le r < R \le +\infty$. In particular, $\mathbb{C}_0 = D(0; 0, +\infty)$.

Definition 1.2 (See [2]) Suppose $f \in H_n^G(D(0; 0, r))$ with $r > 0$. If there exists an integer s such that

$$\limsup_{z \in S_0(0;0,r), z \to 0} |z^{ms} f(z)| = \alpha \quad \text{with} \quad \alpha \in (0, +\infty),$$

then f is said to be order s at the origin, denoted as $\mathrm{Ord}(f, 0) = s$.

The following results can be easily derived from Theorem 2.1 and Corollary 3.2 in [2].

Theorem 1.1 *Let Ω be a rotation-invariant open set under the group G and $0 \notin \Omega$. Then $H_n^G(\Omega) = H_1^G(\Omega) \oplus \cdots \oplus \frac{(|z|^2-1)^j}{j!} H_1^G(\Omega) \oplus \cdots \oplus \frac{(|z|^2-1)^{n-1}}{(n-1)!} H_1^G(\Omega)$ with $\frac{(|z|^2-1)^j}{j!} H_1^G(\Omega) = \left\{\frac{(|z|^2-1)^j}{j!} h(z) : \ h \in H_1^G(\Omega)\right\}$ for $j = 0, 1, \ldots, n-1$.*

Theorem 1.1 implies that $f \in H_n^G(\Omega)$ admits a unique decomposition

$$f(z) = \sum_{j=0}^{n-1} \frac{(|z|^2 - 1)^j}{j!} f_j(z) \quad \text{with} \ f_j \in H_1^G(\Omega). \tag{4}$$

Theorem 1.2 *If* $f \in H_n^G(D(0;0,r))$ *with* $r > 0$, *then*

$$\mathrm{Ord}(f,0) = \max\{\mathrm{Ord}(f_0,0),\ \mathrm{Ord}(f_k,0)+1,\ k=1,2,\ldots,n-1\},$$

where f_k *is the k-component determined by (4).*

Let $\mathbb{D} = \{z : |z| < 1\}$, $\mathbb{T} = \{z : |z| = 1\}$, $\mathbb{T}_0 = \left\{e^{i\theta} : \theta \in \left[0, \frac{2\pi}{m}\right]\right\}$ and $\mathbb{T}_\ell = \rho_\ell(\mathbb{T}_0)$ for $\ell = 1,\ldots,m-1$. Obviously $\mathbb{T} = \cup_{j=0}^{m-1}\mathbb{T}_j$.

2 Rotation-Invariant Schwarz Operator

Now we consider the preliminary problem: Find a function $\Phi \in H_1^G(\mathbb{D}\setminus\{0\}) \cap C(\overline{\mathbb{D}}\setminus\{0\})$ satisfying

$$\begin{cases} \mathrm{Re}\{[a(t)+ib(t)]\Phi^+(t)\} = c(t), & t \in \mathbb{T}, \\ \mathrm{Ord}(\Phi,0) \le \mu, \end{cases} \tag{5}$$

where a,b,c are rotation-invariant, i.e., $a(\rho_1(t)) = a(t)$, $b(\rho_1(t)) = b(t)$, and $c(\rho_1(t)) = c(t)$ on $t \in \mathbb{T}$. The real-valued functions a,b,c are also Hölder-continuous on $\mathbb{T}$, denoted as $a,b,c \in H(\mathbb{T})$. In addition, $a^2(t)+b^2(t) = 1$, $t \in \mathbb{T}$ and $\mu \in \mathbb{Z}$. If $a(t)+ib(t) \equiv 1$, $t \in \mathbb{T}$, the Hilbert problem (5) is reduced to the simplest problem, usually called Schwarz problem. Similarly to [2], one has the following lemma.

Lemma 2.1 *The Schwarz problem* $\mathrm{Re}\{\Phi^+(t)\} = c(t)$, $t \in \mathbb{T}$, *under the condition* $\mathrm{Im}\Phi(0) = 0$, $\Phi \in H_1^G(\mathbb{D}\setminus\{0\}) \cap C(\overline{\mathbb{D}}\setminus\{0\})$ *is uniquely solvable by*

$$\Phi(z) = \frac{m}{2\pi i}\int_{\mathbb{T}_0} c(\tau)\frac{\tau^m + z^m}{\tau^m - z^m}\frac{\mathrm{d}\tau}{\tau} = \mathbf{S}_m[c](z),\ z \in \mathbb{D}, \tag{6}$$

where $\mathbf{S}_m$ *is called the rotation-invariant Schwarz operator,* $m = |G|$ *and* $\Phi(0) = \lim_{z\to 0}\Phi(z)$.

When $m = 1$, the operator $\mathbf{S}_m$ is the classical Schwarz operator. Let

$$X(z) = iz^{m\kappa}\exp\{i\mathbf{S}_m[\Theta](z)\} \quad \text{with} \quad \Theta(t) = \arg\{t^{-m\kappa}[a(t)-ib(t)]\}, \tag{7}$$

where $\kappa = \frac{1}{2\pi}\{\arg[a(t)-ib(t)]\}_{\mathbb{T}_0}$ is the index. Then, by Plemelj's formula [3], $X^+(t) = iR(t)[a(t)-ib(t)]$ with

$$R(t) = \exp\left\{\frac{m}{2\pi}\int_0^{2\pi}\Theta(e^{i\phi})\frac{\sin[m(\phi-\theta)]}{1-\cos[m(\phi-\theta)]}\,\mathrm{d}\phi\right\} > 0.$$

Here R is also called the regularized factor as in [4]. The function X defined by (7) is also called the canonical function.

If $c(t) \equiv 0$, then the boundary condition in (5) turns into the following:

$$\mathrm{Re}\{[a(t) + ib(t)]\Phi^+(t)\} = 0, \quad t \in \mathbb{T}, \tag{8}$$

which is equivalent to

$$\mathrm{Re}\left\{ i\frac{\Phi^+(t)}{X^+(t)} \right\} = 0, \quad t \in \mathbb{T}_0, \tag{9}$$

where X is the canonical function defined by (7). Let $\varphi(z) = \frac{\Phi(z)}{X(z)}$, $z \in \mathbb{D} \setminus \{0\}$. Then $\varphi \in H_1^G(\mathbb{D} \setminus \{0\})$ and $\varphi(z) = \sum_{j=-\kappa}^{+\infty} a_j z^{mj}$, $z \in \mathbb{D} \setminus \{0\}$. Now introduce the rotation-invariant symmetric operator with respect to G

$$\mathbf{L}_m[\varphi](z) = \begin{cases} \mathrm{Re} a_0 + \displaystyle\sum_{j=-\kappa}^{-1} (a_j z^{mj} + \overline{a_j} z^{-mj}) & \text{if } \kappa \geq 0, \\ 0, & \text{if } \kappa < 0, \end{cases} \tag{10}$$

where $z \neq 0$. Clearly, $\mathrm{Re}\,\{i\mathbf{L}_m[\varphi](t)\} = 0$, $t \in \mathbb{T}_0$. By Lemma 2.1, one has

$$\Phi(z) = X(z)\mathbf{L}_m[\varphi](z), \quad z \in \mathbb{D} \setminus \{0\}. \tag{11}$$

Introduce the set of the rotation-invariant symmetric Laurent polynomials

$$S\Pi_{m,k} = \begin{cases} \left\{ \displaystyle\sum_{j=-k}^{k} c_j z^{mj} : \ c_j = \overline{c}_{-j} \ \text{ for } j = 0, 1, 2, \ldots, k \right\} & \text{if } k \geq 0, \\ 0, & \text{if } k < 0. \end{cases} \tag{12}$$

Clearly $S\Pi_{1,k} = S\Pi_k$, where $S\Pi_k$ is defined in [4]. To sum up the above discussion, one has the following.

Lemma 2.2 *The homogeneous Hilbert problem (8) under* $\mathrm{Ord}(\Phi, 0) \leq \mu$ *is solvable by*

$$\Phi(z) = X(z) q_{\mu+\kappa}(z), \tag{13}$$

where $q_{\mu+\kappa} \in S\Pi_{m,\mu+\kappa}$ *and* X *is the canonical function defined by (7).*

By Lemmas 2.1 and 2.2, one immediately has the following result.

Theorem 2.1 *When $\kappa \geq 0$, Hilbert problem (5) is solvable and its solution may be written as*

$$\Phi(z) = \frac{mX(z)}{2\pi i}\left[\int_{\mathbb{T}_0} \frac{c(t)}{[a(t)+ib(t)]X^+(t)} \frac{t^m + z^m}{t^m - z^m} \frac{dt}{t} + 2\pi i q_{\mu+\kappa}(z)\right] \tag{14}$$

with $q_{\mu+\kappa}(z) \in S\Pi_{m,\mu+\kappa}$; When $\kappa < 0$, if and only if the condition

$$\int_{\mathbb{T}_0} \frac{c(t)}{[a(t)+ib(t)]X^+(t)} \frac{dt}{t^{mj+1}} = 0,\ j = 0, 1, \ldots, -\kappa - 1 \tag{15}$$

is satisfied, Hilbert problem (5) is solvable and its solution may be written as

$$\Phi(z) = \frac{mY(z)}{2\pi i} \int_{\mathbb{T}_0} \frac{c(t)}{[a(t)+ib(t)]Y^+(t)} \frac{t^{m-1}dt}{t^m - z^m}, \tag{16}$$

where $Y = z^{-m\kappa}X$.

3 Hilbert BVP for Rotation-Invariant Polyanalytic Functions

As in [2], a rotation-invariant operator is introduced as follows:

$$\mathcal{K}_{z,\bar{z}} = z^{-1}\partial_{\bar{z}}, \tag{17}$$

where $\partial_{\bar{z}}$ is the Cauchy-Riemann operator.

Our first problem is to find a function $V \in H_n^G(\mathbb{D} \setminus \{0\})$ satisfying n boundary conditions on $\mathbb{T}$ and a growth condition at the origin

$$\begin{cases} \operatorname{Re}\left[\left(\mathcal{K}_{z,\bar{z}}^j V\right)^+(t)\right] = 0, \quad t \in \mathbb{T},\ j = 0, 1, \ldots, n-1, \\ \operatorname{Ord}(V, 0) \leq \mu, \end{cases} \tag{18}$$

where the operator $\mathcal{K}_{z,\bar{z}}$ is defined by (17).

Theorem 3.1 *The homogeneous Schwarz problem (18) is solvable and its solution can be written as $V(z) = \sum_{j=0}^{n-1} \frac{(|z|^2-1)^j}{j!} q_j(z)$ with $q_0 \in S\Pi_{m,\mu}$ and $q_j \in S\Pi_{m,\mu-1}$ for $j = 1, 2, \ldots, n-1$.*

Next, we begin to discuss the following problem: find a function $V \in H_n^G(\mathbb{D}\setminus\{0\})$ satisfying n boundary conditions on $\mathbb{T}$ and a growth condition at the origin

$$\begin{cases} \operatorname{Re}\left\{[a(t)+ib(t)]\left(\mathcal{K}_{z,\bar{z}}^j V\right)^+(t)\right\} = 0, \quad t \in \mathbb{T},\ j = 0, 1, \ldots, n-1, \\ \operatorname{Ord}(V, 0) \leq \mu, \end{cases} \tag{19}$$

where the operator $\mathcal{K}_{z,\bar{z}}$ is defined by (17), two given functions a, b satisfy the Hölder condition ($a, b \in H(\mathbb{T})$) and $a^2(t) + b^2(t) = 1$, $t \in \mathbb{T}$. Problem (19) is called the homogeneous Hilbert problem.

Theorem 3.2 *The homogeneous Hilbert problem (19) is solvable and its solution can be written as*

$$V(z) = iX(z) \sum_{j=0}^{n-1} \frac{(|z|^2 - 1)^j}{j!} q_j(z) \tag{20}$$

where $q_0 \in S\Pi_{m,\mu+\kappa}$ and $q_j \in S\Pi_{m,\mu+\kappa-1}$ for $j = 1, 2, \ldots, n-1$, and X is the canonical function defined by (7).

Finally, one comes to investigate the problem: find a function $V \in H_n^G(\mathbb{D} \setminus \{0\})$ satisfying n boundary conditions and a growth condition

$$\begin{cases} \operatorname{Re}\left\{[a(t) + ib(t)] \left(\mathcal{K}_{z,\bar{z}}^j V\right)^+ (t)\right\} = c_j(t), \quad t \in \mathbb{T},\ j = 0, 1, \ldots, n-1, \\ \operatorname{Ord}(V, 0) \le \mu, \end{cases} \tag{21}$$

where the operator $\mathcal{K}_{z,\bar{z}}$ is defined by (17), all the given functions a, b, c_j ($j = 0, 1, \ldots, n-1$) satisfy the Hölder condition on $\mathbb{T}$ and $a^2(t) + b^2(t) = 1$, $t \in \mathbb{T}$. Problem (21) is called the nonhomogeneous Hilbert problem.

The rotation-invariant poly-Schwarz operator on the circumference $\mathbb{T}$ is introduced as follows:

$$S[\gamma_0, \ldots, \gamma_{n-1}](z) = \sum_{j=0}^{n-1} \frac{(|z|^2 - 1)^j}{j!} \mathbf{S}_m[\gamma_j](z) \tag{22}$$

in $z \in \mathbb{D} \setminus \{0\}$ with $\gamma_j \in H(\mathbb{T})$ for $j = 0, 1, 2, \ldots, n-1$, where $\mathbf{S}_m$ is the rotation-invariant Schwarz operator defined by (6). By the boundary behavior of the rotation-invariant Schwarz operator described in Sect. 2,

$$\operatorname{Re}\left\{\left(\mathcal{K}_{z,\bar{z}}^j S[\gamma_0, \ldots, \gamma_{n-1}]\right)^+ (t)\right\} = \gamma_j(t),\ t \in \mathbb{T},\ j = 0, 1, \ldots, n-1. \tag{23}$$

Next, the boundary condition in (21) is reduced to

$$\operatorname{Re}\left\{\left(\mathcal{K}_{z,\bar{z}}^j \left[\frac{V}{iX}\right]\right)^+ (t)\right\} = \frac{c_j(t)}{i[a(t) + ib(t)]X^+(t)}, t \in \mathbb{T},\ j = 0, 1, \ldots, n-1, \tag{24}$$

where X is the canonical function defined in (7). Let

$$W(z) = X(z)S\left[\frac{c_0}{(a+ib)X^+}, \ldots, \frac{c_{n-1}}{(a+ib)X^+}\right](z), \quad z \in \mathbb{D} \setminus \{0\}. \tag{25}$$

From (23), W defined by (25) satisfies the relation (24), or say

$$\mathrm{Re}\left\{\left(\mathcal{K}^j_{z,\bar{z}}\left[\frac{W}{iX}\right]\right)^+(t)\right\} = \frac{c_j(t)}{i[a(t)+ib(t)]X^+(t)}, \; t \in \mathbb{T}, \;\; j = 0, 1, \ldots, n-1. \tag{26}$$

Subtracting (26) from (24), one has

$$\mathrm{Re}\left\{\left(\mathcal{K}^j_{z,\bar{z}}\left[\frac{V-W}{iX}\right]\right)^+(t)\right\} = 0, \;\; t \in \mathbb{T}, \;\; j = 0, 1, \ldots, n-1. \tag{27}$$

Now we discuss the problem (21) in two cases.

Case 1: When $\mu + \kappa \geq 0$, by Theorem 3.2, the solution of the nonhomogeneous Hilbert problem (21) can be written as

$$V(z) = W(z) + iX(z)\sum_{j=0}^{n-1}\frac{(|z|^2-1)^j}{j!}q_j(z) \tag{28}$$

with $q_0 \in S\Pi_{m,\mu+\kappa}$ and $q_j \in S\Pi_{m,\mu+\kappa-1}$ for $j = 1, 2, \ldots, n-1$, where W is given in (25).

Case 2: When $\mu + \kappa < 0$, if the problem (21) is solvable, the solution can be expressed as (28), or say

$$V(z) = iX(z)\left\{[W_0(z)+c_0] + \sum_{j=0}^{n-1}\frac{(|z|^2-1)^j}{j!}W_j(z)\right\}, \quad z \in \mathbb{D} \setminus \{0\} \tag{29}$$

with $c_0 \in \mathbb{C}$ and

$$W_j(z) = -\frac{1}{2\pi}\int_{\mathbb{T}_0}\frac{c_j(\tau)}{[a(\tau)+ib(\tau)]X^+(\tau)}\frac{\tau^m+z^m}{\tau^m-z^m}\frac{\mathrm{d}\tau}{\tau}, \;\; j = 0, 1, \ldots, n-1. \tag{30}$$

By Theorem 1.2, V given by (29) is the solution if and only if $\mathrm{Ord}(W_0 + c_0, 0) \leq \mu + \kappa$ and $\mathrm{Ord}(W_j, 0) \leq \mu + \kappa - 1$ for $j = 1, \ldots, n$. Thus,

$$c_0 = \frac{1}{2\pi}\int_{\mathbb{T}_0}\frac{c_j(t)}{[a(t)+ib(t)]X^+(t)}\frac{\mathrm{d}t}{t} \tag{31}$$

and

$$\begin{cases} \displaystyle\int_{\mathbb{T}_0} \frac{c_0(t)}{[a(t)+ib(t)]X^+(t)} \frac{\mathrm{d}t}{t^{m\ell+1}} = 0, \quad \ell = 1, 2, \ldots, \mu+\kappa-1, \\ \displaystyle\int_{\mathbb{T}_0} \frac{c_j(t)}{[a(t)+ib(t)]X^+(t)} \frac{\mathrm{d}t}{t^{m\ell+1}} = 0, \quad \ell = 0, 1, 2, \ldots, \mu+\kappa-2. \end{cases} \tag{32}$$

In general, the following result is obtained.

Theorem 3.3 *For the nonhomogeneous Hilbert problem (21), there are two cases: If $\mu + \kappa \geq 0$, the solution of problem (21) can be written as (28); If $\mu + \kappa < 0$, if and only if all the conditions (32) are satisfied, problem (21) is solvable and its solution can be written as (29), where c_0 is given by (31).*

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Riemann Boundary Value Problem with Square Roots on the Real Axis

Shouguo Zhong, Ying Wang, and Pei Dang

Abstract In this paper, we consider the Riemann boundary value problem with square roots on the real axis X

$$\sqrt{\Psi^+(x)} = G(x)\sqrt{\Psi^-(x)} + g(x), \quad x \in X,$$

in which a sectionally holomorphic unknown function $\Psi(z)$ having some zeros in the upper and lower half-planes, and we obtain the solution and solvability condition explicitly.

Keywords Branch • Real axis • Riemann boundary value problem • Sectionally holomorphic function • Square root

Mathematics Subject Classification (2010) 45G05; 30E25

1 Introduction

The Riemann boundary value problem with square roots for a sectionally holomorphic unknown function $\Psi(z)$

$$\sqrt{\Psi^+(t)} = G(t)\sqrt{\Psi^-(t)} + g(t), \quad t \in L$$

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has been solved in [1, 2], where L is a smooth closed contour, $G(t), g(t) \in H(L)$ (Hölder continuous) and $G(t) \neq 0$. In this paper, the boundary curve is replaced by the real axis X (including the point ∞), and we investigate the corresponding Riemann boundary value problem with square roots on the real axis, obtaining the solutions and solvability conditions explicitly.

The difference to former investigations lies in the following. (1) The assumption of zeros for the unknown function $\Psi(z)$ in the upper and lower half-planes $Z^{\pm}$ is different from the case of a closed contour. (2) When $\Psi^{\pm}(z)$ has an odd number of zeros of odd order in $Z^{\pm}$, $\Psi^{\pm}(z)$ must have a zero-point on X in order that $\sqrt{\Psi^{\pm}(z)}$ can be single-valued in $Z^{\pm}$, respectively. Since the zero-point is either a finite point on the real axis or $x = \infty$, the discussion will be more complicated.

Let $\Psi^{+}(\infty) = \lim\limits_{z \in Z^{+}, z \to \infty} \Psi(z)$, $\Psi^{-}(\infty) = \lim\limits_{z \in Z^{-}, z \to \infty} \Psi(z)$, then our problem is to find a sectionally holomorphic function $\Psi(z)$ with jump curve X, such that $\sqrt{\Psi^{\pm}(z)}$ are single-valued in $Z^{\pm}$, respectively, and $\Psi^{\pm}(\infty)$ are finite. Moreover, $\sqrt{\Psi^{\pm}(x)}$ are continuous and single-valued on X, satisfying the following condition

$$\sqrt{\Psi^{+}(x)} = G(x)\sqrt{\Psi^{-}(x)} + g(x), \quad x \in X \tag{1}$$

where $G(x) \neq 0$ and $G(x),\ g(x) \in \widehat{H}(X)$ (see [3]).

2 Structure of $\Psi^{\pm}(z)$ in $Z^{\pm}$

Case 1: If $\Psi^{+}(z)$ has an even number of zeros of odd order in Z^{+}, say, $a_1, a_2, \ldots, a_{2m}$. Let

$$\Pi_a(z) = \frac{\prod\limits_{j=1}^{2m}(z - a_j)}{(z + i)^{2m}}, \quad m \geq 0 \tag{2}$$

with $\Pi_a(z) = 1$ for $m = 0$, then we may write

$$\Psi^{+}(z) = \Pi_a(z)\left[\Phi_0^{+}(z)\right]^2 \tag{3}$$

or

$$\sqrt{\Psi^{+}(z)} = \sqrt{\Pi_a(z)}\Phi_0^{+}(z), \quad z \in Z^{+}, \tag{4}$$

where $\sqrt{\Pi_a(z)}$ takes a definite single-valued branch for m non-intersecting cuts in Z^{+}, connected by m pairs of points arbitrarily taken from $a_1, a_2, \ldots, a_{2m}$, $\Phi_0^{+}(z)$ is analytic in Z^{+} and continuous to X. Moreover, $\Phi_0^{+}(\infty)$ is finite.

Now we prove that (4) holds. Actually, $\Psi^+(z)/\Pi_a(z)$ probably has zero-points of even order in Z^+. Suppose $\rho(z)$ is the product of all probable zero-divisors of even order in Z^+, then $\sqrt{\rho(z)}$ is a single-valued analytic function in Z^+. Therefore, $\Psi^+(z)/[\Pi_a(z)\rho(z)]$ is analytic and not equal to 0 in Z^+. By the monodromy theorem in [4], $\Omega(z) = \sqrt{\Psi^+(z)/[\Pi_a(z)\rho(z)]}$ can be taken as a single-valued branch in Z^+. Let $\Phi_0^+(z) = \sqrt{\rho(z)}\Omega(z)$, then (4) is true.

Similarly, if $\Psi^-(z)$ has an even number of zeros of odd order in Z^-, say, $b_1, b_2, \ldots, b_{2n}$. Suppose

$$\Pi_b(z) = \frac{\prod_{k=1}^{2n}(z-b_k)}{(z-i)^{2n}}, \quad n \geq 0 \tag{5}$$

with $\Pi_b(z) = 1$ for $n = 0$. Then we can write

$$\Psi^-(z) = \Pi_b(z)\left[\Phi_0^-(z)\right]^2 \tag{6}$$

or

$$\sqrt{\Psi^-(z)} = \sqrt{\Pi_b(z)}\Phi_0^-(z), \quad z \in Z^-, \tag{7}$$

where $\sqrt{\Pi_b(z)}$ takes a definite single-valued branch for n proper cuts in Z^-, $\Phi_0^-(z)$ is analytic in Z^- and continuous to X. Moreover, $\Phi_0^-(\infty)$ is finite.

Remark 2.1 According to the result in [1, 2], the corresponding expression of $\Pi_a(z)$ and $\Pi_b(z)$ are taken by $\widetilde{\Pi_a}(z) = \prod_{j=1}^{2m}(z-a_j)$ and $\widetilde{\Pi_b}(z) = \prod_{k=1}^{2n}(z-b_j)$, respectively. Then $\widetilde{\Pi_a}(z)$ ($\widetilde{\Pi_b}(z)$) has a singularity of order $2m$ ($2n$) at ∞, which involves us in trouble when we apply the classical results in [3] for the following discussion. Here we add the factor $\dfrac{1}{(z+i)^{2m}}$ and $\dfrac{1}{(z-i)^{2n}}$, but $\Pi_a(z)$ and $\Pi_b(z)$ are still analytic and have only zeros $a_1, a_2, \ldots, a_{2m}$ and $b_1, b_2, \ldots, b_{2n}$ in $Z^{\pm}$, respectively, and also $\Pi_a(\infty) = \Pi_b(\infty) = 1$. Furthermore, $\Psi^+(z)/\Pi_a(z)$ ($\Psi^-(z)/\Pi_b(z)$) is analytic and has no zero-point of odd order in Z^+ (Z^-), and $\Psi^+(\infty)/\Pi_a(\infty)$ ($\Psi^-(\infty)/\Pi_b(\infty)$) is finite.

Case 2: If $\Psi^+(z)$ has an odd number of zeros of odd order in Z^+, say, $a_1, a_2, \ldots, a_{2m-1}$. In order that $\sqrt{\Psi^+(z)}$ is single-valued in Z^+, there must exist $a_{2m} \in X$ such that $\Psi^+(a_{2m}) = 0$. If $a_{2m} \neq \infty$, $\Pi_a(z)$ is still given by (2). If $a_{2m} = \infty$, $\Pi_a(z)$ is written as

$$\Pi_a(z) = \frac{\prod_{j=1}^{2m-1}(z-a_j)}{(z+i)^{2m}}, \quad m > 0, \tag{8}$$

which already has a zero-point of order one at ∞. By the similar reason as in case 1, $\Psi^+(z)$ can be expressed as (3) and (4), with $\Pi_a(z)$ given by (2) or (8). In this case, $\Phi_0^+(z)$ is still analytic in Z^+ and continuous to X from Z^+ except for possibly a singularity of order less than $1/2$ (at most) at a_{2m}, in which a_{2m} is finite on X or ∞.

Similarly, if $\Psi^-(z)$ has an odd number of zeros of odd order in Z^-, say, $b_1,\ b_2,\dots,b_{2n-1}$, there must exist $b_{2n} \in X$ such that $\Psi^-(b_{2n}) = 0$. If $b_{2n} \neq \infty$, $\Pi_b(z)$ is still given by (5). If $b_{2n} = \infty$, then $\Pi_b(z)$ is defined by

$$\Pi_b(z) = \frac{\prod_{k=1}^{2n-1}(z-b_k)}{(z-i)^{2n}}, \quad n > 0. \tag{9}$$

Consequently, $\Psi^-(z)$ can be expressed as (6) and (7), with $\Pi_b(z)$ given by (5) or (9). In this case, $\Phi_0^-(z)$ is analytic in Z^- and continuous to X from Z^- except for possibly a singularity of order less than $1/2$ (at most) at b_{2n}, in which b_{2n} may be finite on X or ∞.

3 Solution of the Problem

Substituting (4),(7) into (1) and let $\Pi(z) = \Pi_a(z)\Pi_b(z)$, (1) can be rewritten as

$$\Phi^+(x) = G(x)\Phi^-(x) + \frac{g(x)}{\sqrt{\Pi(x)}} \tag{10}$$

where $\sqrt{\Pi(x)} = \sqrt{\Pi_a(x)}\,\sqrt{\Pi_b(x)}$ has taken a definite single-valued branch, and

$$\Phi(z) = \begin{cases} \Phi_0^+(z)/\sqrt{\Pi_b(z)}, & z \in Z^+ \\ \Phi_0^-(z)/\sqrt{\Pi_a(z)}, & z \in Z^- \end{cases} \tag{11}$$

which is analytic in $Z^\pm$, continuous to X from Z^+ and Z^-, except for $a_{2m} \in X$ or $b_{2n} \in X$ possibly having a singularity of order $< 1/2$ (when $a_{2m} \neq b_{2n}$) or order < 1 (when $a_{2m} = b_{2n}$).

Since ∞ is possibly a singular point, $\Phi(z)$ may be finite or has a singularity of order less than $1/2$ (or < 1) at ∞. Due to $G(x) \in \widehat{H}(X)$, we still take the canonical function $Y(z)$ as in [3], i.e.,

$$Y(z) = \begin{cases} e^{\Gamma(z)}, & z \in Z^+, \\ \left[\dfrac{z+i}{z-i}\right]^\kappa e^{\Gamma(z)}, & z \in Z^-, \end{cases} \tag{12}$$

where $\kappa = \frac{1}{2\pi}[\arg G(x)]_X$, and

$$\Gamma(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\log\{[(x+i)/(x-i)]^{\kappa} G(x)\}}{x-z}\mathrm{d}x, \quad z \notin X. \tag{13}$$

Then we have the following result.

(**i**) When $\kappa \geq 0$, (10) is always solvable with the general solution

$$\Phi(z) = \frac{Y(z)}{2\pi i}\int_{-\infty}^{+\infty} \frac{g(x)}{\sqrt{\Pi(x)}Y^{+}(x)(x-z)}\mathrm{d}x + Y(z)P_k\left(\frac{1}{z+i}\right), \tag{14}$$

where $P_k\left(\frac{1}{z+i}\right)$ is an arbitrary polynomial of degree κ with respect to the variable $1/(z+i)$. By (4), (7), (11), the general solution to (1) is

$$\sqrt{\Psi(z)} = \sqrt{\Pi(z)}Y(z)\left[\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{g(x)}{\sqrt{\Pi(x)}Y^{+}(x)(x-z)}\mathrm{d}x + P_k\left(\frac{1}{z+i}\right)\right]. \tag{15}$$

(**ii**) When $\kappa < 0$, if and only if

$$\int_{-\infty}^{+\infty} \frac{g(x)}{\sqrt{\Pi(x)}Y^{+}(x)(x+i)^{k+1}}\mathrm{d}x = 0, \quad k = 1, 2, \ldots, -\kappa - 1, \tag{16}$$

is satisfied, (10) has the unique solution

$$\Phi(z) = Y(z)\left[\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{g(x)\mathrm{d}x}{\sqrt{\Pi(x)}Y^{+}(x)(x-z)} - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{g(x)\mathrm{d}x}{\sqrt{\Pi(x)}Y^{+}(x)(x+i)}\right], \tag{17}$$

thus, the unique solution to (1) is

$$\sqrt{\Psi(z)} = \sqrt{\Pi(z)}Y(z)\left[\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{g(x)\mathrm{d}x}{\sqrt{\Pi(x)}Y^{+}(x)(x-z)} - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{g(x)\mathrm{d}x}{\sqrt{\Pi(x)}Y^{+}(x)(x+i)}\right]. \tag{18}$$

4 Proof of $\Psi^+(a_{2m}) = 0$ and $\Psi^-(b_{2n}) = 0$ for $a_{2m}, b_{2n} \in X$

By the solution in the last section, we need to check $\Psi^+(a_{2m}) = 0$ and $\Psi^-(b_{2n}) = 0$ for $a_{2m}, b_{2n} \in X$. From (15) and (18), it is enough to prove that

$$I(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{g(x)\mathrm{d}x}{\sqrt{\Pi(x)}Y^+(x)(x-z)}, \quad z \notin X \tag{19}$$

has a singularity of order $< 1/2$ (when $a_{2m} \neq b_{2n}$) or < 1 (when $a_{2m} = b_{2n}$) at a_{2m} and b_{2n}.

(**i**) When $a_{2m} \neq b_{2n},\ a_{2m}, b_{2n} \neq \infty$, let

$$g_a(x) = \frac{g(x)}{\sqrt{\Pi_a(x)}Y^+(x)}, \quad g_b(x) = \frac{g(x)}{\sqrt{\Pi_b(x)}Y^+(x)}. \tag{20}$$

By applying the extended residue theorem in [5] to Z^+ and Z^-, respectively, we get

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{\sqrt{\Pi_a(x)}(x-z)} = \frac{1}{2}, \quad z \in Z^+, \tag{21}$$

and

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{\sqrt{\Pi_b(x)}(x-z)} = -\frac{1}{2}, \quad z \in Z^-, \tag{22}$$

where $\Pi_a(z), \Pi_b(z)$ are given by (2) and (5), respectively.

Substituting (20)–(22) into (19), we have

$$I(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{g_b(x) - g_b(a_{2m})}{\sqrt{\Pi_a(x)}(x-z)}\mathrm{d}x + \frac{1}{2}g_b(a_{2m}), \quad z \in Z^+, \tag{23}$$

and

$$I(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{g_a(x) - g_a(b_{2n})}{\sqrt{\Pi_b(x)}(x-z)}\mathrm{d}x - \frac{1}{2}g_a(b_{2n}), \quad z \in Z^-. \tag{24}$$

From (23) and (24), we know $I(z)$ has a singularity of order $< 1/2$ at a_{2m} and b_{2n}.

(ii) When $a_{2m} = \infty$, $b_{2n} \neq \infty$, the proof is the same as **(i)** for b_{2n}. For $a_{2m} = \infty$, by (20), we obtain

$$\begin{aligned} I(z) &= \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{g_b(x) - g_b(\infty)}{\sqrt{\Pi_a(x)}(x-z)}\mathrm{d}x + \frac{g_b(\infty)}{2\pi i}\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{\sqrt{\Pi_a(x)}(x-z)} \\ &:= J_1(z) + g_b(\infty)J_2(z), \end{aligned}$$

where $\Pi_a(z)$ is defined by (8).

Putting (8) into $J_2(z)$, the integrand of $J_2(z)$ has order $-1/2$ at ∞ with respect to x. Therefore, by the extended residue theorem[5], $J_2(z) \equiv 0$ for $z \in Z^+$. Taking the transformation

$$t + i = -\frac{1}{x+i}, \quad w + i = -\frac{1}{z+i} \tag{25}$$

and denoting the images of $X,\ g_b(x),\ a_j$ by $\Gamma,\ g_b^*(t),\ a_j^*$ $(j = 1, 2, \ldots, 2m-1)$, respectively, in particular, $a_j^* + i = -\dfrac{1}{a_j + i}$, then we have

$$\begin{aligned} I(z) = J_1(z) = (-1)^m \sqrt{\prod_{j=1}^{2m-1}(a_j^* + i)}\Bigg[&\frac{1}{2\pi i}\int_\Gamma \frac{\left[g_b^*(t) - g_b^*(-i)\right]\mathrm{d}t}{\sqrt{(t+i)\prod_{j=1}^{2m-1}(t-a_j^*)(t-w)}} \\ &-\frac{1}{2\pi i}\int_\Gamma \frac{\left[g_b^*(t) - g_b^*(-i)\right]\mathrm{d}t}{\sqrt{(t+i)\prod_{j=1}^{2m-1}(t-a_j^*)(t+i)}}\Bigg]. \end{aligned} \tag{26}$$

The second integral in (26) is a singular integral of higher order (order $3/2$ at $-i$, see [3, 5]) and furthermore is a constant. The first integral has a singularity of order $< 1/2$ at $-i$, in other words, $I(z)$ has a singularity of order $< 1/2$ at ∞.

(iii) When $a_{2m} \neq \infty$, $b_{2n} = \infty$, the proof is similar to case (ii). Nevertheless, for $b_{2n} = \infty$, we can also take the following transformation instead of (25),

$$t - i = -\frac{1}{x-i}, \quad w - i = -\frac{1}{z-i}. \tag{27}$$

(iv) When $a_{2m} = b_{2n} \neq \infty$. If $\Psi^+(a_{2m}) = \Psi^-(b_{2n}) = 0$, then $g(a_{2m}) = 0$ by (1). Conversely, when $g(a_{2m}) = 0$, from (19) we get that $I(z)$ has a singularity of order < 1 at a_{2m} and b_{2n}, respectively.

(**v**) When $a_{2m} = b_{2n} = \infty$. Taking the transformation (25) and denoting the images of $g(x)$, $Y^+(x)$, b_j by $g^*(t)$, $Y^*(t)$, b_j^* $(j = 1, 2, \ldots, 2n-1)$, respectively, in particular, $b_j^* + i = -\dfrac{1}{b_j + i}$, we get

$$\begin{aligned} I(z) = (-1)^{m+n} &\sqrt{\prod_{j=1}^{2m-1}(a_j^* + i)\prod_{k=1}^{2n-1}(b_k^* + i)} \\ &\times \left[\frac{1}{2\pi i}\int_{\Gamma} \frac{(2it-1)^n g^*(t)\mathrm{d}t}{\sqrt{\prod_{j=1}^{2m-1}(t - a_j^*)\prod_{k=1}^{2n-1}(t - b_k^*) Y^*(t)(t+i)(t-w)}} \right. \\ &\left. - \frac{1}{2\pi i}\int_{\Gamma} \frac{(2it-1)^n g^*(t)\mathrm{d}t}{\sqrt{\prod_{j=1}^{2m-1}(t - a_j^*)\prod_{k=1}^{2n-1}(t - b_k^*) Y^*(t)(t+i)^2}} \right]. \end{aligned} \tag{28}$$

The second integral in (28) (which is a singular integral of higher order, see [3, 5]) is a constant. Since $g^*(-i) = g(\infty) = 0$, the first integral in (28) has a singularity of order < 1 at $-i$, in other words, $I(z)$ has a singularity of order < 1 at ∞.

Remark 4.1 In order to prove (**v**), we can also take the transformation (27) instead of (25).

Remark 4.2 Combining this article with [6], we can consider the boundary value problem with radicals

$$\sqrt[p]{\Psi^+(x)} = G(x)\sqrt[q]{\Psi^-(x)} + g(x), \quad x \in X$$

for arbitrary positive integers p, q.

Remark 4.3 If we take the hypotheses on $\Pi_a(z)$ and $\Pi_b(z)$ as in [1, 2], we have to extend the classical Riemann problem on X for further discussion.

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Part III
Functions Theory of One and Several Complex Variables

Wenjun Yuan and Mingsheng Liu

Picard Values and Some Normality Criteria of Meromorphic Functions

Zhixue Liu and Tingbin Cao

Abstract In this paper, by making use of Nevanlinna theory and the Zalcman-Pang lemma, we obtain some interesting results of normal criteria relating to the type of differential polynomials $f^m(z) + a(f^{(k)}(z))^n$, which may be seen as some significant generalizations of normal criteria for meromorphic functions in a domain G.

Keywords Meromorphic functions • Normal family • Picard values

Mathematics Subject Classification (2010) Primary 30D35; Secondary 30D45

1 Introduction

At the beginning of the twentieth century, P. Montel introduced the concept of normal families and built the theory of normal families. One major study of normal families theory is to seek normality criteria. Corresponding to the famous Picard's theorem which says that a nonconstant entire function can omit at most one value, Montel obtained the following result called later as the Montel's theorem (see [7]): *Let $\mathcal{F}$ be a family of holomorphic functions in $G \subseteq \mathbb{C}$. If $f(z) \neq 0, f(z) \neq 1$ for all $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in G.*

In recent years, various interesting results of normal criteria (for example, see works [1–5, 8, 11]) including the form of differential polynomials for a family of holomorphic or meromorphic functions have been established, which are benefited from Zalcman's lemma. One may ask whether there exist normal criteria of a family of meromorphic functions with respect to more general differential polynomials such as $f^m + a(f^{(k)})^n$? In this paper, we mainly consider this and get some results as showed in Theorems 2.1–2.3. In order to prove our theorems, we need the following lemma by using the method of Xu [7, Lemma 1].

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Lemma 1.1 *Let $a(\neq 0), b$ be two finite complex numbers. Suppose that f is a meromorphic function on the complex plane with poles and zeros of orders at least $l(\geq 1)$ and $t(\geq 1)$, respectively. If there exist three positive integers m, n, k such that $f^m(z) + a(f^{(k)}(z))^n \neq b(m > \frac{nk+1}{l} + n + \frac{1}{t})$ or $f^m + a(f^{(k)})^n \equiv b(m > \frac{nk}{l} + n)$, then f is a constant identically.*

Proof First we consider the case of $f^m(z) + a(f^{(k)}(z))^n \neq b$ for the condition $m > \frac{nk+1}{l} + n + \frac{1}{t}$. Assume that f is not a constant, it follows from $f^m(z) + a(f^{(k)}(z))^n \neq b$ that f is not a polynomial. Let

$$w(z) := \frac{a(f^{(k)}(z))^n - b}{-f^m(z)} \neq 1.$$

Noting that $w \not\equiv 0$, we have

$$\begin{aligned} m \cdot m(r,f) = m(r,f^m) &= m\left(r, \frac{a(f^{(k)})^n - b}{-w}\right) \\ &\leq m\left(r, \frac{1}{w}\right) + n \cdot m\left(r, \frac{f^{(k)}}{f}\right) + n \cdot m(r,f) + O(1). \end{aligned}$$

Thus, we can get

$$(m-n) \cdot m(r,f) \leq m\left(r, \frac{1}{w}\right) + S(r,f) \qquad (m > n). \tag{1}$$

On the other hand,

$$\begin{aligned} N\left(r, \frac{1}{w}\right) &= N\left(r, \frac{f^m}{a(f^{(k)})^n - b}\right) \\ &= (m-n)N(r,f) - nk\overline{N}(r,f) + N\left(r, \frac{1}{a(f^{(k)})^n - b}\right). \end{aligned}$$

Thus, we can get

$$\left(m - n - \frac{nk}{l}\right) N(r,f) \leq N\left(r, \frac{1}{w}\right) - N\left(r, \frac{1}{a(f^{(k)})^n - b}\right) \quad \left(m > \frac{nk}{l} + n\right). \tag{2}$$

Combining (1), (2) and using the first and second main theorem, we have

$$\begin{aligned}\left(m-n-\frac{nk}{l}\right)T(r,f) &\leq T(r,w)-N\left(r,\frac{1}{a(f^{(k)})^n-b}\right)+S(r,f)\\ &\leq \overline{N}\left(r,\frac{1}{w}\right)+\overline{N}(r,w)+\overline{N}\left(r,\frac{1}{w-1}\right)\\ &\quad -N\left(r,\frac{1}{a(f^{(k)})^n-b}\right)+S(r,f).\end{aligned}$$

Noting that $f^m(z)+a(f^{(k)}(z))^n \neq b$ and $m > \frac{n(k+1)}{l}+n+\frac{1}{t}$, we know that

$$\begin{aligned}\left(m-n-\frac{nk}{l}\right)T(r,f) &\leq \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+S(r,f)\\ &\leq \left(\frac{1}{t}+\frac{1}{l}\right)T(r,f)+S(r,f)\end{aligned}$$

However, according to the given condition that $m > \frac{nk+1}{l}+n+\frac{1}{t}$, it follows from the above inequality that $T(r,f) = S(r,f)$, a contradiction. Hence, f should be a constant.

Next we consider the other case of $f^m+a(f^{(k)})^n \equiv b$ for the condition $m > \frac{nk}{l}+n$. Then f must be a entire function. Suppose that f is not a constant. It follows from the fact

$$|f|^{m-n} \leq |b|+|a|\cdot\left|\frac{f^{(k)}}{f}\right|^n+1 \qquad (m>n), \tag{3}$$

for all f that

$$(m-n)T(r,f) = (m-n)\cdot m(r,f) \leq S(r,f),$$

a contradiction for $m > \frac{nk}{l}+n$. Hence, f should be a constant. □

As showed in the proof of Lemma 1.1, it is easy to verify that if f is an entire function or $f(z)\neq 0$, then the restriction of m in Lemma 1.1 is shown in Remarks 1.2 and 1.3, respectively, which will be used frequently and plays a vital role in the proof of latter results of this paper.

Remark 1.2 Let $a(\neq 0), b$ be two finite complex numbers. Suppose that f is an entire function on the complex plane with zeros of orders at least $t(\geq 1)$. If there exist three positive integers m, n, k such that $f^m(z)+a(f^{(k)}(z))^n \neq b(m > n+\frac{1}{t})$ or $f^m+a(f^{(k)})^n \equiv b(m>n)$, then f is a constant identically.

Remark 1.3 Let $a(\neq 0), b$ be two finite complex numbers. Suppose that f is a meromorphic function on the complex plane with poles of orders at least $l(\geq 1)$.

If f has no zero and there exist three positive integers m, n, k such that $f^m(z) + a(f^{(k)}(z))^n \neq b(m > \frac{nk+1}{l} + n)$, then f is a constant identically.

2 Main Results and Proofs

Theorem 2.1 *Let $\mathcal{F}$ be a family of meromorphic functions in a domain G and let $a(\neq 0), b$ be two finite complex numbers. For each $f \in \mathcal{F}$, all zeros of every $f \in \mathcal{F}$ have multiplicity at least $k+1$, $k \in \mathbb{N}^+$. If there exist two positive integers m, n such that $n \geq m + 1$ and $f^m(z) + a(f^{(k)}(z))^n \neq b$ in G, then $\mathcal{F}$ is normal in G.*

Proof of Theorem 2.1 Assume that $\mathcal{F}$ is not normal in the domain $G \subset \mathbb{C}$. Without loss of generality, we may assume that G is the unit disc. If $b = 0$, then we can obtain $f^m(z) + a(f^{(k)}(z))^n \neq 0$. It follows from the zeros of every $f \in \mathcal{F}$ have multiplicity at least $k + 1$ that $f(z) \neq 0$. Then by Pang-Zalcman Lemma (see works [9, 11]), for $-1 < \alpha = \frac{nk}{n-m} < \infty$, there exist a sequence of points $z_j \in G, z_j \to z_0$, a sequence of positive numbers $\rho_j \to 0$ and a sequence of functions $f_j \in \mathcal{F}$ such that

$$g_j(\zeta) = \frac{f_j(z_j + \rho_j\zeta)}{\rho_j^{\frac{nk}{n-m}}} \to g(\zeta)$$

locally uniformly with respect to the spherical metric, where $g(z) \neq 0$ is a non-constant meromorphic function on $\mathbb{C}$. In addition, we know that $g^m(\zeta) + a(g^{(k)}(\zeta))^n$ is the uniform limit of

$$g_j^m(\zeta) + a(g_j^{(k)}(\zeta))^n = \rho_j^{\frac{mnk}{m-n}}(f_j^m(z_j + \rho_j\zeta) + a(f_j^{(k)}(z_j + \rho_j\zeta))^n) \neq 0,$$

for all $\zeta \in \mathbb{C}$. By Hurwitz's theorem, $g^m(z) + a(g^{(k)}(z))^n \neq 0$ or $g^m + a(g^{(k)})^n \equiv 0$ holds in $\mathbb{C}$. Let's divide into two cases:

- If $g^m(z) + a(g^{(k)}(z))^n \neq 0$, then g must be a rational function (see work [10]) and we may assume $g(\zeta) = \frac{1}{P(\zeta)}$ where $P(\zeta)$ is a polynomial of degree p. Thus we have

$$0 \neq g^m(\zeta) + a(g^{(k)}(\zeta))^n = \frac{1}{P^m(\zeta)} + a\left(\left(\frac{1}{P(\zeta)}\right)^{(k)}\right)^n = \frac{1}{Q(\zeta)}.$$

  where $Q(\zeta)$ is a polynomial of degree q. Hence,

$$a\left(\left(\frac{1}{P(\zeta)}\right)^{(k)}\right)^n = \frac{P^m(\zeta) - Q(\zeta)}{P^m(\zeta)Q(\zeta)}, \tag{4}$$

It is easy to verify that the difference between the degree of the numerator and denominator of $a\left(\left(\frac{1}{P(\zeta)}\right)^{(k)}\right)^n$ is $n(p+k)$, and the difference between the degree of the numerator and denominator of $\frac{P^m(\zeta)-Q(\zeta)}{P^m(\zeta)Q(\zeta)}$ is less than $\min\{mp, q\}$. It is impossible for (4) since $n \geq m+1$.

- If $g^m + a(g^{(k)})^n \equiv 0$, then g is an entire function. Note that $g(z) \neq 0$, by Pang-Zalcman Lemma (see work [9, 11]), we can conclude that $g(\zeta) = e^{c\zeta+d}$ (note here $\sigma(g) \leq 1$), where $c(\neq 0), d$ are two complex numbers. Thus

$$e^{mc\zeta+md} + a(c^k e^{c\zeta+d})^n = e^{m(c\zeta+d)}(1 + ac^{nk}e^{(n-m)(c\zeta+d)}) \equiv 0.$$

It is impossible since $n > m$, so $\mathcal{F}$ is normal in G.

Next we consider the case of $b \neq 0$. By Pang-Zalcman Lemma (see works [9, 11]), for $\alpha = k$, there exist a sequence of points $z_j \in G, z_j \to z_0$, a sequence of positive numbers $\rho_j \to 0$ and a sequence of functions $f_j \in \mathcal{F}$ such that

$$g_j(\zeta) = \frac{f_j(z_j + \rho_j\zeta)}{\rho_j^k} \to g(\zeta)$$

locally uniformly with respect to the spherical metric, where g is a non-constant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least $k+1$. In addition, we know that $a(g^{(k)}(\zeta))^n - b$ is the uniformly limit of

$$\rho_j^{mk} g_j^m(\zeta) + a(g_j^{(k)}(\zeta))^n - b = f_j^m(z_j + \rho_j\zeta) + a(f_j^{(k)}(z_j + \rho_j\zeta))^n - b \neq 0.$$

By Hurwitz's theorem, $a(g^{(k)}(z))^n - b \neq 0$ or $a(g^{(k)})^n - b \equiv 0$ holds in $\mathbb{C}$. If $a(g^{(k)})^n - b \equiv 0$, noting that $g(\zeta)$ is nonconstant, then $g(\zeta)$ must be a polynomial of degree k. It is impossible since the zeros of g have multiplicity at least $k+1$. Hence, $a(g^{(k)}(z))^n - b \neq 0$. Based on Nevanlinna's second main theorem, we have

$$\begin{aligned}
T(r, g^{(k)}) &\leq (n-1)T(r, g^{(k)}) \\
&\leq \overline{N}(r, g^{(k)}) + \overline{N}\left(r, \frac{1}{g^{(k)} - a_1}\right) + \cdots + \overline{N}\left(r, \frac{1}{g^{(k)} - a_n}\right) + S(r, g^{(k)}) \\
&\leq \frac{1}{k+1}T(r, g^{(k)}) + S(r, g^{(k)}),
\end{aligned}$$

where $a_1, a_2, \ldots, a_n$ are n distinct solutions of $z^n = \frac{b}{a}$(note here $a \neq 0$). It follows that $T(r, g^{(k)}) = S(r, g^{(k)})$, a contradiction. In summary, $\mathcal{F}$ is normal in G. This complete the proof of Theorem 2.1. $\square$

To show the conditions that $n \geq m+1$ and all the zeros of f have multiplicity $k+1$ are necessary and best choices in Theorem 2.1, see the following examples.

Example Take the family of meromorphic functions $\mathcal{F} = \{f_j(z) = e^{jz} | z \in G, j \in \mathbb{N}^+\}$ in $G = \{z : |z| < 1\}$. Obviously, $f(z) \neq 0$ and $f_j^m(z) + a(f_j^{(k)}(z))^m = e^{mjz}(1 + aj^{mk}) \neq 0$ in G, but $\mathcal{F}$ is not normal at the point $z = 0$. This shows the condition that $n \geq m + 1$ in Theorem 2.1 is necessary.

Example Take the family of meromorphic functions $\mathcal{F} = \{f_j(z) = jz^k | z \in G, j \in \mathbb{N}^+\}$ in $G = \{z : |z| < 1\}$. Noting that all the zeros of f have multiplicity k, and we know that in G, $f_j^m + a(f_j^{(k)})^n = j^m z^{mk} + a(jk!)^n \neq 1$ for j large enough, but $\mathcal{F}$ is not normal at the point $z = 0$. This shows the condition that all the zeros of f have multiplicity $k + 1$ in Theorem 2.1 is necessary.

Theorem 2.2 *Let $\mathcal{F}$ be a family of meromorphic functions in a domain G and let $a(\neq 0), b$ be two finite complex numbers. For each $f \in \mathcal{F}$, if there exist three positive integers m, n, k such that $m \geq n(k + 1) + 3$ and $f^m(z) + a(f^{(k)}(z))^n \neq b$ in G, then $\mathcal{F}$ is normal in G.*

Proof of Theorem 2.2 Assume that $\mathcal{F}$ is not normal in the domain $G \subset \mathbb{C}$. In view of Pang-Zalcman Lemma (see works [9, 11]), we may choose $-1 < \alpha = \frac{nk}{n-m} < 1$. It follows from Remarks 1.2 and 1.3 that g must be a constant (note here $m \geq n(k + 1) + 1$), similarly as in the proof of Theorem 2.1. This is a contradiction and we complete the proof of Theorem 2.2. □

Theorem 2.3 *Let $\mathcal{F}$ be a family of meromorphic functions in a domain G and let $a(\neq 0), b$ be two finite complex numbers. There exist three positive integers m, n, k such that $m \geq n(k + 1) + 1$. For each $f \in \mathcal{F}$, if f has no simple pole and $f^m(z) + a(f^{(k)}(z))^n \neq b$ in G, then $\mathcal{F}$ is normal in G.*

Proof of Theorem 2.3 For the case of $n = k = 1$, we can see [6]. Next we consider the fact that $n(k+1)+1 > \frac{nk+1}{2} + n + 1$ for the case of $nk \geq 2$. Assume that $\mathcal{F}$ is not normal in the domain $G \subset \mathbb{C}$. Then by Pang-Zalcman Lemma (see works [9, 11]), we may choose $-2 < \alpha = \frac{nk}{n-m} < 1$ and in view of Remarks 1.2 and 1.3, g must be a constant (note here $m \geq n(k + 1) + 3$), a contradiction. Thus, we completes the proof of Theorem 2.3. □

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Rational Approximation of Functions in Hardy Spaces

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Abstract We present some results on rational approximation, Laplace integral representation and Fourier spectrum characterization of functions in the Hardy Spaces. These generalize the results of Stein and Weiss in the same context for $p = 2$, as well as the Poisson and the Cauchy integral representation formulas for the H^2 spaces to the H^p spaces on tubes for $p \in [1, \infty]$.

Keywords Hardy spaces • Rational approximation

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1 Hardy Space in the Half Plane

The classical Hardy space $H^p(\mathbb{C}_k),\ 0 < p < +\infty, k = \pm 1$, consists of the functions f analytic in the half plane $\mathbb{C}_k = \{z = x + iy : ky > 0\}$. They are Banach spaces under the norms

$$\|f\|_{H_k^p} = \sup_{ky>0} \left(\int_{-\infty}^{\infty} |f(x+iy)|^p dx \right)^{\frac{1}{p}}$$

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is finite. A function $f \in H^p(\mathbb{C}_k)$ has non-tangential boundary limits (*NTBL*s) $f(x)$ for almost all $x \in \mathbb{R}$, and the boundary function belongs to $L^p(\mathbb{R})$, and

$$\|f\|_p = \left(\int_{-\infty}^{\infty} |f(x)|^p dx\right)^{\frac{1}{p}} = \|f\|_{H^p_k}.$$

For $p = \infty$ the Hardy spaces $H^\infty(\mathbb{C}_k)$ $(k = \pm 1)$ are defined to be the set of bounded analytic functions in $\mathbb{C}_k$. They are Banach spaces under the norms

$$\|f\|_{H^\infty_k} = \sup\{|f(z)| \,:\, z \in \mathbb{C}_k\}.$$

As for the finite indices p cases any $f \in H^\infty(\mathbb{C}_k)$ has non-tangential boundary limit (NTBL) $f(x)$ for almost all $x \in \mathbb{R}$. Similarly, we have

$$\|f\|_\infty = \text{ess sup}\{|f(x)| \,:\, x \in \mathbb{R}\} = \|f\|_{H^\infty(\mathbb{C}_k)}.$$

We note that $g(z) \in H^p(\mathbb{C}_{-1})$ if and only if the function $f(z) = \overline{g(\bar{z})} \in H^p(\mathbb{C}_{+1})$. The correspondence between their non-tangential boundary limits and the functions themselves in the Hardy spaces is an isometric isomorphism. We denote by $H^p_k(\mathbb{R})$ the spaces of the non-tangential boundary limits, or, precisely,

$$H^p_k(\mathbb{R}) = \left\{f \,:\, \mathbb{R} \to \mathbb{C}, f \text{ is the NTBL of a function in } H^p(\mathbb{C}_k)\right\}.$$

For $p = 2$ the Boundary Hardy spaces $H^2_k(\mathbb{R})$ are Hilbert spaces. We will need some very smooth classes of analytic functions that are dense in $H^p(\mathbb{C}_{+1})$ and will play the role of the polynomials in the disc case. Garnett [1] has shown the following results.

Theorem A ([1]) *Let N be a positive integer. For $0 < p < \infty$, $pN > 1$, the class $\mathfrak{A}_N$ is dense in $H^p(\mathbb{C}_{+1})$, where $\mathfrak{A}_N$ is the family of $H^p(\mathbb{C}_{+1})$ functions satisfying*

(i) *$f(z)$ is infinity differentiable in the closed upper plane $\overline{\mathbb{C}_{+1}}$,*
(ii) *$|z|^N f(z) \to 0$ as $z \to \infty, z \in \overline{\mathbb{C}_{+1}}$.*

We shall notice that the condition $pN > 1$ implies the class $\mathfrak{A}_N$ is contained in $H^p(\mathbb{C}_{+1})$. Let α be a complex number and $\mathfrak{R}_N(\alpha)$ the family of rational functions $f(z) = (z + \alpha)^{-N-1} P((z + \alpha)^{-1})$, $P(w)$ are polynomials. We notice that the class $\mathfrak{R}_N(\alpha)$ is contained in the class $\mathfrak{A}_N$ for $\text{Im}\alpha > 0$.

The first, replaced the class $\mathfrak{A}_N$ by the class $\mathfrak{R}_N(i)$, we will generalize Theorem A to as follows.

Theorem 1 *Let N be a positive integer. For $0 < p < \infty$, $Np > 1$, the class $\mathfrak{R}_N(i)$ is dense in $H^p(\mathbb{C}_{+1})$.*

Corollary 2 *Let N be a positive integer. For $0 < p < \infty$, $Np > 1$, the class $\mathfrak{R}_N(-i)$ is dense in $H^p(\mathbb{C}_{-1})$.*

The second is decomposition of functions in $L^p(\mathbb{R}), 0 < p < 1$, into sums of the corresponding Hardy space functions in $H^p_{+1}(\mathbb{R})$ and in $H^p_{-1}(\mathbb{R})$ for the same range of the indices p through rational functions approximation, and, in fact, rational atoms.

Theorem 3 (Hardy Spaces Decomposition of L^p Functions for Index Range $0 < p < 1$) *Suppose that $0 < p < 1$ and $f \in L^p(\mathbb{R})$. Then, there exist two sequences of rational functions $\{P_k(z)\}$ and $\{Q_k(z)\}$ such that $P_k \in H^p(\mathbb{C}_{+1})$, $Q_k \in H^p(\mathbb{C}_{-1})$ and*

$$\sum_{k=1}^{\infty}\left(\|P_k\|^p_{H^p_{+1}} + \|Q_k\|^p_{H^p_{-1}}\right) \leq A_p\|f\|^p_p, \tag{1}$$

$$\lim_{n\to\infty}||f - \sum_{k=1}^{n}(P_k + Q_k)||_p = 0, \tag{2}$$

where $A_p = 2 + 4\pi(1-p)^{-1}$. Moreover,

$$g(z) = \sum_{k=1}^{\infty} P_k(z) \in H^p(\mathbb{C}_{+1}), \quad h(z) = \sum_{k=1}^{\infty} Q_k(z) \in H^p(\mathbb{C}_{-1}), \tag{3}$$

and $g(x)$ and $h(x)$ are the non-tangential boundary values of functions for $g \in H^p(\mathbb{C}_{+1})$ and $h \in H^p(\mathbb{C}_{-1})$, respectively, $f(x) = g(x) + h(x)$ almost everywhere, and

$$\|f\|^p_p \leq \|g\|^p_p + \|h\|^p_p \leq A_p\|f\|^p_p,$$

that is, in the sense of $L^p(\mathbb{R})$,

$$L^p(\mathbb{R}) = H^p_{+1}(\mathbb{R}) + H^p_{-1}(\mathbb{R}).$$

For the uniqueness of the decomposition, we can ask the following question: what is the intersection space $H^p_{+1}(\mathbb{R}) \bigcap H^p_{-1}(\mathbb{R})$? Aleksandrov [2, 3] has given an answer for this problem.

Theorem B ([2] and [3]) *Let $0 < p < 1$ and X^p denote the L^p closure of the set of $f \in L^p(\mathbb{R})$ which can be written in the form*

$$f(x) = \sum_{j=1}^{N} \frac{c_j}{x - a_j}, \quad a_j \in \mathbb{R}, \; c_j \in \mathbb{C}.$$

Then

$$X^p = H^p_{+1}(\mathbb{R}) \bigcap H^p_{-1}(\mathbb{R}).$$

Aleksandrov's proof [2, 3] is quite complicated and makes use of the vanishing moments and the Hilbert transform, so it is necessary to give more simple proof for his result.

Schwarz's class $\mathbb{S}(\mathbb{R}^n)$ is defined to be the class of all those C^∞ function φ on $\mathbb{R}^n$(i.e., all the partial derivatives of φ exist and are continuous) such that

$$\sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta \varphi(x)| < \infty,$$

for all n-tuples $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ of nonnegative integers. The Fourier transform of a function $f \in L^1(\mathbb{R})$ is defined, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, by

$$\hat{f}(x) = \int_{\mathbb{R}} f(t)e^{-2\pi i x\cdot t}dt, \quad x \cdot t = \sum_{k=1}^{n} x_k t_k.$$

We recall that the Fourier transformation of a tempered distribution T is defined through the relation $(\widehat{T}, \varphi) = (T, \hat{\varphi})$ for φ in the Schwarz class $\mathbb{S}(\mathbb{R}^n)$. This coincides with the traditional definition of Fourier transformation for functions in $L^p(\mathbb{R}^n), 1 \le p \le 2$. A measurable function f such that $f(x)(1 + |x|^2)^{-m} \in L^p(\mathbb{R}^n)$ $(1 \le p \le \infty)$ for some positive integer integer m is called a tempered L^p function (when $p = \infty$ such a function is often called a slowly increasing function). The Fourier transform is a one to one mapping from $\mathbb{S}(\mathbb{R}^n)$ onto $\mathbb{S}(\mathbb{R}^n)$. It is proved in T. Qian's paper [4], that a function in $H^p_{+1}(\mathbb{R}), 1 \le p \le \infty$, induces a tempered distribution T_f such that $\mathrm{supp}\hat{T}_f \subset [0, \infty)$. In T. Qian, Y. S. Xu, D. Y. Yan, L. X. Yan and B. Yu's paper [5], the converse of the result is proved: Let T_f be the tempered distribution induced by f in $L^p(\mathbb{R}), 1 \le p \le \infty$. If $\mathrm{supp}\hat{T}_f \subset [0, \infty)$, then $f \in H^p_{+1}(\mathbb{R})$. The classical Paley-Wiener Theorem deals with the Hardy $H^2(\mathbb{C}_{+1})$ space asserts that $f \in L^2(\mathbb{R})$ is the NTBL of a function in $H^2(\mathbb{C}_{+1})$ if and only if $\mathrm{supp}\hat{f} \subset [0, \infty)$. Moreover, in such case, the integral representation

$$f(z) = \int_0^\infty e^{2\pi i t z}\hat{f}(t)dt \tag{4}$$

holds. The third is to extend the formula (4) to $0 < p < 1$.

Theorem 4 (Integral Representation Formula for Index Range $0 < p \le 1$) *If* $0 < p \le 1$, $f \in H^p(\mathbb{C}_{+1})$, *then there exist a positive constant* A_p, *depending only on p, and a slowly increasing continuous function F whose support is contained in* $[0, \infty)$, *satisfying that, for* φ *in the Schwarz class* $\mathbb{S}$,

$$(F, \varphi) = \lim_{y>0, y\to 0} \int_{\mathbb{R}} f(x + iy)\hat{\varphi}(x)dx,$$

and that

$$|F(t)| \le A_p \|f\|_{H^p_{+1}} |t|^{\frac{1}{p}-1}, \quad (t \in \mathbb{R}) \tag{5}$$

and

$$f(z) = \int_0^{\infty} F(t) e^{2\pi i t z} dt \qquad (z \in \mathbb{C}_{+1}). \tag{6}$$

At page 197 of the book [6], P. Duren explicitly says the argument of the integral representation (4) can be generalized to give a similar representation for $H^p(\mathbb{C}_{+1})$ functions, $1 \le p < 2$. But the detained argument is not given in his book [6], only one conclusion for $p = 1$ is given.

Theorem C ([6], Integral Representation Formula for Index Range $1 \le p \le 2$) *Suppose* $1 \le p \le 2$, $f \in L^p(\mathbb{R})$. *Then* $f \in H^p_{+1}(\mathbb{R})$ *if and only if* $\mathrm{supp}\hat{f} \subset [0, +\infty)$. *If the condition is satisfied, then the integral representation (4) holds.*

We, in fact, prove analogous formulas for all the cases $0 < p \le 2$. For the range $0 < p < 1$ we need to prove extra estimates to guarantee the integrability. The idea of using rational approximation is motivated by the studies of Takenaka-Malmquist systems in Hardy H^p spaces for $1 \le p \le \infty$ [7, 8]. For the range of $1 \le p \le \infty$ this aspect is related to the Plemelj formula in terms of Hilbert transform that has immediate implication to Fourier spectrum characterization in the case. For the range of $0 < p < 1$ the Plemelj formula approach is not available.

2 H^p Space on Tube

The Fourier spectrum properties of the functions as non-tangential boundary limits of those in the classical Hardy spaces $H^p(\mathbb{C}_k)$, $1 \le p \le \infty$, are completely characterized. The characterization is in terms of the location of the supports of the classical or distributional Fourier transforms of those boundary limit functions. It is shown that, for $f \in L^p(\mathbb{R})$, $1 \le p \le \infty$, f is the non-tangential boundary limit of a function in such a $H^p(\mathbb{C}_{+1})$ if and only if $\mathrm{supp}\hat{f}$, for $1 \le p \le 2$, or, alternatively, the distributional $\mathrm{supp}\hat{f}$, for $2 < p \le \infty$, is contained in $[0, \infty)$. For $p = 2$ this property of the Hardy space functions is known as one of the two types of the Paley-Wiener Theorems.

Let B be an open subset of $\mathbb{R}^n$. Then, the tube, T_B, with base B is the subset of all

$$z = (z_1, z_2, \ldots, z_n) = (x_1 + iy_1, \ldots, x_n + iy_n) = x + iy \in \mathbb{C}^n$$

such that $y \in B$. In other words, tube can be represented as follows:

$$T_B = \{z = x + iy \in \mathbb{C}^n; x \in \mathbb{R}^n, y \in B \subset \mathbb{R}^n\}.$$

For example, $\mathbb{C}_+$ is the tube in $\mathbb{C}$ with base $B = \{y \in \mathbb{R}; y > 0\}$. Obviously, the tube T_B is the generalization of $\mathbb{C}_+$.

A function holomorphic on the tube T_B is said to belong to the space $H^p(T_B), 0 < p < \infty$, if there exists a positive constant A such that

$$\int_{\mathbb{R}^n} |F(x+iy)|^p dx \leq A^p,$$

for all $y \in B$. That is to say, $H^p(T_B)$ can be defined as follows:

$$H^p(T_B) = \left\{ F : F \in H(T_B), \sup \left\{ \int_{\mathbb{R}^n} |F(x+iy)|^p dx : y \in B \right\} < \infty \right\}.$$

Let F be a distribution and $f(x+iy)$ an analytic function in T_Γ, where Γ is the first octant:

$$\Gamma = \{y = (y_1, \ldots, y_n) \in \mathbb{R}^n : y_k > 0, k = 1, \ldots, n\}.$$

If for any φ in the Schwarz class $\mathbb{S}(\mathbb{R}^n)$,

$$F_f(\varphi) = (F, \varphi) = \lim_{y>0, y\to 0} \int_{\mathbb{R}^n} f(x+iy)\varphi(x)dx$$

exists, then we say F is a Hardy distribution and $f(x+iy)$ is an analytic representation of F, where $y = (y_1, \ldots, y_n) > 0$ means that each $y_k > 0, k = 1, \ldots, n$.

Theorem 5 *If Γ is the first octant, F is the Hardy distribution represented by the boundary value of a function in $H^p(T_\Gamma), 1 < p < \infty$, then* $\mathrm{supp}\hat{F} \subset \Gamma$*, that is $(\hat{F}, \varphi) = 0$, for all $\varphi \in \mathbb{S}(\mathbb{R}^n)$, such that* $\mathrm{supp}\varphi \subset (\Gamma)^c = \mathbb{R}^n \setminus \Gamma$.

Theorem 6 *If Γ is the first octant and $f \in L^p(\mathbb{R}^n), 1 < p < \infty$, and $(\hat{f}, \varphi) = 0$, for for any φ in the Schwarz class $\mathbb{S}(\mathbb{R}^n)$ with* $\mathrm{supp}\phi \subset (\Gamma)^c$*. Then f is the boundary value of a function in $H^p(T_\Gamma), 1 < p < \infty$.*

Let Γ be a open set in $\mathbb{R}^n$ satisfying the following two conditions:

(1) 0 does not belong to Γ;
(2) For any $x,\ y \in \Gamma$, and any $\alpha,\ \beta > 0$, there holds $\alpha x + \beta y \in \Gamma$. then we call Γ. *open convex cone* with vertex at 0, we note that such a Γ is convex.

We need to introduce the *dual cone of* Γ, denoted by Γ^*, which is defined as $\Gamma^* = \{y \in \mathbb{R}^n : y \cdot x \geq 0,\ \text{for any } x \in \Gamma\}$. An open convex cone Γ is said to be *regular* if the interior of its dual cone Γ^* is nonempty. An open subset Γ of $\mathbb{R}^n$ is called a *polygonal cone* if it is the interior of the convex hull of a finite number of rays meeting at the origin among which we can find at least n sides that are linearly independent. We call a polygonal cone an *n-sided polygonal cone* if the minimum number of linearly independent rays convex-spanning Γ is exactly n. There are two important kernels related to Hardy spaces $H^p(T_\Gamma)$. They are Cauchy kernel and Poisson kernel, whose definitions and the relationship between them had been given by some reference. If Γ is a regular open convex cone of $\mathbb{R}^n$, then the function

$$K(z) = \int_{\Gamma^*} e^{2\pi i z\cdot t}\, dt \tag{7}$$

for $z \in T_\Gamma$ acts as the Cauchy kernel in the tube T_Γ . The function $P(x, y) = |K(z)|^2/K(2iy)$ ($z = x + iy \in T_\Gamma$) may be verified to play the role of Poisson kernel in the tube T_Γ. For the classical one-dimensional upper-half plane case the Poisson kernel $P(x, y) = \frac{1}{\pi}\frac{y}{x^2+y^2}$ coincides with the one obtained from the above definition.

As mentioned previously, if Γ is the first octant of $\mathbb{R}^n$, then its dual cone is the closure of Γ, viz., $\Gamma^* = \overline{\Gamma}$. The Cauchy kernel associated with the first octant of $\mathbb{C}^n$ can be computed explicitly. We have [9] $K(z) = \prod_{j=1}^{n} \frac{-1}{2\pi i z_j}$ for $z \in T_\Gamma$. That is, the Cauchy kernel $K(z)$ associated with the first octant is simply the product of n copies of the one-dimensional Cauchy kernels associated with the upper-half plane.

It follows that the Poisson kernel associated with the tube T_Γ, that is the first octant of $\mathbb{C}^n$, is the product of n copies of the one-dimensional Poisson kernels associated with the upper-half plane:

$$P(x, y) = |K(z)|^2/K(2iy) = \prod_{j=1}^{n} \frac{1}{\pi}\frac{y_j}{x_j^2 + y_j^2}. \tag{8}$$

Theorem 7 *Let Γ be a regular open cone in $\mathbb{R}^n$ and $F(x) \in L^p(\mathbb{R}^n)$, $1 \le p \le 2$. Then $F(x)$ is the boundary limit function of some $F(x + iy) \in H^p(T_\Gamma)$ if and only if the support* $\mathrm{supp}\hat{F} \subset \Gamma^*$. *Moreover, if the condition is satisfied, then*

$$F(z) = \int_{\Gamma^*} e^{2\pi i z\cdot t}\hat{F}(t)\, dt = \int_{\mathbb{R}^n} F(t)K(z - t)\, dt = \int_{\mathbb{R}^n} P(x - t, y)F(t)\, dt,$$

where $K(z)$ and $P(x, y)$ are, respectively, the Cauchy kernel and the Poisson kernel for the tube T_Γ.

Theorem 8 *Let Γ be a regular open cone in $\mathbb{R}^n$, and $F(z) \in H^p(T_\Gamma)$, $2 < p \le \infty$. Then $F(x)$, as the boundary limit of $F(z)$, is the tempered holomorphic distribution represented by the function $F(z)$. Moreover, d-*$\mathrm{supp}\hat{F} \subset \Gamma^*$, *that is $(\hat{F}, \varphi) = 0$ for all $\varphi \in \mathbb{S}(\mathbb{R}^n)$ with* $\mathrm{supp}\varphi \subset \overline{\mathbb{R}^n \setminus \Gamma^*}$.

Theorem 9 *Let Γ be a regular open cone in $\mathbb{R}^n$. Suppose that $F(z) \in H^p(T_\Gamma)$, $1 \le p \le \infty$, and $F(x)$ is the boundary limit function of the function $F(z)$. If $F(x) \in L^q(\mathbb{R}^n)$, $1 \le q \le \infty$, then $F(z) \in H^q(T_\Gamma)$.*

Theorem 10 *Let Γ be a regular open cone in $\mathbb{R}^n$. If $F \in L^p(\mathbb{R}^n)$, $2 < p \leq \infty$, and $(\hat{F}, \varphi) = 0$, for all $\varphi \in \mathbb{S}(\mathbb{R}^n)$ with $\operatorname{supp} \varphi \subset \overline{\mathbb{R}^n \setminus \Gamma^*}$, then F is the boundary value of a function in $H^p(T_\Gamma)$.*

Theorem 11 *Let $K(z)$ be the Cauchy kernel associated with the tube T_Γ, where Γ is the first octant of $\mathbb{R}^n$, $F(x) \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, and*

$$C(F)(z) = \int_{\mathbb{R}^n} F(t)K(z-t)\,dt.$$

Then $C(F)(z) \in H^p(T_\Gamma)$, and there exists a finite constant C_p, depending only on p and the dimension n such that

$$\int_{\mathbb{R}^n} |C(F)(x+iy)|^p\,dx \leq C_p \int_{\mathbb{R}^n} |F(x)|^p\,dx.$$

Theorem 12 *Let Γ be the fist octant of $\mathbb{R}^n$. If $F(z) \in H^p(T_\Gamma)$, $1 \leq p < \infty$, then $F(z)$ is the Cauchy integral of its boundary limit $F(x)$, that is,*

$$F(z) = \int_{\mathbb{R}^n} F(t)K(z-t)\,dt, \quad z \in T_\Gamma. \tag{9}$$

where $K(z)$ is the Cauchy kernel of the tube T_Γ.

Corollary 13 *Let $K(z)$ be the Cauchy kernel associated with the tube T_Γ, where Γ is a polygonal cone in $\mathbb{R}^n$, $F(x) \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, and*

$$C(F)(z) = \int_{\mathbb{R}^n} F(t)K(z-t)\,dt.$$

Then $C(F)(z) \in H^p(T_\Gamma)$, and there exists a finite number N depending on Γ, and a constant C_p, the same as in Theorem 11, such that

$$\int_{\mathbb{R}^n} |C(F)(x+iy)|^p\,dx \leq NC_p \int_{\mathbb{R}^n} |F(x)|^p\,dx.$$

Corollary 14 *Let Γ be a polygonal cone in $\mathbb{R}^n$. If $F(z) \in H^p(T_\Gamma)$, $1 \leq p < \infty$, then $F(z)$ is the Cauchy integral of its boundary limit $F(x)$, that is,*

$$F(z) = \int_{\mathbb{R}^n} F(t)K(z-t)\,dt, \quad z \in T_\Gamma. \tag{10}$$

where $K(z)$ is the Cauchy kernel of the tube T_Γ.

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Suita Conjecture for a Punctured Torus

Robert Xin Dong

To my family

Abstract For a once-punctured complex torus, we compare the Bergman kernel and the fundamental metric, by constructing explicitly the Evans-Selberg potential and discussing its asymptotic behaviors. This work aims to generalize the Suita type results to potential-theoretically parabolic Riemann surfaces.

Keywords Arakelov-Green function • Arakelov metric • Bergman kernel • Evans-Selberg potential • Fundamental metric • Suita conjecture

Mathematics Subject Classification (2010) Primary 32A25; Secondary 14K25, 31A05

1 Introduction

The Suita conjecture [12] asks about the precise relations between the Bergman kernel and the logarithmic capacity. For potential-theoretically hyperbolic Riemann surfaces, it was conjectured that the Gaussian curvature of the Suita metric (induced from the logarithmic capacity) is bounded from above by -4. The relations between the Suita conjecture and the L^2 extension theorem were first observed in [7] and later contributed by several mathematicians. The Suita conjecture proved to be true for the hyperbolic case (see [1, 2, 5, 8]), and it might be interesting to generalize similar results to non-hyperbolic cases. In this article, for a once-punctured complex torus $X_{\tau,u} := X_\tau \backslash \{u\}$, which is a typical potential-theoretically parabolic Riemann surface, we construct a so-called Evans-Selberg potential and further derive a so-called fundamental metric.

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Theorem 1.1 *On $X_{\tau,u}$, let $K_{\tau,u}$ and $c_{\tau,u}$ be the Bergman kernel and the fundamental metric, respectively. In the local coordinate w, write $K_{\tau,u} = k_{\tau,u}(w)|dw|^2$ and $c_{\tau,u} = c_{\tau,u}(w)|dw|^2$. Then, as $w \to u$,*

$$\frac{\pi k_{\tau,u}(w)}{c_{\tau,u}^2(w)} \sim \frac{\pi \cdot |w-u|^2}{2 \cdot \operatorname{Im} \tau} \to 0^+.$$

We also study the degenerate case when a once-punctured complex torus becomes a singular curve and obtain the following result.

Theorem 1.2 *Under the same assumptions as in Theorem 1.1, as* $\operatorname{Im} \tau \to +\infty$, *it follows that*

$$\frac{\pi K_{\tau,u}(w)}{c_{\tau,u}^2(w)} \to 0^+.$$

As we can see, either Theorem 1.1 or Theorem 1.2 will imply that the Gaussian curvature of the fundamental metric on $X_{\tau,u}$ can be arbitrarily close to 0^-, which is different from the hyperbolic case.

Corollary 1.3 *The Gaussian curvature of the fundamental metric on a once-punctured complex torus cannot be bounded from above by a negative constant.*

2 Preliminaries

On a potential-theoretically parabolic Riemann surface R, there exists a so-called Evans-Selberg potential, which is a counterpart of the Green function for the hyperbolic case. Let us recall the definition of an Evans-Selberg potential and the so-called fundamental metric (cf. [10, p. 351], [11, p. 114], [6]).

Definition 2.1 On an open Riemann surface Σ, an Evans-Selberg potential $E_q(p)$ with a pole $q \in \Sigma$ is a real-valued function such that:

(*i*) For all $p \in \Sigma \setminus \{q\}$, $E_q(p)$ is harmonic with respect to p,
(*ii*) $E_q(p) \to +\infty$, as $p \to a_\infty$ (the Alexandroff ideal boundary point),
(*iii*) $E_q(p) - \log |\varphi(p) - \varphi(q)|$ is bounded near q, with φ being the local coordinate.

Definition 2.2 On a potential-theoretically parabolic Riemann surface Σ, the fundamental metric under the local coordinate $z = \varphi(p)$ is defined as

$$c(z)|dz|^2 := \exp \lim_{q \to p} \left(E_q(p) - \log |\varphi(p) - \varphi(q)| \right) |dz|^2.$$

The fundamental metric is a non-compact counterpart of the Arakelov metric, and it coincides at the hyperbolic case with the Suita metric. The Gaussian curvature form of the fundamental metric is

$$-4\frac{\partial^2}{\partial z \partial \bar{z}} \log c(z) = -4\pi k(z), \tag{1}$$

where $k(z)$ (≥ 0) is the coefficient of the Bergman kernel $(1, 1)$-form in the local coordinate z. For a compact complex torus $X_\tau := \mathbb{C}/\left(\mathbb{Z} + \tau\mathbb{Z}\right)$, where $\tau \in \mathbb{C}$ and $\operatorname{Im}\tau > 0$, its Bergman kernel by definition is $K_\tau(z) = (\operatorname{Im}\tau)^{-1} dz \wedge d\bar{z}$, where z is the local coordinate induced from the complex plane $\mathbb{C}$.

Definition 2.3 A once-punctured complex torus $X_{\tau,u} := X_\tau \backslash \{u\}$ is an open Riemann surface obtained by removing one single point u from a compact complex torus X_τ.

Proposition 2.4 *There exists no non-constant subharmonic function which is bounded from above on* $X_{\tau,u}$.

On the one hand, the parabolicity of X_τ follows straightforwardly from Removable Singularity Theorem, which also applies to finitely punctured Riemann surfaces. On the other hand, there exists a proof (by using Maximum Principle and finding a harmonic function, as we will see below) that works for an open Riemann surface X of infinitely many genus. Equivalently, we could say that X admits an Evans-Selberg potential.

Proposition 2.5 *There exists no non-constant subharmonic function which is bounded from above on the algebraic curve*

$$X := \left\{ (y, x) \in \mathbb{C}^2 \;\middle|\; y^2 = x \prod_{n=1}^{\infty} \left(1 - x^2/n^2\right) \right\}.$$

Sketch of proof The idea is by contradiction and suppose there exists a subharmonic function u which is bounded from above on X. On X, we define

$$f(y, x) := \begin{cases} 0, & |x| \leq 1 \\ \log|x|, & |x| > 1, \end{cases}$$

and consider $v := u - \epsilon f$. Taking a limit as ϵ tends to 0, one will prove that the maximum is attainable at the "lift" of the unit circle (away from the boundary). By Maximum Principle, one concludes that u is constant. □

Explicit formulas of Evans-Selberg potentials and fundamental metrics on planar domains are provided in [4].

3 A Compact Torus

The Arakelov-Green function on a complex torus X_τ with a pole w satisfies

$$\frac{\partial^2 g_w(z)}{\partial z \bar{\partial} z} = \frac{\pi}{2}\left(\delta(z-w) - \frac{1}{\operatorname{Im}\tau}\right), \tag{2}$$

and can be expressed via the theta function as

$$g_w(z) = \log\left|\frac{\theta_1(z-w;q)}{\eta(\tau)}\right| - \frac{\pi \cdot (\operatorname{Im}(z-w))^2}{\operatorname{Im}\tau}. \tag{3}$$

Here $\eta(\tau) = q^{\frac{1}{12}} \cdot \prod_{m=1}^{\infty}(1-q^{2m})$ and

$$\theta_1(z;q) := 2q^{1/4}\sin(\pi z)\prod_{n=1}^{\infty}(1-q^{2n})(1-2\cos(2\pi z)q^{2n}+q^{4n}),$$

for $q = \exp(\pi i \tau)$. In this case, it is possible to compare the Bergman kernel and the Arakelov metric. For X_τ, the author in [3] numerically obtained a sharp negative upper bound for the Arakelov metric by computing via elliptic functions.[1] Alternatively, the Gaussian curvature of the Arakelov metric can be computed by *Mathematica* (Version 10.3). Recall that for the hyperbolic case, the Green function is always less than 0 in the interior. However, the supremum of an Arakelov-Green function on a torus can be positive at some points, illustrated in Fig. 1 (plotted by *Mathematica*).

4 A Once-Punctured Torus

Theorem 4.1 *There exists an Evans-Selberg potential on $X_{\tau,u}$ with a pole w given by*

$$E_w^{\tau,u}(z) = \log\left|\frac{\theta_1(z-w;q)}{\theta_1(z-u;q)}\right|,$$

for $z \in X_{\tau,u} \setminus \{w\}$.

Proof of Theorem 4.1 We see that the two terms on the right-hand side of (3) are responsible for the two terms on the right-hand side of (2), respectively. Keeping

[1]The author apologizes for several mistakes contained in [3].

```
Plot3D[Log [
  Abs [ EllipticTheta[1, x + y I, Exp[-Pi 0.9]]]/
   Abs [DedekindEta [0.9 I]]] - Pi/0.9 y^2, {x, -5, 5}, {y, -1, 1},
 AspectRatio -> 1, PlotRange -> {0.236, 0.229}]
```

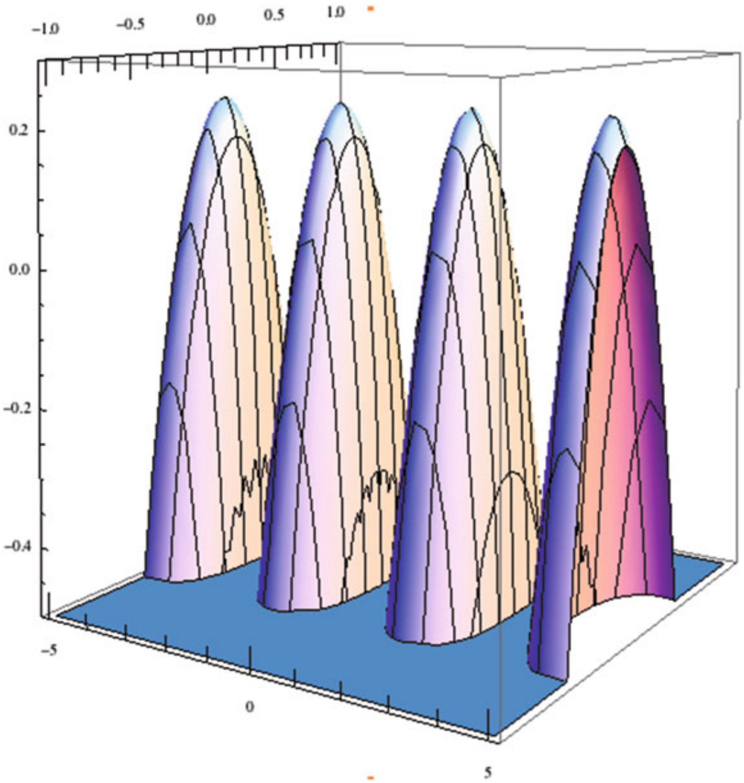

Fig. 1 An Arakelov-Green function on a torus

this in mind, we can construct the Evans-Selberg potential by attaching physics meanings. We regard the potential as an electric flux generated at the pole w and terminates at the boundary point u (see [9] for detailed physics explanations). So, the Evans-Selberg potential $E_w^{\tau,u}(z)$ with a pole w satisfies

$$\frac{\partial^2 E_w^{\tau,u}(z)}{\partial z \bar{\partial} z} = \frac{\pi}{2} \left(\delta(z-w) - \delta(z-u)\right),$$

and can be expressed via the theta function as

$$E_w^{\tau,u}(z) = \log\left|\frac{\theta_1(z-w;q)}{\eta(\tau)}\right| - \log\left|\frac{\theta_1(z-u;q)}{\eta(\tau)}\right| = \log\left|\frac{\theta_1(z-w;q)}{\theta_1(z-u;q)}\right|$$
$$= \log\left|\frac{\sin(\pi(z-w))\cdot\prod_{m=1}^{\infty}(1-2\cos(2\pi(z-w))\cdot q^{2m}+q^{4m})}{\sin(\pi(z-u))\cdot\prod_{m=1}^{\infty}(1-2\cos(2\pi(z-u))\cdot q^{2m}+q^{4m})}\right|.$$

□

Corollary 4.2 *There exists a fundamental metric $c_{\tau,u}$ on $X_{\tau,u}$ in the local coordinate w given by*

$$c_{\tau,u}(w)|dw|^2 = \frac{2\pi\cdot|\eta(\tau)|^3}{|\theta_1(w-u;q)|}|dw|^2.$$

Proof This can be verified by definition, since

$$c_{\tau,u}(w) = \exp\lim_{z\to w}\left(\log\left|\frac{\theta_1(z-w;q)}{\theta_1(z-u;q)}\right| - \log|z-w|\right)$$
$$= \left|\frac{\pi\cdot\prod_{m=1}^{\infty}(1-q^{2m})^2}{\sin(\pi(w-u))\cdot\prod_{m=1}^{\infty}(1-2\cos(2\pi(w-u))\cdot q^{2m}+q^{4m})}\right|$$
$$= \left|\frac{\pi\cdot\eta(\tau)^2}{q^{\frac{1}{6}}\cdot\sin(\pi(w-u))\cdot\prod_{m=1}^{\infty}(1-2\cos(2\pi(w-u))\cdot q^{2m}+q^{4m})}\right|$$
$$= \left|\frac{\pi\cdot\eta(\tau)^2\cdot 2q^{\frac{1}{4}}\cdot\prod_{m=1}^{\infty}(1-q^{2m})}{q^{\frac{1}{6}}\cdot\theta_1(w-u;q)}\right|$$
$$= \left|\frac{\pi\cdot\eta(\tau)^2\cdot 2q^{\frac{1}{4}}\cdot\frac{\eta(\tau)}{q^{\frac{1}{12}}}}{q^{\frac{1}{6}}\cdot\theta_1(w-u;q)}\right| = \left|\frac{2\pi\cdot\eta(\tau)^3}{\theta_1(w-u;q)}\right|.$$

□

By the second equality above, $c_{\tau,u}$ has the following asymptotic behavior, which will yield Theorem 1.2 for any fixed τ.

Corollary 4.3 *Under the same assumptions as in Corollary 4.2, as $w \to u$, it follows that*

$$c_{\tau,u}(w) \sim \frac{1}{|w-u|} \to +\infty.$$

5 The Degenerate Case

By studying the asymptotic behaviors of the fundamental metric under degeneration with respect to the complex structure, we will prove Theorem 1.2. Relating Theorem 1.2 with (1), we further get Corollary 1.3.

Proof of Theorem 1.2 Let $\operatorname{Im}\tau \to +\infty$, and $q \equiv \exp(\pi i \tau)$ will tend to 0. Then, it holds that

$$c_{\tau,u}(w) \to \left| \frac{\pi \cdot \prod_{m=1}^{\infty}(1-0^{2m})^2}{\sin(\pi(w-u)) \cdot \prod_{m=1}^{\infty}(1-2\cos(2\pi(w-u)) \cdot 0^{2m} + 0^{4m})} \right|$$
$$\to \frac{\pi}{|\sin(\pi(w-u))|}.$$

Therefore, it follows that

$$\frac{\pi K_{\tau,u}(w)}{c^2_{\tau,u}(w)} \to \frac{|\sin(\pi(w-u))|^2}{2 \cdot \operatorname{Im}\tau \cdot \pi} \to 0^+,$$

since the denominator is uniformly bounded by 1 for any fixed w. □

On the one hand, at the degenerate case of potential-theoretically hyperbolic Riemann surfaces, we are not sure whether Gaussian curvatures of the Suita metrics are still bounded from above by -4. On the other hand, for a compact complex torus, the Gaussian curvature of the Arakelov metric is always 0 by the genus reason, although our earlier result in [3] shows that as $\operatorname{Im}\tau \to +\infty$,

$$\frac{\pi K_\tau(w)}{c^2_\tau(w)} \to +\infty.$$

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Plemelj Formula for Bergman Integral on Unit Ball in Hermitean Clifford Analysis

Min Ku and Fuli He

Abstract In this paper, we construct the Bergman kernel on the unit ball of $\mathbb{R}^{2n}$ in the setting of Hermitean Clifford analysis, and then derive the Plemelj formula for the Bergman integral on the unit ball.

Keywords Bergman kernel • Hermitean Clifford analysis • Plemelj formula

Mathematics Subject Classification (2010) Primary 15A66; Secondary 30C40

1 Introduction

The well-known Bergman kernel arose from the works of Bergman. It is a reproducing kernel of the Hilbert space of all square integrable holomorphic functions defined in a domain of $\mathbb{C}^n$ [1]. To our knowledge, the study of the Bergman kernel has deeply influenced the development of the theory of functions of several complex variables [2, 3]. Moreover, the Bergman integral has provided a tool to solve partial differential equations of several complex variables [2, 3]. However, because of the dependence on the domains considered of the Bergman kernel, in general it is difficult to give an explicit and closed formula. Nowadays, such problems are being widely investigated by many scholars. As a principal reference for this theory we mention [4].

In parallel to the theory of functions of several complex variables, Clifford analysis is an elegant generalization of the classical complex analysis into higher dimensions [5, 6]. In the past decays, the theory of the Bergman space has been

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extended to the setting of Clifford analysis. In [7], the authors constructed the explicit formulas for the monogenic Bergman kernel on the unit ball and on the half-space using the general approach of constructing the reproducing kernels in a Hilbert space. In [8], the Bergman kernel function in Clifford analysis had been studied on the unit ball using the Bergman projection. In [9, 10] the authors developed the Bergman kernel and the Bergman integral in Clifford analysis in virtue of the Green function, analogous to the relationship between the Green function and the Bergman kernel function in the classical complex analysis. More results involving the construction of monogenic Bergman kernel functions in an explicit form on the special domains in the framework of hyper-complex function theory can be also found in references, e.g., [11, 12]. In the recent years, as a refinement of the Clifford analysis, the Hermitean Clifford analysis centres on the h-monogenic functions (see Sect. 2), and links to the holomorphic function theory of several complex variables, see, e.g., [13–15]. As far as we know, although the Cauchy formula, the Hilbert transform and Taylor series for h-monogenic functions (c.f. [14–16]) were gotten, so far few results about the explicit integral formulae for the h-monogenic functions defined in a domain of the higher dimensional Euclidean space of even dimension. Thus, it is natural to know do such these problems look like. In this paper we will devote to find the Bergman kernel on the unit ball in the setting of the Hermitean Clifford analysis, and get an explicit Plemelj formula for the Bergman integral on the unit ball.

2 Bergman Kernel in Hermitean Clifford Analysis

Let $e_1, \dots, e_{2n}$ be an orthogonal basis of the Euclid space $\mathbb{R}^{2n}$. $\mathbb{C}_{2n}$ is the complex associate but non-communicative Clifford algebra constructed over $\mathbb{R}^{2n}$. The Euclidean space $\mathbb{R}^{2n}$ is embedded in the Clifford algebra $\mathbb{C}_{2n}$ by identifying $(x_1, \dots, x_{2n})$ with the real Clifford vector $\underline{x}$ given by $\underline{x} = \sum_{i=1}^{2n} e_i x_i$. For $j = 1, \dots, n$, the Witt basis $(\mathfrak{e}_j, \mathfrak{e}_j^\dagger)_{j=1}^n$ for the algebra $\mathbb{C}_{2n}$ are obtained through the action of $\pm(\mathbf{1}_{2n} \pm iJ)$ on the orthogonal basis elements

$$\mathfrak{e}_j = (\mathbf{1} + iJ)[e_j] = (e_{2j-1} - ie_{2j}), \quad \mathfrak{e}_j^\dagger = -(\mathbf{1} - iJ)[e_j] = -(e_{2j-1} + ie_{2j}),$$

where the complex structure J is a particular element of $\mathrm{SO}(2n)$, satisfying $J^2 = -\mathbf{1}_{2n}$. The Hermitean Clifford variables $\underline{z}$ and its conjugate $\underline{z}^\dagger$ are

$$\underline{z} = (\mathbf{1} + iJ)[\underline{x}] = \underline{x} + i\underline{x}| = \sum_{j=1}^{n} \mathfrak{e}_j z_j, \ \underline{z}^\dagger = -(\mathbf{1} - iJ)[\underline{x}] = -(\underline{x} - i\underline{x}|) = \sum_{j=1}^{n} \mathfrak{e}_j^\dagger z_j^c,$$

where n complex variables $z_j = x_{2j-1} + ix_{2j}$ have been introduced, with complex conjugates $z_j^c = x_{2j-1} - ix_{2j}, j = 1, \dots, n$. The Hermitean Dirac operators $\partial_{\underline{z}}$ and $\partial_{\underline{z}^\dagger}$

are derived from the following orthogonal Dirac operator $\partial_{\underline{x}} = \sum_{i=1}^{m} e_i \partial_{x_i}$, given by

$$\partial_{\underline{z}} = \frac{1}{2}(\mathbf{1} + iJ)[\partial_{\underline{x}}] = \sum_{j=1}^{n} \mathfrak{e}_j^{\dagger} \partial_{z_j}, \quad \partial_{\underline{z}^{\dagger}} = -\frac{1}{2}(\mathbf{1} - iJ)[\partial_{\underline{x}}] = \sum_{j=1}^{n} \mathfrak{e}_j^{\dagger} \partial_{z_j^c},$$

involving the classical Cauchy-Riemann operators $\partial_{z_j} = \frac{1}{2}(\partial_{x_{2j-1}} - i\partial_{x_{2j}})$ and their conjugate $\partial_{z_j^c} = \frac{1}{2}(\partial_{x_{2j-1}} + i\partial_{x_{2j}})$ on the complex z_j-plane. The (left) h-monogenic function in $\Omega \subset \mathbb{R}^{2n}$ is a function $f(\underline{x}) = f(\underline{z}, \underline{z}^{\dagger})$(or $f(\underline{z})$ for short) taking values in $\mathbb{C}_{2n}$ for which $\partial_{\underline{z}} f = \partial_{\underline{z}^{\dagger}} f = 0$ in Ω. The space of all h-monogenic functions in Ω is denoted by $M_h(\Omega)$. More details about the Hermitean Clifford analysis can be seen in references, e.g., [12–16].

In this context the Bergman space of the square integrable h-monogenic functions is the space $A^2(\Omega) = M_h(\Omega) \cap L^2(\Omega)$. Given the complex Clifford inner product $< f, g >= \int_{\Omega} \left[f(\underline{z}) g^{\dagger}(\underline{z})\right]_0 dV(\underline{z})$, it is a Hilbert space, where $[\cdot]_0$ denotes the scalar part. Its reproducing kernel, the so-called Bergman kernel, is uniquely defined, and satisfies $f(\underline{z}) = \int_{\Omega} K_{\Omega}(\underline{z}, \underline{w}) f(\underline{w}) dV(\underline{w})$ for any square integrable h-monogenic functions defined in $\Omega \subset \mathbb{R}^{2n}$. Moreover, its kernel is Hermitean conjugate symmetric, i.e., $K_{\Omega}(\underline{z}, \underline{w}) = K_{\Omega}^{\dagger}(\underline{w}, \underline{z})$.

The unit ball is $B_n = \left\{\underline{z} = \underline{x} + iJ[\underline{x}] \in \mathbb{R}^{2n} \cong \mathbb{C}^n : |\underline{z}| < 1\right\}$. Restricted attention to the unit ball $B_n \subset \mathbb{R}^{2n} \cong \mathbb{C}^n$, the functions $\underline{z}^{\beta} = \sum_{j=1}^{n} \mathfrak{e}_j z_j^{\beta_j}, \beta = (\beta_1, \ldots, \beta_n)$ belong to $A^2(B_n)$, and they are pairwise orthogonal by the symmetry of the ball. Following the same arguments contained in [2–4], we announce the following conclusions without proof.

Lemma 2.1 *Let $\underline{z} \in B_n \subset \mathbb{R}^{2n} \cong \mathbb{C}^n$. Then*

$$K_{B_n}(\underline{z}, r\mathfrak{e}_1) = \frac{n!}{\pi^n} \frac{1}{(1 - rz_1)^{n+1}}, 0 < r < 1,$$

where $\mathfrak{e}_1 = (e_1 - ie_2)$ is the first element of the Witt basis, and $z_1 = x_1 + ix_2$ is the first complex variable.

Theorem 2.2 *If $\underline{z}, \underline{w} \in B_n$, then*

$$K_{B_n}(\underline{z}, \underline{w}) = \frac{n!}{\pi^n} \frac{1}{(1 - \underline{z} \cdot \underline{w}^{\dagger})^{n+1}},$$

where $\underline{z} \cdot \underline{w}^{\dagger} = z_1 w_1^c + \cdots + z_n w_n^c$.

3 Plemelj Formula of Bergman Integral

Let $f(\underline{z})$ be a $\mathbb{C}_{2n}$-valued function defined in B_n, for $0 < \alpha \leq 1$. We call $f \in \mathrm{H}^\alpha(B_n, \mathbb{C}_{2n})$ if $\left|f(\underline{z}) - f(\underline{w})\right| = O\big([(\underline{z}-\underline{w})\cdot(\underline{z}-\underline{w})^\dagger]^{\frac{\alpha}{2}}\big)$, $\forall \underline{z}, \underline{w} \in B_n$. For arbitrary $\underline{t} \in \partial(B_n)$, we define the principal value of the Bergman integral

$$p.v. \int_{B_n} K_{B_n}(\underline{t}, \underline{w}) f(\underline{w}) dV(\underline{w}) = \lim_{\varepsilon \to 0} \int_{\substack{\underline{w} \in B_n, \\ |\underline{w}-\underline{t}| \geq \varepsilon}} K_{B_n}(\underline{t}, \underline{w}) f(\underline{w}) dV(\underline{w}).$$

Lemma 3.1 *If* $n = 1$, *then*

$$p.v. \int_{B_n} K_{B_n}(\mathfrak{e}, w) dV(w) = \frac{1}{2},$$

where $\mathfrak{e} = (e_1 - ie_2)$ *and* $\mathfrak{e}^\dagger = -(e_1 + ie_2)$ *are the Witt basis.*

Proof $\forall \varepsilon > 0$, write

$$I(\varepsilon) = \frac{1}{\pi} \int_{\substack{|w|<1 \\ |w-1| \geq \varepsilon}} \frac{dV(w)}{(1-w^c)^2} = \frac{1}{2\pi i} \int_{\substack{|w|<1 \\ |w-1| \geq \varepsilon}} \frac{dw^c \wedge dw}{(1-w^c)^2},$$

we get by the Stokes theorem

$$I(\varepsilon) = \frac{1}{2\pi i} \int_{\substack{|w|=1 \\ |w-1| \geq \varepsilon}} \frac{dw}{1-w^c} + \frac{1}{2\pi i} \int_{\substack{|w|<1 \\ |w-1| = \varepsilon}} \frac{dw}{1-w^c} = I_1(\varepsilon) + I_2(\varepsilon).$$

Noticing that the intersections of curve $|w| = 1$ and curve $|w-1| = \varepsilon$ are $\mu = 1 - \varepsilon^2/2 + i\sqrt{\varepsilon^2 - \varepsilon^4/4}$ and μ^c, we derive the result from

$$\lim_{\varepsilon \to 0} I_1(\varepsilon) = p.v. \frac{1}{2\pi i} \int_{|\underline{w}|=1} \frac{w dw}{w-1} = \frac{1}{2},$$

$$\lim_{\varepsilon \to 0} I_2(\varepsilon) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\mu^c}^{\mu} \frac{1-w}{\varepsilon^2} dw = \lim_{\varepsilon \to 0} \frac{\sqrt{\varepsilon^2 - \varepsilon^4/4}}{2\pi} = 0.$$

□

Lemma 3.2 *If* $n > 1$, *then*

$$p.v. \int_{B_n} K_{B_n}(\mathfrak{e}_1, \underline{w}) dV(\underline{w}) = 1.$$

Proof Denote

$$I(\varepsilon) = \frac{n!}{\pi^n} \int_{\substack{|\underline{w}|<1 \\ |\underline{w}-\underline{t}| \geq \varepsilon}} \frac{dV(\underline{w})}{(1-w_1^c)^{n+1}}, \varepsilon > 0.$$

Let $\underline{w} = \mathfrak{e}_1 w_1 + \mathfrak{e}_2 w_2 + \cdots + \mathfrak{e}_n w_n = \mathfrak{e}_1 w_1 + \underline{w}'$, $\underline{w}' \in \mathbb{R}^{2n-2} \cong \mathbb{C}^{n-1}$, $\underline{w}' = r\widetilde{\underline{w}'}$, $r = |\underline{w}'|$, $|\underline{w}'| = 1$, $w_1 = x_{w_1} + iy_{w_1}$, then

$$I(\varepsilon) = \frac{n!\omega_{2n-3}}{\pi^n} \int_{\substack{r^2+|w_1|^2<1 \\ r^2+|w_1-1|^2\geq\varepsilon^2}} \frac{r^{2n-3}}{(1-w_1^c)^{n+1}} dx_{w_1} dy_{w_1} dr$$

$$= \frac{2n(n-1)}{2\pi i} \left[\int_0^\delta r^{2n-3} \int_{\substack{|w_1|^2<1-r^2 \\ |w_1-1|^2\geq\varepsilon^2-r^2}} \frac{dw_1^c \wedge dw_1 dr}{(1-w_1^c)^{n+1}} + \int_\delta^1 r^{2n-3} \int_{|w_1|^2<1-r^2} \frac{dw_1^c \wedge dw_1 dr}{(1-w_1^c)^{n+1}} \right],$$

where $\delta = \sqrt{\varepsilon^2 - \varepsilon^4/4}$, and $\omega_{2n-3} = \frac{2\pi^{n-1}}{\Gamma(n-1)}$ is the area of the unit sphere of $\mathbb{R}^{2n-2} \cong \mathbb{C}^{n-1}$. Observing that

$$\frac{dw_1^c \wedge dw_1}{(1-w_1^c)^{n+1}} = \frac{1}{n} d\left[\frac{dw_1}{(1-w_1^c)^n}\right],$$

by the Stokes theorem, we have $I(\varepsilon) = I_1(\varepsilon) + I_2(\varepsilon) - I_3(\varepsilon)$, where

$$I_1(\varepsilon) = 2(n-1) \int_0^1 r^{2n-3} dr \cdot \frac{1}{2\pi i} \int_{|w_1|^2=1-r^2} \frac{dw_1}{(1-w_1^c)^n},$$

$$I_2(\varepsilon) = 2(n-1) \int_0^\delta r^{2n-3} dr \cdot \frac{1}{2\pi i} \int_{\substack{|w_1|^2<1-r^2 \\ |w_1-1|^2=\varepsilon^2-r^2}} \frac{dw_1}{(1-w_1^c)^n},$$

$$I_3(\varepsilon) = 2(n-1) \int_0^\delta r^{2n-3} dr \cdot \frac{1}{2\pi i} \int_{\substack{|w_1|^2=1-r^2 \\ |w_1-1|^2\leq\varepsilon^2-r^2}} \frac{dw_1}{(1-w_1^c)^n}.$$

Set $s = \sqrt{1-r^2}$, we get

$$\frac{1}{2\pi i} \int_{|w_1|=s} \frac{dw_1}{(1-w_1^c)^n} = \frac{1}{2\pi i} \int_{|w_1|=s} \frac{dw_1}{(1-s^2/w_1)^n} = \frac{1}{2\pi i} \int_{|w_1|=s} \left(1 + \frac{s^2}{w_1 - s^2}\right)^n dw_1$$

$$= \sum_{k=1}^n \binom{n}{k} s^{2k} \int_{|w_1|=s} \frac{dw_1}{(w_1 - s^2)^k} = ns^2.$$

Hence, $I_1(\varepsilon) = 2(n-1) \int_0^1 r^{2n-3} n(1-r^2) dr = 1$.

Noting that

$$\left|\frac{1}{2\pi i}\int_{\substack{|w_1|^2<1-r^2\\|w_1-1|^2=\varepsilon^2-r^2}}\frac{dw_1}{(1-w_1^c)^n}\right|\leq\frac{1}{2\pi}\int_{|w_1-1|^2=\varepsilon^2-r^2}\frac{d|w_1|}{|1-w_1^c|^n}=(\varepsilon^2-r^2)^{-\frac{n-1}{2}},$$

$$|I_2(\varepsilon)|\leq 2(n-1)\int_0^\delta r^{2n-4}(\varepsilon^2-r^2)^{-\frac{n-2}{2}}\frac{rdr}{\sqrt{\varepsilon^2-r^2}},$$

$$r^{2n-4}(\varepsilon^2-r^2)^{-\frac{n-2}{2}}\leq\delta^{2n-4}(\varepsilon^2-\delta^2)^{-\frac{n-2}{2}}=(\varepsilon^2-\varepsilon^4/4)^{n-2}(2/\varepsilon^2)^{n-2}\leq 2^{n-2},$$

one has

$$|I_2(\varepsilon)|\leq 2(n-1)\int_0^\delta 2^{n-2}\frac{rdr}{\sqrt{\varepsilon^2-r^2}}=2^{n-1}(n-1)(\varepsilon-\frac{\varepsilon^2}{2}),$$

hence $\lim_{\varepsilon\to 0}I_2(\varepsilon)=0$. Notice that

$$\frac{1}{2\pi i}\int_{\substack{|w_1|=s\\|w_1-1|^2\leq\varepsilon^2-r^2}}\frac{dw_1}{(1-w_1^c)^n}=\frac{1}{2\pi i}\int_{\substack{|w_1|=\rho\\|w_1-1|^2\leq\varepsilon^2-r^2}}\left(\frac{w_1}{w_1-s^2}\right)^n dw_1,$$

it easy to calculate that the distance of the point s^2 to the two endpoints of the integral path is $d=s\sqrt{\varepsilon^2-r^2}$. Since the integrand is holomorphic in $D=\{w_1:|w_1|>s\}$, by the Cauchy theorem in one complex analysis, we obtain

$$\left|\frac{1}{2\pi i}\int_{\substack{|w_1|=s\\|w_1-1|^2\leq\varepsilon^2-r^2}}\left(\frac{w_1}{w_1-s^2}\right)^n dw_1\right|=\left|\frac{1}{2\pi i}\int_{\substack{|w_1-s^2|=d\\|w_1|\geq s}}\left(\frac{w_1}{w_1-s^2}\right)^n dw_1\right|\leq 2^n d^{-n+1}.$$

Therefore, $|I_3(\varepsilon)|\leq C\int_0^\delta r^{2n-3}(\varepsilon^2-r^2)^{-\frac{n-1}{2}}dr$, where $C\in\mathbb{R}^+$ is a constant. Similarly, we derive $\lim_{\varepsilon\to 0}I_3(\varepsilon)=0$. Thus, the proof of the result completes. □

Theorem 3.3 *If $f\in \mathrm{H}^\alpha(B_n,\mathbb{C}_{2n})$, $0<\alpha\leq 1$, $\forall \underline{t}\in\partial(B_n)$, then the principal value of the Bergman integral exists, and*

$$\lim_{\substack{\underline{z}\in B_n\\\underline{z}\to\underline{t}}}\int_{B_n}K_{B_n}(\underline{z},\underline{w})f(\underline{w})dV(\underline{w})=\begin{cases}p.v.\int_{B_n}K_{B_n}(\underline{t},\underline{w})f(\underline{w})dV(\underline{w})+\frac{1}{2}f(\underline{t}), & n=1,\\ p.v.\int_{B_n}K_{B_n}(\underline{t},\underline{w})f(\underline{w})dV(\underline{w}), & n>1.\end{cases}$$

Proof $p.v.\int_{B_n}K_{B_n}(\underline{t},\underline{w})f(\underline{w})dV(\underline{w})$, $\underline{t}\in\partial(B_n)$ can be decomposed into

$$\int_{B_n}K_{B_n}(\underline{t},\underline{w})[f(\underline{w})-f(\underline{t})]dV(\underline{w})+f(\underline{t})\,p.v.\int_{B_n}K_{B_n}(\underline{t},\underline{w})dV(\underline{w}),$$

and $\int\limits_{B_n} K_{B_n}(\underline{z}, \underline{w})f(\underline{w})dV(\underline{w}) = \Gamma_{\underline{z}}(\underline{w}) + f(\underline{z}), \underline{z} \in B_n$. Moreover, it is easy to verify that $\Gamma_{\underline{z}}(\underline{w}) = \int_{B_n} K_{B_n}(\underline{z}, \underline{w})[f(\underline{w}) - f(\underline{z})]dV(\underline{w})$ is uniformly integrable due to $f \in \mathrm{H}^{\alpha}(B_n)$. Then the map $T : \overline{B}_n \to L^1(B_n),\ T(\underline{z}) = \Gamma_{\underline{z}}$ is uniformly continuous on $\overline{B}_n$. Taking the limit on both sides, we have

$$\lim_{\substack{\underline{z}\in B_n \\ \underline{z}\to\underline{t}}} \int_{B_n} K_{B_n}(\underline{z}, \underline{w})f(\underline{w})dV(\underline{w}) = \int_{B_n} K_{B_n}(\underline{t}, \underline{w})[f(\underline{w}) - f(\underline{t})]dV(\underline{w}) + f(\underline{t})$$

$$= p.v. \int_{B_n} K_{B_n}(\underline{t}, \underline{w})f(\underline{w})dV(\underline{w}) + f(\underline{t}) \left[1 - p.v. \int_{B_n} K_{B_n}(\underline{t}, \underline{w})dV(\underline{w})\right].$$

In terms of Lemmas 3.1 and 3.2,

$$1 - p.v. \int\limits_{B_n} K_{B_n}(\underline{t}, \underline{w})dV(\underline{w}) = \begin{cases} \frac{1}{2}, & n = 1, \\ 0, & n > 1. \end{cases}$$

So, the result follows. □

Remark 3.4 Theorem 3.3 is the Plemelj formula of the Bergman integral in the Hermitean Clifford analysis. If $f \in L^1(B_n)$, similar to [3, 4], it is easy to prove that $F(\underline{z}) = \int_{B_n} K_{B_n}(\underline{z}, \underline{w})f(\underline{w})dV(\underline{w}) \in M_h(B_n)$. Moreover, if $f \in M_h(B_n) \cap L^1(B_n)$, then $F = f$.

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Meromorphic Functions Sharing Four Real Values

Xiao-Min Li, Cui Liu, and Hong-Xun Yi

Abstract We study the uniqueness question of transcendental meromorphic functions that share four distinct finite values $a_1,\ a_2,\ a_3,\ a_4$, where all the a_j-points of the transcendental meromorphic functions are real numbers for $1 \leq j \leq 4$. The results in this paper improve and extend the corresponding results from Czubiak–Gundersen (Proc Am Math Soc 82:393–397, 1981) and Yi (Proc Am Math Soc 124:585–590, 1996). An example is provided to show that the results in this paper, in a sense, are the best possible.

Keywords Meromorphic functions • Nevanlinna theory • Shared values • Uniqueness theorems

Mathematics Subject Classification (2010) Primary 30D35; Secondary 30D30

1 Introduction and Main Results

In 1992, Czubiak-Gundersen [2] began to study the uniqueness question of entire functions that share 0 and 1 IM, where all the zeros and 1-points are real numbers. Later on, Yi [8] corrected the corresponding result from Czubiak-Gundersen [2]. We will prove the following result deduced by Lemma 2.8 in Section 2 of the present paper:

Theorem 1.1 (Main Theorem) *Suppose that f and g share four distinct finite values $a_1,\ a_2,\ a_3$, and a_4 IM, that f has only real a_j-points for* $1 \leq j \leq 4$,

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and that there exists some value $a \in \mathbb{C} \cup \{\infty\}$ *such that* $a \notin \{a_1, a_2, a_3, a_4\}$ *and* $\delta = \delta(a,f) > 0$. *If* $f \not\equiv g$, *then* $\rho(f) \leq 1$.

By Theorem 1.1 we get the following uniqueness result:

Corollary 1.2 *Suppose that f and g are two meromorphic functions such that they share four distinct finite values* $a_1,\ a_2,\ a_3,\ a_4$ *IM, where f has only real* a_j*-points for* $1 \leq j \leq 4$, *and that there exists some value* $a \in \mathbb{C} \cup \{\infty\}$ *such that* $a \notin \{a_1, a_2, a_3, a_4\}$ *and* $\delta = \delta(a,f) > 0$. *If* $\rho(f) > 1$, *then* $f = g$.

The following example shows that the assumptions "$a \notin \{a_1, a_2, a_3, a_4\}$ and $\delta = \delta(a,f) > 0$" and "$\rho(f) > 1$" in Corollary 1.2 are necessary:

Example Let $f(z) = \frac{1}{e^{iz}-2}$ and $g(z) = \frac{1}{e^{-iz}-2}$. Then, we can verify that f and g share $0,\ -\frac{1}{2},\ -1$, and $-\frac{1}{3}$ CM. Moreover, all the a_j-points of f and g are real numbers for $1 \leq j \leq 4$, where $a_1,\ a_2,\ a_3,\ a_4$ are four distinct finite real numbers such that $\{a_1, a_2, a_3, a_4\} = \{0, -1/2, -1, -1/3\}$. Moreover, for any $a \in \mathbb{C} \cup \{\infty\}$ satisfying $a \notin \{a_1, a_2, a_3, a_4\}$, we have $\delta(a,f) = \delta(a, g) = 0$ and $\rho(f) = \rho(g) = 1$. But $f \not\equiv g$.

2 Preliminaries

In this section, we introduce some important lemmas to prove the main results in this paper. We first introduce Nevanlinna theory on an angular domain, which can be found in [5, pp. 23–26]: Let f be a meromorphic function on the angular domain $\overline{\Omega}(\alpha, \beta) = \{z : \alpha \leq \arg z \leq \beta\}$, where $\alpha, \beta \in [0, 2\pi]$ and so $0 \leq \beta - \alpha < 2\pi$. Following Nevanlinna in [5, pp. 23–26], we define

$$A_{\alpha,\beta}(r,f) = \frac{\omega}{\pi} \int_1^r \left(\frac{1}{t^\omega} - \frac{t^\omega}{r^{2\omega}} \right) \{\log^+ |f(te^{i\alpha})| + \log^+ |f(te^{i\beta})|\} \frac{dt}{t}, \tag{1}$$

$$B_{\alpha,\beta}(r,f) = \frac{2\omega}{\pi r^\omega} \int_\alpha^\beta \log^+ |f(re^{i\theta})| \sin \omega(\theta - \alpha) d\theta \tag{2}$$

and

$$C_{\alpha,\beta}(r,f) = 2 \sum_{1<|b_m|<r} \left(\frac{1}{|b_m|^\omega} - \frac{|b_m|^\omega}{r^{2\omega}} \right) \sin \omega(\theta_m - \alpha) \tag{3}$$

respectively, where $\omega = \pi/(\beta - \alpha),\ 1 \leq r < +\infty$ and $b_m = |b_m| e^{i\theta_m}$ are the poles of f on $\overline{\Omega}(\alpha, \beta)$ appearing often according to their multiplicities. $C_{\alpha,\beta}$ is called the angular counting function of the poles of f on $\overline{X}(\alpha, \beta)$ and the Nevanlinna angular characteristic function is defined as $S_{\alpha,\beta}(r,f) = A_{\alpha,\beta}(r,f) + B_{\alpha,\beta}(r,f) + C_{\alpha,\beta}(r,f)$. Similarly, for any finite value a, we define $A_{\alpha,\beta}(r,f_a),\ B_{\alpha,\beta}(r,f_a),\ C_{\alpha,\beta}(r,f_a)$, and $S_{\alpha,\beta}(r,f_a)$, where $f_a = 1/(f - a)$.

Lemma 2.1 ([5, pp. 23–26] and [5, Theorem 3.1]) *Let f be meromorphic on $\overline{\Omega}(\alpha,\beta)$. Then, for arbitrary complex number $a \in \mathbb{C}$ we have $S_{\alpha,\beta}\left(r, \frac{1}{f-a}\right) = S_{\alpha,\beta}(r,f) + O(1)$.*

Lemma 2.2 ([5, p.112, Theorem 3.3]) *Let f be meromorphic on $\overline{\Omega}_{(}\alpha,\beta)$. Then for arbitrary q distinct values $a_j \in \mathbb{C}\cup\{\infty\}$ $(1 \le j \le q)$ we have $(q-2)S_{\alpha,\beta}(r,f) \le \sum_{j=1}^{q} \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-a_j}\right) + R_{\alpha,\beta}(r,f)$, where $R_{\alpha,\beta}(r,f)$ is such a quantity that if f is a meromorphic function in the complex plane satisfying $\rho(f) < \infty$, then $R_{\alpha,\beta}(r,f) = O(1)$, as $r \to \infty$, if f is a meromorphic function in the complex plane satisfying $\rho(f) = \infty$, then $R_{\alpha,\beta}(r,f) = O(\log(rT(r,f)))$, as $r \notin E$ and $r \to \infty$, where and in what follows, E denotes a set of positive real numbers with finite linear measure. It is not necessarily the same for every occurrence in the context.*

Lemma 2.3 ([3]) *Let f be a meromorphic function with $\delta(\infty,f) = \delta > 0$. Then, given $\varepsilon > 0$, we have $mesE(r,f) > \frac{1}{(T(r,f))^{\varepsilon}(\log r)^{1+\varepsilon}}$, $\quad r \notin F$, where $E(r,f) = \left\{\theta \in [-\pi,\pi) : \log^+ |f(re^{i\theta})| > \frac{\delta}{4}T(r,f)\right\}$. Here F is a set of positive real numbers with finite logarithmic measure depending on ε.*

Lemma 2.4 ([6]) *Let f be a transcendental meromorphic function in $\mathbb{C}$. Then, for each $K > 1$, there exists a set $M(K)$ of the lower logarithmic density at most $d(K) = 1 - (2e^{K-1} - 1)^{-1} > 0$, that is $\underline{\log\mathrm{dens}}M(K) = \liminf_{r\to\infty} \frac{1}{\log r}\int_{M(K)\cap[1,r]} \frac{dt}{t} \le d(K)$, such that, for every positive integer k, we have $\limsup_{\substack{r\to\infty\\ r\notin M(K)}} \frac{T(r,f)}{T(r,f^{(k)})} \le 3eK$.*

Lemma 2.5 ([4] or [7]) *Let f be transcendental and meromorphic in $\mathbb{C}$ with the lower order $0 \le \mu < \infty$ and the order $0 < \rho < \infty$. Then for arbitrary positive number σ satisfying $\mu \le \sigma \le \rho$ and a set E with finite linear measure, there exist a sequence of positive numbers $\{r_n\}$ such that (i) $r_n \notin E$ and $\lim_{n\to\infty} \frac{r_n}{n} = \infty$, (ii) $\liminf_{r\to\infty} \frac{\log T(r_n,f)}{\log r_n} \ge \sigma$ and (iii) $T(t,f) < (1+o(1))\left(\frac{t}{r_n}\right)^{\sigma} T(r_n,f)$.*

A sequence $\{r_n\}$ satisfying (i)–(iii) in Lemma 2.5 is called a Pólya peak of order σ outside E in this paper. For $r > 0$ and $a \in \mathbb{C}$, we define (iv) $D(r,a) := \left\{\theta \in [-\pi,\pi) : \log^+ \frac{1}{|f(re^{i\theta})-a|} > \frac{1}{\log r}T(r,f)\right\}$ and $D(r,\infty) := \left\{\theta \in [-\pi,\pi) : \log^+ |f(re^{i\theta})| > \frac{1}{\log r}T(r,f)\right\}$.

Lemma 2.6 ([1]) *Let f be transcendental and meromorphic in $\mathbb{C}$ with the finite lower order μ and the order $0 < \rho \le \infty$, and for some $a \in \mathbb{C}\cup\{\infty\}$, $\delta(a,f) = \delta > 0$. Then for arbitrary Pólya peak $\{r_n\}$ of order $\sigma > 0$, $\mu \le \sigma \le \rho$, we have $\liminf_{n\to\infty} mesD(r_n,a) \ge \min\left\{2\pi, \frac{4}{\sigma}\arcsin\sqrt{\frac{\delta}{2}}\right\}$.*

Remark 2.7 Lemma 2.6 was proved in [1] for the Pólya peak of order μ, the same argument of Baernstein [1] can derive Lemma 2.6 for the Pólya peak of order σ, $\mu \le \sigma \le \rho$. Next we consider q real numbers α_j $(1 \le j \le q)$ satisfying

$$-\pi \le \alpha_1 < \alpha_2 < \cdots < \alpha_q < \pi, \quad \alpha_{q+1} = \alpha_1 + 2\pi, \tag{4}$$

and define

$$\omega = \max\left\{\frac{\pi}{\alpha_2-\alpha_1}, \frac{\pi}{\alpha_3-\alpha_2}, \ldots, \frac{\pi}{\alpha_{q+1}-\alpha_q}\right\}. \tag{5}$$

Lemma 2.8 *Let f and g be two distinct transcendental meromorphic functions such that for some $a \in \mathbb{C}\cup\{\infty\}$ we have $\delta = \delta(a,f) > 0$. Assume that q radii* $\arg z = \alpha_j$ $(1 \le j \le q)$ *satisfies (4), and assume that f and g share $a_1,\ a_2,\ a_3,\ a_4$ IM in $X = \mathbb{C}\setminus\bigcup_{j=1}^{q}\{z : \arg z = \alpha_j\}$, where $a_1,\ a_2,\ a_3,\ a_4$ are four distinct values in the extended complex plane such that $a \notin \{a_1,a_2,a_3,a_4\}$. Then $\rho(f) \le \frac{\pi}{\min\limits_{1\le j\le q}\{\alpha_{j+1}-\alpha_j\}}$.*

Proof Without loss of generality, we suppose that $a_1, a_2, a_3, a_4 \in \mathbb{C}$. Then, by the assumption $f \not\equiv g$, Lemmas 2.1 and 2.2 we deduce

$$S_{\alpha_j,\alpha_{j+1}}(r,f) \le S_{\alpha_j,\alpha_{j+1}}(r,g) + R_{\alpha_j,\alpha_{j+1}}(r,f), \quad 1 \le j \le q,$$
$$S_{\alpha_j,\alpha_{j+1}}(r,g) \le S_{\alpha_j,\alpha_{j+1}}(r,f) + R_{\alpha_j,\alpha_{j+1}}(r,g), \quad 1 \le j \le q.$$

Therefore,

$$S_{\alpha_j,\alpha_{j+1}}(r,f) = S_{\alpha_j,\alpha_{j+1}}(r,g), \quad R_{\alpha_j,\alpha_{j+1}}(r,f) = R_{\alpha_j,\alpha_{j+1}}(r,g), \quad 1 \le j \le q. \tag{6}$$

Again by (6) and Lemma 2.2 we deduce

$$\begin{aligned} &A_{\alpha_j,\alpha_{j+1}}\left(r,\frac{1}{f-a}\right) + B_{\alpha_j,\alpha_{j+1}}\left(r,\frac{1}{f-a}\right) \\ &= R_{\alpha_j,\alpha_{j+1}}\left(r,\frac{1}{f-a}\right) \le O(\log r + \log T(r,f)), \quad 1 \le j \le q, \end{aligned} \tag{7}$$

as $r \notin E$ and $r \to \infty$. Next we prove $\mu(f) < \infty$. Indeed, for the exceptional set F in Lemma 2.3 and the exceptional set E in (7), we have $\overline{\log\text{dens}}(F\cup E) = 0$, and so for $M(K)$ in Lemma 2.4, where $K \ge 2$ is a positive number, we have $\overline{\log\text{dens}}(M(K)\cup F\cup E) \le \overline{\log\text{dens}}(M(K)) \le d(K)$, here $d(K) = 1-(2e^{K-1}-1)^{-1}$. Applying this and Lemma 2.3 to f, we can find that there exist a sequence of positive numbers $r_n \notin M(K)\cup F\cup E$ such that $\text{meas}\,E\left(r_n,\frac{1}{f-a}\right) > \frac{1}{(T(r_n,f))^{\varepsilon}(\log r_n)^{1+\varepsilon}}$, as $r_n \to \infty$. Set $\varepsilon_n = \frac{1}{2q+1}\frac{1}{(T(r_n,f))^{\varepsilon}(\log r_n)^{1+\varepsilon}}$. Then,

$$\begin{aligned} &\text{meas}\left(E\left(r_n,\frac{1}{f-a}\right)\cap\bigcup_{j=1}^{q}(\alpha_j+\varepsilon_n,\alpha_{j+1}-\varepsilon_n)\right) \ge \text{meas}E\left(r_n,\frac{1}{f-a}\right) \\ &-\text{meas}\left(\bigcup_{j=1}^{q}(\alpha_j-\varepsilon_n,\alpha_j+\varepsilon_n)\right) > (2q+1)\varepsilon_n - 2q\varepsilon_n = \varepsilon_n, \end{aligned}$$

which implies that there exists some j_0 satisfying $1 \le j_0 \le q$ such that

$$\operatorname{meas}\left(E\left(r_n, \frac{1}{f-a}\right) \cap \left(\alpha_{j_0} + \varepsilon_n, \alpha_{j_0+1} - \varepsilon_n\right)\right) \ge \frac{\varepsilon_n}{q}. \tag{8}$$

Without loss of generality, we can assume that (8) holds for all n. Next we set

$$\tilde{E}_n = E\left(r_n, \frac{1}{f-a}\right) \cap \left(\alpha_{j_0} + \varepsilon_n, \alpha_{j_0+1} - \varepsilon_n\right). \tag{9}$$

By (9) and the definition of $E\left(r_n, \frac{1}{f-a}\right)$ we have

$$\begin{aligned} \int_{\alpha_{j_0}+\varepsilon_n}^{\alpha_{j_0+1}-\varepsilon_n} \log^+ \frac{1}{|f(r_n e^{i\theta}) - a|} d\theta &\ge \int_{\tilde{E}_n} \log^+ \frac{1}{|f(r_n e^{i\theta}) - a|} d\theta \\ &\ge \operatorname{meas}(\tilde{E}_n) \frac{\delta(a,f)}{4} T(r_n,f) \ge \frac{\varepsilon_n \delta(a,f)}{4q} T(r_n,f). \end{aligned} \tag{10}$$

On the other hand, by (7), Lemmas 2.1, 2.4 and the definition of $B_{\alpha,\beta}(r,f)$ in (2) we have

$$\begin{aligned} \int_{\alpha_{j_0}+\varepsilon_n}^{\alpha_{j_0+1}-\varepsilon_n} \log^+ \frac{1}{|f(r_n e^{i\theta}) - a|} d\theta &\le \frac{\pi r_n^{\omega_{j_0}} B_{\alpha_{j_0},\alpha_{j_0+1}}\left(r_n, \frac{1}{f(r_n e^{i\theta})-a}\right)}{2\omega_{j_0} \sin(\varepsilon_n \omega_{j_0})} \\ &\le \tilde{K}_{j_0,\varepsilon} r_n^{\omega_{j_0}} \log(r_n T(r_n,f)) = \tilde{K}_{j_0,\varepsilon} r_n^{\omega_{j_0}} (\log r_n + \log T(r_n,f)), \end{aligned} \tag{11}$$

as $r_n \notin M(K) \cup F \cup E$ and $r_n \to \infty$, where $\omega_{j_0} = \frac{\pi}{\alpha_{j_0+1} - \alpha_{j_0}}$, $\tilde{K}_{j_0,\varepsilon}$ is a positive constant depending only on j_0 and ε. By (10) and (11) we have

$$\delta(a,f)(T(r_n,f))^{1-\varepsilon} \le 4q(2q+1)\tilde{K}_{j_0,\varepsilon} r_n^{\omega_{j_0}} (\log r_n)^{1+\varepsilon} (\log(r_n T(r_n,f))) + O(1), \tag{12}$$

as $r_n \notin M(K) \cup F \cup E$ and $r_n \to \infty$. By (12) we derive $\mu(f) \le \omega_{j_0} \le \omega$, which implies $\mu(f) < \infty$. Combining this with (7), we will prove $\rho(f) \le \omega$. Suppose, on the contrary, that

$$\rho(f) > \omega. \tag{13}$$

Then, by (13) and the assumptions of Lemma 2.8 we have a contradiction. To do this, we consider the following two cases:

Case 1. Suppose that $\rho(f) > \mu(f)$. Then, by the fact $\sigma = \max\{\omega, \mu\}$ we have

$$\rho(f) > \sigma \ge \mu(f). \tag{14}$$

By (4), (5) and (14) we can find some sufficiently small positive number ε such that

$$\sum_{j=1}^{q}(\alpha_{j+1}-\alpha_j)+2\varepsilon>\frac{4}{\sigma+2\varepsilon}\arcsin\sqrt{\frac{\delta}{2}},\quad \rho(f)>\sigma+2\varepsilon>\mu(f). \tag{15}$$

Applying Lemma 2.5 to f, we can find that there exists a Pólya peak of order $\sigma+2\varepsilon$ outside E. Combining this with Lemma 2.6 and

$$\sigma+2\varepsilon\geq\omega+2\varepsilon\geq\omega_j+2\varepsilon\geq1+2\varepsilon, \tag{16}$$

we have

$$\mathrm{meas}D(r_n,a)\geq\frac{4}{\sigma+2\varepsilon}\arcsin\sqrt{\frac{\delta}{2}}-\varepsilon. \tag{17}$$

Without loss of generality, we can assume that (16) holds for all the n. Set

$$K_n=\mathrm{meas}\left(D(r_n,a)\cap\bigcup_{j=1}^{q}(\alpha_j+\varepsilon,\alpha_{j+1}-\varepsilon)\right). \tag{18}$$

Then, by (15), (17), and (18) we have

$$K_n\geq\mathrm{meas}D(r_n,a)-\mathrm{meas}\left([0,2\pi)\setminus\bigcup_{j=1}^{q}(\alpha_j+\varepsilon,\alpha_{j+1}-\varepsilon)\right)=\mathrm{meas}D(r_n,a)$$
$$-\mathrm{meas}\left(\bigcup_{j=1}^{q}(\alpha_{j+1}-\varepsilon,\alpha_{j+1}+\varepsilon)\right)=\mathrm{meas}D(r_n,a)-2q\varepsilon\geq\varepsilon. \tag{19}$$

By (19) we can find that there exists some positive integer j_0 satisfying $1\leq j_0\leq q$ such that for infinitely many positive integers n, we have

$$\mathrm{meas}\left(D(r_n,a)\cap(\alpha_{j_0}+\varepsilon,\alpha_{j_0+1}-\varepsilon)\right)\geq\frac{K_n}{q}>\frac{\varepsilon}{q}. \tag{20}$$

Without loss of generality, we can assume that (20) holds for all the positive integers n. Next we set $E_n=D(r_n,a)\cap(\alpha_{j_0}+\varepsilon,\alpha_{j_0+1}-\varepsilon)$. Then, by (20) and the definition of $D(r,a)$ in (iv) of Lemma 2.5 we have

$$\int_{\alpha_{j_0}+\varepsilon}^{\alpha_{j_0+1}-\varepsilon}\log^{+}\frac{1}{|f(r_ne^{i\theta})-a|}d\theta\geq\frac{T(r_n,f)}{\log r_n}\mathrm{meas}E_n>\frac{\varepsilon}{q}\frac{T(r_n,f)}{\log r_n}. \tag{21}$$

On the other hand, by (7), Lemmas 2.1, 2.4 and the definition of $B_{\alpha,\beta}(r,f)$ in (2) we have

$$\int_{\alpha_{j_0}+\varepsilon}^{\alpha_{j_0+1}-\varepsilon} \log^+ \frac{1}{|f(r_n e^{i\theta})-a|} d\theta \le \frac{\pi r_n^{\omega_{j_0}}}{2\omega_{j_0}\sin(\varepsilon\omega_{j_0})} B_{\alpha_{j_0},\alpha_{j_0+1}}\left(r_n, \frac{1}{f(r_n e^{i\theta})-a}\right)$$
$$\le K_{j_0,\varepsilon} r_n^{\omega_{j_0}} \log(r_n T(r_n,f)) = K_{j_0,\varepsilon} r_n^{\omega_{j_0}} (\log r_n + \log T(r_n,f)), \tag{22}$$

where $r_n \notin E$ and $\omega_{j_0} = \frac{\pi}{\alpha_{j_0+1}-\alpha_{j_0}}$, $K_{j_0,\varepsilon}$ is a positive constant depending only on j_0 and ε. By (21) and (22) we have

$$\log T(r_n,f) \le \log\log T(r_n,f) + \omega_{j_0} \log r_n + 3\log\log r_n + O(1), \tag{23}$$

where $r_n \notin E$ and $r_n \to \infty$. Noting that $\{r_n\}$ is a Pólya peak of order $\sigma + 2\varepsilon$ of f outside E, we can get by (23) that $\sigma + 2\varepsilon \le \lim\limits_{r_n\to\infty} \frac{\log T(r_n,f)}{\log r_n} \le \omega_{j_0} \le \omega$, which contradicts the assumption $\sigma = \max\{\omega, \mu\}$, and so we have the conclusion of Lemma 2.8.

Case 2. Suppose that $\rho(f) = \mu(f)$. By the same argument as in Case 1 with all $\sigma + 2\varepsilon$ replaced with $\sigma = \mu(f) = \rho(f)$, we can derive $\rho(f) = \sigma \le \omega$, which contradicts (13). This completes the proof of Lemma 2.8. □

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The Minimal Cycles over Isolated Brieskorn Complete Intersection Surface Singularities

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Abstract In this paper, we study a complete intersection surface singularity of Brieskorn type, and provide a condition for the coincidence of fundamental cycle and minimal cycle on minimal resolution space.

Keywords Cyclic quotient singularities • Fundamental cycle • Minimal cycle • Surface singularities

Mathematics Subject Classification (2010) Primary 14J17; 32S25

1 Introduction

Let (X, o) be a germ of a normal complex surface singularity and $\pi : (\widetilde{X}, E) \to (X, o)$ be a resolution, where $E = \pi^{-1}(o)$ denotes the exceptional divisor. Let $E = \bigcup_{i=1}^{r} E_i$ be the irreducible decomposition of E. A formal sum $D = \sum_{i=1}^{r} d_i E_i$ ($d_i \in \mathbb{Z}$) is called a cycle on E. For any effective cycle D (i.e., $d_i \geq 0$ for any i) on E, the

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arithmetic genus $p_a(D)$ of D is defined by

$$p_a(D) = 1 + \frac{1}{2}(D^2 + K_{\widetilde{X}}D), \tag{1}$$

where $K_{\widetilde{X}}$ is the canonical divisor on $\widetilde{X}$. If B, C are cycles, we have

$$p_a(B + C) = p_a(B) + p_a(C) - 1 + BC. \tag{2}$$

The fundamental cycle Z_E is the smallest one among the cycles $F > 0$ such that $FE_i \leq 0$ for every E_i of E. The arithmetic genus of Z_E is called the fundamental genus of (X, o) and denoted by $p_f(X, o)$. The minimal cycle A on E is the smallest one among the cycles $D > 0$ such that $p_a(D) = p_a(Z_E),\ D \leq Z_E$. Clearly, we always have $A \leq Z_E$. It sometimes happens that $A = Z_E$. This equality holds on the minimal resolution for minimal Kulikov singularities (cf. [6]), hypersurface singularities of Brieskorn type with certain conditions (cf. [7]). However, even for a particular class of singularities, a more systematic study will be required in order to classify when such a coincidence of important cycles occurs.

In this paper, we consider a germ $(W, o) \subset (\mathbb{C}^4, o)$ of an isolated complete intersection surface singularity of Brieskorn type defined by

$$W = \{(x_i) \in \mathbb{C}^4 | q_{j1}x_1^{a_1} + q_{j2}x_2^{a_2} + q_{j3}x_3^{a_3} + q_{j4}x_4^{a_4} = 0,\ j = 3, 4\},$$

where $a_i \geq 2$ are integers. By Serre's criterion for normality, (W, o) is a normal surface singularity. The aim of this paper is to give a condition for the coincidence of the fundamental cycle and the minimal cycle over (W, o).

This paper is organized as follows. In Sect. 2, we mention fundamental facts on cycles over a cyclic quotient singularity, and the minimal cycles over normal surface singularities. In Sect. 3, we consider the minimal cycle over (W, o) and give a condition for the coincidence of the fundamental cycle and the minimal cycle on the minimal resolution space.

2 Preliminaries

For $1 \leq i \leq 4$, we define positive integers $d_{i4},\ n_{i4}$ and e_{i4} as follows:

$$d_{i4} := \mathrm{lcm}(a_1, \ldots, \hat{a}_i, \ldots, a_4), n_{i4} := \frac{a_i}{\gcd(a_i, d_{i4})}, e_{i4} := \frac{d_{i4}}{\gcd(a_i, d_{i4})}.$$

(The symbol ^ in the definition of d_{i4} indicates an omitted term.) In addition, we define integers μ_{i4} by the condition:

$$e_{i4}\mu_{i4} + 1 \equiv 0 \pmod{n_{i4}},\ 0 \leq \mu_{i4} < n_{i4}.$$

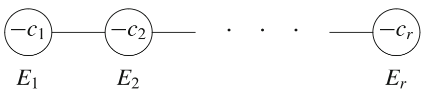
Furthermore, for $1 \le i \le 4$, we define integers $\hat{g}$ and $\hat{g}_i$ as follows:

$$\hat{g} := \frac{a_1 \cdots a_4}{\mathrm{lcm}(a_1, \dots, a_4)}, \ \hat{g}_i := \frac{a_1 \cdots \hat{a}_i \cdots a_4}{\mathrm{lcm}(a_1, \dots, \hat{a}_i, \dots, a_4)}.$$

2.1 Cyclic Quotient Singularities

For any $x \in \mathbb{R}$, we put $\lfloor x \rfloor = \max\{n \in \mathbb{Z} | n \le x\}$, and $\lceil x \rceil = \min\{n \in \mathbb{Z} | n \ge x\}$. For integers $c_i \ge 2, \ i = 1, \dots, r$, we put $[[c_1, \dots, c_r]] := c_1 - \cfrac{1}{c_2 - \cfrac{1}{\ddots - \cfrac{1}{c_r}}}$.

Let n and μ be positive integers that are relatively prime and $\mu < n$. Let ϵ_n denote the primitive n-th root of unity $\exp(2\pi\sqrt{-1}/n)$. Then the singularity of the quotient $\mathbb{C}^2 \Big/ \left\langle \begin{pmatrix} \epsilon_n & 0 \\ 0 & \epsilon_n^{\mu} \end{pmatrix} \right\rangle$ is called the cyclic quotient singularity of type $C_{n,\mu}$. It is known (cf. [1]) that if $E = \bigcup_{i=1}^{r} E_i$ is the exceptional divisor of the minimal resolution of $C_{n,\mu}$, then $E_i \simeq \mathbb{P}^1$ and the weighted dual graph of E is chain-shaped as the following picture, where $n/\mu = [[c_1, \dots, c_r]]$.



Lemma 2.1 ([2, Lemma 1.2]) *Let $e_i = [[c_i, \dots, c_r]]$. Take a positive integer λ_0 and define the sequence $\{\lambda_i\}_{i=0}^{r}$ by the recurrence formula $\lambda_i = \lceil \lambda_{i-1}/e_i \rceil$ for $1 \le i \le r$. Take relatively prime positive integers n_i and μ_i satisfying $n_i/\mu_i = e_i$ for $1 \le i \le r$. Put $\lambda_{r+1} := \lambda_r c_r - \lambda_{r-1}$.*

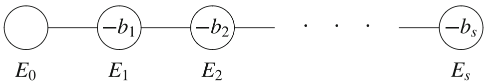
1. *If $\lambda_{i-1} = \lambda_i c_i - \lambda_{i+1}$ holds for $1 \le i \le r$, then $\lambda_1 = (\mu_1 \lambda_0 + \lambda_{r+1})/n_1$.*
2. *If either $\lambda_0 \equiv 0 \pmod{n_1}$ or $\mu_1 \lambda_0 + 1 \equiv 0 \pmod{n_1}$, then $\lambda_{i-1} = \lambda_i c_i - \lambda_{i+1}$ holds for $1 \le i \le r$. Furthermore, $\lambda_{r+1} = 0$ when $\lambda_0 \equiv 0 \pmod{n_1}$, and $\lambda_{r+1} = 1$ when $\mu_1 \lambda_0 + 1 \equiv 0 \pmod{n_1}$.*
3. *If $\lambda_0 \equiv 0 \pmod{n_1}$, then $\lambda_r = \lambda_0/n_1$. If $\mu_1 \lambda_0 + 1 \equiv 0 \pmod{n_1}$, then $\lambda_r = \lceil \lambda_0/n_1 \rceil$.*

2.2 Minimal Cycles over Normal Surface Singularities

Let (X, o) be a normal complex surface singularity, $\pi : (\widetilde{X}, E) \to (X, o)$ be a resolution of (X, o), where $\pi^{-1}(o) = E = \bigcup_{i=1}^{r} E_i$ is the decomposition of E. Let Z_E be the fundamental cycle on E.

Definition 2.2 ([7, Definition 1.2]) Let A be a cycle on E satisfying $0 < A \le Z_E$. Suppose $p_f(X, o) \ge 1$. Then A is said to be a minimal cycle on E if $p_a(A) = p_f(X, o)$ and $p_a(D) < p_f(X, o)$ for any cycle D with $D < A$.

The existence and the uniqueness of the minimal cycle A can be shown as in [3]. Suppose that $E = \bigcup_{i=0}^{N} E_i$ whose dual graph is star-shaped with central curve E_0. We consider a cyclic branch $\bigcup_{i=1}^{s} E_i$ with $E_0 \bigcap E_1 \neq \emptyset$. Suppose that the weighted dual graph of $\bigcup_{i=1}^{s} E_i$ is as the following picture, where $E_i^2 = -b_i$ for $i = 1, \ldots, s$.



Let d, e be positive integers and $d/e = [[b_1, \ldots, b_s]]$ satisfying $\gcd(d, e) = 1$. Let $c_0 = d$, $c_1 = e$ and $c_2, c_3, \ldots, c_s$ be the integers which are inductively defined by the relation $c_{i+1} = b_i c_i - c_{i-1}$ $(1 \le i \le s-1)$, then $c_s = 1$ by Lemma 2.1. Let l, μ be the integers defined by $\mu d - el = 1, 0 < \mu < d$. Then $l/\mu = [[b_1, \ldots, b_{s-1}, b_s - 1]]$. Put $\gamma_0 = l$, $\gamma_1 = \mu$ and define $\gamma_2, \ldots, \gamma_s$ inductively by $\gamma_i = b_{i-1}\gamma_{i-1} - \gamma_{i-2}$ $(i = 2, \ldots, s)$. Then we have the following two lemmas.

Lemma 2.3 ([7, Lemma 3.2]) *Suppose that the coefficient of E_0 in Z_E is dt, where t is a positive integer. Then the coefficient of E_i in Z_E is given by tc_i, $i = 1, \ldots, s$. In particular, $Z_E E_i = 0$ for $i = 1, \ldots, s$.*

Lemma 2.4 ([7, Lemma 3.3]) *If the coefficient of E_0 in Z_E is l, then the coefficient of E_i in Z_E is given by γ_i, $1 \le i \le s$. In particular, $Z_E E_i = 0$ for $i = 1, \ldots, s-1$ and $Z_E E_s = -1$. Furthermore, if $\lfloor \frac{d}{l} \rfloor = 1$, then $b_s \ge 3$.*

3 The Minimal Cycle over (W, o)

In this section, we consider the minimal cycle over (W, o) defined as in Sect. 1, and provide a condition for the coincidence of the fundamental cycle and the minimal cycle on the minimal resolution space.

Let $\pi : (\widetilde{W}, E) \to (W, o)$ be the minimal good resolution of (W, o). Let $\alpha_i := n_{i4}$, $\beta_i := \mu_{i4}$ for $i = 1, \ldots, 4$ and $d_4 = \mathrm{lcm}(a_1, a_2, a_3, a_4)$.

Let $l = \gcd(a_1, a_2, a_3, a_4)$, $l_i = \gcd(a_1, \ldots, \hat{a}_i, \ldots, a_4)/l$ for $i = 1, 2, 3, 4$. Then the Demazure's divisor F is given as follows (cf. (4.3) in [7]):

$$F = F_0 - \sum_{i=1}^{4} \sum_{j=1}^{ll_i} \frac{\beta_i}{\alpha_i} P_{ij}, P_{ij} \in E_0, \tag{3}$$

where F_0 is a divisor on E_0 such that $\mathcal{O}_{E_0}(F_0)$ is the restriction to E_0 of the conormal sheaf of E_0 in $\widetilde{W}$ and $\deg D_0 = \frac{l}{\alpha_1\alpha_2\alpha_3\alpha_4} + \sum_{i=1}^{4} \frac{ll_i\beta_i}{\alpha_i}$ following Theorem 3.6.1 in [5].

Let $[kF]$ be a $\mathbb{Z}$-coefficient divisor on E_0 defined by:

$$[kF] := kF_0 - \sum_{i=1}^{4}\sum_{j=1}^{ll_i}\left\lceil \frac{k\beta_i}{\alpha_i}\right\rceil P_{ij},\ P_{ij}\in E_0, \tag{4}$$

where k is a non-negative integer. Let m be an integer defined by $\min\{k \in \mathbb{Z} | \deg[kF] \geq 0\}$. Thus, due to Theorem 3.1 in [7] and Riemann-Roch Theorem, we have $p_f(W,o) = m(g-1) + 1 - \sum_{k=1}^{m-1} \deg[kF]$.

Theorem 3.1 *If $a_4 \geq \max\{a_1, a_2, a_3\}$, then $m = \min\{\alpha_1\alpha_2\alpha_3\alpha_4, \alpha_1\alpha_2\alpha_3 l_4\}$.*

Proof Assume $\alpha_4 \geq l_4$, we show that $m = \alpha_1\alpha_2\alpha_3 l_4$. From (3) and (4), $\deg[kF] = l\left\{\frac{k}{\alpha_1\alpha_2\alpha_3\alpha_4} - \sum_{i=1}^4 l_i\left(\left\lceil\frac{k\beta_i}{\alpha_i}\right\rceil - \frac{k\beta_i}{\alpha_i}\right)\right\}$. Since $\alpha_1\alpha_2\alpha_3 l_4\beta_4 + 1 \equiv 0 \pmod{\alpha_4}$, it is easy to see that

$$\deg[\alpha_1\alpha_2\alpha_3 l_4 F] = l\left\{\frac{\alpha_1\alpha_2\alpha_3 l_4}{\alpha_1\alpha_2\alpha_3\alpha_4} - l_4\left\lceil\frac{\alpha_1\alpha_2\alpha_3 l_4\beta_4}{\alpha_4}\right\rceil + \frac{l_4\alpha_1\alpha_2\alpha_3 l_4\beta_4}{\alpha_4}\right\} = 0.$$

Hence it suffices to prove that $\deg[kF] < 0$ for any k with $0 < k < \alpha_1\alpha_2\alpha_3 l_4$. If α_4 does not divide k with $k < \alpha_1\alpha_2\alpha_3 l_4$, then

$$\frac{1}{l}\deg[kF] = \frac{k}{\alpha_1\alpha_2\alpha_3\alpha_4} - \sum_{i=1}^{4} l_i\left(\left\lceil\frac{k\beta_i}{\alpha_i}\right\rceil - \frac{k\beta_i}{\alpha_i}\right) < 0.$$

In other words, suppose $\alpha_4 | k$, let $k = \alpha_4 t$ for some positive integer t, then

$$\frac{1}{l}\deg[kF] = \frac{t}{\alpha_1\alpha_2\alpha_3} - \sum_{i=1}^{3} l_i\left(\left\lceil\frac{\alpha_4\beta_i t}{\alpha_i}\right\rceil - \frac{\alpha_4\beta_i t}{\alpha_i}\right).$$

If $(\alpha_1\alpha_2\alpha_3)|t$, then $(\alpha_1\alpha_2\alpha_3\alpha_4)|k$ following the fact that $(\alpha_i, \alpha_j) = 1$ for $i \neq j$, which contradicts the inequality $k < \alpha_1\alpha_2\alpha_3 l_4$. Suppose that α_j does not divide t for some $j \in \{1, 2, 3\}$, then

$$\begin{aligned}\frac{1}{l}\deg[kF] &\leq \frac{t}{\alpha_1\alpha_2\alpha_3} - l_j\left(\left\lceil\frac{\alpha_4\beta_j t}{\alpha_j}-\right\rceil - \frac{\alpha_4\beta_j t}{\alpha_j}\right) \leq \frac{t}{\alpha_1\alpha_2\alpha_3} - \frac{l_j}{\alpha_j}\\ &= \frac{\alpha_4 t - l_j\prod_{i=1,i\neq j}^4\alpha_i}{\alpha_1\alpha_2\alpha_3\alpha_4} = \frac{k - l_j\prod_{i=1,i\neq j}^4\alpha_i}{\alpha_1\alpha_2\alpha_3\alpha_4}.\end{aligned}$$

Following the assumption $a_j \leq a_4$ $(j = 1, 2, 3)$, $\alpha_1\alpha_2\alpha_3 l_4 \leq l_j\prod_{i=1,i\neq j}^4\alpha_i$. It follows that $k - l_j\prod_{i=1,i\neq j}^4\alpha_i < 0$, which implies that $\deg[kF] < 0$.

In the next, we show that $m = \alpha_1\alpha_2\alpha_3\alpha_4$ if $l_4 > \alpha_4$. Since $\deg[\alpha_1\alpha_2\alpha_3\alpha_4 F] = l > 0$, it suffices to prove that $\deg[kF] < 0$ for any k with $0 < k < \alpha_1\alpha_2\alpha_3\alpha_4$.

Following the above proof, we know that for the integer k, α_i does not divide k for some $i \in \{1, 2, 3, 4\}$. Without loss of generality, we may assume that α_1 does not divide k. Then

$$\frac{1}{l}\deg\,[kF] = \frac{k}{\alpha_1\alpha_2\alpha_3\alpha_4} - \sum_{i=1}^{4} l_i \left(\left\lceil \frac{k\beta_i}{\alpha_i} \right\rceil - \frac{k\beta_i}{\alpha_i} \right) \leq \frac{k - \alpha_2\alpha_3\alpha_4 l_1}{\alpha_1\alpha_2\alpha_3\alpha_4}.$$

Since $\alpha_4 < l_4$ and $a_1 \leq a_4$, we have $k < \alpha_1\alpha_2\alpha_3\alpha_4 < \alpha_1\alpha_2\alpha_3 l_4 \leq \alpha_2\alpha_3\alpha_4 l_1$. It follows that $\deg[kF] < 0$. Thus we obtain the assertion. □

Theorem 3.2 ([4, Theorem 5.1]) *Let* $\epsilon_{w,\nu} = [[c_{w,\nu}, \ldots, c_{w,s_w}]]$ *if* $s_w > 0$, *and* $Z_E = \theta_0 E_0 + \sum_{w=1}^{4}\sum_{\nu=1}^{s_w}\sum_{\xi=1}^{\hat{g}_w} \theta_{w,\nu,\xi} E_{w,\nu,\xi}$. *Then* θ_0 *and the sequence* $\{\theta_{w,\nu,\xi}\}$ *are determined by the following:* $\theta_{w,0,\xi} := \theta_0 := \min\,(e_{44}, \alpha_1\alpha_2\alpha_3\alpha_4)$, $\theta_{w,\nu,\xi} = \lceil \theta_{w,\nu-1,\xi}/\epsilon_{w,\nu} \rceil$ $(1 \leq \nu \leq s_w)$.

Theorem 3.3 *Let* $\pi' : (\hat{W}, E) \to (W, o)$ *be the minimal resolution. Assume* $\mathrm{lcm}(a_1, a_2, a_3) \leq a_4 < 2 \cdot \mathrm{lcm}(a_1, a_2, a_3)$, *then* $Z_E = A$ *on* E.

Proof We need only to prove that $p_a(Z_E - E_i) < p_f(W, o)$ for any irreducible component E_i of E. By (1) and (2), we have $p_a(Z_E) = p_a(Z_E - E_i + E_i) = p_a(Z_E - E_i) + p_a(E_i) - 1 + (Z_E - E_i)E_i$, which implies that

$$p_a(Z_E - E_i) = p_a(Z_E) - p_a(E_i) + 1 - Z_E E_i + E_i^2. \tag{5}$$

Assume that π' is the minimal good resolution, then $E_0^2 \leq -2$ (or $E_0^2 = -1$ and $g(E_0) \geq 1$) and the weighted dual graph of the minimal resolution of (W, o) is given as in Figure 3 in [4]. Let B be any irreducible component of $E - E_0 - \bigcup_{w=1}^{4}(\bigcup_{\xi=1}^{\hat{g}_w} E_{w,s_w,\xi})$, by Lemma 2.1, Theorem 3.2, and (5), we have $Z_E B = 0$ and $p_a(Z_E - B) < p_f(W, o)$. Since $\mathrm{lcm}(a_1, a_2, a_3) \leq a_4$, $e_{44} \leq \alpha_4 \leq \alpha_1\alpha_2\alpha_3\alpha_4$. From Theorem 3.2, the coefficient of E_0 in Z_E is e_{44}. It follows from Theorem 3.2, Lemma 2.3 and Lemma 2.1 (3) that for $w \in \{1, 2, 3, 4\}$ and $\xi \in \{1, \ldots, \hat{g}_w\}$,

$$Z_E E_{w,s_w,\xi} = \begin{cases} 0 & \text{if } w \neq 4, \\ -1 & \text{if } w = 4. \end{cases}$$

Since $\mathrm{lcm}(a_1, a_2, a_3) \leq a_4 < 2 \cdot \mathrm{lcm}(a_1, a_2, a_3)$, we have $e_{44} \leq \alpha_4 < 2e_{44}$, which implies $\lfloor \frac{\alpha_4}{e_{44}} \rfloor = 1$. Following Lemma 2.4, we have $\left(E_{4,s_4,\xi}\right)^2 < -2$, $\xi \in \{1, \ldots, \hat{g}_4\}$. Then by (5), we have $p_a(Z_E - E_{w,s_w,\xi}) < p_f(W, o)$. From Theorem 4.4 in [4],

$$-Z_E E_0 = c_0 e_{44} - \sum_{w=1}^{3} \frac{\hat{g}_w e_{44} \beta_w}{\alpha_w} - \frac{\hat{g}_4(e_{44}\beta_4 + 1)}{\alpha_4} = e_{44}\hat{g}/d_4 - \hat{g}_4/\alpha_4 = 0.$$

Therefore, by (5), $p_a(Z_E - E_0) = p_a(Z_E) - g(E_0) + 1 + E_0^2 < p_f(W, o)$.

In other words, suppose that the minimal resolution does not coincide with the minimal good resolution. Let $\pi := \phi \circ \pi' : (\bar{W}, \bar{E}) \xrightarrow{\phi} (\hat{W}, E) \xrightarrow{\pi'} (W, o)$ be the minimal good resolution, where ϕ is a birational morphism obtained by iterating monoidal transforms centered at a point. We may assume that E has at least two irreducible components, otherwise $Z_E = A$ obviously. So it suffices to show that $p_a(Z_E - E_i) < p_f(W, o)$ for any $E_i \subset E$. Assume that $p_a(Z_E - E_i) = p_f(W, o)$ for some $E_i \subset E$. It follows from Lemma 1.4 in [7] that E_i is a smooth rational curve and

$$Z_E E_i = (Z_E - E_i + E_i)E_i = (Z_E - E_i)E_i + E_i^2 = 1 + E_i^2.$$

Since E_i is smooth, $g(E_i) = 0$. By (1) and the adjunction formula $K_{\hat{W}} E_i = -E_i^2 + 2g(E_i) - 2$ for any $E_i \subset E$, where $K_{\hat{W}}$ is the canonical divisor on $\hat{W}$,

$$\begin{aligned} p_a(Z_E - E_i) - p_a(Z_E) = 1 + \frac{1}{2}\left((Z_E - E_i)^2 + K_{\hat{W}}(Z_E - E_i)\right) \\ + 1 + \frac{1}{2}(Z_E^2 + K_{\hat{W}} Z_E) = -1 - Z_E E_i = 0, \end{aligned}$$

which implies $Z_E E_i = -1$. Thus $E_i^2 = -2$. Let $\bar{E}_i$ be the proper transform of E_i by ϕ. Then $Z_E E_i = Z_{\bar{E}} \bar{E}_i = -1$ by (0.2.2) in [8], which implies that $\bar{E}_i = E_{4,s_4,\xi}$, $\xi \in \{1, \ldots, \hat{g}_4\}$, and the coefficient of $\bar{E}_i$ in $Z_{\bar{E}}$ is equal to 1 by Lemma 2.4. From Proposition 2.9 in [8], the coefficient of E_i in Z_E is equal to 1. It follows that there exists only one irreducible component $E_j \subset E$ that intersects E_i transversely, which implies that ϕ doesn't contain any monoidal transform centered at a point of E_i. Then $E_{4,s_4,\xi}^2 = \bar{E}_i^2 = E_i^2 = -2$, which contradicts Lemma 2.4. Hence the assertion holds. □

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Properties and Characteristics of ε Starlike Functions

Ming-Sheng Liu

Abstract In this note, the author investigates some properties and characteristics of subclass $S^*(U, \varepsilon)$ of ε starlike functions on the unit disk U. In particular, several sufficient criteria for ε starlike functions are provided. From these, we may construct many concrete ε starlike functions on U. Furthermore, a covering theorem is also provided. Some results, presented in this paper, generalize the related results of earlier authors.

Keywords Convex function • Starlike function • ε starlike function

Mathematics Subject Classification (2010) Primary 30C45

1 Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of functions f normalized by $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, which are analytic in the open unit disk $U = \{z \in C : |z| < 1\}$. Let $H(U)$ denote the subclass of $\mathcal{A}$ consisting of functions with $f'(z) \neq 0$ on U. Suppose also that $S^*,\ K$ denote the familiar subclasses of $\mathcal{A}$ consisting of functions which are, respectively, starlike in U and convex in U.

For some recent investigations of these and related function classes, see (for example) the works by Liu et al. [6, 7].

Gong and Liu [1, 2, 4] introduced the notion of ε starlike mappings on C^n, in purpose to treat the family of convex mappings and the family of starlike mappings as one family. Moreover, they [2] also introduced the notion of ε starlike mappings on the complex Banach space, and they gave a criterion for the family of ε starlike mappings on the unit ball B in the complex Banach space. Liu and Zhu [5] proposed the notion of ε quasi-convex mapping on the unit ball B of the complex Banach

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space, in purpose to treat the family of quasi-convex mappings and the family of starlike mappings as one family. We also extended the notion of ε starlike mappings on B from $\varepsilon \in [0, 1]$ to $\varepsilon \in [-1, 1]$.

Setting $X = C, B = U$, both ε starlike mappings and ε quasi-convex mappings are reduced to the following ε starlike functions:

Definition 1.1 Suppose that $f \in H(U)$ and $\varepsilon \in [-1, 1]$. If

$$\mathrm{Re}\Big[\frac{f(z) - \varepsilon f(\xi z)}{zf'(z)}\Big] \geq 0, \ \forall z \in U, \forall \xi \in \overline{U}.$$

Then $f(z)$ is said to be an ε starlike function on U, and let $S^*(U, \varepsilon)$ denote the class of all ε starlike functions on U.

From Theorem 4.1 in [5], we get the geometric explanation of ε starlike function: if $\varepsilon \in [-1, 1]$ and $f \in H(U)$, then f is ε starlike if and only if $\varepsilon f(U) \subset f(U)$ and $f(U)$ is starlike with respect to every point in $\varepsilon f(U)$. This provides a new way to describe the starlike function family how to transit to convex function family, which is different from that of α-convex functions.

From Theorems 3.1–3.3 in [5], we have the following inclusion relations and estimates of coefficient of ε starlike functions on U.

Theorem A ([5])

(1) $S^*(U, \varepsilon) \subset S^*$ *for each* $\varepsilon \in [-1, 1]$*;*
(2) If $1 \geq \varepsilon_1 > \varepsilon_2 \geq 0$ *or* $-1 \leq \varepsilon_1 < \varepsilon_2 \leq 0$*, then* $S^*(U, \varepsilon_1) \subset S^*(U, \varepsilon_2)$*;*

In particular, $K \subset S^*(U, \varepsilon) \subset S^*$ for each $\varepsilon \in [0, 1]$.

Theorem B ([5]) *Suppose* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in S^*(U, \varepsilon)$ *and* $\varepsilon \in [-1, 1]$.

(1) If $0 \leq \varepsilon \leq 1$*, then* $|a_2| \leq \dfrac{2(1 + \varepsilon)}{1 + 3\varepsilon}$.
(2) If $-1 \leq \varepsilon < 0$*, then* $|a_2| \leq \dfrac{2}{\sqrt{1 - \varepsilon}}$.

In our present sequel to the aforementioned works, we investigate several other properties and characteristics of ε starlike functions on the unit disk U. In particular, a covering theorem is proven here. Several sufficient criteria for ε starlike functions are also provided. From these, we may construct many concrete ε starlike functions on U.

2 Main Results

In this section, we first derive a necessary and sufficient criterion for ε starlike functions on the unit disk U, which is one of the main results in this paper.

Theorem 2.1 *Suppose $f \in H(U)$ and $|\varepsilon| < 1$. Then $f \in S^*(U,\varepsilon)$ if and only if f satisfies the following inequality*

$$\Big|f(z) - \varepsilon f(\xi z) - (1-\varepsilon\xi) zf'(z)\Big| \le \Big|f(z) - \varepsilon f(\xi z) + (1-\varepsilon\overline{\xi}) zf'(z)\Big|, \tag{1}$$

for all $z \in U, \forall \xi \in \overline{U}$.

Proof Fix $\varepsilon \in (-1,1)$ and $\xi \in \overline{U}$, then we have $1 - \mathrm{Re}(\varepsilon\xi) > 0$.

Since $f \in H(U)$, we get that $f'(z) \neq 0$ for $z \in U$. Then the inequality (1) is equivalent to

$$\Big|\frac{f(z) - \varepsilon f(\xi z)}{zf'(z)} - (1-\varepsilon\xi)\Big| \le \Big|\frac{f(z) - \varepsilon f(\xi z)}{zf'(z)} + (1-\varepsilon\overline{\xi})\Big|, \quad \forall z \in U\backslash\{0\}, \forall \xi \in \overline{U},$$

or

$$\begin{aligned}&\Big|\frac{1}{1-\mathrm{Re}(\varepsilon\xi)}\Big[\frac{f(z) - \varepsilon f(\xi z)}{zf'(z)} + i\mathrm{Im}(\varepsilon\xi)\Big] - 1\Big| \\ &\quad\le \Big|\frac{1}{1-\mathrm{Re}(\varepsilon\xi)}\Big[\frac{f(z) - \varepsilon f(\xi z)}{zf'(z)} + i\mathrm{Im}(\varepsilon\xi)\Big] + 1\Big|,\end{aligned}$$

for all $z \in U\backslash\{0\}$ and $\xi \in \overline{U}$, which is equivalent to

$$\mathrm{Re}\frac{f(z) - \varepsilon f(\xi z)}{zf'(z)} = \mathrm{Re}\Big[\frac{f(z) - \varepsilon f(\xi z)}{zf'(z)} + i\mathrm{Im}(\varepsilon\xi)\Big] \ge 0, \quad \forall z \in U\backslash\{0\}, \forall \xi \in \overline{U}.$$

Evidently, we have

$$\mathrm{Re}\frac{f(z) - \varepsilon f(\xi z)}{zf'(z)}\Big|_{z=0} = \mathrm{Re}(1-\varepsilon\xi) \ge 1 - |\varepsilon| \ge 0, \quad \forall \xi \in \overline{U}.$$

Hence we conclude the conclusion of this theorem by Definition 1.1. □

Next, we derive several sufficient criteria for ε starlike functions on the unit disk U.

Theorem 2.2 *Let $|\varepsilon| < 1$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ be an analytic function on U. If f satisfies*

$$\sum_{k=2}^{\infty}[(1+|\varepsilon|)k + |\varepsilon|]|a_k| \le 1 - |\varepsilon|. \tag{2}$$

Then $f \in S^*(U,\varepsilon)$.

Proof Since $f'(z) = 1 + \sum_{k=2}^{\infty} ka_k z^{k-1}$, we have

$$|f'(z)| > 1 - \frac{1}{1+|\varepsilon|}\sum_{k=2}^{\infty}[(1+|\varepsilon|)k + |\varepsilon|]|a_k| \geq 1 - \frac{1-|\varepsilon|}{1+|\varepsilon|} = \frac{2|\varepsilon|}{1+|\varepsilon|} \geq 0$$

for all $z \in U$. Thus we get $f \in H(U)$.

On the other hand, by (2) and simple computation, we have

$$\begin{aligned}
&|f(z) - \varepsilon f(\xi z) - (1-\varepsilon\xi)zf'(z)| \\
&= |\sum_{k=2}^{\infty}(1 - \varepsilon\xi^k - k + k\varepsilon\xi)a_k z^k| \leq \sum_{k=2}^{\infty}[k - 1 + (k+1)|\varepsilon|]|a_k||z| \\
&\leq |z|\{2 - 2|\varepsilon| - \sum_{k=2}^{\infty}[k + 1 + (k+1)|\varepsilon|]|a_k|\} \\
&\leq |f(z) - \varepsilon f(\xi z) + (1-\varepsilon\overline{\xi})zf'(z)|, \forall z \in U,\ \xi \in \overline{U},
\end{aligned}$$

hence we conclude from Theorem 2.1 that $f \in S^*(U, \varepsilon)$. □

Notice that

$$k\left(|\varepsilon|k + 1 - \frac{|\varepsilon|}{2}\right) - [(1+|\varepsilon|)k + |\varepsilon|] = [k\left(k - \frac{3}{2}\right) - 1]|\varepsilon| \geq 0,$$

for $k \geq 2$, we get the following corollary from Theorem 2.2.

Corollary 2.3 *Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ be an analytic function on U and $|\varepsilon| < 1$. If f satisfies*

$$\sum_{k=2}^{\infty} k\left(|\varepsilon|k + 1 - \frac{|\varepsilon|}{2}\right)|a_k| \leq 1 - |\varepsilon|, \tag{3}$$

then $f \in S^*(U, \varepsilon)$.

Remark 2.4 Setting $\varepsilon = 0$ in Theorem 2.2 or Corollary 2.3, we get the related result in [3].

Notice that the inequality $\sum_{k=2}^{\infty} k^2|a_k| \leq 1$ implies the analytic function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in K$. We propose the following conjecture:

Conjecture 2.5 Suppose $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is an analytic function on U and $|\varepsilon| \leq 1$. If f satisfies $\sum_{k=2}^{\infty} k(|\varepsilon|k + 1 - |\varepsilon|)|a_k| \leq 1$, then $f \in S^*(U, \varepsilon)$.

Theorem 2.6 *Suppose $f \in H(U)$ and $|\varepsilon| \leq 1$. If f satisfies the following inequality*

$$|f(z) - \varepsilon f(\xi z) - zf'(z)| \leq |zf'(z)|, \quad \forall z \in U, \forall \xi \in \overline{U}, \tag{4}$$

then $f \in S^*(U, \varepsilon)$.

Proof Since $f \in H(U)$, we get that $f'(z) \neq 0$ for $z \in U$. Combining this fact with (4), we obtain

$$\left|\frac{f(z) - \varepsilon f(\xi z)}{zf'(z)} - 1\right| \leq 1, \quad \forall z \in U \backslash \{0\}, \forall \xi \in \overline{U}.$$

This implies that

$$\mathrm{Re}\frac{f(z) - \varepsilon f(\xi z)}{zf'(z)} \geq 1 - 1 = 0, \quad \forall z \in U \backslash \{0\}, \forall \xi \in \overline{U}.$$

Evidently, we have

$$\mathrm{Re}\frac{f(z) - \varepsilon f(\xi z)}{zf'(z)}\bigg|_{z=0} = \mathrm{Re}(1 - \varepsilon \xi) \geq 1 - |\varepsilon| \geq 0, \quad \forall \xi \in \overline{U}.$$

Hence by Definition 1.1, we get that $f \in S^*(U, \varepsilon)$. The proof is complete. □

Theorem 2.7 *Let $|\varepsilon| \leq 1$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ be an analytic function on U. If f satisfies*

$$\sum_{k=2}^{\infty}(2k - 1 + |\varepsilon|)|a_k| \leq 1 - |\varepsilon|. \tag{5}$$

then $f \in S^*(U, \varepsilon)$.

Proof Since $f'(z) = 1 + \sum_{k=2}^{\infty} ka_k z^{k-1}$, we have

$$|f'(z)| \geq 1 - \sum_{k=2}^{\infty} k|a_k| \geq 1 - \frac{2}{3+|\varepsilon|} \sum_{k=2}^{\infty}(2k-1+|\varepsilon|)|a_k|$$
$$\geq 1 - \frac{2-2|\varepsilon|}{3+|\varepsilon|} = \frac{1+3|\varepsilon|}{3+|\varepsilon|} > 0$$

for all $z \in U$. Thus we get $f \in H(U)$.

On the other hand, by (5) and simple computation, we have

$$|f(z) - \varepsilon f(\xi z) - zf'(z)| \leq \left(|\varepsilon| + \sum_{k=2}^{\infty}(k-1+|\varepsilon|)|a_k|\right)|z|$$
$$\leq |z|\left(1 - \sum_{k=2}^{\infty} k|a_k|\right) \leq |zf'(z)|$$

for all $z \in U,\ \xi \in \overline{U}$, we conclude from Theorem 2.6 that $f \in S^*(U, \varepsilon)$. □

Notice that for $k \geq 2$, we have

$$(2k-1+|\varepsilon|) \geq (1+|\varepsilon|)k + |\varepsilon| \Leftrightarrow |\varepsilon| \leq 1 - \frac{1}{k},$$

combining Theorem 2.2 with Theorem 2.7, we get the following corollary.

Corollary 2.8 *Let* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ *be an analytic function on* U *and* $|\varepsilon| < 1$. *If* f *satisfies*

$$\sum_{k=2}^{\infty}[(1+|\varepsilon|)k + |\varepsilon|]|a_k| \leq 1 - |\varepsilon| \quad \text{for} |\varepsilon| \leq \frac{1}{2},$$
$$\sum_{k=2}^{\infty}(2k-1+|\varepsilon|)|a_k| \leq 1 - |\varepsilon| \quad \text{for} \frac{1}{2} < |\varepsilon| < 1,$$

then $f \in S^*(U, \varepsilon)$.

Now we construct some concrete ε starlike functions on U by applying the above sufficient criteria.

Example 1 Let $|\varepsilon| \leq 1/2$ and $f(z) = z + \sum_{k=2}^{\infty} \frac{1-|\varepsilon|}{2^{k-1}((1+|\varepsilon|)k+|\varepsilon|)} z^k$. Then $f \in S^*(U, \varepsilon)$.

Example 2 Let $\frac{1}{2} < |\varepsilon| < 1$ and $f(z) = z + \sum_{k=2}^{\infty} \frac{1-|\varepsilon|}{2^{k-1}(2k-1+|\varepsilon|)} z^k$. Then $f \in S^*(U, \varepsilon)$.

Finally, we establish a covering theorem for ε starlike functions on U, which extends the covering theorem of S^* in [3].

Theorem 2.9 (Covering Theorem) *Suppose that* $\varepsilon \in [-1, 1]$ *and* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in S^*(U, \varepsilon)$. *Then the unit disk U is mapped by $f(z)$ onto a domain that contains the disk $|w| < r_\varepsilon$, where*

$$r_\varepsilon = \begin{cases} \frac{1+3\varepsilon}{4(1+2\varepsilon)}, & 0 \le \varepsilon \le 1, \\ \frac{\sqrt{1-\varepsilon}}{2+2\sqrt{1-\varepsilon}}, & -1 \le \varepsilon < 0. \end{cases} \tag{6}$$

Proof Suppose w_0 is any complex number such that $f(z) \neq w_0$ for $z \in U$, then $w_0 \neq 0$, and

$$\frac{w_0 f(z)}{w_0 - f(z)} = z + (a_2 + \frac{1}{w_0})z^2 + \cdots$$

is univalent in U by Theorem A, this leads to $\left| a_2 + \frac{1}{w_0} \right| \le 2$.

On the other hand, by Theorem B, we get that $|w_0| \ge \frac{1}{2+|a_2|} \ge r_\varepsilon$, and the proof is complete. □

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Generalized Integration Operators from $Q_K(p, q)$ to the Little Zygmund-Type Spaces

Yongmin Liu and Yanyan Yu

Abstract Let φ be an analytic self-map of the unit disk $\mathbb{D}$, $H(\mathbb{D})$ the space of analytic functions on $\mathbb{D}$, $g \in H(\mathbb{D})$ and $n \in \mathbb{N}$. The present paper continues the line of research in Ren (Appl Math Comput 236:27–32, 2014). The boundedness and compactness of the generalized integration operator

$$C^n_{\varphi,g}f(z) = \int_0^z f^{(n)}(\varphi(\xi))g(\xi)d\xi, \ z \in \mathbb{D},$$

from $Q_K(p, q)$ space to the little Zygmund-type space are obtained.

Keywords Generalized integration operators • Little Zygmund-type space • $Q_K(p,q)$ space

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1 Introduction

Let $\mathbb{D}$ be the unit disk in the finite complex plane $\mathbb{C}$, $\partial\mathbb{D}$ boundary of $\mathbb{D}$, $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$, $\mathbb{N}_0$ the set of all nonnegative integers and $\mathbb{N}$ the set of all positive integers.

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Let μ be a normal function on $[0, 1)$ (see, for example, [10]). Let $\mathcal{Z}_\mu$ denote the space of all $f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ such that

$$\sup_{z\in\mathbb{D}} \mu(|z|)|f''(z)| < \infty.$$

Under the norm

$$\|f\|_{\mathcal{Z}_\mu} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} \mu(|z|)|f''(z)|, \tag{1}$$

it is easy to see that $\mathcal{Z}_\mu$ is a Banach space.

The little Zygmund-type space $\mathcal{Z}_{\mu,0}$ is defined to be the subspace of $\mathcal{Z}_\mu$ consisting of those $f \in \mathcal{Z}_\mu$ such that

$$\lim_{|z|\to 1} \mu(|z|)|f''(z)| = 0.$$

It is easy to see that $\mathcal{Z}_{\mu,0}$ is a closed subspace of $\mathcal{Z}_\mu$ and the set of all polynomials is dense in $\mathcal{Z}_{\mu,0}$.

Let $p > 0, q > -2$ and K be a nonnegative nondecreasing function on $[0, \infty)$. Throughout the paper we assume that

$$\int_0^1 (1 - r^2)^q K(-\log r) r dr < \infty.$$

For the definition of the space $Q_K(p, q)$ and $Q_{K,0}(p, q)$, see [8].

In this paper, we consider an integration operator $C^n_{\varphi,g}$ which is defined as

$$C^n_{\varphi,g} f(z) = \int_0^z f^{(n)}(\varphi(\xi))g(\xi)d\xi, \; z \in \mathbb{D},$$

where $n \in \mathbb{N}_0$. This operator is called the generalized integral operator, which was introduced in [9] and studied in [9, 11]. Also, the operator $C^n_{\varphi,g}$ is a generalization of the Riemann-Stieltjes operator I_g induced by g, defined as

$$I_g f(z) = \int_0^z f(\xi)g'(\xi)d\xi, \; z \in \mathbb{D}.$$

Some related results see, for example, [1, 2, 4, 5, 7–9]. Motivated by the results [1, 2, 5, 8, 14], we consider the boundedness and compactness of the operators $C^n_{\varphi,g}$ from $Q_K(p, q)$ (or $Q_{K,0}(p, q)$) to the little Zygmund-type space.

Now we are in a position to characterize the boundedness and compactness of $C^n_{\varphi,g} : Q_K(p, q)$ (or $Q_{K,0}(p, q)$) $\to \mathcal{Z}_{\mu,0}$.

2 The Boundedness and Compactness of $C^n_{\varphi,g}$ from $Q_K(p,q)$ (or $Q_{K,0}(p,q)$) to the Little Zygmund-Type Space

First, we study the boundedness of the operator $C^n_{\varphi,g} : Q_{K,0}(p,q) \to \mathcal{Z}_{\mu,0}$.

Theorem 2.1 *Let $p > 0; q > -2$ and K be a nonnegative nondecreasing function on $[0,\infty)$ such that*

$$\int_0^1 K(-\log r)(1-r)^{\min\{-1,-q\}}\left(\log\frac{1}{1-r}\right)^{\chi_{-1}(q)} rdr < \infty. \tag{2}$$

Assume that φ is an analytic self-map of $\mathbb{D}$, $g \in H(\mathbb{D})$, μ is normal, and n is a positive integer. Then the operator $C^n_{\varphi,g} : Q_{K,0}(p,q) \to \mathcal{Z}_{\mu,0}$ is bounded if and only if the operator $C^n_{\varphi,g} : Q_{K,0}(p,q) \to \mathcal{Z}_{\mu}$ is bounded,

$$\lim_{|z|\to 1} \mu(|z|)|g'(z)| = 0, \tag{3}$$

and

$$\lim_{|z|\to 1} \mu(|z|)\left|\varphi'(z)g(z)\right| = 0. \tag{4}$$

Proof Now assume that $C^n_{\varphi,g} : Q_{K,0}(p,q) \to \mathcal{Z}_{\mu,0}$ is bounded, then $C^n_{\varphi,g} : Q_{K,0}(p,q) \to \mathcal{Z}_{\mu}$ is bounded, and $C^n_{\varphi,g}f \in \mathcal{Z}_{\mu,0}$, $\forall f \in Q_{K,0}(p,q)$. Note that

$$\left|\mu(|z|)(C^n_{\varphi,g}f)''(z)\right| = \mu(|z|)\left|f^{(n+1)}(\varphi(z))\varphi'(z)g(z) + f^{(n)}(\varphi(z))g'(z)\right|.$$

By taking the function $f(z) = \frac{z^n}{n!} \in Q_{K,0}(p,q)$, it follows that

$$\lim_{|z|\to 1} \mu(|z|)|g'(z)| = 0,$$

that is (3) follows. By taking the function $f(z) = \frac{z^{n+1}}{(n+1)!} \in Q_{K,0}(p,q)$, we have

$$\lim_{|z|\to 1} \mu(|z|)|\varphi'(z)g(z) + \varphi(z)g'(z)| = 0, \tag{5}$$

from (3), (5) and the boundedness of φ, we get

$$\lim_{|z|\to 1} \mu(|z|)|\varphi'(z)g(z)| = 0,$$

(4) holds.

Conversely, assume that the operator $C_{\varphi,g}^n : Q_{K,0}(p,q) \to \mathcal{Z}_\mu$ is bounded, (3) and (4) hold. By Ren [8, Theorem 1] we have

$$M_1 := \sup_{z\in\mathbb{D}} \frac{\mu(|z|)|g'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}} < \infty$$

and

$$M_2 := \sup_{z\in\mathbb{D}} \frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n+1}} < \infty.$$

It from [12, Theorem 2.2] and [13, Proposition 8] follows that for any $\epsilon > 0$, there exists a $\delta \in (0;1)$, such that $\delta < |z| < 1$ implies

$$\mu(|z|)|g'(z)| < \epsilon/2, \tag{6}$$

$$\mu(|z|)\left|\varphi'(z)g(z)\right| < \epsilon/2, \tag{7}$$

$$(1-|z|^2)^{\frac{2+q-p}{p}+n}\left|f^{(n)}(z)\right| < \frac{\epsilon}{2M_1}, \tag{8}$$

and

$$(1-|z|^2)^{\frac{2+q-p}{p}+n+1}\left|f^{(n+1)}(z)\right| < \frac{\epsilon}{2M_2}, \tag{9}$$

for each function $f \in Q_{K,0}(p,q)$. Hence writing $\mathbb{D}_1 = \{z \in \mathbb{D} : \delta < |z| < 1\}$, using (6) and (7), we deduce that for each function $f \in Q_{K,0}(p,q)$

$$\begin{aligned}
&\sup_{\{z\in\mathbb{D}_1:|\varphi(z)|\le\delta\}} \left|\mu(|z|)(C_{\varphi,g}^n f)''(z)\right| \\
&= \sup_{\{z\in\mathbb{D}_1:|\varphi(z)|\le\delta\}} \mu(|z|)\left|f^{(n+1)}(\varphi(z))\varphi'(z)g(z) + f^{(n)}(\varphi(z))g'(z)\right| \\
&\le C\left(\mu(|z|)|g(z)||\varphi'(z)| + \mu(|z|)|g'(z)|\right) \\
&\le C\epsilon,
\end{aligned} \tag{10}$$

and using (8), (9) and [8, Theorem 1], we get that

$$\begin{aligned}
&\sup_{\{z\in\mathbb{D}_1:\delta<|\varphi(z)|<1\}} \left|\mu(|z|)(C_{\varphi,g}^n f)''(z)\right| \\
&= \sup_{\{z\in\mathbb{D}_1:\delta<|\varphi(z)|<1\}} \mu(|z|)\left|f^{(n+1)}(\varphi(z))\varphi'(z)g(z) + f^{(n)}(\varphi(z))g'(z)\right|
\end{aligned}$$

$$\leq \frac{\epsilon}{2} \sup_{\{z\in\mathbb{D}_1:\delta<|\varphi(z)|<1\}} \left(\frac{\mu(|z|)|g(z)||\varphi'(z)|}{M_2(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n+1}} + \frac{\mu(|z|)|g'(z)|}{M_1(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}} \right)$$
$$\leq \epsilon. \tag{11}$$

From (10) and (11), we get that $C^n_{\varphi,g}f \in \mathcal{Z}_{\mu,0}$ for each function $f \in Q_{K,0}(p,q)$. The boundedness of the operator $C^n_{\varphi,g} : Q_{K,0}(p,q) \to \mathcal{Z}_\mu$ implies that the operator $C^n_{\varphi,g} : Q_{K,0}(p,q) \to \mathcal{Z}_{\mu,0}$ is bounded. □

Secondly, we characterize the compactness of the operator $C^n_{\varphi,g} : Q_K(p,q) \to \mathcal{Z}_{\mu,0}$.

Theorem 2.2 *Let $p > 0; q > -2$ and K be a nonnegative nondecreasing function on $[0,\infty)$ such that (2) holds. Assume that φ is an analytic self-map of $\mathbb{D}$, $g \in H(\mathbb{D})$, μ is normal and n is a positive integer. Then the following statements are equivalent.*

(i) *The operator $C^n_{\varphi,g} : Q_K(p,q) \to \mathcal{Z}_{\mu,0}$ is compact;*
(ii) *The operator $C^n_{\varphi,g} : Q_{K,0}(p,q) \to \mathcal{Z}_{\mu,0}$ is compact;*
(iii)

$$\lim_{|z|\to 1} \frac{\mu(|z|)|g'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}} = 0, \tag{12}$$

and

$$\lim_{|z|\to 1} \frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n+1}} = 0. \tag{13}$$

Proof $(iii) \Rightarrow (ii)$. Assume that conditions (12) and (13) hold. Then $C^n_{\varphi,g} : Q_K(p,q) \to \mathcal{Z}_\mu$ is bounded by Ren [8, Theorem 1]. Using [12, Theorem 2.2] and [13, Proposition 8], we get for any $f \in Q_K(p,q)$

$$\left|\mu(|z|)(C^n_{\varphi,g}f)''(z)\right| = \mu(|z|)\left|f^{(n+1)}(\varphi(z))\varphi'(z)g(z) + f^{(n)}(\varphi(z))g'(z)\right|$$
$$\leq C\left(\frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n+1}} + \frac{\mu(|z|)|g'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}} \right) \|f\|_{Q_K(p,q)}, \tag{14}$$

thus

$$\left|\mu(|z|)(C^n_{\varphi,g}f)''(z)\right| \to 0 \text{ as } |z| \to 1.$$

From this we see that $C^n_{\varphi,g}f \in \mathcal{Z}_{\mu,0}$ for each $f \in Q_K(p,q)$, it follows that $C^n_{\varphi,g} : Q_K(p,q) \to \mathcal{Z}_{\mu,0}$ is bounded. So $C^n_{\varphi,g} : Q_{K,0}(p,q) \to \mathcal{Z}_{\mu,0}$ is bounded. Taking the supremum in inequality (14) over all $f \in Q_{K,0}(p,q)$ such that $\|f\|_{Q_K(p,q)} \le 1$ and letting $|z| \to 1$ yields

$$\lim_{|z|\to 1} \sup_{\|f\|_{Q_K(p,q)}\le 1} \mu(|z|)|(C^n_{\varphi,g}f)''(z)| = 0.$$

Hence, by Li and Stević [3, Lemma 3.1] (also see [6, Lemma 1]), we see that the operator $C^n_{\varphi,g} : Q_{K,0}(p,q) \to \mathcal{Z}_{\mu,0}$ is compact.

$(ii) \Rightarrow (i)$. This implication is obvious.

$(i) \Rightarrow (iii)$. Now assume that $C^n_{\varphi,g} : Q_K(p,q) \to \mathcal{Z}_{\mu,0}$ is compact. Then $C^n_{\varphi,g} : Q_K(p,q) \to \mathcal{Z}_{\mu,0}$ is bounded. By the proof of Theorem 2.1, (3) and (4) hold. Since $f_{\varphi(z)},\ h_{\varphi(z)} \in Q_K(p,q)$, we have $C^n_{\varphi,g}f_{\varphi(z)}, C^n_{\varphi,g}h_{\varphi(z)} \in \mathcal{Z}_{\mu,0}$. Because $|\varphi(z)| \to 1$ implies $|z| \to 1$, we obtain

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(|z|)|g'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}} = 0, \tag{15}$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n+1}} = 0. \tag{16}$$

We only prove that (4) and (16) imply (13). The proof of (3) and (15) imply (12) is similar, hence it will be omitted.

From (16), it follows that for every $\epsilon > 0$, there exists $\delta \in (0,1)$ such that

$$\frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n+1}} < \epsilon, \tag{17}$$

when $\delta < |\varphi(z)| < 1$. Using (4) we see that there exists $\tau \in (0,1)$ such that

$$\mu(|z|)|\varphi'(z)g(z)| < \epsilon \inf_{t\in[0,\delta]} (1-t^2)^{\frac{2+q-p}{p}+n+1}, \tag{18}$$

when $\tau < |z| < 1$. Therefore, when $\tau < |z| < 1$ and $\delta < |\varphi(z)| < 1$, by (17) we have

$$\frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n+1}} < \epsilon. \tag{19}$$

On the other hand, when $\tau < |z| < 1$ and $|\varphi(z)| \leq \delta$, by (18) we obtain

$$\frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n+1}} \leq \frac{\mu(|z|)|\varphi'(z)g(z)|}{\inf\limits_{t\in[0,\delta]}(1-t^2)^{\frac{2+q-p}{p}+n+1}} < \epsilon. \tag{20}$$

From (19) and (20), we obtain (13), as desired. The proof is completed. □

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The Hyperbolic Metric on the Complement of the Integer Lattice Points in the Plane

Katsuhiko Matsuzaki

Abstract A domain in the plane obtained by removing all integer lattice points admits the hyperbolic metric, which is the rank 2 Abelian cover of the once-punctured square tours. We compare the hyperbolic metric of this domain with a scaled Euclidean metric in the complement of the cusp neighborhoods. They are quasi-isometric. We investigate the best possible quasi-isometry constant relying on numerical experiment by Mathematica.

Keywords Absolute norm • Continued fraction • Hyperbolic metric • Mathematica • Once-punctured torus • Quasi-isometry • Simple closed geodesic

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1 Euclidean Metric vs. Hyperbolic Metric

In this note, we consider metrics on a planar domain

$$\Omega = \mathbb{C} - \mathbb{Z} \times \mathbb{Z}.$$

Take the square torus $T = \mathbb{C}/\langle z \mapsto z+1, z \mapsto z+i \rangle$ and remove the point $[0]$ from T to make an once-punctured torus T^*. It admits a complete hyperbolic metric by the uniformization theorem. The universal cover $\mathbb{C} \to T$ with the deck transformation group $\mathbb{Z} \times \mathbb{Z}$ induces an Abelian cover $\pi : \Omega \to T^*$. The hyperbolic metric on Ω is defined by the pull-back of that on T^*. The Euclidean metric on Ω is the restriction of the Euclidean metric on $\mathbb{C}$ with scaling (defined later).

We compare these two metrics on the complement of the cusp neighborhoods. The hyperbolic metric gets much larger near to the punctures and there is no

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comparison there. For a given open neighborhood A of $[0] \in T^*$, set

$$T_0^* := T^* - A; \quad \Omega_0 := \Omega - \pi^{-1}(A).$$

The hyperbolic metric and the Euclidean metric on T_0^* are comparable (bi-Lipschitz) since T_0^* is compact, and so are on the covering space Ω_0. Hence the inner distances induced by the integration of these metrics along the paths in Ω_0 are also comparable. However, the distances we deal with are just the restriction of the hyperbolic and the Euclidean distances on Ω to Ω_0.

2 A Problem on the Optimal Quasi-Isometry Constant

We denote the hyperbolic distance and the Euclidean distance on Ω by d_H and d_E respectively, and use the same notation for their restriction to Ω_0.

Definition 2.1 For metric spaces (X, d_X) and (Y, d_Y) in general, a map $f : X \to Y$ is called a K-quasi-isometry ($K \geq 1$) if there is a constant $C \geq 0$ such that

$$\frac{1}{K}(d_X(x_1, x_2) - C) \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2) + C$$

for any $x_1, x_2 \in X$.

From the fact that any geodesic curve on (Ω, d_H) connecting any two points in Ω_0 cannot go deeply into the cusp, we see the following.

Proposition 2.2 *The identity map* $\mathrm{id} : (\Omega_0, d_H) \to (\Omega_0, d_E)$ *is a K-quasi-isometry with the constant* $C \geq 0$ *depending on the cusp neighborhood A.*

We try to find the best possible constant K in this proposition. Note that this is independent of the choice of the cusp neighborhood A. We put the following normalization. The Euclidean metric is scaled so that the length of the unit interval is equal to the hyperbolic length of the simple closed geodesic on the punctured torus T^* corresponding to the covering transformation $z \mapsto z + 1$ on Ω. (This also coincides with that for $z \mapsto z + i$ by the symmetry of the square torus.)

Due to the additive constant C, we can ignore small errors in distance without changing the quasi-isometry constant K. Hence we do not have to consider any two points in Ω_0 for the comparison of the distances. Only the following measurement is enough to determine K: for any coprime $p, q \in \mathbb{N}$, the distances between a fixed $z_0 \in \Omega_0$ and $z_0 + pi + q \in \Omega_0$. The Euclidean distance is simply given by $\sqrt{p^2 + q^2}$ without scaling and the hyperbolic distance is comparable with the hyperbolic length of the (p/q)-simple closed geodesic on the once-punctured torus T^*.

3 The Computation of Lengths of Simple Closed Geodesics

It is known that the hyperbolic length of the (p/q)-simple closed geodesic on T^*, which is denoted by Length(p/q), can be computed recursively by the trace identity from the lengths of (1/0)- and (0/1)-simple closed geodesics, which are $2\,\mathrm{arccosh}\,\sqrt{2}$ for the square torus. The information about how many times we should apply the recursive relations alternatively is represented by the coefficients of the regular continued fraction of p/q, which is

$$\frac{p}{q} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots \cfrac{\cdots}{a_{n-1} + \cfrac{1}{a_n}}}}} = [a_0, a_1, a_2, \ldots, a_{n-1}, a_n].$$

The idea of this algorithm can be found in Mumford et al. [2, Chap. 9]. For example, $30/13 = [2, 3, 4]$. Then, in the order of

$$\frac{0}{1} \overset{\frac{1}{0}}{\to} \frac{1}{1} \overset{\frac{1}{0}}{\to} \frac{2}{1}, \quad \frac{1}{0} \overset{\frac{2}{1}}{\to} \frac{3}{1} \overset{\frac{2}{1}}{\to} \frac{5}{2} \overset{\frac{2}{1}}{\to} \frac{7}{3}, \quad \frac{2}{1} \overset{\frac{7}{3}}{\to} \frac{9}{4} \overset{\frac{7}{3}}{\to} \frac{16}{7} \overset{\frac{7}{3}}{\to} \frac{23}{10} \overset{\frac{7}{3}}{\to} \frac{30}{13},$$

we derive the lengths of their simple closed geodesics.

To obtain the desired estimate, we consider when the supremum of

$$\frac{\mathrm{Length}(p/q)}{2\,\mathrm{arccosh}\,\sqrt{2} \cdot \sqrt{p^2 + q^2}}$$

is achieved, where $p, q \in \mathbb{N}$ run over coprime integers. At first, we expected that it should be when p/q tend to the golden ratio $\phi = (1 + \sqrt{5})/2 = [1, 1, 1, \ldots]$ and its inverse $\phi^{-1} = [0, 1, 1, \ldots]$.

However, a numerical experiment tells us that this expectation is false. As Fig. 1 by Mathematica shows, the supremum $1.082085\cdots$ is achieved when p/q converge to $1.62024\cdots$ and its inverse, which is slightly different from the golden ratio $\phi = 1.61803\cdots$.

We also observe the following state (Fig. 2) by Mathematica besides the fact we have mentioned above. In this note, statements of experimental results without rigorous proof are called "Claim."

Claim 3.1 *For a rational number p/q with coprime $p, q \in \mathbb{N}$, define a function*

$$h(p/q) := \frac{\mathrm{Length}(p/q)}{2\,\mathrm{arccosh}\,\sqrt{2} \cdot \sqrt{p^2 + q^2}}.$$

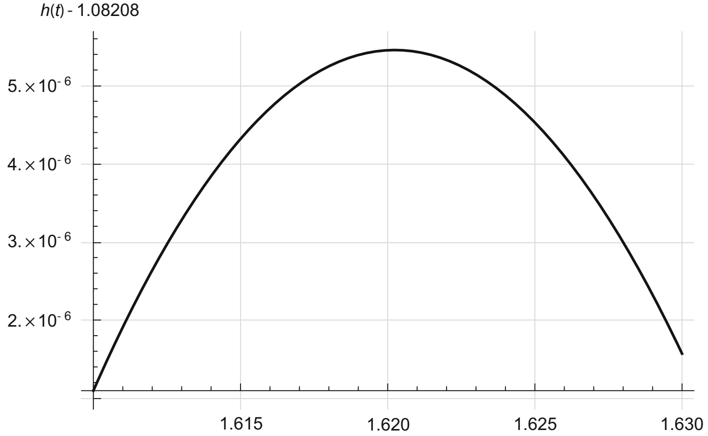

Fig. 1 Maximum of function $h(t)$

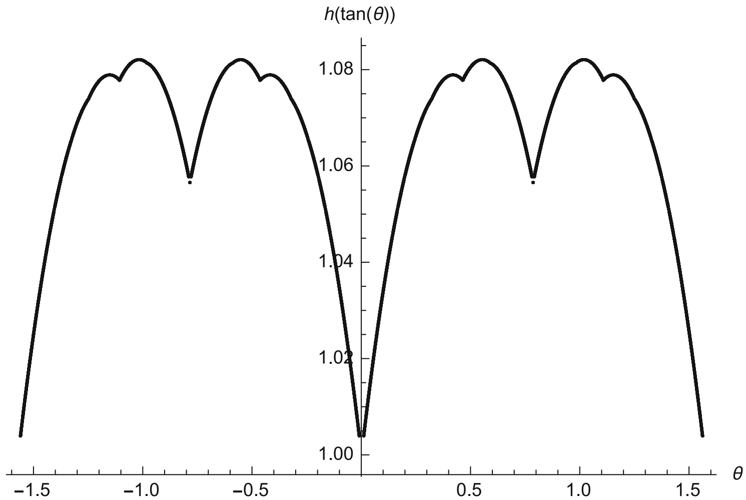

Fig. 2 Graph of $h(\tan(\theta))$ on $[-\pi/2, \pi/2]$ for square torus

Then h satisfies the following:

1. *h is bounded and extends continuously to any* $t \in \mathbb{R} \cup \{-\infty, \infty\}$*;*
2. *the range of h is* $1 \leq h(t) < 1.08209$.

By these numerical experiments, we can obtain the optimal quasi-isometry constant.

Claim 3.2 *There is a constant* $C \geq 0$ *depending on the cusp neighborhood A such that*

$$\widetilde{d}_E(z_1, z_2) - C \leq d_H(z_1, z_2) < 1.08209 \cdot \widetilde{d}_E(z_1, z_2) + C$$

for any $z_1, z_2 \in \Omega_0 = \Omega - \pi^{-1}(A)$*, where* $\widetilde{d}_E = (2 \operatorname{arccosh} \sqrt{2}) d_E$.

4 Absolute Norm and Rough-Isometry

We introduce a new real norm to $\mathbb{C} = \mathbb{R}^2$ by using the above function h, which is equivalent to the Euclidean norm $\|\cdot\|_2$. We use a general result concerning absolute norm.

For a positive continuous function $\varphi : [0, \pi/2] \to (0, \infty)$ with $\varphi(0) = \varphi(\pi/2) = 1$, we define

$$\|(x, y)\|_\varphi := \|(x, y)\|_2 \cdot \varphi(\arctan(y/x))$$

for every non-trivial $(x, y) \in \mathbb{R}^2$ with $x, y \geq 0$, and then extend it to $\mathbb{R}^2$ by $\|(x, y)\|_\varphi = \|(|x|, |y|)\|_\varphi$ and $\|(0, 0)\|_\varphi = 0$. The following fact is known by Bonsall and Duncan [1, Sect. 21, Lemma 3].

Proposition 4.1 *Under the above notation, $\|(x, y)\|_\varphi$ gives a real norm on $\mathbb{R}^2$ if and only if $\psi(t) = \|(1-t, t)\|_\varphi$ $(0 \leq t \leq 1)$ is a convex continuous function such that $\psi(0) = \psi(1) = 1$ and $\max\{1-t, t\} \leq \psi(t) \leq 1$.*

Now we set $\varphi(\theta) = h(\tan\theta)$ by using our function h. A numerical experiment by Mathematica gives the following graph (Fig. 3) of $\psi(t) = \|(1-t, t)\|_\varphi$, which satisfies the condition in the above proposition. Then $\|\cdot\|_{h\circ\tan}$ is a norm on $\mathbb{R}^2$.

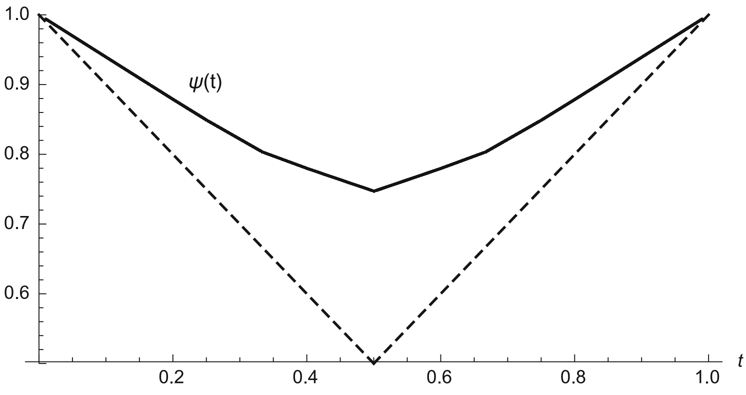

Fig. 3 Graph of $\psi(t) = \|(1-t, t)\|_\varphi$ for $\varphi(\theta) = h(\tan\theta)$

If the above observation is true, we will have another claim from the ones in the previous section.

Claim 4.2 *The hyperbolic distance d_H on Ω_0 is rough-isometric to the distance defined by the norm $(2\operatorname{arccosh}\sqrt{2})\|\cdot\|_{h\circ\tan}$, where rough-isometry means K-quasi-isometry for $K = 1$.*

5 Generalization: Another Example

We can consider the similar problem starting from a torus in general

$$T = \mathbb{C}/\langle z \mapsto z + 1, z \mapsto z + \tau \rangle$$

for $\tau \in \mathbb{H}$. A difficulty in this case is to describe explicitly the correspondence between τ and the hyperbolic structure on T^*. Here, we only deal with another special case: $\tau = (-1 + \sqrt{3}i)/2$.

In this case, our function h becomes

$$h(t) = \frac{\text{Length}(p/q)}{2\,\text{arccosh}(3/2) \cdot \sqrt{p^2 + q^2 - pq}} \qquad (t = \sqrt{3}p/(2q - p))$$

and its range is $1 \leq h(t) < 1.06453$ and the maximum is taken at $t \approx 0.42949$, 0.74692, 8.44047 and their symmetric points (Fig. 4).

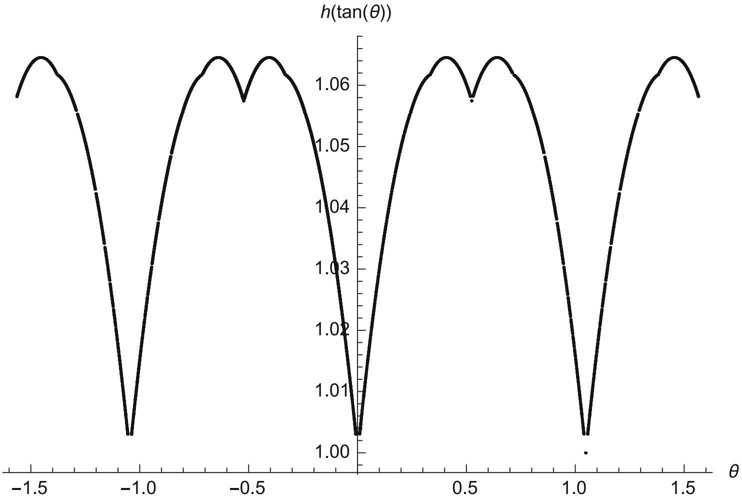

Fig. 4 Graph of $h(\tan(\theta))$ on $[-\pi/2, \pi/2]$ for equilateral torus

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Further Results of a Normality Criterion Related a Famous Hayman Conjecture

Wenjun Yuan, Fanning Meng, and Shengjiang Chen

Dedicated to Professor Yuzan He on the Occasion of his 80th Birthday

Abstract In this paper, we study the normality criterion related a famous Hayman conjecture, and get four normal criteria. Our results improve the related theorems which were obtained by Pang (Kexue Tongbao (in Chinese) 33(22):1690–1693, 1988), Schwick (J Anal Math 52:241–289, 1989) and Xu (J Math 21(4):381–386, 2001), respectively.

Keywords Higher derivative • Meromorphic function • Normal family • Shared value

Mathematics Subject Classification (2010) Primary 30D30; 30D45

1 Introduction and Main Results

The following normality criterion is a famous Hayman conjecture [3].

Theorem 1.1 *Let $\mathcal{F}$ be a family of holomorphic (meromorphic) functions defined in a domain D, $n \in \mathbb{N}, a \neq 0, b \in \mathbb{C}$. If $f'(z) + af^n(z) - b$ does not vanish in D for each function $f(z) \in \mathcal{F}$ and $n \geq 2 (n \geq 3)$, then $\mathcal{F}$ is normal in D.*

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Remark 1.2 Many mathematicians contributed above theorem, the detailed illustration can be found in [2].

If f' is substituted by $f^{(k)}$ in above theorem. Pang [4] and Schwick [5] got the following theorem, respectively.

Theorem 1.3 *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, $n, k \in \mathbb{N}, a \neq 0, b \in \mathbb{C}$. If $f^{(k)}(z) + af^n(z) - b$ does not vanish in D for each function $f(z) \in \mathcal{F}$ and $n \geq k + 4$, then $\mathcal{F}$ is normal in D.*

In 1995, Chen and Fang [1] proposed the following conjecture.

Conjecture CF *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, $k, n \in \mathbb{N}, a \neq 0, b \in \mathbb{C}$. If $f^{(k)}(z) - af^n(z) - b$ does not vanish in D for each function $f(z) \in \mathcal{F}$ and $n \geq k + 2$ and $k \geq 2$, then $\mathcal{F}$ is normal in D.*

In response to Conjecture CF, Xu [6] proved the following result.

Theorem 1.4 *Let $\mathcal{F}$ be a family of meromorphic functions in D, k and $n(\geq k + 2)$ be two positive integers. Let $a(\neq 0)$ and b be two finite complex numbers. If for every function $f \in \mathcal{F}$*

(i) *$f^{(k)} - af^n - b$ has no zero in D,*
(ii) *f has no simple pole in D,*

then $\mathcal{F}$ is normal in D.

In this paper, we relax conditions of Theorems 1.3 and 1.4, and get the following results.

Theorem 1.5 *Let D be a domain in $\mathbb{C}$ and let $\mathcal{F}$ be a family of meromorphic functions in D. Let $k, n \in \mathbb{N}^+$ such that $n \geq k + 5$, and let a and b be two finite complex numbers with $a \neq 0$. If $f^{(k)} - af^n - b$ has at most one distinct zero, and any zero of f has multiplicity at least k in D for every function $f \in \mathcal{F}$, then $\mathcal{F}$ is a normal family in D.*

Theorem 1.6 *Let D be a domain in $\mathbb{C}$ and let $\mathcal{F}$ be a family of meromorphic functions in D. Let $k, n \in \mathbb{N}^+$ such that $n \geq k + 3$ and $k \geq 2$, and let a and b be two finite complex numbers with $a \neq 0$. If for every function $f \in \mathcal{F}$,*

(i) *$f^{(k)} - af^n - b$ has at most one distinct zero,*
(ii) *f has no simple pole, and any zero of f has multiplicity at least k in D,*

then $\mathcal{F}$ is a normal family in D.

Theorem 1.7 *Let D be a domain in $\mathbb{C}$ and let $\mathcal{F}$ be a family of meromorphic functions in D. Let $k, n \in \mathbb{N}^+$ such that $n \geq k + 5$, and let a and b be two finite complex numbers with $a \neq 0$. Suppose that any zero of f has multiplicity at least k in D for every function $f \in \mathcal{F}$. If $f^{(k)} - af^n$ and $g^{(k)} - ag^n$ share the value b IM for every pair of functions (f, g) of $\mathcal{F}$, then $\mathcal{F}$ is a normal family in D.*

Theorem 1.8 *Let D be a domain in $\mathbb{C}$ and let $\mathcal{F}$ be a family of meromorphic functions in D. Let $k, n \in \mathbb{N}^+$ such that $n \geq k+3$ and $k \geq 2$, and let a and b be two finite complex numbers with $a \neq 0$. If*

(i) *$f^{(k)} - af^n$ and $g^{(k)} - ag^n$ share the value b IM in D for every pair of functions (f, g) of $\mathcal{F}$,*
(ii) *f has no simple pole, and any zero of f has multiplicity at least k in D for every function $f \in \mathcal{F}$,*

$\mathcal{F}$ is a normal family in D.

2 Preliminary Lemmas

In order to prove our results, we need the following lemmas.

Lemma 2.1 ([7]) *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc satisfying all zeros of functions in $\mathcal{F}$ have multiplicity $\geq p$ and all poles of functions in $\mathcal{F}$ have multiplicity $\geq q$. Let α be a real number satisfying $-q < \alpha < p$. Then $\mathcal{F}$ is not normal at 0 if and only if there exist*

a) *a number $0 < r < 1$;*
b) *points z_n with $|z_n| < r$;*
c) *functions $f_n \in \mathcal{F}$;*
d) *positive numbers $\rho_n \to 0$*

such that $g_n(\zeta) := \rho^{-\alpha} f_n(z_n + \rho_n \zeta)$ converges spherically uniformly on each compact subset of $\mathbb{C}$ to a non-constant meromorphic function $g(\zeta)$, whose all zeros have multiplicity $\geq p$ and all poles have multiplicity $\geq q$ and order is at most 2.

Lemma 2.2 *Let $f(z)$ be a meromorphic function such that $f^{(k)}(z) \not\equiv 0$ and $c \in \mathbb{C}\backslash\{0\}$, $k, n \in \mathbb{N}^+$ with $n \geq 2$. Then*

$$(n-1)T(r,f) \leq (k+1)\overline{N}(r,f) + \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{f^{(k)} - cf^n}) + S(r,f), \tag{1}$$

where $S(r,f) = o(T(r,f))$, as $r \to \infty$, possibly outside a set with finite linear measure.

Proof Set

$$\Phi(z) := \frac{f^{(k)}(z)}{cf^n(z)}.$$

Since $f^{(k)}(z) \not\equiv 0$, we have $\Phi(z) \not\equiv 0$. Thus

$$f^n(z) = \frac{f^{(k)}(z)}{c\Phi(z)}. \tag{2}$$

Hence

$$\begin{aligned} nm(r,f) = m(r,f^n) &\leq m\left(r, \frac{f^{(k)}}{\Phi}\right) + \log^+ \frac{1}{|c|} \\ &\leq m\left(r, \frac{1}{\Phi}\right) + m(r,f^{(k)}) + \log^+ \frac{1}{|c|} \\ &\leq m\left(r, \frac{1}{\Phi}\right) + m\left(r, \frac{f^{(k)}}{f}\right) + m(r,f) + \log^+ \frac{1}{|c|}. \end{aligned}$$

So that

$$(n-1)m(r,f) \leq m\left(r, \frac{1}{\Phi}\right) + m\left(r, \frac{f^{(k)}}{f}\right) + \log^+ \frac{1}{|c|}. \tag{3}$$

On the other hand, (2) gives

$$\begin{aligned} nN(r,f) &\leq N(r,f^n) = N\left(r, \tfrac{f^{(k)}}{\Phi}\right) \\ &\leq N(r,f^{(k)}) + N(r, \tfrac{1}{\Phi}) - \overline{N}(r, \Phi = f^{(k)} = 0), \end{aligned} \tag{4}$$

where $\overline{N}(r, \Phi = f^{(k)} = 0)$ denotes the counting function of zeros of both Φ and $f^{(k)}$.

We obtain

$$nN(r,f) \leq N(r,f) + k\overline{N}(r,f) + N\left(r, \frac{1}{\Phi}\right) - \overline{N}(r, \Phi = f^{(k)} = 0),$$

$$(n-1)N(r,f) \leq k\overline{N}(r,f) + N\left(r, \frac{1}{\Phi}\right) - \overline{N}(r, \Phi = f^{(k)} = 0). \tag{5}$$

By (2), we have

$$\overline{N}\left(r, \frac{1}{\Phi}\right) + \overline{N}(r, \Phi) = \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, \Phi = f^{(k)} = 0). \tag{6}$$

From (3) to (6), we obtain

$$\begin{aligned}(n-1)T(r,f) &\le k\overline{N}(r,f) + T\left(r, \frac{1}{\Phi}\right) - \overline{N}(r, \Phi = f^{(k)} = 0) + S(r,f) \\ &\le k\overline{N}(r,f) + T(r, \Phi) - \overline{N}(r, \Phi = f^{(k)} = 0) + S(r,f) \\ &\le k\overline{N}(r,f) + \overline{N}(r, \frac{1}{\Phi}) + \overline{N}(r, \Phi) + \overline{N}\left(r, \frac{1}{\Phi - 1}\right) \\ &\quad - \overline{N}(r, \Phi = f^{(k)} = 0) + S(r,f) \\ &\le (k+1)\overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^{(k)} - cf^n}\right) + S(r,f),\end{aligned}$$

(1) holds. □

3 Proofs of Theorems

Proof (The Proof of Theorem 1.5) Suppose that $\mathcal{F}$ is not normal in D. Then there exists at least one point z_0 such that $\mathcal{F}$ is not normal at the point z_0. Without loss of generality we assume that $z_0 = 0$. By Lemma 2.1, there exist points $z_j \to 0$, positive numbers $\rho_j \to 0$ and functions $f_j \in \mathcal{F}$ such that

$$g_j(\xi) = \rho_j^{\frac{k}{n-1}} f_j(z_j + \rho_j \xi) \Rightarrow g(\xi) \tag{7}$$

locally uniformly with respect to the spherical metric, where g is a non-constant meromorphic function in $\mathbb{C}$ and any zero of g has multiplicity at least k. Obviously, $g^{(k)} \not\equiv 0$.

From (7) and by computation, we have

$$g_j^{(k)}(\xi) = \rho_j^{\frac{nk}{n-1}} f_j^{(k)}(z_j + \rho_j \xi) \Rightarrow g^{(k)}(\xi)$$

and

$$\begin{aligned}&g_j^{(k)}(\xi) - a g_j^n(\xi) - \rho_j^{\frac{nk}{n-1}} b \\ &= \rho_j^{\frac{nk}{n-1}} (f_j^{(k)}(z_j + \rho_j \xi) - a f_j^n(z_j + \rho_j \xi) - b) \\ &\Rightarrow g^{(k)}(\xi) - a g^n(\xi)\end{aligned} \tag{8}$$

also locally uniform with respect to the spherical metric.

If $g^{(k)}(\xi) - ag^n(\xi) \equiv 0$, then

$$\begin{aligned} nT(r,g) &= T(r,g^n) = T(r,g^{(k)}) + O(1) \\ &= m(r,g^{(k)}) + N(r,g^{(k)}) + O(1) \\ &\le m(r,g) + N(r,g) + k\overline{N}(r,g) + S(r,g) \\ &\le (k+1)T(r,g) + S(r,g). \end{aligned} \tag{9}$$

Note that $n \ge k+5$, therefore, (9) gives that $g(\xi)$ is a constant, a contradiction. So $g^{(k)}(\xi) - ag^n(\xi) \not\equiv 0$. Furthermore, by (1) of Lemma 2.2, we have

$$T(r,g) \le \frac{1}{2}\overline{N}(r, \frac{1}{g^{(k)} - ag^n}) + S(r,g). \tag{10}$$

If $g^{(k)}(\xi) - ag^n(\xi) \neq 0$, then (10) implies that $g(\xi)$ is a constant. Hence, $g^{(k)}(\xi) - ag^n(\xi)$ is a nonconstant meromorphic function and has at least one distinct zero.

Next we prove that $g^{(k)}(\xi) - ag^n(\xi)$ has at most one distinct zero. To the contrary, let ξ_0 and ξ_0^* be two distinct zeros of $g^{(k)}(\xi) - ag^n(\xi)$, and choose $\delta(> 0)$ small enough such that $D(\xi_0,\delta) \cap D(\xi_0^*,\delta) = \phi$ where $D(\xi_0,\delta) = \{\xi : |\xi - \xi_0| < \delta\}$ and $D(\xi_0^*,\delta) = \{\xi : |\xi - \xi_0^*| < \delta\}$. From (8), by Hurwitz's theorem, there exist points $\xi_j \in D(\xi_0,\delta)$, $\xi_j^* \in D(\xi_0^*,\delta)$ such that for sufficiently large j

$$f_j^{(k)}(z_j + \rho_j\xi_j) - af_j^n(z_j + \rho_j\xi_j) - b = 0,$$
$$f_j^{(k)}(z_j + \rho_j\xi_j^*) - af_j^n(z_j + \rho_j\xi_j^*) - b = 0.$$

Note that $z_j + \rho_j\xi_j \to 0$ and $z_j + \rho_j\xi_j^* \to 0$ as $j \to \infty$, then $f_j^{(k)}(z) - af_j^n(z) - b$ has at least two distinct zeros $z_j + \rho_j\xi_j$ and $z_j + \rho_j\xi_j^*$ for each large enough j, a contradiction with hypothesis.

Thus (10) deduces that $g(\xi)$ is also a constant, a contradiction.

This completes the proof of Theorem 1.5. □

Proof (The Proof of Theorem 1.6) The proof is the same as the proof of Theorem 1.5 except for $g(\xi)$ having no simple pole and (10). Here we omit the detailed statements and only deduce an inequality (11) substituting into (10).

Since f has no simple pole, we get

$$\overline{N}(r,f) \le \frac{1}{2}N(r,f) \le \frac{1}{2}T(r,f).$$

Note that $n \geq k + 3$ and $k \geq 2$, (1) of Lemma 2.2 gives that

$$T(r,f) \leq \frac{2}{3}\overline{N}\left(r, \frac{1}{f^{(k)} - cf^n}\right) + S(r,f). \tag{11}$$

This completes the proof of Theorem 1.6. □

Proof (The Proof of Theorem 1.7) We are proceeding the proof of Theorem 1.5 to obtain (7)–(10).

If $g^{(k)}(\xi) - ag^n(\xi) \neq 0$, then (10) gives that $g(\xi)$ is a constant. Hence, $g^{(k)}(\xi) - ag^n(\xi)$ is a non-constant meromorphic function and has at least one zero.

Next we prove that $g^{(k)}(\xi) - ag^n(\xi)$ has just a unique zero. To the contrary, let ξ_0 and ξ_0^* be two distinct zeros of $g^{(k)}(\xi) - ag^n(\xi)$, and choose $\delta(> 0)$ small enough such that $D(\xi_0, \delta) \cap D(\xi_0^*, \delta) = \phi$ where $D(\xi_0, \delta) = \{\xi : |\xi - \xi_0| < \delta\}$ and $D(\xi_0^*, \delta) = \{\xi : |\xi - \xi_0^*| < \delta\}$. From (8), by Hurwitz's theorem, there exist points $\xi_j \in D(\xi_0, \delta)$, $\xi_j^* \in D(\xi_0^*, \delta)$ such that for sufficiently large j

$$f_j^{(k)}(z_j + \rho_j\xi_j) - af_j^n(z_j + \rho_j\xi_j) - b = 0,$$
$$f_j^{(k)}(z_j + \rho_j\xi_j^*) - af_j^n(z_j + \rho_j\xi_j^*) - b = 0.$$

By the hypothesis that for each pair of functions f and g in $\mathcal{F}$, $f^{(k)} - af^n$ and $g^{(k)} - ag^n$ share b in D, we know that for any positive integer m

$$f_m^{(k)}(z_j + \rho_j\xi_j) - af_m^n(z_j + \rho_j\xi_j) - b = 0,$$
$$f_m^{(k)}(z_j + \rho_j\xi_j^*) - af_m^n(z_j + \rho_j\xi_j^*) - b = 0.$$

Fix m, take $j \to \infty$, and note $z_j + \rho_j\xi_j \to 0$, $z_j + \rho_j\xi_j^* \to 0$, then $f_m^{(k)}(0) - af_m^n(0) - b = 0$. Since the zeros of $f_m^{(k)} - af_m^n - b$ has no accumulation point, we know

$$z_j + \rho_j\xi_j = 0, z_j + \rho_j\xi_j^* = 0,$$

for large enough j. Hence $\xi_j = -\frac{z_j}{\rho_j}, \xi_j^* = -\frac{z_j}{\rho_j}$. This contradicts with $\xi_j \in D(\xi_0, \delta), \xi_j^* \in D(\xi_0^*, \delta)$ and $D(\xi_0, \delta) \cap D(\xi_0^*, \delta) = \phi$. So $g^{(k)}(\xi) - ag^n(\xi)$ has just a unique zero.

Thus (10) deduces that $g(\xi)$ is also a constant, a contradiction.

This completes the proof of Theorem 1.7. □

Proof (The Proof of Theorem 1.8) The proof is the same as the proof of Theorem 1.7 except for $g(\xi)$ having no simple pole and (10) being replaced by (11). Here we omit the detail statements.

This completes the proof of Theorem 1.8. □

Remark 3.1 Obviously, by analyzing above proofs, we see that if $\mathcal{F}$ be a family of holomorphic functions in D in Theorems 1.5–1.8, then the condition $n \geq k + x$ can be replaced by $n \geq x - 1$.

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Variable Integral Exponent Besov and Triebel-Lizorkin Spaces Associated with Non-negative Self-Adjoint Operators

Jingshi Xu and Xiaodi Yang

Abstract In this paper, variable integral exponent Besov and Triebel-Lizorkin spaces associated with a non-negative self-adjoint operator are introduced. Then equivalent norms and atomic decompositions of these new spaces are given.

Keywords Besov space • Non-negative self-adjoint operators • Triebel-Lizorkin space • Variable exponent

Mathematics Subject Classification (2010) Primary 46E30; Secondary 42B25

1 Introduction

In recent years, function spaces associated with non-negative self-adjoint operators have attracted many authors' attention. Indeed, G. Kerkyacharian and P. Petrushev introduced Besov and Triebel-Lizorkin spaces associated with non-negative self-adjoint operators and gave their Heat kernel characterization and frame decomposition in [15]. In [13] Hu gave their equivalent quasi-norms by Peetre type maximal functions and atomic decompositions.

In recent decades variable exponent function spaces have been developed and proved to be useful tools in the study of ordinary and partial differential equations and image restoration. For details one can see [1–3, 5, 6, 8–12, 14, 16–20] and the references therein.

Inspired by the mentioned references, in this paper we aim to introduce variable integral exponent Besov and Triebel-Lizorkin spaces associated with non-negative self-adjoint operators. The structure of the paper is as follows. In Sect. 2, we shall

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give the definitions of these spaces and equivalent quasi-norms by the Peetre type maximal functions. In Sect. 3, we shall give atomic decompositions of these new spaces.

2 Equivalent Norms

In the sequel, we assume $(\mathcal{X}, \rho, \mu)$ is a metric measure space satisfying the conditions: $(\mathcal{X}, \rho)$ is a locally compact metric space with distance $\rho(\cdot, \cdot)$ and μ is a positive Radon measure obeying the volume doubling condition

$$0 < \mu(B(x, 2r)) \leqslant c_0\mu(B(x, r)) < \infty \text{ for all } x \in \mathcal{X} \text{ and } r > 0,$$

where $B(x, r)$ is the open ball centered at x with radius r and c_0 is a constant.

Let $\mathcal{L}$ be a self-adjoint non-negative operator on $L^2(\mathcal{X}, \mathrm{d}\mu)$ such that the associated semigroup $P_t = e^{-t\mathcal{L}}$ consists of integral operators with heat kernel $p_t(x, y)$ obeying the following conditions:

(a) Gaussian upper bound: for $x, y \in \mathcal{X},\ t > 0$,

$$|p_t(x, y)| \leqslant \frac{C_1 \exp\{-c_2\rho^2(x, y)/t\}}{\sqrt{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}}.$$

(b) Hölder continuity: There exists a constant $\alpha > 0$ such that for $x, y \in \mathcal{X},\ t > 0$ and $\rho(y, y') \leqslant \sqrt{t}$

$$|p_t(x, y) - p_t(x, y')| \leqslant C_1 \left(\frac{\rho(y, y')}{\sqrt{t}}\right)^{\alpha} \frac{\exp\{-c_2\rho^2(x, y)/t\}}{\sqrt{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}}.$$

We denote the domain of $\mathcal{L}$ by $D(\mathcal{L})$. We also denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Definition 2.1

(i) If $\mu(\mathcal{X}) < \infty$, the test function class $\mathcal{D}$ is defined as the set of all functions $\phi \in \cap_{n\in\mathbb{N}_0} D(\mathcal{L}^n)$ with the topology induced by the family of seminorms

$$\mathcal{P}_n(\phi) := \|\mathcal{L}^n\phi\|_{L^2(\mathcal{X},\mathrm{d}\mu)},\ n \in \mathbb{N}_0.$$

(ii) If $\mu(\mathcal{X}) = \infty$, the class $\mathcal{D}$ is defined as the set of all functions $\phi \in \cap_{n\in\mathbb{N}_0} D(\mathcal{L}^n)$ with the topology induced by the family of seminorms

$$\mathcal{P}_{n,l}(\phi) := \sup_{x\in\mathcal{X}} (1 + \rho(x, x_0))^l |\mathcal{L}^n\phi(x)| < \infty,\ n, l \in \mathbb{N}_0,$$

where $x_0 \in \mathcal{X}$ is a fixed point.

In the case $\mu(\mathcal{X}) = \infty$, the class $\mathcal{D}$ is independent of the choice of x_0. Therefore, we choose a point $x_0 \in \mathcal{X}$ and fix it in the sequel.

The set of all continuous linear functionals on $\mathcal{D}$ is denoted by $\mathcal{D}'$. The duality between the spaces is denoted by the map $(\cdot,\cdot) : \mathcal{D}' \times \mathcal{D} \to \mathbb{C}$.

Let $p(\cdot)$ be a measurable function on $\mathcal{X}$ with range in $[1,\infty)$. The variable Lebesgue space $L^{p(\cdot)}(\mathcal{X})$ denotes the set of measurable functions f on $\mathcal{X}$ such that

$$\|f\|_{L^{p(\cdot)}(\mathcal{X})} := \inf\left\{\lambda > 0 : \int_{\mathcal{X}} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \mathrm{d}\mu(x) \leqslant 1\right\} < \infty.$$

If $p(\cdot)$ is measurable function on $\mathcal{X}$ with values in $(0,\infty)$, we denote $p_- := \operatorname{ess\,inf}_{x\in\mathcal{X}} p(x)$ and $p_+ := \operatorname{ess\,sup}_{x\in\mathcal{X}} p(x)$. Let $\mathcal{P}(\mathcal{X})$ be the set of measurable functions $p(\cdot)$ on $\mathcal{X}$ such that $p_- > 1$ and $p_+ < \infty$. Define $\mathcal{P}^0(\mathcal{X})$ to be the set of measurable functions $p(\cdot)$ on $\mathcal{X}$ such that $p_- > 0$ and $p_+ < \infty$. Given $p(\cdot) \in \mathcal{P}^0(\mathcal{X})$, one can define the space $L^{p(\cdot)}(\mathcal{X})$ as above. Indeed, a quasi-norm on this space is $\|f\|_{L^{p(\cdot)}} = \||f|^{p_0}\|_{L^{q(\cdot)}}^{1/p_0}$, where $0 < p_0 < p_-$.

Let $\mathcal{S}([0,\infty))$ denote the Schwartz class on $[0,\infty)$.

Definition 2.2 Let (ϕ_0,ϕ) be a pair of functions in $\mathcal{S}([0,\infty))$ and M be an integer. The pair (ϕ_0,ϕ) is said to be in the class $\mathcal{A}_M([0,\infty))$ if

$$|\phi_0(\lambda)| > 0 \ \text{ on } [0,4\epsilon), \tag{1}$$

$$|\phi(\lambda)| > 0 \ \text{ on } (\epsilon/4,4\epsilon) \tag{2}$$

for some $\epsilon > 0$, and if $(\cdot)^{-M}\phi(\cdot) \in \mathcal{S}([0,\infty))$.

In the sequel, given any pair (ϕ_0,ϕ) of functions in $\mathcal{S}([0,\infty))$, we denote the system $\{\phi_j\}$ of functions in $\mathcal{S}([0,\infty))$ by setting

$$\phi_j(\lambda) := \phi(2^{-2j}\lambda) \text{ for } j \geqslant 1.$$

Let $\varphi \in \mathcal{S}([0,\infty))$, by Proposition 5.3 in [15], the integral kernel $K_{\varphi(\mathcal{L})}(x,y)$ of the operator $\varphi(\mathcal{L})$ belongs to $\mathcal{D}$ as a function of x and as a function of y. Thus, for $f \in \mathcal{D}'$ and $\varphi \in \mathcal{S}([0,\infty))$, one can define

$$\varphi(\mathcal{L})f(x) := (f, K_{\varphi(\mathcal{L})}(x,\cdot)),$$

which extends the action of the operator $\varphi(\mathcal{L})$ from $f \in L^2(\mathcal{X},\mathrm{d}\mu)$ to $f \in \mathcal{D}'$.

After these preparations, we have the following definition.

Definition 2.3 Let $s \in \mathbb{R}$, $0 < q \leqslant \infty$, $p(\cdot) \in \mathcal{P}^0(\mathcal{X})$. Let ϕ_0, ϕ be functions in $\mathcal{S}([0,\infty))$.

(i) The variable integral exponent Besov space $B^s_{p(\cdot),q}(\mathcal{X},\mathcal{L})$ is the set of all $f \in \mathcal{D}'$ such that

$$\|f\|_{B^s_{p(\cdot),q}(\mathcal{X},\mathcal{L})} := \left\| \{2^{sj}\phi_j(\mathcal{L})f\}_{j=0}^{\infty} \right\|_{\ell^q(L^{p(\cdot)})} < \infty;$$

(ii) The variable integral exponent Triebel-Lizorkin space $F^s_{p(\cdot),q}(\mathcal{X},\mathcal{L})$ is the set of all $f \in \mathcal{D}'$ such that

$$\|f\|_{F^s_{p(\cdot),q}(\mathcal{X},\mathcal{L})} := \left\| \{2^{sj}\phi_j(\mathcal{L})f\}_{j=0}^{\infty} \right\|_{L^{p(\cdot)}(\ell^q)} < \infty.$$

Here to make these spaces definite, the primary point is to show them independent of the choice of functions ϕ_0 and ϕ. To this objective we need more notation. Let $\theta_0, \theta \in \mathcal{S}([0,\infty))$, and a positive number a. For any $f \in \mathcal{D}'$ denote the system of the Peetre type maximal functions by

$$\theta^*_{j,a}(\mathcal{L})f(x) := \sup_{y\in\mathcal{X}} \frac{|\theta_j(\mathcal{L})f(y)|}{(1+2^j\rho(x,y))^a}, \ \ j \in \mathbb{N}_0.$$

In the following, our key tool is the boundedness of Hardy-Littlewood maximal operator on variable Lebesgue spaces. For a local integrable function f, the Hardy-Littlewood maximal function $\mathcal{M}f$ is defined by

$$\mathcal{M}f(x) := \sup_{B\ni x} \frac{1}{\mu(B)} \int_B |f(y)|\mathrm{d}\mu(y), \ \forall x \in \mathcal{X},$$

where B is any ball containing x. Let $\mathscr{B}(\mathcal{X})$ be the set of $p(\cdot) \in \mathcal{P}(\mathcal{X})$ such that the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded on $L^{p(\cdot)}(\mathcal{X})$. For the detail, see [1].

Now we state our results in this section.

Theorem 2.4 *Let ϕ_0, ϕ be functions in $\mathcal{S}([0,\infty))$ satisfying (1) and (2), $0 < q \le \infty$, and $p(\cdot) \in \mathcal{P}^0(\mathcal{X})$.*

(i) *If there exists $p_0 < \min\{p_-, 1\}$ such that $p(\cdot)/p_0 \in \mathscr{B}(\mathcal{X})$ and $ap_0 > 2d$, then there exists a constant $C > 0$ such that for all $f \in \mathcal{D}'$*

$$\|\{2^{sj}\phi^*_{j,a}(\mathcal{L})f\}_{j=0}^{\infty}\|_{\ell^q(L^{p(\cdot)})} \leqslant C\|\{2^{js}\phi_j(\mathcal{L})f\}_{j=0}^{\infty}\|_{\ell^q(L^{p(\cdot)})}.$$

(ii) *If there exists $p_0 < \min\{p_-, 1, q\}$ such that $p(\cdot)/p_0 \in \mathscr{B}(\mathcal{X})$ and $ap_0 > 2d$, then there exists a constant $C > 0$ such that for all $f \in \mathcal{D}'$*

$$\|\{2^{sj}\phi^*_{j,a}(\mathcal{L})f\}_{j=0}^{\infty}\|_{L^{p(\cdot)}(\ell^q)} \leqslant C\|\{2^{js}\phi_j(\mathcal{L})f\}_{j=0}^{\infty}\|_{L^{p(\cdot)}(\ell^q)}.$$

Theorem 2.5 *Let $a > 0, M > s/2$ and let $(\phi_0, \phi), (\theta_0, \theta) \in \mathcal{A}_M([0, \infty))$. Let $0 < q \le \infty$, and $p(\cdot) \in \mathcal{P}^0(\mathcal{X})$.*

(i) *If there exists $p_0 < \{p_-, 1\}$ such that $ap_0 > 2d$, then there exists a constant $C > 0$ such that for all $f \in \mathcal{D}'$*

$$\|\{2^{sj}\phi^*_{j,a}(\mathcal{L})f\}_{j=0}^{\infty}\|_{\ell^q(L^{p(\cdot)})} \leqslant C\|\{2^{js}\theta^*_{j,a}(\mathcal{L})f\}_{j=0}^{\infty}\|_{\ell^q(L^{p(\cdot)})}.$$

(ii) *If there exists $p_0 < \{p_-, 1, q\}$ such that $ap_0 > 2d$, then there exists a constant $C > 0$ such that for all $f \in \mathcal{D}'$*

$$\|\{2^{sj}\phi^*_{j,a}(\mathcal{L})f\}_{j=0}^{\infty}\|_{L^{p(\cdot)}(\ell^q)} \leqslant C\|\{2^{js}\theta^*_{j,a}(\mathcal{L})f\}_{j=0}^{\infty}\|_{L^{p(\cdot)}(\ell^q)}.$$

From Theorems 2.4 and 2.5, we have the following result.

Corollary 2.6 *Let $a > 0, M > s/2$ and let $(\phi_0, \phi), (\theta_0, \theta) \in \mathcal{A}_M([0, \infty))$. Let $0 < q \le \infty$, and $p(\cdot) \in \mathcal{P}^0(\mathcal{X})$.*

(i) *If there exists $p_0 < \min\{p_-, 1\}$ such that $p(\cdot)/p_0 \in \mathscr{B}(\mathcal{X})$ and $ap_0 > 2d$, then*

$$\|\{2^{sj}\phi^*_{j,a}(\mathcal{L})f\}_{j=0}^{\infty}\|_{\ell^q(L^{p(\cdot)})}, \|\{2^{js}\theta^*_{j,a}(\mathcal{L})f\}_{j=0}^{\infty}\|_{\ell^q(L^{p(\cdot)})}, \|\{2^{js}\phi_j(\mathcal{L})f\}_{j=0}^{\infty}\|_{\ell^q(L^{p(\cdot)})}$$
$$\text{and } \|\{2^{js}\theta_j(\mathcal{L})f\}_{j=0}^{\infty}\|_{\ell^q(L^{p(\cdot)})}$$

are equivalent quasi-norms on $B^s_{p(\cdot),q}(\mathcal{X}, \mathcal{L})$.

(ii) *If there exists $p_0 < \min\{p_-, 1, q\}$ such that $p(\cdot)/p_0 \in \mathscr{B}(\mathcal{X})$ and $ap_0 > 2d$, then*

$$\|\{2^{sj}\phi^*_{j,a}(\mathcal{L})f\}_{j=0}^{\infty}\|_{L^{p(\cdot)}(\ell^q)}, \|\{2^{js}\theta^*_{j,a}(\mathcal{L})f\}_{j=0}^{\infty}\|_{L^{p(\cdot)}(\ell^q)}, \|\{2^{js}\phi_j(\mathcal{L})f\}_{j=0}^{\infty}\|_{L^{p(\cdot)}(\ell^q)}$$
$$\text{and } \|\{2^{js}\theta_j(\mathcal{L})f\}_{j=0}^{\infty}\|_{L^{p(\cdot)}(\ell^q)}$$

are equivalent quasi-norms on $F^s_{p(\cdot),q}(\mathcal{X}, \mathcal{L})$.

Theorems 2.4 and 2.5 can be proved by using the idea in [13, 20] and the following lemma, which can be proved similarly to that of Corollary 2.1 in [7]. So we omit the detail here.

Lemma 2.7 *If $p(\cdot) \in \mathscr{B}(\mathcal{X})$, and $1 < q \leqslant \infty$, then there is a constant C such that*

$$\|\{\mathcal{M}f_j\}_{j=0}^{\infty}\|_{L^{p(\cdot)}(\ell^q)} \leqslant C\|\{f_j\}_{j=0}^{\infty}\|_{L^{p(\cdot)}(\ell^q)}$$

holds for all locally integrable functions $\{f_j\}_{j=0}^{\infty}$ on $\mathcal{X}$.

3 Atomic Decompositions

To give atomic decompositions, we first recall the so-called dyadic type structure of the space $\mathcal{X}$.

Lemma 3.1 (See Theorem 11 in [4]) *There exists a collection of open subsets* $\{Q_\alpha^k : k \in \mathbb{Z}, \alpha \in I_k\}$ *of* $\mathcal{X}$, *where* I_k *is some index set (possibly finite), and constants* $\delta \in (0,1)$ *and* $A_1, A_2 > 0$, *such that*

(i) $\mu(\mathcal{X} \setminus \bigcup_{\alpha \in I_k} Q_\alpha^k) = 0$ *for each fixed* k *and* $Q_\alpha^k \bigcap Q_\beta^k = \emptyset$ *if* $\alpha \neq \beta$;
(ii) *for any* α, β, k, l *with* $l \geqslant k$, *either* $Q_\alpha^k \subset Q_\beta^l$ *or* $Q_\alpha^k \bigcap Q_\beta^l = \emptyset$;
(iii) *for each* (k, α) *and* $l < k$, *there exists a unique* β *such that* $Q_\alpha^k \subset Q_\beta^l$;
(iv) $\operatorname{diam}(Q_\alpha^k) \leqslant A_1 \delta^k$, *where* $\operatorname{diam}(Q_\alpha^k) := \sup\{\rho(x,y) : x, y \in Q_\alpha^k\}$;
(v) *each* Q_α^k *contains some ball* $B(z_\alpha^k, A_2\delta^k)$, *where* $z_\alpha^k \in \mathcal{X}$.

The set Q_α^k in Lemma 3.1 is thought of as a dyadic cube on $\mathcal{X}$ with diameter roughly δ^k and centered at z_α^k. We denote by $\mathscr{D}$ the family of all dyadic cubes on $\mathcal{X}$. For $k \in \mathbb{Z}$, we set $\mathscr{D}_k = \{Q_\alpha^k : \alpha \in I_k\}$, so that $\mathscr{D} = \bigcup_{k \in \mathbb{Z}} \mathscr{D}_k$. For any dyadic cube $Q = Q_\alpha^k$, we denote by $z_Q := z_\alpha^k$ the "center" of Q. In the sequel, without loss of generality, we assume that $\delta = \frac{1}{2}$. In fact, if this is not the case, we only need to replace 2^j in the definition of $B_{p,q}^s(\mathcal{X}, L)$ by δ^{-j} and make some other necessary changes.

Definition 3.2 Let $K, S \in \mathbb{N}_0$, and let Q be a dyadic cube in $\mathscr{D}_k$, with $k \in \mathbb{N}_0$. When $k \geqslant 1$, a function $a_Q \in L^2(\mathcal{X}, \mathrm{d}\mu)$ is called a (K,S)-atom for Q if a_Q satisfies the following condition for $m = K$ and also for $m = -S$.

(i) $a_Q \in D(\mathcal{L}^m)$;
(ii) $\operatorname{supp}(\mathcal{L}^m a_Q) \subset B(z_Q, (A_1 + 1)2^{-k})$;
(iii) $\sup_{x \in \mathcal{X}} |\mathcal{L}^m a_Q(x)| \leqslant 2^{2km}[\mu(Q)]^{-1/2}$.

When $k = 0$, a function a_Q is called a (K,S)-atom for Q if it satisfies (i)–(iii) only for $m = K$. Here, for $m \in \mathbb{N}$, $\mathcal{L}^{-m}$ is defined via the spectral resolution associated with the self-adjoint positive operator $\mathcal{L}$.

Definition 3.3

(i) Let $-\infty < s < \infty$, $p(\cdot) \in \mathcal{P}^0(\mathcal{X})$, and $0 < q \leqslant \infty$. The sequence space $b_{p(\cdot),q}^s$ consists all complex sequence $w = \{w_Q\}_{Q \in \cup_{k \geqslant 0} \mathscr{D}_k}$ such that

$$\|w\|_{b_{p(\cdot),q}^s} := \left\| \left\{ 2^{ks} \left\| \sum_{Q \in \mathscr{D}_k} (|w_Q| [\mu(Q)]^{-1/2}) \right\|_{L^{p(\cdot)}} \right\}_{k=0}^{\infty} \right\|_{\ell^q}.$$

(ii) Let $-\infty < s < \infty$, $p(\cdot) \in \mathcal{P}^0(\mathcal{X})$, and $0 < q < \infty$. The sequence space $f^s_{p(\cdot),q}$ consists all complex sequence $w = \{w_Q\}_{Q\in\cup_{k\geq 0}\mathscr{D}_k}$ such that

$$\|w\|_{f^s_{p(\cdot),q}} := \left\| \left\| \left\{ 2^{ks} \sum_{Q\in\mathscr{D}_k} (|w_Q|[\mu(Q)]^{-1/2}) \right\}_{k=0}^{\infty} \right\|_{\ell^q} \right\|_{L^{p(\cdot)}}.$$

Here χ_Q is the characteristic function of Q.

Next we give the atomic decomposition of the Bosov and Triebel-Lizorkin spaces, respectively.

Theorem 3.4 *Let $-\infty < s < \infty$, and $K \in \mathbb{N}_0$ such that $K > \frac{s}{2}$.*

(i) *Let $0 < q \leqslant \infty$, and $p(\cdot) \in \mathcal{P}^0(\mathcal{X})$ with $p_0 < p_-$ such that $p(\cdot)/p_0 \in \mathscr{B}(\mathcal{X})$. Let $S \in \mathbb{N}_0$ such that $S > \frac{d}{2\min(1,p_0)} - \frac{s}{2}$.*

(ii) *Let $0 < q < \infty$, and $p(\cdot) \in \mathcal{P}^0(\mathcal{X})$ with $p_0 < \min\{p_-, q\}$ such that $p(\cdot)/p_0 \in \mathscr{B}(\mathcal{X})$. Let $S \in \mathbb{N}_0$ such that $S > \frac{d}{2\min(1,p_0,q)} - \frac{s}{2}$.*

Then there is a constant $C > 0$ such that for every sequence (K, S)-atoms $\{a_Q\}_{Q\in\cup_{k\geq 0}\mathscr{D}_k}$ we have

$$\left\| \sum_{k=0}^{\infty} \sum_{Q\in\mathscr{D}_k} w_Q a_Q \right\|_A \leqslant C\|w\|_\Gamma, \; w = \{w_Q\}_{Q\in\cup_{k\geq 0}\mathscr{D}_k}.$$

Conversely, there is a constant C' such that given any distribution $f \in A$ and any $K, S \in \mathbb{N}_0$, there exist a sequence of (K, S)-atoms $\{a_Q\}_{Q\in\cup_{k\geq 0}\mathscr{D}_k}$ and a sequence of complex numbers $w = \{w_Q\}_{Q\in\cup_{k\geq 0}\mathscr{D}_k}$ such that

$$f = \sum_{k=0}^{\infty} \sum_{Q\in\mathscr{D}_k} w_Q a_Q,$$

where the sum converge in $\mathcal{D}'$, and moreover, $\|w\|_\Gamma \leqslant C'\|f\|_A$.

Here in cases (i) *and* (ii)*, $A = B^s_{p(\cdot),q}(\mathcal{X}, \mathcal{L})$ and $F^s_{p(\cdot),q}(\mathcal{X}, \mathcal{L})$, $\Gamma = b^s_{p(\cdot),q}$ and $f^s_{p(\cdot),q}$, respectively.*

Theorem 3.4 can be proved by using the method in [13] and Lemma 2.7. We omit the details here.

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A Decomposition Theorem for Null-Solutions to Polynomial Slice Dirac Operator

Hongfen Yuan, Tieguo Ji, and Hongyan Ji

Abstract In this paper, a decomposition theorem for null-solutions to the polynomial slice Dirac operator is established by the generalized Euler operator in $\mathbb{R}^{m+1}$. This is a generalization of the well-known Almansi decomposition theorem. In the sequel our decomposition will be used in the study of boundary value problems for slice monogenic functions.

Keywords Almansi decomposition • Euler operator • Polynomial slice Dirac operator

Mathematics Subject Classification (2010) Primary 30G35; Secondary 33C45

1 Introduction

The classical Almansi decomposition theorem is a decomposition for poly-harmonic functions in terms of harmonic functions defined in a star-like domain centred at the origin, see, e.g., [1]. Nowadays, it has been extended to the case of complex analysis, Clifford analysis and Clifford analysis in superspace, see, e.g., [2–5]. Recently, Cerejeiras, Kähler, Ku and other scholars have studied the Riemann and Hilbert boundary value problems in Clifford analysis making full use of the Almansi type decomposition theorems for the null-solutions to the iterated Dirac and Cauchy-Riemann operators defined in a domain, see, e.g., [6, 7]. In 2016, the first author investigated the Riquier's problem in superspace in terms of the Almansi type decompositions for null-solutions to super Dirac operator (c.f. [8]). However, up to now, the Almansi type decomposition for the kernel of the polynomial slice Dirac operator, i.e., null-solutions to the polynomial slice Dirac operator, has not been considered. In this paper, we will focus on this.

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From [9], the slice Dirac operator below allows us to establish the Lie superalgebra structure behind the theory of slice monogenic functions, see [10, 11].

Let Ω be a bounded subdomain of R^{m+1}. Then the slice Dirac operator is defined as

$$D = e_0\partial_{x_0} + \frac{\underline{x}}{|\underline{x}|^2}\mathbb{E},$$

where the Euler operator $\mathbb{E} = \sum_{i=1}^{m} x_i\partial_{x_i}$, $\underline{x} = \sum_{i=1}^{m} x_i e_i$, and $e_i e_j + e_j e_i = -2\delta_{ij}, i, j = 0, \ldots, m$. Functions belonging to the kernel of the slice Dirac operator are called slice monogenic functions. In this paper, we investigate the Almansi-type decomposition for null-solutions to polynomial slice Dirac operators in a starlike domain. This will be a starting point for our further research, in particular on boundary value problems for null-solutions to polynomial slice Dirac operators.

2 A Decomposition for the Kernel of the Operator D_λ^k

Definition 2.1 Let $\Omega \subset R^{m+1}$. Let Cl_{m+1} be the $m+1$-dimensional real Clifford algebra. The generalized Euler operator defined on the space $C^1(\Omega) \otimes Cl_{m+1}$ is given by

$$\mathbb{E}_s = s\mathbf{I} + \mathbb{E} = s\mathbf{I} + \sum_{i=0}^{m} x_i\partial_{x_i},$$

where s is a complex number, $\mathbf{I}$ is the identity operator, and $\mathbb{E}$ is the Euler operator in R^{m+1}.

Lemma 2.2 ([9]) *The operators $D, \mathbb{E}$ have the following property*

$$[\mathbb{E} + 1, D] = -D. \tag{1}$$

Lemma 2.3 *Let $\Omega \subset R^{m+1}$. If $f(x) \in C^2(\Omega) \otimes Cl_{m+1}$, then*

$$D\mathbb{E}_s f(x) = \mathbb{E}_{s+1} Df(x), \tag{2}$$

where s is a complex number.

Proof It follows from Definition 2.1 and Lemma 2.2 that

$$D\mathbb{E}_s f(x) = D(s\mathbf{I} + \mathbb{E})f(x) = sDf(x) + \mathbb{E}Df(x) + Df(x) = \mathbb{E}_{s+1}Df(x).$$

□

Definition 2.4 We define the generalized slice Dirac operator by

$$D_\lambda = D - \lambda,$$

where D is the slice Dirac operator and λ is a complex number.

Denote $\ker D_\lambda^k = \{f|(D-\lambda)^k f = 0,\ \ f \in C^k(\Omega)\otimes Cl_{m+1},\ \ k \in \mathbf{N}\}$.

Lemma 2.5 *If $f \in \ker(D_\lambda)$, then*

$$C_k D_\lambda^k \mathbb{E}_\lambda^k f = f, \tag{3}$$

where $C_k = \frac{1}{k!\lambda^k}$ and $k \in \mathbf{N}$.

Proof Let $f \in \ker(D_\lambda)$. Then for $k=1$, it follows from Lemma 2.3 that

$$D_\lambda \mathbb{E}_\lambda f = (D-\lambda)\mathbb{E}_\lambda f = D\mathbb{E}_\lambda f - \lambda \mathbb{E}_\lambda f = \mathbb{E}_{\lambda+1} Df - \lambda \mathbb{E}_\lambda f = \lambda f.$$

Suppose that for $k=l$, $C_l D_\lambda^l \mathbb{E}_\lambda^l f = f$, where $C_l = \frac{1}{l!\lambda^l}$. For $k = l+1$,

$$D_\lambda^{l+1}\mathbb{E}_\lambda^l f = D_\lambda D_\lambda^l \mathbb{E}_\lambda^l f = \frac{1}{C_l} D_\lambda f = 0.$$

Then by Lemma 2.3, we get

$$\begin{aligned}
D_\lambda^{l+1}\mathbb{E}_\lambda^{l+1} f &= D_\lambda^l D_\lambda \mathbb{E}_\lambda \mathbb{E}_\lambda^l f \\
&= D_\lambda^l (\mathbb{E}_{\lambda+1} D_\lambda + \lambda)\mathbb{E}_\lambda^l f \\
&= D_\lambda^{l-1} D_\lambda \mathbb{E}_{\lambda+1} D_\lambda \mathbb{E}_\lambda^l f + \frac{\lambda}{C_l} f \\
&= D_\lambda^{l-1} \mathbb{E}_{\lambda+2} D_\lambda^2 \mathbb{E}_\lambda^l f + \frac{2\lambda}{C_l} f \\
&= \cdots \\
&= \mathbb{E}_{\lambda+l+1} D_\lambda^{l+1} \mathbb{E}_\lambda^l f + \frac{(l+1)\lambda}{C_l} f = \frac{1}{C_{l+1}} f.
\end{aligned}$$

Therefore, we have (3) by induction. □

Theorem 2.6 *If $f(x) \in \ker D_\lambda^k$, then there exist unique functions $f_0, \ldots, f_{k-1} \in \ker D_\lambda$ such that*

$$f(x) = f_0(x) + \mathbb{E}_\lambda f_1(x) + \mathbb{E}_\lambda^2 f_2(x) + \cdots + \mathbb{E}_\lambda^{k-1} f_{k-1}(x), \tag{4}$$

where $f_0, \ldots, f_{k-1}$ *are given as follows:*

$$\begin{cases} f_0(x) = (\mathbf{I} - C_1\mathbb{E}_\lambda D_\lambda)\left(\mathbf{I} - C_2\mathbb{E}_\lambda^2 D_\lambda^2\right)\cdots\left(\mathbf{I} - C_{k-1}\mathbb{E}_\lambda^{k-1}D_\lambda^{k-1}\right)f(x), \\ f_1(x) = C_1 D_\lambda\left(\mathbf{I} - C_2\mathbb{E}_\lambda^2 D_\lambda^2\right)\cdots\left(\mathbf{I} - C_{k-1}\mathbb{E}_\lambda^{k-1}D_\lambda^{k-1}\right)f(x), \\ \quad\vdots \\ f_{k-2}(x) = C_{k-2}D_\lambda^{k-2}(\mathbf{I} - C_{k-1}\mathbb{E}_\lambda^{k-1}D_\lambda^{k-1})f(x), \\ f_{k-1}(x) = C_{k-1}D_\lambda^{k-1}f(x), \end{cases} \tag{5}$$

and $C_k = \frac{1}{k!\lambda^k}$.

Conversely, if functions $f_0, \ldots, f_{k-1} \in$ *ker* D_λ, *then* $f(x)$ *given by* (4) *satisfies the equation* $D_\lambda^k f = 0$.

Proof If we let the operator D_λ^{k-1} act on Eq. (4), then using Lemma 2.5, one has

$$D_\lambda^{k-1}f(x) = D_\lambda^{k-1}\left(f_0(x) + \sum_{i=1}^{k-1}(\mathbb{E}_\lambda)^i f_i(x)\right) = D_\lambda^{k-1}\mathbb{E}_\lambda^{k-1}f_{k-1}(x) = \frac{f_{k-1}(x)}{C_{k-1}},$$

which implies that

$$f_{k-1}(x) = C_{k-1}D_\lambda^{k-1}f(x).$$

Similarly, if D_λ^{k-2} acts on $f(x) - \mathbb{E}_\lambda^{k-1}f_{k-1}(x)$, then we have

$$f_{k-2}(x) = C_{k-2}D_\lambda^{k-2}(\mathbf{I} - C_{k-1}\mathbb{E}_\lambda^{k-1}D_\lambda^{k-1})f(x).$$

Thus, one gets (5) by induction.

Conversely, suppose that $f_0, \ldots, f_{k-1} \in \ker D_\lambda$. Applying Lemma 2.5, we obtain

$$D_\lambda^k f(x) = D_\lambda^k\left[f_0(x) + \sum_{i=1}^{k-1}(\mathbb{E}_\lambda)^i f_i(x)\right] = 0,$$

which completes the proof. □

3 A Decomposition for the Kernel of the Operator *P*(*D*)

Let $P(\lambda) = \lambda^k + b_0\lambda^{k-1} + \cdots + b_{k-1}$, with $b_l \in \mathbb{C}$, and $l = 0, \ldots, k-1$. The polynomial slice Dirac operator is defined as

$$P(D) = D^k + b_0 D^{k-1} + \cdots + b_{k-1}. \tag{6}$$

Denote $\ker P(D) = \{f | P(D)f = 0, \quad f \in C^k(\Omega) \otimes Cl_{m+1}, \quad k \in \mathbf{N}\}$.

If $P(\lambda)$ has the decomposition

$$P(\lambda) = (\lambda - \lambda_0)^{n_0} \cdots (\lambda - \lambda_{l-1})^{n_{l-1}}, \tag{7}$$

where $\lambda_i \in \mathbb{C}$, and $\lambda_i \neq 0$, $i = 0, \ldots, l-1$, then $P(D)$ has the following decomposition

$$P(D) = (D - \lambda_0)^{n_0} \cdots (D - \lambda_{l-1})^{n_{l-1}}. \tag{8}$$

Lemma 3.1 ([4]) *Let* $\pi(\lambda) = \prod_{k=0}^{l-1} (\lambda - \lambda_k)^{n_k}$ *be a polynomial of* λ, *with* $\lambda_k \in \mathbb{C}$, $n_k \in \mathbf{N}$, *and* $n_0 + \cdots + n_{l-1} = s$. *Then*

$$\frac{1}{\pi(\lambda)} = \sum_{k=0}^{l-1} \sum_{j=1}^{n_k} \frac{1}{(n_k - j)!} \left[\frac{d^{n_k - j}}{d\lambda^{n_k - j}} \frac{(\lambda - \lambda_k)^{n_k}}{\pi(\lambda)} \right]_{\lambda = \lambda_k} \frac{1}{(\lambda - \lambda_k)^j}. \tag{9}$$

Lemma 3.2 *If* $P(D)$ *in* (6) *has the decomposition* (8), *then*

$$\ker P(D) = \ker D_{\lambda_0}^{n_0} \oplus \cdots \oplus \ker D_{\lambda_{l-1}}^{n_{l-1}},$$

where $\ker D_{\lambda_i}^{n_i} = \{f | (D - \lambda_i)^{n_i} f = 0, \quad f \in C^{n_i}(\Omega) \otimes Cl_{m+1}, \quad n_i \in \mathbf{N}\}$.

Proof Inspired by Gong et al. [4], Cerejeiras et al. [6] and Ku et al. [7], we derive

$$\ker P(D) = \ker D_{\lambda_0}^{n_0} + \cdots + \ker D_{\lambda_{l-1}}^{n_{l-1}}$$

by Lemma 3.1.

Then, it is easy to prove that

$$\ker P(D) = \ker D_{\lambda_0}^{n_0} \oplus \cdots \oplus \ker D_{\lambda_{l-1}}^{n_{l-1}}$$

by the division algorithm. □

Theorem 3.3 *If* $f \in \ker P(D)$, *then there exist unique functions* $f_{i,j} \in \ker D_{\lambda_i}$, $i = 0, \ldots, l-1$, $j = 0, \ldots, n_i - 1$, *such that*

$$f = \sum_{i=0}^{l-1} f_{i,0} + \sum_{i=0}^{l-1} \sum_{j=1}^{n_i - 1} \mathbb{E}_\lambda^j f_{i,j}.$$

where $f_{i,0}, \ldots, f_{i,n_i-1}$ *are given as follows:*

$$\begin{cases} f_{i,0}(x) = (\mathbf{I} - C_1\mathbb{E}_\lambda D_\lambda)\left(\mathbf{I} - C_2\mathbb{E}_\lambda^2 D_\lambda^2\right)\cdots\left(\mathbf{I} - C_{n_i-1}\mathbb{E}_\lambda^{n_i-1} D_\lambda^{n_i-1}\right)f(x), \\ f_{i,1}(x) = C_1 D_\lambda\left(\mathbf{I} - C_2\mathbb{E}_\lambda^2 D_\lambda^2\right)\cdots\left(\mathbf{I} - C_{n_i-1}\mathbb{E}_\lambda^{n_i-1} D_\lambda^{n_i-1}\right)f(x), \\ \quad\vdots \\ f_{i,n_i-2}(x) = C_{n_i-2} D_\lambda^{n_i-2}(\mathbf{I} - C_{n_i-1}\mathbb{E}_\lambda^{n_i-1} D_\lambda^{n_i-1})f(x), \\ f_{i,n_i-1}(x) = C_{n_i-1} D_\lambda^{n_i-1} f(x), \end{cases} \tag{10}$$

and $C_k = \frac{1}{k!\lambda^k}$.

Proof Let $f(x) \in kerP(D)$. Then it follows by Lemma 3.2 that there exist unique functions f_i such that

$$f = \sum_{i=0}^{l-1} f_i,$$

where $f_i \in \ker D_{\lambda_i}^{n_i}$. Theorem 2.6 shows that there exist unique functions $f_{i,j}$, $i = 0, \ldots, l-1$, $j = 1, \ldots, n_i - 1$, such that

$$f_i = f_{i,0} + \sum_{j=1}^{n_i-1} \mathbb{E}_\lambda^j f_{i,j},$$

where $f_{i,j}$ are given in (5). Therefore, the proof completes. □

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Further Results on Meromorphic Functions Sharing a Set with Their Derivatives

Shengjiang Chen, Weichuan Lin, and Wenjun Yuan

Dedicated to Professor Yuzan He on the Occasion of his 80th Birthday

Abstract In this paper, we consider some properties of meromorphic functions sharing a set with their first derivatives. Our results extend and improve the related theorems which were obtained by Lü and Xu (Houst J Math 34:1213–1223, 2008) and Qi and Zhu (Math Slovaca 64(6):1421–1436, 2014). Moreover, examples show that the condition is necessary and sharp.

Keywords Derivative • Meromorphic function • Shared set

Mathematics Subject Classification (2010) Primary 30D35; 30D45

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1 Introduction and Main Results

In this paper, a meromorphic function will always mean meromorphic in the whole complex plane. We assume that the reader is familiar with the fundamental concepts of Nevanlinna's value distribution theory (see [3, 7]) and in particular with the most usual of its symbols: $m(r,f)$, $N(r,f)$, $T(r,f)$.

For f a nonconstant meromorphic function and S a set of complex numbers, let

$$E(S,f) = \bigcup_{a\in S}\{z \in \mathbb{C} : f(z) - a = 0\},$$

where a zero of multiplicity m is counted m times in the set $E(S,f)$. Similarly, let

$$\overline{E}(S,f) = \bigcup_{a\in S}\{z \in \mathbb{C} : f(z) - a = 0\},$$

where a zero of multiplicity m is counted only once in the set $\overline{E}(S,f)$.

For two nonconstant meromorphic functions f and g, we say that f and g share the set S CM when $E(S,f) = E(S,g)$. If $\overline{E}(S,f) = \overline{E}(S,g)$, it is said to be that f and g share the set S IM.

The unicity theory of meromorphic functions sharing values with their derivatives is the special case of uniqueness of meromorphic functions, and has found a wide utilization in many fields. The classical result in the respect was obtained by Rubel and Yang (see [7], Theorem 8.1) in 1976, which says that if f is an entire function and shares two finite values CM with f', then $f \equiv f'$. Later, many mathematicians (see [7], Chap. 8) contributed to this issue in the view of sharing values.

From the point of view of sharing sets, Fang and Zalcman [2] obtained a result in 2003 that there exists a finite set S containing three elements such that if f is a nonconstant entire function and $E(S,f) = E(S,f')$, then $f = f'$. In 2007, Chang et al. [1] extended the above result to an arbitrary set having three elements as follows.

Theorem 1.1 *Let f be a nonconstant entire function and let $S = \{a,b,c\}$, where a, b, and c are distinct complex numbers. If $E(S,f) = E(S,f')$, then either*

(i) $f(z) = Ce^{z}$; *or*
(ii) $f(z) = Ce^{-z} + \frac{2}{3}(a+b+c)$ *and* $(2a-b-c)(2b-a-c)(2c-a-b) = 0$; *or*
(iii) $f(z) = Ce^{\frac{1\pm i\sqrt{3}}{2}z} + \frac{3\pm i\sqrt{3}}{6}(a+b+c)$ *and* $a^2+b^2+c^2-ab-bc-ca = 0$, *where C is a nonzero constant.*

Later, Lü and Xu [5] obtained a result as follows when the set S contains two complex numbers by using the theory of normal families.

Theorem 1.2 *Let a and b be two distinct finite complex numbers with $a+b \neq 0$, and let f be a nonconstant entire function. If f and f' share the set $\{a,b\}$ CM, then f has the form $f(z) = Ae^{z}$ or $f(z) = Ae^{-z} + a + b$ where A is a nonzero constant.*

It is natural to ask: does the conclusions of Theorems 1.1 and 1.2 still hold for meromorphic functions? In this direction, Lü [4] proved the following theorem which improved Theorem 1.1 partially in 2011.

Theorem 1.3 *Let f be a transcendental meromorphic function with at most finitely many poles, and let $S = \{a, b, c\}$, where a, b, and c are distinct complex numbers. If f and its derivative f' satisfy $E(S, f) = E(S, f')$, then the conclusion of Theorem 1.1 holds.*

In 2014, Qi and Zhu [6] improved Theorem 1.2 to the next result.

Theorem 1.4 *Let f be a transcendental meromorphic function with at most finitely many poles, and let $S = \{a, b\}$ where $a + b \neq 0$ and a, b are two distinct complex numbers. If f and f' share S CM, then the conclusion of Theorem 1.2 holds.*

In [4, 6], they both said that they do not know whether the Theorems 1.3 and 1.4 still hold or not if f is a rational function, respectively. In this direction, Yuan et al. [8] proved the following result very recently.

Theorem 1.5 *Let f be a nonconstant meromorphic function with at most finitely many poles, and let $S = \{a, b, c\}$, where a, b, and c are distinct complex numbers. If f and its derivative f' satisfy $E(S, f) = E(S, f')$, then the conclusions of Theorem 1.1 hold.*

In this paper, we will show that Theorem 1.4 does not hold for rational functions and prove the following result firstly.

Theorem 1.6 *Let f be a nonconstant meromorphic function with at most finitely many poles, and let b be a nonzero finite complex number. If f and f' share $S_2 = \{0, b\}$ CM, then f has the form $f(z) = Ae^z$ or $f(z) = Ae^{-z} + b$ or $f(z) = \frac{b}{2} - \frac{b}{4(z-z_0)}$, where $A(\neq 0)$, z_0 are some constants.*

For the case $ab \neq 0$, we obtain the following result.

Theorem 1.7 *Let a, b be two distinct finite complex numbers with $ab \neq 0$, and let $R(z) = \frac{P(z)}{Q(z)}$ be a nonconstant irreducible rational function. If $R(z)$ and $R'(z)$ share $S_2 = \{a, b\}$ CM, then $\deg P = \deg Q + 1$. Furthermore, if $\lim_{z\to\infty} \frac{P(z)}{zQ(z)} \neq a, b$, then $R(z)$ assume the form*

$$R(z) = \frac{ab}{a+b}(z - z_0) + \frac{a+b}{2} - \frac{a+b}{4(z-z_0)}$$

for some constant z_0.

Remark 1.8 The following Example 1 shows that the conclusions of Theorem 1.7 could happen and hence the Theorem 1.4 does not hold for rational functions. In addition, Example 2 as follows shows that the condition $\lim_{z\to\infty} \frac{P(z)}{zQ(z)} \neq a, b$ is necessary.

Example 1 Let $S_2 = \{1, 2\}$ and let $R(z) = \frac{2}{3}z + \frac{3}{2} - \frac{3}{4z}$. By a simple calculation, we have $E(S_2, R) = E(S_2, R') = \left\{\pm\frac{3}{2}, \pm\frac{3}{4}\right\}$.

Example 2 Set $a \neq 0$, $b = 2a$, $S_2 = \{a, b\}$. Let $R(z) = \frac{P(z)}{Q(z)}$, where

$$P(z) = az^3 + \frac{a}{2}z^2 - \frac{a}{4}z - \frac{a}{8}, \quad Q(z) = z\left(z - \frac{1}{2}\right).$$

By a direct calculation, we have

$$\begin{aligned}(R-a)(R-b) &= \frac{az^3 + \frac{a}{2}z^2 - \frac{a}{4}z - \frac{a}{8} - az(z-\frac{1}{2})}{z(z-\frac{1}{2})} \cdot \frac{az^3 + \frac{a}{2}z^2 - \frac{a}{4}z - \frac{a}{8} - bz(z-\frac{1}{2})}{z(z-\frac{1}{2})} \\ &= \frac{a^2z^6 - 2a^2z^5 + \frac{7a^2}{4}z^4 - a^2z^3 + \frac{7a^2}{16}z^2 - \frac{a^2}{8}z + \frac{a^2}{64}}{z^2(z-\frac{1}{2})^2}\end{aligned}$$

and

$$\begin{aligned}(R'-a)(R'-b) &= \frac{az^4 - az^3 + \frac{a}{4}z - \frac{a}{16} - az^2(z-\frac{1}{2})^2}{z^2(z-\frac{1}{2})^2} \\ &\quad \cdot \frac{az^4 - az^3 + \frac{a}{4}z - \frac{a}{16} - bz^2(z-\frac{1}{2})^2}{z^2(z-\frac{1}{2})^2} \\ &= \frac{a^2z^6 - 2a^2z^5 + \frac{7a^2}{4}z^4 - a^2z^3 + \frac{7a^2}{16}z^2 - \frac{a^2}{8}z + \frac{a^2}{64}}{4z^4(z-\frac{1}{2})^4}\end{aligned}$$

Thus $E(R, S_2) = E(R', S_2)$. But $R(z)$ has two distinct poles.

2 Proofs of Theorems

Proof (The Proof of Theorem 1.6) From Theorem 1.4, we only need to deal with the case that f is rational. We may assume that

$$f(z) = H_1(z) + \frac{H_2(z)}{Q(z)}, \tag{1}$$

where the polynomial Q has $q(\geq 1)$ distinct zeros and satisfies $\deg Q(z) = n$, $H_1(z), H_2(z)$ are two polynomials with $\deg H_2(z) \leq \deg Q(z) - 1$. Clearly, $f'(z) = H_1'(z) + \frac{H_3(z)}{Q_1(z)}$, where $\deg Q_1(z) = n+q$ and $\deg H_3(z) = \deg H_2(z) + q - 1 \leq n+q-2$.

Next, we claim that $f'(z) \neq 0$. Otherwise, if there exist a point z_0 such that $f'(z_0) = 0$, then we can deduce from $E(S_2, f) = E(S_2, f')$ that both $f(z_0) = 0$ and $f(z_0) = b$ hold. But this is impossible. Thus, we have $H_1'(z) \equiv 0$ and $\deg H_3 =$

$\deg H_2 + q - 1 = 0$, which imply H_1, H_2 are constants, saying d and c , respectively, and hence $q = 1$. Without loss of generality, set $Q(z) = (z - z_0)^n$. So, (1) becomes

$$f(z) = d + \frac{c}{(z - z_0)^n}, \quad c \neq 0. \tag{2}$$

If $d = 0$, then we have $\#_{\mathrm{CM}}\{E(S_2,f)\} = n$, here and throughout this paper, we denote by $\#_{\mathrm{CM}}\{E\}$ the number of elements of the finite set E counting multiplicities. On the other hand, we have $\#_{\mathrm{CM}}\{E(S_2,f')\} = n+1$. This contradicts with $E(S_2,f) = E(S_2,f')$. Similarly, we can deduce that $d \neq b$.

So, $d \neq 0, b$. Then, we have $\#_{\mathrm{CM}}\{E(S_2,f)\} = 2n$ and $\#_{\mathrm{CM}}\{E(S_2,f')\} = n + 1$. It follows from $E(S_2,f) = E(S_2,f')$ that $n = 1$ and that $\frac{f(z)[f(z)-b]}{f'(z)[f'(z)-b]} = A(z - z_0)^2$ holds for some nonzero constant A. That is,

$$[d(z - z_0) + c][(d - b)(z - z_0) + c] \equiv Ac[c + b(z - z_0)^2]. \tag{3}$$

Comparing the coefficients of terms $(z-z_0)^0$, $(z-z_0)$ and $(z-z_0)^2$ in (3), we deduce that $A = 1$, $d = \frac{b}{2}$ and $c = -\frac{b}{4}$.

This completes the proof of Theorem 1.6. □

Proof (The Proof of Theorem 1.7) We may assume that the leading coefficient of $Q(z)$ is equal to 1, $Q(z)$ has $q(\geq 1)$ distinct zeros, $\deg Q(z) = n$ and $\deg P(z) = m$. First of all, we suppose contrary that $m \neq n + 1$. We consider the following two cases.

Case 1. Suppose that $m \leq n$. Then, we have $\#_{\mathrm{CM}}\{E(S_2, R)\} \leq 2n$. Set

$$R' = \frac{P_1}{Q_1}, \tag{4}$$

where $P_1(z)$ and $Q_1(z)$ are two mutually prime polynomials in z. Obviously, we can see that $\deg P_1 \leq m + q - 1 < n + q = \deg Q_1$ in this case. Thus, we have $\#_{\mathrm{CM}}\{E(S_2, R')\} = 2(n + q)$. But this contradicts with $E(S_2, R) = E(S_2, R')$.

Case 2. Suppose that $m \geq n + 2$. Then, we have $\#_{\mathrm{CM}}\{E(S_2, R)\} = 2m$. Setting R' as (4), we have $\deg P_1 = m + q - 1 > n + q = \deg Q_1$ in this case. Thus, we have $\#_{\mathrm{CM}}\{E(S_2, R')\} = 2(m + q - 1)$. It follows from $E(S_2, R) = E(S_2, R')$ that $q = 1$. Without loss of generality, we can assume that $Q(z) = (z - z_0)^n$. Clearly, $\frac{[R(z)-a][R(z)-b]}{[R'(z)-a][R'(z)-b]} = A(z - z_0)^2$ holds for some nonzero constant A. Namely,

$$\left(1 - \frac{a}{R(z)}\right)\left(1 - \frac{b}{R(z)}\right) = \left(\frac{R'(z)}{R(z)} - \frac{a}{R(z)}\right)\left(\frac{R'(z)}{R(z)} - \frac{b}{R(z)}\right)A(z - z_0)^2. \tag{5}$$

Let $z \to \infty$ and $z \to z_0$ in both sides of (5), respectively, it follows that $1 = A(m - n)^2$ and $1 = An^2$ in this case, respectively. Thus, we can obtain $m = 2n$

and $n \geq 2$ (if $n = 1$, then $m = n + 1$). Further, set $P(z) = \sum_{k=0}^{2n} a_k(z - z_0)^k$, where

$$a_{2n} \neq 0, \quad a_0 = P(z_0) \neq 0. \tag{6}$$

Hence, from $E(S_2, R) = E(S_2, R')$, we have

$$\begin{aligned}
&\left[\sum_{j=1}^{n} a_{n+j}(z-z_0)^{n+j} + (a_n - a)(z-z_0)^n + \sum_{t=0}^{n-1} a_t(z-z_0)^t\right] \\
&\quad \times \left[\sum_{j=1}^{n} a_{n+j}(z-z_0)^{n+j} + (a_n - b)(z-z_0)^n + \sum_{t=0}^{n-1} a_t(z-z_0)^t\right] \\
&\equiv A\left[\sum_{j=2}^{n} j a_{n+j}(z-z_0)^{n+j} + (a_{n+1} - a)(z-z_0)^{n+1} - \sum_{t=0}^{n-1}(n-t)a_t(z-z_0)^t\right] \\
&\quad \times \left[\sum_{j=2}^{n} j a_{n+j}(z-z_0)^{n+j} + (a_{n+1} - b)(z-z_0)^{n+1} - \sum_{t=0}^{n-1}(n-t)a_t(z-z_0)^t\right].
\end{aligned} \tag{7}$$

Comparing the coefficient of term $(z - z_0)^0$ and $(z - z_0)^{4n}$ in (7), we can obtain from $n \geq 2$ that $An^2 = 1$ and that $2a_{2n}a_0 = \frac{1}{n^2}[-2n^2 a_{2n}a_0]$, which contradicts with $a_{2n}a_0 \neq 0$ in (6). Hence, case 2 can not occur too.

Thus, we obtain the desired conclusion $\deg P = \deg Q + 1$ firstly.

Now, we will prove the latter work of Theorem 1.7. We may also assume that $Q(z)$ has $q(\geq 1)$ distinct zeros $\{z_0, z_1, \ldots, z_{q-1}\}$. Similarly, we can arrive at (5), where A is polynomial. Let $z \to \infty$ in both sides of (5) [in fact, here $A = A_1 \sum_{j=1}^{q-1}(z - z_j)^2$ $(A_1 \neq 0)$ if $q \geq 2$], it follows from the assumption $\lim_{z\to\infty} \frac{P(z)}{zQ(z)} \neq a, b$ that $q = 1$.

Next, we assume that $Q(z) = (z - z_0)^n$ and $P(z) = \sum_{k=0}^{n+1} a_k(z - z_0)^k$, where $a_{n+1} \neq 0$, $a_0 = P(z_0) \neq 0$. Similarly, we have

$$\begin{aligned}
&\left[a_{n+1}(z-z_0)^{n+1} + (a_n - a)(z-z_0)^n + \textstyle\sum_{t=0}^{n-1} a_t(z-z_0)^t\right] \\
&\quad \times \left[a_{n+1}(z-z_0)^{n+1} + (a_n - b)(z-z_0)^n + \textstyle\sum_{t=0}^{n-1} a_t(z-z_0)^t\right] \\
&\equiv A\left[(a_{n+1} - a)(z-z_0)^{n+1} - \textstyle\sum_{t=0}^{n-1}(n-t)a_t(z-z_0)^t\right] \\
&\quad \times \left[(a_{n+1} - b)(z-z_0)^{n+1} - \textstyle\sum_{t=0}^{n-1}(n-t)a_t(z-z_0)^t\right].
\end{aligned} \tag{8}$$

Comparing the coefficient of term $(z-z_0)^0$, $(z-z_0)$, ..., $(z-z_0)^{n-1}$ in (8) one by one, we can obtain $An^2 = 1$ and $a_1 = \cdots = a_{n-1} = 0$. Furthermore, by comparing the coefficients of terms $(z-z_0)^n$ and $(z-z_0)^{2n}$ in (8), we deduce that $n = 1$ (otherwise, if $n \geq 2$, then we have $a = b$, which is a contradiction). Finally, by comparing the coefficients of terms $(z-z_0)$, $(z-z_0)^2$ and $(z-z_0)^4$ in (8), we deduce from $ab \neq 0$ that

$$a+b \neq 0, \quad a_2 = \frac{ab}{a+b}, \quad a_1 = \frac{a+b}{2}, \quad a_0 = -\frac{a+b}{4},$$

which imply the assertion of Theorem 1.7 follows.

This completes the proof of Theorem 1.7. □

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Normal Criterion on Differential Polynomials

Cuiping Zeng

Abstract Let $k, q(\geq 2)$ be two positive integers, $b \neq 0$ be a complex number, and let $H(f,f',\ldots,f^{(k)})$ be a differential polynomial with $\frac{\Gamma}{\gamma}|_H < k+1$. Let $\mathcal{F}$ be a family of meromorphic functions defined in D, all of whose zeros have multiplicity at least $k+1$. If for each pair of functions f and g in $\mathcal{F}$, $(f^{(k)})^q + H(f,f',\ldots,f^{(k)})$ and $(g^{(k)})^q + H(g,g',\ldots,g^{(k)})$ shared b in D, then $\mathcal{F}$ is normal in D.

Keywords Meromorphic functions • Normal families • Shared values

Mathematics Subject Classification (2010) Primary 30D45

1 Introduction

Let f and g be meromorphic functions on a domain D in $\mathbb{C}$, and let a and b be complex numbers. If $g(z) = b$ whenever $f(z) = a$, we write $f(z) = a \Rightarrow g(z) = b$. If $f(z) = a \Rightarrow g(z) = a$ and $g(z) = a \Rightarrow f(z) = a$, we say that f and g share a in D.

Let k be a positive integer, $n_i (i = 0, 1, \ldots, k)$ be non-negative integers. A differential monomial of f is defined by $M(f,f',\ldots,f^{(k)}) = \prod_{i=0}^{k} f^{(i)^{n_i}}$. $\gamma_M = \sum_{i=0}^{k} n_i$ is called the degree of $M(f,f',\ldots,f^{(k)})$ and $\Gamma_M = \sum_{i=0}^{k}(i+1)n_i$ is called the weight of $M(f,f',\ldots,f^{(k)})$.

Let $H(f,f',\ldots,f^{(k)}) = \sum_{i=1}^{m} a_i(z)M_i(f,f',\ldots,f^{(k)})$, then $H(f,f',\ldots,f^{(k)})$ is called a differential polynomial of f. $\gamma_H = \max\limits_{1\leq i\leq k}\{\gamma_{M_i}\}$ is called the degree of $H(f,f',\ldots,f^{(k)})$ and $\Gamma_H = \max\limits_{1\leq i\leq k}\{\Gamma_{M_i}\}$ is called the weight of $H(f,f',\ldots,f^{(k)})$. The ratio of the weight to degree of H is denoted by $\frac{\Gamma}{\gamma}|_H = \max\limits_{1\leq i\leq k}\left\{\frac{\Gamma_{M_i}}{\gamma_{M_i}}\right\}$.

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Fang and Hong [1] considered a differential polynomial of f which omitted a value and proved the following theorem.

Theorem 1.1 *Let $\mathcal{F}$ be a family of meromorphic functions defined in D, k, $q(\geq 2)$ be two positive integers, and $H(f, f', \ldots, f^{(k)})$ be a differential polynomial with $\frac{\Gamma}{\gamma}|_H < k+1$. If the zeros of $f(z)$ are of multiplicity at least $k+1$ and $(f^{(k)})^q + H(f, f', \ldots, f^{(k)}) \neq 1$ for each $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

In this paper, we improve Theorem 1.1 as follows

Theorem 1.2 (Main Theorem) *Let $k, q(\geq 2)$ be two positive integers, $b \neq 0$ be a complex number, and let $H(f, f', \ldots, f^{(k)})$ be a differential polynomial with $\frac{\Gamma}{\gamma}|_H < k+1$. Let $\mathcal{F}$ be a family of meromorphic functions defined in D, all of whose zeros have multiplicity at least $k+1$. If for each pair of functions f and g in $\mathcal{F}$, $(f^{(k)})^q + H(f, f', \ldots, f^{(k)})$ and $(g^{(k)})^q + H(g, g', \ldots, g^{(k)})$ shared b in D, then $\mathcal{F}$ is normal in D.*

Example Let $D = \{z : |z| < 1\}$, $\mathcal{F} = \{f_n\}$, where $f_n(z) = nz^{k+1}$. Then

$$(f_n^{(k)}(z))^{k+1} + f_n(z) = [(n(k+1)!)^{k+1} + n]z^{k+1}.$$

We can see that for each pair of functions f_n and f_m in $\mathcal{F}$, $(f_n^{(k)})^{k+1} + f_n$ and $(f_m^{(k)})^{k+1} + f_m$ share 0, but $\mathcal{F}$ fails to be normal in D. This shows that $b \neq 0$ is necessary in Theorem 1.2.

2 Some Lemmas

For the proof of Theorem 1.2, we require the following results.

Lemma 2.1 ([2]) *Let k be a positive integer, let $\mathcal{F}$ be a family of functions meromorphic on the unit disc Δ, all of whose zeros have multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. Then if $\mathcal{F}$ is not normal at z_0, there exist, for each $0 \leq \alpha \leq k$,*

(a) points $z_n \in \Delta$, $z_n \to z_0$;
(b) functions $f_n \in \mathcal{F}$; and
(c) positive numbers $\rho_n \to 0^+$

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n \zeta) = g_n(\zeta) \to g(\zeta)$ locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathcal{C}$, all of whose zeros have multiplicity at least k, such that $g^{\#}(\zeta) \leq g^{\#}(0) = kA + 1$. In particular, g has order at most 2.

Lemma 2.2 ([3]) *Let $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 + q(z)/p(z)$, where $a_0, a_1, \ldots, a_n$ are constants with $a_n \neq 0$, and q and p are two co-prime polynomials, neither of which vanishes identically, with $\deg q < \deg p$; and let k be a positive* *integer and b a nonzero complex number. If* $f^{(k)} \neq b$, *and the zeros of* f *all have multiplicity at least* $k+1$, *then*

$$f(z) = \frac{b(z-d)^{k+1}}{k!(z-c)},$$

where c and d are distinct complex numbers.

3 Proof of Theorem 1.1

Proof Let $z_0 \in D$, we will show that $\mathcal{F}$ is normal at z_0. Let $D_\delta(z_0) = \{z : |z-z_0| < \delta\}$, $D_\delta^0(z_0) = \{z : 0 < |z-z_0| < \delta\}$. For $f \in \mathcal{F}$, we consider two cases.

Case 1. $[(f^{(k)})^q + H(f,f',\ldots,f^{(k)})](z_0) \neq b$. Then there exists a $\delta > 0$ such that $[(f^{(k)})^q + H(f,f',\ldots,f^{(k)})](z) \neq b$ in $D_\delta(z_0)$. Thus, for every $g \in \mathcal{F}$, the zeros of g have multiplicity at least $k+1$ and $[(g^{(k)})^q + H(g,g',\ldots,g^{(k)})] \neq b$ in D_δ. From the proof of Theorem 1.1 in [1], it is easy to see that the conclusion still hold for $[(g^{(k)})^q + H(g,g',\ldots,g^{(k)})] \neq b(\neq 0)$. Therefore, $\mathcal{F}$ is normal in D_δ, so $\mathcal{F}$ is normal at z_0.

Case 2. $[(f^{(k)})^q + H(f,f',\ldots,f^{(k)})](z_0) = b$. Next we consider two subcases.

Case 2.1 There exists a $\delta > 0$ such that $(f^{(k)})^q + H(f,f',\ldots,f^{(k)}) \neq b$ in D_δ^0. Then, by the condition of Theorem 1.2, for every $f_n \in \mathcal{F}$, we have $(f_n^{(k)})^q + H(f_n,f_n',\ldots,f_n^{(k)}) \neq b$ in D_δ^0, and $[(f_n^{(k)})^q + H(f_n,f_n',\ldots,f_n^{(k)})](z_0) = b$.

We may assume that $z_0 = 0$ and $\delta = 1$. Then for $z_n + \rho_n\xi \neq 0$

$$(f_n^{(k)}(z_n+\rho_n\xi))^q + H(f_n,f_n',\ldots,f_n^{(k)})(z_n+\rho_n\xi) \neq b,$$

and

$$[(f_n^{(k)})^q + H(f_n,f_n',\ldots,f_n^{(k)})](0) = b.$$

We claim that $\mathcal{F}$ is normal in the unit disc Δ.

Suppose, on the contrary, that $\mathcal{F}$ is not normal in Δ. Then by Lemma 2.1, we can find a subsequence of $\mathcal{F}$, which we may denote by $\{f_n\}$, $z_n \in \Delta$, $z_n \to 0$ and $\rho_n \to 0^+$ such that $g_n(\xi) = \rho_n^{-k} f_n(z_n+\rho_n\xi)$ converges locally uniformly with respect to the spherical metric to a nonconstant meromorphic function g on $\mathbb{C}$, all of whose zeros have multiplicity at least $k+1$.

Taking note of that

$$
\begin{aligned}
&(f_n^{(k)}(z_n+\rho_n\xi))^q+H(f_n,f_n',\ldots,f_n^{(k)})(z_n+\rho_n\xi)-b\\
&\quad=(g_n^{(k)}(\xi))^q+\sum_{i=1}^{m}a_i(z_n+\rho_n\xi)M_i(f_n,f_n'\ldots,f_n^{(k)})(z_n+\rho_n\xi)-b\\
&\quad=(g_n^{(k)}(\xi))^q+\sum_{i=1}^{m}a_i(z_n+\rho_n\xi)\rho_n^{(k+1)\gamma_{M_i}-\Gamma_{M_i}}M_i(g_n,g_n',\ldots,g_n^{(k)})(\xi)-b
\end{aligned}
$$

Considering $a_i(z)(i=1,2,\ldots,m)$ are analytic on D , we have

$$|a_i(z_n+\rho_n\xi)|\le M\left(\frac{1+r}{2},a_i(z)\right)<\infty,(i=1,2,\ldots,m),$$

for sufficiently large n.

Hence we deduce from $\frac{\Gamma}{\gamma}|_H<k+1$ that

$$\sum_{i=1}^{m}a_i(z_n+\rho_n\xi)\rho_n^{(k+1)\gamma_{M_i}-\Gamma_{M_i}}M_i(g_n,g_n',\ldots,g_n^{(k)})(\xi)$$

converges uniformly to 0 on $D_{\frac{1}{2}}(0)$.

Thus we know that

$$(g_n^{(k)}(\xi))^q+\sum_{i=1}^{m}a_i(z_n+\rho_n\xi)\rho_n^{(k+1)\gamma_{M_i}-\Gamma_{M_i}}M_i(g_n,g_n',\ldots,g_n^{(k)})(\xi)-b$$

converges uniformly to $(g^{(k)}(\xi))^q-b$ on $D_{\frac{1}{2}}(0)$.

Now we consider two subcases.

Case 2.1.1 $\frac{z_n}{\rho_n}\to\infty$. Then $(g^{(k)})^q\neq b$. For $q\ge 2$, so by Nevanlinna Theory, we have

$$
\begin{aligned}
T(r,g^{(k)})&\le\overline{N}(r,g^{(k)})+\overline{N}\left(r,\frac{1}{g^{(k)}-b_1}\right)+\cdots+\overline{N}\left(r,\frac{1}{g^{(k)}-b_q}\right)+S(r,g^{(k)})\\
&\le\frac{1}{k+1}N(r,g^{(k)})+S(r,g^{(k)})\\
&\le\frac{1}{k+1}T(r,g^{(k)})+S(r,g^{(k)}).
\end{aligned}
$$

where $b_i(i=1,2,\ldots,q)$ are solutions of $\omega^q=b$. Hence, we get that $T(r,g^{(k)})=S(r,g^{(k)})$. It follows that $g^{(k)}$ is a constant. Together with the zeros of g have multiplicity at least $k+1$, we get that g is a constant, a contradiction.

Case 2.1.2 $\frac{z_n}{\rho_n} \to -\alpha$. By Hurwitz's theorem, we deduce that $(g^{(k)}(\xi))^q \neq b$ for $\xi \neq \alpha$ and $(g^{(k)}(\alpha))^q = b$. Let $b_i (i = 1, 2, \ldots, q)$ be solutions of $\omega^q = b$. Without loss of generality, we may assume that $g^{(k)}(\alpha) = b_1$, then $g^{(k)}(\xi) \neq b_1$ for $\xi \neq \alpha$, and $g^{(k)}(\xi) \neq b_2$.

Firstly, we will show that $g(\xi)$ is not a transcendental meromorphic function. By Nevanlinna Theory, we have

$$\begin{aligned} T(r, g^{(k)}) &\le \overline{N}(r, g^{(k)}) + \overline{N}\left(r, \frac{1}{g^{(k)} - b_1}\right) + \overline{N}\left(r, \frac{1}{g^{(k)} - b_2}\right) + S(r, g^{(k)}) \\ &\le \frac{1}{k+1} N(r, g^{(k)}) + O(\log r) + S(r, g^{(k)}) \\ &\le \frac{1}{k+1} T(r, g^{(k)}) + O(\log r) + S(r, g^{(k)}). \end{aligned}$$

Hence, we get that $T(r, g^{(k)}) = O(\log r) + S(r, g^{(k)})$, it follows that $g(\xi)$ is not a transcendental meromorphic function. Obviously, $g(\xi)$ cannot be a polynomial, so $g(\xi)$ is a rational function. For $b_2 \neq 0$, by Lemma 2.2 we have

$$g(\xi) = \frac{b_2(\xi - d)^{k+1}}{k!(\xi - c)},$$

where c, d are distinct complex numbers. Thus

$$g^{(k)} = b_2 + \frac{A}{(\xi - c)^{k+1}},$$

where A is a nonzero complex number.

Obviously, $g^{(k)}(\xi) = b_1$ has $k + 1$ distinct solutions, which contradicts to the fact that $g^{(k)}(\xi) = b_1$ has only the solution $\xi = \alpha$.

Hence $\mathcal{F}$ is normal in Δ and so $\mathcal{F}$ is normal at z_0.

Case 2.2 $[(f^{(k)})^q + H(f, f', \ldots, f^{(k)})](z) \equiv b$ for all z in $D_\delta(z_0)$. Then, by the condition of Theorem 1.2, for every $f_n \in \mathcal{F}$, we have $(f_n^{(k)})^q + H(f_n, f_n', \ldots, f_n^{(k)}) \equiv b$ in $D_\delta(z_0)$.

Without loss of generality, we may again let $z_0 = 0$ and $\delta = 1$. We claim that $\mathcal{F}$ is normal in the unit disc Δ. Suppose, on the contrary, that $\mathcal{F}$ is not normal in Δ. Then by Lemma 2.1, we can find a subsequence of $\mathcal{F}$, which we may denote by $\{f_n\}$, $z_n \in \Delta$, $z_n \to 0$ and $\rho_n \to 0^+$ such that $g_n(\xi) = \rho_n^{-k} f_n(z_n + \rho_n \xi)$ converges locally uniformly with respect to the spherical metric to a nonconstant meromorphic function g on $\mathbb{C}$, all of whose zeros have multiplicity at least $k + 1$. As in case 2.1, we have $(g^{(k)})^q \equiv b$ in $D_{\frac{1}{2}}(0)$. Therefore, g is polynomial with $\deg(g) \le k$. This contradicts to the fact that the zeros of g have multiplicity at least $k + 1$. Hence $\mathcal{F}$ is normal in Δ and so $\mathcal{F}$ is normal at z_0.

Therefore $\mathcal{F}$ is normal in D. The proof of Theorem 1.2 is complete. □

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Part IV
Harmonic Analysis and Nonlinear PDEs

Jens Wirth and V. Georgiev

A Conjecture Regarding Optimal Strichartz Estimates for the Wave Equation

Neal Bez, Chris Jeavons, Tohru Ozawa, and Hiroki Saito

Abstract We propose a conjecture concerning the shape of initial data which extremise the classical Strichartz estimates for the wave propagator with initial data of Sobolev regularity $\frac{d-1}{4}$ in all spatial dimensions $d \geq 3$, complementing an earlier conjecture of Foschi in the critical case of $\frac{1}{2}$ regularity. Some supporting evidence for the conjectures is given.

Keywords Extremisers • Strichartz estimates • Wave equation

Mathematics Subject Classification (2010) Primary 35B45; Secondary 35L05, 42B37

1 Introduction

The classical Strichartz estimates for the (one-sided) wave propagator state that

$$\|e^{itD}f\|_{L^p(\mathbb{R}\times\mathbb{R}^d)} \leq \mathbf{W}(d,s)\|f\|_{\dot{H}^s(\mathbb{R}^d)} \tag{1}$$

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where $d \geq 2$, $s \in [\frac{1}{2}, \frac{d}{2})$, $p = \frac{2(d+1)}{d-2s}$, and $\mathbf{W}(d,s)$ is some finite constant which we take to be the *optimal* constant. The initial data f lies in the homogeneous Sobolev space with norm $\|f\|_{\dot{H}^s} = \|D^s f\|_{L^2}$ and $\widehat{D^s f} = |\cdot|^s \widehat{f}$. A difficult open problem is to determine for all such admissible (d,s) the exact shape of nontrivial extremal initial data which attain equality in (1). The existence of such extremisers is known for all admissible exponents as can be found in the work of Fanelli–Vega–Visciglia [5] for $s \in (\frac{1}{2}, \frac{d}{2})$ and Ramos [9] at the critical exponent $s = \frac{1}{2}$.

The value of $\mathbf{W}(d,s)$ and a characterisation of extremisers have been obtained only when (d,s) is either $(2, \frac{1}{2})$, $(3, \frac{1}{2})$, $(4, \frac{3}{4})$ or $(5, 1)$. In each case, the initial data $f_\star$ given by

$$\widehat{f_\star}(\xi) = |\xi|^{-1} e^{-|\xi|}$$

is an extremiser, and uniquely so up to the action of a group of symmetries. For $(2, \frac{1}{2})$ and $(3, \frac{1}{2})$, this is due to Foschi [6] and, furthermore, he conjectured that in the critical case $s = \frac{1}{2}$, the initial data $f_\star$ is an extremiser for all $d \geq 2$; this is currently still open for $d \geq 4$. Notice that $p = \frac{2(d+1)}{d-1}$ when $s = \frac{1}{2}$ and for $d \geq 4$ this exponent ceases to be an even integer, which gives rise to the apparent difficulty in resolving Foschi's conjecture.

For $(4, \frac{3}{4})$ and $(5, 1)$, it was shown in [1] and [2], respectively, that $f_\star$ is an extremiser (essentially uniquely). These cases (as well as $(3, \frac{1}{2})$) each satisfy $s = \frac{d-1}{4}$ or, equivalently, $p = 4$. Moreover, it was shown in [4] that amongst *radially symmetric* initial data, $f_\star$ is an extremiser when $s = \frac{d-1}{4}$. Our main purpose here is to propose a stronger version of Foschi's conjecture which identifies the exponent $s = \frac{d-1}{4}$ as another special case for which we expect $f_\star$ to be an extremiser for all $d \geq 3$, and moreover, $f_\star$ is not expected to be an extremiser all for other regularity exponents.

Conjecture 1.1 Suppose $d \geq 2$ and $s \in [\frac{1}{2}, \frac{d}{2})$.

(1) Then $f_\star$ is an extremiser for (1) if and only if $s \in \{\frac{1}{2}, \frac{d-1}{4}\}$.
(2) When $s = \frac{1}{2}$, the initial data f is an extremiser if and only if

$$\widehat{f}(\xi) = |\xi|^{-1} e^{a|\xi| + b\cdot\xi + c} \quad (|\mathrm{Re}(b)| < -\mathrm{Re}(a), c \in \mathbb{C}),$$

and when $s = \frac{d-1}{4}$ with $d \geq 4$, the initial data f is an extremiser if and only if

$$\widehat{f}(\xi) = |\xi|^{-1} e^{a|\xi| + b\cdot\xi + c} \quad (\mathrm{Re}(a) < 0, \mathrm{Re}(b) = 0, c \in \mathbb{C}).$$

To reiterate, all statements in this conjecture concerning the critical case $s = \frac{1}{2}$ were made by Foschi in [6] (see Conjecture 1.11). Part (1) is an "existence" claim concerning the existence of the particular initial data $f_\star$ as an extremiser, and Part (2) is a counterpart "uniqueness" claim. We anticipate that the claims in Conjecture 1.1

regarding the case $s = \frac{d-1}{4}$ are rather more tractable than $s = \frac{1}{2}$ simply because we always have $p = 4$ in the former case.

The presentation in terms of the Fourier transform of the initial data is to facilitate a comparison with, say, the analogous questions for the free Schrödinger propagator and the corresponding estimates

$$\|e^{it\Delta} f\|_{L^q(\mathbb{R}\times\mathbb{R}^d)} \leq \mathbf{S}(d,s)\|f\|_{\dot{H}^s(\mathbb{R}^d)}$$

where $d \geq 1$, $s \in [0, \frac{d}{2})$, $q = \frac{2(d+2)}{d-2s}$. At the critical exponent $s = 0$ it is conjectured that gaussian initial data should be extremal; that is,

$$\widehat{f}(\xi) = e^{a|\xi|^2 + b\cdot\xi + c}$$

for $\mathrm{Re}(a) < 0$, $b \in \mathbb{C}^d$ and $c \in \mathbb{C}$ (see [6] and [8] for the conjectures, and verification for $d = 1, 2$). It was observed in [3] that *only* when $s = 0$ can gaussians be extremal, contrasting with the two special values $s = \frac{1}{2}, \frac{d-1}{4}$ expected to be associated with the extremality of $f_\star$ for (1).

2 Further Supporting Evidence

In order to show that $f_\star$ is not an extremiser for (1) when $s \notin \{\frac{1}{2}, \frac{d-1}{4}\}$, it may be more fruitful to prove the stronger statement that $f_\star$ is not a critical point of the functional

$$f \mapsto \frac{\|e^{itD} f\|_{L^p(\mathbb{R}\times\mathbb{R}^d)}}{\|f\|_{\dot{H}^s(\mathbb{R}^d)}} \tag{2}$$

for such s. Based on this, we present the following partial result.

Theorem 2.1 *Suppose $s \in [\frac{1}{2}, \frac{d}{2}) \setminus \{\frac{1}{2}, \frac{d-1}{4}\}$ and $p \in 2\mathbb{N}$. Then $f_\star$ is not a critical point of the functional (2) and hence not an extremiser for (1).*

Proof We assume $f_\star$ is a critical point of (2). Equivalently, $f_\star$ is a solution of the Euler–Lagrange equation

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d} e^{-i(x\cdot\xi + t|\xi|)} e^{itD} f_\star(x)|e^{itD} f_\star(x)|^{p-2}\, \mathrm{d}x\mathrm{d}t = \lambda \widehat{D^{2s} f_\star}(\xi) \tag{3}$$

for almost every $\xi \in \mathbb{R}^d$, where λ is some nonzero constant. Writing $p = 2m$ for some integer m, we may write

$$e^{itD} f_\star(x)|e^{itD} f_\star(x)|^{p-2} = C \int\int e^{i(x,t)\cdot\vec{v}}\, \frac{e^{-(\sum_{j=1}^{m}|\xi_j| + \sum_{k=1}^{m-1}|\eta_k|)}}{\prod_{j=1}^{m}|\xi_j| \prod_{k=1}^{m-1}|\eta_k|}\, \mathrm{d}\xi\mathrm{d}\eta$$

where $\xi = (\xi_1, \dots, \xi_m) \in (\mathbb{R}^d)^m$, $\eta = (\eta_1, \dots, \eta_{m-1}) \in (\mathbb{R}^d)^{m-1}$ and

$$\vec{v} := \Big(\sum_{j=1}^{m} \xi_j - \sum_{k=1}^{m-1} \eta_k, \sum_{j=1}^{m} |\xi_j| - \sum_{k=1}^{m-1} |\eta_k| \Big).$$

Here C is a constant which depends on (d, m); such notation is used throughout this proof, although the precise value may change at each occurrence. The left-hand side of (3) with the variable ξ replaced by η_m becomes

$$C \int \int \delta\Big(\sum_{j=1}^{m} (\xi_j, |\xi_j|) - \sum_{k=1}^{m} (\eta_k, |\eta_k|) \Big) \frac{e^{-(\sum_{j=1}^{m} |\xi_j| + \sum_{k=1}^{m-1} |\eta_k|)}}{\prod_{j=1}^{m} |\xi_j| \prod_{k=1}^{m-1} |\eta_k|} \,\mathrm{d}\xi \mathrm{d}\eta$$

after performing the spatial and temporal integrations. Using the singular support of the delta distribution, this can be written as

$$C e^{-|\eta_m|} \int_{(\mathbb{R}^d)^{m-1}} \frac{e^{-2\sum_{k=1}^{m-1} |\eta_k|}}{\prod_{k=1}^{m-1} |\eta_k|} \int_{(\mathbb{R}^d)^m} \delta\Big(\xi_\eta - \sum_{j=1}^{m} \xi_j, \tau_\eta - \sum_{j=1}^{m} |\xi_j| \Big) \frac{\mathrm{d}\xi \mathrm{d}\eta}{\prod_{j=1}^{m} |\xi_j|}$$

where $(\xi_\eta, \tau_\eta) := (\sum_{k=1}^{m} \eta_k, \sum_{k=1}^{m} |\eta_k|)$. The inner ξ-integral may be computed exactly; this is done in Lemma 3.1 in [2] and we directly obtain

$$\int_{(\mathbb{R}^d)^m} \delta\Big(\xi_\eta - \sum_{j=1}^{m} \xi_j, \tau_\eta - \sum_{j=1}^{m} |\xi_j| \Big) \frac{\mathrm{d}\xi}{\prod_{j=1}^{m} |\xi_j|} = C(\tau_\eta^2 - |\xi_\eta|^2)^\alpha$$

where $\alpha := \frac{1}{2}(d-1)(m-1) - 1$. Hence (3) is equivalent to

$$\int_{(\mathbb{R}^d)^{m-1}} \frac{e^{-2\sum_{k=1}^{m-1} |\eta_k|}}{\prod_{k=1}^{m-1} |\eta_k|} \Big(\sum_{1 \le j < k \le m} (|\eta_j||\eta_k| - \eta_j \cdot \eta_k) \Big)^\alpha \mathrm{d}\eta = C|\eta_m|^{2s-1} \tag{4}$$

for each $\eta_m \in \mathbb{R}^d$. However it is easily checked that $m \geq 3$ and $\alpha > 0$ for $s \notin \{\frac{1}{2}, \frac{d-1}{4}\}$, and therefore, by letting $|\eta_m| \to 0$, we obtain a contradiction. □

We conclude by proving the following complementary result.

Theorem 2.2 *The initial data $f_\star$ is a critical point of the functional (2) if $s = \frac{d-1}{4}$ and any $d \geq 3$, or if $s = \frac{1}{2}$ and any $d \geq 3$ which is odd.*

Proof When $s = \frac{d-1}{4}$, if we continue to use the notation from the above proof, then we have $m = 2$ and $\alpha = \frac{d-3}{2}$. Thus, the left-hand side of (4) is

$$\int_{\mathbb{R}^d} |\eta_1|^{-1} e^{-2|\eta_1|} (|\eta_1||\eta_2| - \eta_1 \cdot \eta_2)^{\frac{d-3}{2}} \mathrm{d}\eta_1$$

and after changing to polar coordinates this is easily shown to be a constant multiple of $|\eta_2|^{\frac{d-3}{2}}$. Based on this, it follows that $f_\star$ is a critical point of (2).

Now suppose $s = \frac{1}{2}$ and $d = 2n - 1$ with $n \geq 2$. Since $p = 2(1 + \frac{1}{n-1}) \notin 2\mathbb{N}$ for $n \geq 3$, we are forced to adopt a different approach and we proceed making use of an explicit formula for $e^{itD}f_\star(x)$. For $n = 2$, Lemma 5.3 in [6] provides the expression

$$e^{itD}f_\star(x) = C(|x|^2 + (1 - it)^2)^{-(n-1)} \qquad \text{for } (t, x) \in \mathbb{R} \times \mathbb{R}^{2n-1} \tag{5}$$

and where C is some constant. For each $t \in \mathbb{R}$, if we write

$$\Phi_t(\xi) = \varphi_t(|\xi|) = |\xi|^{-1}e^{-(1-it)|\xi|}$$

then $e^{itD}f_\star(-x) = \widehat{\Phi}_t(x)$, and there is an amusing trick which allows us to conclude that (5) is in fact valid for all $n \geq 2$. Indeed, since Φ_t is radially symmetric we know that

$$\widehat{\Phi}_t(x) = (2\pi)^{n-\frac{1}{2}} \int_0^\infty \varphi_t(s)\widetilde{J}_{n-\frac{3}{2}}(s|x|)\, s^{2(n-1)}\mathrm{d}s =: \mathcal{F}_{2n-1}(\varphi_t)(|x|)$$

where $\widetilde{J}_\nu(x) = x^{-\nu}J_\nu(x)$, and J_ν is the Bessel function of the first kind of order ν. Using Theorem 1.1 in [7], which ultimately rests on the identity $\widetilde{J}_\nu'(r) = -r\widetilde{J}_{\nu+1}(r)$, we immediately obtain the relation

$$\mathcal{F}_{2n-1}(\varphi_t)(r) = -\frac{2\pi}{r}\frac{\mathrm{d}}{\mathrm{d}r}\mathcal{F}_{2n-3}(\varphi_t)(r)$$

for $n \geq 3$. Thus, from the fact that (5) holds for $n = 2$, we obtain the veracity of (5) for all $n \geq 2$, as claimed.

For the rest of the proof, assume $n \geq 3$ (as mentioned, when $n = 2$ we know from [6] the stronger fact that $f_\star$ is a global extremiser for (2)). In order to show that (3) holds, we pass to polar coordinates in the spatial integration, use (5) and the fact that for surface measure on $\mathbb{S}^{2(n-1)}$,

$$\widehat{\mathrm{d}\sigma} = (2\pi)^{n-\frac{1}{2}}|\cdot|^{-(n-\frac{3}{2})}J_{n-\frac{3}{2}}(|\cdot|),$$

to obtain the equivalent goal of

$$\int_0^\infty \int_{\mathbb{R}} \frac{e^{-ita}r^{n-\frac{1}{2}}J_{n-\frac{3}{2}}(ar)}{(r^2 + (1 - it)^2)^n(r^2 + (1 + it)^2)}\,\mathrm{d}t\mathrm{d}r = Ca^{n-\frac{3}{2}}e^{-a} \tag{6}$$

for all $a > 0$. We observe that it is possible to write

$$z^{\frac{1}{2}}J_{n-\frac{3}{2}}(z) = P_n(z^{-1})\sin(z) - Q_n(z^{-1})\cos(z) \tag{7}$$

for polynomials P_n and Q_n of degree $n-2$ and $n-3$, respectively, satisfying $P_n(-z) = (-1)^n P_n(z)$ and $Q_n(-z) = (-1)^{n-1} Q_n(z)$. One consequence is that $r \mapsto r^{n-\frac{1}{2}} J_{n-\frac{3}{2}}(ar)$ is even.

For each $r \in \mathbb{R} \setminus \{0\}$, the integrand

$$F_r(t) = (r^2 + (1-it)^2)^{-n}(r^2 + (1+it)^2)^{-1} e^{-ita}$$

is a holomorphic function in t in the lower half of the complex plane except for poles of order n at $t = \pm r - i$ and a computation shows that

$$\operatorname{res}(F_r; -r-i) = e^{-a} e^{iar} \sum_j \frac{C_j a^{j_1}}{r^{n+j_2}(r+i)^{j_4+1}},$$

where the summation is taken over $0 \le j_\ell \le n-1$, $j_1 + j_2 + j_3 + j_4 = n-1$, and each $C_j \in \mathbb{C}$ is some constant. Since $F_r = F_{-r}$, we immediately obtain a similar expression for $\operatorname{res}(F_r; r-i)$. Using Cauchy's residue theorem we obtain an expression for the integrable function $r \mapsto \int_{\mathbb{R}} F_r(t)\,\mathrm{d}t$ (which can be continued to $r = 0$). Hence (6) is equivalent to

$$\sum_j C_j a^{j_1} \int_{\mathbb{R}} \Phi_j(r)\,\mathrm{d}r = C a^{n-\frac{3}{2}} \tag{8}$$

with the r-integral taken in the principal value sense, and where

$$\Phi_j(r) := \frac{e^{iar} J_{n-\frac{3}{2}}(ar)}{r^{j_2+\frac{1}{2}}(r+i)^{j_4+1}}.$$

If Γ_ε denotes the semi-circle in the upper half of the complex plane centred at the origin and radius $\varepsilon > 0$, then using the asymptotics $J_{n-\frac{3}{2}}(z) \sim z^{n-\frac{3}{2}}$ as $|z| \to 0$, one can show that

$$\lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} \Phi_j(z)\,\mathrm{d}z = \begin{cases} C a^{n-\frac{3}{2}} & \text{if } \vec{j} = (0, n-1, 0, 0) \\ 0 & \text{otherwise.} \end{cases}$$

In order to show that the integral of Φ_j over Γ_R vanishes in the limit $R \to \infty$, we observe from (7) that

$$|e^{iaz} z^{\frac{1}{2}} J_{n-\frac{3}{2}}(az)| \le C R^{-(n-3)}$$

for $z \in \Gamma_R$ with $R > 0$ sufficiently large. It follows that $|\Phi_j(z)| \le C R^{-(n-1)}$ for such R, and hence $\int_{\Gamma_R} \Phi_j(z)\,\mathrm{d}z \to 0$ as $R \to \infty$. This gives (8) and completes our proof that $f_\star$ is a critical point of (2) when $s = \frac{1}{2}$ and d is odd. $\square$

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Uniform Regularity for the Time-Dependent Ginzburg-Landau-Maxwell Equations

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Abstract We study global weak solutions to the 3D time-dependent Ginzburg-Landau-Maxwell equations with the Coulomb gauge. We obtain uniform bounds of solutions with respect to the dielectric constant $\epsilon > 0$. Consequently, the existence of global weak solutions to the Ginzburg-Landau equations follows by a compactness argument.

Keywords Coulomb gauge • Superconductivity • Weak solutions

Mathematics Subject Classification (2010) 35A05, 35A40, 82D55

1 Introduction

We consider the 3D time-dependent Ginzburg-Landau-Maxwell system in superconductivity [1, 2]:

$$\eta\partial_t\psi + i\eta\kappa\phi\psi + \left(\frac{i}{\kappa}\nabla + A\right)^2\psi + (|\psi|^2 - 1)\psi = 0, \quad (1)$$

$$\epsilon(\partial_t^2 A + \partial_t\nabla\phi) + \partial_t A + \nabla\phi + \mathrm{rot}^2 A + \mathrm{Re}\left\{\left(\frac{i}{\kappa}\nabla\psi + \psi A\right)\overline{\psi}\right\} = 0, \quad (2)$$

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in $Q_T := (0,T) \times \Omega$, with boundary and initial conditions

$$\nabla\psi \cdot \nu = 0,\ A \cdot \nu = 0,\ \mathrm{rot}A \times \nu = 0,\ \nabla\phi \cdot \nu = \ \text{ on } \ (0,T) \times \partial\Omega, \tag{3}$$

$$(\psi, A, \partial_t A, \phi)(0,\cdot) = (\psi_0, A_0, A_1, \phi_0)(\cdot) \ \text{ in } \ \Omega \subset \mathbb{R}^3. \tag{4}$$

Here, the unknowns ψ, A, and ϕ are $\mathbb{C}$-valued, $\mathbb{R}^d$-valued, and $\mathbb{R}$-valued functions, respectively, and they stand for the order parameter, the magnetic potential, and the electric potential, respectively. η and κ are Ginzburg-Landau positive constants, ϵ is the dielectric constant and is supposedly very small in superconductors, and $i := \sqrt{-1}$. Ω is a simply connected and bounded domain with smooth boundary $\partial\Omega$ and ν is the outward unit normal to $\partial\Omega$. $\overline{\psi}$ denotes the complex conjugate of ψ, $\mathrm{Re}\psi := (\psi + \overline{\psi})/2$ is the real part of ψ and $|\psi|^2 := \psi\overline{\psi}$ is the density of superconductivity carriers. T is any given positive constant.

It is well known that the Ginzburg-Landau-Maxwell system is gauge invariant, that is, if (ψ, A, ϕ) is a solution of (1)–(4), then for any real valued smooth function χ, $(\psi e^{i\kappa\chi}, A + \nabla\chi, \phi - \partial_t\chi)$ is also a solution of (1)–(4). So in order to obtain the well-posedness of the problem, we need to impose the gauge condition. From physical point of view, one usually has four types of the gauge condition:

(1) Coulomb gauge: $\mathrm{div}\, A = 0$ in Ω and $\int \phi dx = 0$.
(2) Lorentz gauge of type I: $\phi = -\mathrm{div}\, A$ in Ω.
(3) Lorentz gauge of type II: $\partial_t\phi = -\mathrm{div}\, A$ in Ω.
(4) Temporal gauge: $\phi = 0$ in Ω.

In 1999, Tsutsumi and Kasai [3] proved the existence and uniqueness of global weak solutions to the problem with the Coulomb gauge under the assumption that $\psi_0 \in H^1 \cap L^\infty, A_0 \in H^1, A_1 \in L^2$, and $\phi_0 \in H^1$. Very recently, Fan and Ozawa [4] prove a similar result as that in [3] with initial data belonging to a larger space $\psi_0 \in L^\infty \cap W^{\frac{2}{3},\frac{3}{2}}$. The aim of this paper is to give a further result, we will prove

Theorem 1.1 *Suppose that* $0 < \epsilon < 1,\ \psi_0 \in L^4 \cap W^{2-\frac{2}{r},r}$ *for some* r *with* $\frac{5}{3} < r \leq 2,\ \mathrm{div}\, A_0 = 0,\ \ A_0 \in H^1,\ \ A_1 \in L^2$, *and* $\phi_0 \in H^1$. *Then for any* $T > 0$ *there exists a unique weak solution* $(\psi_\epsilon, A_\epsilon, \phi_\epsilon)$ *of* (1)–(4) *satisfying*

$$\begin{aligned}
&\|\psi_\epsilon\|_{L^\infty(0,T;L^4)\cap L^6(Q_T)} \leq C,\\
&\|\partial_t\psi_\epsilon\|_{L^2(0,T;L^p)} + \|\psi_\epsilon\|_{L^2(0,T;W^{2,p})} \leq C,\\
&\|\psi_\epsilon\|_{L^2(0,T;L^\infty)\cap L^2(0,T;W^{1,3})} \leq C,\\
&\|A_\epsilon\|_{L^\infty(0,T;H^1)} \leq C,\ \ \|\partial_t A_\epsilon\|_{L^2(0,T;L^2)} \leq C,\\
&\|\phi_\epsilon\|_{L^2(0,T;H^1)} \leq C,
\end{aligned} \tag{5}$$

where C *is independent of* $\epsilon > 0$ *and* $p = \frac{3r}{5-r}$.

Remark 1.1 Our proof is different from that of [3]. Our key estimate is to obtain $L^2(0,T;W^{2,p})$ estimates of ψ ($L^p - L^q$ theory), while their proof in [3] is to get $W_2^{2,1}$ estimates of ψ (L^2 theory). Thus our assumption on the initial data ψ_0 is weaker than that in [3]. By the definition of p, r is rewritten as $r = \frac{5p}{p+3}$ and p ranges over $\frac{3}{2} < p \leq 2$ if $\frac{5}{3} < r \leq 2$.

Remark 1.2 As soon as the uniform a priori estimates with respect to $\epsilon > 0$ such as (5) are established, the standard compactness arguments show the existence of a convergent subsequence $(\psi_{\epsilon_j}, A_{\epsilon_j}, \phi_{\epsilon_j})$ with $\epsilon_1 > \epsilon_2 > \cdots > \epsilon_j \downarrow 0$ as $j \to \infty$ for (1)–(4). When $\epsilon = 0$, the Ginzburg-Landau-Maxwell system reduces to the well-known Ginzburg-Landau equations, which have received many studies [5–13].

2 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. The proof of uniqueness part has been given in [4] and thus we omit the details here. To prove the existence part, we only need to prove a priori estimates (5). From now on, we drop the subscript ϵ for simplicity. In the following calculations, we need to keep track of the independence of ϵ on constants C.

Multiplying (1) by $\overline{\psi}$ and integrating by parts, then taking the real part, we see that

$$\frac{\eta}{2}\frac{d}{dt}\int |\psi|^2 dx + \int \left|\frac{i}{\kappa}\nabla\psi + \psi A\right|^2 dx + \int |\psi|^4 dx = \int |\psi|^2 dx,$$

which gives

$$\int_0^T \int \left|\frac{i}{\kappa}\nabla\psi + \psi A\right|^2 dxdt \leq C. \tag{6}$$

Similarly, multiplying (1) by $|\psi|^2\overline{\psi}$ and integrating by parts, then taking the real part, we find that

$$\frac{\eta}{4}\frac{d}{dt}\int |\psi|^4 dx + \int \left|\frac{i}{\kappa}\nabla\psi + \psi A\right|^2 |\psi|^2 dx + \int |\psi|^6 dx = \int |\psi|^4 dx,$$

which implies

$$\|\psi\|_{L^\infty(0,T;L^4)} + \|\psi\|_{L^6(Q_T)} \leq C, \tag{7}$$

$$\int_0^T \int \left|\frac{i}{\kappa}\nabla\psi + \psi A\right|^2 |\psi|^2 dxdt \leq C. \tag{8}$$

Testing (2) by $\partial_t A + \nabla\phi$ and using (8), we derive

$$\frac{1}{2}\frac{d}{dt}\int(\epsilon|\partial_t A|^2 + \epsilon|\nabla\phi|^2 + |\mathrm{rot}\,A|^2)dx + \int(|\partial_t A|^2 + |\nabla\phi|^2)dx$$
$$= -\int \mathrm{Re}\left\{\left(\frac{i}{\kappa}\nabla\psi + \psi A\right)\overline{\psi}\right\}(\partial_t A + \nabla\phi)dx$$
$$\leq \frac{1}{2}\int(|\partial_t A|^2 + |\nabla\phi|^2)dx + C\int\left|\frac{i}{\kappa}\nabla\psi + \psi A\right|^2|\psi|^2 dx,$$

which yields

$$\|\partial_t A\|_{L^2(Q_T)} + \|\phi\|_{L^2(0,T;H^1)} \leq C, \tag{9}$$
$$\|A\|_{L^\infty(0,T;H^1)} \leq C, \tag{10}$$

where we have used the well-known Poincaré inequality for vector-valued functions of the form

$$\|A\|_{L^2} \leq C\|\mathrm{rot}\,A\|_{L^2}. \tag{11}$$

Inequalities (7), (8) and (10) lead to

$$\|\psi\|_{L^2(0,T;H^1)} \leq C. \tag{12}$$

Equation (1) can be rewritten as

$$\eta\partial_t\psi - \frac{1}{\kappa^2}\Delta\psi = f := -i\eta\kappa\phi\psi - \frac{2i}{\kappa}A\nabla\psi - |A|^2\psi - (|\psi|^2 - 1)\psi. \tag{13}$$

By the well-known $L^2(0,T;W^{2,p})$-regularity theory of the heat equation [14], and using (9), (10), (12) and (7), we have

$$\int_0^T \|\partial_t\psi\|_{L^p}^2 dt + \int_0^T \|\psi\|_{W^{2,p}}^2 dt$$
$$\leq C\|\psi_0\|_{W^{2-\frac{2}{r},r}}^2 + C\int_0^T \|f\|_{L^p}^2 dt$$
$$\leq C + C\int_0^T \|\phi\|_{L^6}^2\|\psi\|_{L^3}^2 dt + C\int_0^T \|A\|_{L^6}^2\|\nabla\psi\|_{L^q}^2 dt$$
$$+ C\int_0^T \|A\|_{L^6}^4\|\psi\|_{L^6}^2 dt \quad \left(\frac{1}{p} = \frac{1}{6} + \frac{1}{q}\right)$$

$$
\begin{aligned}
&\leq C + C\int_0^T \|\nabla\psi\|_{L^q}^2 dt \\
&\leq C + C\int_0^T \|\nabla\psi\|_{L^2}^{2(1-\theta)} \|\psi\|_{W^{2,p}}^{2\theta} dt \quad \left(\theta = \frac{4p-6}{5p-6}\right) \\
&\leq C + C\int_0^T \|\nabla\psi\|_{L^2}^2 dt + \frac{1}{2}\int_0^T \|\psi\|_{W^{2,p}}^2 dt,
\end{aligned}
$$

which yields

$$
\|\partial_t\psi\|_{L^2(0,T;L^p)} + \|\psi\|_{L^2(0,T;W^{2,p})} \leq C. \tag{14}
$$

By the Sobolev embedding, we arrive at

$$
\|\psi\|_{L^2(0,T;L^\infty)} + \|\psi\|_{L^2(0,T;W^{1,3})} \leq C. \tag{15}
$$

This completes the proof.

□

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L^p-boundedness of Functions of Schrödinger Operators on an Open Set of $\mathbb{R}^d$

Tsukasa Iwabuchi, Tokio Matsuyama, and Koichi Taniguchi

Abstract The purpose of this paper is to prove L^p-boundedness of an operator $\varphi(H_V)$, where $H_V = -\Delta + V(x)$ is the Schrödinger operator on an open set Ω of $\mathbb{R}^d$ ($d \geq 1$). Moreover, we prove uniform L^p-estimates for $\varphi(\theta H_V)$ with respect to a parameter $\theta > 0$. This paper will give an improvement of our previous paper (Iwabuchi et al., L^p-mapping properties for Schrödinger operators in open sets of $\mathbb{R}^d$, submitted); assumptions of potential V and space dimension.

Keywords Functional calculus • L^p-estimates • Schrödinger operators

Mathematics Subject Classification (2010) Primary 47F05; Secondary 26D10

1 Introduction

Let Ω be an open set in $\mathbb{R}^d$ ($d \geq 1$). We consider the Schrödinger operator

$$H_V = -\Delta + V(x) = -\sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} + V(x),$$

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where $V(x)$ is a real-valued measurable function on Ω. When H_V is self-adjoint on $L^2(\Omega)$, the operator $\varphi(H_V)$ can be defined on $L^2(\Omega)$ by

$$\varphi(H_V) := \int_{-\infty}^{\infty} \varphi(\lambda)\, dE_{H_V}(\lambda)$$

for a Borel measurable function φ on $\mathbb{R}$, where $\{E_{H_V}(\lambda)\}_{\lambda\in\mathbb{R}}$ is the spectral resolution of the identity for H_V. This paper is devoted to obtaining L^p-boundedness of $\varphi(H_V)$ for $1 \le p \le \infty$, and to proving uniform L^p-estimates for $\varphi(\theta H_V)$ with respect to a parameter $\theta > 0$. In our previous work [5], we established these estimates when the potential V is of the Kato class in the case $d \ge 3$. The aim of this paper is to improve the results in [5] under a weaker assumption on V in all space dimensions.

Throughout this paper we always assume that V satisfies

$$V = V_+ - V_-, \quad V_\pm \ge 0, \quad V_+ \in L^1_{\mathrm{loc}}(\Omega), \quad V_- \in K_d(\Omega). \tag{1}$$

Here $K_d(\Omega)$ is the Kato class. More precisely, we say that V_- belongs to $K_d(\Omega)$ if

$$\begin{cases} \displaystyle\lim_{r\to 0}\sup_{x\in\Omega}\int_{\Omega\cap\{|x-y|<r\}} \frac{|V_-(y)|}{|x-y|^{d-2}}\, dy = 0, & d \ge 3, \\ \displaystyle\lim_{r\to 0}\sup_{x\in\Omega}\int_{\Omega\cap\{|x-y|<r\}} \log\{|x-y|^{-1}\}|V_-(y)|\, dy = 0, & d = 2, \\ \displaystyle\sup_{x\in\Omega}\int_{\Omega\cap\{|x-y|<1\}} |V_-(y)|\, dy < \infty, & d = 1. \end{cases}$$

Then it is well known from Theorem VIII.15 in [10] (see also [5, 9]) that $-\Delta + V$ has a self-adjoint realization on $L^2(\Omega)$, and we denote by H_V its realization with the domain

$$\mathcal{D}(H_V) = \left\{ u \in H^1_0(\Omega) \,\middle|\, \sqrt{V_+}u \in L^2(\Omega),\ H_V u \in L^2(\Omega) \right\},$$

where $H^1_0(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with $H^1(\Omega)$-norm. Moreover H_V is semi-bounded, and the infimum of the spectrum of H_V is finite. Hence

$$\int_\Omega (H_V u)\overline{u}\, dx \ge -M_0 \|u\|^2_{L^2(\Omega)}, \quad \forall\, u \in \mathcal{D}(H_V),$$

where

$$M_0 \text{ is the infimum of the spectrum of } H_V. \tag{2}$$

Here and below, we denote by $\mathcal{S}(\mathbb{R})$ the space of rapidly decreasing functions on $\mathbb{R}$. We write $\|\cdot\|_{\mathcal{B}(X,Y)}$ as the operator norm from X to Y, and in particular, $\|\cdot\|_{\mathcal{B}(X)}$ when $X = Y$.

We shall prove the following.

Theorem 1.1 *Let $1 \le p \le \infty$ and $\varphi \in \mathcal{S}(\mathbb{R})$. Assume that the measurable potential V satisfies assumption (1). Then the following assertions hold:*

(i) *There exists a constant $C = C(d, \varphi, V) > 0$ such that*

$$\|\varphi(\theta H_V)\|_{\mathcal{B}(L^p(\Omega))} \le C \quad \textit{for any } 0 < \theta \le 1. \tag{3}$$

(ii) *Assume that V further satisfies*

$$\begin{cases} V_- = 0, & \textit{if } d = 1, 2, \\ \sup\limits_{x\in\Omega} \displaystyle\int_\Omega \frac{|V_-(y)|}{|x-y|^{d-2}}\, dy < \frac{\pi^{d/2}}{\Gamma(d/2-1)}, & \textit{if } d \ge 3, \end{cases} \tag{4}$$

where $\Gamma(\cdot)$ is the Gamma function. Then (3) holds for any $\theta > 0$.

When $\Omega = \mathbb{R}^d$, there are several known results. For example, Jensen and Nakamura proved the assertion (i) in Theorem 1.1 (see [6, 7]). Georgiev and Visciglia proved the assertion (ii) under $V \ge 0$ (see [3]), and then, D'Ancona and Pierfelice extended [3] to the operators with Kato class potentials satisfying (4) (see [1]). When Ω is an open subset of $\mathbb{R}^d$ or of the metric measure space with doubling volume property, several authors studied L^p-boundedness of $\varphi(L)$ for any non-negative self-adjoint operator L such that the integral kernel of semigroup e^{-tL} satisfies Gaussian upper bound (see, e.g., [2, 8, 9]). Duong, Ouhabaz, and Sikora also proved the estimate (3) of $\varphi(\theta L)$ for any $\theta > 0$ by using several estimates for the integral kernel of $\varphi(\sqrt{L})$. However they need non-negativity of L and the condition of compact support of φ to prove (3) (see [2]). In Theorem 1.1 we obtain (3) under the assumption that H_V admits negative eigenvalues and φ does not necessarily have compact support. The strategy is different from them. In our previous work [5], we proved the assertion (ii) for Kato class potentials on open sets of $\mathbb{R}^d$ ($d \ge 3$) satisfying assumption (4). Theorem 1.1 will be proved by some slight modifications of argument in [5]. Later on, we will use the scaled amalgam spaces to prove uniform L^p-estimates. So, let us define the spaces $\ell^p(L^q)_\theta$ as follows:

Definition 1.2 (Scaled Amalgam Spaces $\ell^p(L^q)_\theta$) Let $1 \le p, q \le \infty$ and $\theta > 0$. The space $\ell^p(L^q)_\theta$ is defined as

$$\ell^p(L^q)_\theta = \ell^p(L^q)_\theta(\Omega) := \Big\{ f \in L^q_{\mathrm{loc}}(\overline{\Omega}) \;\Big|\; \sum_{n\in\mathbb{Z}^d} \|f\|^p_{L^q(C_\theta(n))} < \infty \Big\}$$

with norm

$$\|f\|_{\ell^p(L^q)_\theta} = \Big(\sum_{n\in\mathbb{Z}^d} \|f\|^p_{L^q(C_\theta(n))} \Big)^{1/p},$$

where $C_\theta(n)$ is the intersection of Ω and the cube centered at $\theta^{1/2}n (n \in \mathbb{Z}^d)$ with side length $\theta^{1/2}$.

When $\Omega = \mathbb{R}^d$ and $\theta = 1$, the scaled amalgam spaces $\ell^p(L^q)_\theta$ coincide with the classical amalgam spaces due to Holland and Wiener (see [4, 12, 13]). It can be checked that $\ell^p(L^q)_\theta$ is a Banach space with norm $\|\cdot\|_{\ell^p(L^q)_\theta}$ and has the property that

$$\ell^p(L^q)_\theta \hookrightarrow L^p(\Omega) \cap L^q(\Omega) \quad \text{for } 1 \le p \le q \le \infty.$$

This paper is organized as follows. In Sect. 2 we will prepare key lemmas to prove Theorem 1.1, which state the estimates for the resolvent of H_V and $\varphi(\theta H_V)$ on the amalgam spaces. In Sect. 3 we will prove Theorem 1.1.

2 Key Lemmas

In this section we prepare L^p-$\ell^p(L^q)_\theta$-estimates for the resolvent of H_V and uniform $\ell^p(L^2)_\theta$-estimates for $\varphi(\theta H_V)$ with respect to θ. These estimates play an important role in proving Theorem 1.1.

L^p-$\ell^p(L^q)_\theta$-estimates for the resolvent of H_V can be proved by using the following Gaussian upper bound for the integral kernel of semigroup e^{-tH_V} generated by H_V.

Proposition 2.1 *Assume that V satisfies assumption (1). Let $K(t, x, y)$ be the integral kernel of semigroup $\{e^{-tH_V}\}_{t>0}$ generated by H_V. Then there exists a positive constant $C = C(d, V)$ such that*

$$0 \le K(t, x, y) \le Ct^{-d/2}e^{-|x-y|^2/8t}, \quad 0 < t \le 1, \quad x, y \in \Omega. \tag{5}$$

If we further assume that V_- satisfies assumption (4) in Theorem 1.1, then the estimate (5) holds for any $t > 0$.

Proposition 2.1 can be proved along the same argument as Proposition 3.1 in [5] (see also Ouhabaz [9]).

As a consequence of Proposition 2.1, we have the following.

Corollary 2.2 *Let $1 \le p \le q \le \infty$ and $\beta > d(1/p - 1/q)/2$. Assume that V satisfies assumption (1). Then for any $M > M_0$, there exists a positive constant*

$C = C(d, p, q, \beta, M)$ such that

$$\|(\theta H_V + M)^{-\beta}\|_{\mathcal{B}(L^p(\Omega), \ell^p(L^q)_\theta)} \le C\theta^{-d(1/p-1/q)/2}, \quad 0 < \theta \le 1, \tag{6}$$

where M_0 is defined by (2)*. If we further assume that V_- satisfies assumption* (4)*, then the estimate* (6) *holds for any $\theta > 0$.*

Outline of proof of Corollary 2.2 The following formula is well known (see, e.g., (A9) in page 449 of Simon [11]): For any $M > M_0$ and $\beta > 0$,

$$(H_V + M)^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} e^{-Mt} e^{-tH_V}\, dt.$$

Combining this formula with Proposition 2.1, we can prove Corollary 2.2 along the completely same argument of proof of Theorem 4.1 in [5] (see also [7]). □

Next, we prepare uniform $\ell^p(L^2)_\theta$-estimates for $\varphi(\theta H_V)$ with respect to the parameter θ.

Lemma 2.3 *Let $1 \le p \le 2$. Assume that V satisfies assumption* (1)*. Then there exists a positive constant $C = C(d, p, V)$ such that*

$$\|\varphi(\theta H_V)\|_{\mathcal{B}(\ell^p(L^2)_\theta)} \le C, \quad 0 < \theta \le 1. \tag{7}$$

If we further assume that V_- satisfies assumption (4)*, then the estimate* (7) *holds for any $\theta > 0$.*

For the details of proof of Lemma 2.3, see [5, Section 6] (see also [7, Section 2.B, Section 3]).

3 Proof of Theorem 1.1

We have only to prove the assertion (i) in Theorem 1.1, since the assertion (ii) can be also proved in the same way. It is sufficient to show that there exists a positive constant C such that

$$\|\varphi(\theta H_V) f\|_{L^1(\Omega)} \le C\|f\|_{L^1(\Omega)}, \quad f \in L^1(\Omega),\ 0 < \theta \le 1. \tag{8}$$

In fact, once (8) is proved, then L^∞-estimate for $\varphi(\theta H_V)$ is obtained by duality argument, and hence, we obtain L^p-estimates for $1 \le p \le \infty$ by applying the Riesz-Thorin interpolation theorem.

Recalling the definition (2) of the constant M_0, let $M > M_0$ and $\beta > d/4$. Let us take $\tilde{\varphi} \in \mathcal{S}(\mathbb{R})$ as

$$\tilde{\varphi}(\lambda) = (\lambda + M)^\beta \varphi(\lambda), \quad \lambda > -M.$$

Then, by using Schwarz inequality, Corollary 2.2, and Lemma 2.3, we obtain for any $f \in L^1(\Omega)$,

$$\begin{aligned}\|\varphi(\theta H_V)f\|_{L^1(\Omega)} &\leq \theta^{d/4}\|\varphi(\theta H_V)f\|_{\ell^1(L^2)_\theta} \\ &= \theta^{d/4}\|\tilde{\varphi}(\theta H_V)(\theta H_V + M)^{-\beta}f\|_{\ell^1(L^2)_\theta} \\ &\leq C\theta^{d/4}\|(\theta H_V + M)^{-\beta}f\|_{\ell^1(L^2)_\theta} \\ &\leq C\theta^{d/4}\cdot\theta^{-d/4}\|f\|_{L^1(\Omega)} \\ &= C\|f\|_{L^1(\Omega)},\end{aligned}$$

where the constant C is independent of θ. The proof of Theorem 1.1 is finished.

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The Kirchhoff Equation with Gevrey Data

Tokio Matsuyama and Michael Ruzhansky

Abstract In this article the Cauchy problem for the Kirchhoff equation is considered, and the almost global existence of Gevrey space solutions is described.

Keywords Gevrey spaces • Kirchhoff equation

Mathematics Subject Classification (2010) Primary 35L40, 35L30; Secondary 35L10, 35L05, 35L75

1 Introduction

In this article we shall describe the recent result on the almost global existence of Gevrey space solutions to the Cauchy problem for the Kirchhoff equation of the form

$$\begin{cases} \partial_t^2 u - \left(1 + \int_{\mathbb{R}^n} |\nabla u(t,y)|^2 \, dy\right) \Delta u = 0, & t > 0, \quad x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), & \qquad\quad x \in \mathbb{R}^n. \end{cases} \tag{1}$$

G. Kirchhoff proposed Eq. (1) in his book on mathematical physics in 1876, as a model equation for transversal motion of the elastic string, where Ω is a bounded interval of $\mathbb{R}^1$ (see [16], and for a finite dimensional approximation problem, see Nishida [24]). Since then, it was first in 1940 that Bernstein proved the existence of global in time analytic solutions on an interval of real line in his celebrated paper [2].

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After him, Arosio and Spagnolo discussed the global existence of analytic solutions in higher spatial dimensions (see [1]), and D'Ancona and Spagnolo proved analytic well-posedness for the degenerate Kirchhoff equation (see [5], and also Kajitani and Yamaguti [15]).

As it is well known, Eq. (1) has a Hamiltonian structure. More precisely, let us define the energy

$$H(u;t) := \frac{1}{2}\left\{\|\nabla u(t)\|_{L^2}^2 + \|\partial_t u(t)\|_{L^2}^2\right\} + \frac{1}{4}\|\nabla u(t)\|_{L^2}^4.$$

Then we have

$$H(u;t) = H(u;0)$$

as long as a solution exists. Nevertheless it involves a challenging problem *whether or not, one can prove the existence of time global solutions corresponding to data in Gevrey classes, H^∞-class or standard Sobolev spaces without smallness condition.* Up to now, there is no solution to these problems.

The global existence of quasi-analytic solutions is known, see Ghisi and Gobbino [10], Nishihara [25], and Pohožhaev [26]. Here quasi-analytic classes are intermediate ones between the analytic class and the C^∞-class. Manfrin discussed the time global solutions in Sobolev spaces corresponding to non-analytic data having a spectral gap (see [18]), and a similar result is obtained by Hirosawa (see [13]). For the local existence in Geveray spaces, see [9].

On the other hand, global well-posedness in Sobolev space $H^{3/2}$, or H^2 with *small data* is well established in [3, 6–8, 11, 14, 19, 20, 28, 30, 31]. There, the classes of small data consist of compactly supported functions (see [11]), or more generally, they are characterised by some weight conditions (see [3, 6–8]) or oscillatory integrals (see [12, 14, 17, 19, 20, 27–31]). Recently, the authors studied the global well-posedness for Kirchhoff systems with small data (see [21]), and generalised all the previous results in the framework of small data. Here, the class of data in [21] consists of Sobolev space $(H^1)^m$, m being the order of system, and is characterised by some oscillatory integrals. The precise statements of the known results can be found in the survey paper [22].

We shall now recall the definition of Gevrey class of L^2 type. For $s \geq 1$, we denote by $\gamma^s_{L^2} = \gamma^s_{L^2}(\mathbb{R}^n)$ the Roumieu-Gevrey class of order s on $\mathbb{R}^n$:

$$\gamma^s_{L^2} = \bigcup_{\eta>0} \gamma^s_{\eta,L^2},$$

where f belong to γ^s_{η,L^2} if

$$\int_{\mathbb{R}^n} e^{\eta|\xi|^{1/s}} |\hat{f}(\xi)|^2 \, d\xi < \infty.$$

Here $\hat{f}(\xi)$ stands for the Fourier transform of $f(x)$. The class $\gamma^s_{L^2}$ is endowed with the inductive limit topology. In particular, if $s = 1$, then $\gamma^1_{L^2}(\mathbb{R}^n)$ is the class $\mathcal{A}_{L^2}$ of the analytic functions on $\mathbb{R}^n$. We will use the norm

$$\|f\|_{\gamma^s_{\eta,L^2}} = \left[\int_{\mathbb{R}^n} e^{\eta|\xi|^{1/s}} |\hat{f}(\xi)|^2 \, d\xi\right]^{1/2}$$

and

$$\|(f,g)\|_{\gamma^s_{\eta,L^2}\times\gamma^s_{\eta,L^2}} = \left[\int_{\mathbb{R}^n} e^{\eta|\xi|^{1/s}} \left\{|\hat{f}(\xi)|^2 + |\hat{g}(\xi)|^2\right\} d\xi\right]^{1/2}$$

for $\eta > 0$.

We have the following:

Theorem 1.1 *Let $T > 0$ and $s > 1$. Let $A > 0$, $R > 0$ and denote*

$$\eta_0(A; R, T) = 2sC_1(A)RT^{1+\frac{1}{s}} + C_2(A)$$

with certain large constants $C_i(A)(i = 1, 2)$ depending only on A. If the functions $u_0, u_1 \in \gamma^s_{L^2}$, for some $\eta > \eta_0(A; R, T)$, satisfy conditions

$$H(u; 0) < A,$$

$$\left\|((-\Delta)^{3/4}u_0, (-\Delta)^{1/4}u_1)\right\|^2_{\gamma^s_{\eta,L^2}\times\gamma^s_{\eta,L^2}} \le R,$$

then the Cauchy problem (1) *admits a unique solution $u \in C^1([0, T]; \gamma^s_{L^2})$.*

We note that Theorem 1.1 does not seem to require the smallness of data. In fact, A and R (measuring the size of the data) are allowed to be large. However, it follows that η (measuring the regularity of the data) then also have to be big. So, we can informally describe conditions of Theorem 1.1 that ‘the larger the data is the more regular it has to be (but still within the same class $\gamma^s_{L^2}$)’.

We can also make the following observation concerning the statement of Theorem 1.1. The formula for $\eta_0(A; R, T)$ in Theorem 1.1 comes from condition

$$\eta > \frac{2C_1(A)RT^q}{q-1} + C_2(A), \tag{2}$$

where s and q related by

$$s = \frac{1}{q-1} \quad \text{and} \quad 1 < q < 2. \tag{3}$$

The proof actually yields a more precise conclusion, namely, that the solution u from Theorem 1.1 satisfies

$$u \in \bigcap_{j=0}^{1} C^j \left([0,T]; (-\Delta)^{-(3/4)+(j/2)} \gamma^s_{\eta',L^2} \cap (-\Delta)^{-(1/2)+(j/2)} \gamma^s_{\eta',L^2} \right),$$

with

$$\eta' = \eta - \eta_0(A; R, T) > 0. \tag{4}$$

This and the order η' in (4) can be found from the class of data of linear equation and (8) in Proposition 1.2 below with s and q related by (3). Let us consider the linear Cauchy problem

$$\begin{cases} \partial_t^2 u - c(t)^2 \Delta u = 0, & t \in (0,T), \quad x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), & x \in \mathbb{R}^n. \end{cases} \tag{5}$$

The assumptions for the following estimates are related with Theorem 2 from Colombini et al. [4]. However, here we need more precise conclusions on the behaviour of constants.

Proposition 1.2 *Let $\sigma \geq 1$ and $1 \leq s < q/(q-1)$ for some $q > 1$. Assume that $c = c(t) \in \mathrm{Lip}_{\mathrm{loc}}([0,T))$ satisfies*

$$1 \leq c(t) \leq M, \quad t \in [0,T], \tag{6}$$

$$\left| c'(t) \right| \leq \frac{K}{(T-t)^q}, \quad a.e.\, t \in [0,T), \tag{7}$$

for some $M > 1$ and $K > 0$. If $((-\Delta)^{\sigma/2} u_0, (-\Delta)^{(\sigma-1)/2} u_1) \in \gamma^s_{\eta,L^2} \times \gamma^s_{\eta,L^2}$ for some η satisfying

$$\eta > \frac{2K}{q-1} + 4M^2, \tag{8}$$

then the Cauchy problem (5) admits a unique solution

$$u \in \bigcap_{j=0}^{1} C^j \left([0,T]; (-\Delta)^{-(\sigma-j)/2} \gamma^s_{\eta',L^2} \right)$$

such that

$$\|(-\Delta)^{\sigma/2}u(t)\|^2_{\gamma^s_{\eta',L^2}} + \|(-\Delta)^{(\sigma-1)/2}\partial_t u(t)\|^2_{\gamma^s_{\eta',L^2}} \tag{9}$$
$$\leq M^2 e^{4M^2 \max\{1,T^{1-(qs-s)}\}} \|((-\Delta)^{\sigma/2}u_0, (-\Delta)^{(\sigma-1)/2}u_1)\|^2_{\gamma^s_{\eta,L^2}\times\gamma^s_{\eta,L^2}}$$

for $t \in [0, T]$, *where*

$$\eta' = \eta - \left(\frac{2K}{q-1} + 4M^2\right) > 0.$$

For the proof of Proposition 1.2, one can refer to Proposition 2.1 from our recent paper [23].

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On Inverse Scattering on a Sun-Type Graph

Kiyoshi Mochizuki and Igor Trooshin

Abstract We treat an inverse scattering problem on a graph with an infinite rays and a loop joined at different points. Our problem amounts to the reconstruction of potential on the basis of the scattering data of operator.

Keywords Scattering • Schrödinger operator • Sun-type graph

Mathematics Subject Classification (2010) 34L25, 81Q35

1 Introduction and Main Results

Differential equations on graphs arise as simplified models in mathematics, physics, chemistry and engineering (nanotechnology), when one considers the propagation of waves of different natures in thin, tube-like domains (for more details, see Exner and Seba [3], Pokornyi et al. [17], the papers of Kuchment [8, 9] and the references within). Among several problems in this field, the scattering problems have been studied by many authors (e.g. Pavlov [6], Gerasimenko [5, 6], Harmer [7], Kurasov and Stenberg [10], Boman and Kurasov [2], Latushkin, Pivovarchik [11]), Pivovarchik [16] because of the general importance of their applications.

Let Γ be a graph which consists of a loop $\kappa = \{z \mid 0 < z < 2\pi\}$ and N half lines $\gamma_i = \{x_i \mid 0 < x_i < \infty\},\ i = 1, \ldots, N$, joined at the points $\{x_i = 0\} = \{z = \alpha_i\}$, where $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_N < 2\pi$. (We call such points the vertices of the graph.)

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We consider on Γ the following spectral problem:

$$-u'' + \left\{q(x) - \lambda^2\right\} u = 0, \quad x \in \Gamma, \tag{1}$$

$$u(x_i = 0) = u(z = \alpha_i),\ i = 1, \ldots, N, \tag{2}$$

$$u'(x_i = 0 + 0) + u'(z = \alpha_i + 0) - u'(z = \alpha_i - 0) = 0,\ i = 1, \ldots, N. \tag{3}$$

Here differentiation with respect to the variable x is understood as differentiation with respect to x_i, when $x \in \gamma_i$, and as differentiation with respect to z, when $x \in \kappa$. Differentiation is not defined at the vertices. The potential $q(x)$ is real-valued, $(1+x)q(x) \in L^1(\gamma_i)$ and $q(x) \in L^2_{\text{loc}}(\Gamma)$. Later on we will associate $x \in \gamma_i$ with $x \in \Re_+$ and write also $q_i(x), u_i(x), \ldots, x \in \Re_+$ instead of $q(x), u(x), \ldots, x \in \gamma_i$. The parameter λ is a complex number such that $\text{Im}\lambda \geq 0$.

The matching conditions (2)–(3) at the vertices guarantee the self-adjointness of the resulting Schrödinger operator

$$\mathcal{L}(u) = -u''(x) + q(x)u(x), \quad x \in \Gamma. \tag{4}$$

Moreover, the essential spectrum of $\mathcal{L}$ consists of the half line $[0, \infty)$.

There exist N solutions $\phi_1(x,\lambda), \ldots, \phi_N(x,\lambda)$ to the problem (1)–(3) behaving asymptotically

$$\phi_l(x,\lambda) = e^{-i\lambda x} + s_{ll}(\lambda)e^{i\lambda x} + o(1),\ x \in \gamma_l$$
$$\phi_l(x,\lambda) = s_{jl}(\lambda)e^{i\lambda x} + o(1),\ x \in \gamma_j,\ j \neq l$$

for any real $\lambda \neq 0$.

Such solutions define the functions $s_{ij}(\lambda), i,j = 1, \ldots, N,\ \lambda \in \Re \setminus \{0\}$ uniquely. By drawing an analogy with classical scattering on a half line (e.g. [1]), the matrix function $S(\lambda) = (s_{ij}(\lambda))_{i,j=1}^N$ is called the scattering matrix for the boundary value problem (1)–(3).

The matrix function $S(\lambda)$ is unitary and continuous on the whole line $-\infty < \lambda < \infty$ (except, possibly, at the point $\lambda = 0$) and admits a representation

$$S(\lambda) = T(\lambda)^{-1}T(-\lambda), \tag{5}$$

$$T(\lambda) = E(0,\lambda) + G(\lambda)E'(0,\lambda). \tag{6}$$

We defined here

$$E(x,\lambda) = \text{diag}(e_i(x,\lambda))_{i=1}^N, \tag{7}$$

where $e_i(x,\lambda),\ i = 1, \ldots, N$, are the so-called Jost solutions to the problem (1) on the parts of the graph γ_i, which behave as

$$e_i(x,\lambda) = e^{i\lambda x}\{1 + o(1)\},\ x \in \gamma_i,\ |x| \to \infty. \tag{8}$$

on the closed upper half-plane of the spectral parameter λ and $G(\lambda)$ is an $N \times N$-matrix valued entire function.

The scattering matrix $S(\lambda)$ has an asymptotic behaviour

$$S(\lambda) = S_0(\lambda) + O\left(\frac{1}{\lambda}\right), \ |\lambda| \to \infty, \tag{9}$$

where $S_0(\lambda)$ is the scattering matrix of (1)–(3) in the case $q(x) \equiv 0, \ x \in \Gamma$.

Later we suppose that

$$\det T(\lambda) \neq 0, \ \mathrm{Im}\lambda \geq 0.$$

This implies that the problem (1)–(3) can have eigenvalues only with compact supported eigenfunctions.

Our inverse scattering problem is the following:

IScP Given the scattering matrix $S(\lambda), \ \lambda \in \Re \setminus \{0\}$, recover the potential $q(x), \ x \in \Gamma \setminus K$.

As is well known (see, for instance, [1], Chap. 1), that the Jost solutions $e_i(x, \lambda)$ of Eq. (1) in γ_i can be represented as

$$e_i(x, \lambda) = e^{i\lambda x} + \int_x^\infty K_i(x, t)e^{i\lambda t}dt, \tag{10}$$

where the kernel $K_i(x, t)$ is continuous on $0 \leq x \leq t < \infty$ and satisfies the equation

$$K_i(x, x) = \frac{1}{2}\int_x^\infty q_i(t)dt, \quad x > 0. \tag{11}$$

As a function of the variable λ, $e_i(x, \lambda)$ is analytic in the open half-plane $\mathrm{Im}\lambda > 0$ and continuous on $\mathrm{Im}\lambda \geq 0$.

We denote $K(x, t) = \mathrm{diag}(K_i(x, t))$. Then we can write

$$E(x, \lambda) = e^{i\lambda x}I_N + \int_x^\infty K(x, t)e^{i\lambda t}dt \tag{12}$$

The following theorem allows us to reconstruct the potential $q(x)$ on the half-lines $\gamma_i, \ i = 1, \ldots, N$.

We define the function

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^\infty (S_0(\lambda) - S(\lambda))e^{i\lambda x}d\lambda. \tag{13}$$

which is understood as the Fourier transform of $S_o - S \in L^2(-\infty, \infty)$.

In the case $\det E(0, 0) \neq 0$ and $\det T(\lambda)) \neq 0, \ \Im\lambda \geq 0$ we have proved the following theorem.

Theorem 1.1

(i) For any fixed $x \geq 0$ *the integral kernel* $K(x,t)$ *of the operator (8) satisfies the equation*

$$F(x+t) + K(x,t) + \int_x^\infty K(x,y)F(t+y)dy = 0, \ \ 0 < x < t < \infty. \qquad (14)$$

(ii) If the function $F(t)$ *is constructed from the scattering data of problem (1)–(3) according to (13), then Eq. (14) has a unique solution* $K(x,t)$*, belonging to* $L^1(x,\infty)$ *for each fixed* $x \geq 0$.

Theorem 1.1 allows us to prove the uniqueness of the solution to the inverse scattering problem (IScP) on the semi-lines γ_i in the following sense.

Let us consider a second boundary value problem

$$-u'' + \left\{\tilde{q}(x) - \lambda^2\right\} u = 0, \quad x \in \Gamma, \qquad (15)$$

subject to matching conditions (2)–(3). Here function $\tilde{q}(x)$ is real-valued, required to satisfy $(1+x)\tilde{q}(x) \in L^1(\gamma)_i$ and $\tilde{q}(x) \in L^2_{\mathrm{loc}}(\Gamma)$.

This second boundary-value problem possesses the scattering function $\tilde{S}(\lambda)$.

Theorem 1.2 *Let us suppose that the scattering matrices of problems (1)–(3) and (15), (2)–(3) coincide, i.e.,* $S(\lambda) = \tilde{S}(\lambda), \ \lambda \in \Re \setminus \{0\}$. *Then* $q(x) = \tilde{q}(x), \ x \in \gamma_i, \ i = 1, \ldots, N.$

Theorem 1.1 allows us also to reconstruct the potential $q(x)$ for any $x \in \gamma_i, \ i = 1, \ldots, N$, based on the following procedure:

Reconstruction Procedure Given scattering matrix $S(\lambda), \ \lambda \in \Re \setminus \{0\}$

Step 1. Construct $F(x)$ via formula (13)
Step 2. Find $K(x,t), \quad 0 \leq x \leq t < \infty$ by solving main Eq. (14).
Step 3. Recover the potential according to formula

$$q_i(x) = -2\frac{d}{dx}K_i(x,x), \ x > 0. \qquad (16)$$

Remark Knowledge of the scattering matrix $S(\lambda), \lambda \in \Re \setminus \{0\}$, allows us to reconstruct the potential $q(x)$ only on semiinfinite lines γ_i. However, given some additional information we can reconstruct the potential $q(x)$ on the whole graph Γ. For example, in the case of a potential which is polynomial of degree $\leq N-1$ on the loop and continuous on γ in the neighbourhood of each vertex, we can extend the reconstruction procedure to the whole graph.

The case of the "loop-shaped" graph ("sun-type graph" with $N = 1$) was previously investigated by the authors in [12, 13]. They also investigated the inverse scattering problem on the star-shaped graph, containing a compact part [14, 15].

Now we mention some of the results closely related to ours.

Pavlov and Gerasimenko [5, 6] started the rigorous investigation of scattering problems on graphs. M. Harmer had deduced the Marchenko equation in the case of a graph consisting of a finite set of semiinfinite lines, joined at one point. Kurasov and Boman [2] had proved the existense of graphs equipped with different potentials, which possesses the same scattering matrix. (Their results do not mean a non-uniqueness in our case under consideration.) Pivovarchik and Latushkin [11, 16] had investigated cases of loop-shaped and fork-shaped graphs with potentials vanishing on semi-lines. They used a connection with the Regge-type spectral problem on a finite interval to investigate spectral and scattering properties of problem on graphs and they proved the existense of loop-shaped graphs equipped with different potentials on the loop, which possesses the same scattering function. Freiling and Ignatyev [4] had used the method of V. Yurko (see, e.g., [18]) to prove the uniqueness result on the whole sun-type graph on the basis of the scattering data and a-priori knowledge on the spectral properties on compact part of the graph.

2 Proof of the Theorems

(i) The function $\Phi(x,\lambda) = E(x,-\lambda) - S(\lambda)E(x,\lambda)$, defined for any real $\lambda \neq 0$, can be represented as

$$\Phi(x,\lambda) = -2i\lambda E^{-1}(0,\lambda)\left[\omega(x,\lambda) - T^{-1}(\lambda)G(\lambda)E(x,\lambda)\right], \tag{17}$$

where the function $\omega(x,\lambda) = \operatorname{diag}(\omega_i(x,\lambda))$, $\omega_i(x,\lambda)$ is the solution to Eq. (1) on γ_i, satisfying initial conditions $\omega_i(0,\lambda) = 0$, $\omega_i'(0,\lambda) = 1$. The function $\Phi(x,\lambda)$ can be analytically extended to a function, which is meromorphic function in the half-plane $\mathrm{Im}\lambda > 0$, continuous up to the real axis, except for its poles and, probably, $\lambda = 0$. Using formula (12) we come to the following equation.

$$\begin{aligned} &-2i\lambda E^{-1}(0,\lambda)\left(\omega(x,\lambda) + T^{-1}(\lambda)G(\lambda)E(x,\lambda)\right) - e^{-i\lambda x}I_N \\ &+ S_0(\lambda)E(x,\lambda) = \int_x^{\infty} K(x,t)e^{-i\lambda t}dt \\ &+ (S_0(\lambda) - S(\lambda))\left(e^{i\lambda x}I_N + \int_x^{\infty} K(x,t)e^{i\lambda t}dt\right) \end{aligned} \tag{18}$$

We fix $x > 0$ and let $\chi(t)$ be a continuously differentiable compactly supported function such that $\sup\chi(t) \in (x,\infty)$.Then we denote by $\hat{\chi}(\lambda)$ a Fourier transform of $\chi(t)$ and multiply both sides of (18) by $\dfrac{1}{2\pi}\hat{\chi}(\lambda)$ and integrate over $(-\infty,\infty)$. The left side of (18) is analytic in the half-plane $\mathrm{Im}\lambda > 0$ except, probably, zeros of $\det(E(0,\lambda))$. But we can show that $\mathrm{Res}_{\lambda=\mu}\Phi(x,\lambda) = 0$ where μ is zero of $\det(E(0,\lambda))$ in the upper half-plane. As $S_0(\lambda) - S(\lambda) =$

$\int_{-\infty}^{\infty} F(t)e^{-i\lambda t}dt$, we can use the differentiability of $\chi(t)$ to see by the use of Jordan lemma that

$$\int_{y}^{\infty} \chi(t)\left[F(x+t) + \int_{-\infty}^{\infty} F(t-\tau)K(x,-\tau)d\tau + K(x,t)\right]dt = 0$$

Since the function $\chi(t)$ is arbitrary, the assertion (i) follows from these equations.

(ii) The function $F(x)$ possesses the same properties as the corresponding function in the case of scattering problem on a semi-axis, which allows us to repeat arguments from [1], Chap. III, to prove the unique solvability of the Marchenko Eq. (14).

Proof of the Theorem 1.2. In the case under consideration $F(x) = \tilde{F}(x)$ and, as a result of unique solvability of the main Eq. (14), $K(x,t) = \tilde{K}(x,t),\ 0 < x < t < \infty$ and, by continuity, $K(x,x) = \tilde{K}(x,x),\ 0 \le x < \infty$. Then, according to (16), $q(x) = \tilde{q}(x),\ x \in \gamma$.

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On Certain Exact Solutions for Some Equations in Field Theory

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Abstract Some exact solutions (functionally invariant solutions) of self-dual Yang-Mills equations in $SU(2)$ case, and generalized Yang's equations, have been presented.

Keywords Charap's equations • Functionally invariant solutions • Generalized Yang's equations • R gauge • SDYM equations • Self-dual Yang-Mills equations • Yang gauge

Mathematics Subject Classification (2010) Primary 99Z99; Secondary 00A00

1 Motivation

Self-dual Yang-Mills (SDYM) equations play very important role in many branches of mathematics and physics. Some important subclass of the solutions of Yang-Mills equations (also of SDYM equations) are "nonabelian wave solutions." They were obtained and/or studied in many papers, among others, in: [4, 7, 11, 12, 21, 24, 29, 32]. Nonabelian waves are important, in the context of the color radiation in the classical YM theory, [29, 39]. Moreover, some of these solutions are so-called *functionally invariant solutions* and this paper is devoted to presentation of some of them. Functionally invariant solutions were obtained and investigated firstly (for wave equation) in Sobolev's papers and next in Erugin's papers [14] and in many other papers (also for other PDEs), among others, in: [3, 16]. In [23] certain functionally invariant solutions were obtained in the context of dynamics of the classical continuous *XY* model. In this paper, we are dealt with SDYM equations in R gauge (or Yang gauge). The SDYM equations (not only in R gauge) and their solutions were discussed and/or investigated and/or solved in many books and papers, among others, in: [4, 5, 7, 10, 11, 14, 16, 18, 19, 21, 23, 24, 29, 32, 34, 35, 39].

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In [28, 34], the so-called, generalized Yang's equations were investigated, which are some more general form than the form of SDYM equations in R-gauge in SU(2) case. Some generalization of SDYM equations is also studied in [17].

In this paper we obtain some new classes of solutions (functionally invariant solutions) of: self-dual Yang Mills equations (among others, in R gauge) and generalized Yang equations. We find these solutions, by applying, by the so-called, *decomposition method* (firstly presented and developed correspondingly, in the first and the second paper in [38]). We give here a definition of functionally invariant solution (basing on [26]):

Definition We call a solution $f(x), x \in \mathbb{R}^n$ of a PDE, as functionally invariant solution, if for any function $F : \mathbb{R} \longrightarrow \mathbb{R}$, the composition $F(f(x))$ is also a solution of the same PDE.

If $f, F \in \mathbb{C}$, then this definition is analogical.

This paper is organized as follows. In Sect. 2, we briefly describe SDYM equations in the SU(2) case, for the cases: R gauge and non-R gauge. In the next section, we shortly describe the decomposition method, which allows to obtain functionally invariant solutions. In the next section we present the new solutions of the self-dual Yang-Mills equations, in the case SU(2). We investigate the equations derived in [41], studied in [40] and in the second and the third paper in [20]. As one can see, our approach, applied here for SDYM equations and generalized Yang's equations, *differs* from the approaches presented in: [17, 21], in [26] and in other papers devoted to finding the exact solutions of these equations. In these papers, the group analysis were applied to investigated equations. The solutions of SDYM equations in R-gauge, found in these above papers, are similar to these ones presented in this talk, but they possess different form or the solutions presented in this talk are more general than those.

2 Investigated Equations

2.1 SDYM Equations in the $SU(2)$ Case

2.1.1 The *R* Gauge Case

Self-dual Yang-Mills (SDYM) equations in the so-called Yang (or R) gauge were described by Yang in [41]. There he has made an analytic continuation of gauge potentials a^i_μ into complex space with complex coordinates: $y = \frac{x_1+ix_2}{\sqrt{2}}$, $\bar{y} = \frac{x_1-ix_2}{\sqrt{2}}, z = \frac{x_3-ix_4}{\sqrt{2}}$, $\bar{z} = \frac{x_3+ix_4}{\sqrt{2}}$. Of course, [41]: $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} - A_\mu A_\nu + A_\nu A_\mu, A_\mu = -\frac{1}{2} i a^i_\mu \sigma_i$, $i = 1, 2, 3$, where σ_i are Pauli matrices. The self-duality condition: $2F_{\alpha\beta} = \varepsilon_{\alpha\beta\mu\nu} F_{\mu\nu}$, has the form, [41]: $F_{yz} = F_{\bar{y}\bar{z}} = 0, F_{y\bar{y}} + F_{z\bar{z}} = 0$. Yang has obtained that the gauge potentials have the form, [41]: $A_y = D^{-1}\, \partial_y D$,

$A_z = D^{-1}\partial_z D, A_{\bar{y}} = \bar{D}^{-1}\partial_{\bar{y}}\bar{D}, A_{\bar{z}} = \bar{D}^{-1}\partial_{\bar{z}}\bar{D}$, where $\det(D) = 1$ and for the SU(2) case, we may write (this is just R or Yang gauge), [41]: $D = R = \frac{1}{\sqrt{u}}\begin{pmatrix} 1 & 0 \\ v & u \end{pmatrix}$. If $x_\mu \in R^4, \mu = 1,2,3,4$, then : $\bar{D} = (D^\dagger)^{-1}, u \doteq \text{real}, \bar{v} \doteq v^*$ and: $A_\mu = -A_\mu^\dagger$. The notation "$\doteq$" means here "equal to" for real x_μ. When we write the equations following from the duality condition, in four-dimensional Euclidean space (with x_μ), they are of the form, (the first paper in [40]):

$$u\triangle u - (\nabla u)^2 + \nabla w \nabla w^* = -iN(w, w^*), c.c. \tag{1}$$

$$u\triangle w - 2\nabla u \nabla w = 2iN(w, u), c.c. \tag{2}$$

$$N(p_1, p_2) = (p_{1,x_1}p_{2,x_2} - p_{1,x_2}p_{1,x_1}) - (p_{1,x_3}p_{2,x_4} - p_{1,x_4}p_{2,x_3}), \tag{3}$$

where ∇ and Δ denote also, correspondingly, the "nabla" operator and Laplace operator, but in four-dimensional space. Takeno in the first paper in [40], has showed that Eqs. (1)–(3) are identical to Ernst equations in 4d Euclidean space (original Ernst equations were derived in [13]), if $w = e^{ia}v, \omega = u+iv, a = \text{const}, N(v, u) = N(-\frac{i}{2}(\omega - \omega^*), \frac{1}{2}(\omega + \omega^*)) = 0, a \in R$:

$$(\Re(\omega))\triangle\omega = (\nabla\omega)^2. \tag{4}$$

The connection between Einstein equations, SDYM equations and equations of nonlinear "sigma" model, was also investigated by Sanchez in the second paper in [38]. Some results concerning an analogical connection between (1)–(3) and (4), but in cylindrical coordinates, are included in [1].

2.1.2 The Non-*R* Gauge Case

Let us come back to general case (without R gauge), when the SDYM equations are: $F_{yz} = F_{\bar{y}\bar{z}} = 0, F_{y\bar{y}} + F_{z\bar{z}} = 0,\ F_{ij} \in sl(n; \mathcal{C}),\ i, j \in \{y, \bar{y}, z, \bar{z}\}$, (first paper in [31]). However, here the "bar" does not denote the complex conjugation "$*$". If $n \to \infty$, then the potentials have the form $A_i = \Lambda_{i,s}\frac{\partial}{\partial r} - \Lambda_{i,r}\frac{\partial}{\partial s}$, where: $\Lambda_i = \Lambda_i(y, z, \bar{y}, \bar{z}, r, s)$, $\Lambda_{i,s} \equiv \frac{\partial\Lambda_i}{\partial s}$, etc. ($r, s$—coordinates on $\mathcal{N}^2$), and the $sl(n \to \infty, \mathcal{C})$ limit of the SDYM equations has the form, [31]: $\Lambda_{y,z} - \Lambda_{z,y} + (\Lambda_{y,r}\Lambda_{z,s} - \Lambda_{y,s}\Lambda_{z,r}) + \mathcal{J}_1(y, z, \bar{y}, \bar{z}) = 0$, $\Lambda_{\bar{y},\bar{z}} - \Lambda_{\bar{z},\bar{y}} + (\Lambda_{\bar{y},r}\Lambda_{\bar{z},s} - \Lambda_{\bar{y},s}\Lambda_{\bar{z},r} + \bar{\mathcal{J}}_1(y, z, \bar{y}, \bar{z}) = 0$, $\Lambda_{y,\bar{y}} - \Lambda_{\bar{y},y} + \Lambda_{z,\bar{z}} - \Lambda_{\bar{z},z} + (\Lambda_{y,r}\Lambda_{\bar{y},s} - \Lambda_{y,s}\Lambda_{\bar{y},r} + \Lambda_{z,r}\Lambda_{\bar{z},s} - \Lambda_{z,s}\Lambda_{\bar{z},r}) + \mathcal{J}_2(y, z, \bar{y}, \bar{z}) = 0$, where $\mathcal{J}_i$, $(i = 1, 2)$ - arbitrary holomorphic functions of their arguments. If $\Lambda_y = \theta_{,s}, \Lambda_z = -\theta_{,r}, \Lambda_{\bar{y}} = \Lambda_{\bar{z}} = 0, \bar{\mathcal{J}}_1 = \mathcal{J}_2 = 0, \mathcal{J}_1 = \mathcal{J}_1(y, z), \theta = \theta(y, z, r, s)$ and: $v = \theta + r\cdot f(y, z), f_y = \mathcal{J}_1(y, z)$ and $\mathcal{J}_1$ is an arbitrary holomorphic function of its arguments, then one can reduce SDYM equations to second heavenly equation of Plebański: $v_{,rr}v_{,ss} - v_{,rs}^2 + v_{,ry} + v_{,sz} = 0$, [31].

2.2 Generalized Yang's Equations and Extended Charap's Equations

If we write the system (1)–(3), in real terms p, v: $w = p + i \cdot v$, and include there some constants κ, ξ, we have some generalization of the system (1)–(3), [34]:

$$u\triangle_{(\kappa)}u = \xi \cdot [(\nabla_{(\kappa)}u)^2 - (\nabla_{(\kappa)}p)^2 - (\nabla_{(\kappa)}v)^2 - 2N(p,v)],$$
$$\frac{u}{2}\triangle_{(\kappa)}p = \xi \cdot [u_{,\rho}p_{,\rho} + N(u,v)], \;\; \frac{u}{2}\triangle_{(\kappa)}v = \xi \cdot [u_{,\rho}v_{,\rho} - N(u,p)], \tag{5}$$

where: $\nabla_{(\kappa)} = [\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}, \kappa\frac{\partial}{\partial x^4}], \rho = 1, \ldots, 4, \kappa < 0, \xi \neq 0$ (in [34]: $\kappa = \pm 1, \xi = 1$ or $\xi = \frac{1}{2}$). Equation (5) are called as generalized Yang's equations (GYE) [34]. One can show that their structure is similar to the so-called extended Charap's equations (ECE), [34], which are some extension of Charap's equations of pion dynamics [8] (first paper).

3 The Concept of the Decomposition Method

The (non-Bogomolny) decomposition method can be used to find functionally invariant solutions of some given PDE, which can be decomposed into some fragments, homogeneous of the derivatives of the unknown functions ω_j:

$F(x^\mu, \omega_1, \ldots, \omega_m, \omega_{1,x^\mu}, \omega_{1,x^\mu x^\nu}, \ldots, \omega_{m,x^\mu}, \omega_{m,x^\mu x^\nu}) = 0, \quad \mu, \nu = 0, \ldots, 3,$ [38]. For such equations, we apply the ansatz (where $a_n, b_n, c_n, d_n (n = 1, \ldots, 5)$—in general complex coefficients)

$$\omega_j = \beta_1 + f_j(k_n^1 x^n + \beta_2, \ldots, k_n^5 x^n + \beta_6), \; k_n^1 x^n = a_1 x_1 + \ldots + d_1 x_4, \tag{6}$$

$\beta_p = \text{const}, p = 1, \ldots, 6$ (obviously, one can consider the term $k_n^1 x^n$ as something similar to quaternion—the solutions written by using quaternions and biquaternions, correspondingly, for SDYM equations, were found in the papers [10]). We require vanishing of some algebraic terms, which appear by the terms consisting of $\omega_j, \omega_{j,x_1}, \omega_{j,x_1 x_1}, \ldots$ etc., after inserting the ansatz. Thus, the problem of solving the given PDE (or of the system of PDEs) becomes the problem of solving of some (very often nonlinear) algebraic equation (or system of them). We call such system as determining algebraic system.

We stress here that many authors, engaged in looking for functionally invariant solutions of investigated (non)linear PDEs, have looked for the solutions, where the coefficients by the independent variables are equal to "1", or if they have looked for the solutions with the coefficients, they have not been engaged in solving of such systems of algebraic equations, which must be satisfied by the coefficients. For example, the functionally invariant solutions for nonlinear sigma model were obtained in the first paper in [38], independent the analogical results obtained in [16].

4 Some Solutions for SDYM Equations, GYE and ECE

4.1 *The Solutions for SDYM Equations*

4.1.1 The *R* Gauge Case

We apply here the ansatz (6) for (4) and for the condition $N(v, u) = 0$, and we obtain rather complicated, algebraic determining system of nonlinear equations, which must be satisfied by the coefficients. The non-zero solutions of this system are: $a_3 = i, a_5 = -\gamma, d_3 = 1, d_5 = i\gamma$, hence, an example of the exact solution, is: $\omega = f(ix_1 + x_4 + \beta_4, -\gamma x_1 + i\gamma x_4 + \beta_6)$, where f–an arbitrary, twice differentiable function. Then the functions $u = \Re(\omega)$ and $w = e^{ia} * \Im(\omega)$ are the new exact solutions of the SDYM Eqs. (1)–(3). The ansatz (6) can give in general, the functionally invariant solutions for SU(2) SDYM equations in *R* gauge (when the algebraic determining system is satisfied), possessing more general form than the functionally invariant solutions (depending on the arguments of the kind given in (6)) for SU(2) SDYM equations in *R* gauge, known until now, to the author's knowledge.

Obviously, the search for the solutions of other form is in proceed.

4.1.2 The Non-*R* Gauge Case

Because of limited numbers of pages, we mention here only that by using the fact of reduction of SDYM equation to second heavenly equation of Plebański [30] (paragraph 2.1.2), we can apply here the results from the third paper in [38], where the non-invariant functionally invariant solutions for among others, second heavenly equation, have been obtained, among others, in the form of *infinite series*, (after linearization of Legendre transformed second heavenly equation this was done in the paper of Malykh, Nutku and Sheftel in [15]): $\vartheta = \sum_{j=1}^{n} g_j(\Sigma_j)$, where g_j is some arbitrary holomorphic function of $\Sigma_j = \alpha_j r + \gamma_j q + \zeta_j t + \lambda_j z + \beta_j, (\beta_j = const.)$ Legendre transformation: $\vartheta = v - yv_{,y} - sv_{,s}, v_y = t, v_{,s} = q, y = -\vartheta_{,t}, s = -\vartheta_{,q}$ and $\zeta_j = \frac{\gamma_j^2}{\alpha_j}, \lambda_j = -\frac{\alpha_j^2}{\gamma_j}$ (first subclass) and $\alpha_j = \frac{\gamma_j^2}{\zeta_j}, \lambda_j = -\frac{\gamma_j^3}{\zeta_j^2}$ (second subclass), β_j—arbitrary constants. The series obtained by differentiation of this above series need to be uniformly convergent, but they cannot be absolutely summable (third paper in [36]). Some other solutions of SDYM equations, including certain infinite series (but for $sl(2, C)$), were obtained by J. Schiff in [39].

4.2 *The Solutions of GYE and ECE*

The Case I: The Solutions of GYE We substituted for each of the functions u, p, w in (5), the ansatz (6), where the real functions $f_m \in \mathcal{C}^2, (m = 1, 2, 3)$, are some

arbitrary functions of their arguments. Of course, we are interested only in real solutions of the algebraic determining system. Here we present an example of such solutions of this system ($\kappa = -\frac{a_2^2+b_2^2}{d_2^2}, f_1 = f, f_2 = g, f_3 = h$—arbitrary functions of their arguments): $u = \beta_1 + f(a_2x_1 + b_2x_2 + d_2x_4 + \beta_2)$,
$p = \beta_3 + g\left(A_3x_1 + \frac{A_3b_2}{a_2}x_2 + \frac{A_3d_2}{a_2}x_4 + \beta_4, A_4x_1 + \frac{A_4b_2}{a_2}x_2 + \frac{A_4d_2}{a_2}x_4 + \beta_5\right)$,
$v = \beta_6 + h\left(\frac{a_2\lambda_1}{d_2}x_1 + \frac{b_2\lambda_1}{d_2}x_2 + \lambda_1x_4 + \beta_7, \frac{a_2\lambda_2}{d_2}x_1 + \frac{b_2\lambda_2}{d_2}x_2 + \lambda_2x_4 + \beta_8,\right.$
$\left.\frac{a_2\lambda_4}{d_2}x_1 + \frac{b_2\lambda_4}{d_2}x_2 + \lambda_4x_4 + \beta_9\right)$, $\beta_k =$ const. The forms of these above solutions are different from the solutions found in [28] and [34].

The Case II: The Solutions of GYE and ECE If $u = u(\phi), p = p(\phi), v = v(\phi)$, then one can find the exact solutions of GYE and ECE, by solving the nonlinear $O(3)$ "sigma" model-like equations: $\Box\phi = 0, \phi_{,\mu}\phi^{,\mu} = 0, \phi \in \mathcal{C}^2$. This possibility was already mentioned in [6] (first paper) and in [34]. Then, the solutions have the form: $\phi = \beta_1 + f(\frac{a_3d_1}{d_3}x_1 + \frac{b_3d_1}{d_3}x_2 + \frac{c_3d_1}{d_3}x_3 + d_1x_4 + \beta_2, \frac{a_3d_2}{d_3}x_1 + \frac{b_3d_2}{d_3}x_2 + \frac{c_3d_2}{d_3}x_3 + d_2x_4 + \beta_3, a_3x_1 + b_3x_2 + c_3x_3 + d_3x_4 + \beta_4)$ and $\kappa = -\frac{a_3^2+b_3^2+c_3^2}{d_3^2}$. These solutions have other form than these ones found in [6] (first paper), [8] and in [27].

The solutions found in these two above cases can be considered as describing nonlinear waves or nonlinear superposition of the solutions of the investigated equations. One can also notice that we can impact the presence of certain variable(s) in one of these superposed solution(s), by putting a zero coefficient in other superposed solution(s). This property has been mentioned in [38] (fourth paper), in the case of the exact solutions of the so-called extended Skyrme-Faddeev model. The ansatz (6) and these solutions can be regarded as somewhat more general ansatz and solution, correspondingly, applied and obtained in [8] (third paper). The searching of more general solutions, than presented in this paper, and the investigation of the physical properties of these solutions are in proceed [38].

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Elliptic Equations, Manifolds with Non-smooth Boundaries, and Boundary Value Problems

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To ISAAC

Abstract We discuss basic principles for constructing the theory of boundary value problems on manifolds with non-smooth boundaries. It includes studying local situations related to model pseudo-differential equations in canonical domains. The technique consists of Fourier transform, multi-dimensional Riemann boundary value problem, wave factorization, and multi-variable complex analysis.

Keywords Elliptic symbol • Multi-dimensional Riemann boundary value problem • Pseudo-differential equation • Singularities • Wave factorization

Mathematics Subject Classification (2010) Primary 35S15; Secondary 42B30

1 Introduction

One considers a general elliptic pseudo differential equation

$$(Au)(x) = f(x), \quad x \in M, \tag{1}$$

in Sobolev–Slobodetskii spaces $H^s(M)$, where M is a smooth manifold with non-smooth boundary, i.e. its boundary has some singularities like a cone, a wedge, etc., and the unknown function u is defined on M.

If $A(x,\xi)$, $(x,\xi) \in \mathbb{R}^m \times \mathbb{R}^m$, is a symbol (in local coordinates of the co-tangent bundle T^*M) of a pseudo-differential operator A, then to obtain a Fredholm property for the operator A we need to describe invertibility conditions for some classes of its local representatives.

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Such operators are defined by a well-known formula if M is a compact smooth manifold because one can use the "freezing coefficients principle" or, in other words, "local principle." For a manifold with a smooth boundary we need a new local formula for defining the operator A: more precisely near inner points of M we use an usual formula, but near boundary points we need another formula

$$u(x) \longmapsto \int\limits_{\mathbb{R}^m_+} \int\limits_{\mathbb{R}^m} A(x,\xi)u(y)e^{i(x-y)\cdot\xi}d\xi dy. \tag{2}$$

For invertibility of such an operator with symbol $A(\cdot,\xi)$ not depending on a spatial variable x one can apply the theory of the classical Riemann boundary value problem for upper and lower complex half-planes with a parameter $\xi' = (\xi_1,\dots,\xi_{m-1})$. This step was systematically studied in the book [2]. But if the boundary ∂M has at least one conical point, this approach is not effective.

A conical point at the boundary is such a point for which its neighborhood is diffeomorphic to the cone $C^a_+ = \{x \in \mathbb{R}^m : x_m > a|x'|,\ x' = (x_1,\dots,x_{m-1}),\ a > 0\}$, hence the local definition for pseudo-differential operator near the conical point is the following

$$u(x) \longmapsto \int\limits_{C^a_+} \int\limits_{\mathbb{R}^m} A(x,\xi)u(y)e^{i(x-y)\cdot\xi}d\xi dy. \tag{3}$$

To study an invertibility property for the operator (3) the author has introduced the concept of wave factorization for an elliptic symbol near a singular boundary point [5, 7, 9] and using this property has described Fredholm properties for Eq. (1).

Other approaches to the theory of boundary value problems one can find in papers of V.G. Mazya, B.A. Plamenevskii, B.-W. Schulze, R.B. Melrose, M. Taylor, V. Nistor, and many others. I cannot enumerate all authors but in author's book [6] very large survey of these approaches with names and papers is given.

2 Studying Model Operators

To describe Fredholm properties for a general pseudo-differential operator on the manifold M one needs to study local situations separately. These correspond to *model operators in canonical domains.*

2.1 Simple and Complicated Singularities

2.1.1 Simple Singularities

A simple standard singularity in m-dimensional space is the cone C_+^a.

Example 1 A conical singularity can be stratified, i.e. for example the cone $C_+^a \times C_+^b \subset \mathbb{R}^{n+m}$, where $C_+^a \subset \mathbb{R}^n$, $C_+^b \subset \mathbb{R}^m$, is a stratified cone

Example 2 A quadrant on the plane $\mathbb{R}^2$ is represented as a direct product of two half-axes.

Example 3 Octant in the space $\mathbb{R}^3$ is a cone of 3-wedged angle type which can be represented as a direct product of a quadrant (i.e., two-dimensional cone) and a half-axis (one-dimensional cone).

Example 4 A wedge of codimension k in m-dimensional space is the set $\{x \in \mathbb{R}^m : x = (x', x'', x_m), x' \in \mathbb{R}^{m-k}, x' = (x_1, \dots, x_{m-k}), x_m > a|x''|, x'' = (x_{m-k+1}, \dots, x_{m-1}), a > 0\}$.

Example 5 A multi-wedged angle in m-dimensional space is the set $P_m = \{x \in \mathbb{R}^m : x_m > \sum\limits_{k=1}^{m-1} a_k|x_k|, a_k > 0\}$.

2.1.2 Complicated Singularities

Such singularities arise if a singularity's type cannot be described by the standard cone C_+^a.

Example 6 A variant of the thin cone $T_{m-k} = \{x \in \mathbb{R}^m : x_m > a|x''|, x'' = (x_1, \dots, x_{m-k}), x_{m-k+1} = \cdots = x_{m-1} = 0\}$.

Example 7 A union of m-dimensional cones with a common origin.

Example 8 A union of cones with distinct dimensions with a common origin.

2.2 Local Index and Local Solvability

Here we consider Eq. (1) for a model operator with the elliptic symbol $A(\xi)$ in a canonical m-dimensional domain D (Examples 1–5). For this case we deal with a convex cone which does not contain a whole straight line. For Example 4 we have $D = C_+^a \times \mathbb{R}^{m-k}$, $C_+^a \subset \mathbb{R}^k$, and the variable $x'' \in \mathbb{R}^{m-k}$ will be a parameter. Thus a principal case is that **the set** D **is a convex sharp cone** in m-dimensional space $\mathbb{R}^m$. If so one needs to describe invertibility conditions for the model operator A for this canonical domain. For this purpose the author has introduced a special variant of a multi-dimensional Riemann boundary value problem which is distinct from all known ones. This problem can be solved by using a wave factorization concept, moreover one can obtain an integral representation for the solution of the model Eq. (1). For a model equation we use "local" constructions of Sobolev–Slobodetskii spaces in $\mathbb{R}^m$.

2.2.1 Spaces

By definition the space $H^s(D)$ consists of distributions from the space $H^s(\mathbb{R}^m)$ [2] for which their supports belong to $\overline{D}$. A norm in the space $H^s(D)$ is induced by the norm of the space $H^s(\mathbb{R}^m)$. The right-hand side f is chosen from the space $H_0^{s-\alpha}(D)$ which consists of distributions from $S'(D)$ admitting a continuation into the whole space $H^{s-\alpha}(\mathbb{R}^m)$. A norm in the space $H_0^{s-\alpha}(D)$ is defined by the formula

$$||f||^+_{s-\alpha} = \inf ||lf||_{s-\alpha},$$

where *infimum* is taken over all continuations l. Here we use notations like ones in the Eskin's book [2].

2.2.2 Wave Factorization

Let us denote $\overset{*}{D}$ a conjugate cone [1, 15]

$$\overset{*}{D} = \{x \in \mathbb{R}^m : x \cdot y > 0, y \in D\},$$

where $x \cdot y$ denotes an inner product.

Example 9 If $D = C^a_+$, then $\overset{*}{D} = \{x \in \mathbb{R}^m : ax_m > |x'|\}$.

Definition 2.1 A radial tube domain $T(D)$ over the cone D is called a subset of m-dimensional complex space $\mathbb{C}^m$ of the type $\mathbb{R}^m + iD$.

Definition 2.2 The symbol $A(\xi)$ is called an elliptic symbol of order $\alpha \in \mathbb{R}$ if $\exists c_1, c_2 > 0$ such that

$$c_1 \le |A(\xi)(1 + |\xi|)^{-\alpha}| \le c_2, \quad \forall \xi \in \mathbb{R}^m.$$

Definition 2.3 Wave factorization with respect to the cone D for the elliptic symbol $A(\xi)$ is called a representation in the form

$$A(\xi) = A_{\neq}(\xi)A_{=}(\xi),$$

where the factors $A_{\neq}(\xi), A_{=}(\xi)$ must satisfy the following conditions:

1) $A_{\neq}(\xi), A_{=}(\xi)$ are defined for all admissible values $\xi \in \mathbb{R}^m$, without possibility, the points $\xi \in \partial(\overset{*}{D} \bigcup (-\overset{*}{D}))$;
2) $A_{\neq}(\xi), A_{=}(\xi)$ admit an analytical continuation into radial tube domains $T(\overset{*}{D})$, $T(-\overset{*}{D})$ respectively with estimates

$$|A_{\neq}^{\pm 1}(\xi + i\tau)| \leq c_1(1 + |\xi| + |\tau|)^{\pm æ},$$

$$|A_{=}^{\pm 1}(\xi - i\tau)| \leq c_2(1 + |\xi| + |\tau|)^{\pm(\alpha - æ)}, \ \forall \tau \in \overset{*}{C_+^a}.$$

The number $æ \in \mathbb{R}$ is called index of wave factorization.

2.2.3 Multi-Dimensional Riemann Problem

Taking into account that we will use the Fourier transform let us introduce the following notations. We use notation $\tilde{u}$ for the Fourier transform of function u, and the notation $\widetilde{H}$ for Fourier image of the Hilbert space H.

For small s, $|s| < 1/2$, we denote by $A(\mathbb{R}^m)$ a subspace in the space $\widetilde{H}^s(\mathbb{R}^m)$ of functions $u(x)$ which admit an analytical continuation into radial tube domain $T(\overset{*}{D})$ over conjugate cone $\overset{*}{D}$, the subspace $B(\mathbb{R}^m)$ is a direct complement of the subspace $A(\mathbb{R}^m)$ in the space $\widetilde{H}^s(\mathbb{R}^m)$, so that $\widetilde{H}^s(\mathbb{R}^m) = A(\mathbb{R}^m) \oplus B(\mathbb{R}^m)$.

The mentioned multi-variable Riemann problem is formulated as follows. One seeks two functions $\Phi^+(x) \in A(\mathbb{R}^m)$, $\Phi^-(x) \in B(\mathbb{R}^m)$ which satisfy the linear relation

$$\Phi^+(x) = W(x)\Phi^-(x) + w(x). \tag{4}$$

almost everywhere on $\mathbb{R}^m$, where $W(x), w(x)$ are given.

The transfer from Eq. (1) to the problem (4) is very simple. If we will apply the Fourier transform to the model Eq. (1) we obtain a certain multi-dimensional singular integral equation with the kernel $B(x)$ like a characteristic one-dimensional singular integral equation [3, 4]. This kernel $B(z), z \in \overset{*}{D}$, is the Bochner kernel for the cone D [1, 15] and

$$B(z) = \int_D e^{iy \cdot z} dy, \quad z = x + i\tau, \quad z \in T(\overset{*}{D}),$$

and a corresponding integral operator is the following

$$(B\tilde{u})(\xi) = \lim_{\tau \to 0, \tau \in \overset{*}{D}} \int_{\mathbb{R}^m} B(\xi - y + i\tau)\tilde{u}(y)dy. \tag{5}$$

Remark 2.4 The needed variant of Paley–Wiener theorem for this situation one can find in the book [15], Chap. 5, Sect. 26. Principal point here is that representation

$$\tilde{u} = \tilde{u}_1 + \tilde{u}_2, \quad \forall \tilde{u} \in \widetilde{H}^s(\mathbb{R}^m),$$

where $\tilde{u}_1 \in A(\mathbb{R}^m), \tilde{u}_2 \in B(\mathbb{R}^m)$, is unique for $|s| < 1/2$ only (see also [2, 5]).

Example 10 If D is a one-dimensional cone $\mathbb{R}_+$, then [1, 2, 15] $B(z)$ is the Cauchy kernel $i(2\pi z)^{-1}$, and the corresponding one-dimensional analogue of the singular integral operator (5) is the following

$$\tilde{u}(\xi) \longmapsto \frac{1}{2}\tilde{u}(\xi) + \frac{1}{\pi i} v.p. \int\limits_{-\infty}^{+\infty} \frac{\tilde{u}(\eta)d\eta}{\xi - \eta}.$$

This follows from Plemelj–Sokhotskii formulas [3, 4].

2.2.4 Solvability and Boundary Conditions

The operator B and wave factorization give a possibility to describe solvability of the model Eq. (1).

Proposition 2.5 *If the elliptic symbol $A(\xi)$ admits wave factorization with respect to the cone D with index æ, then*

1) for $|æ - s| < 1/2$ there exists a unique solution $u \in H^s(D)$ of the model Eq. (1) for arbitrary right-hand side $f \in H_0^{s-\alpha}(D)$, and we have

$$\tilde{u}(\xi) = A_{\neq}^{-1}(\xi)(B(A_{=}^{-1}\widetilde{lf}))(\xi),$$

where $A_{=}^{-1}\widetilde{lf}$ means the function $A_{=}^{-1}(\xi)\widetilde{lf}(\xi)$, lf is an arbitrary continuation of $f \in H_0^{s-\alpha}(D)$ on the whole $H^{s-\alpha}(\mathbb{R}^m)$;

2) for $æ - s = n + \delta, n \in \mathbb{N}, |\delta| < 1/2$, there are a lot of solutions depending on n arbitrary functions $c_k \in H^{s_k}(\mathbb{R}^{m-1}), s_k = s - æ + k - 1/2, k = 1, \dots, n$;

3) for $æ - s = -n + \delta, n \in \mathbb{N}, |\delta| < 1/2$, then a solution from $H^s(D)$ exists iff certain n additional integral conditions on right-hand side $f \in H_0^{s-\alpha}(D)$ hold.

Remark 2.6 Two-dimensional variant of the proposition was proved by the author many years ago [5]. Some multi-dimensional constructions are described in [10–12].

Some Comments to the Proposition 2.5. Indeed functions c_k appear after wave factorization and change of variables reducing the cone into a half-space. A certain special operator similar to a pseudo-differential one takes part in this construction. All details are given in [10–12].

For the case 2 of the Proposition 2.5 one needs some additional conditions to extract a unique solution. These may be boundary conditions as usual or conditions of another type. Some preliminary considerations are given in [10, 12].

3 Partition of Unity and Transfer to Manifolds

These ideas lead to many interesting deductions. To define correctly a pseudo differential operator on a manifold with non-smooth boundary one needs to choose a partition of unity and to consider boundary neighborhoods in dependence on the type of singular point. Since pseudo differential operators are operators of a local type, the Fredholm property will be conserved. It means the following. If we use a change of variables diffeomorphic transforming singular neighborhood onto certain cone we locally obtain an operator of the type (3) plus some compact operator. Since the index of an operator is stable under compact perturbations we obtain operators with same indices.

4 Conclusion

There are a lot of singularities types in a manifold with a non-smooth boundary. For example, the author's papers [8, 14] are related to thin singularities, and the paper [13] concerns to the union of cones. The author hopes that the developed methods will be useful for the general theory of boundary value problems.

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Part V
Integral Transforms and Reproducing Kernels

Juri Rappoport

Learning Coefficients and Reproducing True Probability Functions in Learning Systems

Miki Aoyagi

Abstract Recently, the widely applicable information criterion (WAIC) model selection method has been considered for reproducing and estimating a probability function from data in a learning system. The learning coefficient in Bayesian estimation serves to measure the learning efficiency in singular learning models, and has an important role in the WAIC method. Mathematically, the learning coefficient is the log canonical threshold of the relative entropy. In this paper, we consider the Vandermonde matrix-type singularity learning coefficients in statistical learning theory.

Keywords Generalization error • Learning coefficient • Resolution of singularities • Training error

Mathematics Subject Classification (2010) Primary 62D05; Secondary 62M20, 32S10, 14Q15

1 Introduction

Let $q(x)$ be a true probability density function of variables $x \in \mathbf{R}^N$ and let $x^n := \{x_i\}_{i=1}^n$ be n training samples independently and identically selected from $q(x)$. Consider a learning model that is written in probabilistic form as $p(x|w)$, where $w \in W \subset \mathbf{R}^d$ is a parameter. The purpose of the learning system is to estimate the unknown true density function $q(x)$ from x^n using $p(x|w)$. This procedure aims to reproduce the true probability density function $q(x)$ by n training samples x^n. The generalization errors relate to the generalization losses via the entropy of the true distribution. Hence, it is important to estimate the generalization errors from the training errors that are calculated from the training samples x^n using learning

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model p. The widely applicable information criterion (WAIC) model selection method can estimate these training errors, and thus, we can use this method to select a suitable model from among several statistical models. The WAIC method is a generalized form of the Akaike information criterion (AIC) [1, 16]. For instance, WAIC can be applied to singular learning models, whereas AIC cannot. The learning coefficient in Bayesian estimation measures the learning efficiency in singular learning models and has an important role in WAIC. Mathematically, the learning coefficient is the log canonical threshold of the Kullback function (relative entropy). Hironaka's Theorem indicates a way to obtain such thresholds; however, additional theorems are required to obtain these within learning theory. In this paper, we consider a learning coefficient based on Vandermonde matrices-type singularities in statistical learning theory. The Vandermonde matrix type is a generic and essential concept in learning theory. These log canonical thresholds provide the learning coefficients of normal mixture models, three-layered neural networks, and mixtures of binomial distributions, which are widely used and effective learning models.

The learning coefficients for the restricted Boltzmann machine [3] have also been considered recently. The authors of [15, 17], respectively, obtained these learning coefficients for naive Bayesian networks and directed tree models with hidden variables.

2 Log Canonical Threshold

The log canonical threshold is defined with respect to the complex or real field as follows.

Definition 2.1 Let f be a nonzero holomorphic function on $\mathbb{C}^d$ or an analytic function on $\mathbb{R}^d$ in a neighborhood U of w^*. Let ψ be a C^∞ function with a compact support. The log canonical threshold is defined as follows:

$$c_{w^*}(f,\psi) = \sup\{c : |f|^{-c} \text{ is locally } L^2 \text{ in a neighborhood of } w^*\}$$

over $\mathbb{C}$, and

$$c_{w^*}(f,\psi) = \sup\{c : |f|^{-c} \text{ is locally } L^1 \text{ in a neighborhood of } w^*\}$$

over $\mathbb{R}$. In addition, $\theta_{w^*}(f,\psi)$ is defined to be its order.

If $\psi(w^*) \neq 0$, then $c_{w^*}(f) = c_{w^*}(f,\psi)$ and $\theta_{w^*}(f) = \theta_{w^*}(f,\psi)$, because the log canonical threshold and its order are independent of ψ.

It is known that if f is a polynomial or a convergent power series, $c_0(\mathbb{C}^d)$ is the largest root of the Bernstein-Sato polynomial $b(s) \in \mathbb{C}[s]$ of f, where $b(s)f^s = Pf^{s+1}$ for linear differential operator P [6, 7, 10]. The log canonical threshold $c_{w^*}(f)$ also corresponds to the largest pole of $\int_U |f|^{2z}\psi(w)dw$ over $\mathbb{C}$ or $\int_U |f|^z\psi(w)dw$ over $\mathbb{R}$.

Hironaka's Theorem [9] enables us to obtain the log canonical thresholds. In algebraic geometry and algebraic analysis, these studies are usually done over an algebraically closed field [11, 13]. However, many differences exist for real and complex fields. For example, log canonical thresholds over the complex field are less than one, whereas those over the real field are not necessarily so. In addition, the following theorem over the complex field and a counterexample (Example 1) over the real field also highlight these differences.

Theorem 2.2 ([11, 14]) *Let $f(w_1,\ldots,w_d,w_{d+1})$ be a **holomorphic** function in a neighborhood of the origin. Let $g=f|_{w_{d+1}=0}$ (or $g=f_H$ for a hyperplane H) denote the restriction of f to $w_{d+1}=0$ (or H). Then, $c_0(g)\le c_0(f)$.*

Example 1 Consider the function $f=(w_1^2+w_2^2+w_3^2+w_4^2+w_5^2+w_6-1)^2$. We have $c_{(0,0,0,0,0,1)}(f)=1/2$, whereas $c_0(f(w_1,w_2,w_3,w_4,w_5,1))=5/4$.

Therefore, we cannot apply results over an algebraically closed field directly to our problems in learning theory. Moreover the Kullback functions are degenerated with respect to their Newton polyhedra and these singularities are non-isolated [8]. It is a new problem in algebraic geometry to obtain the log canonical threshold of the Kullback functions.

The next two theorems for the homogeneous cases are useful for obtaining the log canonical thresholds of Vandermonde matrix-type singularities.

Theorem 2.3 ([4]) *Let $f_1(w_1,\ldots,w_d)$, …, $f_m(w_1,\ldots,w_d)$ be homogeneous functions of $w_1,\ldots,w_d$ of degree n_i in $w_1,\ldots,w_d$. Set $f_1'(w_2,\ldots,w_d)=f_1(1,w_2,\ldots,w_d)$, …, $f_m'(w_2,\ldots,w_d)=f_m(1,w_2,\ldots,w_d)$. If $w_1^*\ne 0$, then we have $c_{(w_1^*,\ldots,w_d^*)}(f_1^2+\cdots+f_m^2)=c_{(w_2^*/w_1^*,\ldots,w_d^*/w_1^*)}(f_1'^2+\cdots+f_m'^2)$.*

This theorem shows that Example 1 is valid if functions are homogeneous over the real field.

Theorem 2.4 ([4]) *Let $f_1(w_1,\ldots,w_d)$, …, $f_m(w_1,\ldots,w_d)$ be homogeneous functions of $w_1,\ldots,w_j (j\le d)$ with degree n_i of $w_1,\ldots,w_j$. Furthermore, let ψ be a C^∞ function such that $\psi_{(0,\ldots,0,w_{j+1}^*,\ldots,w_d^*)}\ge\psi_{(w_1^*,\ldots,w_d^*)}$ and ψ_w is homogeneous for $w_1,\ldots,w_j$ in a small neighborhood of $(0,\ldots,0,w_{j+1}^*,\ldots,w_d^*)$.*

We then have

$$c_{(0,\ldots,0,w_{j+1}^*,\ldots,w_d^*)}(f_1^2+\cdots+f_m^2,\psi)\le c_{(w_1^*,\ldots,w_j^*,w_{j+1}^*,\ldots,w_d^*)}(f_1^2+\cdots+f_m^2,\psi).$$

In general, it is not true that $c_{w_0}(f_1^2+\cdots+f_m^2,\psi)\le c_{w^*}(f_1^2+\cdots+f_m^2,\psi)$ even if $w_0\in\mathbb{R}^d$ satisfies $f_i(w_0)=\frac{\partial f_i}{\partial w_j}(w_0)=0, 1\le i\le m, 1\le j\le d$.

Example 2 Let $f_1=x(x-1)^2$, $f_2=(y^2+(x-1)^2)((y-1)^6+x)$, and $f_3=(z^2+(x-1)^2)((z-1)^6+x)$. We then have $f_1=f_2=f_3=\frac{\partial f_1}{\partial x}=\frac{\partial f_2}{\partial y}=\frac{\partial f_2}{\partial x}=\frac{\partial f_3}{\partial z}=\frac{\partial f_3}{\partial x}=0$ if and only if $x=1, y=0$, and $z=0$. In this case, we have $c_{(1,0,0)}(f_1^2+f_2^2+f_3^2)=3/4>c_{(0,1,1)}(f_1^2+f_2^2+f_3^2)=2/3$.

3 Main Theorems

3.1 Vandermonde Matrix-Type Singularities

We denote constants such as a^*, b^*, and w^* by the suffix $*$.

Lemma 3.1 ([2, 12]) *Let U be a neighborhood of $w^* \in \mathbb{R}^d$. Let $\mathbf{J}$ be the ideal generated by $f_1, \ldots, f_n$, which are analytic functions defined on U.*

(1) *If $g_1^2 + \cdots + g_m^2 \leq f_1^2 + \cdots + f_n^2$, then $c_{w^*}(g_1^2 + \cdots + g_m^2) \leq c_{w^*}(f_1^2 + \cdots + f_n^2)$.*
(2) *If $g_1, \ldots, g_m \in \mathbf{J}$, then $c_{w^*}(g_1^2 + \cdots + g_m^2) \leq c_{w^*}(f_1^2 + \cdots + f_n^2)$. In particular, if $g_1, \ldots, g_m$ generate the ideal $\mathbf{J}$, then $c_{w^*}(f_1^2 + \cdots + f_n^2) = c_{w^*}(g_1^2 + \cdots + g_m^2)$.*

Definition 3.1 Assume $b_1^* = \cdots = b_{i-1}^* = 0, b_i^* \neq 0$. Let $\gamma_i = \begin{cases} 1 & \text{if } Q \text{ is odd,} \\ |\frac{b_i^*}{b_i^*}| & \text{if } Q \text{ is even.} \end{cases}$ Define $[b_1^*, b_2^*, \ldots, b_N^*]_Q = \gamma_i(0, \ldots, 0, b_i^*, \ldots, b_N^*)$.

Definition 3.2 Let $A_{M,H,r} = \begin{pmatrix} a_{11} & \cdots & a_{1H} & a_{1,H+1}^* & \ldots & a_{1,H+r}^* \\ & \vdots & & & \vdots & \\ a_{M1} & \cdots & a_{MH} & a_{M,H+1}^* & \ldots & a_{M,H+r}^* \end{pmatrix}$,

$I = (\ell_1, \ldots, \ell_N) \in (\mathbf{N} \cup \{0\})^N$,

$$B_{N,H,r,I} = \left(\prod_{j=1}^N b_{1j}^{\ell_j}, \prod_{j=1}^N b_{2j}^{\ell_j}, \ldots, \prod_{j=1}^N b_{Hj}^{\ell_j}, \prod_{j=1}^N {b_{H+1,j}^*}^{\ell_j}, \ldots, \prod_{j=1}^N {b_{H+r,j}^*}^{\ell_j} \right)^t,$$

and $B_{N,H,r}^{(Q)} = (B_I)_{\ell_1 + \cdots + \ell_N = Qn+1, 0 \leq n \leq H+r-1}$

$$= (B_{(1,0,\ldots,0)}, B_{(0,1,\ldots,0)}, \ldots, B_{(0,0,\ldots,1)}, B_{(1+Q,0,\ldots,0)}, \ldots).$$

(The superscript t denotes matrix transposition.)

Variables a_{ki} and $b_{ij}(1 \leq k \leq M, 1 \leq i \leq H, 1 \leq j \leq N)$ are in a neighborhood of a_{ki}^* and b_{ij}^*, where a_{ki}^* and b_{ij}^* are fixed constants.

We call the singularities of the ideal generated by the elements of AB, Vandermonde matrix-type singularities.

To simplify, we usually assume that $(a_{1,H+j}^*, a_{2,H+j}^*, \ldots, a_{M,H+j}^*)^t \neq 0$, $(b_{H+j,1}^*, b_{H+j,2}^*, \ldots, b_{H+j,N}^*) \neq 0$ and $[b_{H+j,1}^*, b_{H+j,2}^*, \ldots, b_{H+j,N}^*]_Q \neq [b_{H+j',1}^*, b_{H+j',2}^*, \ldots, b_{H+j',N}^*]_Q$ for $1 \leqq j \neq j' \leqq r$.

In [4, 5], bounds were derived on the learning coefficients for the Vandermonde matrix-type singularities. The next theorem shows that we need to obtain $c_0(||A_{M,H,0}B_{H,N,0}^{(Q)}||^2)$ and $c_{w^*}(||A_{M,H,1}B_{H,N,0}^{(Q)}||^2)$.

Theorem 3.4 ([4]) *Consider a sufficiently small neighborhood U of $w^* = \{a_{ki}^*, b_{ij}^*\}$ and variables $w = \{a_{ki}, b_{ij}\}$ in the set U. Set $(b_{01}^{**}, b_{02}^{**}, \ldots, b_{0N}^{**}) = (0, \ldots, 0)$.*

Let each $(b^{**}_{11}, b^{**}_{12}, \ldots, b^{**}_{1N}), \ldots, (b^{**}_{r'1}, b^{**}_{r'2}, \ldots, b^{**}_{r'N})$ *be a different real vector in* $[b^*_{i1}, b^*_{i2}, \ldots, b^*_{iN}]_Q \neq 0$, *for* $i = 1, \ldots, H + r$. *Further, set* $(b^{**}_{i1}, \ldots, b^{**}_{iN}) = [b^*_{H+i,1}, \ldots, b^*_{H+i,N}]_Q$, *for* $1 \leq i \leq r$.

Assume that

$$[b^*_{i1}, \ldots, b^*_{iN}]_Q = \begin{cases} 0, & 1 \leq i \leq H_0 \\ (b^{**}_{11}, \ldots, b^{**}_{1N}), & H_0 + 1 \leq i \leq H_0 + H_1, \\ \vdots & \\ (b^{**}_{r'1}, \ldots, b^{**}_{r'N}), & H_0 + \cdots + H_{r'-1} + 1 \leq i \leq H_0 + \cdots + H_{r'} \end{cases}$$

and $H_0 + \cdots + H_{r'} = H$.

We then have $c_{w^*}(||A_{M,H,r}B^{(Q)}_{H,N,r}||^2) = \frac{Mr'}{2} + c_{w_1^{(0)*}}(||A_{M,H_0,0}B^{(Q)}_{H_0,N,0}||^2) + \sum_{\alpha=1}^{r} c_{w_1^{(\alpha)*}}(||A_{M,H_\alpha-1,1}B^{(1)}_{H_\alpha,N,0}||^2) + \sum_{\alpha=r+1}^{r'} c_{w_1^{(\alpha)*}}(||A_{M,H_\alpha-1,0}B^{(1)}_{H_\alpha-1,N,0}||^2)$

where $w_1^{(0)^*} = \{a^*_{k,i}, 0\}_{1 \leq i \leq H_0}$, $w_1^{(\alpha)^*} = \{a^*_{k,H_0+\cdots+H_{\alpha-1}+i}, 0\}_{2 \leq i \leq H_\alpha}$.

In this paper, we obtain explicit values for $H = 1, 2, 3$.

Let $\lambda = c_0(\|A_{M,H,0}B^{(Q)}_{H,N,0}\|^2)$, and θ be its order. Further, let $\lambda' = c_0(\|A_{M,H-1,1}B^{(Q)}_{H,N,0}\|^2)$, and θ' be its order.

Theorem 3.5

Case 1 $H = 1$*:*

1. $\lambda = \min\{\frac{M}{2}, \frac{N}{2}\}$, *and its order* $\theta = \begin{cases} 1, \text{ if } M \neq N, \\ 2, \text{ if } M = N. \end{cases}$
2. $\lambda' = \frac{N}{2}$, *and* $\theta' = 1$.

Case 2 $H = 2$*:*

1. *If* $M > N + 1$, *then* $\lambda = \lambda' = N$ *and* $\theta = \theta' = 1$.
2. *If* $M = N + 1$, *then* $\lambda = \lambda' = N$ *and* $\theta = \theta' = 2$.
3. *If* $M = N$, *then* $\lambda = \lambda' = \frac{2N+Q(2N-1)}{2(Q+1)}$ *and* $\theta = \theta' = 1$.
4. *If* $M \leq N - 1$, *then* $\lambda = M$ *and* $\theta = 1$.
5. *If* $N - Q + 1 \leqq M \leq N - 1$, *then* $\lambda' = \frac{2N+Q(2N-1)}{2(Q+1)}$ *and* $\theta' = 1$.
6. *If* $M = N - Q$, *then* $\lambda' = \frac{N+M}{2}$ *and* $\theta' = 2$.
7. *If* $M \leq N - Q - 1$, *then* $\lambda' = \frac{N+M}{2}$ *and* $\theta' = 1$.

Case 3 $H = 3$*:*

1. *If* $M > N + 2$, *then* $\lambda = \lambda' = \frac{3N}{2}$ *and* $\theta = \theta' = 1$.
2. *If* $M = N + 2$, *then* $\lambda = \lambda' = \frac{3N}{2}$ *and* $\theta = \theta' = 2$.
3. *If* $M = N + 1$, *then* $\lambda = \lambda' = \frac{3N+(3N-1)Q}{2(Q+1)}$ *and* $\theta = \theta' = 1$.
4. *If* $M = N$, *then* $\lambda = \lambda' = \frac{3N+(3N-2)Q}{2(Q+1)}$ *and* $\theta = \theta' = 2$.

5. *If* $M = N - 1$, *then* $\begin{cases} \lambda = \frac{3-Q+3M(Q+1)}{2(Q+1)} \text{ and } \theta = 1 \text{ for } Q > 3, \\ \lambda = \frac{3M}{2} \text{ and } \theta = 2 \text{ for } Q = 3, \\ \lambda = \frac{3M}{2} \text{ and } \theta = 1 \text{ for } Q < 3. \end{cases}$
6. *If* $M < N - 1$, *then* $\lambda = \frac{3M}{2}$ *and* $\theta = 1$.
7. *If* $M = N - S$ *for* $S = 1, 2, \ldots$, *then*
$\begin{cases} \lambda' = \frac{S(3+Q)-2Q+3M(Q+1)}{2(Q+1)} \text{ and } \theta' = 1 \text{ for } Q > S, \\ \lambda' = \frac{2M+N}{2} \text{ and } \theta' = 2 \text{ for } Q = S, \\ \lambda' = \frac{2M+N}{2} \text{ and } \theta' = 1 \text{ for } Q < S. \end{cases}$

We already have exact values for $N = 1$.

Theorem 3.6 ([2]) *If* $N = 1$, *we have* $\lambda = \lambda' = \frac{MQk(k+1)+2H}{4(1+kQ)}$ *where* $k = \max\{i \in \mathbb{Z}; 2H \geq M(i(i-1)Q + 2i)\}$,

$$\theta = \begin{cases} 1, \text{ if } 2H > M(k(k-1)Q + 2k), \\ 2, \text{ if } 2H = M(k(k-1)Q + 2k), \end{cases} \text{ and}$$

$$\theta' = \begin{cases} 1, \text{ if } M = H = 1, \\ 1, \text{ if } 2H > M(k(k-1)Q + 2k), \\ 2, \text{ if } 2H = M(k(k-1)Q + 2k), H > 1. \end{cases}$$

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A Survey of Change of Scale Formulas on an Analogue of Wiener Space

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Dedicated to Jerry

Abstract Let $(C[0,t], w_\varphi)$ denote an analogue of Wiener space, that is, the space of real-valued continuous paths on $[0,t]$. The measure w_φ and w_φ-measurability behave badly under change of scale, and under translation. In this paper we introduce several change of scale formulas on $C[0,t]$ for the generalized analytic conditional Wiener integrals of the cylinder functions and the functions in a Banach algebra which corresponds to the Cameron-Storvick's Banach algebra.

Keywords Analogue of Wiener space • Change of scale formula • Conditional Wiener integral • Simple formula for conditional Wiener integral • Wiener measure

Mathematics Subject Classification (2010) Primary 28C20; Secondary 60G05, 60G15, 60H05

1 Introduction and Preliminaries

Let $(C[0,t], \mathcal{B}(C[0,t]), w_\varphi)$ be the analogue of Wiener space associated with a probability measure φ on the Borel class of $\mathbb{R}$, where $\mathcal{B}(C[0,t])$ denotes the Borel class of $C[0,t]$ [1–4]. For $v \in L_2[0,t]$ and $x \in C[0,t]$ let (v,x) denote the Paley–Wiener–Zygmund integral of v according to x. The inner product on the real Hilbert space $L_2[0,t]$ is denoted by $\langle \cdot, \cdot \rangle$.

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Let $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = t$ be a partition of $[0, t]$, where n is a fixed nonnegative integer. Let $h \in L_2[0, t]$ be of bounded variation with $h \neq 0$ a.e. on $[0, t]$. For $j = 1, \ldots, n+1$ let $\alpha_j = \frac{1}{\|\chi_{(t_{j-1},t_j]}h\|}\chi_{(t_{j-1},t_j]}h$ and let $V^\perp$ be the orthogonal complement of V which is the subspace of $L_2[0, t]$ generated by $\{\alpha_1, \ldots, \alpha_{n+1}\}$. Let $\mathcal{P} : L_2[0, t] \to V^\perp$ be the orthogonal projection. Let a be absolutely continuous on $[0, t]$ and define stochastic processes $X, Z : C[0, t] \times [0, t] \to \mathbb{R}$ by $X(x, s) = (\chi_{[0,s]}h, x)$ and $Z(x, s) = X(x, s) + x(0) + a(s)$ for $x \in C[0, t]$ and for $s \in [0, t]$. Define random vectors Z_n and Z_{n+1} on $C[0, t]$ by $Z_n(x) = (Z(x, t_0), Z(x, t_1), \ldots, Z(x, t_n))$ and $Z_{n+1}(x) = (Z(x, t_0), Z(x, t_1), \ldots, Z(x, t_n), Z(x, t_{n+1}))$ for $x \in C[0, t]$. Let $b(s) = \|\chi_{[0,s]}h\|^2$ and for any function f on $[0, t]$ define the polygonal function $P_{b,n+1}(f)$ of f by $P_{b,n+1}(f)(s) = \sum_{j=1}^{n+1}\chi_{(t_{j-1},t_j]}(s)[\frac{b(t_j)-b(s)}{b(t_j)-b(t_{j-1})}f(t_{j-1}) + \frac{b(s)-b(t_{j-1})}{b(t_j)-b(t_{j-1})}f(t_j)] + \chi_{\{0\}}(s)f(0)$ for $s \in [0, t]$. For $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \ldots, \xi_n, \xi_{n+1}) \in \mathbb{R}^{n+2}$ define the polygonal function $P_{b,n+1}(\vec{\xi}_{n+1})$ of $\vec{\xi}_{n+1}$ as above, where $f(t_j)$ is replaced by ξ_j for $j = 0, 1, \ldots, n, n+1$. If $\vec{\xi}_n = (\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1}$, $P_{b,n}(\vec{\xi}_n)$ is interpreted as $\chi_{[0,t_n]}P_{b,n+1}(\vec{\xi}_{n+1})$ on $[0, t]$. Let $A(s) = a(s) - P_{b,n+1}(a)(s)$, $X_{b,n+1}(x, s) = X(x, s) - P_{b,n+1}(X(x, \cdot))(s)$ and $Z_{b,n+1}(x, s) = Z(x, s) - P_{b,n+1}(Z(x, \cdot))(s)$. For $\alpha, \beta, u \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ let $\Psi(\lambda, u, \alpha, \beta) = (\frac{\lambda}{2\pi\beta})^{\frac{1}{2}} \exp\{-\frac{\lambda}{2\beta}(u - \alpha)^2\}$ with $\beta \neq 0$.

For a function $F : C[0, t] \to \mathbb{C}$ let $F_Z(x) = F(Z(x, \cdot))$ for $x \in C[0, t]$. For $\lambda > 0$ let $F_Z^\lambda(x) = F_Z(\lambda^{-\frac{1}{2}}x)$ and $Z_{n+1}^\lambda(x) = Z_{n+1}(\lambda^{-\frac{1}{2}}x)$. Suppose that $E[F_Z^\lambda]$ exists. By Lemma 2.1 of [5]

$$E[F_Z^\lambda|Z_{n+1}^\lambda](\vec{\xi}_{n+1}) = E[F(\lambda^{-\frac{1}{2}}X_{b,n+1}(x, \cdot) + A + P_{b,n+1}(\vec{\xi}_{n+1}))] \tag{1}$$

for $P_{Z_{n+1}^\lambda}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, where $P_{Z_{n+1}^\lambda}$ is the probability distribution of Z_{n+1}^λ on $(\mathbb{R}^{n+2}, \mathcal{B}(\mathbb{R}^{n+2}))$. Moreover, by Lemma 2.2 of [5], we have for $P_{Z_n^\lambda}$-a.e. $\vec{\xi}_n = (\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1}$

$$E[F_Z^\lambda|Z_n^\lambda](\vec{\xi}_n) = \int_{\mathbb{R}} \Psi(\lambda, \xi_{n+1} - \xi_n, a(t) - a(t_n), b(t) - b(t_n))E[F(\lambda^{-\frac{1}{2}} \times X_{b,n+1}(x, \cdot) + A + P_{b,n+1}(\vec{\xi}_{n+1}))]d\xi_{n+1}, \tag{2}$$

where $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \ldots, \xi_n, \xi_{n+1})$. If the right-hand side of (1) has an analytic extension $J_\lambda^*(F_Z)(\vec{\xi}_{n+1})$ on $\mathbb{C}_+ \equiv \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$, then it is called the conditional analytic Wiener w_φ-integral of F_Z given Z_{n+1} with the parameter λ and denoted by $E^{\mathrm{anw}_\lambda}[F_Z|Z_{n+1}](\vec{\xi}_{n+1}) = J_\lambda^*(F_Z)(\vec{\xi}_{n+1})$ for $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$. Moreover if for nonzero real q, $E^{\mathrm{anw}_\lambda}[F_Z|Z_{n+1}](\vec{\xi}_{n+1})$ has a limit as λ approaches to $-iq$ through $\mathbb{C}_+$, then it is called the conditional analytic Feynman w_φ-integral of F_Z given Z_{n+1} with the parameter q and denoted by $E^{\mathrm{anf}_q}[F_Z|Z_{n+1}](\vec{\xi}_{n+1}) = \lim_{\lambda\to -iq} E^{\mathrm{anw}_\lambda}[F_Z|Z_{n+1}](\vec{\xi}_{n+1})$. $E^{\mathrm{anw}_\lambda}[F_Z|Z_n](\vec{\xi}_n)$ and $E^{\mathrm{anf}_q}[F_Z|Z_n](\vec{\xi}_n)$ are similarly understood with the right-hand side of (2).

2 The One-Dimensional Change of Scale Formulas

Let e be in $L_2[0,t]$ with $\|e\| = 1$. For $1 \leq p \leq \infty$ let $\mathcal{A}^{(p)}$ be the space of the cylinder functions F having the following form

$$F(x) = f((e,x)) \tag{3}$$

for w_φ-a.e. $x \in C[0,t]$, where $f \in L_p(\mathbb{R})$.

Throughout this paper, $\{\lambda_m\}_{m=1}^{\infty}$ denotes any sequence in $\mathbb{C}_+$ converging to $-iq$ as m approaches ∞.

Theorem 2.1 *Let $1 \leq p \leq \infty$ and $F \in \mathcal{A}^{(p)}$ be given by (3). Then for $\lambda \in \mathbb{C}_+$, for a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$ and for a.e. $\vec{\xi}_n = (\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1}$,*

$$E^{\mathrm{anw}_\lambda}[F_Z|Z_{n+1}](\vec{\xi}_{n+1}) = \int_{\mathbb{R}} f(u)\Psi(\lambda, u, (e, A + P_{b,n+1}(\vec{\xi}_{n+1})), \|\mathcal{P}(eh)\|^2)du$$

if $eh \notin V$, and letting $(e, P_{b,n}(\vec{\xi}_n)) = \sum_{j=1}^{n}\langle e\alpha_j, \alpha_j\rangle(\xi_j - \xi_{j-1})$

$$\begin{aligned}
&E^{\mathrm{anw}_\lambda}[F_Z|Z_n](\vec{\xi}_n)\\
&\quad= \int_{\mathbb{R}} f(u)\Psi(\lambda, u, (e,A) + (e, P_{b,n}(\vec{\xi}_n)) + \langle e\alpha_{n+1}, \alpha_{n+1}\rangle[a(t)\\
&\qquad -a(t_n)], \|\mathcal{P}(eh)\|^2 + \langle e\alpha_{n+1}, \alpha_{n+1}\rangle^2[b(t) - b(t_n)])du
\end{aligned}$$

if $eh \notin V$ or $\langle e\alpha_{n+1}, \alpha_{n+1}\rangle \neq 0$. Furthermore if $p = 1$, then $E^{\mathrm{anf}_q}[F_Z|Z_{n+1}]$ and $E^{\mathrm{anf}_q}[F_Z|Z_n]$ are given by the right-hand sides of the above equalities, respectively, replacing λ by $-iq$. If $eh \in V$, then

$$E^{\mathrm{anw}_\lambda}[F_Z|Z_{n+1}](\vec{\xi}_{n+1}) = E^{\mathrm{anf}_q}[F_Z|Z_{n+1}](\vec{\xi}_{n+1}) = f((e, A + P_{b,n+1}(\xi_{n+1}))),$$

and if $eh \in V$ and $\langle e\alpha_{n+1}, \alpha_{n+1}\rangle = 0$, then

$$E^{\mathrm{anw}_\lambda}[F_Z|Z_n](\vec{\xi}_n) = E^{\mathrm{anf}_q}[F_Z|Z_n](\vec{\xi}_n) = f((e,A) + (e, P_{b,n}(\xi_n))).$$

Let $\{e_1, e_2, \ldots\}$ be any complete orthonormal basis of $L_2[0,t]$. For $m \in \mathbb{N}$, $\lambda \in \mathbb{C}$ and $x \in C[0,t]$ let $K_m(\lambda, x) = \exp\{\frac{1-\lambda}{2}\sum_{j=1}^{m}(e_j, x)^2\}$.

Theorem 2.2 *Let $1 \leq p \leq \infty$ and $F \in \mathcal{A}^{(p)}$ be given by (3). Then for $\lambda \in \mathbb{C}_+$, for a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$ and for a.e. $\vec{\xi}_n = (\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1}$,*

$$\begin{aligned}
&E^{\mathrm{anw}_\lambda}[F_Z|Z_{n+1}](\vec{\xi}_{n+1})\\
&\quad= \lim_{m\to\infty} \lambda^{\frac{m}{2}} E[K_m(\lambda, x)F(Z_{b,n+1}(x,\cdot) + P_{b,n+1}(\vec{\xi}_{n+1}))] \qquad (4)\\
&\quad= \lim_{m\to\infty} \lambda^{\frac{m}{2}} E[K_m(\lambda,\cdot)f((v,\cdot)\|\mathcal{P}(eh)\| + (e, A + P_{b,n+1}(\vec{\xi}_{n+1})))]
\end{aligned}$$

and letting $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \ldots, \xi_n, \xi_{n+1})$ *for* $\xi_{n+1} \in \mathbb{R}$

$$\begin{aligned} &E^{\mathrm{anw}_\lambda}[F_Z|Z_n](\vec{\xi}_n) \\ &= \lim_{m\to\infty} \lambda^{\frac{m}{2}} \int_{\mathbb{R}} \Psi(\lambda, \xi_{n+1} - \xi_n, a(t) - a(t_n), b(t) - b(t_n)) E[K_m(\lambda, x) \\ &\quad \times F(Z_{b,n+1}(x, \cdot) + P_{b,n+1}(\vec{\xi}_{n+1}))] d\xi_{n+1} \qquad (5) \\ &= \lim_{m\to\infty} \lambda^{\frac{m}{2}} \int_{\mathbb{R}} \Psi(\lambda, \xi_{n+1} - \xi_n, a(t) - a(t_n), b(t) - b(t_n)) E[K_m(\lambda, \cdot) \\ &\quad \times f((v, \cdot)\|\mathcal{P}(eh)\| + (e, A + P_{b,n+1}(\vec{\xi}_{n+1})))] d\xi_{n+1} \end{aligned}$$

for any unit element $v \in L_2[0,t]$. *Moreover if* $p = 1$, *then* $E^{\mathrm{anf}_q}[F_Z|Z_{n+1}]$ *and* $E^{\mathrm{anf}_q}[F_Z|Z_n]$ *are given by the right-hand sides of the above equalities, respectively, replacing* λ *by* λ_m.

Let $\mathcal{M}(L_2[0,t])$ be the class of all complex Borel measures of bounded variation on $L_2[0,t]$ and let $\mathcal{S}_{w_\varphi}$ be the space of all functions F which have the form

$$F(x) = \int_{L_2[0,t]} \exp\{i(v, x)\} d\sigma(v) \qquad (6)$$

for $\sigma \in \mathcal{M}(L_2[0,t])$ and for w_φ-a.e. $x \in C[0,t]$.

Theorem 2.3 *Let* F *be given by* (6). *Then for* $\lambda \in \mathbb{C}_+$, *for a.e.* $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$ *and for a.e.* $\vec{\xi}_n = (\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1}$

$$\begin{aligned} &E^{\mathrm{anw}_\lambda}[F_Z|Z_{n+1}](\vec{\xi}_{n+1}) \\ &\quad = \int_{L_2[0,t]} \exp\left\{ -\frac{1}{2\lambda}\|\mathcal{P}(vh)\|^2 + i(v, P_{b,n+1}(\vec{\xi}_{n+1}))\right\} d\sigma_A(v), \end{aligned}$$

and letting $(v, P_{b,n}(\vec{\xi}_n)) = \sum_{j=1}^n \langle v\alpha_j, \alpha_j\rangle(\xi_j - \xi_{j-1})$

$$\begin{aligned} &E^{\mathrm{anw}_\lambda}[F_Z|Z_n](\vec{\xi}_n) \\ &\quad = \int_{L_2[0,t]} \exp\left\{ -\frac{1}{2\lambda}[\|\mathcal{P}(vh)\|^2 + [b(t) - b(t_n)]\langle v\alpha_{n+1}, \alpha_{n+1}\rangle^2] \right. \\ &\qquad \left. + i[(v, P_{b,n}(\vec{\xi}_n)) + [a(t) - a(t_n)]\langle v\alpha_{n+1}, \alpha_{n+1}\rangle] \right\} d\sigma_A(v), \end{aligned}$$

where

$$d\sigma_A(v) = \exp\{i(v, A)\} d\sigma(v).$$

Moreover $E^{\mathrm{anf}_q}[F_Z|Z_{n+1}]$ *and* $E^{\mathrm{anf}_q}[F_Z|Z_n]$ *are given by the right-hand sides of the above equalities, respectively, replacing* λ *by* $-iq$.

Theorem 2.4 *Let* F *be given by* (6)*. Then for* $\lambda \in \mathbb{C}_+$, $E_\lambda^{anw}[F_Z|Z_{n+1}]$ *and* $E^{\mathrm{anw}_\lambda}[F_Z|Z_n]$ *are given by* (4) *and* (5)*, respectively. Moreover for any nonzero real* q, $E_q^{anf}[F_Z|Z_{n+1}]$ *and* $E^{\mathrm{anf}_q}[F_Z|Z_n]$ *are given by the right-hand sides of the same equalities, respectively, replacing* λ *by* λ_m.

3 The *r*-Dimensional Change of Scale Formulas

Let $\{v_1, v_2, \ldots, v_r\}$ be an orthonormal subset of $L_2[0,t]$ such that $\{\mathcal{P}(hv_1), \ldots, \mathcal{P}(hv_r)\}$ is an independent set. Let $\{e_1, \ldots, e_r\}$ be the orthonormal set obtained from $\{\mathcal{P}(hv_1), \ldots, \mathcal{P}(hv_r)\}$ by the Gram-Schmidt orthonormalization process. Now for $l = 1, \ldots, r$ let $\mathcal{P}(hv_l) = \sum_{j=1}^r \alpha_{lj} e_j$ and let M be the transpose of the coefficient matrix.

For $1 \le p \le \infty$ let $\mathcal{A}_r^{(p)}$ be the space of the cylinder functions F_r having the following form

$$F_r(x) = f_r((\vec{v}, x)) \tag{7}$$

for w_φ-a.e. $x \in C[0,t]$, where $(\vec{v}, x) = ((v_1, x), \ldots, (v_r, x))$ and $f_r \in L_p(\mathbb{R}^r)$.

Theorem 3.1 *Let* $1 \le p \le \infty$ *and let* K_m *be as given in the previous section with the orthonormal set* $\{e_1, \ldots, e_r, e_{r+1}, \ldots\}$ *in this section. For an orthonormal set* $\{h_1, \ldots, h_r\}$ *in* $L_2[0,t]$ *let* $H_r(\lambda, x) = \exp\{\frac{1-\lambda}{2}\sum_{j=1}^r (h_j, x)^2\}$*. Then for* $\lambda \in \mathbb{C}_+$ *and for a.e.* $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$

$$\begin{aligned} &E^{\mathrm{anw}_\lambda}[(F_r)_Z|Z_{n+1}](\vec{\xi}_{n+1}) \\ &= \lambda^{\frac{r}{2}} E[K_r(\lambda, x) F_r(Z_{b,n+1}(x,\cdot) + P_{b,n+1}(\vec{\xi}_{n+1}))] \\ &= \lambda^{\frac{r}{2}} E[H_r(\lambda, x) f_r((\vec{h}, x)M^T + (\vec{v}, A + P_{b,n+1}(\vec{\xi}_{n+1})))], \end{aligned}$$

where $(\vec{h}, x) = ((h_1, x), \ldots, (h_r, x))$*. Moreover if* $p = 1$*, then*

$$\begin{aligned} &E^{\mathrm{anf}_q}[(F_r)_Z|Z_{n+1}](\vec{\xi}_{n+1}) \\ &= \lim_{m\to\infty} \lambda_m^{\frac{r}{2}} E[K_r(\lambda_m, x) F_r(Z_{b,n+1}(x,\cdot) + P_{b,n+1}(\vec{\xi}_{n+1}))] \\ &= \lim_{m\to\infty} \lambda_m^{\frac{r}{2}} E[H_r(\lambda_m, x) f_r((\vec{h}, x)M^T + (\vec{v}, A + P_{b,n+1}(\vec{\xi}_{n+1})))]. \end{aligned}$$

Theorem 3.2 *Let $G_r = FF_r$, where $F \in \mathcal{S}_{w_\varphi}$ and $F_r \in \mathcal{A}_r^{(p)} (1 \le p \le \infty)$ is given by (7). Then for $\lambda \in \mathbb{C}_+$, $E_\lambda^{anw}[(G_r)_Z|Z_{n+1}]$ is given by the right-hand side of (4) replacing F by G_r. For a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$ it also can be expressed by*

$$E^{\mathrm{anw}_\lambda}[(G_r)_Z|Z_{n+1}](\vec{\xi}_{n+1})$$
$$= \lim_{m\to\infty} \lambda^{\frac{m}{2}} \int_{L_2[0,t]} \exp\{i(v, P_{b,n+1}(\vec{\xi}_{n+1}))\} E[K_m(\lambda, x)\exp\{i(\mathcal{P}(vh), x)\}$$
$$\times f_r((\vec{e}, x)M^T + (\vec{v}, A + P_{b,n+1}(\vec{\xi}_{n+1})))] d\sigma_A(v), \tag{8}$$

where $(\vec{e}, x) = ((e_1, x), \ldots, (e_r, x))$. If $p = 1$, then $E^{\mathrm{anf}_q}[(G_r)_Z|Z_{n+1}]$ can be given by the right-hand sides of (4) and (8), where λ and F are replaced by λ_m and G_r, respectively.

Let $\hat{\mathrm{M}}(\mathbb{R}^r)$ be the space of all functions ϕ on $\mathbb{R}^r$ defined by

$$\phi(\vec{u}) = \int_{\mathbb{R}^r} \exp\{i\langle \vec{u}, \vec{z}\rangle\} d\rho(\vec{z}), \tag{9}$$

where ρ is a complex Borel measure of bounded variation on $\mathbb{R}^r$.

Theorem 3.3 *Let $\Phi(x) = \phi((\vec{v}, x))F(x)$ for w_φ-a.e. $x \in C[0, t]$, where ϕ is given by (9). Then $E^{\mathrm{anf}_q}[\Phi_Z|Z_{n+1}]$ is given by the right-hand side of (4) replacing λ and F by λ_m and Φ, respectively. For a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$ it also can be expressed by*

$$E^{\mathrm{anf}_q}[\Phi_Z|Z_{n+1}](\vec{\xi}_{n+1})$$
$$= \lim_{m\to\infty} \lambda_m^{\frac{m}{2}} \int_{L_2[0,t]} \int_{\mathbb{R}^r} A_1(\vec{\xi}_{n+1}, v, \vec{z}) E[K_m(\lambda_m, x)$$
$$\times \exp\{i[(\mathcal{P}(vh), x) + \langle (\vec{e}, x), \vec{z}M\rangle]\}] d\rho_A(\vec{z}) d\sigma_A(v),$$

where $\rho_A(\vec{z}) = \exp\{i\langle(\vec{v}, A), \vec{z}\rangle\}$ and $A_1(\vec{\xi}_{n+1}, v, \vec{z}) = \exp\{i[(v, P_{b,n+1}(\vec{\xi}_{n+1})) + \langle(\vec{v}, P_{b,n+1}(\vec{\xi}_{n+1})), \vec{z}\rangle]\}$.

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Integrability and Uniform Convergence of Multiplicative Transforms

B.I. Golubov and S.S. Volosivets

Abstract For multiplicative Fourier transforms the analogues of the results of the papers (Dyachenko et al., J. Math. Anal. Appl., 372:328–338, 2010; Sampson and Tuy, Pac. J. Math., 75:519–537, 1978; Moricz, Stud. Math., 199:199–205, 2010; Liflyand and Tikhonov, C.R. Acad. Sci. Paris, Ser. 1, 346:1137–1142, 2008) on uniform convergence and integrability with power weights of classical Fourier transform are obtained. Some generalizations of the results of Onneweer (Lect. Notes., Math 939:106–121, 1981; Monatsh. Math., 97:297–310, 1984) on belonging of multiplicative Fourier transforms to Besov-Lipschitz or Herz spaces are stated.

Keywords Besov-Lipschitz space • Herz space • Modulus of continuity • Multiplicative Fourier transform • Uniform convergence

Mathematics Subject Classification (2010) Primary 43A25; Secondary 43A15, 42C99.

1 Introduction

For the function f locally integrable on $\mathbb{R}_+ = [0, \infty)$ that is $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ let us introduce the cosine Fourier transform

$$\widehat{f}_c(x) = \int_0^\infty f(y) \cos xy \, dy,$$

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where the integral is considered in improper sense with singular point $+\infty$. By $\int_a^b |df(t)|$ we denote the variation of the function f on the interval $[a, b]$. We say that the pair of functions (f, β) defined on $(0, +\infty)$ belongs to the class GM, if $f \in V[a, b]$ for any interval $[a, b] \subset (0, +\infty)$ and the inequality $\int_x^{2x} |df(t)| \le C\beta(x)$ holds for each $x \in (0, +\infty)$, where the constant C does not depend on x. Let us set

$$S_t(\widehat{f}_c)(x) = \int_0^t f(y) \cos xy \, dy$$

In the paper [1] the following theorem was proved.

Theorem 1.1 *If* $f(x) > 0$ *on* $(0, +\infty)$ *or* $(f, \beta) \in GM$ *and* $\lim_{x\to+\infty} x\beta(x) = 0$*, then the integral* $\int_0^\infty f(y) \cos xy \, dy$ *converges uniformly on* $\mathbb{R}_+$ *iff the integral* $\int_0^\infty f(x) \, dx$ *converges. In the last case the following estimate*

$$\|\widehat{f}_c - S_M(\widehat{f}_c)\|_\infty \le \sup_{x\ge M} \left| \int_M^x f(t) \, dt \right| + C \sup_{x\ge M/2} \int_x^{2x} |df(t)|$$

holds, where the constant $C > 0$ *does not depend on* $M > 0$.

The similar result for the sine Fourier transform holds [1]. In the papers [2] and [3] the authors investigated the conditions on the functions under which their Fourier transforms

$$F(f)(x) = \int_{\mathbb{R}} f(y) \exp(-ixy) \, dy$$

belong to the class $\operatorname{Lip} \alpha$ on $\mathbb{R}$ for $\alpha \in (0, 1)$. We combine their results in the following theorem.

Theorem 1.2

1) Let $\alpha \in (0, 1)$ *and the function* $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ *satisfies the conditions*

$$\int_{1/h}^{+\infty} f(y) \exp(-ixy) \, dy = O(h^\alpha), \quad \int_{-\infty}^{-1/h} f(y) \exp(-ixy) \, dy = O(h^\alpha), \quad h > 0, \tag{1}$$

uniformly on $\mathbb{R}$*, where the integrals in* (1) *are considered in improper sense. Then* $F(f) \in \operatorname{Lip} \alpha$ *on* $\mathbb{R}$.

2) Let $\alpha \in (0, 1)$, $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ *and*

$$\int_{|x|<y} |xf(x)| \, dx = O(y^{1-\alpha}), \quad y > 0. \tag{2}$$

Then $f \in L^1(\mathbb{R})$ *and* $F(f) \in \operatorname{Lip} \alpha$ *on* $\mathbb{R}$.

3) If $F(f) \in \operatorname{Lip} \alpha$ *on* $\mathbb{R}$ *and* $xf(x) \ge 0$ *for all* $x \in \mathbb{R}$*, then the condition* (2) *holds.*

The statement of the item 1) of this theorem was proved by Sampson and Tuy [2] and the items 2) and 3) were proved by Moricz [3].

In the paper [4] E. Liflyand and S. Tikhonov introduced the class GM^* of functions $f \in V_{\mathrm{loc}}(0, +\infty)$ vanishing at $+\infty$ and satisfying the condition

$$\int_x^{2x} |df(t)| \leq C \int_{x/b}^{bx} |f(u)| \frac{du}{u}, \quad x > 0$$

for some $b > 1$ and $C > 0$. They proved the following theorem.

Theorem 1.3 *Let* $1 < p, q < \infty$, $-1/p' < \gamma < 1/p$, *where* $1/p + 1/p' = 1$, *the function* $f \in GM^*$ *is nonnegative and the integral* $F_+(f)(x) = \int_0^{+\infty} f(y) \exp(-ixy)\, dy$ *is considered in improper sense with singular points* 0 *and* $+\infty$. *Then the following statements are valid:*

1) if $q \leq p$ *and* $x^{1+\gamma-1/p-1/q} f(x) \in L^q(\mathbb{R}_+)$, *then* $x^{-\gamma} F_+(f)(x) \in L^p(\mathbb{R}_+)$;
2) if $p \leq q$ *and* $x^{-\gamma} F_+(f)(x) \in L^p(\mathbb{R}_+)$, *then* $x^{1+\gamma-1/p-1/q} f(x) \in L^q(\mathbb{R}_+)$.

In the paper [5] we proved an analog of Theorem 1.3 for multiplicative Fourier transforms of monotone functions. In this paper we generalize that result using functions f satisfying the condition

$$\int_x^{+\infty} |df(t)| \leq C x^{\theta-1} \int_{x/b}^{+\infty} u^{-\theta} |f(u)|\, du, \quad x > 0, \tag{3}$$

where $b > 1$, $\theta \in (0, 1)$ and the constant $C > 0$ does not depend on $x > 0$.

In this report we formulate some analogues of Theorems 1.1, 1.2 and a generalization of Theorem 1.3 and some results of the papers [6, 7] for multiplicative Fourier transforms.

2 Basic Definitions

Multiplicative Fourier transform was introduced by Vilenkin [8] as a generalization of the Walsh transform, which has been defined by Fine [9]. We shall consider only symmetric multiplicative Fourier transforms (see [10, p. 127]).

Let be given two-sided symmetric sequence of natural numbers $\mathbf{P} = \{p_j\}_{j \in \mathbb{N}}$, where $p_j \in \mathbb{N}$, $p_j \geq 2$ and $p_{-j} = p_j$ for $j \in \mathbb{N}$. We set $m_j = p_1 \cdot \dots \cdot p_j$, $m_{-j} = 1/m_j$ for $j \in \mathbb{N}$ and $p_0 = 1$. Then each number $x \in \mathbb{R}_+$ can be expressed in the form

$$x = \sum_{j=1}^{k(x)} x_{-j} m_{j-1} + \sum_{j=1}^{\infty} x_j m_{-j}, \quad x_j \in \mathbb{Z} \cap [0, p_j) =: Z(p_j), \ |j| \in \mathbb{N}. \tag{4}$$

For $x = k/m_n$, $k \in \mathbb{Z}_+ = \mathbb{Z} \cap [0, +\infty)$ we take the expansion (4) with finite number of non-zero coordinates x_j. Then the expansion (4) is unique for all $x \in \mathbb{R}_+$. Below we shall suppose that the sequence $\mathbf{P} = \{p_j\}_{j \in \mathbb{N}}$ is bounded, that is $p_j \le C$, $|j| \in \mathbb{N}$. For the numbers $x, y \in \mathbb{R}_+$ with the expansions of the type (4) we set $x \oplus y = z$, where $z_j = x_j + y_j \pmod{p_j}$, $|j| \in \mathbb{N}$. The inverse operation $\ominus$ is defined in a similar way. Moreover we introduce the kernel $\chi(x, y)$ by the equality

$$\chi(x,y) = \exp\left\{2\pi i \left(\sum_{j=1}^{\infty} (x_j y_{-j} + x_{-j} y_j)/p_j\right)\right\}.$$

The multiplicative Fourier $\mathbf{P}$-transform $\widehat{f}$ for the function $f \in L^1(\mathbb{R}_+)$ is defined by the equality $\widehat{f}(x) = \int_{\mathbb{R}_+} f(y)\overline{\chi(x,y)}\,dy$. For the functions $f \in L^p(\mathbb{R}_+)$, $1 < p \le 2$, we set $\widehat{f}(x) = (L^{p'}) \lim_{a \to +\infty} \int_0^a f(y)\overline{\chi(x,y)}\,dy$, where $1/p + 1/p' = 1$. The existence of this limit is well known (see, for example, [10, p. 132]).

For functions $f \in L^p(\mathbb{R}_+)$, $1 \le p < \infty$, the L^p-modulus of continuity is defined by the equality $\omega^*(f, \delta)_p = \sup_{0<h<\delta} \|f(\cdot) - f(\cdot \oplus h)\|_p$. Let $\omega = \{\omega_n\}_{n=0}^{\infty}$ be non-negative sequence tending to zero. By H_p^{ω} we denote the class of functions $f \in L^p(\mathbb{R}_+)$, $1 \le p < \infty$, for which $\omega_n(f)_p = O(\omega_n)$, $n \in \mathbb{Z}_+$, where $\omega_n(f)_p = \omega^*(f, 1/m_n)_p$. The class $H^{\omega} = H_{\infty}^{\omega}$ is defined in a similar way. We shall say that non-negative tending to zero sequence $\omega = \{\omega_n\}_{n=0}^{\infty}$ belongs to the class B, if $\sum_{k=n}^{\infty} \omega_k = O(\omega_n)$ for $n \in \mathbb{N}$ and it belongs to the class B_l, $l > 0$, if $\sum_{k=0}^{n} m_k^l \omega_k = O(m_n^l \omega_n)$, $n \in \mathbb{Z}_+$.

Let us introduce the multiplicative Dirichlet kernel $D_y(x) = \int_0^y \chi(x,t)\,dt$.

Definition 2.1 Let $\alpha \in \mathbb{R}$, $1 \le p \le \infty$ and $0 < q < +\infty$. We say that the function $f \in L^p(\mathbb{R}_+)$ belongs to Besov-Lipschitz $\mathbf{P}$-space $\Lambda(\alpha, p, q)$, if

$$\|f\|_{\Lambda(\alpha,p,q)} = \|f\|_p + \left(\sum_{k\in\mathbb{Z}} \|m_k^{\alpha}\,(D_{m_k} - D_{m_{k-1}}) * f\|_p^q\right)^{1/q} < \infty,$$

where

$$g * f(x) = \int_{\mathbb{R}_+} g(y) f(x \ominus y)\,dy$$

is $\mathbf{P}$-convolution of the functions g and f. For $q = +\infty$ it is assumed that

$$\|f\|_{\Lambda(\alpha,p,\infty)} = \|f\|_p + \sup_{k\in\mathbb{Z}} \|m_k^{\alpha}(D_{m_k} - D_{m_{k-1}}) * f\|_p < \infty.$$

Definition 2.2 Let $\alpha \in \mathbb{R}$, $1 \le p \le \infty$ and $0 < q < +\infty$. We say that the function $f \in L^p(\mathbb{R}_+)$ belongs to Herz **P**-space $K(\alpha, p, q)$, if

$$\|f\|_{K(\alpha,p,q)} = \left(\sum_{k \in \mathbb{Z}} \|m_k^\alpha X_{[m_{k-1}, m_k)} f\|_p^q \right)^{1/q} < \infty,$$

where X_E is the indicator function of a set E. For $q = +\infty$ it is assumed that

$$\|f\|_{K(\alpha,p,\infty)} = \sup_{k \in \mathbb{Z}} \|m_k^\alpha X_{[m_{k-1}, m_k)} f\|_p < \infty.$$

Let $|x|_{\mathbf{P}} = m_k$ for $m_{k-1} \le x < m_k$, $k \in \mathbb{Z}$. Then it is easy to see that in the case $q = p$ we have

$$\|f\|_{K(\alpha,p,p)} = \left(\int_{\mathbb{R}_+} |x|_{\mathbf{P}}^{\alpha p} |f(x)|^p \, dx \right)^{1/p}.$$

3 Main Results

Theorem 3.1 *If $(f, \beta) \in GM$ and $\lim_{t \to +\infty} t\beta(t) = 0$, then the improper integral $\int_0^{+\infty} f(y)\overline{\chi(x, y)}\, dy$ converges uniformly on $\mathbb{R}_+$ if and only if the integral $\int_0^{\infty} f(t)\, dt$ converges. In the last case the estimate*

$$\left| \int_A^{+\infty} f(t)\overline{\chi(x, t)}\, dt \right| \le B \left(\sup_{y \ge A} \left| \int_A^y f(t)\, dt \right| + \sup_{s > A/(2C)} s \int_s^{2s} |df(t)| \right)$$

is valid, where C is the upper bound of $\{p_n\}_{n \in \mathbb{N}}$, the constant $B > 0$ does not depend on A and $x \in \mathbb{R}_+$.

This theorem be an analog of the result stated in the paper [1] for classic Fourier transform.

Theorem 3.2

1) Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and the sequence $\{\omega_n\}_{n=0}^{\infty}$ decrease to zero while for $n \in \mathbb{N}$ the estimate $\left| \int_{m_n}^{+\infty} f(y)\overline{\chi(x, y)}\, dy \right| = O(\omega_n)$ holds uniformly in $x \in \mathbb{R}_+$. Then $f \in L^1(\mathbb{R}_+)$ and $\widehat{f} \in H^\omega$. (Here the integral defining $\widehat{f}$ is assumed as improper with singular point $+\infty$).

2) Let $f(t) \ge 0$, $t \in \mathbb{R}_+$ and $\widehat{f} \in H^\omega$. Then $\left| \int_{m_n}^{+\infty} f(y)\overline{\chi(x, y)}\, dy \right| = O(\omega_n)$ for $n \in \mathbb{N}$ uniformly in $x \in \mathbb{R}_+$.

Corollary 3.3 *Let $f(t) \geq 0$, $t \in \mathbb{R}_+$, $f \in L^1(\mathbb{R}_+)$ and $\omega \in B \cap B_l$ for some $l \in (0, +\infty)$. Then the following three conditions are equivalent: 1) $f \in H^\omega$; 2) $\int_{m_n}^{+\infty} f(t)\,dt = O(\omega_n)$, $n \in \mathbb{Z}_+$; 3) $\int_0^{m_n} t^l f(t)\,dt = O(m_n^l \omega_n)$, $n \in \mathbb{Z}_+$.*

The function f is said to be admissible on $\mathbb{R}_+$, if: 1) $f \in L^1[0,1]$; 2) $f \in V[1,+\infty)$, i.e. f has bounded variation on $[1,+\infty)$; 3) $\lim_{t\to+\infty} f(t) = 0$. For the function f admissible on $\mathbb{R}_+$ the multiplicative Fourier **P**-transform $\widehat{f}(x) = \int_{\mathbb{R}_+} f(y)\overline{\chi(x,y)}\,dy$ exists as improper integral (see [11]). The following theorem generalizes Theorems 4 and 5 from our paper [5].

Theorem 3.4 *Let f is an admissible on $\mathbb{R}_+$ non-negative function satisfying the condition (3) for some $\theta \in (0,1]$.*

1) *If $1 < q \leq p < +\infty$, $1/p - \theta < \gamma < 1/p$ and $f(x)x^{\gamma+1-1/p-1/q} \in L^q(\mathbb{R}_+)$, then $x^{-\gamma}\widehat{f}(x) \in L^p(\mathbb{R}_+)$.*
2) *If $f \in L^r(\mathbb{R}_+)$ for some $r \in [1,2]$, $1 < p \leq q < +\infty$, $1/p - \theta < \gamma < 1/p$ and $x^{-\gamma}\widehat{f}(x) \in L^p(\mathbb{R}_+)$, then $f(x)x^{\gamma+1-1/p-1/q} \in L^q(\mathbb{R}_+)$.*

Let us consider Besov-Lipschitz **P**-space $\Lambda(\alpha,p,q,\beta)$ with power weight $w_a(x) = |x|_{\mathbf{P}}^a$, where $|x|_{\mathbf{P}} = m_k$ for $m_{k-1} \leq x < m_k$, $k \in \mathbb{Z}$. The space $\Lambda(\alpha,p,q,\beta)$, where $1 \leq p < \infty$, $0 < q \leq \infty$, $\alpha, \beta \in \mathbb{R}$, consists of functions $f \in L^p_{w_{\beta p}}(\mathbb{R}_+)$ with finite norm

$$\|f\|_{\Lambda(\alpha,p,q,\beta)} = \|f\|_{p,w_{\beta p}} + \left(\sum_{k\in\mathbb{Z}} \left(\int_{\mathbb{R}_+} \left|m_k^\alpha (D_{m_k} - D_{m_{k-1}}) * f(x)\right|^p w_{\beta p}(x)\,dx\right)^{q/p}\right)^{1/q} < \infty.$$

Here $D_y(x) = \int_0^y \chi(x,t)\,dt$ is multiplicative Dirichlet kernel and

$$\|f\|_{p,w_{\beta p}} = \left(\int_{\mathbb{R}_+} |f(x)|^p w_{\beta p}(x)\,dx\right)^{1/p}.$$

Theorem 3.5 *Let $\alpha \in \mathbb{R}$, $1 \leq p \leq 2$, $0 < q \leq \infty$, $\beta \in [0, 1-1/p]$. Then the following statements are valid: 1) if $f \in \Lambda(\alpha,p,q,\beta)$, then $\widehat{f} \in K(\alpha-\beta,p',q)$; 2) if $f \in L^p_{w_{\beta p}} \cap K(\alpha,p,q)$, then $\widehat{f} \in \Lambda(\alpha,p',q,-\beta)$.*

Theorem 3.6 *Let $1 \leq p \leq 2$, $0 < q < \infty$, $\alpha \in \mathbb{R}$. Then the following statements are valid. 1) If $p' \leq r < \infty$, $\beta \in [0,1/r]$ and $f \in K(\alpha+\beta,p,q) \cap L^{r'}_{w_{\beta r'}}$, then $\widehat{f} \in \Lambda(\alpha-1/p'+1/r, r, q, -\beta)$. 2) If $1 \leq r \leq p'$, $\beta \in [0,1/p']$ and $f \in \Lambda(\alpha,p,q,\beta)$, then $\widehat{f} \in K(\alpha+1/p'-1/r'-\beta, r, q)$.*

The proofs of all results from this section were published in [12].

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Shifting for the Fourier–Feynman Transform on Wiener Space

Byoung Soo Kim

Abstract In this paper we survey results on the shifting for the Fourier–Feynman transform. In particular we introduce some results on the shifting, scaling and modulation proprerties for Fourier–Feynman transform of functionals in a Banach algebra $\mathcal{S}$.

Keywords Analytic Feynman integral • Convolution • Fourier–Feynman transform • Modulation • Shifting property

Mathematics Subject Classification (2010) Primary 28C20; Secondary 60J25, 60J65.

1 Introduction

In a 1945 paper [3], Cameron defined a “transform” of a functional which was somewhat analogous to the “Fourier transform” of a function. Since then, many results based on or inspired by this definition have appeared in the literature.

The concept of an L_1 analytic Fourier–Feynman transform for functionals on Wiener space was introduced by Brue in [2]. In [4], Cameron and Storvick introduced an L_2 analytic Fourier–Feynman transform. In [8], Johnson and Skoug developed an L_p analytic Fourier–Feynman transform for $1 \leq p \leq 2$ that extended the results in [4].

In [6, 7], Huffman, Park and Skoug defined a convolution product for functionals on Wiener space and showed that the Fourier–Feynman transform of a convolution product is a product of Fourier–Feynman transforms. Recently Kim et al. [11] obtained change of scale formulas for Wiener integrals related to Fourier–Feynman transform and convolution.

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Let $C_0[0,T]$ denote the Wiener space, that is, the space of real valued continuous functions x on $[0,T]$ with $x(0)=0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$ and let m denote Wiener measure. Then $(C_0[0,T],\mathcal{M},m)$ is a complete measure space and we denote the Wiener integral of a functional F by $\int_{C_0[0,T]} F(x)\,dm(x)$.

A subset E of $C_0[0,T]$ is said to be scale-invariant measurable [9] provided ρE is measurable for each $\rho>0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $m(\rho N)=0$ for each $\rho>0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.).

Let $\mathbb{C}_+$ denote the set of complex numbers with positive real part. Let F be a complex valued measurable functional on $C_0[0,T]$ such that the Wiener integral

$$J_F(\lambda)=\int_{C_0[0,T]} F(\lambda^{-1/2}x)\,dm(x)$$

exists as a finite number for all $\lambda>0$. If there exists a function $J_F^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J_F^*(\lambda)=J_F(\lambda)$ for all $\lambda>0$, then $J_F^*(\lambda)$ is defined to be the analytic Wiener integral of F over $C_0[0,T]$ with parameter λ, and for $\lambda\in\mathbb{C}_+$ we write

$$\int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F(x)\,dm(x)=J_F^*(\lambda).$$

If the following limit exists for nonzero real q, then we call it the analytic Feynman integral of F over $C_0[0,T]$ with parameter q and we write

$$\int_{C_0[0,T]}^{\mathrm{anf}_q} F(x)\,dm(x)=\lim_{\lambda\to -iq}\int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F(x)\,dm(x) \tag{1}$$

where λ approaches $-iq$ through $\mathbb{C}_+$.

Now we briefly describe the class of functionals that we work with in this paper. The Banach algebra $\mathcal{S}$, which was introduced by Cameron and Storvick [5], consists of functionals expressible in the form

$$F(x)=\int_{L_2[0,T]}\exp\{i\langle v,x\rangle\}\,df(v) \tag{2}$$

for s-a.e. x in $C_0[0,T]$, where f is a complex Borel measure on $L_2[0,T]$ and $\langle v,x\rangle$ denote the Paley–Wiener–Zygmund stochastic integral $\int_0^T v(t)\,dx(t)$.

2 Fourier–Feynman Transform on Wiener Space

In this section we survey some of important properties on the Fourier–Feynman transform of functionals in the Banach algebra $\mathcal{S}$. Let $1 \le p < \infty$ and let q be a nonzero real number throughout this paper.

Definition 2.1 Let F be a functional on $C_0[0,T]$. For $\lambda \in \mathbb{C}_+$ and $y \in C_0[0,T]$, let

$$T_\lambda[F](y) = \int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F(x+y)\,dm(x). \tag{3}$$

For $1 < p < \infty$, we define the L_p analytic Fourier–Feynman transform $T_q^{(p)}[F]$ of F on $C_0[0,T]$ by the formula ($\lambda \in \mathbb{C}_+$)

$$T_q^{(p)}[F](y) = \underset{\lambda\to -iq}{\mathrm{l.i.m.}}\, T_\lambda[F](y), \tag{4}$$

whenever this limit exists; that is, for each $\rho > 0$,

$$\lim_{\lambda\to -iq}\int_{C_0[0,T]} |T_\lambda[F](\rho x) - T_q^{(p)}[F](\rho x)|^{p'}\,dm(x) = 0$$

where $1/p+1/p' = 1$. We define the L_1 analytic Fourier–Feynman transform $T_q^{(1)}[F]$ of F by ($\lambda \in \mathbb{C}_+$)

$$T_q^{(1)}[F](y) = \lim_{\lambda\to -iq} T_\lambda[F](y), \tag{5}$$

for s-a.e. $y \in C_0[0,T]$, whenever this limit exists [1, 4, 6–8].

Huffman, Park and Skoug established the existence of Fourier–Feynman transform on $C_0[0,T]$ for functionals in $\mathcal{S}$.

Theorem 2.2 (Theorem 3.1 of [7]) *Let $F \in \mathcal{S}$ be given by (2). Then the Fourier–Feynman transform $T_q^{(p)}[F]$ exists, belongs to $\mathcal{S}$ and is given by*

$$T_q^{(p)}[F](y) = \int_{L_2[0,T]} \exp\Big\{i\langle v,y\rangle - \frac{i}{2q}\|v\|^2\Big\}\,df(v) \tag{6}$$

for s-a.e. $y \in C_0[0,T]$.

The most important property of the Fourier–Feynman transform is that the Fourier–Feynman transform of the convolution product is equal to the product of the Fourier–Feynman transforms.

Theorem 2.3 (Theorem 3.3 of [7]) *Let F and G be elements of $\mathcal{S}$ with corresponding finite Borel measures f and g. Then,*

$$T_q^{(p)}(F * G)_q(z) = T_q^{(p)}(F)\Big(\frac{z}{\sqrt{2}}\Big)T_q^{(p)}(G)\Big(\frac{z}{\sqrt{2}}\Big) \tag{7}$$

for s-a.e. $y \in C_0[0,T]$.

In Theorem 2.3 above, $(F * G)_q$ denotes the convolution product of F and G. For a definition and properties of convolution product, see [1, 6, 7, 13].

Theorem 2.4 (Theorem 3.4 of [7]) *Let F and G be elements of $\mathcal{S}$ with corresponding finite Borel measures f and g. Then, the Parseval's identity*

$$\int_{C_0[0,T]}^{\text{anf}_{-q}} T_q^{(p)}(F * G)_q(z)\,dm(z) = \int_{C_0[0,T]}^{\text{anf}_q} F(z/\sqrt{2})G(z/\sqrt{2})\,dm(z) \tag{8}$$

holds.

Some of the other classes of functionals for which the relationship (7) holds are

(1) $F(x) = f(\langle\alpha_1, x\rangle, \ldots, \langle\alpha_n, x\rangle)$, where $f \in L_p(\mathbb{R}^n)$ and $\{\alpha, \ldots, \alpha_n\}$ is an orthonormal set of functionals in $L_2[0,T]$, and
(2) $F(x) = \exp\{\int_0^T f(t, x(t))\,dt\}$, where $f \in L_{pr}([0,T] \times \mathbb{R})$.

For a detailed survey of the previous work on Fourier–Feynman transform and related topics, see [14].

3 Shifting Properties for Fourier–Feynman Transform

The Fourier transform $\mathcal{F}$ turns a function f into a new function $\mathcal{F}[f]$. Because the transform is used in signal analysis, we usually use t for time as the variable with f, and ω as the variable of the transform $\mathcal{F}[f]$. Engineers refer to the variable ω in the transformed function as the frequency of the signal f [12].

We will use the same convention in this paper, that is, for a Fourier–Feynman transform $T_q^{(p)}[F](y)$ of $F(x)$, we call the variable x as a time and the variable y as a frequency.

In this section, we give a very brief description of the shifting, scaling, and modulation properties for the Fourier–Feynman transform of functionals in $\mathcal{S}$. For details, see [10].

By the definition of $T_\lambda[F]$, the time shifting and frequency shifting of $T_\lambda[F]$ are expressed as $\int_{C_0[0,T]} F(\lambda^{-1/2}x - x_0 + y)\,dm(x)$. Hence we have the following theorem.

Theorem 3.1 *Let F be a functional on $C_0[0,T]$ and let $x_0 \in C_0[0,T]$. Then we have*

$$T_q^{(p)}[F(\cdot - x_0)](y) = T_q^{(p)}[F](y - x_0) \tag{9}$$

if each sides exist.

The following theorem is reminiscent of the time shifting theorem for the Fourier transform. Hence we call the following theorem the time shifting formula for Fourier–Feynman transform on Wiener space.

Theorem 3.2 (Time Shifting) *Let $F \in \mathcal{S}$ be given by (2) and let $x_0 \in C_0[0,T]$. Then we have*

$$T_q^{(p)}[F(\cdot - x_0)](y) = \exp\left\{-iq\langle x_0, y\rangle + \frac{iq}{2}\|x_0\|^2\right\} T_q^{(p)}[F(\cdot)\exp\{iq\langle x_0, \cdot\rangle\}](y) \tag{10}$$

for s-a.e. $y \in C_0[0,T]$.

Proof Let $G(x) = F(x)\exp\{iq\langle x_0, x\rangle\} = \int_{L_2[0,T]} \exp\{i\langle w, x\rangle\}\, dg(w)$, where $g(E) = f(E - qx_0)$ for a Borel subset E of $L_2[0,T]$. Then by Theorem 2.2 we have

$$\begin{aligned} T_q^{(p)}[G](y) &= \int_{L_2[0,T]} \exp\left\{i\langle w, y\rangle - \frac{i}{2q}\|w\|^2\right\} dg(w) \\ &= \exp\left\{iq\langle x_0, y\rangle - \frac{iq}{2}\|x_0\|^2\right\} T_q^{(p)}[F](y - x_0) \end{aligned}$$

which completes the proof. □

The next theorem is reminiscent of the frequency shifting theorem for the Fourier transform. Using Theorem 3.2 we have the following property for the frequency shifting of the Fourier–Feynman transform.

Theorem 3.3 (Frequency Shifting) *Let $F \in \mathcal{S}$ be given by (2) and let $y_0 \in C_0[0,T]$. Then we have*

$$T_q^{(p)}[F](y - y_0) = \exp\left\{-iq\langle y_0, y\rangle + \frac{iq}{2}\|y_0\|^2\right\} T_q^{(p)}[F(\cdot)\exp\{iq\langle y_0, \cdot\rangle\}](y) \tag{11}$$

for s-a.e. $y \in C_0[0,T]$.

The following theorem is called a scaling theorem because we want the transform not of $F(x)$, but of $F(ax)$. It can be proved by a similar method as in Theorem 3.2.

Theorem 3.4 (Scaling) *Let $F \in \mathcal{S}$ be given by (2) and let a be a nonzero real number. Then we have*

$$T_q^{(p)}[F(a\cdot)](y) = T_{q/a^2}^{(p)}[F](ay) \tag{12}$$

for s-a.e. $y \in C_0[0,T]$.

Putting $a = -1$ in (12), we have the following corollary.

Corollary 3.5 (Time Reversal) *Let $F \in \mathcal{S}$ be given by* (2). *Then we have*

$$T_q^{(p)}[F(-\cdot)](y) = T_q^{(p)}[F](-y) \tag{13}$$

for s-a.e. $y \in C_0[0,T]$.

Our final theorem is useful in obtaining the Fourier–Feynman transforms of new functionals from the Fourier–Feynman transforms of old functionals for which we know their Fourier–Feynman transform.

Theorem 3.6 (Modulation) *Let $F \in \mathcal{S}$ be given by* (2) *and let $x_0 \in C_0[0,T]$. Then we have*

$$T_q^{(p)}[F(\cdot)\cos(q\langle x_0,\cdot\rangle)](y) = \frac{1}{2}(K[F](x_0,y) + K[F](-x_0,y)) \tag{14}$$

and

$$T_q^{(p)}[F(\cdot)\sin(q\langle x_0,\cdot\rangle)](y) = \frac{1}{2i}(K[F](x_0,y) - K[F](-x_0,y)), \tag{15}$$

where

$$K[F](x_0,y) = \exp\Big\{iq\langle x_0,y\rangle - \frac{iq}{2}\|x_0\|^2\Big\}T_q^{(p)}[F(\cdot - x_0)](y) \tag{16}$$

for s-a.e. $y \in C_0[0,T]$.

Proof Putting $\cos(q\langle x_0,\cdot\rangle) = \frac{1}{2}(\exp\{iq\langle x_0,\cdot\rangle\} + \exp\{-iq\langle x_0,\cdot\rangle\})$ and using the linearity of the Fourier–Feynman transform and the time shifting theorem we obtain (14). The second conclusion is proved similarly. □

Since the Dirac measure concentrated at $v = 0$ in $L_2[0,T]$ is a complex Borel measure, the constant function $F \equiv 1$ belongs to $\mathcal{S}$. Hence, as a corollary of Theorem 3.6, we have

$$T_q^{(p)}[\cos(q\langle x_0,\cdot\rangle)](y) = \cos(q\langle x_0,y\rangle)\exp\Big\{-\frac{iq}{2}\|x_0\|^2\Big\} \tag{17}$$

and

$$T_q^{(p)}[\sin(q\langle x_0,\cdot\rangle)](y) = \sin(q\langle x_0,y\rangle)\exp\Big\{-\frac{iq}{2}\|x_0\|^2\Big\} \tag{18}$$

for s-a.e. $y \in C_0[0,T]$.

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General Integral Transforms by the Concept of Generalized Reproducing Kernels

T. Matsuura and S. Saitoh

Abstract In Saitoh (Proc Am Math Soc 89:74–78, 1983), the general integral transforms in the framework of Hilbert spaces were combined with the general theory of reproducing kernels by Aronszajn (Trans Am Math Soc 68:337–404, 1950) and many applications were developped, for example, in Saitoh (Integral transforms, reproducing kernels and their applications, vol 369, Addison Wesley Longman, Harlow, 1997). The basic assumption here that the integral kernels belong to some Hilbert spaces. However, as a very typical integral transform, in the case of Fourier integral transform, the integral kernel does not belong to $L_2(\mathbf{R})$ and, however, we can establish the isometric identity and inversion formula.

On the above situations, we will develop some general integral transform theory containing the Fourier integral transform case that the integral kernel does not belong to any Hilbert space, based on the recent general concept of generalized reproducing kernels in Saitoh and Sawano (Generalized delta functions as generalized reproducing kernels, manuscript; General initial value problems using eigenfunctions and reproducing kernels, manuscript).

Keywords Fourier transform • Initial value problem • Integral transform • Inversion formula • Isometric mapping • Monotone sequence of reproducing kernels • Reproducing kernel

Mathematics Subject Classification (2010) Primary 42A38; Secondary 32A30.

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1 Introduction

In order to fix our background in this paper, following [1, 4–8], we recall a general theory for linear mappings in the framework of Hilbert spaces using the general theory of reproducing kernels.

We assume that $\mathcal{H} = L^2(I, dm)$ and that $H_K(E)$ is a closed subspace of $L^2(E, d\mu)$. For a simplicity statement we assume that I is an interval on the real line. Furthermore, below we assume that $(I, \mathcal{I}, dm)$ and $(E, \mathcal{E}, d\mu)$ are both σ-finite measure spaces and that

$$H_K(E) \hookrightarrow L^2(E, d\mu). \tag{1}$$

Suppose that we are given a measurable function $h : I \times E \to \mathbb{C}$ satisfying $h_y = h(\cdot, y) \in L^2(I, dm)$ for all $y \in E$. Let us set

$$K(x, y) \equiv \langle h_y, h_x \rangle_{L^2(I,dm)}. \tag{2}$$

Then, for the reproducing kernel Hilbert space $H_K(E)$ admitting the kernel $K(x, y)$, we have:

$$H_K(E) \equiv \{f \in \mathcal{F}(E) \, : f(x) = \langle F, h_x \rangle_{L^2(I,dm)} \text{ for } F \in \mathcal{H}\}. \tag{3}$$

Let us now define

$$L : \mathcal{H} \to H_K(E)(\hookrightarrow L^2(E, d\mu)) \tag{4}$$

by

$$LF(x) \equiv \langle F, h_x \rangle_{L^2(I,dm)} = \int_I F(\lambda)\overline{h(\lambda, x)}\, dm(\lambda), \quad x \in E \tag{5}$$

for $F \in \mathcal{H} = L^2(I, dm)$. Observe that $LF \in H_K(E)$.

The next result will give the inversion formula.

Proposition 1.1 *Assume that $\{E_N\}_{N=1}^{\infty}$ is an increasing sequence of measurable subsets in E such that*

$$\bigcup_{N=1}^{\infty} E_N = E \tag{6}$$

and that

$$\int_{I \times E_N} |h(\lambda, x)|^2 \, dm(\lambda)\, d\mu(x) < \infty \tag{7}$$

for all $N \in \mathbb{N}$. Then we have

$$L^*f(\lambda) \left(= \lim_{N\to\infty} (L^*[\chi_{E_N} f])(\lambda) \right) = \lim_{N\to\infty} \int_{E_N} f(x)h(\lambda, x)\, d\mu(x) \tag{8}$$

for all $f \in L^2(I, d\mu)$ in the topology of $\mathcal{H} = L^2(I, dm)$. Here, L^*f is the adjoint operator of L, but it represents the inversion with the minimum norm for $f \in H_K(E)$.

2 Formulation of a Fundamental Problem

Our basic assumption is that $h : I \times E \to \mathbb{C}$ satisfies $h_y = h(\cdot, y) \in L^2(I, dm)$ for all $y \in E$; that is, the integral kernel or linear mapping is in the framework of Hilbert spaces. In this paper, we assume that the integral kernel $h_y = h(\cdot, y)$ does not belong to $L^2(I, dm)$, however, for any exhaustion $\{I_t\}_{t>0}$ such that $I_t \subset I_{t'}$ for $t \le t'$, $\bigcup_{t>0} I_t = I$, $h_y = h(\cdot, y) \in L^2(I_t, dm)$ for all $y \in E$ and $\{h_y; y \in E\}$ is complete in $L^2(I_t, dm)$ for any $t > 0$.

We will consider the integral transform

$$f_t(x) = \langle F, h_x \rangle_{L^2(I_t, dm)} \text{ for } \quad F \in L^2(I, dm) \tag{9}$$

and the corresponding reproducing kernel

$$K_t(x, y) = \langle h_y, h_x \rangle_{L^2(I_t, dm)}. \tag{10}$$

Here, we assume that $\mathcal{H}_t$ is the Hilbert space $L^2(I_t, dm)$ and $h_x \in \mathcal{H}_t$ for any x. We assume that the non-decreasing reproducing kernels $K_t(x, y)$, in the sense: for any $t' > t$, $K_{t'}(y, x) - K_t(y, x)$ is a positive definite quadratic form function, do, in general, not converge, when $\lim_{t\uparrow\infty} K_t(x, y)$. We write, however, the limit by $K_\infty(x, y)$ formally, that is,

$$\begin{aligned} K_\infty(x, y) &:= \lim_{t\uparrow\infty} K_t(x, y) \\ &= \langle h_y, h_x \rangle_{L^2(I, dm)}. \end{aligned} \tag{11}$$

This integral does, in general, not exist and the limit is a special meaning. We are interested, however, in the relationship between the spaces $L^2(I_t, dm)$ and $L^2(I, dm)$ by associating the kernels $K_t(x, y)$ and $K_\infty(x, y)$, respectively.

At first, for the space $\mathcal{H}_t$ and the reproducing kernel Hilbert space $H_{K_t}(E)$, we recall the isometric identity in (9), by assuming that $\{h_x : x \in E\}$ is complete in the space $\mathcal{H}_t$

$$\|f_t\|_{H_{K_t}(E)} = \|F\|_{L^2(I_t,dm)}. \tag{12}$$

Next note that for any $F \in L^2(I, dm)$,

$$\lim_{t\uparrow\infty} \|F\|_{L^2(I_t,dm)} = \|F\|_{L^2(I,dm)}. \tag{13}$$

As the corresponding function to $f_t \in H_{K_t}(E)$, we consider the function, in the viewpoint of (9)

$$f(x) = \langle F, h_x\rangle_{L^2(I,dm)} \text{ for } \quad F \in L^2(I, dm). \tag{14}$$

However, this function is not defined, because the above integral does, in general, not exist. So, we consider the function formally, tentatively. However, we are considering the correspondings

$$f_t \longleftrightarrow f \tag{15}$$

and

$$H_{K_t}(E) \longleftrightarrow H_{K_\infty}(E), \tag{16}$$

however, for the space $H_{K_\infty}(E)$, we have to give its meaning; here, when the kernel $K_\infty(x, y)$ exists by the condition $h_x \in L^2(I, dm), x \in E$, $H_{K_\infty}(E)$ is the reproducing kernel Hilbert space admitting the kernel $K_\infty(x, y)$.

In this paper, we will give the natural and precise theory for the above formal idea.

3 Completion Property

We introduce a preHilbert space by

$$H_{K_\infty} := \bigcup_{t>0} H_{K_t}(E).$$

For any $f \in H_{K_\infty}$, there exists a space $H_{K_t}(E)$ containing the function f for some $t > 0$. Then, for any t' such that $t < t'$,

$$H_{K_t}(E) \subset H_{K_{t'}}(E)$$

and, for the function $f \in H_{K_\infty}$,

$$\|f\|_{H_{K_t}(E)} \geq \|f\|_{H_{K_{t'}}(E)}.$$

(Here, inequality holds, in general, however, in this case, equality, indeed, holds, for the sake of the completeness of the integral kernel.) Therefore, there exists the limit:

$$\|f\|_{H_{K_\infty}} := \lim_{t'\uparrow\infty} \|f\|_{H_{K_{t'}}(E)}.$$

Denote by H_∞ the completion of H_{K_∞}. Then we obtain:

Theorem 3.1 *For the general situation such that $K_t(x, y)$ exists for all $t > 0$ and $K_\infty(x, y)$ does, in general, not exist, and for any function $f \in H_\infty$*

$$\lim_{t\uparrow\infty} \left(f(x'), K_t(x', x)\right)_{H_\infty} = f(x), \tag{17}$$

in the space H_∞.

Theorem 3.1 may be looked as a reproducing kernel in the natural topology and by the sense of Theorem 3.1, and the reproducing property may be written as follows:

$$f(x) = \langle f, K_\infty(\cdot, x)\rangle_{H_\infty},$$

with (17). Here **the limit $\mathbf{K}_\infty(\cdot, \mathbf{x})$ does, in general, not need to exist**.

4 Convergence of $f_t(x) = \langle F, h_x\rangle_{L^2(I_t,dm)}$; $F \in L^2(I, dm)$

As in the case of Fourier integral, we can prove the convergence of (9) in the completion space H_∞.

In this sense, as in the Fourier integral of the cace $L^2(\mathbf{R}, dx)$ we will write, for

$$\lim_{t\uparrow\infty} f_t = f \quad \text{in} \quad H_\infty$$

as follows:

$$\begin{aligned} f(x) &= \lim_{t\uparrow\infty} (F(\cdot), h(\cdot, x))_{L^2(I_t,dm)} \\ &= (F(\cdot), h(\cdot, x))_{L^2(I,dm)}. \end{aligned} \tag{18}$$

5 Inversion of the Integral Transforms

We will consider the inversion of the integral transform (18) from the space H_∞ onto $L^2(I, dm)$. For any $f \in H_\infty$, we take functions $f_t \in H_{K_t}(E)$ such that

$$\lim_{t\uparrow\infty} f_t = f$$

in the space H_∞. For the functions $f_t \in H_{K_t}(E)$, we can construct the inversion in the following way:

$$F_t(\lambda) = \lim_{N\to\infty} \int_{E_N} f_t(x) h(\lambda, x)\, d\mu_t(x) \tag{19}$$

in the topology of $L^2(I, dm)$ satisfying

$$\begin{aligned} f_t(x) &= (F_t(\cdot), h_x(\cdot))_{L^2(I,dm)} \\ &= (F_t, h_x)_{L^2(I_t,dm)}. \end{aligned} \tag{20}$$

Here, of course, the function F_t of $L^2(I, dm)$ is the zero extension of a function F_t of $L^2(I_t, dm)$. Note that the isometric relation that for any $t < t'$

$$\|f_t - f_{t'}\|_{H_0} = \|F_t - F_{t'}\|_{L^2(I,dm)}. \tag{21}$$

Then, we see the desired result: The functions F_t converse to a function F in $L^2(I, dm)$ and

$$f(x) = (F, h_x)_{L^2(I,dm)} \tag{22}$$

in our sense. We can write down the inversion formula as follows:

$$F(\lambda) = \lim_{t\uparrow\infty} \lim_{N\to\infty} \int_{E_N} \big(f(x'), K_t(x', x)\big)_{H_\infty} h(\lambda, x)\, d\mu_t(x), \tag{23}$$

where the both limits $\lim_{N\to\infty}$ and $\lim_{t\uparrow\infty}$ are taken in the sense of the space $L^2(I, dm)$.

Of course, the correspondence $f \in H_\infty$ and $F \in L^2(I, dm)$ is one to one.

6 Conclusion

When we consider the integral transform

$$LF(x) = \int_I F(\lambda)\overline{h(\lambda, x)}\, dm(\lambda), \quad x \in E \tag{24}$$

for $F \in \mathcal{H} = L^2(I, dm)$, indeed, the integral kernel $h(\lambda, x)$ does not need to belong to the space $L^2(I, dm)$ and with the very general assumptions that for any exhaustion $\{I_t\}$ of I,

$$h(\lambda, x) \quad \text{belongs to } L^2(I_t, dm) \text{ for any} \quad x \quad \text{of} \quad E$$

and

$$\{h(\lambda, x); x \in E\} \text{ is complete in } L^2(I_t, dm),$$

we can establish the isometric identity and inversion formula of the integral transform (24) by giving the natural interpretation of the integral transform (24), as in the Fourier transform.

In particular, note that recently, we obtained a very general inversion formula based on the Aveiro Discretization Method in Mathematics [2, 3] by using the ultimate realization of reproducing kernel Hilbert spaces.

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On Some Applications of Kontorovich–Lebedev Transforms

Juri M. Rappoport

To my mother

Abstract The application of the Kontorovich–Lebedev integral transforms and dual integral equations to the solution of some mixed boundary value problems is considered. We reduce the diffusion and elastic problems to the solution of the proper mixed boundary value problem for the Helmholtz equation.

The solution of this problem as derived by Lebedev is determined in the form of the Kontorovich–Lebedev integral transform from the solution of dual integral equation with modified Bessel function of pure imaginary order in the kernel.

It is shown that we can resolve the above-mentioned problem for the Helmholtz equation in the form of single quadrature from the solution of the Fredholm integral equation. The dimension of the problem is lowered on unit by this, which is the essential advantage of this method. The examples permitting the complete analytical solution of the problem are given.

Keywords Dual integral equations • Fredholm integral equation • Helmholtz equation • Kontorovich–Lebedev integral transforms • Modified Bessel functions

Mathematics Subject Classification (2010). Primary 44A15; Secondary 65L10

The application of the Kontorovich–Lebedev integral transforms and dual integral equations to the solution of the mixed boundary value problems is considered. The diffusion, elastic and other physical problems reduced to the solution of the proper mixed boundary value problems for the Helmholtz equation in the wedge domains.

The solution of stationary and nonstationary diffusion and heat mass transfer problems is given under the conditions that the concentration of the substance is known on the part of the boundary and the flow of the substance is known on

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the other part of the boundary. Some properly mixed boundary value problems for the complementary differential equation of plate's deformation are considered. The solution of model combustion problem in the sectorial domain is carried out. The electrostatics problem on the point charge field near the boundary of fine composite plate reducing to the proper mixed boundary value problem for the spatial Laplace equation $\Delta u = 0$ in the wedge domains is analyzed.

The mixed boundary value problems for the Helmholtz equation [1, 2]

$$\Delta u - k^2 u = 0, \tag{1}$$

are arised in some fields of mathematical physics.

The solution of this type of problems in the wedge domains is determined in the next way in the form of the Kontorovich–Lebedev integral transform [1, 2]

$$u(r,\varphi) = \int_0^\infty M(\tau) \frac{\cosh \varphi\tau}{\cosh \alpha\tau} K_{i\tau}(kr) d\tau,$$

where $M(\tau)$ is the solution of dual integral equation.

It is shown that the above-mentioned problems solution for the Helm-holtz equation is present in the form of single quadrature from the solution of Fredholm integral equation type. The dimension of the problem is lowered on unit by this, which is the essential advantage of this method. The examples permitting the complete analitical solution of the problem are given.

The numerical solution of the mixed boundary value problems and received dual integral equations is carried out. It consists of two parts. Firstly, the numerical solution of the second kind Fredholm integral equation with symmetric kernels. Secondly, the followed taking of quadratures from their solution. The estimation of error is given. The control calculations results give the precision for the solution in 6–7 digits after comma. The considered examples demonstrate the efficiency of the dual integral method in the solution of the mixed boundary value problems for the Helmholtz equation in the wedge domains.

Let's use the following notations here and further: r, φ—polar coordinates of the point; α—angle of the sectorial domain; u—desired function; η—normal to the boundary.

The numerical solution of some boundary value problems for the equation of the form (1) in arbitrary sectorial domains is considered in our work under the assumption that the function $u|_\Gamma$ is known on the part of the boundary and the normal derivative $\frac{\partial u}{\partial \eta}|_\Gamma$ is known on the other part of the boundary. The Kontorovich–Lebedev integral transforms [1, 2] and dual integral equations method [3–5] are used for searching of the solution.

Let's consider the symmetric case for the simplicity of the calculations

$$\begin{cases} \Delta u - k^2 u = 0, \\ \dfrac{\partial u}{\partial \eta}|_{\varphi=\pm\alpha}(r) = g(r), \quad 0 < r < a, \\ u|_{\varphi=\pm\alpha}(r) = f(r), \quad r > a, \\ u|_{r\to 0} - \text{restricted}, \\ u|_{r\to\infty} - \text{restricted}. \end{cases} \tag{2}$$

The solution of (2) is determined by the following way in the form of Kontorovich-Lebedev integral transforms [1, 2]

$$u(r,\varphi) = \int_0^\infty M(\tau)\frac{\cosh\varphi\tau}{\cosh\alpha\tau}K_{i\tau}(kr)d\tau, \tag{3}$$

where $M(\tau)$ is the solution of dual integral equation

$$\begin{gathered} \textstyle\int_0^\infty M(\tau)\tau\tanh(\alpha\tau)K_{i\tau}(kr)d\tau = rg(r),\ 0 < r < a, \\ \textstyle\int_0^\infty M(\tau)K_{i\tau}(kr)d\tau = f(r),\ r > a, \end{gathered} \tag{4}$$

where $g(r)$ and $f(r)$—given functions and $K_\nu(z)$—modified Bessel function (Macdonald function) of imaginary order.

The dual integral equations with Macdonald's function of the imaginary order $K_{i\tau}(x)$ in the kernel of the following form were introduced by Lebedev and Skalskaya [1, 2]. It was shown in [1, 2] that the solutions of these equations may be determined in the form of single quadratures from auxiliary functions satisfying the second kind Fredholm integral equations with symmetric kernel containing MacDonald's function of the complex order $K_{1/2+i\tau}(x)$.

The economical methods of the evaluation of kernels of the integral equations based on Gauss quadrature formulas on Laguerre polynomial's knots are proposed. The procedures of the preliminary transformation of integrals and extraction of the singularity in the integrand are used for the increase of accuracy and speed of algorithms. The cases of dual integral equations admitting complete analytical solution are considered. Observed examples demonstrate the efficiency of this approach in the numerical solution of the mixed boundary value problems of elasticity and combustion in the wedge domains [5].

Let's

$$P(\tau) = \frac{2\sinh(\pi\tau)\cosh(\alpha\tau)}{\pi^2\sinh(\alpha\tau)}\int_0^a g(r)K_{i\tau}(kr)dr,$$

$$f^-(r) = f(r) - \int_0^\infty P(\tau)K_{i\tau}(kr)d\tau,$$

$$h(t) = -\frac{\sqrt{k}e^{kt}}{\pi}\frac{d}{dt}\int_t^\infty \frac{e^{-kr}f^-(r)}{\sqrt{r-t}}dr,$$

$$K(s,t) = \frac{4}{\pi}\int_0^\infty \frac{\sinh[(\pi-\alpha)\tau]}{\sinh(\alpha\tau)}\mathrm{Re}K_{1/2+i\tau}(ks)\mathrm{Re}K_{1/2+i\tau}(kt)d\tau.$$

Then we obtain on the basis of [1, 2] the following formulas for $M(\tau)$

$$M(\tau) = N(\tau) + P(\tau), \tag{5}$$

where

$$N(\tau) = \frac{2\sqrt{2}\sinh(\pi\tau)\cosh(\alpha\tau)}{\pi\sqrt{\pi}\sinh(\alpha\tau)}\int_a^\infty \psi(t)\mathrm{Re}K_{\frac{1}{2}+i\tau}(kt)dt,$$

and $\psi(t)$ is the solution of Fredholm integral equation of the second kind

$$\psi(t) = h(t) - \frac{k}{\pi}\int_a^\infty K(s,t)\psi(s)ds, a \le t < \infty. \tag{6}$$

The kernel $K(s,t)$ is the analytic function on every variable s and t in the domain $a \le s < \infty, a \le t < \infty$.

The integrals which give the expressions for the kernels $K(s,t)$ may be analytically calculated for special values of α, in particular for $\alpha = \frac{\pi}{n}, n = 1, 2, \ldots$

We have for $n = 1$:

$$K(s,t)|_{\alpha=\pi} = 0,$$

for $n = 2$:

$$K(s,t)|_{\alpha=\frac{\pi}{2}} = K_0(k(s+t)) + K_1(k(s+t)),$$

for $n = 3$:

$$K(s,t)|_{\alpha=\frac{\pi}{3}} = \sqrt{3}K_0(k\sqrt{s^2+t^2+st}) + \frac{\sqrt{3}(s+t)}{\sqrt{s^2+t^2+st}}K_1(k\sqrt{s^2+t^2+st}),$$

and so on.

We obtain for the case $g(r) = 0$ (impenetrable boundary)

$$P(\tau) = 0, f^-(r) = f(r)$$

and

$$M(\tau) = \frac{2\sqrt{2}\sinh(\pi\tau)\cosh(\alpha\tau)}{\pi\sqrt{\pi}\sinh(\alpha\tau)}\int_a^\infty \psi(t)\mathrm{Re}K_{\frac{1}{2}+i\tau}(kt)dt. \tag{7}$$

The general case is reduced to the case $g(r) = 0$ as it follows from [1, 2]. Let's consider this case for the simplicity further in this paper.

Let's denote

$$h(t) = -\frac{\sqrt{k}e^{kt}}{\pi}\frac{d}{dt}\int_0^\infty \frac{e^{-kr}f(r)}{\sqrt{r-t}}dr,$$
$$K(s,t) = \frac{4}{\pi}\int_0^\infty \frac{\sinh[(\pi-\alpha)\tau]}{\sinh(\alpha\tau)}\mathrm{Re}K_{1/2+i\tau}(ks)\mathrm{Re}K_{1/2+i\tau}(kt)d\tau, \tag{8}$$

where $\mathrm{Re}K_{1/+i\tau}(z)$—real part of MacDonald's function of complex order $1/2 + i\tau$.

Then we obtain the following procedure for the determination of $M(\tau)$ on the basis of [1, 2]

$$M(\tau) = \frac{2\sqrt{2}\sinh(\pi\tau)\cosh(\alpha\tau)}{\pi\sqrt{\pi}\sinh(\alpha\tau)}\int_a^\infty \psi(t)\mathrm{Re}K_{1/2+i\tau}(kt)dt, \tag{9}$$

where $\psi(t)$—solution of the integral Fredholm equation of the second kind

$$\psi(t) = h(t) - \frac{k}{\pi}\int_a^\infty K(s,t)\psi(s)ds, a \le t < \infty. \tag{10}$$

It's useful under the decision of boundary value problems to find the solution u on the boundary of sectorial domain

$$u|_\Gamma(r) = \int_0^\infty M(\tau)K_{i\tau}(kr)d\tau. \tag{11}$$

Substituting expression (5) for $M(\tau)$ in (11) and transposing the order of the integration we obtain

$$u|_\Gamma(r) = \frac{2\sqrt{2}}{\pi\sqrt{\pi}}\int_a^\infty \psi(t)G_r(t)dt, \tag{12}$$

where

$$G_r(t) = \int_0^\infty \frac{\sinh(\pi\tau)\cosh(\alpha\tau)}{\sinh(\alpha\tau)}K_{i\tau}(kr)\mathrm{Re}K_{1/2+i\tau}(kt)d\tau. \tag{13}$$

Let's denote

$$F_{r\varphi}(t) = \int_0^\infty \frac{\sinh(\pi\tau)\cosh(\varphi\tau)}{\sinh(\alpha\tau)} K_{i\tau}(kr)\mathrm{Re}K_{1/2+i\tau}(kt)d\tau. \tag{14}$$

Then

$$u(r,\varphi) = \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \int_a^\infty \psi(t)F_{r\varphi}(t)dt. \tag{15}$$

So the values of the solution inside the wedge domain and on its boundary may be obtained in the form of the single quadratures from the solution of integral Fredholm equation as it may be seen from (12) and (15).

The integrals (13), (14) may be expressed from the known functions for the special values of the angle α, in particular $\alpha = \frac{\pi}{n},\ n = 1, 2, \ldots$

So the numerical solution of the boundary value problem (2) consists from two parts. One is the numerical solution of integral Fredholm equation of the second kind with symmetric kernel. The other is from the consequent taking of the quadratures from its solution.

Let's truncate the integral equation (10) by the following way

$$\psi(t) = h(t) - \frac{k}{\pi}\int_a^b K(s,t)\psi(s)ds, a \le t \le b. \tag{16}$$

The conducted estimations show that we don't obtain any loss of accuracy in the bounds $10^{-7} - 10^{-8}$ under the truncation of the integral equation (10) for $b \ge 10$ in view of fast decrease of the kernels $K(s,t)$ for $s,t \to \infty$.

The method of mechanical quadratures with the use of combined Simpson formula with instant integration step is one of the most convenient methods of numerical solution of Fredholm integral equation of the second kind. It's necessary to compute N^2 values $K_{ij} = K(s_i, t_j), i = 1, \ldots, N, j = 1, \ldots, N$, under the solution of the system of algebraic equations of this form.

It's convenient to use Gauss quadrature formulas by Laguerre polynomials knots and to perform the computations of N integrand for one fixed variable s or t by parallel for the economy of computer time under the integrals computation. Let's note moreover the symmetry $K(s,t) = K(t,s)$ which gives the possibility to decrease the number of computed integrals twice.

The numerical solution is conducted and the problems of the computational methodology are discussed [5–7]. Examples demonstrate the efficiency of the Kontorovich–Lebedev integral transform and dual integral methods in the numerical solution of the mixed boundary value problems of elasticity, combustion, and electrostatics in the wedge domains.

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Generalized Reproducing Kernels and Generalized Delta Functions

S. Saitoh and Y. Sawano

Abstract In this paper, we shall state a simple and general meaning for reproducing kernels. We would like to answer a general and essential question that: what are reproducing kernels? By considering this basic problem, we were able to obtain a general concept of the generalized delta function as a generalized reproducing kernel and, as a general reproducing kernel Hilbert space, we can consider all separable Hilbert spaces consisting of functions.

Keywords Complete orthonormal basis • Completion • Generalized delta function • Generalized reproducing kernel • Initial value problem • Positive definite quadratic form function • Positive matrix • Reproducing kernel

Mathematics Subject Classification (2010) Primary 30C40; Secondary 44A05

1 Introduction

We would like to introduce the concept of general reproducing kernels and at the same time, we would like to answer clearly a general and essential question that: what are reproducing kernels? By considering this basic problem, we obtained a general concept of the generalized delta function as a generalized reproducing kernel and, as a general reproducing kernel Hilbert space, we can consider all separable Hilbert spaces consisting of functions.

For the background motivation and ideas, see [1, 2].

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2 What is a Reproducing Kernel?

We shall consider a family of *any complex-valued functions* $\{U_n(x)\}_{n=0}^{\infty}$ defined on an abstract set E that are linearly independent. Then, we consider the form:

$$K_N(x, y) = \sum_{n=0}^{N} U_n(x)\overline{U_n(y)}. \tag{1}$$

Then, $K_N(x, y)$ is a *reproducing kernel* in the following sense:

We will consider the family of all the functions, for arbitrary complex numbers $\{C_n\}_{n=0}^{N}$

$$F(x) = \sum_{n=0}^{N} C_n U_n(x) \tag{2}$$

and we introduce the norm

$$\|F\|^2 = \sum_{n=0}^{N} |C_n|^2. \tag{3}$$

The function space forms a Hilbert space $H_{K_N}(E)$ determined by the kernel $K_N(x, y)$ with the inner product induced from the norm (3), as usual. Then, we note that, for any $y \in E$

$$K_N(\cdot, y) \in H_{K_N}(E) \tag{4}$$

and for any $F \in H_{K_N}(E)$ and for any $y \in E$

$$F(y) = (F, K_N(\cdot, y))_{H_{K_N}(E)} = \sum_{n=0}^{N} C_n U_n(y). \tag{5}$$

The properties (4) and (5) are called a *reproducing property* of the kernel $K_N(x, y)$ for the Hilbert space $H_{K_N}(E)$, because the functions F in the inner product (5) are appeared on the left-hand side. From this formula we learn that the functions F may be represented by the kernel $K_N(x, y)$ and that all the members of the Hilbert space $H_{K_N}(E)$ are represented by the kernel $K_N(x, y)$.

3 A General Reproducing Kernel

We wish to introduce a pre-Hilbert space by

$$H_{K_\infty} := \bigcup_{N=0}^{\infty} H_{K_N}(E).$$

For any $F \in H_{K_\infty}$, there exists a space $H_{K_M}(E)$ containing the function F for some $M \geq 0$. Then, for any N such that $M < N$,

$$H_{K_M}(E) \subset H_{K_N}(E)$$

and, for the function $F \in H_{K_M}(E)$, by linearly independence of the functions $\{U_n(x)\}_{n=0}^{\infty}$,

$$\|F\|_{H_{K_M}(E)} = \|F\|_{H_{K_N}(E)}. \tag{6}$$

Therefore, there exists the limit:

$$\|F\|_{H_{K_\infty}} := \lim_{N\to\infty} \|F\|_{H_{K_N}(E)}.$$

Denote by H_∞ the completion of H_{K_∞} with respect to this norm.

Note that for any $M < N$, and for any $F_M \in H_{K_M}(E)$, $F_M \in H_{K_N}(E)$ and furthermore, in particular, that

$$\langle f, g\rangle_{H_{K_M(E)}} = \langle f, g\rangle_{H_{K_N(E)}}$$

for all $N > M$ and for any $f, g \in H_{K_M}(E)$.

Theorem 3.1 *Under the above conditions, for any function $F \in H_\infty$ and for the function F_N^* defined by*

$$F_N^*(x) = \langle F, K_N(\cdot, x)\rangle_{H_\infty},$$

$F_N^* \in H_{K_N}(E)$ for all $N > 0$, and as $N \to \infty$, $F_N^* \to F$ in the topology of H_∞.

Proof Just observe that the identity

$$K_N(x, y) = \langle K_N(\cdot, y), K_N(\cdot, x)\rangle_{H_\infty},$$

as we see from (6). Then, we see immediately that

$$F_N^* \in H_{K_N}(E)$$

and

$$\|F_N^*\|_{H_{K_N}(E)} \leq \|F\|_{H_\infty}.$$

The mapping $F \mapsto F_N^*$ being uniformly bounded, and so, we can assume that $F \in H_{K_L}(E)$ for any fixed L. However, in this case, the result is clear, because, since, $F \in H_{K_N}(E)$ for $L < N$ and

$$\lim_{N\to\infty} F_N^*(x) = \lim_{N\to\infty} \langle F, K_N(\cdot, x)\rangle_{H_\infty} = \lim_{N\to\infty} \langle F, K_N(\cdot, x)\rangle_{H_{K_N}(E)} = F(x).$$

□

Theorem 3.1 may be regarded as a reproducing kernel in the natural topology and by the sense of Theorem 3.1, and the reproducing property may be written as follows:

$$F(x) = \langle F, K_\infty(\cdot, x)\rangle_{H_\infty},$$

with

$$K_\infty(\cdot, x) \equiv \lim_{N\to\infty} K_N(\cdot, x) = \sum_{n=0}^{\infty} U_n(\cdot)\overline{U_n(x)}. \tag{7}$$

Here *the limit does, in general, not need to exist*, however, the series are non-decreasing, in the sense: for any $N > M$, $K_N(y, x) - K_M(y, x)$ is a positive definite quadratic form function.

4 Conclusion

Any reproducing kernel (separable case) may be considered as the form (7) by arbitrary linear independent functions $\{U_n(x)\}$ on an abstract set E, here, the sum does not need to converge. Furthermore, the property of linear independent is not essential.

Recall the *double helix structure of gene* for the form (7).

The completion H_∞ may be found, in concrete cases, from the realization of the spaces $H_{K_N}(E)$.

The typical case is that the family $\{U_n(x)\}_{n=0}^{\infty}$ is a complete orthonormal system in a Hilbert space with the norm

$$\|F\|^2 = \int_E |F(x)|^2 dm(x) \tag{8}$$

with a dm measurable set E in the usual form $L_2(E, dm)$. Then, the functions (2) and the norm (3) are realized by this norm and the completion of the space $H_{K_\infty}(E)$ is given by this Hilbert space with the norm (8).

For any separable Hilbert space consisting of functions, there exists a complete orthonormal system, and so, by our generalized sense, for the Hilbert space there exist approximating reproducing kernel Hilbert spaces and so, the Hilbert space is the generalized reproducing kernel Hilbert space in the sense of this paper.

This will mean that we were able to extend the classical reproducing kernels [3–5], beautifully and completely.

The form (7) may be considered as a *generalized delta function* in the very general situation.

The fundamental applications to initial value problems using eigenfunctions and reproducing kernels in the framework of Hilbert spaces, see [1, 2].

5 Remarks

The common fundamental definitions and results on reproducing kernels are given in [3–5] as follows:

Definition 5.1 Let E be an arbitrary abstract (non-void) set. Denote by $\mathcal{F}(E)$ the set of all complex-valued functions on E. A reproducing kernel Hilbert spaces on the set E is a Hilbert space $\mathcal{H} \subset \mathcal{F}(E)$ coming with a function $K : E \times E \to \mathcal{H}$, which is called the reproducing kernel, having *the reproducing property* that

$$K_p \equiv K(\cdot, p) \in \mathcal{H} \text{ for all } p \in E \tag{9}$$

and that

$$f(p) = \langle f, K_p \rangle_{\mathcal{H}} \tag{10}$$

holds for all $p \in E$ and all $f \in \mathcal{H}$.

Definition 5.2 A complex-valued function $k : E \times E \to \mathbb{C}$ is called a *positive definite quadratic form function* on the set E, or shortly, *positive definite function*, when it satisfies the property that, for an arbitrary function $X : E \to \mathbb{C}$ and for any finite subset F of E,

$$\sum_{p,q \in F} \overline{X(p)} X(q) k(p, q) \geq 0. \tag{11}$$

Then, the fundamental result is given by: *a reproducing kernel and a positive definite quadratic form function are the same and are one to one correspondence* with the reproducing kernel Hilbert space.

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Pasting Reproducing Kernel Hilbert Spaces

Yoshihiro Sawano

Abstract The aim of this article is to find the necessary and sufficient condition for the mapping

$$H_K(E) \ni f \mapsto (f|E_1, f|E_2) \in H_{K|E_1 \times E_2}(E_1) \oplus H_{K|E_2 \times E_2}(E_2)$$

to be isomorphic, where K is a positive definite function on $E = E_1 + E_2$. As an application, the Binet-Cauchy equality and its variant are considered.

Keywords Pasting • Reproducing kernel Hilbert spaces • Restriction

Mathematics Subject Classification (2010) Primary 47B32; 46E22

1 Introduction

Let E be a set and $K : E \times E \to \mathbb{C}$ be a positive definite function. For $f \in H_K(E)$, we can easily check that $f|E_0 \in H_{K|E_0 \times E_0}(E_0)$ since

$$f|E_0 \otimes f|E_0 \ll \|f\|_{H_K(E)}{}^2 K|E_0 \times E_0 \otimes K|E_0 \times E_0$$

in the sense that

$$\sum_{j,k=1,2,\ldots,n} \left(\|f\|_{H_K(E)}{}^2 K|E_0 \times E_0(p_j, p_k) - f|E_0 \otimes f|E_0(p_j, p_k)\right) z_j \overline{z_k}$$
$$= \sum_{j,k=1,2,\ldots,n} \left(\|f\|_{H_K(E)}{}^2 K(p_j, p_k) - f(p_j, p_k)\right) z_j \overline{z_k} \geq 0$$

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for any finite set $\{p_1, p_2, \dots, p_k\} \subset E_0$ and $\{z_1, z_2, \dots, z_k\} \subset \mathbb{C}$. Therefore, when E is partitioned into the sum $E = E_1 + E_2$, the operation the mapping $R : H_K(E) \ni f \mapsto (f|E_1, f|E_2) \in H_{K|E_1 \times E_2}(E_1) \oplus H_{K|E_2 \times E_2}(E_2)$ makes sense. Note that R is injection, since $f|E_1 = 0$ and $f|E_2 = 0$ imply $f = 0$.

2 Main Result

We show the necessary and sufficient condition for R to be isomorphic.

Theorem 2.1 *The mapping R is isomorphic if and only if $K|E_1 \times E_2 = 0$.*

Proof Assume first that $K|E_1 \times E_2 = 0$. Let us first show that R is surjection. To this end, given $g_1 \in H_{K|E_1 \times E_1}(E_1)$ and $g_2 \in H_{K|E_2 \times E_2}(E_2)$, we define a function f on E by $f(p) = g_1(p)$ on E_1 and $f(p) = g_2(p)$ on E_2. Let us check that $f \in H_K(E)$. To this end, we set $f_1 = \chi_{E_1} f$ and $f_2 = \chi_{E_2} f$. Then for $l = 1, 2$, we have

$$\sum_{j,k=1,2,\dots,n} \left(\|f_l\|_{H_K(E)}{}^2 K(p_j, p_k) - f_l \otimes f_l(p_j, p_k)\right) z_j \overline{z_k}$$
$$\geq \sum_{j,k=1,2,\dots,n\ ,\, p_j, p_k \in E_l} \left(\|f_l\|_{H_K(E)}{}^2 K(p_j, p_k) - f_l \otimes f_l(p_j, p_k)\right) z_j \overline{z_k}$$

by assumption. Since

$$\|g_l\|_{H_{K|E_l \times E_l}(E_l)} = \inf\{\|h\|_{H_K(E)} \ :\ h|E_1 = g_l\} \leq \|f_l\|_{H_K(E)}$$

from a general result on the reproducing kernel Hilbert spaces [2, 3], we have

$$\sum_{j,k=1,2,\dots,n} \left(\|f_l\|_{H_K(E)}{}^2 K(p_j, p_k) - f_l \otimes f_l(p_j, p_k)\right) z_j \overline{z_k}$$
$$\geq \sum_{\substack{j,k=1,2,\dots,n \\ p_j, p_k \in E_l}} \left(\|g_l\|_{H_{K|E_l \times E_l}(E_l)}{}^2 K(p_j, p_k) - g_l \otimes g_l(p_j, p_k)\right) z_j \overline{z_k} \geq 0.$$

Thus, $f_l \in H_{K_l}(E_l)$.

It remains to show that R is an isomorphism. In fact, $\{K(\cdot, p)\}_{p \in E_l}$ is a dense subspace in $H_{K|E_l \times E_l}(E_l)$, we have only to show that

$$\left(\left\| \sum_{m=1}^{L} (z_1^m K(\cdot, p_1^m) + z_2^m K(\cdot, p_2^m)) \right\|_{H_K(E)} \right)^2$$

$$= \left(\left\| \sum_{m=1}^{L} z_1^m K(\cdot, p_1^m) \right\|_{H_{K|E_1 \times E_1}(E_1)} \right)^2$$

$$+ \left(\left\| \sum_{m=1}^{L} z_2^m K(\cdot, p_2^m) \right\|_{H_{K|E_2 \times E_2}(E_2)} \right)^2$$

for any $p_1^m \in E_1, p_2^m \in E_2, z_1^m, z_2^m \in \mathbb{C}$ with $m = 1, 2, \ldots, L$.

Conversely, if R is an isomorphism, then

$$\left(\|K(\cdot, p_1) + zK(\cdot, p_2))\|_{H_K(E)} \right)^2$$
$$= \left(\|K(\cdot, p_1)\|_{H_{K|E_1 \times E_1}(E_1)} \right)^2 + |z|^2 \left(\|K(\cdot, p_2)\|_{H_{K|E_2 \times E_2}(E_2)} \right)^2$$

for any $p_1 \in E_1$ and $p_2 \in E_2$ and $z \in \mathbb{C}$. Thus $K|E_1 \times E_2 = 0$. □

This result is based on [3, 4].

We can generalize Theorem 2.1 as follows:

Theorem 2.2 *Let $E = E_1 + E_2 + \cdots + E_M$ be a partition. Then*

$$f \in H_K(E) \mapsto \{f|E_j\}_{j=1}^M \in \oplus_{j=1}^M H_{K|E_j \times E_j}(E_j)$$

is an isomorphism if and only if $K|E_p \times E_q = 0$ whenever $1 \le p < q \le M$.

3 Applications

3.1 Sobolev Type Spaces

Let $H_{K_1}(0, \infty)$ be the set of all absolutely continuous functions f on $(0, \infty)$ such that f and its derivative f' satisfy

$$\lim_{x \downarrow 0} f(x) = 0$$

and

$$\int_0^\infty |f'(x)|^2 e^x \, dx < \infty.$$

Then a direct calculation shows that $H_{K_1}(0, \infty)$ is a reproducing kernel Hilbert space with reproducing kernel $K_1(x, y) = 1 - e^{-\min(x,y)}$. Sometimes $H_{K_1}(0, \infty)$ is called the Sobolev-type reproducing kernel Hilbert spaces. Likewise let $H_{K_2}(0, \infty)$ be the set of all absolutely continuous functions g on $(-\infty, 0)$ such that g and its derivative g' satisfy

$$\lim_{x \uparrow 0} g(x) = 0$$

and

$$\int_{-\infty}^{0} |g'(x)|^2 e^{-x}\, dx < \infty.$$

Then $H_{K_2}(0, \infty)$ is also a reproducing kernel Hilbert space with kernel $K_2(x, y) = 1 - e^{\max(x,y)}$. Note that

$$K(x, y) = \begin{cases} 1 - e^{-\min(|x|,|y|)} & xy \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

is a reproducing kernel on $\mathbb{R}$. The space $H_K((-\infty, \infty) \setminus \{0\})$ is given as the set of all absolutely continuous function h except at 0 such that h and its derivative h' satisfy

$$h(0+) = 0$$

and

$$\int_{-\infty}^{\infty} |h'(x)|^2 e^{-|x|}\, dx < \infty$$

according to Theorem 2.1. We refer to [1] for an example of application of reproducing kernel Hilbert space $H_{K_1}(0, \infty)$ to the real inversion formula.

3.2 The Binet-Cauchy Equality and Its Variant

Let $\wedge^n \mathbb{R}^m$ be the n-fold wedge product of $\mathbb{R}^m$, where $m \geq n$. Let

$$L = (\vec{L}_1, \ldots, \vec{L}_n),$$

where each $\vec{L}_j$ is a vector in $\mathbb{R}^m$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ be elementary vectors in $\mathbb{R}^n$. Note that $\wedge^n \mathbb{R}^m$ is an inner product space with the inner product

$$\langle \mathbf{e}_{j_1} \wedge \mathbf{e}_{j_2} \wedge \cdots \wedge \mathbf{e}_{j_n}, \mathbf{e}_{k_1} \wedge \mathbf{e}_{k_2} \wedge \cdots \wedge \mathbf{e}_{k_n} \rangle_{\wedge^n \mathbb{R}^m} = \det(\{\langle \mathbf{e}_{j_p}, \mathbf{e}_{k_q} \rangle\}_{p,q=1}^n).$$

Observe that

$$\det(L^*L) = \det(\langle \vec{L}_p, \vec{L}_q \rangle_{p,q=1}^n)$$
$$= \langle \vec{L}_1 \wedge \vec{L}_2 \wedge \cdots \wedge \vec{L}_n, \vec{L}_1 \wedge \vec{L}_2 \wedge \cdots \wedge \vec{L}_n \rangle_{\wedge^n \mathbb{R}^m}.$$

Next, we let

$$\mathfrak{E} = \{(j_1, j_2, \ldots, j_n) \in \{1, 2, \ldots, m\}^n \,:\, \sharp\{j_1, j_2, \ldots, j_n\} = n\}/\sim,$$

where

$$(j_1, j_2, \ldots, j_n) \sim (k_1, k_2, \ldots, k_n)$$

if and only if

$$\{j_1, j_2, \ldots, j_n\} = \{k_1, k_2, \ldots, k_n\}.$$

When we are given a vector $(j_1, j_2, \ldots, j_n) \in \{1, 2, \ldots, m\}^n$ such that

$$\sharp\{j_1, j_2, \ldots, j_n\} = n$$

we denote by

$$[j_1, j_2, \ldots, j_n]$$

the class to which $(j_1, j_2, \ldots, j_n)$ belongs. Write

$$\vec{L}_j = \begin{pmatrix} L_{j1} \\ L_{j2} \\ \vdots \\ L_{jm} \end{pmatrix} = L_{j1}\mathbf{e}_1 + L_{j2}\mathbf{e}_2 + \cdots + L_{jm}\mathbf{e}_m.$$

Then we have

$$\vec{L}_1 \wedge \vec{L}_2 \wedge \cdots \wedge \vec{L}_m$$
$$= \sum_{j_1, j_2, \ldots, j_n = 1}^{m} L_{1j_1} L_{2j_2} \cdots L_{nj_n} \mathbf{e}_{j_1} \wedge \mathbf{e}_{j_2} \wedge \cdots \wedge \mathbf{e}_{j_n}.$$

Let $\mathfrak{F} \subset \{1, 2, \ldots, m\}^n$ be chosen so that

$$\mathfrak{E} = \{[j_1, j_2, \ldots, j_n] \,:\, (j_1, j_2, \ldots, j_n) \in \mathfrak{F}\}$$

Then we have

$$\begin{aligned}
&\vec{L}_1 \wedge \vec{L}_2 \wedge \cdots \wedge \vec{L}_m \\
&= \sum_{(j_1,j_2,\dots,j_n)\in\mathfrak{F}} \sum_{\sigma\in S_n} L_{1\sigma_1(j_1)} L_{2\sigma_2(j_2)} \cdots L_{n\sigma_n(j_n)} \\
&\qquad \times \mathbf{e}_{\sigma_1(j_1)} \wedge \mathbf{e}_{\sigma_2(j_2)} \wedge \cdots \wedge \mathbf{e}_{\sigma_n(j_n)} \\
&= \sum_{(j_1,j_2,\dots,j_n)\in\mathfrak{F}} \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) L_{1\sigma_1(j_1)} L_{2\sigma_2(j_2)} \cdots L_{n\sigma_n(j_n)} \\
&\qquad \times \mathbf{e}_{j_1} \wedge \mathbf{e}_{j_2} \wedge \cdots \wedge \mathbf{e}_{j_n} \\
&= \sum_{(j_1,j_2,\dots,j_n)\in\mathfrak{F}} \det(\{L_{pj_q}\}_{p,q=1}^n) \mathbf{e}_{j_1} \wedge \mathbf{e}_{j_2} \wedge \cdots \wedge \mathbf{e}_{j_n}.
\end{aligned}$$

Using this expression, we obtain

$$\det(L^* L) = \sum_{(j_1,j_2,\dots,j_n)\in\mathfrak{F}} |\det(\{L_{pj_q}\}_{p,q=1}^n)|^2.$$

This equality is known as the Binet-Cauchy formula.

Let $\mathbf{e}_1^\dagger, \dots, \mathbf{e}_n^\dagger$ be a linearly independent sequence of vectors in $\mathbb{R}^m$. Write

$$\vec{L}_j^\dagger = L_{j1}\mathbf{e}_1^\dagger + L_{j2}\mathbf{e}_2^\dagger + \cdots + L_{jn}\mathbf{e}_n^\dagger$$

and define

$$L^\dagger = (\vec{L}_1^\dagger, \vec{L}_2^\dagger, \dots, \vec{L}_n^\dagger).$$

Then we have

$$\begin{aligned}
\det((L^\dagger)^* L^\dagger) &= \det(\langle \vec{L}_p^\dagger, \vec{L}_q^\dagger \rangle_{p,q=1}^n) \\
&= \langle \vec{L}_1^\dagger \wedge \vec{L}_2^\dagger \wedge \cdots \wedge \vec{L}_n^\dagger, \vec{L}_1^\dagger \wedge \vec{L}_2^\dagger \wedge \cdots \wedge \vec{L}_n^\dagger \rangle_{\wedge^n \mathbb{R}^m}
\end{aligned}$$

and

$$\begin{aligned}
&\vec{L}_1^\dagger \wedge \vec{L}_2^\dagger \wedge \cdots \wedge \vec{L}_m^\dagger \\
&= \sum_{(j_1,j_2,\dots,j_n)\in\mathfrak{F}} \det(\{L_{pj_q}\}_{p,q=1}^n) \mathbf{e}_{j_1}^\dagger \wedge \mathbf{e}_{j_2}^\dagger \wedge \cdots \wedge \mathbf{e}_{j_n}^\dagger
\end{aligned}$$

as before. Let H denote the set of all real-valued functions defined on $\mathfrak{F}$ with the bilinear form:

$$\langle f_1, f_2\rangle = \sum_{(j_1,j_2,\dots,j_n)\in\mathfrak{F}} f_1(j_1,j_2,\dots,j_n)f_2(j_1,j_2,\dots,j_n)\det(\langle \mathbf{e}_{j_p}, \mathbf{e}_{j_q}\rangle_{p,q=1}^n).$$

Then H is a Hilbert space with the reproducing kernel $K : \mathfrak{F}\times\mathfrak{F}\to\mathbb{R}$ satisfying

$$K((j_1,j_2,\dots,j_n),(k_1,k_2,\dots,k_n))\det(\langle \mathbf{e}_{j_1}, \mathbf{e}_{l_q}\rangle_{p,q=1}^n) = \delta_{k_1,l_1}\delta_{k_2,l_2}\cdots\delta_{k_n,l_n}.$$

Choose a partition $\mathfrak{F} = \mathfrak{F}_1+\mathfrak{F}_2+\cdots+\mathfrak{F}_M$ so that $K|\mathfrak{F}_p\times\mathfrak{F}_q = 0$ for $1\le p<q\le M$. Then we have

$$\det((L^\dagger)^*L^\dagger) = \sum_{k=1}^M \left\| \sum_{(j_1,j_2,\dots,j_n)\in\mathfrak{F}} \det(\{L_{pj_q}\}_{p,q=1}^n)\mathbf{e}_{j_1}^\dagger\wedge\mathbf{e}_{j_2}^\dagger\wedge\cdots\wedge\mathbf{e}_{j_n}^\dagger \right\|_{\wedge^n\mathbb{R}^n}.$$

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Fundamental Solutions of Hyperbolic System of Second Order and Their Graphical Representations

G.K. Zakir'yanova

Abstract The system of hyperbolic equations of second order is considered. By using the Fourier transform of generalized functions the fundamental and generalized solutions are constructed. Are given the conditions on the wavefront, that simulate the shock waves in continuous media. The results of the numerical calculations which show the existence of lacunas for hyperbolic equations with constant coefficients are presented.

Keywords Anisotropy • Continuous media • Fundamental solution • Hyperbolic system

Mathematics Subject Classification (2010) Primary 35E05; Secondary 74E10

An investigation of the waves propagation in continuous media under the influence of various external and internal sources of natural or synthetic origin refers to the actual problems of mechanics and mathematical physics, and is associated with the solving of boundary value problems for systems of equations of the hyperbolic and mixed types. Solutions of these equations can have characteristic surface on which themselves solutions, or their derivatives are discontinuous [1]. In the study of wave processes the case of wave propagation from point source takes a special place. Fundamental solutions obtained in this are important, because they can be used to receive the solution for various mass forces. In addition, they needed to build kernels of singular boundary integral equations for solving boundary value problems. Here the fundamental and generalized solutions of the system of hyperbolic equations of second order are constructed and results of the numerical calculations for some constant coefficients of hyperbolic equations corresponding to elastic anisotropic medium are presented.

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1 Equations of Motion, Conditions on the Fronts

Consider the system of the hyperbolic equations with second order derivatives with constant coefficients:

$$L_{ij}(\partial_x, \partial_t)u_j(x,t) + G_i(x,t) = 0,\ (x,t) \in R^{N+1} \tag{1}$$

$$L_{ij}\left(\partial_x, \partial_t\right) = C_{ij}^{ml}\partial_m\partial_l - \delta_{ij}\partial_t^2, \quad i,j,m,l = \overline{1,N}, \tag{2}$$

$$C_{ij}^{ml} = C_{ij}^{lm} = C_{ji}^{ml} = C_{ml}^{ij}, \tag{3}$$

where $\partial_x = (\partial_1, \ldots, \partial_N) = (\partial/\partial x_1, \ldots, \partial/\partial x_N)$, $\partial_t = \partial/\partial t$, $\delta_{ij} = \delta_j^i$ is Kronecker symbol, C_{ij}^{ml} is the matrix of constants satisfying the properties of symmetry under the index permutation (3) and the strict hyperbolicity condition:

$$W(n, v) = C_{ij}^{ml} n_m n_l v^i v^j > 0 \quad \forall n \neq 0, \quad v \neq 0 \tag{4}$$

(everywhere the summation over repeated indices in the indicated range is assumed). The system of Eq. (1) describes, for example, the movement of anisotropic elastic medium. In physical problems $N = 2$ or 3 and G corresponds to the mass force which acts in considered medium.

Further u is twice differentiated vector function almost everywhere by exception characteristic surfaces—wavefronts F_t, on which the following conditions on gaps are executed [2]:

$$[u_i(x,t)]_{F_t} = 0, \tag{5}$$

$$[u_{i,t}\, m_l + c u_{i,l}]_{F_t} = 0, \tag{6}$$

$$\left[\sigma_i^l m_l + c u_{i,t}\right]_{F_t} = 0 \tag{7}$$

where $[f\,(x,t)]_{F_t} = \lim_{\varepsilon \to +0} (f\,(x + \varepsilon n, t) - f\,(x - \varepsilon n, t))$, $x \in F_t$, $n(x,t)$ is unit normal to F_t, $\sigma_i^l = C_{ij}^{ml} u_{j,m}$, c is the velocity of a wavefront motion determined by the solution of a characteristic equation of the system (1):

$$\det\{C_{ij}^{ml} n_m n_l - c^2 \delta_{ij}\} = 0$$

By virtue of (4) this equation has $2N$ real roots: $c = \pm c_k(n)$, $0 < c_k \leq c_{k+1}$, $k = \overline{1,N}$ and in general case depends on direction of motion of wavefront F_t. Note that condition (6) is a consequence of continuity condition (5) and means that the tangent derivatives of u on the front wave are continuous. In physical problems, condition (7) corresponds to the conservation of the momentum on wavefronts. On wavefronts, the derivatives of functions and even the functions themselves can have jump discontinuities. It is assumed that the number of wavefronts is finite and each front is almost everywhere a Lyapunov surface of dimension $N-1$. In [3] conditions (1.5)–(1.7) were received by using the generalized functions theory.

2 Fundamental Solutions

Consider system (1) in the distribution space $D'_M(R^{N+1})$. A fundamental solution (Green tensor) of system (1) is a solution $U_j^k(x,t)$ corresponding to $G_i = \delta_i^k \delta(x,t) = \delta_i^k \delta(x)\delta(t)$ and satisfying conditions

$$U_j^k(x,t) = 0 \text{ for } t \le 0 \text{ and } \|x\| \ge c_{\max} t$$

Here $\delta(x,t)$ is the Dirac δ-function, $\left(\delta_i^k \delta(x,t), \varphi_i(x,t)\right) = \varphi_k(0,0)\ \forall \varphi \in D'_N(R^{N+1})$. Using Fourier transform leads our system to the system of linear equations of the form

$$L_{jk}(i\xi, i\omega) U_l^k(\xi,\omega) + \delta_j^l = 0,\ j,k,l = \overline{1,N} \tag{8}$$

where $(\xi,\omega) = (\xi_1,\ldots,\xi_N,\omega)$ are the Fourier transformation parameters corresponding to (x,t), $L_{jk}(\xi,\omega)$ are the homogeneous second-order polynomials corresponding to the differential operators (2). Solving the system (8), we obtain the transform of the Green's matrix, which, in view of the homogeneity of differential polynomials has the form

$$U_j^k(i\xi, i\omega) = -\frac{Q_{jk}(\xi,\omega)}{Q(\xi,\omega)}$$

Here $Q_{jk}(\cdot)$ are the cofactor of the element with the index (k,j) of $\{L(i\xi, i\omega)\}$, $Q(\cdot)$ is the symbol of L (2): $Q(i\xi, i\omega) = (-1)^N \det\{L_{jk}(\xi,\omega)\}$.

Polynomials Q_{jk}, Q satisfy the conditions of symmetry by ξ, ω and to the homogeneity conditions. In view of the strict hyperbolicity (1) the characteristic equation $Q(\xi,\omega) = 0$ has $2N$ roots that can be represented as

$$\omega_q = \|\xi\| c_q(e),\ \omega_{2q} = -\omega_q,\ q = \overline{1,N},\ e = \xi/\|\xi\|$$

Using Lemma Jordan residue, we find the inverse Fourier transform of U_j^k by time. In [3] it is shown that the construction of the Green tensor reduces to the calculation of integrals over the unit sphere. For odd N the above theorem allows to build only approach of the Green tensor. For even N to determine approach must be multidimensional integration of the surface integral over the unit sphere. However, in some cases, this procedure could be simplified.

2.1 Green Tensor for $N = 2$

Let N equal to 2 in the system (1). In physical problems it takes place for anisotropic medium in the case of plane strain: $u_1 = u_1(x_1, x_2)$, $u_2 = u_2(x_1, x_2)$, $u_3 = 0$. The symmetry relations (3) allow the tensor C_{ij}^{ml} to introduce as a square matrix $C_{\alpha,\beta}(\alpha, \beta = \overline{1, 6})$ so the correspondence between the pairs of indexes (i, j), (m, l) and indexes α, β, established by the scheme $(11) \leftrightarrow 1$, $(22) \leftrightarrow 2$, $(33) \leftrightarrow 3$, $(23) = (32) \leftrightarrow 4$, $(31) = (13) \leftrightarrow 5$, $(12) = (21) \leftrightarrow 6$. In this case it is convenient to use polar coordinates. Using Lemma Jordan residue and taking the limit in the formula for the inverse Fourier transform in the sense of convergence of generalized functions, we obtain the Green tensor, which is the sum of the residues of rational functions [4, 5]:

$$U_j^k(x, t) = \frac{1}{\pi t} \mathrm{Im} \sum_{q=1}^{2} \frac{Q_{jk}(\zeta_q, 1, (x_1\zeta_q + x_2)/t)}{Q_{,\zeta}(\zeta_q, 1, (x_1\zeta_q + x_2)/t)} \tag{9}$$

for $\mathrm{Im}\xi_q > 0$, where ζ_q are the roots of the equation

$$Q(\zeta, 1, (x_1\zeta_q + x_2) = 0, \ \ Q = Q_{11}Q_{22} - Q_{12}^2 \tag{10}$$

For anisotropic medium the wave propagation velocity depends on the direction of wave propagation, and the shape of the wavefronts depends significantly on the coefficients of (1). In the expression for the Green's tensor (9) are summed residues of rational functions in the upper half, which requires knowledge of the values of the roots of the polynomial (10). The roots of this equation of the fourth degree are complex conjugate, so we always have two roots satisfying $\mathrm{Im}\zeta \geq 0$.

3 Graphical Representations of Fundamental Solutions

The existence of lacunas for hyperbolic equations with constant coefficients, which include the equations of motion of the anisotropic elastic medium, was detected by Petrovsky [6]. He introduced the necessary and sufficient conditions for the existence of lacunas, components of the complement to the surface of the wavefront in which the fundamental solution vanishes (strong lacunas). Below the results of the numerical calculations of fundamental solutions of (1) for some constant coefficients corresponding to elastic anisotropic (orthotropic) medium are presented. Note that anisotropic medium with characteristics closest to the real environment, in particular rocks. The stress-strain state of the medium depends strongly on the

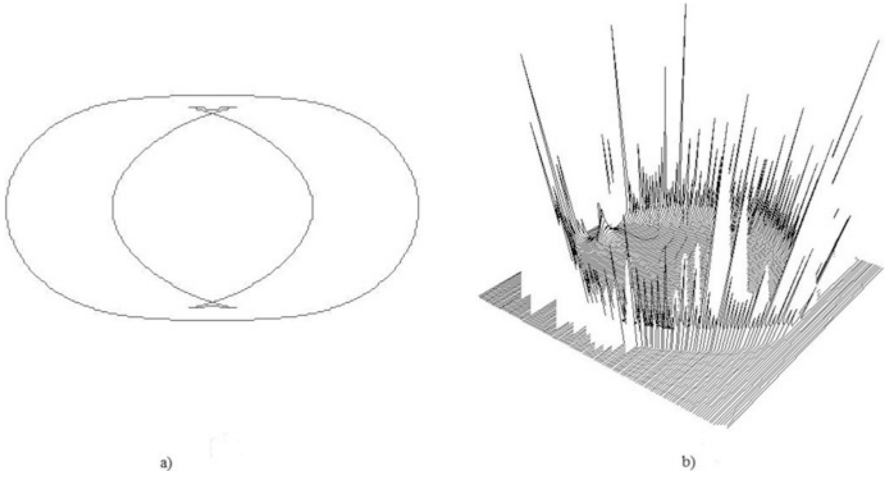

Fig. 1 Picture wavefronts (**a**) and the amplitude of movements (**b**) for Zn under the action of a concentrated force

degree of anisotropy. For example, in medium with strong anisotropy of the elastic properties there lacunas—moving unperturbed regions bounded by the wavefronts and expanding over time, and the front of the wave is very different from the classic, has a complex non-smooth shape. Here the calculations for $C_{11} = 4,219$, $C_{12} = 0,59$, $C_{22} = 1,645$, $C_{66} = 1,0$ (Zn), $C_{11} = 28,2$, $C_{12} = 13,1$, $C_{22} = 34,9$, $C_{66} = 12,6$ (topaz), and $C_{11} = 5,82$, $C_{12} = 2,29$, $C_{22} = 3,59$, $C_{66} = 0,57$ (potassium pentaborate) are presented.

The location of lacuna depends on the matrix of constants. Denote $A_1 = (C_{11}^{11} - C_{12}^{12})(C_{22}^{22} - C_{12}^{12}) - (C_{11}^{22} + C_{12}^{12})^2$, $A_2 = (C_{11}^{11} - C_{12}^{12})C_{22}^{22} - (C_{11}^{22} + C_{12}^{12})^2$, $A_3 = (C_{22}^{22} - C_{12}^{12})C_{11} + C_{12}^{12})^2$. Computing show that the case $A_1 < 0$, $A_2 > 0$, $A_3 < 0$ (Zn) corresponds to the existence of lacunas on the coordinate axis x_2 (Fig. 1). For topaz $A_i < 0$, $i = 1, 2, 3$ lacunas lie on both coordinate axes (Fig. 2). For potassium pentaborate $A_i > 0$, $i = 1, 2, 3$ lacunas lie between coordinate axes (Fig. 3).

4 Generalized Solutions

A study of the propagation of waves from the earthquake source is connected with the study of stress-strain state of the medium under the influence of distributed mass forces $G_k(x, t)$. For regular mass forces we have $u_j(x, t) = \int_0^\infty d\tau \int_{R^N} U_j^k(x - y, t - \tau)G_k(y, \tau)dV(y)$. For remote source of earthquake, the distance to which substantially exceeds its size, are used the models concentrated sources as singular

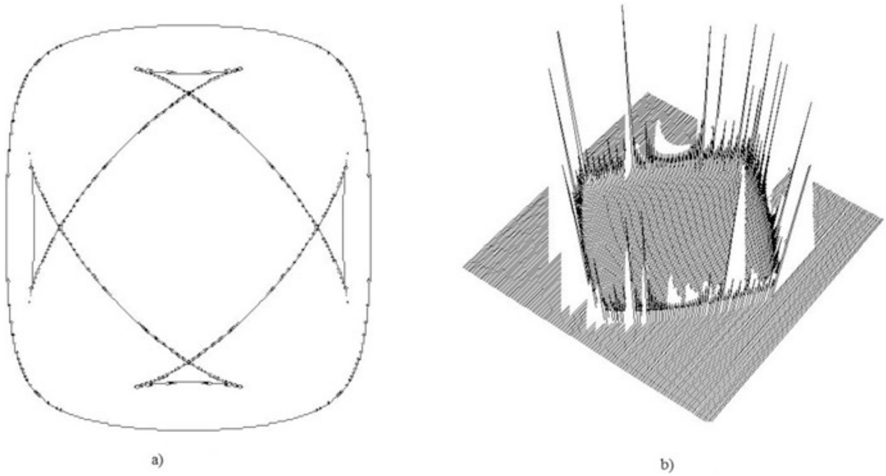

Fig. 2 Picture wavefronts (**a**) and the amplitude of movements (**b**) for topaz under the action of a concentrated force

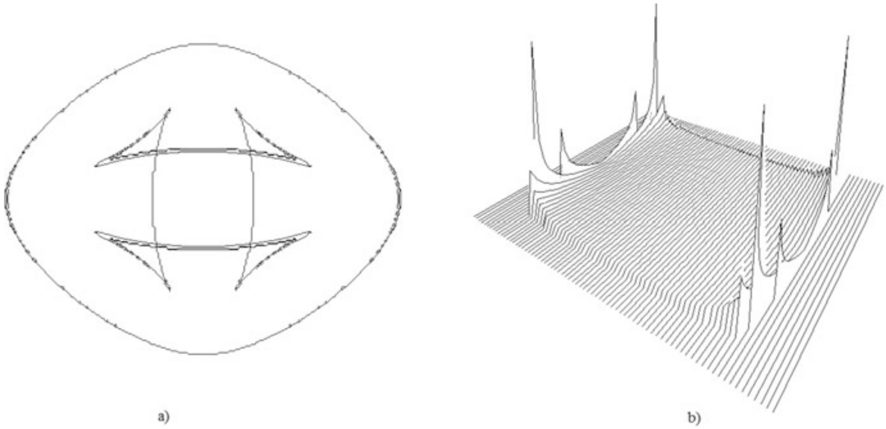

Fig. 3 Picture wavefronts (**a**) and the amplitude of movements (**b**) for potassium pentaborate under the action of a concentrated force

distributions with point support (dipole, multipole, etc.) [7]. The displacement field is then given by the convolution U_j^k with the appropriate G_k :

$$u_j(x,t) = U_j^k(x,t) * G_k(x,t)$$

which should be taken according to the rules determining the convolution of distributions. Figure 4 shows the components of the Green tensor for topaz under action of the dipole.

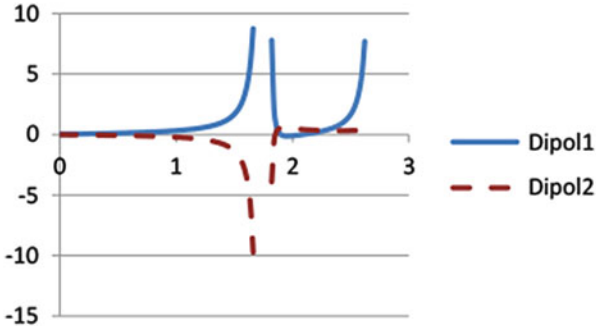

Fig. 4 Components of the Green tensor for topaz under action of the dipole

5 Conclusion

Here presents the fundamental and generalized solutions and their graphical representations showing the presence of lacunas for solutions for system with constant coefficients (1). The results of the numerical calculations of displacements for elastic orthotropic medium under the influence of different types of sources can be seen in [8].

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Learning with Reproducing Kernel Banach Spaces

Haizhang Zhang and Jun Zhang

Abstract The major obstacle in building Banach space methods for machine learning is the lack of an inner product. We give justifications of substituting inner products with semi-inner-products in Banach spaces as a remedy. By using semi-inner-products, we are able to establish the notion of reproducing kernel Banach spaces (RKBS), and develop regularized learning schemes in the spaces.

Keywords Banach spaces • Machine learning • Reproducing kernels • Sparse approximation

Mathematics Subject Classification (2010) Primary 46E15; Secondary 46E22

1 Introduction

In the task of learning a function dependency from finite sample data, patterns are usually preprocessed in order to obtain their features. In machine learning, extracting features by mapping the patterns into a Hilbert space is dominant. There are many advantages of this approach, thanks to the existence of an inner product in a Hilbert space. In particular, the similarity between patterns can be measured by the inner product of their features in the Hilbert space. This leads to reproducing kernels and gives birth to the popular and successful kernel methods for machine learning [3, 12, 13].

There are some occasions where it might be more appropriate to use a Banach space, a generalization of Hilbert spaces. Firstly, Hilbert spaces constitute a very limited class of Banach spaces. Any two Hilbert spaces over a common number

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field with the same dimension are isometrically isomorphic. By reaching out to other Banach spaces, one obtains more variety in geometric structures and norms that are potentially useful for learning and approximation. Secondly, many training data come with intrinsic structures that make them impossible to be embedded into a Hilbert space. Learning algorithms based on features in a Hilbert space may not work well for them. Thirdly, in some applications, a norm from a Banach space is invoked without being induced from an inner product for some particular purpose. A typical example is the linear programming regularization in coefficient-based regularization for machine learning [12]:

$$\min_{\boldsymbol{c}\in\mathbb{R}^n} Q(K^{\mathbf{x}}(x_1)\boldsymbol{c}, K^{\mathbf{x}}(x_2)\boldsymbol{c}, \cdots, K^{\mathbf{x}}(x_n)\boldsymbol{c}) + \lambda\|\boldsymbol{c}\|_1, \tag{1}$$

where $\mathbf{x} := (x_j : j \in \mathbb{N}_n)$ with $\mathbb{N}_n := \{1, 2, \ldots, n\}$ is a given sequence of sampling points, K is a positive-definite reproducing kernel, and $K^{\mathbf{x}}(x_j)$ denotes the row vector $(K(x_i, x_j) : i \in \mathbb{N}_n)$, Q is a loss function, λ is a positive regularization parameter, and $\|\boldsymbol{c}\|_1 := \sum_{j=1}^{n} |c_j|$ is employed to obtain sparsity in the resulting minimizer.

There has been research in understanding learning of functions in Banach spaces. Minimizing a loss function subject to a regularization condition on a norm in Banach space was studied by Bennett and Bredensteiner [1], Micchelli and Pontil [10], Micchelli and Pontil [11], and Zhang [15]. Gentile [4] and Kimber and Long [7] considered on-line learning in finite-dimensional Banach spaces, and learning of an L^p function, respectively. Classifications in Banach spaces, and more generally in metric spaces were discussed in [1, 2, 6, 14].

The major obstacle in learning with Banach spaces is caused by the absence of an inner product. As a consequence, kernel methods were not developed in those studies. In particular, it is unknown whether the linear programming regularization (1) results from a minimization problem in an infinite-dimensional Banach space. As a consequence, in the learning rate estimates, the hypothesis error will not go away automatically as it does in the reproducing kernel Hilbert space (RKHS) case. In Banach spaces, semi-inner-products [5, 9] in mathematics seem to be a natural substitute for the inner product. The purpose of this note is to introduce this useful tool for developing Banach space methods for machine learning. To this end, we shall give the definition of semi-inner-products and justify their capabilities of substituting the important roles of inner products in Sect. 2. The notion of reproducing kernel Banach spaces (RKBS) was recently established in [16, 17]. We shall review the main results in Sect. 3.

2 Semi-Inner-Products

Semi-inner-products were introduced for the purpose of extending Hilbert space type arguments to Banach spaces [5, 9]. A semi-inner-product on a Banach space $\mathcal{B}$ is a function, usually denoted by $[\cdot,\cdot]_{\mathcal{B}}$, from $\mathcal{B}\times\mathcal{B}$ to $\mathbb{R}$ such that for all $f, g, h \in \mathcal{B}$ and $\alpha, \beta \in \mathbb{R}$

1. (linearity with respect to the first variable) $[\alpha f+\beta g,h]_{\mathcal{B}}=\alpha[f,h]_{\mathcal{B}}+\beta[g,h]_{\mathcal{B}}$;
2. (norm compatibility) $[f,f]=\|f\|_{\mathcal{B}}^2$, where $\|f\|_{\mathcal{B}}$ denotes the norm of f in $\mathcal{B}$;
3. (homogeneity with respect to the second variable) $[f,\alpha g]_{\mathcal{B}}=\alpha[f,g]_{\mathcal{B}}$;
4. (Cauchy-Schwartz inequality) $|[f,g]_{\mathcal{B}}|\leq[f,f]_{\mathcal{B}}^{1/2}[g,g]_{\mathcal{B}}^{1/2}$.

A Banach space always has a semi-inner-product [5, 9]. We see that the only property of inner products that a semi-inner-product is not required to possess is symmetry, that is, $[f,g]\neq[g,f]$ in general. Semi-inner-products were invoked in the machine learning context by Der and Lee [2] for the study of large margin hyperplane classification in Banach spaces. Below we give justifications for the significant roles that semi-inner-products could play in learning and approximation in Banach spaces.

The classical Riesz representation theorem states that every continuous linear functional on a Hilbert space is representable as an inner product. This theoretical result is of fundamental importance to kernel methods for machine learning, in which finite sample data are usually modeled as the point evaluations of the desired function at some inputs. In the general regularization framework for machine learning, the desired function is approximated by functions from an RKHS through a regularized minimization problem. In an RKHS, point evaluations are continuous linear functionals. It follows by the Riesz representation theorem that the point evaluation can be represented by the inner product with a kernel function. This is the starting point for the development of kernel methods for machine learning. For Banach spaces with appropriate conditions, we have an analogue to the classical Riesz representation theorem. Some definitions and notations are needed to present this fundamental fact.

A Banach space $\mathcal{B}$ is *uniformly convex* if for all $\varepsilon>0$ there exists a $\delta>0$ such that

$$\|f+g\|_{\mathcal{B}}\leq 2-\delta \text{ for all } f,g\in\mathcal{B} \text{ with } \|f\|_{\mathcal{B}}=\|g\|_{\mathcal{B}}=1 \text{ and } \|f-g\|_{\mathcal{B}}\geq\varepsilon.$$

We also say that $\mathcal{B}$ is *uniformly Fréchet differentiable* if for all $f,g\in\mathcal{B}\setminus\{0\}$

$$\lim_{t\in\mathbb{R},\,t\to 0}\frac{\|f+tg\|_{\mathcal{B}}-\|f\|_{\mathcal{B}}}{t} \tag{2}$$

exists and the limit is approached uniformly for all f,g in the unit ball of $\mathcal{B}$. For simplicity, we call a Banach space *uniform* if it is both uniformly convex and uniformly Fréchet differentiable.

Let $\mathcal{B}$ be a Banach space with the semi-inner-product $[\cdot,\cdot]_{\mathcal{B}}$. For each $f\in\mathcal{B}$, the mapping sending $g\in\mathcal{B}$ to $[g,f]_{\mathcal{B}}$ is a continuous linear functional on $\mathcal{B}$. We denote this linear functional by f^* and call it the *dual element* of f. The mapping $f\to f^*$ is called the *duality mapping* from $\mathcal{B}$ to $\mathcal{B}^*$ and is denoted by $\mathcal{J}_{\mathcal{B}}$.

Lemma 2.1 ([5]) *Let $\mathcal{B}$ be a uniform Banach space. Then it has a unique semi-inner product $[\cdot,\cdot]_{\mathcal{B}}$ and the duality mapping $\mathcal{J}_{\mathcal{B}}$ is bijective and norm-preserving from $\mathcal{B}$ to $\mathcal{B}^*$. In other words, for each μ in the dual space $\mathcal{B}^*$ there exists a unique*

$f \in \mathcal{B}$ such that

$$\mu(g) = [g,f]_{\mathcal{B}} \text{ for all } g \in \mathcal{B}.$$

and

$$\|f^*\|_{\mathcal{B}^*} = \|f\|_{\mathcal{B}} \text{ for all } f \in \mathcal{B}. \tag{3}$$

Moreover,

$$[f^*, g^*]_{\mathcal{B}^*} := [g,f], \;\; f, g \in \mathcal{B} \tag{4}$$

defines a semi-inner-product on $\mathcal{B}^*$.

3 RKBS

In this section, we give a brief introduction to RKBS established in [16, 17]. This class of Banach spaces of functions are applicable to learning a single task.

In machine learning, we are concerned with Banach space of functions with bounded point evaluation functionals. Let X be an input space. We call $\mathcal{B}$ a Banach space of functions on X if it is a Banach space consisting of certain functions on X such that for every $f \in \mathcal{B}$, $\|f\|_{\mathcal{B}} = 0$ if and only if f vanishes everywhere on X. We stress this definition to make sure that point evaluations are well defined. For instance, the space of continuous functions on a compact metric space with the usual maximum norm is a Banach space of functions, while $L^p([0, 1])$ is not. We call $\mathcal{B}$ a *pre-RKBS* on X if it is a Banach space of functions on X and for each $x \in X$, the point evaluation functional

$$\delta_x(f) := f(x), \;\; f \in \mathcal{B} \tag{5}$$

is continuous on $\mathcal{B}$. When $\mathcal{B}$ is also a Hilbert space, it is well known that it possesses a reproducing kernel. In this case, $\mathcal{B}$ is actually an RKHS. The term “pre” is used because there might not exist a reproducing kernel for Banach spaces $\mathcal{B}$. However, when $\mathcal{B}$ is uniform, it does have a reproducing kernel induced by the semi-inner-product. In the following, we always denote by $[\cdot, \cdot]_V$ the unique semi-inner-product on a uniform Banach space V.

Theorem 3.1 ([16, 17]) *Let $\mathcal{B}$ be a uniform pre-RKBS on X. Then there exists a unique function $K : X \times X \to \mathbb{R}$ such that $K(x, \cdot) \in \mathcal{B}$ for all $x \in X$ and*

$$f(x) = [f, K(x, \cdot)]_{\mathcal{B}} \text{ for all } f \in \mathcal{B} \text{ and } x \in X.$$

In view of the above theorem, we call a uniform pre-RKBS an RKBS and regard the unique function K as the reproducing kernel of $\mathcal{B}$. When $\mathcal{B}$ is also a Hilbert space, it coincides with the reproducing kernel in the usual sense. Similar to the RKHS case, we also have a feature map characterization of reproducing kernels for RKBS.

Theorem 3.2 ([17]) *A function $K : X \times X \to \mathbb{R}$ is the reproducing kernel of some RKBS on X if and only if there exists a mapping Φ from X to a uniform Banach space $\mathcal{W}$ such that*

$$K(x, y) = [\Phi(x), \Phi(y)]_{\mathcal{W}} \text{ for all } x, y \in X. \tag{6}$$

The function Φ and the space $\mathcal{W}$ are called a pair of *feature map* and *feature space* for K. Feature map representations lead to useful constructions of explicit examples of RKBS. For a mapping Φ from X to a uniform Banach space $\mathcal{W}$, we denote by Φ^* the associated mapping from X to $\mathcal{W}^*$ defined by $\Phi^*(x) := (\Phi(x))^*$, $x \in X$.

Theorem 3.3 ([17]) *Let $\mathcal{W}$ be a uniform Banach space and Φ a mapping from X to $\mathcal{W}$ such that*

$$\overline{\operatorname{span}}\, \Phi(X) = \mathcal{W}, \quad \overline{\operatorname{span}}\, \Phi^*(X) = \mathcal{W}^*.$$

Then $\mathcal{B} := \{[u, \Phi(\cdot)]_{\mathcal{W}} : u \in \mathcal{W}\}$ equipped with the semi-inner-product

$$\Big[[u, \Phi(\cdot)]_{\mathcal{W}}, [v, \Phi(\cdot)]_{\mathcal{W}}\Big]_{\mathcal{B}} := [u, v]_{\mathcal{W}}$$

and norm

$$\Big\| [u, \Phi(\cdot)]_{\mathcal{W}} \Big\|_{\mathcal{B}} := \|u\|_{\mathcal{W}}$$

is an RKBS on X. Moreover, the reproducing kernel K of $\mathcal{B}$ is given by (6).

With the above theoretical preparations, regularized learning schemes were investigated in [16, 17]. We shall present the major result on the representer theorem of the minimizer. Consider a general regularized learning scheme in an RKBS $\mathcal{B}$ on X:

$$\inf_{f \in \mathcal{B}} Q(f(\mathbf{x})) + \lambda \phi(\|f\|_{\mathcal{B}}), \tag{7}$$

where $\mathbf{x} := (x_j : j \in \mathbb{N}_n)$ is a sequence of sampling points in X, Q and ϕ are nonnegative loss function and regularization function, and λ is a positive regularization parameter. The loss function Q and the regularization function ϕ should satisfy some minimal requirements for (7) to be useful. For this consideration, the learning scheme (7) is said to be *acceptable* if both Q and ϕ are continuous and

$$\lim_{t\to\infty} \phi(t) = +\infty. \tag{8}$$

The above condition is imposed on ϕ to ensure that it can really put a constraint on the complexity of functions in $\mathcal{B}$ used for learning.

Theorem 3.4 ([16]) *Let $\mathcal{B}$ be an RKBS on X. Then every acceptable regularized learning scheme (7) has at least one minimizer f_0 of the form*

$$f_0^* = \sum_{j=1}^{n} c_j (K(x_j, \cdot))^* \tag{9}$$

for some constants $c_j \in \mathbb{R}$. If additionally, ϕ is strictly increasing, then every minimizer of (7) must have the form (9). Furthermore, if Q is convex and ϕ is strictly increasing and strictly convex, then an acceptable (7) has a unique minimizer, which satisfies (9).

When $\mathcal{B}$ is an RKHS then the dual element of a function in $\mathcal{B}$ is itself. Therefore, in this case, Theorem 3.4 recovers the classical representer theorem [8].

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Part VI
Recent Advances in Sequence Spaces

Binod Chandra Tripathy

Mappings of Orlicz Type

Awad A. Bakery

Abstract Let E be the sequence space defined and studied by Tripathy and Mahanta (Soochow J Math 29:379–391, 2003) which is invariant under the doubling operator $D : x = (x_0, x_1, x_2, \dots) \mapsto y = (x_0, x_0, x_1, x_1, x_2, x_2, \dots)$. Using the approximation numbers $(\alpha_n(T))_{n=0}^{\infty}$ of operators from a Banach space X into a Banach space Y, we give the sufficient conditions on E such that the finite rank operators are dense in the complete space of operators $U_E^{\text{app}}(X, Y)$, where $U_E^{\text{app}}(X, Y) := \Big\{T \in L(X, Y) : ((\alpha_n(T))_{n=0}^{\infty} \in E\Big\}$. When $M(t) = t^p$, $1 \le p < \infty$ with $\sup_s \phi_s < \infty$ our results coincide with that known for the space ℓ_p.

Keywords Approximation numbers • Operator ideal • Orlicz sequence space

Mathematics Subject Classification (2010) Primary 46B70; Secondary 47L20

1 Introduction and Basic Definitions

By $L(X, Y)$ we denote the space of all bounded linear operators from a normed space X into a normed space Y and by w, we denote the space of all real sequences. In [1], Pietsch by using the approximation numbers and p-absolutely summable sequences of real numbers $\ell^p (0 < p < \infty)$ form the operator ideals. In [2], Faried and Bakery have considered the space ℓ_M, when $M(t) = t^p$, $(0 < p < \infty)$, which match in special with ℓ^p. A map which assigns to every operator $T \in L(X, Y)$ a unique sequence $(\alpha_n(T))_{n=0}^{\infty}$ is called an α-function of T and the number $\alpha_n(T)$ is called the nth approximation of T which is defined by: $\alpha_n(T) = \inf \Big\{ \|T - A\| : A \in$

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$L(X,Y)$ and $rank(A) \le n\Big\}$. An operator ideal U is a subclass of $L = \{L(X,Y)$: X, Y are Banach Spaces $\}$ such that its components satisfy the following conditions:

(i). $F(X,Y) \subseteq U(X,Y)$, where $F(X,Y)$ is the space of all operators of finite rank from the Banach space X into the Banach space Y.
(ii). If $T_1, T_2 \in U(X,Y)$, then $\lambda_1 T_1 + \lambda_2 T_2 \in U(X,Y)$ for any scalars λ_1, λ_2.
(iii). If $T \in L(X_0, X)$, $S \in U(X,Y)$ and $R \in L(Y, Y_0)$, then $RST \in U(X_0, Y_0)$. See [3].

An Orlicz function [4] is a function $M : [0,\infty) \to [0,\infty)$ which is convex, positive, non-decreasing, continuous with $M(0) = 0$ and $\lim_{x\to\infty} M(x) = \infty$. An Orlicz function M is said to satisfy Δ_2-condition for all values of $x \ge 0$, if there exists a constant $k > 0$, such that $M(2x) \le kM(x)$.

Remark 1.1 An Orlicz function M satisfies the following inequality $M(\lambda x) \le \lambda M(x)$, for all λ with $0 < \lambda < 1$.

Let P_s be the class of all subsets of $\mathbb{N}$ those do not contain more than s number of elements and $\{\phi_n\}$ be a non-decreasing sequence of positive real numbers such that $n\phi_{n+1} \le (n+1)\phi_n$, for all $n \in \mathbb{N}$. Tripathy and Mahanta [5] defined and studied the following sequence space

$$m(\phi, M) = \Big\{(x_k) \in \omega : \sup_{s\ge 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M(\frac{|x_k|}{\zeta}) < \infty, \text{ for some } \zeta > 0\Big\}$$

with $\rho(x) = \inf\Big\{\zeta > 0 : \sup_{s\ge 1,\sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M(\frac{|x_k|}{\zeta} \le 1\Big\}$.

Lemma 1.2

(i). $\ell_M \subseteq m(\phi, M)$*, where* ℓ_M *is Orlicz sequence space defined by Lindentrauss and Tzafriri* [6].
(ii). $\ell_M = m(\phi, M)$ *if and only if* $\sup_s \phi_s < \infty$.

Definition 1.3 ([2]) A class of linear sequence spaces E, called a special space of sequences(sss) has three properties:

(1). $e_n \in E$, for all $n \in \mathbb{N}$. We denote $e_n = \{0, 0, \dots, 1, 0, 0, \dots\}$ where 1 appears at nth place for all $n \in \mathbb{N}$.
(2). If $x = (x_n) \in w$, $y = (y_n) \in E$ and $|x_n| \le |y_n|$, for all $n \in \mathbb{N}$, then $x \in E$ "i.e. E is solid".
(3). If $(x_n)_{n=0}^{\infty} \in E$, then $(x_{[\frac{n}{2}]})_{n=0}^{\infty} \in E$, where $[\frac{n}{2}]$ denotes the integral part of $\frac{n}{2}$.

And we call such space E_ρ a pre-modular special space of sequences if there exists a function $\rho : E \to [0,\infty[$, satisfies the following conditions:

(i). $\rho(x) \ge 0$, for each $x \in E$ and $\rho(\theta) = 0$, where θ is the zero of E,
(ii). there exists a constant $l \ge 1$ such that $\rho(\lambda x) \le l|\lambda|\rho(x)$ for all $x \in E$, and for any scalar λ,

(iii). for some numbers $k \geq 1$ we have the inequality $\rho(x+y) \leq k(\rho(x) + \rho(y))$, for all $x, y \in E$,

(iv). if $|x_n| \leq |y_n|$, for all $n \in \mathbb{N}$, then $\rho((x_n)) \leq \rho((y_n))$,

(v). for some numbers $k_0 \geq 1$ we have the inequality,
$\rho((x_n)) \leq \rho((x_{[\frac{n}{2}]})) \leq k_0\rho((x_n))$,

(vi). the set of all finite sequences is ρ-dense in E. This means for each $x = (x(i))_{i=o}^{\infty} \in E$ there exists $s \in \mathbb{N}$ such that $\rho((x(i))_{i=s}^{\infty}) < \infty$,

(vii). for any positive real number λ there exists a constant $\xi > 0$ such that $\rho(\lambda, 0, 0, 0, \ldots) \geq \xi\lambda\rho(1, 0, 0, 0, \ldots)$.

Definition 1.4 ([2]) $U_E^{\text{app}} := \Big\{U_E^{\text{app}}(X, Y);\ X \text{ and } Y \text{ are Banach Spaces}\Big\}$, where $U_E^{\text{app}}(X, Y) := \Big\{T \in L(X, Y) : ((\alpha_n(T))_{n=0}^{\infty} \in E\Big\}$.

Theorem 1.5 ([2]) *U_E^{app} is an operator ideal, if E is a (sss).*

Theorem 1.6 ([7]) *Let E_ρ be a pre-modular (sss). Then the linear space $F(X, Y)^g$ is dense in $U_{E_\rho}^{\text{app}}(X, Y)$, where $g(T) = \rho(\alpha_n(T)_{n=0}^{\infty})$.*

Theorem 1.7 ([7]) *Let X and Y be Banach spaces and E_ρ be a pre-modular (sss), then $U_{E_\rho}^{\text{app}}(X, Y)$ is complete.*

2 Main Results

We give here the sufficient conditions on $m(\phi, M)$ such that the class of all bounded linear operators between any arbitrary Banach spaces with nth approximation numbers of the bounded linear operators in $m(\phi, M)$ form an operator ideal, the ideal of the finite rang operators in the class of Banach spaces is dense in $U_{m(\phi,M)}^{\text{app}}(X, Y)$.

Theorem 2.1 *Let M be an Orlicz function satisfying Δ_2-condition. Then*

(a). *$U_{m(\phi,M)}^{\text{app}}$ is an operator ideal,*

(b). *the linear space $F(X, Y)$ is dense in $U_{m(\phi,M)}^{\text{app}}(X, Y)$.*

Proof We first prove that the space $m(\phi, M)$ is a special space of sequences(sss).

(1) Let $\lambda_1, \lambda_2 \in \mathbb{R}$, and $x, y \in m(\phi, M)$, then there exists $\zeta_1 > 0$, $\zeta_2 > 0$ be such that

$$\sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\Big(\frac{|x_k|}{\zeta_1}\Big) < \infty \text{ and } \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M|\Big(\frac{|x_k|}{\zeta_2}\Big) < \infty.$$

Let $\zeta_3 = \max\Big(2|\lambda_1|\zeta_1, 2|\lambda_2|\zeta_2\Big)$. Since M is non-decreasing convex function with Δ_2-condition, we have

$$\sum_{k\in\sigma} M\Big(\frac{|\lambda_1 x_k + \lambda_2 y_k|}{\zeta_3}\Big) \leq \frac{1}{2}\Big[\sum_{k\in\sigma} M\Big(\frac{|x_k|}{\zeta_1}\Big) + \sum_{k\in\sigma} M\Big(\frac{|y_k|}{\zeta_2}\Big)\Big].$$

So, we get

$$\sup_{s\geq 1,\sigma\in P_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\Big(\frac{|\lambda_1 x_k + \lambda_2 y_k|}{\zeta_3}\Big) \leq \frac{1}{2}\Big[\sup_{s\geq 1,\sigma\in P_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\Big(\frac{|x_k|}{\zeta_1}\Big) + \sup_{s\geq 1,\sigma\in P_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\Big(\frac{|y_k|}{\zeta_2}\Big)\Big].$$

Thus, $\lambda_1 x + \lambda_2 y \in m(\phi, M)$. Hence $m(\phi, M)$ is a linear space over the field of numbers. Also since $e_n \in \ell_M$ and $\ell_M \subseteq m(\phi, M)$, we have $e_n \in m(\phi, M)$ for all $n \in \mathbb{N}$.

(2) Let $x \in \omega$, $y = (y_k)_{k=0}^{\infty} \in m(\phi, M)$ with $|x_k| \leq |y_k|$ for each $k \in \mathbb{N}$, since M is non-decreasing, then we get $\sup_{s\geq 1,\sigma\in P_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\Big(\frac{|x_k|}{\zeta}\Big) \leq \sup_{s\geq 1,\sigma\in P_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\Big(\frac{|y_k|}{\zeta}\Big) < \infty$,
then $x = (x_k)_{k=0}^{\infty} \in m(\phi, M)$.

(3) Let $x = (x_k)_{k=0}^{\infty} \in m(\phi, M)$, we have

$$\sup_{s\geq 1,\sigma\in P_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\Big(\frac{|x_{[\frac{k}{2}]}|}{\zeta}\Big) \leq 2\sup_{s\geq 1,\sigma\in P_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\Big(\frac{|x_k|}{\zeta}\Big) < \infty,$$

then $x = (x_{[\frac{k}{2}]})_{k=0}^{\infty} \in m(\phi, M)$.

Finally we have proven that the space $m(\phi, M)$ with $\rho(x)$ is a pre-modular special space of sequences.

(i) Clearly $\rho(x) \geq 0$ for all $x \in m(\phi, M)$ and $\rho(\theta) = 0$,

(ii) Let $\lambda \in \mathbb{R}$, $x \in m(\phi, M)$ without loss of generality, take $\lambda \neq 0$ then

$$\begin{aligned}\rho(\lambda x) &= \inf\Big\{\zeta > 0 : \sup_{s\geq 1,\sigma\in P_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\Big(\frac{|\lambda x_k|}{\zeta}\Big) \leq 1\Big\} \\ &= \inf\Big\{|\lambda|\mu > 0 : \sup_{s\geq 1,\sigma\in P_s} \frac{1}{\phi_s}\sum_{k\in\sigma} M\Big(\frac{|x_k|}{\mu}\Big) \leq 1\Big\},\end{aligned}$$

where $\mu = \frac{\zeta}{|\lambda|}$. Thus

$$\rho(\lambda x) = |\lambda| \inf\Big\{\mu > 0 : \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\Big(\frac{|x_k|}{\mu}\Big) \leq 1\Big\} = |\lambda|\rho(x).$$

(iii) Let $x, y \in m(\phi, M)$, then there exists $\zeta_1 > 0$ and $\zeta_2 > 0$ be such that

$$\sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\Big(\frac{|x_k|}{\zeta_1}\Big) \leq 1 \text{ and } \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\Big(\frac{|x_k|}{\zeta_2}\Big) \leq 1.$$

Let $\zeta = \zeta_1 + \zeta_2$, since M is non-decreasing and convex, then we have

$$\begin{aligned}
&\sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\Big(\frac{|x_k + y_k|}{\zeta}\Big) \leq \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\Big(\frac{|x_k| + |y_k|}{\zeta_1 + \zeta_2}\Big) \\
&\quad \leq \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} \Big[\Big(\frac{\zeta_1}{\zeta_1 + \zeta_2}\Big) M\Big(\frac{|x_k|}{\zeta_1}\Big) + \Big(\frac{\zeta_2}{\zeta_1 + \zeta_2}\Big) M\Big(\frac{|y_k|}{\zeta_2}\Big)\Big] \\
&\quad \leq \Big(\frac{\zeta_1}{\zeta_1 + \zeta_2}\Big) \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\Big(\frac{|x_k|}{\zeta_1}\Big) + \Big(\frac{\zeta_2}{\zeta_1 + \zeta_2}\Big) \sum_{k\in\sigma} M\Big(\frac{|y_k|}{\zeta_2}\Big) \leq 1.
\end{aligned}$$

Since the ζ's are nonnegative, so we have

$$\begin{aligned}
\rho(x + y) &= \inf\Big\{\zeta > 0 : \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\Big(\frac{|x_k + y_k|}{\zeta}\Big) \leq 1\Big\} \\
&\leq \inf\Big\{\zeta_1 > 0 : \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\Big(\frac{|x_k|}{\zeta_1}\Big) \leq 1\Big\} \\
&\quad + \inf\Big\{\zeta_2 > 0 : \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\Big(\frac{|y_k|}{\zeta_2}\Big) \leq 1\Big\} = \rho(x) + \rho(y).
\end{aligned}$$

(iv) Let $|x_k| \leq |y_k|$ for each $k \in \mathbb{N}$, since M is non-decreasing, then we get

$$\sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\Big(\frac{|x_k|}{\zeta}\Big) \leq \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\Big(\frac{|y_k|}{\zeta}\Big),$$

thus

$$\inf\Big\{\zeta > 0 : \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\Big(\frac{|x_k|}{\zeta}\Big)\Big\} \leq \inf\Big\{\zeta > 0 : \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\Big(\frac{|y_k|}{\zeta}\Big)\Big\}.$$

So, $\rho(x) \leq \rho(y)$.

(v) Since

$$\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\Big(\frac{|x_{[\frac{k}{2}]}|}{\zeta}\Big)\leq 2\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\Big(\frac{|x_k|}{\zeta}\Big),$$

we have

$$\inf\Big\{\zeta>0:\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\Big(\frac{|x_{[\frac{k}{2}]}|}{\zeta}\Big)\Big\}\leq 2\inf\Big\{\zeta>0:\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\Big(\frac{|x_k|}{\zeta}\Big)\Big\}.$$

So, $\rho((x_k))\leq\rho((x_{[\frac{k}{2}]}))\leq 2\rho((x_k))$.

(vi) For each $x=(x_k)_{k=0}^{\infty}\in m(\phi,M)$ then

$$\rho(x_k)_{k=0}^{\infty}=\inf\Big\{\zeta>0:\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\Big(\frac{|x_{[\frac{k}{2}]}|}{\zeta}\Big)<\infty\Big\},$$

we can find $t\in\mathbb{N}$ such that $\rho(x_k)_{k=t}^{\infty}<\infty$.

(vii) For any $\lambda>0$ there exists a constant $\zeta\in]0,1]$ such that

$$\rho(\lambda,0,0,0,\ldots)\geq\zeta\lambda\rho(1,0,0,0,\ldots).$$

By using Theorems (1.5) and (1.6) we get

(a). $U^{\text{app}}_{m(\phi,M)}$ is an operator ideal,
(b). the linear space $F(X,Y)$ is dense in $U^{\text{app}}_{m(\phi,M)}(X,Y)$.

□

As a special cases of the above theorem we can also obtain the following corollaries:

Corollary 2.2 *If* $\sup_s\phi_s<\infty$, *we get*

(a). $U^{\text{app}}_{\ell_M}$ *is an operator ideal,*
(b). *the linear space* $F(X,Y)$ *is dense in* $U^{\text{app}}_{\ell_M}(X,Y)$.

Corollary 2.3 *If* $\sup_s\phi_s<\infty$ *and* $M(t)=t^p$ *with* $1\leq p<\infty$, *we get*

(a). $U^{\text{app}}_{\ell^p}$ *is an operator ideal,*
(b). *the linear space* $F(X,Y)$ *is dense in* $U^{\text{app}}_{\ell^p}(X,Y)$. *See* [1]

By applying Theorems (1.7) and (2.1) on $m(\phi,M)$, we can easily conclude the next corollaries:

Corollary 2.4 *If* X *and* Y *are Banach spaces and* M *be an Orlicz function such that* M *satisfies* Δ_2*-condition. Then* M *is continuous from right at* 0 *and* $U^{\text{app}}_{m(\phi,M)}(X,Y)$ *is complete.*

Corollary 2.5 *If X and Y are Banach spaces and M be an Orlicz function such that M satisfies Δ_2-condition with $\sup_s \phi_s < \infty$. Then M is continuous from right at* 0 *and $U_{l_M}^{\text{app}}(X, Y)$ is complete.*

Corollary 2.6 *If X and Y are Banach spaces and $M(t) = t^p$ with $1 \leq p < \infty$. Then $U_{\ell^p}^{\text{app}}(X, Y)$ is complete.*

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On Rate of Convergence of Sequences

Binod Chandra Tripathy

Abstract

Definition 0.1 The sequence (x_n) converges to σ ***faster than*** the sequence (y_n) converges to λ, defined by $(x_n) < (y_n)$ if $\lim_{n\to\infty} \frac{x_n-\sigma}{y_n-\lambda} = 0$, provided $y_n - \lambda \neq 0$ for all $n \in N$.

Definition 0.2 The regular matrix $A = (a_{nk})$ ***accelerates*** the convergence of the sequence $x = (x_k)$ if $Ax < x$. The ***acceleration field*** of the matrix A is the class of sequences $\{x = (x_k) \in \omega : Ax < x\}$.

At the initial stage, works on acceleration convergence sequences were done by Smith and Ford (SIAM J Numer Anal 16(2):223–240, 1979), Keagy and Ford (Pac J Math 132(2):357–362, 1988), Salzer (J Math Phys 33:356–359, 1955), Dawson (Pac J Math 24(1):51–56, 1968), Brezinski et al. (SIAM J Numer Anal 20:1099–1105, 1983), Brezinski (Rend Math 7(6):303–316, 1974), and many others.

The notion of acceleration convergence depends on the convergence of sequences. Over the years, different types of convergent sequences such as statistically convergent, I-convergent, etc. have been introduced. Accordingly, Tripathy and Sen (Ital J Pure Appl Math 17:151–158, 2005) have introduced the notion of statistical acceleration convergence. Tripathy and Mahanta (J Frankl Inst 347:591–598, 2010) have investigated about the ideal acceleration convergence of sequences. Patterson and Savas (Hacettepe J Math Stat 41(4):487–497, 2012) have studied about the acceleration convergence with respect to four dimensional matrix maps.

The notion of fuzzy sets attracted research worker on sequence spaces to study about the sequences of fuzzy real numbers. Tripathy and Dutta (Math Modell Anal 17(4):549–557, 2012) have studied about acceleration convergence of sequences of fuzzy numbers. In this talk we shall discuss about the different developments on rate of convergence of sequences.

Keywords Acceleration convergence • Filter • I-convergence • Ideal • Statistical convergence

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1 Introduction and Basic Definitions

Faster convergence of sequences particularly the acceleration of convergence of sequence of partial sums of series via linear and nonlinear transformations are widely used in finding solutions of mathematical as well as different scientific and engineering problems (one may refer to Brezinski [1] and Brezinski et.al. [2]). The problem of acceleration convergence often occurs in numerical analysis. To accelerate the convergence, the standard interpolation and extrapolation methods of numerical mathematics are quite helpful. It is useful to study about the acceleration of convergence methods (we shall focus on matrix transformations), which transform a slowly converging sequence into a new sequence, converging to the same limit faster than the original sequence. The speed of convergence of sequences is of the central importance in the theory of subsequence transformation.

There are many sequences (or series) those converge very slowly. Again the rate of convergence of two convergent sequences (series) may not be equal. In order to speed the convergence of slowly convergent sequences (series) many well-known mathematicians like Salzer [8], Smith and Ford [10] established different methods. Dawson [3] has studied matrix summability over certain classes of sequences ordered with respect to rate of convergence.

Definition 1.1 The sequence (x_n) ***converges*** to σ ***at the same rate as*** the sequence (y_n) converges to λ, written as $(x_n) \approx (y_n)$ if

$$0 < \lim - \inf \left| \frac{x_n - \sigma}{y_n - \lambda} \right| \leq \lim - \sup \left| \frac{x_n - \sigma}{y_n - \lambda} \right| < \infty.$$

Example 1.1 Consider the sequences (x_k) and (y_k) defined by $x_k = 1 + k^{-1}$ and $y_k = 23 + 123k^{-1}$, for all $k \in N$.

It can be easily verified that (y_k) converges to 23 faster than (x_k) converges to 1.

Definition 1.2 Let $A = (a_{nk})$ be an infinite matrix. For a sequence $x = (x_k)$, then the A transform of X is defined by $Ax = (A_n x)$, where $A_n x = \sum_{k=1}^{\infty} a_{nk} x_k$, for all $n \in N$.

Definition 1.3 The subsequence $x = (x_{n_i})$ of (x_k) can be represented as a regular matrix transformation $A = (a_{nk})$ times (x_k) by defining $a_{i,n_i} = 1$, for all $i \in N$ and $a_{pq} = 0$, otherwise.

Definition 1.4 The convergence field of the matrix $A = (a_{nk})$ is defined by $\{x = (x_k) : Ax \in c\}$, where c denotes the class of all convergent sequences.

Definition 1.5 The matrix $A = (a_{nk})$ accelerates the convergence of x if $Ax < x$. The acceleration field of A is defined by $\{(x_k) \in \omega : Ax < x\}$.

Let S_0 denote the set of all sequences in c_0 with non-zero terms. If $a \in S_0$, let $[a] = \{x \in S_0 : x \approx a\}$. Also let $E_0 = \{[x] : x \in S_0\}$. If $[a], [b] \in E_0$, then we say $[a]$

is less than $[b]$, $[a] <^{/} [b]$, provided $a < b$. Then E_0 is partially ordered with respect to $<^{/}$.

Open intervals in S_0 will be denoted by $(a, b), (a, -), (-, b)$, where $(a, -) = \{x \in S_0 : a < x\}$ and $(-, b) = \{x \in S_0 : x < b\}$. On combining these two we have $(a, b) = \{x = (x_k) \in S_0 : a < x < b\}$.

The necessary and sufficient conditions for a matrix $A = (a_{pq})$ to be convergence preserving over (abbreviated c.p.o.) S_0 are (deduced from the Silverman and Toeplitz conditions)

(i) $(a_{pq})_{p=1}^{\infty}$ converges for each $q = 1, 2, 3, \ldots$ and

(ii) there exists K such that $\sum\limits_{q=1}^{\infty} a_{pq} < K$, for each $p = 1, 2, 3, \ldots$.

Dawson [3] has characterized the summability field of a matrix A by showing A is convergence preserving over the set of all sequences which converges faster than some fixed sequence. The following results are due to him.

Theorem 1.6 *If A is c.p.o.* $[b]$, *then there exists* $b^{/} \in S_0$ *such that* $b < b^{/}$ *and A is c.p.o.* $[b^{/}]$.

Theorem 1.7 *If A is c.p.o. each of the sets* $[b^{(1)}], [b^{(2)}], [b^{(3)}], \ldots$, *then there exists* $d \in S_0$ *such that* $b^{(p)} < d$, $p = 1, 2, 3, \ldots$ *and A is c.p.o.* $[d]$.

It follows that A is convergence preserving over a set of the type $(-, x)$.

Keagy and Ford [5] proved that if a subsequence transformation A accelerates $x \in S_0$, then it accelerates each $y \in S_0$ which converge at the same rate as x. They also proved the following two results.

Theorem 1.8 *If A is a subsequence transformation and* $x \in S_0$, *then there exists* $y, z \in S_0$, *such that* $y < x < z$ *and A does not accelerate y or z.*

Theorem 1.9 *If A is a subsequence transformation and* $x \in S_0$, *then there exists* $y \in S_0$ *such that* $y < x$ *and A accelerates y.*

It is also proved by Keagy and Ford [5] that an analog to the above theorem does not exist for $x < z$, that is the acceleration field of a subsequence transformation cannot be any of the forms $(x, -)$, $[x, -)$, $(-, x]$, (x, y), $[x, y]$, $(x, y]$ or $[x, y)$; nor it include any of the first four of these forms. They showed that the acceleration field for each subsequence transformation A is the union of collection of sets of the form (x, y). The result is given below.

Theorem 1.10 *If* $x \in S_0$ *and A is a subsequence matrix that accelerates x, then there exist y and z such that* $y < x < z$ *and A accelerates each* $r \in (y, z)$.

They also proved that this algorithm cannot be extended to a larger class of sequences defined in terms of rate of convergence.

2 On Statistical Acceleration Convergence of Sequences

The notion of statistical convergence of sequences was introduced by Fast [4] and Schoenberg [9] independently. Later on it was further investigated from different aspects of sequence spaces and summability theory by many research workers.

A subset E of N is said to have *asymptotic density* $\delta(E)$ if $\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{\infty} \chi_E(k)$ exists, where χ_E is the *characteristic function* of E. Clearly all finite subsets of N have zero asymptotic density and $\delta(E^c) = \delta(N - E) = 1 - \delta(E)$.

A sequence (x_k) is said to be *statistically convergent* to L if for every $\varepsilon > 0$, $\delta(\{k \in N : |x_k - L| \geq \varepsilon\}) = 0$. We write stat$-\lim x_k = L$.

Throughout ω, ℓ_∞, c, c_0, $\bar{c}$, $\bar{c_0}$, m_0, represent the spaces of *all*, *bounded*, *convergent*, *null*, *statistically convergent*, *statistically null* and *bounded statistically null* sequences, respectively. Further S_0, $\bar{S_0}$ denote the subsets of the spaces c_0 and m_0, respectively, with non-zero terms.

Example 2.1 The sequence (x_k) defined by $x_k = i$, for $k = i^2, i \in N$ and $x_k = k^{-2}$, otherwise, is statistically convergent to 0 and is unbounded.

Tripathy and Sen [13] introduced the notion of statistical acceleration convergence of sequences as follows.

Definition 2.1 Let the sequence (x_k) be statistically convergent to σ and the sequence (y_k) statistically convergent to λ with $(x_k - \sigma) \notin \bar{S_0}$ and $(y_k - \sigma) \notin \bar{S_0}$, then the sequence (x_k) statistically converges to σ statistically faster than (y_k) statistically convergent to λ, written as $(x_k) <^{\text{stat}} (y_k)$ if
stat $- \lim \frac{x_k - \sigma}{y_k - \sigma} = 0$, provided $(y_k - \sigma) \neq 0$ for all $k \in N$.

Definition 2.2 The sequence (x_k) statistically converges to σ statistically at the same rate as the sequence (y_k) statistically converges to λ, written as $(x_k) \approx (y_k)$ if

$$0 < \text{stat} - \lim - \inf \left| \frac{x_k - \sigma}{y_k - \lambda} \right| \leq \text{stat} - \lim - \sup \left| \frac{x_k - \sigma}{y_k - \lambda} \right| < \infty.$$

Tripathy and Sen [13] proved the statistical analogue of most of the above results as well as they established the following decomposition theorem for acceleration convergence.

Theorem 2.3 *Let* (x_k), $(y_k) \in \bar{S_0}$, *then the following are equivalent.*

(i) $(x_k) <^{\text{stat}} (y_k)$.
(ii) there exist $(x_k^/)$ *and* $(y_k^/)$ *in* S_0 *such that* $x_k = x_k^/$ *for a.a.k ,* $y_k = y_k^/$ *for a.a.k and* $(x_k^/) < (y_k^/)$.
(iii) there exists a subset $K = \{k_i : I \in N\}$ *of* N *such that* $\delta(K) = 1$ *and* $(x_{k_i}) < (y_{k_i})$.

Remark 2.4 Keagy and Ford [5] conjectured "If A is any subsequence transformation and $(x_k) \in S_0$, then either $Ax < x$ or $Ax \approx x$". Tripathy and Sen [13] provided the following example, which shows that this conjecture fails.

Example 2.2 Let the sequence (x_k) be defined by the subsequence $(x_{k_i}) = (x_1, x_3, x_5, \ldots)$.

3 On *I*-Acceleration Convergence of Sequences

The notion of I-convergence was introduced by Kostyrko et al. [6]. Later on it was further investigated from sequence space point of view and linked with summability theory by many others.

The notion depends on the notion of ideals. Let X be a non-empty set, then a family of sets $I \subset 2^X$ is an *ideal* if and only if for each $A, B \in I$, we have $A \cup B \in I$ and for $A \in I$ and for each $B \subset A$, we have $B \in I$. A non-empty family of sets $F \subset 2^X$ is a *filter* on X if and only if $\emptyset \notin F$, for each $A, B \in F$, we have $A \cap B \in F$ and for each $A \in F$ and for each $B \supset A$, we have $B \in F$. An ideal I is called *non-trivial* if $I \neq \emptyset$ and $X \notin I$. Hence $I \subset 2^X$ is a non-trivial ideal if and only if $F = F(I) = \{X - A : A \in I\}$ is a filter on X.

A subset E of N is said to have *logarithmic density* $d(E)$ if $d(E) = \lim_{n\to\infty} \frac{1}{s_n} \sum_{k=1}^{n} \frac{\chi_E(k)}{k}$ exists, where $s_n = \sum_{k=1}^{n} \frac{1}{k}$, for all $n \in N$. Clearly all finite subsets of N have zero logarithmic density and $d(E^c) = d(N - E) = 1 - d(E)$.

Let $T = (t_{nk})$ be a regular non-negative matrix. Then for $E \subset N$, if $d_T(E) = \lim_{n\to\infty} \sum_{k=1}^{\infty} t_{nk} \chi_E(k)$ exists, it is called the *T-density* of E. From the regularity of T it follows that $\lim_{n\to\infty} \sum_{k=1}^{\infty} t_{nk} = 1$ and from this and non-negativeness of T it follows that $d_T(E) \in [0, 1]$.

Clearly the asymptotic density and logarithmic density can be obtained as the particular cases of T-density. If one considers $t_{nk} = \frac{1}{n}$, for $k \leq n$ and $t_{nk} = 0$, otherwise, then $d_T(E) = \delta(E)$. If one considers $t_{nk} = \frac{k^{-1}}{s_n}$, for $k \leq n$ and $t_{nk} = 0$, otherwise, then one will get $d_T(E) = d(E)$.

The *uniform density* of a subset E of N is defined as follows: For integers $t \geq 0$ and $s \geq 1$, let $E(t + 1, s + 1) = Card\{n \in E : t + 1 \leq n \leq t + s\}$. Let $\beta_s = \liminf_{t\to\infty} E(t + 1, t + s)$ and $\beta^s = \limsup_{t\to\infty} E(t + 1, t + s)$. Then $\underline{u}(E) = \lim_{s\to\infty} \frac{\beta_s}{s}$ and $\bar{u}(E) = \lim_{s\to\infty} \frac{\beta^s}{s}$ exist. If $\underline{u}(E) = \bar{u}(E)$, then we say that the uniform density of E exists and $u(E) = \underline{u}(E) = \overline{u}(E)$.

Remark 3.1 Throughout we consider I to be a non-trivial ideal of subsets of N, the set of natural numbers

Definition 3.2 A sequence (x_k) is said to be $I -$ *convergent to* L if for each $\varepsilon > 0$, $\{k \in N : x_k - L \geq \varepsilon\} \in I$. We write $I - \lim x_k = L$.

The following are the examples of ideals :

Example 3.1 The class I_f of all finite subsets of N is an ideal of 2^N.

Example 3.2 The class $I_C = \{E \subset N :\}$ is an ideal of 2^N.

Example 3.3 The class $I = \{E \subset N : (E) = 0\}$ is an ideal of 2^N.

Example 3.4 The class $I_d = \{EN : d(E) = 0\}$ is an ideal of 2^N.

Example 3.5 The class $= \{E \subset N : T_d(E) = 0\}$ is an ideal of 2^N.

Example 3.6 The class $I_u = \{E \subset N : u(E) = 0\}$ is an ideal of 2^N.

Remark 3.3 All the ideals considered in the above examples are non-trivial ideals.

Tripathy and Mahanta [12] introduce the following definition on acceleration convergence related to I-convergence of sequences:

Definition 3.4 Let $I - \lim x_k = \sigma$ and $I - \lim y_k = \lambda$ with $(x_k - \sigma), (y_k - \lambda) \in S_0^I$. Then we say (x_k) I-converges to σ, *I-faster than* (y_k) I-converges to λ, written as $(x_k) <^I (y_k)$ if $I - \lim_{k\to\infty} \frac{x_k-\sigma}{y_k-\lambda} = 0$, provided $y_k - \lambda \neq 0$, for all $k \in N$.

Peterson and Savas [7] have studied the acceleration convergence for double sequences, Tripathy and Dutta [11] have studied I-acceleration convergence for sequences of fuzzy real numbers.

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Studies on Bounded Difference Sequence Space $\ell_\infty(\Delta)$ with Statistical Metric

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Abstract In this article we want to introduce the notion of bounded difference sequence space $\ell_\infty(\triangle)$ with the concept of statistical metric and discuss some of its properties such as completeness, solidness, symmetricity and convergence free.

Keywords Convergence free • Solid space • Statistical metric • Symmetric space • *t*-norm

Mathematics Subject Classification (2010) Primary 40A05, 40A30; Secondary 60B10

1 Introduction

The concept of statistical metric space or (briefly, *SM*-space) was introduced by Menger [1] in 1942. This is also termed as probabilistic metric space (briefly, *PM*-space). *SM*-space is the generalization of abstract metric space. It is a space in which the distance notion between two points is a distribution function instead of a single non-negative number and the concept of *SM*-space corresponds to the situations when the distance is inexact. The scope for studies of the *SM*-space was developed by Schweizer-Sklar [2]. Using the concept of *SM*-space, different authors have worked in different fields but, only a very few work has been done on sequences in *SM*-space [3, 4].

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2 Definitions and Preliminaries

Definition 2.1 A real valued function f on the set of real numbers is called a distribution function if it is non-decreasing, left continuous and has $\inf_{t \in R} f(t) = 0$ and $\sup_{t \in R} f(t) = 1$.

In the sequel, H will denote the distribution function defined by

$$H(t) = \begin{cases} 0, & t \leq 0 \\ 1, & t > 0 \end{cases}$$

Let X be a non-empty set and S denote the set of all distribution functions defined on X. Let f be a mapping from $X \times X$ into S and for every pair (p, q) of X, we denote the distribution function $F(p, q)$ by F_{pq} whence the symbol $F_{pq}(t)$ will denote the value of F_{pq} for the real argument t and $F_{pq}(t)$ interprets the probability that the distance from p to q is less than t.

Definition 2.2 An ordered pair (X, F) is called a statistical metric space (briefly, SM-space) if it satisfies the following conditions (refer [2])

(1) $F_{pq}(t) = 1$ for all $t > 0$ if and only if $p = q$.
(2) $F_{pq}(0) = 0$.
(3) $F_{pq}(t) = F_{qp}(t)$.
(4) If $F_{pq}(t_1) = 1$ and $F_{qr}(t_2) = 1$, then $F_{pr}(t_1 + t_2) = 1$ for all p, q, r in X and $t_1, t_2 \geq 0$.

A metric space (X, d) may be regarded as an SM-space, with metric F defined by $F_{pq}(t) = H(t - d(p, q))$, for all p, q in X.

Definition 2.3 A t-norm is a function $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ satisfying the following conditions:

(1) $T(0, 0) = 0$,
(2) $T(a, 1) = a$,
(3) $T(a, b) = T(b, a)$,
(4) $T(c, d) \geq T(a, b)$ for $c \geq a, d \geq b$,
(5) $T(T(a, b), c) = T(a, T(b, c))$ for all a, b, c in $[0, 1]$.

For example: $T(a, b) = ab(Product)$ and $T(a, b) = \text{Min}\ (a, b)$ are t-norms.

Definition 2.4 A Menger space (refer to [2]) is a statistical metric space (X, F) satisfying $F_{pr}(t_1 + t_2) \geq T(F_{pq}(t_1), F_{qr}(t_2))$, for all p, q, r in X; $t_1, t_2 \geq 0$ and T is a t-norm. This inequality is known as Menger's triangle inequality.

Throughout the article, by a statistical metric space we mean that the statistical metric space satisfying the Menger's triangle inequality.

Definition 2.5 A sequence $x = (x_k)$ in a statistical metric space (X, F) is said to converge to a point l in X if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $n_0(\epsilon, \lambda)$ such that $F_{x_k l}(\epsilon) > 1 - \lambda$ for all $k \geq n_0(\epsilon, \lambda)$.

Definition 2.6 A sequence$x = (x_k)$ in a statistical metric space (X, F) is said to be a Cauchy sequence if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $n_0(\epsilon, \lambda)$ such that $F_{x_m x_n}(\epsilon) > 1 - \lambda$ for all $m, n \geq n_0(\epsilon, \lambda)$.

Definition 2.7 A statistical metric space (X, F) with continuous t-norm is said to be complete if every Cauchy sequence in X converges to point in X.

Definition 2.8 A class of sequences E is said to be normal (or solid) if $(y_k) \in E$, whenever $|y_k| \leq |x_k|$, for all k $\in$ N and $(x_k) \in E$.

Definition 2.9 Let $K = \{k_1 < k_2 < k_3 \ldots\} \subseteq N$ and E be a class of sequences. A K-step set of E is a set of sequences $\lambda_K^E = \{(x_{k_n}) \in w : (x_n) \in E\}$.

Definition 2.10 A canonical pre-image of a sequence $(x_{k_n}) \in \lambda_K^E$ is a sequence $(y_n) \in w$ defined as follows:

$$y_n = \begin{cases} x_n, & \text{if } n \in K, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.11 A canonical pre-image of a step set λ_K^E is a set of canonical pre-images of all elements in λ_K^E, i.e., y is in canonical pre-image λ_K^E if and only if y is canonical pre-image of some $x \in \lambda_K^E$.

Definition 2.12 A class of sequences E is said to be monotone if E contains the canonical pre-images of all its step sets.

From the above definitions we have the following well-known Remark.

Remark 2.13 A class of sequences E is solid $\Rightarrow$ E is monotone.

Definition 2.14 A class of sequences E is said to be symmetric if $(x_{\pi(n)}) \in E$, whenever $(x_k) \in E$, where π is a permutation of N.

Definition 2.15 A class of sequences E is said to be convergence free if $(y_k) \in E$, whenever $(x_k) \in E$ and $x_k = 0$ implies $y_k = 0$.

Definition 2.16 A sequence $x = (x_k)$ is called bounded difference sequence (in traditional metric) if the sequence $\triangle x = (\triangle x_k)$, $\triangle x_k = x_k - x_{k+1}$ is bounded.

Throughout w and $\ell_\infty(\triangle)$ denote the classes of *all* and *bounded difference sequences*, respectively.

In this article we want to introduce the notion of the bounded difference sequences in a statistical metric space as follows:

Definition 2.17 A sequence $x = (x_k)$ in a statistical metric space (X, F) is said to be bounded difference sequence if there exists $h > 0$ and $0 < \delta < 1$ such that

$F_{x0}(h) > 1 - \delta$, where 0 (zero) is in x and the distance d between $x = (x_k)$ and 0 (zero) is defined by $d(x, 0) = |x_1| + \sup_k |\triangle x_k|$.

Let $\ell_\infty(\triangle)$ be a class of bounded difference sequences and F be a mapping from $\ell_\infty(\triangle) \times \ell_\infty(\triangle)$ to S(a set of distributive functions) such that

$$F_{xy}(h) = \begin{cases} e^{-\frac{d(x,y)}{h}}, & h > 0 \\ 0, & h = 0, \end{cases} \tag{1}$$

for all $x = (x_k)$ and $y = (y_k)$ in $\ell_\infty(\triangle)$ and $d(x, y) = |x_1 - y_1| + \sup_k |\triangle x_k|$.

It is easy to see that d is a metric on $\ell_\infty(\triangle)$. It can be seen that $(\ell_\infty(\triangle), F)$ is a SM-space with the t-norm $T(a, b) = ab$ (Product).

Let $x = (x_k), y = (y_k)$ in $\ell_\infty(\triangle)$. Then

(i) $x = y \Rightarrow d(x, y) = 0$ so we have, $F_{xy}(h) = 1$, for all $h > 0$.
Conversely,
$F_{xy}(h) = 1$, for all $h > 0$. i.e., $e^{-\frac{d(x,y)}{h}} = 1 = e^0$ and we get, $x = y$.

(ii) $F_{xy}(h) = 0$, for $h = 0$ (by given condition).

(iii) $F_{xy}(h) = F_{yx}(h)$ as $d(x, y) = d(y, x)$, for $h \geq 0$.

(iv) For all $x = (x_k), y = (y_k)$ and $z = (z_k)$ in $\ell_\infty(\triangle)$ and for all $h, t > 0$, we have
$d(x, y) \leq (\frac{h+t}{h})d(x, z) + (\frac{h+t}{h})d(z, y)$ implies $e^{\frac{d(x,y)}{h+t}} \leq e^{\frac{d(x,z)}{h}}.e^{\frac{d(z,y)}{t}}$.

Since e^x is an increasing function for $x > 0$. So
$e^{-\frac{d(x,y)}{h+t}} \geq e^{-\frac{d(x,z)}{h}}.e^{-\frac{d(z,y)}{t}}$ and thus $F_{xy}(h + t) \geq F_{xz}(h).F_{zy}(t)$, for $h, t > 0$. i.e., $F_{xy}(h + t) \geq T(F_{xz}(h), F_{zy}(t))$, for $h, t > 0$.

For h or $t = 0$, we have $F_{xy}(h + t) > F_{xz}(h).F_{zy}(t) = 0$
and for $h, t = 0$, $F_{xy}(h + t) = T(F_{xz}(h), F_{zy}(t))$.
Hence $F_{xy}(h + t) \geq T(F_{xz}(h), F_{zy}(t))$, for $h, t \geq 0$.

3 Main Result

Theorem 3.1 *The sequence space $(\ell_\infty(\triangle), F)$ is a complete metric space with the statistical metric F defined by*

$$F_{xy}(h) = \begin{cases} e^{-\frac{d(x,y)}{h}}, & h > 0 \\ 0, & h = 0, \end{cases}$$

under the t-norm $T(a,b) = ab$ *(product), where the metric*
$d(x,y) = |x_1 - y_1| + \sup_k|\triangle x_k - \triangle y_k|$, *for* $x = (x_k), y = (y_k)$ *in* $\ell_\infty(\triangle)$.

Proof Let $(x^{(n)})$ be a Cauchy sequence in $(\ell_\infty(\triangle), F)$ where $x^{(n)} = (x_k^{(n)})_k = (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \ldots) \in (\ell_\infty(\triangle), F)$, for all $n \in N$. Then, for each $\epsilon > 0$, there exists a positive integer n_0 such that for all $m, n \geq n_0$,
$d(x^{(n)}, x^{(m)}) = |x_1^{(n)} - x_1^m| + \sup_k|\triangle x_k^n - \triangle x_k^m|$.

$$\Rightarrow |x_1^{(n)} - x_1^m| < \epsilon, |\triangle x_k^n - \triangle x_k^m| < \epsilon, \text{ for each } k \in N, \text{ and } m, n \geq n_0. \tag{2}$$

Hence we obtain $|x_k^{(n)} - x_k^m| < \epsilon$, for all $m, n \geq n_0$ and for each $k \in N$.

Now for a given $h > 0, 0 < \delta < 1$,

$$F_{x^{(n)}x^{(m)}}(h) = \begin{cases} e^{-\frac{d(x^{(n)},x^{(m)})}{h}}, & h > 0 \\ 0, & h = 0, \end{cases}$$

gives
$F_{x^{(n)}x^{(m)}}(h) > 1 - \delta$, (for $h > 0$), for all $m, n \geq n_0$.
$\Rightarrow G_{x_k^{(n)}x_k^{(m)}}(h) > 1 - \delta$, for $h > 0; m, n \geq n_0$ and each $k \in N$, where G is a statistical metric (can be easily verified as F is verified) on R, the set of real numbers defined by

$$G_{x_k^{(n)}x_k^{(m)}}(h) = \begin{cases} e^{-\frac{|x_k^{(n)} - x_k^{(m)}|}{h}}, & h > 0 \\ 0, & h = 0, \end{cases}$$

Hence for each k, $(x_k^{(n)})_{n=1}^{\infty}$ is a Cauchy sequence in R. Since R is complete, so for each k, $(x_k^{(n)})_{n=1}^{\infty}$ converges to some $x_k \in R$ so that
$\lim_{n\to\infty} G_{x_k^{(n)}x_k}(h) = 1$, for $h > 0$.

Let $x = (x_1, x_2, x_3, \ldots)$. We show that $x \in (\ell_\infty(\triangle), F)$ and $x^{(n)} \to x$.

Now fix $n \geq n_0$ and let $m \to \infty$ in inequalities (2), we have

$$|x_1^{(n)} - x_1| < \epsilon \text{ and } |\triangle x_k^{(n)} - \triangle x_k| < \epsilon, \text{ for each } k \in N. \tag{3}$$

Hence, $d(x^{(n)}, x) = |x_1^{(n)} - x_1| + \sup_k|\triangle x_k^{(n)} - \triangle x_k|$, for all $n \geq n_0$
and we have $\lim_{n\to\infty} F_{x^{(n)}x}(h) = 1$, for $h > 0$. i.e., $x^{(n)} \to x$ as $n \to \infty$.

Since $x^{(n)} = (x_k^{(n)})_k \in \ell_\infty(\triangle)$, for all $n \in N$, there exists a real no. $h > 0$ such that $G_{\triangle x_k^{(n)} 0}(h) > 1 - \delta, 0 < \delta < 1$, for each $k \in N$.

Again, $|\triangle x_k| \leq |\triangle x_k - \triangle x_k^{(n)}| + |\triangle x_k^{(n)}| + < M_n$, for some $M_n \in R, n \geq n_0$ and each $k \in N$ (using the inequalities (3) and from the bounded sequence $(\triangle x_k^{(n)})_k$). Since G is statistical metric, so it obeys the triangle inequality with t-norm as well as the metric other properties. That is, for each $k \in N, n \geq n_0$ and for $h_1, h_2 > 0$, we have

$$G_{\triangle x_k 0}(h_1 + h_2) \geq T\Big(G_{\triangle x_k \triangle x_k^{(n)}}(h_1),\ G_{\triangle x_k^{(n)} 0}(h_2)\Big) \text{ as } T(a,b) = ab \text{ (Product).}$$

Thus, from the inequality $|\triangle x_k| \leq M_n$, for some $M_n \in R$, we have for each $k \in N$ and $n \geq n_0$, there exists $h > 0$ such that $G_{\triangle x_k 0}(h) > 1 - \delta, 0 < \delta < 1$. Since $d(x, 0) = |x_1| + \sup_k |\triangle x_k| < \infty$. So $F_{x0}(h) > 1 - \delta; h > 0, 0 < \delta < 1$ (F is defined in(1)). Thus $x \in (\ell_\infty(\triangle), F)$. Hence the result. □

Theorem 3.2 *The sequence space $(\ell_\infty(\triangle), F)$ is neither monotone nor solid.*

Proof This result follows from the following sequence. Let us consider the sequence $x = (x_k)$, where $x_k = k$. Therefore, $|\triangle x_k| = 1$, for all $k \in N$ and we have $d(x, 0) = |x_1| + \sup_k |\triangle x_k| = 2$. Thus $F_{x0}(h) > 1 - \delta$, for $h > 0, \delta > 0$ (F is defined in (1)). Hence $(x_k) \in (\ell_\infty(\triangle), F)$.

Let $J = \{k \in N : k = 2i - 1, i \in N\}$ be a subset of N and let $\overline{\overline{\ell_\infty(\triangle)_J}}$ be the canonical pre-image of the J-step set $\ell_\infty(\triangle)_J$ of $\ell_\infty(\triangle)$, defined as follows.

$(y_k) \in \overline{\overline{\ell_\infty(\triangle)_J}}$, is the canonical pre-image of $(x_k) \in \ell_\infty(\triangle)$ implies

$$y_k = \begin{cases} x_k, & \text{for } k \in J, \\ 0 & \text{for } k \notin J. \end{cases}$$

i.e., $(y_k) = (1, 0, 3, 0, 5, \ldots)$. Then $|\triangle y_k| = (1, 3, 3, 5, 5, \ldots)$.

We have, $d(y, 0) = |y_1| + \sup_k |\triangle y_k|$ which is unbounded. Therefore, $F_{y0}(h) > 1 - \delta$ does not exist (F is defined in (1)). Thus $(y_k) \notin (\ell_\infty(\triangle), F)$. Hence the space $(\ell_\infty(\triangle), F)$ is not monotone.

The space $(\ell_\infty(\triangle), F)$ is not solid which follows from the Remark 2.13. □

Theorem 3.3 *The space $(\ell_\infty(\triangle), F)$ is not symmetric.*

Proof This result follows from the following sequence. Consider the sequence $x = (x_k)$, where $x_k = k$. Hence $(x_k) \in (\ell_\infty(\triangle), F)$ as shown in the Theorem 3.2.

Let (y_k) be a rearrangement of the sequence (x_k), defined as follows:

$(y_k) = (x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, \ldots)$.

$$\text{i.e., } (y_k) = \begin{cases} x_{\left(\frac{k+1}{2}\right)^2}, & \text{for } k \text{ odd}, \\ x_{\left(n + \frac{k}{2}\right)}, & \text{for } k \text{ even and } n \in N, \\ & \text{satisfying } n(n-1) < \frac{k}{2} \leq n(n+1). \end{cases}$$

Thus $(y_k) = (1, 2, 4, 3, 9, 5, 16, 6, 25, 7, \ldots)$.

Again, for k odd and $n \in N$, satisfying $n(n-1) < \frac{k+1}{2} \le n(n+1)$,

$\Delta y_k = x_{\left(\frac{k+1}{2}\right)^2} - x_{\left(n+\frac{k+1}{2}\right)} = \left(\frac{k+1}{2}\right)^2 - \left(n + \frac{k+1}{2}\right)$ which diverges to ∞ and for k even and $n \in N$, satisfying $n(n-1) < \frac{k}{2} \le n(n+1)$,

$\Delta y_k = x_{\left(n+\frac{k}{2}\right)} - x_{\left(\frac{k+2}{2}\right)^2} = \left(n + \frac{k}{2}\right) - \left(\frac{k+2}{2}\right)^2$ which diverges to $-\infty$.

Thus, $d(y, 0)$ is unbounded. Hence the space $(\ell_\infty(\triangle), F)$ is not symmetric. □

Theorem 3.4 *The space $(\ell_\infty(\triangle), F)$ is not convergence free.*

Proof This result follows from the following two sequences. Consider the sequences $x = (x_k)$ and $y = (y_k)$, defined as follows:

For k even, $x_k = \frac{1}{k}$ and for k odd, $x_k = 0$. Also, for k even, $y_k = k$ and for k odd, $y_k = 0$. Then $|\triangle x_k| = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \ldots\right)$ and thus, $d(x, 0) = 1$. Hence $(x_k) \in (\ell_\infty(\triangle), F)$.

Again, $|\triangle y_k|$ is unbounded and thus $d(y, 0)$ is unbounded. Hence the space $(\ell_\infty(\triangle), F)$ is not convergence free. □

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Rough Convergence in Metric Spaces

Shyamal Debnath and Debjani Rakshit

Abstract In this paper, we have introduced the notion of rough convergence in general metric spaces and the set of rough limit points and proved several results associated with this set.

Keywords Metric space • Rough cauchy sequence • Rough convergence • Rough limit points

Mathematics Subject Classification (2010) Primary 40A35; Secondary 40D25, 03E72

1 Introduction and Preliminaries

The classical analysis is often based on fine behaviors which are valid *for all* points of some subsets, even if some distance tends to zero. Since many things of the material universe and many objects represented by digital computers cannot satisfy such *for all* requirements, the so-called rough analysis is developed as an approach to such rough worlds. The idea of rough convergence was first introduced by Phu [5], in finite dimensional normed linear spaces.

Let (x_n) be a sequence in some normed linear space $(X, \| \, . \, \|)$, and r be a non-negative real number. (x_n) is said to be r-convergent to x^*, denoted by $x_n \xrightarrow{r} x^*$, if given $\epsilon > 0$ there exists a natural no n_0 such that

$$\| x_n - x^* \| < r + \epsilon, \forall n \geq n_0.$$

and the r-limit set of (x_n) is defined as $\mathrm{LIM}^r x_n = \{x^* \in X : x_n \xrightarrow{r} x^*\}$

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They showed that the set $\mathrm{LIM}^r x_n$ is bounded, closed and convex; and introduced the notion of rough Cauchy sequence. Also investigated the relations between rough convergence and other convergence types and the dependence of $\mathrm{LIM}^r x_n$ on the roughness degree r.

Later on several researchers, namely Aytar [1, 2], Pal et. al. [4] have generalized this concept.

We have seen that in a metric space (X, d), if a sequence (x_n) converges to a point $x \in X$ then given $\epsilon > 0$ we can always find natural number n_0 such that

$$d\,(x_n, x) < \epsilon, \forall n \geq n_0$$

and usually denoted as $x_n \to x$ in (X, d). x is called the limit of the sequence (x_n), which is unique also [3].

Let us introduce another notion of convergence.

2 Main Results

Definition 2.1 A sequence (x_n) in (X, d) is said to be r-convergent where $r \geq 0$, to a point $x^* \in X$ if given $\epsilon > 0$, there exists a natural no n_0 such that

$$d\left(x_n, x^*\right) < r + \epsilon, \forall n \geq n_0.$$

This is the rough convergence with r as roughness degree.

For $r = 0$, we have the classical definition of convergence in (X, d). But our proper interest is the case $r > 0$. There are several reasons for this interest. Let (y_n) be a sequence converging to classical sense to x^*. Suppose that we do not set the terms of (y_n) exactly, but we have that the terms of (y_n) differs from another terms of (x_n) in metric (distance) by a quantity less than equal to r, i.e., we have the situation:

$$d\left(y_n, x^*\right) < \epsilon, \forall n \geq n_0 \text{ and } d\,(y_n, x_n) \leq r, \forall n \in N.$$

Thus we have $d\,(x_n, x^*) \leq d\,(x_n, y_n) + d\,(y_n, x^*) < r + \epsilon, \forall n \geq n_0$.

Therefore, we see that (x_n) is r-convergent to the point $x^* \in X$.

Remark 2.2 Unlike the classical case, if (x_n) is a sequence in (X, d), $r > 0$ is a real no and $x_n \xrightarrow{r} x^*$, then $x^*$is not unique.

Thus we have the set

$$\begin{aligned}\mathrm{LIM}^r x_n &= \left\{x^* \in X : x_n \xrightarrow{r} x^*\right\}\\ &= \left\{x^* \in X : d\left(x_n, x^*\right) < r + \epsilon, \forall n \geq n_0\right\}\end{aligned}$$

It is clear that, if $r = 0$ then we have the classical case and therefore $\mathrm{LIM}^r x_n$ is either a singleton set or an empty set.

Example ([5]) Let us consider the sequence $y_n = 0.5 + 2.\frac{(-1)^n}{n}$, it is obvious that (y_n) converges to 0.5. For machine calculation, y_n cannot be calculated exactly for large n. This occurs due to rounding off the numbers. But it can be rounded to some machine number, i.e., to the nearest one.

Let, (x_n) be a sequence defined as $x_n = \text{round}\ (y_n) = z$, where z is an integer lies $z - 0.5 \leq y_n < z + 0.5$.

Then $x_1 = -1, x_2 = 2, x_{2k-1} = 0, x_{2k} = 1$, for $k = 2, 3, \ldots$

It is easy to see that $\limsup x_n = 1$ and $\liminf x_n = 0$, so that the sequence (x_n) does not converge. However by definition $\mathrm{LIM}^r x_n = \begin{cases} \phi, & \text{if } r < 0.5 \\ [1 - r, r] & \text{if } r \geq 0.5 \end{cases}$.

Proof We see that from the definition of rough convergence, infinitely many terms of the sequence (x_n) after a certain n must lie in an interval (for the real line) centred at x^*(a r-limit point) with the interval width $2r + \epsilon$, where ϵ is predetermined.

Here the odd and even position terms are separated (after $n = 2$) by a distance 1. Consequently, if $2r < 1$ i.e., $r < 0.5$, then there exists no integer n_0 such that all the terms of the sequence will lie in the interval $(x^* - r - \epsilon, x^* + r + \epsilon)$, for $n \geq n_0$.

Therefore, $\mathrm{LIM}^r x_n = \phi$, for $r < 0.5$.

On the otherhand, we have the relation if $r \geq 0.5$

$|x_n - x^*| < r + \epsilon$, $\forall n \geq n_0$

i.e., $x^* < x_n + r + \epsilon$ and $x^* > x_n - r - \epsilon$.

(Note that if $r \geq 0.5$, then $r \geq 1 - r$.)

Taking the odd integers after n_0, the less than inequality gives $x^* < r + \epsilon$.

Taking the even integers after n_0, the greater than inequality gives $x^* > 1 - r - \epsilon$.

Since ϵ is arbitrary, we obtain at once $1 - r \leq x^* \leq r$.

i.e., $\mathrm{LIM}^r x_n = [1 - r, r]$, for $r \geq 0.5$. □

Definition 2.3 Sometimes we are concerned about the set $\mathrm{LIM}^r x_n$ lying in a set $S \subset X$.

We define $\mathrm{LIM}^{S,r} x_n = \left\{x^* \in S : x_n \xrightarrow{r} x^*\right\}$.

In the above example, let $S = \{x_1, x_2, x_3, \ldots..\}$ be the set.

Observe that $|x_k - x_j| = 1$ or 0.

Therefore $|x_k - x_j| \leq 1$.

Now for $r < 1$, $|x_k - x^*| = |x_k - x_s| \leq 1$, which cannot be made less than $r + \epsilon$.

(Since, we are interested in the r-limit points lying in the set S, $x^* = x_s \in S$)

But for $r \geq 1$, $|x_k - x_s| \leq 1 \leq r < r + \epsilon$.

Therefore we obtain $\mathrm{LIM}^{S,r} x_n = \begin{cases} \phi, & \text{if } r < 1 \\ \{x_3, x_4, x_5, \ldots\} & \text{if } r \geq 1 \end{cases}$.

Definition 2.4 Let (y_n) be a classical Cauchy sequence in (X, d) and (x_n) be another sequence such that $d(x_n, y_n) \leq \frac{\rho}{2}$, $\forall n \in N$.

Then $d(x_m, x_n) \leq d(x_m, y_m) + d(y_m, y_n) + d(y_n, x_n) < \rho + \epsilon$, for all $m, n \geq n_0$.

Such a sequence (x_n) is said to be a rough Cauchy sequence with ρ as the degree of roughness.

2.1 Some Basic Properties of Rough Limit Sets

Proposition 2.5 *The r-limit set* $\text{LIM}^r x_n$ *of an arbitrary sequence* (x_n) *of* (X, d) *is a closed set.*

Proof Let (y_n) be a sequence in $\text{LIM}^r x_n$ which converges to y. We want to show that $y \in \text{LIM}^r x_n$. Since (y_n) converges to y, we have $d(y_n, y) < \frac{\epsilon}{2}$, $\forall n \geq n_0$.

In particular, $d(y_{n_0}, y) < \frac{\epsilon}{2}$

By the definition of $\text{LIM}^r x_n$, $d(x_n, y_{n_0}) < \frac{\epsilon}{2} + r$, $\forall n \geq n_1$.

Therefore, $d(x_n, y) \leq d(x_n, y_{n_0}) + d(y_{n_0}, y) < r + \frac{\epsilon}{2} + \frac{\epsilon}{2} = r + \epsilon$, $\forall n \geq n_1$.

Hence $y \in \text{LIM}^r x_n$. Thus $\text{LIM}^r x_n$ is a closed set. □

Proposition 2.6 *For any sequence* (x_n) *in* (X, d) *the diameter of* $\text{LIM}^r x_n$ *is not greater than* $2r$.

Proof If possible let $\text{diam}\,(\text{LIM}^r x_n) > 2r$.

Then there exists y, z in $\text{LIM}^r x_n$ such that $d(y, z) > 2r$. Let $d(y, z) = d_1$ (say). Then $d_1 > 2r$.

Since y and z are r-limit point of (x_n), therefore there exists a natural no n_0 such that

$d(x_n, y) < r + \frac{\epsilon}{2}$ and $d(x_n, z) < r + \frac{\epsilon}{2}$, $\forall n \geq n_0$.

Hence $d(y, z) \leq d(x_n, y) + d(x_n, z) < 2r + \epsilon$, $\forall n \geq n_0$.

i.e., $d_1 < 2r + \epsilon$, $\forall n \geq n_0$.

since ϵ is arbitrary, we put $\epsilon = d_1 - 2r$ and hence we obtain $d(y, z) = d_1 < d_1$, a contradiction.

Therefore $\text{diam}\,(\text{LIM}^r x_n) \leq 2r$. □

Proposition 2.7 *A sequence* (x_n) *is bounded if and only if there exists an* $r \geq 0$ *such that* $\text{LIM}^r x_n \neq \emptyset$.

Proof Let, $x = (x_n)$ be bounded. i.e., $\sup\{d(x_n, y_n) : x_n, y_n \in x\} = \bar{r}$ is finite.

Therefore, $\text{LIM}^{\bar{r}} x_n \neq \emptyset$.

Conversely, Let $\text{LIM}^r x_n \neq \emptyset$, for some $r \geq 0$.

i.e., for all but finite element of (x_n) are contained in some ball with any radius greater than r. Therefore the sequence (x_n) is bounded. □

Proposition 2.8 *For all* $r > 0$, *a bounded sequence* (x_n) *always contains a subsequence* (x_{n_i}) *with* $\text{LIM}^{(x_{n_i}),r} x_{n_i} \neq \phi$.

Proof As (x_n) is a bounded sequence, it has a convergent subsequence (x_{n_i}). Let, x^* be its limit point , then $\mathrm{LIM}^r x_{n_i} = \bar{B}_r\,(x^*)$ and for $r > 0$,
$\mathrm{LIM}^{(x_{n_i}),r} x_{n_i} = \{x_{n_i} : d\,(x^*, x_{n_i}) \le r\} \ne \phi$. □

Proposition 2.9 *If* $\left(x'_n\right)$ *is a subsequence of* (x_n)*, then* $\mathrm{LIM}^r x_n \subseteq \mathrm{LIM}^r x'_n$.

Proof Let $y \in \mathrm{LIM}^r x_n$ then $d\,(x_n, y) < r+\epsilon \forall n \ge n_0$ and since $\left(x'_n\right)$ is a subsequence of (x_n) $d\left(x'_n, y\right) < r + \epsilon\ \forall n \ge n_0$.
i.e., $y \in \mathrm{LIM}^r x'_n$. Hence proved. □

Proposition 2.10 *If a sequence* (x_n) *converges to* x^**, then* $\mathrm{LIM}^r x_n = \bar{B}_r\,(x^*)$.

Proof Let (x_n) be a convergent sequence with $lim\, x_i = x^*$.
Then for $\bar{B}_r\,(x^*) = \{y \in X : d\,(x^*, y) \le r\}$.
Now $d\,(x_n, y) \le d\,(x_n, x^*) + d\,(x^*, y) < \epsilon + r$ for $y \in \bar{B}_r\,(x^*)$.
i.e., $y \in \mathrm{LIM}^r x_n$ and therefore $\bar{B}_r\,(x^*) \subseteq \mathrm{LIM}^r x_n$.
Similarly we can show that $\mathrm{LIM}^r x_n \subseteq \bar{B}_r\,(x^*)$.
i.e., $\mathrm{LIM}^r x_n = \bar{B}_r\,(x^*)$.
But the converse is not true.
For example, let (X, d) be a metric space, where $X = \{-1, 0, 1\}$ and $d\,(x, y) = \begin{cases} 0, & \text{if } x = y \\ \max\{x, y\}, & \text{if } x \ne y \end{cases}.$
let $x_n = (-1)^n$ be a sequence in (X, d).
Then for $r = 1$, $\mathrm{LIM}^r x_n = \{-1, 0, 1\} = \bar{B}_r\,(0)$.
But (x_n) does not converge. □

Proposition 2.11 *Every rough convergent sequence is a rough cauchy sequence.*

Proof Let (x_n) be a rough convergent sequence, i.e., for all $\epsilon > 0$ $d\,(x_n, x^*) < r + \frac{\epsilon}{2} \forall n \ge n_0$.
Now $d\,(x_n, x_m) \le d\,(x_n, x^*) + d\,(x^*, x_m) < 2r + \epsilon \forall n, m \ge n_0$.
i.e., $d\,(x_n, x_m) < \rho + \epsilon\ \forall n, m \ge n_0$ where $\rho = 2r$. Hence proved. □

Proposition 2.12 $cl\left(\bigcup\limits_{0 \le r' < r} \mathrm{LIM}^{r'} x_n\right) \subseteq \mathrm{LIM}^r x_n = \bigcap\limits_{r' > r} \mathrm{LIM}^{r'} x_n$.

Proof It follows from the definition that $\mathrm{LIM}^{r_1} x_n \subseteq \mathrm{LIM}_n^{r_2}$ if $r_1 < r_2$.
By the monotonicity and the closedness property of r-limit set we have $\mathrm{cl}\left(\bigcup\limits_{0 \le r' < r} \mathrm{LIM}^{r'} x_n\right) \subseteq \mathrm{LIM}^r x_n \subseteq \bigcap\limits_{r' > r} \mathrm{LIM}^{r'} x_n$.
Now consider an arbitrary $y \in X - \mathrm{LIM}^r x_n$. By definition, there is an $\epsilon > 0$ such that $\forall k \in N \exists n \ge k : d(x_n, y) \ge r + \epsilon$.
This implies for $r' < r + \epsilon$ that $\epsilon' = r + \epsilon - r' > 0$ and $\forall k \in N\ \exists n \ge k$: $d(x_n, y) \ge r' + \epsilon'$.
Thus $y \notin \mathrm{LIM}^{r'} x_n$ for $r' < r + \epsilon$ which implies $y \notin \bigcap\limits_{r' > r} \mathrm{LIM}^{r'} x_n$.
Hence $\mathrm{LIM}^r x_n = \bigcap\limits_{r' > r} \mathrm{LIM}^{r'} x_n$. □

Proposition 2.13 (X, d) *be a bounded metric space if and only if there exist* $r > 0$ *such that* $\mathrm{LIM}^r x_n$ *is dense in* X.

Proof Let X be bounded and $m = \sup_{x, y \in X} d(x, y)$.

From definition we have $\mathrm{LIM}^r x_n \subseteq X$.

Let $x^* \in X$ then for $r = m$, $d(x_n, x^*) < r + \epsilon$.

Therefore $x^* \in \mathrm{LIM}^r x_n$.

i.e., there exist a r for which $\mathrm{LIM}^r x_n$ is dense in X.

Now if $\mathrm{LIM}^r x_n$ is dense in X then $\mathrm{LIM}^r x_n = X$ for some $r > 0$. i.e., $d(x_n, x^*) < r + \epsilon$ for all $x^* \in X$. Therefore X is bounded. □

Proposition 2.14 *Suppose* $r_1 \geq 0$ *and* $r_2 > 0$. *If a sequence* y_n *in* X *such that* $y_n \xrightarrow{r_1} x^*$ *and* $d(x_n, y_n) \leq r_2$, $n = 1, 2, \ldots$. *then* (x_n) *in* X *is* $(r_1 + r_2)$*-convergent to* x^*.

Proof Since $y_n \xrightarrow{r_1} x^*$ then for all $\epsilon > 0$ $\exists$ a n_0 such that $d(y_n, x^*) < r_1 + \epsilon$ for $n \geq n_0$.

So, $d(x_n, x^*) \leq d(x_n, y_n) + d(y_n, x^*) < r_1 + r_2 + \epsilon$ for $n \geq n_0$.

Therefore (x_n) is $(r_1 + r_2)$-convergent to x^*. □

Proposition 2.15 *If* c *is a cluster point of the sequence* (x_n), *then* $\mathrm{LIM}^r x_n \subseteq \bar{B}_r(c)$.

Proof Let, $x^* \in \mathrm{LIM}^r x_n$ then for all $\epsilon > 0 \exists$ a n_0 such that $d(x_n, x^*) < r_1 + \frac{\epsilon}{2}$ for $n \geq n_0 \longrightarrow$(i)

Let c be a cluster point of (x_n) then $d(x_n, c) < \frac{\epsilon}{2}$ for infinite elements (x_n) $\longrightarrow$(ii)

Then there must exist a $x_{n_1} \in (x_n)$ which satisfies both (i) and (ii).

Therefore, $d(x^*, c) \leq d(x_{n_1}, x^*) + d(x_{n_1}, c) < r + \epsilon$

implies $d(x^*, c) \leq r$. Hence proved. □

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Class of p-Absolutely Summable Sequence of Interval Number

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Abstract In this article we have introduced the class of sequence $\ell^i(p)$ of interval numbers. We established some properties like completeness, linearity, symmetric and some inclusion relation.

Keywords Convergence free • Interval number • Sequence algebra • Solid • Symmetric

Mathematics Subject Classification (2010) 40C05; 40J05; 46A45

1 Preliminaries

The idea of interval arithmetic was first used by Dwyer [2, 3]. Ramon E. Moore has applied interval arithmetic as an approach to bound rounding errors in mathematical computation. Further development on interval arithmetic was done by Moore [7, 8], Moore and Yang [9, 10] and Fischer [6].

An interval $\bar{x} = [a, b]$ is the set of real numbers between a and b, i.e. $\bar{x} = [a, b] = \{x : a \leq x \leq b\}$. If R denotes the set of all real valued closed intervals, an interval number is an element of R and a closed subset of the set of real numbers, represented by $\bar{x} = [x_\ell, x_r]$, where x_ℓ and x_r are the left and right points, respectively. Geometrically represents a line segment on the real line. In particular if $x_\ell = x_r = x$, then reduced to a real number $x = [x, x]$, called point interval or singleton. Thus we can say that an interval number is the generalization of the point interval. We define some arithmetic operations with the interval numbers $\bar{x}_1 = [x_{1\ell}, x_{1r}]$ and $\bar{x}_2 = [x_{2\ell}, x_{2r}]$ as follows:

(i) $\bar{x}_1 = \bar{x}_2 \Rightarrow x_{1\ell} = x_{2\ell}$ and $x_{1r} = x_{2r}$.
(ii) $\bar{x}_1 + \bar{x}_2 = [x_{1\ell} + x_{2\ell}, x_{1r} + x_{2r}]$.
(iii) $\bar{x} = [x_\ell, x_r], \Rightarrow -\bar{x} = -[x_\ell, x_r] = [-x_r, -x_\ell]$.

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(iv) $\bar{x}_1 - \bar{x}_2 = [x_{1\ell} + x_{2r}, x_{1r} + x_{2\ell}]$.
(v) $\bar{x}_1.\bar{x}_2 = [\min\{x_{1\ell}.x_{2\ell}, x_{1\ell}.x_{2r}, x_{1r}.x_{2\ell}, x_{1r}.x_{2r}\}, \max\{x_{1\ell}.x_{2\ell}, x_{1\ell}.x_{2r}, x_{1r}.x_{2\ell}, x_{1r}.x_{2r}\}]$.
(vi) $\frac{\bar{x_1}}{\bar{x_2}} = [x_{1\ell}, x_{1r}] \times \frac{1}{[x_{2r}, x_{2\ell}]}$
$= [\min\{x_{1\ell} \div x_{2\ell}, x_{1\ell} \div x_{2r}, x_{1r} \div x_{2\ell}, x_{1r} \div x_{2r}\}$,
$\max\{x_{1\ell} \div x_{2\ell}, x_{1\ell} \div x_{2r}, x_{1r} \div x_{2\ell}, x_{1r} x_{2r}\}], 0 \notin \bar{x_2}$.
(vii) Let $\alpha > 0$, then $\alpha\bar{x} = [\alpha x_\ell, \alpha x_r]$ and if $\alpha < 0$, then $\alpha\bar{x} = [\alpha x_r, \alpha x_\ell]$.
(viii) If $\bar{x}_1 \subset \bar{x}_2$, i.e. $[x_{1\ell}, x_{1r}] \subset [x_{2\ell}, x_{2r}] \Rightarrow x_{2\ell} < x_{1\ell} < x_{1r} < x_{2r}$.

Remark 1.1 This property can be generalized for more than two intervals and often refereed as nesting property of intervals.

(ix) Let $\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4 \in R$, then $\bar{x}_3(\bar{x}_1 + \bar{x}_2) \subset \bar{x}_3\bar{x}_1 + \bar{x}_3\bar{x}_2$.

Remark 1.2 The equality $\bar{x}_3(\bar{x}_1 + \bar{x}_2) = \bar{x}_3\bar{x}_1 + \bar{x}_3\bar{x}_2$ holds with the condition that if $a \in [x_{1\ell}, x_{1r}] = \bar{x}_1$ and $b \in [x_{2\ell}, x_{2r}] = \bar{x}_2$ then $ab \geq 0$. It holds well with the point interval $a = [a, a]$, i.e. $a(\bar{x}_1 + \bar{x}_2) = a\bar{x}_1 + a\bar{x}_2$.

(x) For $\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4 \in R$, if $\bar{x}_1 \subset \bar{x}_2$ and $\bar{x}_3 \subset \bar{x}_4$ then
(a) $\bar{x_1} + \bar{x_3} \subset \bar{x_2} + \bar{x_4}$ (b) $\bar{x_1} - \bar{x_3} \subset \bar{x_2} - \bar{x_4}$ (c) $\bar{x}_1.\bar{x}_3 \subset \bar{x}_2.\bar{x}_4$ (d) $\frac{\bar{x}_1}{\bar{x}_3} \subset \frac{\bar{x}_2}{\bar{x}_4}$, if $0 \notin \bar{x}_3, \bar{x}_4$

The absolute value of $\bar{x} = [x_\ell, x_r]$ is defined by

$$|\bar{x}| = \begin{cases} [\min\{|x_\ell||x_r|\}, \max\{|x_\ell||x_r|\}], & \text{if } x_\ell.x_r \geq 0, \\ 0, \max\{|x_\ell||x_r|\}], & \text{if } x_\ell.x_r < 0. \end{cases}$$

We consider the metric $d(\bar{x}_1, \bar{x}_2) = \max\{|x_{1\ell} - x_{2r}|, |x_{1r} - x_{2r}|\}$. Since R is complete, so it is easy to verify that the set of all interval numbers is a complete metric space with respect to d. In the special case of $\bar{x}_1 = [a, a]$ and $\bar{x}_2 = [b, b]$, we obtain usual metric of the R with $d(\bar{x}_1, \bar{x}_2) = |a - b|$.

2 Introduction

Consider the transformation f from N to R defined by $k \to f(k) = \bar{x}$ then $(\bar{x}_n)$ is called the sequence of interval numbers, where $\bar{x}_n$ is the nth term of the sequence $(\bar{x}_n)$. We denote the set of all sequences of interval number by w^i. The addition and scalar multiplication of $(\bar{x}_n), (\bar{y}_n) \in w^i$ are defined as follows:

$$(\bar{x}_n) + (\bar{y}_n) = [\bar{x}_{n\ell} + \bar{y}_{n\ell}, \bar{x}_{nr} + \bar{y}_{nr}]$$
$$(\alpha\bar{x_n}) = [\alpha\bar{x}_{n\ell}, \alpha\bar{x}_{nr}], \text{ if } \alpha \geq 0$$
$$= [\alpha\bar{x}_{nr}, \alpha\bar{x}_{n\ell}], \text{ if } \alpha < 0$$

Definition 2.1 An interval sequence $\bar{x} = (\bar{x}_n)$ is said to be convergent to the interval number $\bar{x}_0$ if for each $\varepsilon > 0$ there exists a positive integer n_0 such that $d(\bar{x}_n, \bar{x}_0) < \varepsilon$ for all $n \geq n_0$, and we write it as $\lim_n \bar{x}_n = \bar{x}_0$ which imply $\lim_n \bar{x}_{n\ell} = \bar{x}_{0\ell}$ and $\lim_n \bar{x}_{nr} = \bar{x}_{0r}$.

Definition 2.2 An interval sequence $\bar{x} = (\bar{x}_n)$ is said to be interval Cauchy sequence if for every $\varepsilon > 0$ there exists $k_0 \in N$ such that $d(\bar{x}_n), \bar{x}_k)) < \varepsilon$ for $n, k \geq k_0$.

We introduced the following concepts for the classes of sequences of interval numbers.

Definition 2.3 An interval sequence $\bar{x} = (\bar{x}_n)$ is said to be bounded if $d(\bar{x}_n, \theta) < \infty$, equivalently, if there exist $\mu \in R$ such that $|\bar{x}_n| \leq \mu$ for all $n \in N$.

Definition 2.4 An interval sequence space w^i is said to be solid if $(\bar{x}_n) \in w^i$ whenever $(\bar{y}_n) \in w^i$ and $(\bar{x}_n) \leq (\bar{y}_n)$, for all $n \in N$.

Definition 2.5 An interval sequence space w^i is said to be symmetric if $(\bar{x}_{\pi(n)}) \in w^i$, whenever $(\bar{x}_n) \in w^i$, where π is a permutation on N.

Definition 2.6 An interval sequence space w^i is said to be convergence free if $(\bar{y}_n) \in w^i$ whenever $(\bar{x}_n) \in w^i$ and $\bar{x}_n = \bar{0}$ implies $\bar{y}_n = 0$.

Definition 2.7 An interval sequence space w^i is said to be sequence algebra if for $(\bar{x}_n), (\bar{y}_n) \in w^i$, $(\bar{x}_n \otimes \bar{y}_n) \in w^i$.

Chiao [1] introduced sequence of interval numbers and studied the usual convergence. Recently Esi [4, 5] has made several investigations on different classes of sequence of interval numbers. Şengönül and Eryilmaz [11] introduced the following sequence spaces of interval numbers and proved their completeness.

$$c_0^i = \left\{ \bar{x} = (\bar{x}_n) \in w^i : \lim_n \bar{x}_n = \theta, \text{ where } \theta = [0, 0] \right\}$$

$$c^i = \left\{ \bar{x} = (\bar{x}_n) \in w^i : \lim_n \bar{x}_n = \bar{x}_0, \text{ where } \bar{x}_0 \in R \right\}$$

$$\ell_\infty^i = \left\{ \bar{x} = (\bar{x}_n) \in w^i : \sup_n \{|\bar{x}_{n\ell}|, |\bar{x}_{nr}|\} < \infty \right\}$$

We introduced the class of p-absolutely summable sequence $\ell^i(p)$ of interval number, defined by

$$\ell^i(p) = \left\{ \bar{x} = (\bar{x}_n) \in w^i : \sum_{n=1}^{\infty} \{d(\bar{x}_n, \theta)\}^{p_n} < \infty \right\},$$

where $\bar{x} = [x_\ell, x_r]$ and $p = (p_n)$ is a bounded sequence of positive numbers so that $0 < p_n \leq \sup p_n < \infty$.

We consider the following metric to study different properties on the space $\ell^i(p)$

$$d(\bar{x}_n, \bar{y}_n) = \left\{ \sum_k \{\max(|x_{n\ell} - y_{n\ell}|, |x_{nr} - y_{nr}|)\}^{P_n} \right\}^{\frac{1}{M}},$$

where $0 < p_n \le \sup p_n < \infty$ and $M = \max(1, \sup p_n)$.

3 Main Result

Theorem 3.1 *The class of sequence $\ell^i(p)$ is closed with respect to addition and scalar multiplication.*

Proof Let $(\bar{x}_n), (\bar{y}_n) \in \ell^i(p)$ and α, β be scalars such that

$$\sum_{n=1}^{\infty} \{d(\bar{x}_n, \theta)\}^{p_n} < \infty \text{ and } \sum_{n=1}^{\infty} \{d(\bar{y}_n, \theta)\}^{p_n} < \infty.$$

Thus

$$\sum_{n=1}^{\infty} [d\{(\alpha\bar{x}_n + \beta\bar{y}_n), \theta)\}]^{p_n} \le \sum_{n=1}^{\infty} \{d(\bar{x}_n, \theta)\}^{p_n} + \sum_{n=1}^{\infty} \{d(\bar{x}_n, \theta)\}^{p_n} < \infty$$

This completes the proof. □

Theorem 3.2 *The class of sequence $\ell^i(p)$ is a complete metric space with respect to the metric defined by*

$$d(\bar{x}_n, \bar{y}_n) = \left\{ \sum_n [d(\bar{x}_n, \bar{y}_n)]^{P_n} \right\}^{\frac{1}{M}}$$

Proof It is easy to verify that d is a metric on $\ell^i(p)$. Let $\bar{x}^j = (\bar{x}^j_n) = (\bar{x}^j_1, \bar{x}^j_2, \bar{x}^j_3, \ldots ..)$ be a Cauchy sequence in $\ell^i(p)$ for each j. Then for every $\varepsilon > 0$ there exist an $n_0 \in N$ such that

$$d(\bar{x}^j_n, \bar{x}^k_n) = \left\{ \sum_n [d(\bar{x}^j_n, \bar{x}^k_n)]^{p_n} \right\}^{\frac{1}{M}} < \varepsilon, \text{ for } j, k \ge n_0$$

$$\Rightarrow d(\bar{x}^j_n, \bar{x}^k_n) < \varepsilon \text{ for } j, k \ge n_0$$

$$\Rightarrow \left\{ \sum_k [\max |\bar{x}^j_{n\ell} - \bar{x}^k_{n\ell}|, |\bar{x}^j_{nr} - \bar{x}^k_{nr}|]^{p_n} \right\}^{\frac{1}{M}} < \varepsilon$$

This implies $|\bar{x}^j_{n\ell} - \bar{y}^k_{n\ell}|$ and $|\bar{x}^j_{nr} - \bar{y}^k_{nr}| < \varepsilon$. This shows that $(\bar{x}^j_n)$ is a Cauchy sequence in R. Since R is complete, $(\bar{x}^j_n)$ is convergent. Let $\lim\limits_n \bar{x}^j_n = \bar{x}_n$ for each $n \in N$. Thus for each $\varepsilon > 0$, there exists n_0 such that $d(\bar{x}^j_n, \bar{x}_n) < \varepsilon$, for $j \geq n_0$. The proof will complete once we show that $\bar{x}_n \in \ell^i(p)$. We have

$$d(\bar{x}_n, \theta) \leq d(\bar{x}_n, \bar{x}^j_n) + d(\bar{x}^j_n, \theta) < \varepsilon + K < \infty.$$

This completes the proof. □

Theorem 3.3 *The class of sequence $\ell^i(p)$ is solid and hence monotone.*

Proof Let $(\bar{x}_n)$ and $(\bar{y}_n)$ be two sequences of interval numbers such that $|\bar{x}_n| \leq |\bar{y}_n|$, for all $k \in N$. Let $\bar{x}_n \in \ell^i(p)$, then $\sum\limits_{n=1}^{\infty} \{d(\bar{x}_n, \theta)\}^{p_n} = \sum\limits_n [\max\{|x_{n\ell}|, |x_{nr}|\}]^{p_n} < \infty$. Now we have

$$\sum_{n=1}^{\infty} \{d(\bar{y}_n, \theta)\}^{p_n} \leq \sum_{n=1}^{\infty} \{d(\bar{x}_n, \theta)\}^{p_n} < \infty$$

Thus $(\bar{y}_n) \in \ell^i(p)$. This completes the proof. □

Theorem 3.4 *The class of sequence $\ell^i(p)$ is a sequence algebra.*

Proof Let $(\bar{x}_n)$ and $(\bar{y}_n)$ be two sequences of interval numbers taken from. Then we have

$$\sum_{n=1}^{\infty} \{d(\bar{x}_n, \theta)\}^{p_n} < \infty$$

and

$$\sum_{n=1}^{\infty} \{d(\bar{y}_n, \theta)\}^{p_n} < \infty, \text{for all } n \in N.$$

We have

$$\sum_{n=1}^{\infty} \{d(\bar{x}_n \otimes \bar{y}_n, \theta)\}^{p_n} \leq \sum_{n=1}^{\infty} \{d(\bar{x}_n, \theta).d(\bar{y}_n, \theta)\}^{p_n}$$
$$\leq \left[\sum_{n=1}^{\infty} \{d(\bar{x}_n, \theta)\}^{p_n}\right]\left[\sum_{n=1}^{\infty} \{d(\bar{x}_n, \theta)\}^{p_n}\right] < \infty$$

Thus $(\bar{x}_n \otimes \bar{y}_n) \in \ell^i(p)$. This completes the proof. □

Theorem 3.5 *The class of sequence $\ell^i(p)$ is not convergence free.*

Proof We provide the following example in support of the proof.

Example 1 Consider the interval sequence $(\bar{x}_n)$ defined by $\bar{x}_n = \left[\frac{1}{n+1}, \frac{1}{n}\right]$ for $n \in N$ and take $p_k = 1$, for all $k \in N$. Then we have

$$\sum_{n=1}^{\infty} \{d(\bar{x}_n, \theta)\}^{p_n} = \sum_{n=1}^{\infty} \left[\max\left\{\left|\frac{1}{n+1}\right|, \left|\frac{1}{n}\right|\right\}\right]^{p_k} < \infty$$

Now consider the interval sequence $(\bar{y}_n)$ defined by $\bar{y}_n = [n, n+1]$, for $n \in N$ and take $p_n = 1$, for all $k \in N$. Then we have

$$\sum_{n=1}^{\infty} \{d(\bar{y}_n, \theta)\}^{p_k} = \sum_{n=1}^{\infty} [\max\{|n|, |n+1|\}]^{p_k} \to \infty$$

Thus we can conclude that$\ell^i(p)$ is not convergence free. This completes the proof. □

Theorem 3.6 *The class of sequence $\ell^i(p)$ is not symmetric.*

Proof The proof follows from the following example.

Example 2 Consider the interval sequence $(\bar{x}_n)$ defined by

$$\bar{x}_n = \begin{cases} \left[\frac{1}{(n+1)^2}, \frac{1}{n^2}\right], & \text{for } n \text{ odd} \\ \left[\frac{1}{(n+1)}, \frac{1}{n}\right], & \text{for } n \text{ even} \end{cases}$$

Consider $p_k = 1$, for all k odd and $p_k = 2$, for all k even. Then we have

$$\sum_{n=1}^{\infty} \{d(\bar{x}_n, \theta)\}^{p_n} = \sum_{n=1}^{\infty} \left[\max\left\{\left|\frac{1}{(n+1)^2}\right|, \left|\frac{1}{n^2}\right|\right\}\right] < \infty$$

Now consider the rearrangement $(\bar{y}_n)$ of $(\bar{x}_n)$ defined by

$$\bar{y}_n = (\bar{X}_2, \bar{X}_1, \bar{X}_4, \bar{X}_3, \bar{X}_6, \ldots\ldots..)$$

Then we have

$$\sum_{n=1}^{\infty} \{d(\bar{y}_n, \theta)\}^{p_k} = \sum_{n \text{ odd}} \left[\max\left\{\left|\frac{1}{n+1}\right|, \left|\frac{1}{n}\right|\right\}\right] \to \infty$$

Thus it implies $(\bar{y}_n) \notin \ell^i(p)$. This completes the proof. □

Theorem 3.7 *For* $0 < p < q, \ell^i(p) \subset \ell^i(q)$.

Proof The proof is simple, so omitted. □

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Part VII
Recent Progresses in Evolution Equations

Marcello D’Abbicco

Global Existence of Small Data Solutions to the Semilinear Fractional Wave Equation

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Abstract In this paper, we find the critical exponent for the global existence of small data solutions to the semilinear fractional wave equation in low space dimension.

Keywords Critical exponent • Fractional partial differential equation • Global existence • Small data

Mathematics Subject Classification (2010) Primary 35R11; Secondary 35A01

1 Introduction

In this note, we prove the global existence of small data solutions to

$$\begin{cases} \partial_t^{1+\alpha} u - \triangle u = |u|^p \,, \\ u(0,x) = u_0(x) \,, \\ u_t(0,x) = 0 \end{cases} \tag{1}$$

with $\alpha \in (0,1)$, for $p > \bar{p}$, where

$$\bar{p} = \max\left\{ p_\alpha(n) \,,\, \frac{1}{1-\alpha} \right\}, \qquad p_\alpha(n) \doteq 1 + \frac{2(1+\alpha)}{(n-2)(1+\alpha)+2}. \tag{2}$$

In (1), we write $\partial_t^{1+\alpha} u$ to denote

$$\partial_t^{1+\alpha} u = D_{0|t}^{\alpha}(u_t), \qquad \text{with} \quad D_{0|t}^{\alpha} f = \partial_t \left(J_{0|t}^{1-\alpha} f \right),$$

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where $J^\beta_{a|t}$ is the fractional Riemann–Liouville integral defined by

$$J^\beta_{a|t} f = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\beta-1} f(s)\, ds,$$

for $0 < \beta < 1$, $a \in \mathbb{R}$ and Γ is the Euler gamma function.

Thanks to the assumption $u_t(0,x) = 0$, Cauchy problem (1) for the semilinear fractional wave may be written in the form of a Cauchy problem for an integro-differential equation:

$$\begin{cases} \partial_t u = J^\alpha_{0|t}(\triangle u + |u|^p)\,, \\ u(0,x) = u_0(x)\,. \end{cases} \tag{3}$$

If $u_t(0,x)$ does not identically vanish, the equivalence of (1) with (3) is no longer true and the approach employed in this paper to treat the nonlinear problem is no longer valid.

We have the following.

Theorem 1.1 *Let $p > \bar{p}$, with $\bar{p}$ as in (2). Then there exists $\varepsilon > 0$, $\bar{\delta} > 0$ such that for any $u_0 \in L^1 \cap L^\infty$ with*

$$\|u_0\|_{L^1\cap L^\infty} \leq \varepsilon,$$

and for any $\delta \in (0, \bar{\delta})$, there exists a unique global solution

$$u \in \mathcal{C}([0,\infty), L^{1+\delta} \cap L^\infty)$$

to (1). Moreover, it satisfies the following decay estimate:

$$\|u(t,\cdot)\|_{L^q} \leq C\,(1+t)^{-\beta_q+\alpha}\, \|u_0\|_{L^{1+\delta}\cap L^\infty}, \qquad q \in [1+\delta, \infty], \tag{4}$$

for any $t \geq 0$, where

$$\beta_q = \beta_q(\delta) \doteq \min\left\{ \frac{n(1+\alpha)}{2}\left(\frac{1}{1+\delta} - \frac{1}{q}\right), 1 \right\}. \tag{5}$$

Moreover, when $\bar{p} = p_\alpha(n)$ we are able to prove a counter-part of our existence result, so that we can say that $p_\alpha(n)$ is the *critical exponent* to (1) (at least) in space dimension $n \leq 2/(\alpha(1+\alpha))$.

Theorem 1.2 *Let $p \in (1, p_\alpha(n)]$, with $p_\alpha(n)$ as in (2), and $u_0 \in L^1$ in (1), be such that*

$$\int_{\mathbb{R}^n} u_0(x)\, dx > 0. \tag{6}$$

Then there exists no global weak solution $u \in L^p_{\mathrm{loc}}([0,\infty) \times \mathbb{R}^n)$ (see later, Definition 3.1) to (1).

Remark 1.3 In space dimension $n = 2$, the critical exponent $p_\alpha(2) = 2 + \alpha$ for global small data solutions has been previously derived in [5].

Formally setting $\alpha = 0$, (1) becomes the Cauchy problem for the semilinear heat equation, and the exponent in (2) reduces to the well-known Fujita critical exponent $1 + 2/n$. On the other hand, $\bar{p} \to \infty$ as $\alpha \to 1^-$ in Theorem 1.1, in particular, we do not obtain Strauss exponent for the nonlinear wave equation in space dimension $n \geq 2$. This hints to the chance to improve the existence result for large values of α, by using an approach different from the one employed in this paper. It is an open problem to check if Strauss exponent could be approached taking the limit as $\alpha \to 1^+$, in a result obtained for $\alpha > 1$ in (1). We also remark that linear estimates (10), which play a fundamental role to prove Theorem 1.1, are only valid in the special case $r = q = 2$, if we formally set $\alpha = 1$ in (1).

1.1 Loss of Decay in Theorem 1.1

In decay estimate (5) for the nonlinear problem (1), it appears a loss of decay rate which is not lesser than t^α, with respect to the linear problem (see later, (10) and (11)). This loss of decay rate influences the critical exponent, so that it can no longer be determined by scaling arguments. This effect, which is related to the presence of fractional integrals, has been already observed for the heat equation with nonlinear memory [2], namely, for

$$\begin{cases} \partial_t u - \triangle u = J^\alpha_{0|t}(|u|^p)\,, \\ u(0,x) = u_0(x)\,. \end{cases} \tag{7}$$

In this case, the critical exponent is

$$\max\left\{\tilde{p}_\alpha(n)\,,\ \frac{1}{1-\alpha}\right\}, \qquad \tilde{p}_\alpha(n) \doteq 1 + \frac{2(1+\alpha)}{n-2\alpha}. \tag{8}$$

We notice that $\tilde{p}_\alpha(n) > p_\alpha(n)$ for any $\alpha \in (0,1)$.

Similar results about the critical exponent for global small data solutions have been obtained for damped waves with nonlinear memory [3, 4].

Also, in Theorem 1.1, there is a δ-loss of decay, which can be taken arbitrarily small as $\delta \to 0$, described by the difference $\beta_q(0) - \beta_q(\delta) \geq 0$. This arbitrarily small loss of decay is related to the fact that linear $L^r - L^q$ linear estimates to (1) are currently available only for $r > 1$ (see later, (10)), so that our estimates are derived on $L^{1+\delta}$ basis instead of being derived on L^1 basis.

2 Proof of Theorem 1.1

The solution to the linear problem

$$\begin{cases} \partial_t^{1+\alpha} u - \triangle u = 0\,, \\ u(0,x) = u_0(x)\,, \\ u_t(0,x) = 0 \end{cases} \tag{9}$$

satisfies the following estimates (see Lemma 4.3 in [1])

$$\|u(t,\cdot)\|_{L^q} \lesssim t^{-\frac{n(1+\alpha)}{2}\left(\frac{1}{r}-\frac{1}{q}\right)} \|u_0\|_{L^r}, \qquad 1 < r \le q \le \infty, \tag{10}$$

provided that $n\,(1/r - 1/q) < 2$. In particular, for any $\alpha \in (0,1)$ and for any fixed $\delta > 0$, the solution to (9) satisfies the following estimate

$$\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\beta_q}\,(\|u_0\|_{L^{1+\delta}} + \|u_0\|_{L^q}), \quad \forall\, q \in [1+\delta, \infty] \tag{11}$$

where β_q is defined in (5).

Proof of Theorem 1.1 Thanks to Duhamel's principle, the solution to (1) is given by

$$u(t,x) = K(t,x) *_{(x)} u_0(x) + Nu(t,x),$$

where $K(t,x)$ is the fundamental solution to (9) and

$$Nu(t,x) = \int_0^t K(t-\tau,x) *_{(x)} (J^\alpha_{0|\tau}|u|^p)\,d\tau.$$

Assume that $\delta < \alpha$. Then, for any $n \ge 2$, there exists $\bar{q} \in (1+\delta, \infty)$ such that

$$\frac{n(1+\alpha)}{2}\left(\frac{1}{1+\delta} - \frac{1}{\bar{q}}\right) = 1.$$

We define the space

$$X_\delta \doteq \mathcal{C}([0,\infty), L^{1+\delta} \cap L^\infty),$$

with norm

$$\|u\|_{X_\delta} \doteq \sup_{t\ge 0}\big\{(1+t)^{-\alpha}\|u(t,\cdot)\|_{L^{1+\delta}} + (1+t)^{\beta_\infty-\alpha}\|u(t,\cdot)\|_{L^\infty}\big\},$$

if $n = 1$, and

$$\|u\|_{X_\delta} \doteq \sup_{t\geq 0}\big\{(1+t)^{-\alpha}\|u(t,\cdot)\|_{L^{1+\delta}} + (1+t)^{1-\alpha}(\|u(t,\cdot)\|_{L^{\bar q}} + \|u(t,\cdot)\|_{L^\infty})\big\},$$

if $n \geq 2$. For any $u \in X_\delta$, we consider the operator

$$P : X_\delta \to X_\delta, \qquad Pu \doteq K(t,x) *_{(x)} u_0(x) + Nu,$$

and we prove that

$$\|Pu\|_{X_\delta} \lesssim \Big((1+t)^{-\alpha}\|u_0\|_{L^{1+\delta}\cap L^\infty} + \|u\|_{X_\delta}^p\Big). \tag{12}$$

Thanks to linear estimate (11), it only remains to prove $\|Nu\|_{X_\delta} \lesssim \|u\|_{X_\delta}^p$. If $u \in X_\delta$, by interpolation we derive

$$\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\beta_q+\alpha}\,\|u\|_{X_\delta}, \qquad \forall q \in [1+\delta,\infty] \tag{13}$$

so that

$$\||u(t,\cdot)|^p\|_{L^q} \lesssim \|u(t,\cdot)\|_{L^{pq}}^p \lesssim (1+t)^{-p(\beta_p-\alpha)}\,\|u\|_{X_\delta}^p, \tag{14}$$

for any $q \in [1+\delta,\infty]$, due to $\beta_{pq} \geq \beta_p$. Thanks to (11) and to (14), we can now estimate

$$\begin{aligned} &\|Nu(t,\cdot)\|_{L^q} \lesssim \|u\|_{X_\delta}^p\, I_q(t), \qquad \forall q \in [1+\delta,\infty],\\ &I_q(t) \doteq \int_0^t (1+t-\tau)^{-\beta_q} \int_0^\tau (\tau-s)^{\alpha-1}\,(1+s)^{-p(\beta_p-\alpha)}\,ds\,d\tau. \end{aligned} \tag{15}$$

We notice that $p(\beta_p-\alpha) > 1$ for some $\bar\delta > 0$ (and $\bar\delta < \alpha$ if $n \geq 2$) if, and only if,

$$p > \bar p = \max\left\{p_\alpha(n)\,,\ \frac{1}{1-\alpha}\right\}.$$

Therefore, for any $\delta \in (0,\bar\delta)$, we may estimate (see, for instance, Lemma 3.1 in [8]),

$$I_q(t) \lesssim \int_0^t (1+t-\tau)^{-\beta_q}\,(1+\tau)^{\alpha-1}\,d\tau \lesssim (1+t)^{-\beta_q+\alpha},$$

thanks to the fact that $\beta_q \in (0,1]$ and $\alpha \in (0,1)$.

Therefore, (15) gives

$$\|Nu\|_{X_p} \lesssim \|u\|_{X_p}^p$$

and (12) is proved. By standard contraction arguments, the global existence of small data solutions to (1) follows by (12). □

3 Sketch of the Proof of Theorem 1.2

Theorem 1.2 may be proved by using the test function method, as done by the first author in [4] for a damped wave equation with nonlinear memory.

By virtue of integration by parts for fractional integrals (see (2.64), p. 46 in [7] and (2.106) in [6]), it is possible to give the following definition of weak solution to (1).

Definition 3.1 We say that $u \in L^p_{\mathrm{loc}}([0,T)\times\mathbb{R}^n)$, $T \in (0,\infty]$, is a weak solution to (1) if for any test functions $\varphi \in \mathcal{C}^2([0,\infty))$, with $\operatorname{supp}\varphi = [0,T]$, and $\Phi \in \mathcal{C}^2_c(\mathbb{R}^n)$, it holds

$$\begin{aligned} &-\int_0^T \int_{\mathbb{R}^n} u(t,x)\,\Phi(x)\,dx\,(\partial_t D^\alpha_{t|T}\varphi)\,dt - \big(D^\alpha_{0|T}\varphi\big)\int_{\mathbb{R}^n} u_0(x)\,\Phi(x)\,dx \\ &\quad = \int_0^T \int_{\mathbb{R}^n} (u(t,x)\triangle\Phi(x) + |u(t,x)|^p\Phi(x))dx\,\varphi(t)dt\,, \end{aligned} \tag{16}$$

where

$$D^\alpha_{t|T} \doteq -\partial_t J^{1-\alpha}_{t|T}, \qquad J^\alpha_{t|T} f \doteq \frac{1}{\Gamma(\alpha)}\int_t^T (s-t)^{\alpha-1} f(s)\,ds\,.$$

Test function method may then be applied, by choosing a test function φ whose fractional derivative in time is known. In particular, we set $\varphi(t) = \omega(t)^\beta$, for sufficiently large β (more precisely, $\beta > (\alpha+1)p'$), where

$$\omega(t) \doteq \begin{cases} (1-t/T) & \text{if } t\in[0,T], \\ 0 & \text{if } t > T. \end{cases} \tag{17}$$

It follows that $\operatorname{supp}\varphi = [0,T]$ and $\varphi \in \mathcal{C}^k_c([0,\infty))$, $k \geq 0$, for any $\beta > k$. Moreover (see Lemma 4.1 in [4]), for any $\alpha \in (0,1)$, there exists $C = C(\alpha,\beta)$ such that

$$D^\alpha_{t|T}\,\omega(t)^\beta = C(\alpha,\beta)\,T^{-\alpha}\omega(t)^{\beta-\alpha}\,, \qquad \text{for any } \beta > \alpha. \tag{18}$$

We set $\Psi_R(t,x) \doteq \Psi(x/R)$, for any $R > 1$, where $\Psi \in \mathcal{C}_c^2$ is a suitable radial, non-negative, test function, which assumes a positive constant value in a neighborhood of the origin, and $\Phi(x) \doteq \Psi_R(x)^\ell$, for sufficiently large $\ell > 0$ (more precisely, $\ell > 2p'$).

Then, after straightforward calculations, a standard application of the test function method leads to derive that the integral

$$I_{T,R} \doteq \int_0^T \int_{\mathbb{R}^n} |u(t,x)|^p \Phi(x) dx\, \varphi(t) dt$$

verifies

$$I_{T,R} \leq C\, TR^n (T^{-(\alpha+1)p'} + R^{-2p'}). \tag{19}$$

For any $p \in (1, p_\alpha(n))$, setting $R = R(T) \doteq T^{\frac{\alpha+1}{2}}$, it follows that

$$\int_0^\infty \int_{\mathbb{R}^n} |u(t,x)|^p\, dx\, dt = \lim_{T\to\infty} I_{T,R(T)} = 0\,;$$

hence $u \equiv 0$. The critical case $p = p_\alpha(n)$ may be treated with minor modifications. The proof follows by contradiction, since we assumed nontrivial initial data.

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On Strichartz Estimates in an Abstract Form

Andrei V. Faminskii

Abstract Classical Strichartz type argument is applied for an abstract one-parameter set of linear continuous operators and a Strichartz type estimate in a non-endpoint case is rigorously justified.

Keywords One-parameter set of linear operators • Strichartz estimate

Mathematics Subject Classification (2010) Primary 35A99; Secondary 35Q53

The goal of this short paper is to justify rigorously the derivation of the well-known Strichartz estimate for the simple non-endpoint case in an abstract setting.

There is a great amount of papers where estimates of Strichartz type, that is estimates on solutions in space-time norms, based both on general duality arguments and certain specific properties of considered problems, were established for concrete evolution equations (see, for example, [9]). In an abstract setting a good preliminary work was done in [4, 5], where a group of unitary operators in a Hilbert space was considered. However, the final estimates were obtained there not in general case, but for the specific cases of Schrödinger and wave equations.

Strichartz estimates in a general form for the first time were obtained in [6]. In particular, these estimates were established for a one-parameter set of operators without any group structure. However, there were no assumptions even on measurability of this set with respect to the parameter. In particular, it was assumed by default that all the considered integrals in abstract spaces existed. No explicit assumptions of this kind were introduced also in [8], where the results from [6] were generalized for the inhomogeneous case.

In this paper we justify the scheme from [6] in the case of abstract functional spaces. As a first step of the study we do not consider here a more complicated endpoint case. The obtained abstract result is illustrated by two examples for dispersive equations.

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Note that in the books [7, 9] no results for Strichartz type inequalities in an abstract setting are considered.

Let H be a complex Hilbert space, $\mathcal{T} : H^* \to H$ be an isomorphism defined by $\langle f, x\rangle = (x, \mathcal{T}f)$ for all $f \in H^*$, $x \in H$. In particular, $\|f\|_{H^*} = \|\mathcal{T}f\|_H$. Change the standard linear structure of H^* in the following way: $\langle cf, x\rangle = \overline{c}\langle f, x\rangle$ for any $c \in \mathbb{C}$. Then it is easy to see that the space H^* remains to be a linear space and $\mathcal{T}$ becomes a linear isomorphism. Therefore, from now on we identify H^* and H and write $\langle u, x\rangle = (x, u)$ for any $u \in H^* = H$ and $x \in H$.

Consider a certain Banach space B, let D be a dense linear subspace in B, D_a^* its algebraic dual space (then $B^* \subset D_a^*$, we again set $\langle cf, x\rangle = \overline{c}\langle f, x\rangle$ for any $f \in D_a^*$, $x \in D$ and $c \in \mathbb{C}$).

Let $A : D \to H$ be a linear operator, $A^* : H = H^* \to D_a^*$ its algebraic adjoint, that is

$$\langle A^*u, x\rangle = \langle u, Ax\rangle = (Ax, u) \qquad \forall\, x \in D, \quad \forall\, u \in H.$$

Lemma 1 *The following three conditions are equivalent:*

1) *there exists a constant $c \geq 0$ such that $\|Ax\|_H \leq c\|x\|_B$ for all $x \in D$;*
2) *if $u \in H$, then $A^*u \in B^*$ and there exists a constant $c \geq 0$ such that $\|A^*u\|_{B^*} \leq c\|u\|_H$ for all $u \in H$;*
3) *if $x \in D$, then $A^*Ax \in B^*$ and there exists a constant $c \geq 0$ such that $\|A^*Ax\|_{B^*} \leq c^2\|x\|_B$.*

The constant c is the same in all three cases.

Proof See [4]. □

Lemma 2 *Let $\{U(t),\ t \in I\}$ for a certain interval $I \subset \mathbb{R}$ (bounded or unbounded) be a one-parameter set of continuous linear operators in a separable Hilbert space H such that the functions $U(t)u, U^*(t)u \in (I \to H)$ are Bochner measurable for all $u \in H$ and $\|U(t)\|$ are uniformly bounded on I. Consider a Banach space $B = L_1(I; H)$, let D be a dense linear subspace in B. For any $f \in D$ let*

$$Af = \int_I U(\tau)f(\tau)\, d\tau. \tag{1}$$

Then A can be extended to a linear continuous operator from B to H. Moreover, $A^*u = U^*(t)u$, where $A^* : H \to L_\infty(I; H) \subset D_a^*$ is given by

$$\langle A^*u, f\rangle = \int_I \big(f(t), U^*(t)u\big)\, dt \qquad \forall u \in H, \quad \forall\, f \in B.$$

Proof First of all note that $U(\tau)f(\tau) \in L_1(I;H)$. The properties of the operator A is obvious. Note also that $U^*(t)u \in L_\infty(I;H)$. Moreover, for $f \in B$ and $u \in H$

$$\langle A^*u,f\rangle = (Af,u) = \Big(\int_I U(\tau)f(\tau)\,d\tau, u\Big) = \int_I \big(U(\tau)f(\tau),u\big)\,d\tau$$
$$= \int_I \big(f(\tau),U^*(\tau)u\big)\,d\tau = \int_I \langle U^*(\tau)u,f(\tau)\rangle\,d\tau.$$

□

Remark 3 The hypothesis of Lemma 2 are obviously satisfied for $I = \mathbb{R}$ and $U(t) = G(-t)P$, where $\{G(t),\ t \in \mathbb{R}\}$ is a continuous group of unitary linear operators and P is a continuous linear operator in H.

Corollary 0.4 *Under the hypothesis of Lemma 2 for any $f \in B$*

$$A^*Af = \int_I U^*(t)U(\tau)f(\tau)\,d\tau \in L_\infty(I;H). \tag{2}$$

Proof The proof is obvious. □

Theorem 0.5 *Let $\{U(t),\ t \in I\}$ for a certain interval $I \subset \mathbb{R}$ (bounded or unbounded) be a one-parameter set of continuous linear operators in a separable Hilbert space H such that the functions $U(t)u, U^*(t)u \in (I \to H)$ are Bochner measurable for all $u \in H$ and $\|U(t)\|$ are uniformly bounded on I. Let B_0 be a separable Banach space such that both B_0 and H are subspaces of a certain Hausdorff topological space, $B_0 \cap H$ is dense both in B_0 and H. Let there exist a linear subspace D_0 dense in $B_0 \cap H$ such that $U^*(t)U(\tau)x \in B_0^*$ for any $x \in D_0$ and a.e. $t,\tau \in I$, moreover, the mapping $U^*(t)U(\tau)x \in (I^\tau \to B_0^*)$ is Bochner measurable for a.e. $t \in I$. Assume that there exist constants $a \in (0,1)$ and $c_0 \geq 0$ such that for any $x \in D_0$ and a.e. $t,\tau \in I$ the following inequality holds:*

$$\|U^*(t)U(\tau)x\|_{B_0^*} \leq c_0|t-\tau|^{-a}\|x\|_{B_0}. \tag{3}$$

Then $U^*(t)u \in B_0^*$ for any $u \in H$ and a.e. $t \in I$, moreover, there exists a constant $c = c(a,c_0) \geq 0$ such that for any $u \in H$

$$\Big(\int_I \big\|U^*(t)u\big\|_{B_0^*}^{2/a}\,dt\Big)^{a/2} \leq c\|u\|_H. \tag{4}$$

Proof Let $p = \dfrac{2}{2-a}$, then $p' = 2/a$, that is $\dfrac{1}{p} + \dfrac{a}{2} = 1$. Note that $p \in (1,2)$, $p' \in (2,+\infty)$.

Let $B = L_p(I; B_0)$, then $B^* = L_{p',*w}(I; B_0^*)$ — the space of functions $u \in (I \to B_0^*)$ such that the function $\langle u(t), x\rangle$ is Lebesgue measurable for any $x \in B_0$ (further such functions are called $*$-weakly measurable) and $\|u\|_{B_0^*} \in L_p(I)$ (note that $L_{p',*w}(I; B_0^*) = L_{p'}(I; B_0^*)$ if B_0 is reflexive, see [1]).

Let D be the space of step-functions mapping I into D_0, that is $D = \operatorname{span}\{x\chi_{I'}(t)\}$ for all $x \in D_0$ and bounded intervals $I' \subset I$ (the symbol $\chi_{I'}$ denoted the characteristic function of I'). The space D is obviously dense in $B = L_p(I; B_0)$ since $p < +\infty$ and $L_1(I; H)$.

Define the operator A on D by formula (1). Then according to Lemma 2, A is a linear operator mapping D into H and $A^*u = U^*(t)u$ for any $u \in H$, where the operator A^* maps H into $L_\infty(I; H) \subset D_a^*$.

By virtue of (2) for any $f \in L_1(I; H)$

$$A^*Af = \int_I U^*(t)U(\tau)f(\tau)\, d\tau \ \in L_\infty(I; H).$$

Now let $f(\tau) \equiv x\chi_{I'}(\tau)$ for a certain $x \in D_0$ and a bounded interval I'. Then $U^*(t)U(\tau)f(\tau) = \chi_{I'}(\tau)U^*(t)U(\tau)x$ for all $t, \tau \in I$.

Since $U^*(t)U(\tau)x \in B_0^*$ for a.e. $t, \tau \in I$ and for the a.e. fixed value of t the function $U^*(t)U(\tau)x \in (I^\tau \to B_0^*)$ is Bochner measurable, then the function $U^*(t)U(\tau)f(\tau) \in (I^\tau \to B_0^*)$ is also Bochner measurable. Besides that, inequality (3) provides that for the a.e. fixed value of t and a.e. $\tau \in I$

$$\big\|U^*(t)U(\tau)f(\tau)\big\|_{B_0^*} \le \chi_{I'}(\tau)c_0|t-\tau|^{-a}\|x\|_{B_0} \in L_1(I^\tau),$$

since $a < 1$ and the interval I' is bounded.

According to the definition of the space D we have that for any function $f \in D$ for the a.e. fixed value of $t \in I$ the function $U^*(t)U(\tau)f(\tau) \in (I \to B_0^*)$ is Bochner measurable and so $U^*(t)U(\tau)f(\tau) \in L_1(I^\tau; B_0^*)$. Therefore, for a.e $t \in I$

$$F(t) \equiv \int_I U^*(t)U(\tau)f(\tau)\, d\tau \in B_0^* \cap H.$$

Remind that $F = A^*Af \in L_\infty(I; H)$ for any function $f \in D \subset L_1(I; H)$, and so for any $u \in B_0 \cap H$ the function $\langle F(t), u\rangle$ is Lebesgue measurable.

Next, for any $u \in B_0$ choose a sequence $\{u_n \in B_0 \cap H\}_{n\in\mathbb{N}}$ convergent to u in B_0. Then since $F(t) \in B_0^*$, we have that $\langle F(t), u_n\rangle \to \langle F(t), u\rangle$ while $n \to +\infty$ for a.e. $t \in I$. Therefore, the function $\langle F(t), u\rangle$ is Lebesgue measurable on I as a limit of a sequence of Lebesgue measurable functions. In means that the function $F \in (I \to B_0^*)$ is $*$-weakly measurable on I.

In particular, since the space B_0 is separable, the function $\|F(t)\|_{B_0^*}$ is Lebesgue measurable on I (see [1]).

Finally, with the use of Hardy–Littlewood–Sobolev equation we derive that for any function $f \in D$

$$\begin{aligned}\|A^*Af\|_{B^*} &= \|F\|_{B^*} = \Big\| \big\|F(t)\big\|_{B_0^*} \Big\|_{L_{2/a}(I)} \\ &\le \Big[\int_I \Big(\int_I \big\|U^*(t)U(\tau)f(\tau)\big\|_{B_0^*}\, d\tau \Big)^{2/a} dt \Big]^{a/2} \\ &\le c_0 \Big[\int_I \Big(\int_I |t-\tau|^{-a} \|f(\tau)\|_{B_0}\, d\tau \Big)^{2/a} dt \Big]^{a/2} \le c\|f\|_{L_p(I;B_0)} = c\|f\|_B.\end{aligned}$$

Therefore, $A^*Af \in B^*$ and so according to Lemma 1 $A^*u \in B^*$ for any $u \in H$ and

$$\|A^*u\|_{B^*} = \Big(\int_I \big\|U^*(t)u\big\|_{B_0^*}^{2/a} dt \Big)^{a/2} \le c^{1/2}\|u\|_H,$$

which coincides with (4). □

Remark 6 Since $(T^*)^* = T$ and $\|T^*\| = \|T\|$ for any linear continuous operator in H, the operators $U^*(t)$ and $U(t)$ can be interchanged (as in [6, 8]).

Remark 7 Consider the following important example. Let Ω be a domain in $\mathbb{R}^n$. Set $H = L_2(\Omega)$, $B_0 = L_p(\Omega)$ for $1 \le p < +\infty$ (here we consider the spaces of complex-valued functions). Then $H^* = L_2(\Omega)$, $B_0^* = L_{p'}(\Omega)$. One can choose, for example, $D_0 = C_0^\infty(\Omega)$.

Remark 8 As a simple example of an implementation of this result consider the well-known Strichartz estimate for Airy equation (see, for example, [3])

$$u_t + u_{xxx} = 0.$$

Consider the associated continuous group of unitary operators in $H = L_2(\mathbb{R})$:

$$G(t)\varphi = \mathcal{F}^{-1}\Big[e^{it\xi^3} \widehat{\varphi}(\xi) \Big], \quad t \in \mathbb{R}. \tag{5}$$

Let $U(t) \equiv G(-t)$ (then $U^*(t) \equiv G(t)$), $B_0 = L_1(\mathbb{R})$ (then $B_0^* = L_\infty(\mathbb{R})$), $D_0 = C_0^\infty(\mathbb{R})$. With the use of Airy function $\mathcal{A} \equiv \mathcal{F}^{-1}\Big[e^{it\xi^3} \Big]$ for $\varphi \in D_0$ one can write down a formula

$$\big(G(t)\varphi\big)(x) = \frac{1}{\sqrt[3]{t}} \int_{\mathbb{R}} \mathcal{A}\Big(\frac{x-y}{\sqrt[3]{t}} \Big) \varphi(y)\, dy \quad \forall\, x \in \mathbb{R},\ \forall t \ne 0.$$

Since Airy function is bounded on $\mathbb{R}$ it follows that

$$\|G(t)\varphi\|_{B_0^*} \le c_0|t|^{-1/3}\|\varphi\|_{B_0}$$

and, therefore,

$$\|U^*(t)U(\tau)\varphi\|_{B_0^*} \le c_0|t-\tau|^{-1/3}\|\varphi\|_{B_0}, \quad t \ne \tau.$$

Note that since the operators $G(t)$ form the continuous group of unitary operators also, for example, in $H^1(\mathbb{R})$, then $G(t)\varphi \in C(\mathbb{R}^t; H^1(\mathbb{R})) \subset C(\mathbb{R}^t; B_0^*)$ for any $\varphi \in D_0$. In particular, the mapping $U^*(t)U(\tau)\varphi = G(t-\tau)\varphi \in (\mathbb{R}^\tau \to B_0^*)$ is Bochner measurable $\forall t \in \mathbb{R}$.

Therefore, inequality (4) for $a = 1/3$ yields that

$$\big\| \|G(t)\varphi\|_{L_\infty(\mathbb{R}^x)} \big\|_{L_6(\mathbb{R}^t)} \le c\|\varphi\|_{L_2(\mathbb{R})}.$$

Note that it follows from this inequality by density arguments that

$$\|G(t)\varphi\|_{L_6(\mathbb{R}^t; C_b(\mathbb{R}^x))} \le c\|\varphi\|_{L_2(\mathbb{R})}, \tag{6}$$

where the symbol $C_b(\mathbb{R})$ denotes the space of continuous bounded on $\mathbb{R}$ functions.

Remark 9 The situation when the measurability simply follows from the continuity seems to be typical but not necessary. Consider a more general equation

$$u_t + b'(t)u_{xxx} = 0. \tag{7}$$

Let the function b be strictly monotone on a certain interval $I \subset \mathbb{R}$, for example, let it be increasing. It is known that then the function b is differentiable a.e. on I. Without loss of generality assume that $0 \in \overline{I}$ and $b(0) = 0$.

The simple change of variable $\tau = b(t)$ transforms this equation to Airy equation and, therefore, a solution to the initial value problem with initial data $u|_{t=0} = \varphi \in L_2(\mathbb{R})$ can be written in a form $u(t,\cdot) = G(b(t))\varphi$, where G is defined in (5).

Define a one-parameter set of continuous linear operators in $H = L_2(\mathbb{R})$

$$U(t) \equiv G(-b(t)), \quad t \in I. \tag{8}$$

Then $U^*(t) = G(b(t))$. Since the function b is Lebesgue measurable, the functions $U(t)\varphi$ and $U^*(t)\varphi$ are Bochner measurable for all $\varphi \in H$. Moreover, $U^*(t)U(\tau) = G(b(t)-b(\tau))$ and the mapping $G(b(t)-b(\tau))\varphi \in (I^\tau \to B_0^* = L_\infty(\mathbb{R}))$ is Bochner measurable for all $t \in I$ and $\varphi \in C_0^\infty(\mathbb{R})$. Since for $t \ne \tau$

$$\|U^*(t)U(\tau)\varphi\|_{B_0^*} \le c_0\big|b(t)-b(\tau)\big|^{-1/3}\|\varphi\|_{B_0}$$

condition (3) is satisfied if for certain constants $c > 0$ and $a \in [1/3, 1)$

$$b(t) - b(\tau) \ge c(t-\tau)^{3a}, \quad \forall t, \tau \in I,\ t > \tau. \tag{9}$$

By virtue of the properties of monotone functions inequality (9) is provided by the following one:

$$\int_\tau^t b'(\theta)\,d\theta \geq c(t-\tau)^{3a}, \quad \forall t,\tau \in I,\ t > \tau. \tag{10}$$

These conditions establish the upper bound of the degeneration rate of the function b near its stationary points. For example, for $b(t) \equiv t^\alpha$, $t \in (0, T)$, it means that $0 < \alpha < 3$.

Under assumptions (9) or (10) the following analogue of (6) holds

$$\|G(b(t))\varphi\|_{L_{2/a}(I;C_b(\mathbb{R}))} \leq c_1\|\varphi\|_{L_2(\mathbb{R})}. \tag{11}$$

Remark 10 Similar argument can be applied, for example, for the following generalized linearized Zakharov–Kuznetsov equation

$$u_t + b'(t)(u_{xxx} + u_{xyy}) = 0. \tag{12}$$

In [2] for the corresponding continuous group of unitary operators in $L_2(\mathbb{R}^2)$

$$G(t)\varphi = \mathcal{F}^{-1}\left[e^{it(\xi^3+\xi\eta^2)}\widehat{\varphi}(\xi,\eta)\right], \quad t \in \mathbb{R},$$

the following estimate was obtained:

$$\|G(t)\varphi\|_{L_\infty(\mathbb{R}^2)} \leq c_0|t|^{-2/3}\|\varphi\|_{L_1(\mathbb{R}^2)}.$$

Then if again the function b is increasing on a certain interval $I \subset \mathbb{R}$ and for $c > 0$, $a \in [2/3, 1)$

$$b(t) - b(\tau) \geq c(t-\tau)^{3a/2}, \quad \forall t,\tau \in I,\ t > \tau,$$

or

$$\int_\tau^t b'(\theta)\,d\theta \geq c(t-\tau)^{3a/2}, \quad \forall t,\tau \in I,\ t > \tau,$$

the following inequality holds:

$$\|G(b(t))\varphi\|_{L_{2/a}(I;C_b(\mathbb{R}^2))} \leq c_1\|\varphi\|_{L_2(\mathbb{R}^2)}.$$

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A Remark on the Energy Estimates for Wave Equations with Integrable in Time Speed of Propagation

M.R. Ebert, L. Fitriana, and F. Hirosawa

Abstract We consider the energy estimates for the wave equation with time dependent oscillating propagation speed. We expect that the kinetic energy and the elastic energy are estimated by the same order. The main purpose of this paper is to show that if the propagation speed is in $L^1(\mathbb{R}_+)$, then the elastic energy satisfies a better estimate than the kinetic energy.

Keywords Energy estimate • Time dependent coefficient • Wave equations

Mathematics Subject Classification (2010) 35L15; 35B40

1 Introduction

Let us consider the following Cauchy problem for the wave equation with time dependent propagation speed:

$$\begin{cases} \left(\partial_t^2 - a(t)^2\Delta\right) u = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ (u(0,x), (\partial_t u)(0,x)) = (u_0(x), u_1(x)), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

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where $a(t) > 0$. Then the wave type energy of the solution to (1) is given as follows:

$$E(t) = \frac{1}{2}a(t)^2\|\nabla u(t,\cdot)\|_{L^2}^2 + \frac{1}{2}\|u_t(t,\cdot)\|_{L^2}^2, \tag{2}$$

here the first and the second terms are called the elastic energy and the kinetic energy, respectively. One can observe many different effects for the behavior of $E(t)$ as $t \to \infty$ according to the properties of the speed of propagation $a(t)$ (see [3] and [4]).

Let us introduce the following hypothesis to $a(t)$:

Hypothesis 1 $a \in C^m(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$ with $m \geq 2$ and

$$\int_t^\infty a(\tau)d\tau =: A(t) \lesssim^1 (1+t)a(t). \tag{3}$$

Hypothesis 2 There exist two monotone decreasing positive functions $\lambda(t)$ and $\Xi(t)$ satisfying $\lambda(t) \in C^1(\mathbb{R}_+)$, $a(t) \simeq \lambda(t)$ and $\Xi(0) = A(0)$ such that

$$1 \lesssim (1+t)\sqrt{\lambda(t)} \text{ or } (1+t)\sqrt{\lambda(t)} \text{ is monotone decreasing,} \tag{4}$$

and

$$\int_t^\infty |a(s) - \lambda(s)|ds \lesssim \Xi(t) = o(A(t)) \ \ (t \to \infty). \tag{5}$$

Hypothesis 3 The following conditions are valid:

$$\left|a^{(k)}(t)\right| \leq \alpha_k \lambda(t)\eta(t)^k \ \ (k = 1, \ldots, m)$$

with a non-negative function $\eta(t)$ satisfying $\eta(t)\Xi(t) \lesssim \lambda(t)$ and

$$\int_0^t \lambda(s)\left(\frac{\eta(s)}{\lambda(s)}\right)^m ds \lesssim \Xi(t)^{1-m}.$$

Under the assumptions Hypotheses 1–3, it was proved in [2] that the following estimate is established:

$$a(t)\|\nabla u(t,\cdot)\|_{L^2} + \|u_t(t,\cdot)\|_{L^2} \lesssim \|u_0(\cdot)\|_{H^1} + \|u_1(\cdot)\|_{L^2}.$$

[1] Let $f, g : \Omega \to \mathbb{R}$ be two non-negative functions. We use the notation $f \lesssim g$ if there exists a positive constant C such that $f(y) \leq Cg(y)$ for all $y \in \Omega$. Moreover, $f \simeq g$ denotes if $f \lesssim g$ and $g \lesssim f$ hold.

On the other hand, if $a \in L^1(\mathbb{R}_+)$ and monotone decreasing, then there exists a positive function $d(t)$ satisfying $\lim_{t\to\infty} d(t) = 0$ such that the estimate of the elastic energy is developed in [1] as follows:

$$a(t)\|\nabla u(t,\cdot)\|_{L^2} \lesssim d(t)\left(\|u_0(\cdot)\|_{\dot{H}^1} + \|u_1(\cdot)\|_{L^2}\right).$$

The main theorem of this paper is an extension of a result from [1] without the assumption of monotonicity of $a(t)$.

Theorem 1.1 *If Hypotheses 1, 2 and 3 are valid, then there exists a positive constant N such that the following estimate is established:*

$$a(t)\|\nabla u(t,\cdot)\|_{L^2} \lesssim (1+t)\lambda(t)\left(\|F(|\nabla|)u_0(\cdot)\|_{\dot{H}^1} + \|F(|\nabla|)u_1(\cdot)\|_{L^2}\right),$$

where F is defined by

$$F(r) = 1,\ 0 < r < 1 \ \textit{and} \quad F(r) = r\sqrt{\lambda(\Xi^{-1}(Nr^{-1}))},\ r \geq 1 \tag{6}$$

Example 1.2 (i) If $\lambda(t) = (1+t)^{-l}$, $\Xi(t) = (1+t)^{\alpha}$ and $\eta(t) = (1+t)^{-\beta}$ with $l > 1$, $\alpha < -l+1$ and $\beta = l+\alpha-(l+\alpha-1)/m$, then $F(|\xi|) \simeq |\xi|^{1+\frac{l}{2\alpha}}$ for $l < -2\alpha$ and $F(|\xi|) = 1$ for $l \geq -2\alpha$.

(ii) If $\lambda(t) = \exp(-t^{\nu})$ and $\Xi(t) = (1+t)^{-\kappa}\exp(-t^{\nu})$, $\eta(t) = (1+t)^{-\beta}$ with $\nu > 1$, $\kappa > \nu-1$ and $\beta = -\kappa+(\kappa-\nu+1)/m$, then $F(|\xi|) \simeq |\xi|^{1/2}(\log|\xi|)^{\frac{\kappa}{2\nu}}$.

Remark 1.3 The definition of $F(r)$ is independent of N for this example

2 Proofs of Theorem 1.1

Denoting $v(t,\xi) = \hat{u}(t,\xi)$, where $\hat{u}$ is the partial Fourier transformation with respect to the x variable, (1) is rewritten as follows:

$$\begin{cases} v_{tt} + a(t)^2|\xi|^2 v = 0, & (t,\xi) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ (v(0,\xi), v_t(0,\xi)) = (\hat{u}_0(\xi), \hat{u}_1(\xi)), & \xi \in \mathbb{R}^n. \end{cases}$$

For a positive large constant N to be chosen later, we split the extended phase space $\mathbb{R}_+ \times \mathbb{R}^n$ into three zones, the *pseudo differential zone* $Z_{\text{pd}}(N)$, the *stabilized zone* $Z_{\text{st}}(N)$, and the *hyperbolic zone* $Z_{\text{hyp}}(N)$. They are defined as follows:

$$\begin{aligned} Z_{\text{pd}}(N) &= \{(t,\xi) \in \mathbb{R}_+ \times \mathbb{R}^n\ ;\ A(t)|\xi| \leq N\}, \\ Z_{\text{st}}(N) &= \{(t,\xi) \in \mathbb{R}_+ \times \mathbb{R}^n\ ;\ \Xi(t)|\xi| \leq N \leq A(t)|\xi|\}, \\ Z_{\text{hyp}}(N) &= \{(t,\xi) \in \mathbb{R}_+ \times \mathbb{R}^n\ ;\ \Xi(t)|\xi| \geq N\}. \end{aligned}$$

Here we denote by t_ξ and $\tilde{t}_\xi$ the separating lines between $Z_{\rm pd}(N)$ and $Z_{\rm st}(N)$ and between $Z_{\rm st}(N)$ and $Z_{\rm hyp}(N)$, respectively, that is

$$A(t_\xi)|\xi| = \Xi(\tilde{t}_\xi)|\xi| = N.$$

2.1 Estimate in $Z_{\rm pd}(N)$

For $t \geq t_\xi$, that is $(t,\xi) \in Z_{\rm pd}(N)$, we put $V_0(t,\xi) = {}^t(i\theta(t)v(t,\xi), v_t(t,\xi))$, $\theta(t) = a(t)/A(t)$, so that

$$\partial_t V_0 = A_0(t,\xi)V_0, \quad A_0 = \begin{pmatrix} \frac{\theta'(t)}{\theta(t)} & i\theta(t) \\ \frac{i|\xi|^2 a(t)^2}{\theta(t)} & 0 \end{pmatrix}. \tag{7}$$

Let us consider the fundamental solution $E = E(t,s,\xi)$ to (7), that is, the solution of

$$\partial_t E = A_0(t,\xi)E, \quad E(s,s,\xi) = I \tag{8}$$

with $t_\xi \leq s \leq t$, where I is the identity matrix. If we put $E = (E_{ij})_{ij=1,2}$, thanks to (7) we obtain, for $j = 1, 2$, the following integral equations:

$$E_{1j}(t,s,\xi) = \frac{\theta(t)}{\theta(s)}\left(\delta_{1j} + i\int_s^t \theta(s)E_{2j}(\tau,s,\xi)d\tau\right) \tag{9}$$

and

$$E_{2j}(t,s,\xi) = \delta_{2j} + i|\xi|^2\int_s^t \frac{a(\tau)^2}{\theta(\tau)}E_{1j}(\tau,s,\xi)d\tau. \tag{10}$$

By (9), (10) and integrating by parts we get

$$E_{2j}(t,s,\xi) = \delta_{2j} + i|\xi|^2\delta_{1j}\int_s^t \frac{a(\tau)^2}{\theta(s)}d\tau - |\xi|^2\int_s^t E_{2j}(\tau,s,\xi)\left(\int_\tau^t a(\sigma)^2 d\sigma\right)d\tau.$$

By using (3) [1] and Hypothesis 2 we have

$$\int_s^t a(\tau)^2 d\tau \simeq \int_s^t \lambda(\tau)a(\tau)d\tau \leq \lambda(s)\int_s^t a(\tau)d\tau \lesssim a(s)A(s)$$

and

$$\int_s^t \frac{a(\tau)^2}{\theta(s)}d\tau = A(s)\int_s^t \frac{a(\tau)^2}{a(s)}d\tau \simeq A(s) \leq A(s)^2.$$

Taking into account the inequalities

$$
\begin{aligned}
|E_{2j}(t,s,\xi)| &\lesssim 1 + |\xi|^2 A(s) \int_s^t a(\tau) \frac{a(\tau)}{a(s)} d\tau + |\xi|^2 \int_s^t a(\tau) A(\tau) E_{2j}(\tau,s,\xi) d\tau \\
&\lesssim 1 + |\xi|^2 A(s) \int_s^t a(\tau) |E_{2j}(\tau,s,\xi)| d\tau,
\end{aligned}
$$

by Gronwall's inequality, there exists a positive constant C such that

$$
|E_{2j}(t,s,\xi)| \lesssim \exp(C(1 + |\xi|^2 A(s) \int_s^t a(\tau) d\tau)) \le \exp(C(1+N^2)) \lesssim 1 \tag{11}
$$

for $j = 1, 2$ uniformly in $Z_{\mathrm{pd}}(N)$. Therefore, by (9) and (11) we conclude the estimate

$$
|E_{1j}(t,s,\xi)| \lesssim \frac{\theta(t)}{\theta(s)} \big(\delta_{1j} + (1+t)\theta(s)\big) \tag{12}
$$

for $j = 1, 2$ uniformly in $Z_{\mathrm{pd}}(N)$. Summarizing, all the consideration above implies the following estimates:

Lemma 2.1 *In $Z_{\mathrm{pd}}(N)$ the following estimates are established:*

$$
|v(t,\xi)| \lesssim \begin{cases} (1+t)\,(|v(0,\xi)| + |v_t(0,\xi)|) & \textit{for } |\xi| \le N/A(0), \\ (1+t)(\theta(t_\xi)|v(t_\xi,\xi)| + |v_t(t_\xi,\xi)|) & \textit{for } |\xi| \ge N/A(0). \end{cases} \tag{13}
$$

Proof Noting the representation $V_0(t,\xi) = E(t,s,\xi)V_0(s,\xi)$, we have

$$
\begin{pmatrix} i\theta(t)v(t,\xi) \\ v_t(t,\xi) \end{pmatrix} = \begin{pmatrix} E_{11}(t,s,\xi)i\theta(s)v(s,\xi) + E_{12}(t,s,\xi)v_t(s,\xi) \\ E_{21}(t,s,\xi)i\theta(s)v(s,\xi) + E_{22}(t,s,\xi)v_t(s,\xi) \end{pmatrix}. \tag{14}
$$

For $|\xi| \le N/A(0)$, (13) trivially follows by using (12) to (14). For $|\xi| \ge N/A(0)$, by (3) and using (12) to (14) with $s = t_\xi$ we have

$$
\begin{aligned}
|v(t,\xi)| &\lesssim \left(\frac{1}{\theta(t_\xi)} + (1+t) \right) \theta(t_\xi)|v(t_\xi,\xi)| + (1+t)|v_t(t_\xi,\xi)| \\
&\lesssim (1+t)(a(t_\xi)|\xi||v(t_\xi,\xi)| + |v_t(t_\xi,\xi)|).
\end{aligned}
$$

□

2.2 Estimate in $Z_{\mathrm{hyp}}(N)$

We can follow the standard diagonalization procedure and Lemma 3.6 in [2] to get the following estimates in $Z_{\mathrm{hyp}}(N)$.

Lemma 2.2 *There exists a positive constant N such that the following estimate is established in $Z_{\mathrm{hyp}}(N)$:*

$$\lambda(t)|\xi||v(t,\xi)| + |v_t(t,\xi)| \lesssim \sqrt{\lambda(t)}\left(|\xi||v(0,\xi)| + |v_t(0,\xi)|\right).$$

2.3 Estimate in $Z_{\mathrm{st}}(N)$

For any $\tilde{t}_\xi \le s \le t \le t_\xi$, we put $V_1(t,\xi) = {}^t(i\lambda(t)|\xi|v, v_t)$, so that

$$\partial_t V_1(t,\xi) = A_1(t,\xi)V_1(t,\xi), \quad A_1 = \begin{pmatrix} \frac{\lambda'(t)}{\lambda(t)} & i\lambda(t)\xi| \\ \frac{i|\xi|a(t)^2}{\lambda(t)} & 0 \end{pmatrix}.$$

Let M_1 be a diagonalizer of the principal part of A_1 defined by

$$M_1 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

If we put $W = M_1^{-1}V_1$, then we get

$$\partial_t W = \tilde{A}_1(t,\xi)W, \quad \tilde{A}_1 = M_1^{-1}A_1M_1 = \begin{pmatrix} \phi_1 & b_1 \\ \overline{b_1} & \overline{\phi_1} \end{pmatrix},$$

where

$$\phi_1 = \frac{\lambda'(t)}{2\lambda(t)} + \frac{i|\xi|(a^2(t)+\lambda^2(t))}{2\lambda(t)}, \quad b_1 = -\frac{\lambda'(t)}{2\lambda(t)} + \frac{i|\xi|(\lambda^2(t)-a^2(t))}{2\lambda(t)}.$$

Then we have

$$\begin{aligned} \partial_t|W|^2 =& 2\Re\,(W,\partial_t W)_{\mathbb{C}^2} = 2\Re\,(\phi_1)\,|W|^2 + 4\Re\,(b_1 w_1\overline{w_2}) \\ \le& \left(\frac{\lambda'(t)}{\lambda(t)} + 2|b_1|\right)|W|^2 \le \left(\frac{\lambda'(t)}{\lambda(t)} + \left|\frac{\lambda'(t)}{\lambda(t)}\right| + \frac{\left|a^2(t)-\lambda^2(t))\right|\,|\xi|}{\lambda(t)}\right)|W|^2 \\ \le& \frac{\left|a^2(t)-\lambda^2(t)\right|\,|\xi|}{\lambda(t)}|W|^2 \lesssim |a(t)-\lambda(t)||\xi||W|^2. \end{aligned}$$

By (5) and Gronwall's inequality we have

$$|W(t,\xi)|^2 \le \exp\left(C\Xi(\tilde{t}_\xi)|\xi|\right)|W(\tilde{t}_\xi,\xi)|^2 = \exp(CN)\,|W_1(\tilde{t}_\xi,\xi)|^2.$$

Thanks to $|W(t,\xi)| \simeq \lambda(t)|\xi||v(t,\xi)| + |v_t(t,\xi)|$, we have the following lemma:

Lemma 2.3 *In $Z_{\mathrm{st}}(N)$ the following estimate is established:*

$$\lambda(t)|\xi||v(t,\xi)| + |v_t(t,\xi)| \lesssim \lambda(\tilde{t}_\xi)|\xi||v(\tilde{t}_\xi,\xi)| + |v_t(\tilde{t}_\xi,\xi)|.$$

2.4 Proof of Theorem 1.1

The proof of Theorem 1.1 is concluded if the following estimate is proved in all zones:

$$a(t)|\xi||\widehat{u}(t,\xi)| \lesssim (1+t)\lambda(t)\max\left\{1, |\xi|\sqrt{\lambda(\tilde{t}_\xi)}\right\}\left(|\xi||\widehat{u}_0(\xi)| + |\widehat{u}_1(\xi)|\right). \tag{15}$$

If $|\xi| \le N/A(0)$, then the estimate (15) is trivial by Lemma 2.1.

If $(t,\xi) \in Z_{\mathrm{pd}} \cap \{|\xi| \ge N/A(0)\}$, then by using Lemma 2.1 with $t = t_\xi$ and Lemma 2.2 with $t = \tilde{t}_\xi$ we have

$$\begin{aligned} a(t)|\xi||\widehat{u}(t,\xi)| &\lesssim (1+t)a(t)|\xi|\Big(\theta(t_\xi)|\widehat{u}(t_\xi,\xi)| + |\widehat{u}_t(t_\xi,\xi)|\Big) \\ &\lesssim (1+t)a(t)|\xi|\Big(a(t_\xi)|\xi||\widehat{u}(t_\xi,\xi)| + |\widehat{u}_t(t_\xi,\xi)|\Big) \\ &\lesssim (1+t)\lambda(t)|\xi|\sqrt{\lambda(\tilde{t}_\xi)}\left(|\xi||\widehat{u}_0(\xi)| + |\widehat{u}_1(\xi)|\right). \end{aligned}$$

If $(t,\xi) \in Z_{\mathrm{st}}$, then by (3), Lemma 2.2 with $t = \tilde{t}_\xi$ and Lemma 2.3 we have

$$\begin{aligned} a(t)|\xi||\widehat{u}(t,\xi)| &\lesssim \sqrt{\lambda(\tilde{t}_\xi)}\Big(|\xi||\widehat{u}_0(\xi)| + |\widehat{u}_1(\xi)|\Big) \\ &\lesssim (1+t)\lambda(t)|\xi|\sqrt{\lambda(\tilde{t}_\xi)}\Big(|\xi||\widehat{u}_0(\xi)| + |\widehat{u}_1(\xi)|\Big). \end{aligned}$$

Let $(t,\xi) \in Z_{\mathrm{hyp}}$. If $\sqrt{\lambda(t)} \lesssim (1+t)\lambda(t)$, then (15) is trivial by Lemma 2.2. If $(1+t)\sqrt{\lambda(t)}$ is monotone decreasing, then by (3) we have

$$\sqrt{\lambda(t)} \le \frac{(1+t)\lambda(t)}{(1+\tilde{t}_\xi)\sqrt{\lambda(\tilde{t}_\xi)}} \lesssim \frac{(1+t)\lambda(t)\sqrt{\lambda(\tilde{t}_\xi)}}{A(\tilde{t}_\xi)} \lesssim (1+t)\lambda(t)|\xi|\sqrt{\lambda(\tilde{t}_\xi)}.$$

Thus we also have (15) by Lemma 2.2.

Summarizing these estimates in all zones and applying Parseval theorem we conclude the proof of Theorem 1.1. □

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Semilinear Wave Equation in the de Sitter Spacetime with Hyperbolic Spatial Part

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Abstract For the Cauchy problem for the semilinear wave equation in the de Sitter spacetime the global in time existence of the solutions is still an open problem. In this paper we give estimates for the lifespan of the solutions of semilinear wave equation in the de Sitter spacetime with flat and hyperbolic spatial parts under some conditions on the order of the nonlinearity. In the case of hyperbolic spatial part the order of nonlinearity is less than the critical value given by Strauss conjecture.

Keywords de Sitter spacetime • Lifespan • Wave equation

Mathematics Subject Classification (2010) Primary 35Q75; Secondary 35L71

1 Introduction

For the Cauchy problem for the semilinear wave equation in the de Sitter spacetime the global in time existence of the solutions is still an open problem. In this paper we give estimates for the lifespan of the solutions of the semilinear wave equation in the de Sitter spacetime with flat and hyperbolic spatial parts under some conditions on the order of the nonlinearity. In the case of hyperbolic spatial part the order of nonlinearity is less than the critical value given by Strauss conjecture.

We consider the semilinear wave equation in the spacetime, which is produced by an expanding universe, more exactly, in the de Sitter spacetime. The line element of that spacetime is as follows:

$$ds^2 = -c^2dt^2 + e^{2Ht}dr^2 + e^{2Ht}r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,.$$

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Here H is the Hubble constant. The scale factor e^{2Ht} represents an expansion. This spacetime belongs to the family of the Friedmann–Lemaître–Robertson–Walker spacetimes. For simplicity, we set $H = 1$.

The linear wave in the background generated by the metric g obeys the covariant wave equation $\square_g \psi = f$, where

$$\square_g \psi = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|g|} g^{ik} \frac{\partial \psi}{\partial x^k} \right) .$$

The fundamental solutions for the covariant wave equation in the de Sitter spacetime with flat spatial part, as a particular case of Klein–Gordon massless equation, are constructed in [10]. The $L_p - L_q$ and energy estimates for the solutions of the Cauchy problem are obtained in [10] and [2], respectively.

In the present paper we consider the Cauchy problem for the covariant semilinear wave (massless field) equation in the de Sitter spacetime

$$\begin{cases} \psi_{tt} - e^{-2t} \Delta_{\mathbb{H}} \psi + n\psi_t = F(\psi) , & x \in \mathbb{H}^n , \quad t \in [0, \infty), \\ \psi(x, 0) = \psi_0(x) , \quad \psi_t(x, 0) = \psi_1(x), & x \in \mathbb{H}^n , \end{cases} \tag{1}$$

where $\mathbb{H}^n$ is a hyperbolic space and $\Delta_{\mathbb{H}}$ is Laplace–Beltrami operator on $L^2(\mathbb{H}^n)$. The real hyperbolic spaces $\mathbb{H}^n$ are the most simple examples of noncompact Riemannian manifolds with negative curvature. For geometric reasons, one can expect better dispersive properties and, consequently, stronger results than in the Euclidean setting.

Henceforth we assume that $n = 3$. For the Cauchy problem for the semilinear wave equation in Minkowski spacetime in this case,

$$\partial_t^2 u - \Delta u = |u|^{1+\alpha} \quad u(0, x) = u_0(x) , \quad u_t(0, x) = u_1(x) ,$$

the surprising answer, which is due to John, is that for small data a global solution always exists when $\alpha > \sqrt{2}$, but does not, in general, when $\alpha < \sqrt{2}$. For the higher dimensional semilinear wave equations the following conjecture was stated by Strauss [7]: *for $n \geq 2$ blow-up for all data if $p < p_n$ and global existence for all small data, if $p > p_n$*. Here $p = \alpha + 1$, and p_n is the positive root of the equation $(n-1)p_n^2 - (n+1)p_n - 2 = 0$. (For the history of the results which have validated Strauss's conjecture, see, e.g., [3, 6] and the bibliography therein.)

Consider now the semilinear wave equation in the de Sitter spacetime

$$\square_g u = F(u) \quad u(0, x) = u_0(x) , \quad u_t(0, x) = u_1(x) .$$

To the best of our knowledge, the question of the small data global solution is not examined for this equation. To formulate the results of this paper we need the following description of the nonlinear term.

Condition ($\mathcal{L}$) *The function* $F : \mathbb{R} \longrightarrow \mathbb{R}$ *is said to be Lipschitz continuous with exponent* α*, if there exist* $\alpha \geq 0$ *and* $C > 0$ *such that*

$$|F(\psi_1) - F(\psi_2)| \leq C|\psi_1 - \psi_2| \left(|\psi_1|^\alpha + |\psi_2|^\alpha\right) \quad \textit{for all } \psi_1, \psi_2 \in \mathbb{R}\,.$$

First we consider the semilinear covariant wave equation in the de Sitter spacetime with flat spatial part

$$\begin{cases} \psi_{tt} - e^{-2t}\Delta\psi + 3\psi_t = F(\psi), \quad x \in \mathbb{R}^3\,,\ t \in I, \\ \psi(x,0) = \psi_0(x)\,, \quad \psi_t(x,0) = \psi_1(x), \quad x \in \mathbb{R}^3\,. \end{cases} \tag{2}$$

For $\gamma < 2$, $(\gamma - 1)(\alpha + 1) > -1$ define the function

$$R_{\alpha,\gamma}(t) := e^{-1/t} + \max_{0\leq\tau\leq t} \widetilde{R}_{\alpha,\gamma}(\tau)\,, \quad R_{\alpha,\gamma}(0) := 0\,,$$

with the domain $D(R_{\alpha,\gamma}) = (0,\infty)$, where

$$\begin{aligned} \widetilde{R}_{\alpha,\gamma}(t) := (1 - e^{-t})^{\gamma-1} e^{-t} \int_0^t e^{2b}(e^{-b} - e^{-t})^{1-\gamma}(1 - e^{-b})^{(\gamma-1)(\alpha+1)}\, db \\ + (1 - e^{-t})^{\gamma-1} \int_0^t e^{3b}(e^{-b} - e^{-t})^{3-\gamma}(1 - e^{-b})^{(\gamma-1)(\alpha+1)}\, db\,. \end{aligned}$$

We do not know whether the function $\widetilde{R}_{\alpha,\gamma}(t)$ is increasing, but computer simulations suggest that there is $t_0 > 0$ such that the function $\widetilde{R}_{\alpha,\gamma}(t)$ is monotonically increasing function on $[t_0,\infty)$. We denote by $\mathcal{R}^{-1}_{\alpha,\gamma}$ the inverse of $R_{\alpha,\gamma}(t)$ function. The following theorem provides an estimate for the lifespan of the solution of problem (2).

Theorem 1.1 *Assume that the condition* ($\mathcal{L}$) *is fulfilled,* $1 \leq \alpha < 4$ *and* $\gamma = 3\alpha/(\alpha + 2)$*. Then the lifespan* T_{ls} *of the solution* $\psi(t) \in L^q(\mathbb{R}^3)$ *of the Cauchy problem (2), with* $\psi_0, \psi_1 \in C_0^\infty(\mathbb{R}^3)$ *can be estimated as follows*

$$T_{ls} \geq \mathcal{R}^{-1}_{\alpha,\gamma}\left(C\left(\|\psi_0\|_{H^{1,p}} + \|\psi_1\|_{L^p}\right)^{-\alpha}\right)\,,$$

where C *is a positive constant.*

Now we turn to the semilinear wave equation in the de Sitter spacetime with hyperbolic spatial part. Let $\gamma \in (0,1)$, $s = 2\gamma$, $q = \frac{2}{1-\gamma}$, $p = \frac{2}{1+\gamma}$. To formulate the next result we define the function $\omega_\gamma(t)$ as in [4]:

$$\omega_\gamma(t) = \begin{cases} |t|^\gamma + |t|^{3/2}\,, \quad \frac{1}{2} \leq \gamma < 1\,, \\ |t|^\gamma + |t|^{3\gamma}\,, \quad 0 \leq \gamma \leq \frac{1}{2}\,. \end{cases} \tag{3}$$

Consider now the Cauchy problem (1) with the nonlinear term $F(\psi)$ satisfying the condition $(\mathcal{L})$. Define the function

$$R_{\alpha,\gamma,\mathrm{hyp}}(t) := e^{-1/t} + \max_{0\le\tau\le t} \widetilde{R}_{\alpha,\gamma,\mathrm{hyp}}(\tau)\,,\; R_{\alpha,\gamma,\mathrm{hyp}}(0) := 0\,,$$

where

$$\widetilde{R}_{\alpha,\gamma,\mathrm{hyp}}(t) := e^{-1/t} + \omega_\gamma(1-e^{-t})e^{-t}\int_0^t \omega_\gamma^{-1}(e^{-b}-e^{-t})\omega_\gamma^{-1-\alpha}(1-e^{-b})e^{2b}\,db$$
$$+\,\omega_\gamma(1-e^{-t})\int_0^t e^{3b}\omega_\gamma^{-1-\alpha}(1-e^{-b})\,db\int_0^{e^{-b}-e^{-t}} \omega_\gamma^{-1}(r)r\,dr.$$

Denote by $t = \mathcal{R}^{-1}_{\alpha,\gamma,\mathrm{hyp}}(\varepsilon)$ its inverse function. The following theorem provides an estimate for the lifespan of the solution of (1).

Theorem 1.2 *Assume that the condition $(\mathcal{L})$ is fulfilled, $\gamma(1+\alpha) < 1$ and $q = \alpha + 2$. Then the lifespan T_{ls} of the solution $\psi(t) \in L^q(\mathbb{H}^3)$ of the Cauchy problem (1), with $\psi_0, \psi_1 \in C_0^\infty(\mathbb{H}^3)$ can be estimated as follows*

$$T_{ls} \ge \mathcal{R}^{-1}_{\alpha,\gamma,\mathrm{hyp}}\left(C\left(\|\psi_0\|_{H^{1,p}} + \|\psi_1\|_{L^p}\right)^{-\alpha}\right)\,,$$

where C is a positive constant.

The condition $\gamma(1+\alpha) < 1$ on the order of nonlinearity in the last theorem implies $\alpha < \sqrt{2}$, which is the condition due to John. It would be interesting to expand Theorems 1.1 and 1.2 to the higher dimensional case and state an analogue of Strauss's conjecture for the semilinear covariant wave equation in the de Sitter spacetime.

2 Proof of Theorem 1.1

We use Yagdjian's integral transform of [9] to prove Theorems 1.1 and 1.2. That integral transform creates a bridge between solutions of wave equation in the de Sitter spacetime and the wave equation in Minkowski spacetime. Through this transform we derive, in particular, the estimates for the solutions.

Let Ω be a domain in $\mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^n$, while $A(x,\partial_x)$ is a partial differential operator $A(x,\partial_x) = \sum_{|\alpha|\le m} a_\alpha(x) D_x^\alpha$ with smooth coefficients. For $g \in C^\infty(\Omega \times I)$, $I = [0,T]$, $0 < T \le \infty$, and φ_0, $\varphi_1 \in C_0^\infty(\Omega)$, let the function $v_g(x,t;b)$ be a solution to the problem

$$\begin{cases} v_{tt} - A(x,\partial_x)v = 0\,, & x \in \Omega\,, \quad t \in [0, 1-e^{-T}]\,,\\ v(x,0;b) = g(x,b)\,, & v_t(x,0;b) = 0\,, \quad b \in I, \quad x \in \Omega\,, \end{cases}$$

and let the function $v_\varphi = v_\varphi(x,t)$ be a solution of the problem

$$\begin{cases} v_{tt} - A(x,\partial_x)v = 0, & x \in \Omega\,, \quad t \in [0, 1-e^{-T}]\,, \\ v(x,0) = \varphi(x), \quad v_t(x,0) = 0\,, & x \in \Omega\,. \end{cases}$$

Then the function $u = u(x,t)$ defined by

$$\begin{aligned} u(x,t) = {} & 2\int_0^t db \int_0^{\phi(t)-\phi(b)} v_g(x,r;b)\frac{1}{4}e^{\frac{3}{2}(b+t)}\left((e^{-2b}+e^{-2t})-r^2\right) dr \\ & + e^{\frac{t}{2}} v_{\varphi_0}(x,\phi(t)) + 2\int_0^{\phi(t)} v_{\varphi_0}(x,s)\frac{1}{8}e^{-\frac{t}{2}}\left(\left(3s^2+1\right)e^{2t}-3\right) ds \\ & + 2\int_0^{\phi(t)} v_{\varphi_1}(x,s)\frac{1}{4}e^{-\frac{t}{2}}\left(1-e^{2t}\left(s^2-1\right)\right) ds, \quad x \in \Omega,\ t \in I\,, \end{aligned}$$

where $\phi(t) := 1 - e^{-t}$, according to Theorem 2.1 [9] solves the problem

$$\begin{cases} u_{tt} - e^{-2t}A(x,\partial_x)u - \frac{9}{4}u = g, & x \in \Omega\,,\ t \in I, \\ u(x,0) = \varphi_0(x)\,, \quad u_t(x,0) = \varphi_1(x), & x \in \Omega\,. \end{cases}$$

Consequently, the function $\psi(x,t) = e^{-\frac{3}{2}t}u(x,t)$ solves the problem for the covariant wave equation:

$$\begin{cases} \psi_{tt} - e^{-2t}A(x,\partial_x)\psi + 3\psi_t = f, & x \in \Omega\,,\ t \in I, \\ \psi(x,0) = \psi_0(x)\,, \quad \psi_t(x,0) = \psi_1(x), & x \in \Omega\,, \end{cases}$$

where $g = e^{\frac{3}{2}t}f,\ \varphi_0 = \psi_0,\ \varphi_1 = \frac{3}{2}\psi_0 + \psi_1$. Then,

$$\begin{aligned} \psi(x,t) = {} & \frac{1}{2}\int_0^t db \int_0^{\phi(t)-\phi(b)} v_f(x,r;b)e^{3b}\left(e^{-2b}+e^{-2t}-r^2\right) dr \\ & + e^{-t}v_{\psi_0}(x,\phi(t)) + \int_0^{\phi(t)} v_{\psi_0}(x,s)ds \\ & + \frac{1}{2}\int_0^{\phi(t)} v_{\psi_1}(x,s)\left(1+e^{-2t}-s^2\right) ds, \quad x \in \Omega,\ t \in I\,, \end{aligned}$$

where $\phi(t) := 1 - e^{-t}$ (see [8, 9]). In order to reduce the integration in the above formula, we can appeal to the solutions $\mathcal{V}_g = \mathcal{V}_g(x,t;b)$ and $\mathcal{V}_\varphi = \mathcal{V}_\varphi(x,t)$ of the problems

$$\begin{cases} \mathcal{V}_{tt} - A(x,\partial_x)\mathcal{V} = 0\,, & x \in \Omega\,, \quad t \in [0, 1-e^{-T}]\,, \\ \mathcal{V}(x,0;b) = 0\,, \quad \mathcal{V}_t(x,0;b) = g(x,b)\,, & b \in I, \quad x \in \Omega\,, \end{cases}$$

and

$$\begin{cases} \mathcal{V}_{tt} - A(x, \partial_x)\mathcal{V} = 0, \quad x \in \Omega, \quad t \in [0, 1 - e^{-T}], \\ \mathcal{V}(x, 0) = 0, \quad \mathcal{V}_t(x, 0) = \varphi(x), \quad x \in \Omega, \end{cases}$$

respectively. Then $v_g(x, t; b) = \partial_t \mathcal{V}_g(x, t; b)$, $v_\varphi(x, t) = \partial_t \mathcal{V}_\varphi(x, t)$ and

$$\begin{aligned} \psi(x, t) = e^{-t} \int_0^t \mathcal{V}_f(x, e^{-b} - e^{-t}; b) e^{2b} \, db \\ + \int_0^t db \, e^{3b} \int_0^{e^{-b} - e^{-t}} \mathcal{V}_f(x, r; b) r \, dr \\ + e^{-t} v_{\psi_0}(x, 1 - e^{-t}) + \mathcal{V}_{\psi_0}(x, 1 - e^{-t}) \\ + e^{-t} \mathcal{V}_{\psi_1}(x, 1 - e^{-t}) + \int_0^{1 - e^{-t}} \mathcal{V}_{\psi_1}(x, s) s \, ds . \end{aligned} \tag{4}$$

The local existence of the solution of strictly hyperbolic semilinear equation is known (see, e.g., [5, 6]). Using estimates (5) one can easily prove that the solution to (2) can be extended as long as it remains bounded in L^q. For the convergence of the integrals in the definition of the function $\widetilde{R}_{\alpha,\gamma}(t)$ we have assumed $1 - \gamma > -1$, $(\gamma - 1)(\alpha + 1) > -1$ that is $0 < \alpha < 4$. From the representation formula (4) we have

$$\begin{aligned} \|\psi(x, t)\|_{L^q} \le e^{-t} \int_0^t \|\mathcal{V}_f(x, e^{-b} - e^{-t}; b)\|_{L^q} e^{2b} \, db \\ + \int_0^t db \, e^{3b} \int_0^{e^{-b} - e^{-t}} \|\mathcal{V}_f(x, r; b)\|_{L^q} r \, dr \\ + e^{-t} \|v_{\psi_0}(x, 1 - e^{-t})\|_{L^q} + \|\mathcal{V}_{\psi_0}(x, 1 - e^{-t})\|_{L^q} \\ + e^{-t} \|\mathcal{V}_{\psi_1}(x, 1 - e^{-t})\|_{L^q} + \int_0^{1 - e^{-t}} \|\mathcal{V}_{\psi_1}(x, s)\|_{L^q} s ds . \end{aligned} \tag{5}$$

Using $L^p - L^q$ decay estimates for the solutions of the wave equation in the Minkowski spacetime (see, e.g., [1]) with $q = \alpha + 2$, $p = \frac{\alpha+2}{\alpha+1}$, $3\left(\frac{1}{p} - \frac{1}{q}\right) = \frac{3\alpha}{\alpha+2} = \gamma$, we obtain

$$\begin{aligned} \|\psi(x, t)\|_{L^q} \le e^{-t} \int_0^t (e^{-b} - e^{-t})^{1 - 3\left(\frac{1}{p} - \frac{1}{q}\right)} \|F(\psi((x, b))\|_{L^p} e^{2b} \, db \\ + \int_0^t db \, e^{3b} \int_0^{e^{-b} - e^{-t}} r^{1 - 3\left(\frac{1}{p} - \frac{1}{q}\right)} \|F(\psi((x, b))\|_{L^p} r \, dr \end{aligned}$$

$$+\,2(1-e^{-t})^{1-3\left(\frac{1}{p}-\frac{1}{q}\right)}\|\psi_0\|_{H^{1+2s,p}}$$
$$+\,e^{-t}(1-e^{-t})^{1-3\left(\frac{1}{p}-\frac{1}{p}\right)}\|\psi_1\|_{L^p}+\int_0^{1-e^{-t}} r^{1-3\left(\frac{1}{p}-\frac{1}{q}\right)}\|\psi_1\|_{L^p}\,rdr.$$

Since $1-3\left(\frac{1}{p}-\frac{1}{q}\right)>-1$ then

$$\begin{aligned}\|\psi(x,t)\|_{L^q} \le{}& e^{-t}\int_0^t (e^{-b}-e^{-t})^{1-3\left(\frac{1}{p}-\frac{1}{q}\right)}\|\psi(b)\|_{L^q}^{1+\alpha}e^{2b}\,db\\ &+\int_0^t e^{3b}\|\psi(b)\|_{L^q}^{1+\alpha}(e^{-b}-e^{-t})^{3-3\left(\frac{1}{p}-\frac{1}{q}\right)}\,db\\ &+c_0(1-e^{-t})^{1-3\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\|\psi_0\|_{H^{1+2s,p}}+\|\psi_1\|_{L^p}\right).\end{aligned}$$

It follows

$$\begin{aligned}&(1-e^{-t})^{\gamma-1}\|\psi(x,t)\|_{L^q} \le (1-e^{-t})^{\gamma-1}e^{-t}\\ &\quad\times\int_0^t (e^{-b}-e^{-t})^{1-\gamma}(1-e^{-b})^{(\gamma-1)(\alpha+1)}\left[(1-e^{-b})^{\gamma-1}\|\psi(b)\|_{L^q}\right]^{1+\alpha}e^{2b}db\\ &\quad+(1-e^{-t})^{\gamma-1}\\ &\quad\times\int_0^t e^{3b}(1-e^{-b})^{(\gamma-1)(\alpha+1)}\left[(1-e^{-b})^{\gamma-1}\|\psi(b)\|_{L^q}\right]^{1+\alpha}(e^{-b}-e^{-t})^{3-\gamma}db\\ &\quad+c_0\left[\|\psi_0\|_{H^{1+2s,p}}+\|\psi_1\|_{L^p}\right].\end{aligned}$$

For $\gamma\ge 1$ we define $E_q(t):=\sup_{\tau\in[0,t]}(1-e^{-\tau})^{\gamma-1}\|\psi(x,\tau)\|_{L^q}$, then

$$E_q(t)\le E_q(t)^{1+\alpha}R_{\alpha,\gamma}(t)+c_0\left(\|\psi_0\|_{H^{1+2s,p}}+\|\psi_1\|_{L^p}\right).$$

We set

$$T_\varepsilon:=\inf\{t\,:\,E_q(t)\ge 2\varepsilon\},\ \varepsilon:=c_0\left(\|\psi_0\|_{H^{1+2s,p}}+\|\psi_1\|_{L^p}\right).$$

Then for every $\varepsilon>0$ we have

$$2\varepsilon\le\varepsilon+2^{1+\alpha}\varepsilon^{1+\alpha}R_{\alpha,\gamma}(T_\varepsilon)$$

and $T_\varepsilon\ge\mathcal{R}_{\alpha,\gamma}^{-1}(\varepsilon^{-\alpha}2^{-\alpha-1})$. Theorem is proved. □

3 Proof of Theorem 1.2

Consider the Cauchy problem for the linear wave equation

$$(\partial_t^2 - \Delta_{\mathbb{H}})u = g\,, \quad u(0,x) = u_0\,, \quad u_t(0,x) = u_1\,,$$

on $R_+ \times \mathbb{H}^3$, where $\mathbb{H}^3$ is $3D$ hyperbolic space with constant sectional curvature -1, and $\Delta_{\mathbb{H}}$ represents its Laplace–Beltrami operator. Metcalfe and Taylor in [4] proved the following dispersive estimates:

$$\left\| \frac{\sin(t\sqrt{-\Delta_{\mathbb{H}}})}{\sqrt{-\Delta_{\mathbb{H}}}} u_0 \right\|_{H^{1-s,q}} \le \frac{C}{\omega_\gamma(t)} \|u_0\|_{L^p}\,, \tag{6}$$

$$\left\| \cos(t\sqrt{-\Delta_{\mathbb{H}}}) u_1 \right\|_{H^{-s,q}} \le \frac{C}{\omega_\gamma(t)} \|u_1\|_{L^p}\,, \tag{7}$$

where $\gamma \in (0,1)$, $s = 2\gamma$, $q = \frac{2}{1-\gamma}$, $p = \frac{2}{1+\gamma}$, and $\omega_\gamma(t)$ is given by (3).

From the representation (4) and the estimates (6), (7) for $s \le 1$ we have

$$\begin{aligned}
\|\psi(x,t)\|_{H^{1-s,q}} &\le e^{-t} \int_0^t \omega_\gamma^{-1}(e^{-b} - e^{-t}) \|f(b)\|_{L^p} e^{2b}\, db \\
&+ \int_0^t db\, e^{3b} \int_0^{e^{-b}-e^{-t}} \omega_\gamma^{-1}(r) \|f(b)\|_{L^p} r\, dr \\
&+ e^{-t}\omega_\gamma^{-1}(1-e^{-t})\|\psi_0\|_{H^{2s,p}} + \omega_\gamma^{-1}(1-e^{-t})\|\psi_0\|_{L^p} \\
&+ e^{-t}\omega_\gamma^{-1}(1-e^{-t})\|\psi_1\|_{L^p} + \int_0^{1-e^{-t}} \omega_\gamma^{-1}(s)\|\psi_1\|_{L^p}\, s\, ds\,.
\end{aligned}$$

Hence

$$\begin{aligned}
\|\psi(x,t)\|_{L^q} \le \|\psi(x,t)\|_{H^{1-s,q}} &\le e^{-t} \int_0^t \omega_\gamma^{-1}(e^{-b} - e^{-t}) \|f(b)\|_{L^p} e^{2b}\, db \\
&+ \int_0^t db\, e^{3b} \|f(b)\|_{L^p} \int_0^{e^{-b}-e^{-t}} \omega_\gamma^{-1}(r)\, r\, dr \\
&+ e^{-t}\omega_\gamma^{-1}(1-e^{-t})\|\psi_0\|_{H^{2s,p}} + \omega_\gamma^{-1}(1-e^{-t})\|\psi_0\|_{L^p} \\
&+ e^{-t}\omega_\gamma^{-1}(1-e^{-t})\|\psi_1\|_{L^p} + \|\psi_1\|_{L^p} \int_0^{1-e^{-t}} \omega_\gamma^{-1}(s)\, s\, ds\,.
\end{aligned}$$

Denote $\varepsilon := c_0\|\psi_0\|_{H^{2s,p}} + \|\psi_1\|_{L^p}$, and

$$E_{s,p}(t) := \max_{\tau\in[0,t]} \omega_\gamma(1-e^{-\tau})\|\psi(x,\tau)\|_{H^{s,p}}\,.$$

Then

$$
\begin{aligned}
E_{0,q}(t) \le \omega_\gamma(1-e^{-t})e^{-t}\int_0^t \omega_\gamma^{-1}(e^{-b}-e^{-t})\omega_\gamma^{-1-\alpha}(1-e^{-b})E_{0,q}^{1+\alpha}(b)e^{2b}\,db \\
+\,\omega_\gamma(1-e^{-t})\int_0^t db\, e^{3b}\omega_\gamma^{-1-\alpha}(1-e^{-b})E_{0,q}^{1+\alpha}(b) \\
\times\int_0^{e^{-b}-e^{-t}} \omega_\gamma^{-1}(r)r\,dr+\varepsilon \le \varepsilon + E_{0,q}^{1+\alpha}(t)R_{\alpha,\gamma}(t)\,.
\end{aligned}
$$

Now we examine the function $R_{\alpha,\gamma,\mathrm{hyp}}(t)$. Since the arguments of the function ω_γ in those integrals are bounded we can set $\omega_\gamma(t)=t^\gamma$ and consequently

$$
\begin{aligned}
\widetilde{R}_{\alpha,\gamma,\mathrm{hyp}}(t) \simeq (1-e^{-t})^\gamma e^{-t}\int_0^t (e^{-b}-e^{-t})^{-\gamma}(1-e^{-b})^{(-1-\alpha)\gamma}e^{2b}\,db \\
+\frac{1}{2-\gamma}(1-e^{-t})^\gamma\int_0^t (e^{-b}-e^{-t})^{2-\gamma}(1-e^{-b})^{(-1-\alpha)\gamma}e^{3b}db.
\end{aligned}
$$

Here $\gamma=\alpha/(\alpha+2)$. The condition $\gamma(1+\alpha)<1$ of the convergence of the last integral implies $\alpha<\sqrt{2}$. Set

$$
T_\varepsilon := \inf\{t\,:\,E_{0,q}(t)\ge 2\varepsilon\}\,.
$$

Then for $\varepsilon>0$ we have

$$
2\varepsilon \le \varepsilon + 2^{1+\alpha}\varepsilon^{1+\alpha}R_{\alpha,\gamma,\mathrm{hyp}}(T_\varepsilon)\,.
$$

The desired estimate for the lifespan follows from the last inequality. Indeed, there is a $C>0$ such that the inequality $C\varepsilon^{-\alpha}\le R_{\alpha,\gamma,\mathrm{hyp}}(T_\varepsilon)$ implies $T_\varepsilon \ge R_{\alpha,\gamma,\mathrm{hyp}}^{-1}(C\varepsilon^{-\alpha})$. Theorem is proved. □

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Note on Backward Uniqueness for a Class of Parabolic Equations

Christian P. Jäh

Abstract In this note, we review some recent results on the backward uniqueness for solutions of parabolic equations of second and higher order. The main focus is the connection of the backward uniqueness property with the regularity of the principal part coefficients measured by moduli of continuity. We announce a new backward uniqueness result for higher order equations.

Keywords Backward uniqueness • Bony's paraproduct • Carleman estimates • Higher order equations • Parabolic equations • Rough coefficients

Mathematics Subject Classification (2010) Primary 35Bxx, 35Kxx; Secondary 35K25, 35K30

1 Introduction

We consider parabolic equations of the type

$$Pu = \partial_t u + \sum_{0\le|\alpha|,|\beta|\le m} (-1)^{|\alpha|}\partial_x^\alpha \left(a_{\alpha\beta}(t,x)\partial_x^\beta u\right) = 0 \tag{1}$$

on the strip $[0,T]\times\mathbb{R}^n_x$ with $m\in\mathbb{N}$. The α and β are n-multiindices. We assume $a_{\alpha\beta}(t,x) = \overline{a_{\beta\alpha}(t,x)}$ for all $0\le|\alpha|,|\beta|\le m$ and the $a_{\beta\alpha}$ are supposed to be real for $|\alpha|=|\beta|=m$ on $[0,T]\times\mathbb{R}^n_x$. We assume that there exists a $\kappa\in(0,1]$ such that $\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(t,x)\xi^\alpha\xi^\beta \ge \kappa|\xi|^{2m}$ for all $(t,x,\xi)\in[0,T]\times\mathbb{R}^n_x\times\mathbb{R}^n_\xi$.

By saying that P has the *backward uniqueness property*, we mean the following: Given $u\in\mathcal{H}$, $Pu=0$ on $[0,T]\times\mathbb{R}^n_x$ with $u(T,x)=0$ on $\mathbb{R}^n_x$, then it follows that $u=0$ on $[0,T]\times\mathbb{R}^n_x$. The space $\mathcal{H}$ is an appropriate function space for the problem

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at hand. Here, we shall prove the backward uniqueness properties with respect to the space

$$\mathcal{H}^m := H^1([0,T], L^2(\mathbb{R}^n_x)) \cap L^2([0,T], H^{2m}(\mathbb{R}^n_x)).$$

In [10], Lions and Malgrange proved backward uniqueness for P in $\mathcal{H}^m$ under the condition that the $a_{\alpha\beta}$ are Lipschitz continuous with respect to t and sufficiently smooth with respect to x. The latter requirement is to make the operator fall into the abstract framework in which the authors were working. In that paper, the authors also raised the question whether or not the Lipschitz regularity with respect to t is really necessary or could be replaced by simple continuity. In [1], Bardos and Tatar proved essentially the same result as Lions and Malgrange replacing Lipschitz continuity with absolute continuity. In [12], Miller showed that a certain amount of regularity with respect to t is necessary for the backward uniqueness property to hold in the case $m = 1$. He constructed a counterexample with $\frac{1}{6}$-Hölder continuous principal part coefficients.

For $m = 1$, Del Santo and Prizzi proved in [3] the backward uniqueness property for P assuming the so-called Osgood condition for the modulus of continuity with respect to t. More precisely, let $\mu : [0,1] \to [0,1]$ be a modulus of continuity, i.e. a continuous, concave, and increasing function with $\mu(0) = 0$. If the principal part coefficients belong to $C^\mu([0,T], L^\infty(\mathbb{R}^n_x)) \cap L^\infty([0,T], B^2(\mathbb{R}^n_x))$, then uniqueness holds in $\mathcal{H}^2$ if μ satisfies the Osgood condition

$$\int_0^1 \frac{ds}{\mu(s)} = +\infty. \tag{2}$$

The high regularity with respect to x was due to a difficult commutator estimate arising from the use of the Littlewood–Paley decomposition in the proof of a Carleman estimate needed for the uniqueness proof. This was overcome in [5], where the authors assumed that the principal part coefficients belong to the space $C^\mu([0,T], L^\infty(\mathbb{R}^n_x)) \cap L^\infty([0,T], \mathrm{Lip}(\mathbb{R}^n_x))$. The Carleman estimate proved in [5] is on the level of H^{-s}, $s \in (0,1)$ instead of the usual L^2. The precise statement is

Proposition 1.1 (Proposition 3.1 [5]) *Let $s \in (0,1)$ and μ be a modulus of continuity satisfying (2). Assume further that, for all $j,k = 1,\dots,n$,*

$$a_{jk} \in C^\mu([0,T], L^\infty(\mathbb{R}^n_x)) \cap L^\infty([0,T], \mathrm{Lip}(\mathbb{R}^n_x)).$$

Then there exists a strictly increasing C^2-function $\Phi : [0,+\infty) \to [0,+\infty)$ such that there exists a $\gamma_0 \geq 1$ such that

$$\int_0^{T/2} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} \|\partial_t u + \sum_{j,k=1}^n \partial_{x_j}(a_{jk}(t,\cdot)\partial_{x_k} u)\|^2_{H^{-s}} dt$$

$$\gtrsim \gamma^{\frac{1}{2}} \int_0^{T/2} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} (\|\nabla_x u\|^2_{H^{-s}} + \gamma^{\frac{1}{2}} \|u\|^2_{H^{-s}}) dt$$

for all $u \in C_0^\infty(\mathbb{R}_t \times \mathbb{R}^n_x)$ with $\mathrm{supp}(u) \subseteq [0, T/2] \times \mathbb{R}^n_x$ and all $\gamma \geq \gamma_0$.

Remark 1.2 The weight function Φ is coupled to the modulus of continuity as follows. Let μ be the Osgood modulus of continuity. Then set

$$\eta(t) := \int_{\frac{1}{t}}^{1} \frac{1}{\mu(s)} ds, \quad t \geq 1$$

$$\Phi(\tau) := \int_0^{\tau} \eta^{-1}(t) dt, \quad \tau \geq 0.$$

The function Φ is differentiable and, thanks to the Osgood condition, defined on $[0, +\infty)$. It also satisfies the nonlinear ordinary differential equation $\Phi'' = \mu(1/\Phi')(\Phi')^2$. This connection was first described in [14] for a Carleman estimate for a uniqueness result for the solutions of the Cauchy problem for second order elliptic equations with non-Lipschitz coefficients.

Remark 1.3 Replacing Proposition 3.5 in [5] by Theorem 2.5.8 in [11], one can recover the L^2 estimates, i.e. Proposition 1.1 holds true for $s = 0$. In this case, the second order equations with all lower order terms with coefficients can be treated merely in $L^\infty([0, T] \times \mathbb{R}^n_x)$. Compared to that in [5], it is not only the terms of zero order.

Finally, in [6], the authors proved a uniqueness result assuming that the principal part coefficients belong to $C^\mu([0, T], L^\infty(\mathbb{R}^n_x)) \cap L^\infty([0, T], C^\omega(\mathbb{R}^n_x))$, where μ and ω are moduli of continuity, μ satisfies (2), and ω is given by $\omega(s) = \sqrt{\mu(s^2)}$. Unless in the case $\omega(s) = s$, the Carleman estimate will be at the level of a Sobolev space of negative order.

Proposition 1.4 (Proposition 7 in [6]) *Let μ and ω be two moduli of continuity such that $\omega(s) = \sqrt{\mu(s^2)}$. Suppose that μ satisfies the Osgood condition (2). Suppose moreover that there exists a positive constant C such that $\int_0^h \frac{\omega(t)}{t} dt \leq C\omega(h)$, $\frac{\omega(2^{-q})}{\omega(2^{-p})} \leq C\omega(2^{p-q})$ for $1 \leq p \leq q - 1$, and, for all $s \in (0, 1)$, $\sum_{k=0}^{+\infty} 2^{(1-s)k} \omega(2^{-k}) < +\infty$. Assume further that, for all $j, k = 1, \ldots, n$,*

$$a_{jk} \in C^\mu([0, T], L^\infty(\mathbb{R}^n_x)) \cap L^\infty([0, T], C^\omega(\mathbb{R}^n_x)).$$

Let $s \in (0, 1)$. Then there exists a strictly increasing C^2-function $\Phi : [0, +\infty) \to [0, +\infty)$ such that there exists a $\gamma_0 \geq 1$ such that

$$\int_0^{T/2} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} \left\| \partial_t u + \sum_{j,k=1}^{n} \partial_{x_j}(a_{jk}(t, \cdot)\partial_{x_k} u) \right\|^2_{H^{-s}} dt$$

$$\gtrsim \gamma^{1/4} \int_0^{T/2} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} \left(\|\nabla_x u\|^2_{H^{-s}_\omega} + \gamma^{3/4} \|u\|^2_{L^2} \right) dt.$$

for all $u \in C_0^\infty(\mathbb{R}_t \times \mathbb{R}^n_x)$ with $\operatorname{supp}(u) \subseteq [0, T/2] \times \mathbb{R}^n_x$ and all $\gamma \geq \gamma_0$.

The weight function in this Carleman estimate is the same as described in Remark 1.2 and the space $H_\omega^{-s}(\mathbb{R}_x^n)$ is defined by the Littlewood–Paley decomposition (see, e.g., [11]):

$$\|u\|^2_{H_\omega^{-s}} := \sum_{\nu\geq 0} 2^{2(1-s)\nu}\omega^2(2^{-\nu})\|\Delta_\nu u\|^2_{L^2} < +\infty.$$

In this paper, we announce a generalization of the above-mentioned results in [5, 6] to operators of type (1) with $m \geq 2$. The full proof, along with related results, will be published elsewhere [7]. The only existing result of this type, that the author is aware of, is [4], where the operator

$$P = \partial_t + \sum_{0\leq|\alpha|\leq 2m} i^{|\alpha|}\partial_x^\alpha \tag{3}$$

is considered. The precise statement there is

Proposition 1.5 (Proposition 2.1 in [4]) *Let μ be a modulus of continuity satisfying (2) and $a_\alpha \in C^\mu[0,T]$. Then there exists a strictly increasing C^2-function $\Phi : [0,+\infty) \to [0,+\infty)$ such that there exists a $\gamma_0 \geq 1$ such that*

$$\int_0^{T/2} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))}\Big\|\partial_t u - \sum_{0\leq|\alpha|\leq 2m} i^{|\alpha|}a_\alpha(t)\partial_x^\alpha u\Big\|^2_{L^2}dt$$
$$\gtrsim \gamma^{\frac{1}{2}}\int_0^{T/2} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))}\|u\|^2_{H^m}dt$$

for all $u \in C_0^\infty(\mathbb{R}_t \times \mathbb{R}_x^n)$ with $\operatorname{supp}(u) \subseteq [0,T/2]\times\mathbb{R}_x^n$ and all $\gamma \geq \gamma_0$.

From the Carleman estimate follows that P in (3) has the backward uniqueness property. The weight function is again the same as in Remark 1.2.

Our description of the history of the problem that we sketched here is by no means exhaustive. To get a better overview over the literature, the reader may consult the works referenced in the above cited works as well as in [8, 9]. In [15], Tarama proved a similar result to the one in [5] but replacing the regularity measurement of the principal coefficients with respect to t by a modulus of continuity by bounded variation. It would be interesting to see whether this result holds also for higher order operators.

Modifying a well-known counterexample of Pliś [13], Del Santo and Prizzi proved in [3, 4] that the regularity assumptions with respect to t in the above backward uniqueness results are essentially sharp. See Theorem 1.2 and Remark 1.2 in [4]. Up to now there are no counterexamples in the literature that involve x in the principal part to show the sharpness of the assumptions with respect to x.

Notation By $B^2(\mathbb{R}_x^n)$, we denote the twice differentiable functions on $\mathbb{R}_x^n$ which are bounded with all derivatives of order ≤ 2. Given a modulus of continuity μ, $C^\mu[0,T]$

denotes the space of continuous functions f that satisfy $|f(s) - f(t)| \leq C\mu(|t-s|)$. The space $W^{k,\infty}(\mathbb{R}^n_x)$ denotes the space of all functions $f \in L^\infty(\mathbb{R}^n_x)$ such that all distributional derivatives $\partial^\gamma_x u \in L^\infty(\mathbb{R}^n_x)$ for $|\gamma| \leq k$.

2 The Main Result

The main result of this paper is

Theorem 2.1 (Uniqueness) *Consider the operator P, defined by the equation*

$$Pu = \partial_t u + \sum_{|\alpha|,|\beta|=m} (-1)^{|\alpha|} \partial^\alpha_x \left(a_{\alpha\beta}(t,x)\partial^\beta_x u\right) + \sum_{|\gamma|\leq m} b_\gamma(t,x)\partial^\gamma_x = 0 \tag{4}$$

with real $a_{\alpha\beta}$ and possibly complex b_γ. We assume that

- $a_{\alpha\beta} \in C^\mu([0,T], L^\infty(\mathbb{R}^n_x)) \cap L^\infty([0,T], W^{m,\infty}(\mathbb{R}^n_x))$,
- $b_\gamma \in L^\infty([0,T] \times \mathbb{R}^n_x)$,
- *there exists a* $\kappa \in (0,1]$ *such that* $\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(t,x)\xi^\alpha\xi^\beta \geq \kappa|\xi|^{2m}$ *for all* $(t,x,\xi) \in [0,T] \times \mathbb{R}^n_x \times \mathbb{R}^n_\xi$.

Then P has the backward uniqueness property in $\mathcal{H}^m$, i.e. for $u \in \mathcal{H}^m$, $Pu = 0$ on $[0,T] \times \mathbb{R}^n_x$ and $u(T,x) = 0$ on $\mathbb{R}^n_x$ it follows $u = 0$ on $[0,T] \times \mathbb{R}^n_x$.

This result is an extension of [4] generalizing [5] to higher order operators. As in the other cases it follows in a standard way from an appropriate Carleman estimate.

Proposition 2.2 (Carleman Estimate) *Let μ be a modulus of continuity satisfying (2). There exists a strictly increasing C^2-function $\Phi : [0,+\infty) \to [0,+\infty)$ and a $\gamma_0 \geq 1$ such that*

$$\begin{aligned}\int_0^{T/2} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} \left\| \partial_t u - \sum_{|\alpha|,|\beta|=m} (-1)^m \partial^\alpha_x \left(a_{\alpha\beta}(t,\cdot)\partial^\beta_x u\right) \right\|^2_{L^2} dt \\ \gtrsim \gamma^{\frac{1}{2}} \int_0^{T/2} e^{\frac{2}{\gamma}\Phi(\gamma(T-t))} \|u\|^2_{H^m} dt\end{aligned} \tag{5}$$

for all $u \in C^\infty_0(\mathbb{R}_t \times \mathbb{R}^n_x)$ with $\operatorname{supp}(u) \subseteq [0,T/2] \times \mathbb{R}^n_x$ and all $\gamma \geq \gamma_0$.

Remark 2.3 The proof of this result is an extension of the proofs in [5, 6] with weight function Φ from Remark 1.2. To treat different lower order terms in (4), especially terms of the form $\sum_{|\gamma|=2m-1} b_\gamma(t,x)\partial^\gamma_x u$, new ideas are required. This problem will be discussed in a forthcoming paper [7]. Contrary to Proposition 1.5, it is reasonable to expect that the regularity assumption with respect to t can be lowered for the terms of order $\leq 2m-1$.

To prove this Carleman estimate, one follows the strategy of [4–6]. The full proof as well as an extension in the spirit of [6] will appear in a forthcoming paper [7]. Here we line out the main steps of the proof of (5):

1. We replace $v(t,x) = e^{\frac{1}{\gamma}\Phi(\gamma(T-t))}u(t,x)$ and rewrite (4) in terms of v:

$$\int_0^{T/2} \left\| \partial_t v - \sum_{|\alpha|,|\beta|=m} (-1)^m \partial_x^\alpha \left(a_{\alpha\beta}(t,\cdot)\partial_x^\beta v \right) + \Phi'(\gamma(T-t))v \right\|_{L^2}^2 dt$$
$$\gtrsim \gamma^{\frac{1}{2}} \int_0^{T/2} \|v\|_{H^m}^2 dt.$$

2. In this step we use Bony's paraproduct as introduced in [2]. The operator is defined as $T_a u = \sum_{\nu \geq N} S_{\nu-N} a \Delta_\nu u$, where S_ν is an operator localizing to $\{|\xi| \leq 2^{\nu+1}\}$ in the phase space and Δ_ν is localizing to $\{2^{\nu-1} \leq |\xi| \leq 2^{\nu+1}\}$ in the phase space. For more information and the properties of T_a, we refer to Bony's paper [2] and [5, 6, 9]. We replace $a_{\alpha\beta}$ by $T_{a_{\alpha\beta}}$ and use the fact that $a_{\alpha\beta} - T_{a_{\alpha\beta}}$ is m-regularizing for $a_{\alpha\beta} \in W^{m,\infty}$. Thus, the analysis can be reduced to proving

$$\int_0^{T/2} \left\| \partial_t v - \sum_{|\alpha|,|\beta|=m} (-1)^m \partial_x^\alpha \left(T_{a_{\alpha\beta}} \partial_x^\beta v \right) + \Phi'(\gamma(T-t))v \right\|_{L^2}^2 dt$$
$$\gtrsim \gamma^{\frac{1}{2}} \int_0^{T/2} \|v\|_{H^m}^2 dt. \tag{6}$$

3. We microlocalize (6) by writing the norms in terms of Littlewood–Paley decompositions. Using appropriate estimates (similar to the estimates in [5, 6]) for

$$\sum_{|\alpha|,|\beta|=m} (-1)^m \partial_x^\alpha [\Delta_\nu, T_{a_{\alpha\beta}}] \partial_x^\beta u,$$

the analysis is reduced to a term by term analysis of

$$\int_0^{T/2} \left\| \partial_t v_\nu - \sum_{|\alpha|,|\beta|=m} (-1)^m \partial_x^\alpha \left(T_{a_{\alpha\beta}} \partial_x^\beta v_\nu \right) + \Phi'(\gamma(T-t))v_\nu \right\|_{L^2}^2 dt.$$

4. The proof proceeds in performing integration by parts with respect to t on the term

$$2\,\mathrm{Re} \int_0^{T/2} \left\langle \partial_t v_\nu \middle| - \sum_{|\alpha|,|\beta|=m} (-1)^m \partial_x^\alpha \left(T_{a_{\alpha\beta}} \partial_x^\beta v_\nu \right) \right\rangle dt$$

Due to the low-regularity of $a_{\alpha\beta}$ with respect to t, we have to regularize the coefficients. We use a standard mollifying technique and use estimates for $T_{\partial_t a^{\epsilon}_{\alpha\beta}}$ and $T_{a_{\alpha\beta}-a^{\epsilon}_{\alpha\beta}}$. These are the same as in [3, 5, 6].

5. We will estimate the microlocalized pieces term by term and handle low and high frequencies separately to obtain (6) by summing up all pieces.

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Multiplier Method in the Strong Unique Continuation for Electromagnetic Schrödinger Operator

Xiaojun Lu and Xiaofen Lv

Abstract This paper mainly addresses the strong unique continuation property for electromagnetic Schrödinger operator with complex-valued coefficients. Appropriate multipliers with physical backgrounds have been introduced to prove a priori estimates. Moreover, its application in an exact controllability problem has been shown, in which case, the boundary value determines the interior value completely.

Keywords Electromagnetic Schrödinger operator • Multiplier method • Strong unique continuation

Mathematics Subject Classification (2010) Primary 35J10; Secondary 35J25

1 Introduction

Nowadays, quantum studies, especially multiphoton entanglement and interferometry, are attracting many scientists' attention, either theoretically or practically [16]. A few world-famous high-tech companies, such as Apple, Microsoft, are developing new generation of high-performance computers based on the quantum mechanical phenomena.

In our paper, we discuss an important complex-valued operator in this research field. Let $\mathbf{A}(x)$ be the vector potential of the magnetic field $\mathbf{B}$, that is, $\mathbf{B} = \nabla \times \mathbf{A}$. Clearly, $\nabla \cdot \mathbf{B} = \text{div rot}\mathbf{A} = 0$. From one of Maxwell's equations (μ is magnetic permeability) $\nabla \times \mathbf{E} = -\mu\partial\mathbf{B}/\partial t = 0$, we deduce that $\mathbf{E} = -\nabla\phi$, where the scalar ϕ represents the electric potential. We choose an appropriate Lagrangian for the

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non-relativistic charged particle in the electromagnetic field (q is the electric charge of the particle, and $\mathbf{v}$ is its velocity, m is mass), $\mathscr{L} = m\mathbf{v}^2/2 - q\phi + q\mathbf{v}\cdot\mathbf{A}$. Particularly, the canonical momentum is specified by the vector $\mathbf{p} = \nabla_{\mathbf{v}}\mathscr{L} = m\mathbf{v} + q\mathbf{A}$. Next we define the classical Hamiltonian by Legendre transform, $H \triangleq \mathbf{p}\cdot\mathbf{v} - \mathscr{L} = (\mathbf{p} - q\mathbf{A})^2/(2m) + q\phi$. In quantum mechanics, when $\mathbf{p}$ is replaced by $-i\hbar\nabla$,($\hbar$ is the Planck constant), we have the following operator

$$P \triangleq (i\hbar\nabla + q\mathbf{A})^2/(2m) + q\phi : \mathcal{H} \to \mathcal{H}^*, \tag{1}$$

where $\mathcal{H}$ and $\mathcal{H}^*$ are corresponding function spaces [1]. Lots of literature discussed this kind of operator [5, 9, 11, 18].

Let $\Omega \subset \mathbb{R}^N$ be an open, connected, and bounded domain. From the structure of operator P, we define two corresponding simplified operators

$$\mathscr{H}_{\mathbf{A}} \triangleq i\nabla + \mathbf{A}(x) : L^2(\Omega) \to (L^2(\Omega))^N, \tag{2}$$

$$\mathscr{H}_{\mathbf{A}}^2 \triangleq (i\nabla + \mathbf{A}(x))^2 : L^2(\Omega) \to L^2(\Omega), \tag{3}$$

where $\mathbf{A} \in C^1(\overline{\Omega})$ is a real-valued potential vector. Interested readers can refer to [6, 15] for more details concerned with the vector operator $\mathscr{H}_{\mathbf{A}}$ and self-adjoint operator $\mathscr{H}_{\mathbf{A}}^2$. In such a manner, (1) is simplified as

$$\mathscr{H}_{\mathbf{A}}^2 - \phi(x) : L^2(\Omega) \to L^2(\Omega), \tag{4}$$

where the non-positive real-valued function $\phi \in L^\infty(\Omega)$. In this paper, we focus on the strong unique continuation property(SUCP) for the electromagnetic Schrödinger operator (4). First, we introduce the following definitions:

Definition 1.1 A function $u \in L^2_{\text{loc}}(\Omega)$ is said to vanish of infinite order at $x_0 \in \Omega$ if for any sufficiently small $R > 0$, one has

$$\int_{|x-x_0|<R} |u|^2 dx = O(R^M), \text{ for every } M \in \mathbb{N}^+. \tag{5}$$

Definition 1.2 We say that the operator (4) has SUCP if every solution ω of the equation

$$\mathscr{H}_{\mathbf{A}}^2 \omega = \phi\omega,$$

which vanishes of infinite order at x_0 is identically zero in a neighborhood of x_0.

So far, the strong unique continuation problem for second order elliptic operators is well understood. In the case of $\Omega = \mathbb{R}^2$, Carleman proved the SUCP of the elliptic equation with bounded coefficients and $V \in L^\infty_{\text{loc}}(\mathbb{R}^2)$

$$-\Delta u = W \cdot \nabla u + Vu \tag{6}$$

by introducing a weighted L^2-estimate, or the Carleman estimate [4]. For the space dimension $N \geq 3$ with bounded coefficients, Aronszajn [2], and Aronszajn et al. [3] proved the SUCP by means of Carleman type inequalities, namely, observability inequalities. Afterwards, D. Jerison, C.E. Kenig, C.D. Sogge treated Eq. (6) with singular potentials $V \in L^{N/2}_{\mathrm{loc}}(\mathbb{R}^N)$ and $W \in L^\infty(\mathbb{R}^N)$, $N \geq 3$, by the approach of $L^p - L^q$ Carleman estimate involving sharp exponents [10, 11, 17]. And N. Garofalo and F.H. Lin gave a new proof for the SUCP of the elliptic operator $-\Delta u = Vu$ with bounded potential by applying a variational method in [7, 8].

There is a large body of work on SUCP for (6) with real-valued coefficients. In this paper, we investigate the complex-valued case. As a matter of fact, the operator $\mathscr{H}^2_{\mathbf{A}}$ can be decomposed into

$$\mathscr{H}^2_{\mathbf{A}}\omega = -\Delta\omega + i\mathbf{A}\cdot\nabla\omega + i\nabla\cdot(\mathbf{A}\omega) + \mathbf{A}\mathbf{A}^T\omega. \tag{7}$$

In [12, 13], K. Kurata proved the SUCP for (4) with $\mathbf{A}\mathbf{A}^T \in \mathscr{K}^{\mathrm{loc}}_N(\Omega)$, where $\mathscr{K}^{\mathrm{loc}}_N(\Omega)$ denotes the Kato class. When the potential $\mathbf{A} \in (L^\infty(\Omega))^N$, in effect, it does not belong to the Kato class. As a result, we cannot deduce corresponding results directly from K. Kurata's work. In this manuscript, we intend to provide a much simpler proof of SUCP for (4) with complex-valued coefficients by developing new multipliers. At the moment one is ready to state the main results.

Theorem 1.3 *For $N \geq 2$, let complex-valued $\omega \in H^2(\mathbb{B}_1)$ be a solution of the problem*

$$-\Delta\omega + i\mathbf{A}\cdot\nabla\omega + i\nabla\cdot(\mathbf{A}\omega) + \mathbf{A}\mathbf{A}^T\omega = \phi(x)\omega \quad in\ \mathbb{B}_1, \tag{8}$$

where $\mathbb{B}_1$ is a unit ball and the non-positive real-valued function $\phi \in L^\infty(\mathbb{R}^N)$. If ω vanishes of infinite order at $x_0 \in \mathbb{B}_1$, then $\omega \equiv 0$ in $\mathbb{B}_1$.

By virtue of Theorem 1.3, one is able to prove the following statement for a mixed boundary value problem which is of great importance in the discussion of exact controllability through boundary control [14, 15].

Corollary 1.4 *Let Ω be a bounded, open, and connected domain in $\mathbb{R}^N$ with the boundary $\Gamma \in C^2$. Let $\omega \in H^2(\Omega)$ be the solution of the mixed boundary problem*

$$-\Delta\omega + i\mathbf{A}\cdot\nabla\omega + i\nabla\cdot(\mathbf{A}\omega) + \mathbf{A}\mathbf{A}^T\omega = \phi(x)\omega \quad in\ \Omega,$$
$$\omega = \partial\omega/\partial\nu = 0\ on\ \Gamma.$$

Then ω is identically 0 in Ω.

Remark 1.5 Let $\mathbb{B}$ be an arbitrarily small open ball such that $\Gamma \cap \mathbb{B} \neq \varnothing$. Set $\Omega^1 \triangleq \Omega \cup \mathbb{B}$, and define $\omega^1 \triangleq \begin{cases} \omega \text{ in } \Omega; \\ 0 \text{ in } \mathbb{B}\backslash\Omega. \end{cases}$ Indeed, to prove Corollary 1.4, it is sufficient to verify that $\omega^1 \in H^2$. Thus, the result is concluded due to the connectness of Ω.

Remark 1.6 Theorem 1.3 demonstrates that the asymptotic behavior of the solution ω at an interior point x_0 determines the interior value of ω in $\mathbb{B}_1$. In contrast with Theorem 1.3, Corollary 1.4 indicates that the behavior of the solution ω on the boundary determines the interior value of ω in Ω.

2 Sketch of Proof of Theorem 1.3: Multiplier Method

First, we introduce several quantities which will serve as useful tools for our purposes. For every $r \in (0, 1)$, we define the following two quantities

$$\Phi(r) \triangleq \int_{\partial\mathbb{B}_r} |\omega|^2 dS_x, \tag{9}$$

where dS_x denotes $(N-1)$-dimensional Hausdorff measure on $\partial\mathbb{B}_r$.

$$\Psi(r) \triangleq \int_{\mathbb{B}_r} (|\mathscr{H}_{\mathbf{A}}\omega|^2 - \phi|\omega|^2) dV_x. \tag{10}$$

Actually, we have

Lemma 2.1 *By virtue of divergence theorem, the following identity holds,*

$$-\operatorname{Re}\int_{\partial\mathbb{B}_r} \Big(\nabla|\omega|^2 - i\mathbf{A}|\omega|^2\Big) \cdot x/r dS_x = \int_{\mathbb{B}_r} \Big(-2|\mathscr{H}_{\mathbf{A}}\omega|^2 + 2\phi|\omega|^2\Big) dV_x. \tag{11}$$

Next we calculate the derivatives of $\Phi(r)$ and $\Psi(r)$ with respect to r.

Lemma 2.2 *The derivatives of $\Phi(r)$ and $\Psi(r)$ with respect to r are presented as follows*

$$\Phi'(r) = (N-1)\Phi(r)/r + 2\Psi(r). \tag{12}$$

$$\begin{aligned}
\Psi'(r) = {} & (N-2)\Psi(r)/r + (N-2)/r \textstyle\int_{\mathbb{B}_r} \phi|\omega|^2 dV_x + 2/r \operatorname{Re} \textstyle\int_{\mathbb{B}_r} \mathbf{A}\omega \cdot \overline{\mathscr{H}_{\mathbf{A}}\omega} dV_x \\
& + 2/r \operatorname{Re} \textstyle\int_{\mathbb{B}_r} (x \cdot \nabla\omega) \cdot \phi\overline{\omega} dV_x + 2/r \operatorname{Re} \textstyle\int_{\mathbb{B}_r} \omega x \Theta_{\mathbf{A}} \overline{\mathscr{H}_{\mathbf{A}}\omega}^T dV_x \\
& + 2 \textstyle\int_{\partial\mathbb{B}_r} |\nu \cdot (i\nabla\omega + \mathbf{A}\omega)|^2 dS_x - 2 \operatorname{Re} \textstyle\int_{\partial\mathbb{B}_r} (\mathbf{A}\omega \cdot \nu)(\overline{\mathscr{H}_{\mathbf{A}}\omega \cdot \nu}) dS_x \\
& - \textstyle\int_{\partial\mathbb{B}_r} \phi|\omega|^2 dS_x,
\end{aligned} \tag{13}$$

where

$$\nabla_i a_j \triangleq \partial a_j/\partial x_i, \quad i,j = 1,2,3,\ldots,N,$$

and the Jacobi matrix

$$\Theta_{\mathbf{A}} \triangleq \begin{pmatrix} \nabla_1 a_1 & \nabla_1 a_2, & \cdots & \nabla_1 a_N \\ \nabla_2 a_1 & \nabla_2 a_2 & \cdots & \nabla_2 a_N \\ \vdots & \vdots & \cdots & \vdots \\ \nabla_N a_1 & \nabla_N a_2 & \cdots & \nabla_N a_N \end{pmatrix}.$$

Next we show an important comparison lemma.

Lemma 2.3 *There exists an $r_0 \in (0,1)$ such that for every $r \in (0, r_0)$, we have*

$$\int_{\mathbb{B}_r} |\omega|^2 dV_x \leq r \int_{\partial\mathbb{B}_r} |\omega|^2 dS_x. \tag{14}$$

Assume that there exists a small $r_1 \in (0,1)$ such that

$$\Phi(r) \neq 0 \ \text{ for } \ \forall\ r \in (0, r_1). \tag{15}$$

Define the frequency function

$$F(r) \triangleq r\Psi(r)/\Phi(r), \quad r \in (0, r_1). \tag{16}$$

Let $r^* \triangleq \min\{r_0, r_1\}$, and we set

$$\beth_{r^*} \triangleq \Big\{ r \in (0, r^*) : F(r) > 1 \Big\}. \tag{17}$$

With the above definitions, we have the following inequality for the frequency function.

Lemma 2.4 *Under the assumptions (15)–(17), there exists a positive constant $\tau = \tau(N, \phi)$ which is independent of r such that $F'(r)$ is estimated in a uniform fashion,*

$$F'(r) \geq -F(r)\tau.$$

It follows that $\exp(\tau r)F(r)$ is monotonously increasing on $(0, r^*)$, that is to say,

$$\exp(\tau r)F(r) \leq \exp(\tau r^*)F(r^*).$$

Keeping in mind the case $F \leq 1$, we know that, $F(r)$ is bounded on $(0, r^*)$. Since

$$\Phi'(r) = (N-1)/r\Phi(r) + 2\Psi(r),$$

then

$$\left(\log(\Phi(r)/r^{N-1})\right)' = 2\Psi(r)/\Phi(r) = 2F(r)/r \leq C(\tau)/r.$$

We integrate from γ to 2γ, then

$$\log(2^{1-N}\Phi(2\gamma)/\Phi(\gamma)) \leq C(\tau)\log 2.$$

It follows that

$$\Phi(2\gamma) \leq 2^{C(\tau)+N-1}\Phi(\gamma).$$

Finally, integrating with respect to γ gives

$$\int_{\mathbb{B}_{2\gamma}} |\omega|^2 dV_x \leq 2^{C(\tau)+N} \int_{\mathbb{B}_{\gamma}} |\omega|^2 dV_x.$$

Since $\mathbb{B}_1$ is connected, then our theorem follows immediately.

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The Cauchy Problem for Nonlinear Complex Ginzburg–Landau Type Equations

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Abstract The Cauchy problem of nonlinear complex Ginzburg–Landau type equations is considered in Sobolev spaces under the variance of the space. Some properties of the spatial variance on the problem are remarked.

Keywords Cauchy problem • Nonrelativistic limit

Mathematics Subject Classification (2010) Primary 35Q55; Secondary 35L71, 35Q75

When we consider a line element which has complex coefficients in a uniform and isotropic space, and we use the Einstein equation to set its scale-function which describes the spatial variance, we are able to consider nonlinear scalar-field equations. Taking their nonrelativistic limits, we obtain nonlinear complex Ginzburg–Landau type equations. In this paper, we consider the Cauchy problem of them, and we show global and blow-up solutions in Sobolev spaces.

Let us start from the introduction of the nonlinear complex Ginzburg–Landau type equations in this paper. We denote the spatial dimension by $n \geq 1$, the Planck constant by $\hbar := h/2\pi$, the mass by $m > 0$. Let $\sigma \in \mathbb{R}$, $a_0 > 0$, $a_1 \in \mathbb{R}$. We put $T_0 := \infty$ when $(1+\sigma)a_1 \geq 0$, $T_0 := -2a_0/n(1+\sigma)a_1 (> 0)$ when $(1+\sigma)a_1 < 0$. We define a scale-function $a(t)$ for $t \in [0, T_0)$ by

$$a(t) := \begin{cases} a_0 \left(1 + \frac{n(1+\sigma)a_1 t}{2a_0}\right)^{2/n(1+\sigma)} & \text{if } \sigma \neq -1, \\ a_0 \exp\left(\frac{a_1 t}{a_0}\right) & \text{if } \sigma = -1, \end{cases} \tag{1}$$

where we note that $a_0 = a(0)$ and $a_1 = \partial_t a(0)$. We define a weight function $w(t) := (a_0/a(t))^{n/2}$, and a change of variable $s = s(t) := \int_0^t a(\tau)^{-2} d\tau$. We put $S_0 := s(T_0)$. We use conventions $a(s) := a(t(s))$ and $w(s) := w(t(s))$ for $s \in [0, S_0)$ as far as

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there is no fear of confusion. A direct computation shows

$$S_0 = \begin{cases} \frac{2}{a_0 a_1 (4-n(1+\sigma))} & \text{if } a_1(4-n(1+\sigma)) > 0 \\ \infty & \text{if } a_1(4-n(1+\sigma)) \le 0. \end{cases}$$

For $\lambda \in \mathbb{C}$, $1 \le p < \infty$, $-\pi/2 < \omega \le \pi/2$, $0 \le \mu_0 < n/2$ and $0 < S \le S_0$, we consider the Cauchy problem given by

$$\begin{cases} \pm i \dfrac{2m}{\hbar} \partial_s u(s,x) + e^{-2i\omega} \Delta u(s,x) - \lambda e^{-2i\omega} a(s)^2 \left(|uw|^{p-1} u \right)(s,x) = 0, \\ u(0,\cdot) = u_0(\cdot) \in H^{\mu_0}(\mathbb{R}^n) \end{cases} \tag{2}$$

for $(s,x) \in [0,S) \times \mathbb{R}^n$, where $i := \sqrt{-1}$, $\Delta := \sum_{j=1}^n \partial^2/\partial x_j^2$, and $H^{\mu_0}(\mathbb{R}^n)$ denotes the Sobolev space of order $\mu_0 \ge 0$. The double sign $\pm$ is in same order throughout the paper. We say that u is a global solution of (2) if it exists on $[0, S_0)$.

It is well known that the Schrödinger equation

$$i2m\partial_t u/\hbar + \Delta u - \lambda |u|^{p-1} u = 0$$

is derived from the Klein–Gordon equation

$$\partial_t^2 \phi - c^2 \Delta \phi + (mc^2/\hbar)^2 \phi + c^2 \lambda |\phi|^{p-1} \phi = 0$$

by a transform from ϕ to u and the nonrelativistic limit. We consider the spatial variance described by the scale-function $a(\cdot)$, which satisf the Einstein equation with the cosmological constant in a uniform and isotropic space. The study of roles of the cosmological constant and the spatial variance is important to describe the history of the universe, especially, the inflation and the accelerating expansion of the universe. The scale-function (1) follows from the equation of state when we regard the cosmological constant as the dark energy. We study the cosmological constant from the point of view of partial differential equations. We consider the Cauchy problem (2), and we show the well-posedness of the problem, global solutions and blow-up solutions. Especially, we remark that some dissipative properties appear by the spatial variance.

Let us consider the well-posedness of (2). For any real numbers $2 \le q \le \infty$ and $2 \le r < \infty$, we say that the pair (q,r) is admissible if it satisfies $1/r + 2/nq = 1/2$. For $\mu_0 \ge 0$ and two admissible pairs $\{(q_j, r_j)\}_{j=1,2}$, we define a function space

$$X^{\mu_0}([0,S)) := \{u \in C([0,S), H^{\mu_0}(\mathbb{R}^n));\ \max_{\mu=0,\mu_0} \|u\|_{X^\mu([0,S))} < \infty\},$$

where

$$\|u\|_{X^\mu([0,S))} := \begin{cases} \|u\|_{L^\infty((0,S),L^2(\mathbb{R}^n)) \cap \bigcap_{j=1,2} L^{q_j}((0,S),L^{r_j}(\mathbb{R}^n))} & \text{if } \mu = 0, \\ \|u\|_{L^\infty((0,S),\dot H^\mu(\mathbb{R}^n)) \cap \bigcap_{j=1,2} L^{q_j}((0,S),\dot B^\mu_{r_j 2}(\mathbb{R}^n))} & \text{if } \mu > 0. \end{cases}$$

Here, $\dot{H}^{\mu}(\mathbb{R}^n)$ and $\dot{B}^{\mu}_{r_j 2}(\mathbb{R}^n)$ denote the homogeneous Sobolev and Besov spaces, respectively. Since the propagator of the linear part of the first equation in (2) is written as $\exp(\pm i\hbar \exp(-2i\omega)s\Delta/2m)$, we assume $0 \le \pm\omega \le \pi/2$ to define it as a pseudo-differential operator. We note that the scaling critical number of p for (2) is $p(\mu_0) := 1 + 4/(n - 2\mu_0)$ when $a(\cdot) = 1$. We put

$$p_1(\mu_0) := 1 + \frac{4}{n - 2\mu_0} \cdot \left(1 + \frac{4}{n - 2\mu_0} \cdot \frac{2\mu_0}{n(1+\sigma)}\right)^{-1}$$

for $\sigma \neq -1$.

Theorem 1 *Let $n \ge 1$, $\lambda \in \mathbb{C}$, $0 \le \mu_0 < n/2$, and $1 \le p \le p(\mu_0)$. Let ω satisfies $0 \le \pm\omega \le \pi/2$ and $\omega \neq -\pi/2$. Assume $\mu_0 < p$ if p is not an odd number. There exist two admissible pairs $\{(q_j, r_j)\}_{j=1,2}$ with the following properties.*

(1) (Local solutions.) For any $u_0 \in H^{\mu_0}(\mathbb{R}^n)$, there exist $S > 0$ with $S \le S_0$ and a unique local solution u of (2) in $X^{\mu_0}([0, S))$.
(2) (Small global solutions.) Assume that one of the following conditions from (i) to (vi) holds: (i) $\mu_0 = 0$, $p = p(0)$, (ii) $\mu_0 > 0$, $p = p(\mu_0)$, $a_1 \ge 0$, (iii) $1 < p < p(\mu_0)$, $a_1 > 0$, $\sigma < -1$, (iv) $1 < p < p_1(\mu_0)$, $a_1 < 0$, $\sigma > -1$, (v) $p_1(\mu_0) < p < p(\mu_0)$, $a_1 > 0$, $\sigma > -1$, (vi) $\mu_0 > 0$, $1 < p < p(\mu_0)$, $a_1 > 0$, $\sigma = -1$. If $\|u_0\|_{\dot{H}^{\mu_0}(\mathbb{R}^n)}$ is sufficiently small, then the solution u obtained in (1) is a global solution.

The results in Theorem 1 are also valid for the gauge variant equation

$$\pm i\frac{2m}{\hbar}\partial_s u(s,x) + e^{-2i\omega}\,\Delta u(s,x) - \lambda e^{-2i\omega}\,\frac{a(s)^2}{w(s)}\,|uw|^p(s,x) = 0, \tag{3}$$

provided $\mu_0 < p$ when p is not an even number. The result (2) in Theorem 1 especially shows that we always have small global solutions for $1 < p < 1 + 4/n$ when $a(\cdot)$ is not a constant in the conditions (iii) and (vi). This result is much different from the case $a(\cdot) = 1$ in the following sense. In the case $a(\cdot) = 1$ and $\omega = 0$, some weighted spaces, for example, $(1 + |x|)^{-1}L^2(\mathbb{R}^n)$, have been needed for global solutions for $1 + 2/n < p < 1 + 4/n$ (see [3, 9, 10]). There exist blow-up solutions for small initial data for $1 < p \le 1+2/n$ (see [5]). In the case $a(\cdot) = 1$ and $\omega = \pm\pi/4$, there exist blow-up solutions for small initial data for $1 < p \le 1 + 2/n$ (see [2, 4, 7, 11]).

We have the following results for global and blow-up solutions for (2).

Corollary 2 *Let $\mu_0 = 0$ or $\mu_0 = 1$. Let $\lambda > 0$. Let $1 \le p < 1 + 4/n$ when $\mu_0 = 0$. Let $1 \le p < 1 + 4/(n-2)$ and $a_1(p - 1 - 4/n) \ge 0$ when $\mu_0 = 1$. For any $u_0 \in H^{\mu_0}(\mathbb{R}^n)$, the local solution u given by (1) in Theorem 1 is a global solution.*

Corollary 3 *Let $\mu_0 = 1$, $\lambda < 0$, $a_1 \ge 0$ and $1 \le p < 1 + 4/n$. Let $\omega = 0$ or $\omega = \pi/2$. For any $u_0 \in H^1(\mathbb{R}^n)$, the local solution u given by (1) in Theorem 1 is a global solution.*

Corollary 4 *Let $\mu_0 = 1$ and $\lambda < 0$. Let $\omega \neq 0, \pi/2$. Put $p_0 := 2/(\sin 2\omega)^2 - 1$. Let $p_0 < p \leq 1 + 4/(n-2)$. Let $a_1(p - 1 - 4/n) \leq 0$ and $S_0 = \infty$. For any $u_0 \in H^1(\mathbb{R}^n)$ with negative energy*

$$\int_{\mathbb{R}^n} \frac{1}{2}|\nabla u_0(x)|^2 + \frac{\lambda a_0^2 |u_0(x)|^{p+1}}{p+1} dx < 0, \tag{4}$$

the solution u given by (1) in Theorem 1 blows up in finite time.

Corollary 5 *Let $\mu_0 = 1$ and $\lambda < 0$. Let $\omega = 0$ or $\omega = \pi/2$. Let $1 + 4/n \leq p \leq 1 + 4/(n-2)$. Let $a_1 \leq 0$ and $S_0 = \infty$. For any $u_0 \in H^1(\mathbb{R}^n)$ which satisfies $\| |x| u_0(x) \|_{L_x^2(\mathbb{R}^n)} < \infty$ and (4), the solution u given by (1) in Theorem 1 blows up in finite time.*

To prove the above corollaries, we use two dissipative properties. One is from the parabolic structure of the first equation in (2) when $0 < \pm\omega < \pi/2$. The other is from the scale function $a(\cdot)$ when $\partial_t a(0) = a_1 \neq 0$. Even if the equation does not have the parabolic structure when $\omega = 0, \pi/2$, the latter is very effective to obtain the global solutions. The energy estimate shows the dissipative property of the equation when $\lambda a_1(p - 1 - 4/n) > 0$. The properties of semilinear Schrödinger equations of the form $(i\partial_t + \Delta_g)u = |u|^{p-1}u$ have been studied on certain compact or noncompact Riemannian manifold (M, g), where Δ_g is the Laplace–Beltrami operator on (M, g). In the hyperbolic space $\mathbb{H}^n$, the dispersive effect on Schrödinger equations was considered in [1], and the global existence of solutions with finite energy has been shown in [6]. In the de Sitter spacetime, a dissipative effect on Schrödinger equations was shown in [8]. The proofs of the above theorem and corollaries will appear somewhere.

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Uniform Resolvent Estimates for Stationary Dissipative Wave Equations in an Exterior Domain and Their Application to the Principle of Limiting Amplitude

Kiyoshi Mochizuki and Hideo Nakazawa

Abstract The first aim of this work is to prove a uniform resolvent estimate for stationary dissipative wave equations. The second aim is to improve the principle of limiting amplitude for dissipative wave equations proved by S. Mizohata and K. Mochizuki in 1966.

Keywords Dissipative wave equations • The principle of limiting amplitude • Uniform resolvent estimates

Mathematics Subject Classification (2010) Primary 35J05; Secondary 35L05

1 Introduction and Results

Let Ω be an exterior domain in $\mathbb{R}^N$ ($N \geqq 2$) with smooth boundary $\partial\Omega$. We consider in Ω stationary dissipative wave equations of the form

$$\left(-\Delta - i\kappa b(x) - \kappa^2\right) u(x) = f(x), \qquad x \in \Omega \tag{1}$$

with Dirichlet boundary condition

$$u(x) = 0, \qquad x \in \partial\Omega. \tag{2}$$

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Here, $\Delta = \sum_{j=1}^{N} \partial^2/\partial x_j^2$, $\kappa \in \mathbb{C}$, $b(x)$ is a real-valued bounded continuous function of $x \in \Omega \cup \partial\Omega$ and $f(x) \in L^2(\Omega)$.

We denote $r = |x|$, and define the operator $D^{\pm}$ and $D_r^{\pm}$ by

$$D^{\pm}u = \nabla u - \frac{N-1}{2r}u\frac{x}{r} \mp i\kappa u\frac{x}{r} \quad (\pm\Im\kappa \geqq 0),$$

$$D_r^{\pm}u = D^{\pm}u \cdot \frac{x}{r} = u_r - \frac{N-1}{2r}u \mp i\kappa u \quad (\pm\Im\kappa \geqq 0).$$

These two operators are introduced in Morawetz and Ludwig [14] and are used in Ikebe and Saito [2], Mochizuki [8–10], Saito [19, 20], Mochizuki and Nakazawa [11, 12], Nakazawa [15–17]. The weighted L^2-space is defined by

$$L_w^2 = \{f\,;\,||f||_w < \infty\}, \qquad ||f||_w^2 = \int_{\Omega} |wf|^2 dx$$

for a non-negative function w. In the following, we assume that $\mathbb{R}^N \setminus \Omega$ is star-shaped with respect to the origin, i.e., it holds that $(x/r, n) \leqq 0$ for the unit outer normal n of $\partial\Omega$. Moreover, we exclude the case of the whole space, that is, $\Omega \neq \mathbb{R}^N$. Therefore, we assume that there exists $r_0 > 0$ such that

$$\min\{r = |x|; x \in \partial\Omega\} > r_0.$$

We also introduce the following conditions (B) for the function $b(x)$:

$$(B) \qquad |b(x)| \leqq \begin{cases} b_0 r^{-2}\left(1 + \log \dfrac{r}{r_0}\right)^{-2} & (N = 2), \\ b_0 r^{-2-\delta} & (N \geqq 3), \end{cases} \tag{3}$$

for some $\delta > 0$ and for some sufficiently small constant $b_0 \in (0, 1)$. Now we shall state the main result.

Theorem 1.1 *Assume* (B). *Then for a solution* u *of* (1)–(2), *the following uniform resolvent estimates hold for* $\pm\Im\kappa \geqq 0$:

(1) *If* $N = 2$, *then*

$$|\kappa|^2||u||^2_{\frac{1}{r\left(1+\log\frac{r}{r_0}\right)}} + ||u||^2_{\frac{\sqrt{(\pm\Im\kappa)r+1}}{r\left(1+\log\frac{r}{r_0}\right)}} + \int_{\partial\Omega} \{-(n,x)\}\,|u_n|^2 dS$$
$$\leqq C||f||^2_{r\left(1+\log\frac{r}{r_0}\right)}. \tag{4}$$

(2) *If* $N \geqq 3$, *then*

$$
\begin{aligned}
&|\kappa|^2 ||u||^2_{r^{-(1+\delta)/2}} + ||u||^2_{\frac{\sqrt{(\pm\Im\kappa)r+1}}{r}} + \int_{\partial\Omega} \{-(n,x)\}\, |u_n|^2 dS \\
&\leqq C||f||^2_{r^{(1+\delta)/2}}.
\end{aligned}
\tag{5}
$$

As a corollary of Theorem 1.1, we are able to improve a result obtained by Mizohata and Mochizuki [7]. This result is devoted to the asymptotical behavior of solutions to the following mixed problem for dissipative wave equations with time periodic external forcing term:

$$
w_{tt} - \Delta w + b(x) w_t = e^{-i\omega t} f(x), \qquad (x,t) \in \Omega \times (0,\infty), \tag{6}
$$

$$
w(x,0) = w_t(x,0) = 0, \qquad x \in \Omega, \tag{7}
$$

$$
w(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,\infty), \tag{8}
$$

where $\omega \in \mathbb{R}$.

Theorem 1.2 *Assume* (B). *Let* u *be a solution of* (1)–(2) *and let* w *be a solution of* (6)–(8).

(1) *If* $N = 2$, *then*

$$
\lim_{t\to+\infty} ||w - ue^{-i\omega t}||_{r^{-3/2}\left(\log \frac{r}{r_0}\right)^{-1}} = 0.
$$

(2) *If* $N \geqq 3$, *then*

$$
\lim_{t\to+\infty} ||w - ue^{-i\omega t}||_{r^{-3/2}} = 0.
$$

A uniform resolvent estimate was firstly proved by Kato and Yajima [4] (see also Kuroda [5], Watanabe [22]) for Helmholtz equations in $\mathbb{R}^N$ with $N \geqq 3$. Mochizuki [10] extended this to magnetic Schrödinger equations in $\Omega \subseteq \mathbb{R}^N$ with $N \geqq 3$. Refining an inequality presented there, Nakazawa [17] obtained partial results for the exterior problem in $\mathbb{R}^2$. (It is in general known that the uniform resolvent estimate does not hold in $\mathbb{R}^2$. See Yafaev [23].) But the duality relation was violated. This is improved by us in [11] for magnetic Schrödinger equations (its essence is stated in [12]). Applying the methods developed in [15–17] and [11], we obtain Theorem 1.1.

The principle of limiting amplitude states that every solution of the time-dependent problem tends to the steady states $e^{-i\omega t}u(x)$ as t goes to infinity, where $u(x)$ satisfies the corresponding stationary problem with the radiation condition. Many results are known so far, e.g., [1, 3, 6, 7, 13, 18, 21]. For dissipative wave equations in $\mathbb{R}^3$, there are no results other than Mizohata and Mochizuki [7] (and Iwasaki [3]). In [7], this principle was proved under the following assumptions: The function $b(x)$ is Hölder continuous on $\mathbb{R}^3$ and satisfies $0 \leqq b(x) \leqq b_1 r^{-3-\delta_1}$ for sufficiently large r and for some b_1, $\delta_1 > 0$. The function $f(x) \in C^2(\mathbb{R}^3)$ satisfies $|f(x)| \leqq Cr^{-3-\delta_2}$ and $|\nabla f(x)| + |\Delta f(x)| \leqq Cr^{-2-\delta_3}$ for sufficiently large r, and for some C, δ_2 and $\delta_3 > 0$. Then for any bounded set $X \subset \mathbb{R}^3$, it holds that

$$\lim_{t\to+\infty} \max_{x\in X} |e^{i\omega t} w(x,t) - u(x)| = 0.$$

In their result, there are no assumptions on the smallness of $b(x)$. However, we need the smallness to establish Theorems 1.1 and 1.2. To remove the smallness is one of our future problems.

2 Outline of Proofs

[Outline of the Proof of Theorem 1.1] In (1), put

$$v = e^{\rho} u, \qquad g = e^{\rho} f,$$

where the function $\rho(r)$ is defined by

$$\rho(r) = \mp i\kappa r + \frac{N-1}{2} \log r, \quad (\pm \Im \kappa \geqq 0).$$

Then v solves the equation

$$-\Delta v + 2\rho_r v_r + \tilde{b} v = g, \tag{9}$$

where

$$\tilde{b}(x) = -i\kappa b(x) + \frac{(N-1)(N-3)}{4r^2}.$$

Then, multiply by $r\overline{v_r}$ both sides of (9). After using integration by parts, noting (2) and the assumption that the boundary $\partial\Omega$ is star-shaped, we derive some suitable inequality. Making use of it and one more inequality derived from $\Re\left\{(1) \times \overline{\varphi(r) i\kappa u}\right\}$ for $0 \leqq \varphi \in L^1((r_0,\infty))$ satisfying $\varphi_r \leqq 0$, we arrive at the following result.

Proposition 2.1 *Assume* (B) *and* φ *satisfies the* $0 \leqq \varphi \in L^1((r_0, \infty))$ *and* $\varphi_r \leqq 0$. *Then for a solution u of* (1),

$$\begin{aligned} &|\kappa|^2 \int_\Omega \Big(\varphi - |b|g - \frac{|b|r}{2}\Big)|u|^2 dx + \int_\Omega \Big(\varphi - \frac{|b|r}{2}\Big)\Big|u_r + \frac{N-1}{2r}u\Big|^2 dx \\ &\quad + \int_\Omega \Big\{(\pm\Im\kappa)r + \frac{1}{2} - \Big(\varphi + \frac{|b|r}{2}\Big)\Big\}|D^{\pm}u|^2 dx \\ &\quad + \int_\Omega \Big\{(\pm\Im\kappa)r + \frac{1}{2}\Big\}\frac{(N-1)(N-3)}{4r^2}|u|^2 dx + \frac{1}{2}\int_{\partial\Omega}\{-(n,x)\}\,|u_n|^2 dS \\ &\quad + \frac{1}{2}\int_{\partial\Omega}\{-(n,x)\}\,|u_n|^2 dS \leqq \int_\Omega \big|fg\overline{i\kappa u}\big|\,dx + \int_\Omega r\,\big|f\overline{D_r^{\pm}u}\big|\,dx \quad (\pm\Im\kappa \geqq 0). \end{aligned} \tag{10}$$

If $N \geqq 3$, it does not cause any problems. On the other hand, if $N = 2$, the fourth term of left-hand side of (10) is not non-negative. To compensate this, we need the following inequality which is in some sense of Hardy-type related to radiation conditions. Note that this inequality is independent of the dimension $N(\geqq 1)$.

Proposition 2.2 *Suppose that* ξ *and* η *to be both smooth function of* r *satisfying* $\xi \geqq 0$. *Then for any function* $\mu \in C_0^\infty(\Omega)$,

$$\int_\Omega W_{\xi,\eta}|\mu|^2 dx \leqq ||D_r^{\pm}\mu||^2_{\sqrt{\xi}} \qquad (\pm\Im\kappa \geqq 0),$$

where

$$W_{\xi,\eta} = (\pm\Im\kappa)W_1 + W_2 + W_3 + \frac{\xi}{4r^2}$$

with

$$W_1 = 2\xi\Big(\frac{1}{2r} + \eta\Big), \quad W_2 = -\xi_r\Big(\frac{1}{2r} + \eta\Big), \quad W_3 = -\xi\Big(\eta_r + \frac{\eta}{r} + \eta^2\Big).$$

Making use of Propositions 2.1 and 2.2 by choosing φ, ξ and η appropriately, we get Theorem 1.1. □

[Outline of the Proof of Theorem 1.2] To prove Theorem 1.2, we follow the arguments as in Roach and Zhang [18]. Then we obtain the next result.

Proposition 2.3 *Assume* $\kappa = \sigma + i\tau$, *where* $0 < \sigma_1 \leqq \sigma \leqq \sigma_2 < \infty$ *and* $0 < \tau \leqq 1$ *with some* $\sigma_1, \sigma_2 > 0$. *If u is a solution of* (1)–(2), *then the inequality*

$$|\kappa|\,\Big|\Big|\frac{du}{d\kappa}\Big|\Big|_{(\sqrt{r}\psi)^{-1}} \leqq C||f||_{\sqrt{r}\psi}$$

holds for some $C > 0$, *where*

$$\psi(r) = \begin{cases} r\Big(1 + \log \frac{r}{r_0}\Big), & (N = 2), \\ r, & (N \geqq 3). \end{cases}$$

By Proposition 2.3 and the principle of limiting absorption for the operator $\big(-\Delta - i\kappa b(x) - \kappa^2\big)^{-1}$ which is followed from Theorem 1.1 (cf., [15, 16]), we find that for any $0 < \sigma_1 \leqq \sigma, \sigma' \leqq \sigma_2 < \infty$ satisfying $|\sigma - \sigma'| \leqq 1$,

$$\left|\left|\frac{R(\sigma + i0)f - R(\sigma' + i0)f}{\sigma - \sigma'}\right|\right|_{(\sqrt{r}\psi)^{-1}} \leqq C||f||_{\sqrt{r}\psi}.$$

Moreover, it holds that

$$\left|\left|\frac{R(\sigma + i0)f}{\sigma - \omega}\right|\right|_{(\sqrt{r}\psi)^{-1}} \leqq \frac{C||f||_{\sqrt{r}\psi}}{|\sigma - \omega||\sigma|}.$$

Then we can follow the same argument as in Mizohata and Mochizuki [7], §4 to conclude Theorem 1.2. □

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Stabilization of the Fourth Order Schrödinger Equation

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Abstract We study both boundary and internal stabilization problems for the fourth order Schrödinger equation in a smooth bounded domain Ω of $\mathbb{R}^n$. We first consider the boundary stabilization problem. By introducing suitable dissipative boundary conditions, we prove that the solution decays exponentially in an appropriate energy space. In the internal stabilization problem, by assuming that the damping term is effective on the neighborhood of the boundary, we prove the exponential decay of the $L^2(\Omega)$-energy of the solution. Both results are established by using multipliers technique and compactness/uniqueness arguments.

Keywords Boundary stabilization • Exponential stability • Fourth order Schrödinger equation • Internal stabilization

Mathematics Subject Classification (2010) Primary 93D15; Secondary 35Q40

1 Introduction

Let Ω be an open bounded domain of $\mathbb{R}^n$with sufficiently smooth boundary Γ. Let $\{\Gamma_0, \Gamma_1\}$ be a partition of Γ defined by

$$\Gamma_0 = \{x \in \Gamma, m(x) \cdot \nu(x) > 0\} \tag{1}$$

$$\Gamma_1 = \{x \in \Gamma, m(x) \cdot \nu(x) \leq 0\} \tag{2}$$

where $\nu(\cdot)$ is the unit normal vector to Γ pointing towards the exterior of Ω, $m(x) = x - x_0$, x_0 is a fixed point in the exterior of Ω such that

$$\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset \tag{3}$$

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In Ω, we consider the fourth order Schrödinger equation with boundary damping term supported on Γ_0

$$\frac{\partial y(x,t)}{\partial t} = i\Delta^2 y(x,t) \qquad \text{in } \Omega \times (0,+\infty), \tag{4}$$

$$y(x,0) = y_0(x) \qquad \text{in } \Omega, \tag{5}$$

$$y(x,t) = \frac{\partial y(x,t)}{\partial \nu} = 0 \qquad \text{on } \Gamma_1 \times (0,+\infty), \tag{6}$$

$$\Delta y(x,t) = 0 \qquad \text{on } \Gamma_0 \times (0,+\infty), \tag{7}$$

$$\frac{\partial \Delta y(x,t)}{\partial \nu} = m(x)\cdot\nu(x)\frac{\partial y(x,t)}{\partial t} \qquad \text{on } \Gamma_0 \times (0,+\infty), \tag{8}$$

The natural energy space for system (4)–(8) is the space

$$V = \left\{ f \in H^2(\Omega); f = \frac{\partial f}{\partial \nu} = 0 \text{ on } \Gamma_1 \right\}$$

endowed with the norm induced by the inner product

$$\langle f, g\rangle = \int_\Omega \Delta f(x)\Delta\overline{g(x)}dx$$

which in V is equivalent to the H^2-norm. Thus the energy function of a solution of system (4)–(8) is

$$\begin{aligned} E(t) &= \frac{1}{2}\|y(t)\|_V^2 \\ &= \frac{1}{2}\int_\Omega |\Delta y(x,t)|^2\, dx \end{aligned}$$

Semigroups theory may be applied to establish the wellposedness of system (4)–(8).

Theorem 1.1 *For any initial datum $y_0 \in V$, system (4)–(8) has a unique solution*

$$y \in C([0,+\infty);V) \cap C^1([0,+\infty),V')$$

Here V' is the dual of V. Moreover if $y_0 \in H^6(\Omega)\cap V$, and

$$\begin{aligned} \Delta y_0(x) &= 0 \;\; on\; \Gamma_0, \\ \frac{\partial \Delta y_0(x)}{\partial \nu} &= \mathrm{im}(x)\cdot\nu(x)\Delta^2 y_0(x) \;\; on\; \Gamma_0 \end{aligned}$$

then $y \in C^1([0,+\infty);V) \cap C([0,+\infty);H^6(\Omega) \cap V)$ *and satisfies*

$$\Delta y(x,t) = 0 \quad \textit{on } \Gamma_0 \times (0,+\infty)$$
$$\frac{\partial \Delta y(x,t)}{\partial \nu} = \mathrm{im}(x) \cdot \nu(x) \Delta^2 y(x,t) \;\; \textit{on } \Gamma_0 \times (0,+\infty)$$

In the following theorem we state an exponential stability result for system (4)–(8).

Theorem 1.2 *There exist positive constants M and δ such that for any initial datum $y_0 \in V$, the energy $E(\cdot)$ of the solution of the system* (4)–(8) *where Γ_0 and Γ_1 are given by* (1) *and* (2) *satisfies the inequality*

$$E(t) \leq Me^{-\delta t} E(0) \tag{9}$$

for all $t \geq 0$.

In this paper, we also study the stability problem for the fourth order Schrödinger equation with an internal damping term. To this aim, let $\omega \subset \Omega$ be a neighborhood of $\overline{\Gamma}_0$ and let $a(\cdot)$ be an $L^\infty(\Omega)$-function such that

$$\begin{cases} a(x) \geq 0 \text{ a.e. in } \Omega, \\ \exists a_0 > 0 : a(x) \geq a_0 \text{ a.e. in } \omega. \end{cases} \tag{10}$$

Consider the following internally damped fourth order Schrödinger equation

$$\frac{\partial y(x,t)}{\partial t} = i\Delta^2 y(x,t) - a(x) y(x,t) \qquad \text{in } \Omega \times (0,+\infty), \tag{11}$$

$$y(x,0) = y_0(x) \qquad \text{in } \Omega, \tag{12}$$

$$y(x,t) = \frac{\partial y(x,t)}{\partial \nu} = 0 \qquad \text{on } \Gamma \times (0,+\infty), \tag{13}$$

It is easy to see that system (11)–(13) has a unique solution in the class

$$y \in C([0,+\infty);L^2(\Omega)) \cap C^1([0,+\infty);(H^4(\Omega) \cap H_0^2(\Omega))')$$

if $y_0 \in L^2(\Omega)$, and in the class

$$y \in C([0,+\infty);H^4(\Omega) \cap H_0^2(\Omega)) \cap C^1([0,+\infty);L^2(\Omega))$$

if $y_0 \in H^4(\Omega) \cap H_0^2(\Omega)$.

Let us define the energy of a solution of system (11)–(13) as

$$F(t) = \frac{1}{2} \|y(t)\|^2_{L^2(\Omega)}$$
$$= \frac{1}{2} \int_\Omega |y(x,t)|^2 \, dx$$

We have the following exponential decay result for system (11)–(13).

Theorem 1.3 *Let $\omega \subset \Omega$ be a neighborhood of $\overline{\Gamma}_0$. Assume that the $L^\infty(\Omega)$-function $a(.)$ satisfies (10). Then, there exist positive constants M and $\delta > 0$ such that*

$$F(t) \leq Me^{-\delta t} F(0), \quad \forall t > 0$$

for every solution of (11)–(12) with initial datum $y^0 \in L^2(\Omega)$.

Control problems for the fourth order Schrödinger equation have been recently considered [1, 2]. In [2], it was shown that (4) with L^2-Neumann boundary control is exactly controllable in an arbitrarily short time $T > 0$ in $H^{-2}(\Omega)$. In [1], the authors showed that (4) with either Dirichlet or Neumann boundary control and associated colocated observation is well-posed in the sense of Salamon and regular in the sense of Weiss (see the references of [1]). They also established an exact controllability result for (4) with L^2-Dirichlet control. These results together with the one of [2] enabled them to deduce exponential stability of (4) with a dissipative feedback acting either in the Dirichlet or in the Neumann boundary conditions.

The rest of the paper is organized as follows. In Sect. 2, we sketch the general lines of the proof of Theorem 1.2, and in Sect. 3 we outline the proof of Theorem 1.3.

2 Sketch of the Proof of Theorem 1.2

We prove the theorem for regular solutions. The general case follows by a density argument. We proceed in several steps.

Step 1. We differentiate the energy function $E(\cdot)$ and apply Green's Theorem. We obtain

$$E(T) - E(0) = -\int_0^T \int_{\Gamma_0} m(x) \cdot \nu(x) \left| \frac{\partial y(x,t)}{\partial t} \right|^2 d\Gamma dt$$

for all $T > 0$.

Step 2. We multiply both sides of (4) by $m(x) \cdot \nabla \overline{y(x,t)}$ and integrate by parts over $\Omega \times (0,T)$. We obtain

$$
\begin{aligned}
&4\int_0^T \int_\Omega |\Delta y(x,t)|^2 \, dxdt \\
&= \mathrm{Im} \int_\Omega y(x,t) m(x) \cdot \nabla \overline{y(x,t)} \Big|_0^T dx - 2\mathrm{Re} \int_0^T \int_{\Gamma_0} \frac{\partial \Delta y(x,t)}{\partial \nu} m(x) \\
&\cdot \nabla \overline{y(x,t)} d\Gamma dt - n\mathrm{Re} \int_0^T \int_{\Gamma_0} y(x,t) \frac{\partial \Delta \overline{y(x,t)}}{\partial \nu} d\Gamma dt \\
&- \mathrm{Im} \int_0^T \int_{\Gamma_0} y(x,t) \frac{\partial \overline{y(x,t)}}{\partial t} m(x) \cdot \nu(x) d\Gamma dt \\
&+ \int_0^T \int_{\Gamma_1} |\Delta y(x,t)|^2 m(x) \cdot \nu(x) d\Gamma dt
\end{aligned}
$$

Since $m(x) \cdot \nu(x) \leq 0$ on Γ_1, then we have

$$
\begin{aligned}
&4\int_0^T \int_\Omega |\Delta y(x,t)|^2 \, dxdt \\
&\leq \mathrm{Im} \int_\Omega y(x,t) m(x) \cdot \nabla \overline{y(x,t)} \Big|_0^T dx - 2\mathrm{Re} \int_0^T \int_{\Gamma_0} \frac{\partial \Delta y(x,t)}{\partial \nu} m(x) \\
&\cdot \nabla \overline{y(x,t)} d\Gamma dt - n\mathrm{Re} \int_0^T \int_{\Gamma_0} y(x,t) \frac{\partial \Delta \overline{y(x,t)}}{\partial \nu} d\Gamma dt \\
&- \mathrm{Im} \int_0^T \int_{\Gamma_0} y(x,t) \frac{\partial \overline{y(x,t)}}{\partial t} m(x) \cdot \nu(x) d\Gamma dt
\end{aligned}
$$

Step 4. Using the Cauchy inequality, the Poincaré's inequality, and the trace theorem, we get (here and throughout the rest of the paper C is some positive constant different at different occurences)

$$
\begin{aligned}
4\int_0^T \int_\Omega |\Delta y(x,t)|^2 \, dxdt \leq{} & C \int_\Omega (|\nabla y(x,0)|^2 + |\nabla y(x,T)|^2 \, dx \\
&+ C \int_0^T \int_{\Gamma_0} m(x) \cdot \nu(x) \left| \frac{\partial y(x,t)}{\partial t} \right|^2 d\Gamma dt \\
&+ \epsilon(\mu_1 + \mu_2) \int_0^T \int_0^T \int_\Omega |\Delta y(x,t)|^2 \, dxdt
\end{aligned}
$$

where ϵ is a positive constant to be fixed later, μ_1 and μ_2 are such that

$$\int_{\Gamma_0} |\nabla\psi(x)|^2\, d\Gamma \le \mu_1 \int_\Omega |\Delta\psi(x)|^2\, dx$$

$$\int_{\Gamma_0} |\psi(x)|^2\, dx \le \mu_2 \int_\Omega |\Delta\psi(x)|^2\, dxdt$$

for all $\psi \in V$.

Choosing ϵ sufficiently small so that $4 - \epsilon(\mu_1 + \mu_2) > 0$, we arrive at

$$E(T) \le C \int_0^T \int_{\Gamma_0} m(x)\cdot\nu(x) \left|\frac{\partial y(x,t)}{\partial t}\right|^2 d\Gamma dt + C\,\|y\|^2_{C(0,T;H^1_{\Gamma_1}(\Omega))} \tag{14}$$

where

$$H^1_{\Gamma_1}(\Omega) = \{f \in H^1(\Omega); f = 0 \text{ on } \Gamma_1\}$$

Step 5. We drop the lower order term on the right-hand side of (14) by compactness-uniqueness arguments to obtain

$$E(T) \le C \int_0^T \int_{\Gamma_0} m(x)\cdot\nu(x) \left|\frac{\partial y(x,t)}{\partial t}\right|^2 d\Gamma dt$$

from which follows the desired stability estimate.

3 Sketch of the Proof of Theorem 1.3

We prove the theorem for smooth initial data. The general case follows by a density argument.

Step 1. Differentiating the energy function $F(\cdot)$ and applying Green's Theorem, we obtain

$$F(T) - F(0) = -\int_0^T \int_\Omega a(x)\,|y(x,t)|^2\, dx$$

Step 2. We rewrite the solution y of (11)–(13) as $y = \varphi + \psi$ where $\varphi = \varphi(x,t)$ solves

$$\begin{aligned}
&\frac{\partial\varphi(x,t)}{\partial t} = i\Delta^2\varphi(x,t) && \text{in } \Omega\times(0,+\infty)\\
&\varphi(x,0) = y_0(x) && \text{in } \Omega\\
&\varphi(x,t) = \frac{\partial\varphi(x,t)}{\partial\nu} = 0 && \text{on } \Gamma\times(0,+\infty)
\end{aligned}$$

and $\psi = \psi(x,t)$ satisfies

$$\begin{aligned} &\frac{\partial \psi(x,t)}{\partial t} = i\Delta^2\psi(x,t) - a(x)y(x,t) && \text{in } \Omega \times (0,+\infty) \\ &\psi(x,0) = 0 && \text{in } \Omega \\ &\psi(x,t) = \frac{\partial \psi(x,t)}{\partial \nu} = 0 && \text{on } \Gamma \times (0,+\infty) \end{aligned}$$

Using multipliers techniques and compactness-arguments, we establish the following observability estimate for the φ-problem

$$\|y_0\|^2_{L^2(\Omega)} \le C \int_0^T \int_\omega |\varphi(x,t)|^2 \, dxdt$$

Combining this inequality with the decomposition of y and the assumptions made on $a(\cdot)$, we get

$$F(T) \le \frac{C}{2a_0} \int_0^T \int_\Omega a(x)\,|y(x,t)|^2 \, dxdt + \frac{C\,\|a\|_{L^\infty(\Omega)}}{2a_0} \int_0^T \int_\Omega |\psi(x,t)|^2 \, dxdt$$

Step 3. From standard energy estimates on the fourth order Schrödinger equation we have

$$\|\psi\|^2_{L^\infty(0,T;L^2(\Omega))} \le C\,\|a\|_{L^\infty(\Omega)} \int_0^T \int_\Omega a(x)\,|y(x,t)|^2 \, dxdt$$

Inserting this estimate into the previous one, we obtain the sought-after stability result.

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The External Damping Cauchy Problems with General Powers of the Laplacian

Pham Trieu Duong and Michael Reissig

Abstract We present in this note the survey on the following external damping problem

$$
\begin{aligned}
u_{tt} + (-\Delta)^{\sigma} u + u_t &= ||D|^a u|^p, \\
u(0,x) &= u_0(x), \\
u_t(0,x) &= u_1(x),
\end{aligned}
$$

with the assumption: $0 < a < \sigma$, $p > 1$, the parameter σ is from the range $\sigma \in (1, \infty)$. In many literatures that are devoted to the similar models, the range $0 < \sigma < 1$ is stated frequently by the default, since only inside this restricted interval, the operator $(-\Delta)^{\sigma}$ is *stable*. We present the research approach that shows which essential conditions for the differential operators that may influence on the decay estimates and the global existence of solution as well for these initial value problems.

Keywords Cauchy problem • Critical exponent • External damping

Mathematics Subject Classification (2010) 35B40; 35B44; 35E15

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1 The Notions and Results for the Structural Damping Models

Our proposed problem is one special case of more general semi-linear structurally damping models

$$
\begin{aligned}
&u_{tt} + (-\Delta)^\sigma u + (-\Delta)^\delta u_t = F(u, u_t, |D|^\alpha u),\\
&u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x).
\end{aligned} \tag{1}
$$

The parameter $\delta \in [0,\sigma]$ generates a family of structurally damped models interpolating between exterior damping $\delta = 0$ and visco-elastic type damping $\delta = \sigma$. The range of α is the interval $(0,\sigma)$. We will draw some main comparisons between the posed models. We recall here for the reference our previous results with the general power exponents σ, δ.

Proposition 1.1 *Let us consider the Cauchy problem*

$$
v_{tt} + (-\Delta)^\sigma v + (-\Delta)^\delta v_t = 0, \; v(0,x) = v_0(x), \; v_t(0,x) = v_1(x), \tag{2}
$$

for $\delta \in \big(0, \dfrac{\sigma}{2}\big)$ *and data* $(v_0, v_1) \in (L^1 \cap H^\sigma) \times (L^1 \cap L^2)$. *Then the solution and its energy satisfy in arbitrary dimensions* n *the following* $(L^1 \cap L^2) - L^2$ *estimates*:

$$
\|v(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4(\sigma-\delta)}} \|v_0\|_{L^1\cap L^2} + (1+t)^{1-\frac{n}{4(\sigma-\delta)}} \|v_1\|_{L^1\cap L^2}, \tag{3}
$$

$$
\|v_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n+4\delta}{4(\sigma-\delta)}} \|v_0\|_{L^1\cap H^\sigma} + (1+t)^{1-\frac{n+4\delta}{4(\sigma-\delta)}} \|v_1\|_{L^1\cap L^2}, \tag{4}
$$

$$
\||D|^\sigma v(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n+2\sigma}{4(\sigma-\delta)}} \|v_0\|_{L^1\cap H^\sigma} + (1+t)^{1-\frac{n+2\sigma}{4(\sigma-\delta)}} \|v_1\|_{L^1\cap L^2}, \tag{5}
$$

and the following $L^2 - L^2$ *estimates*:

$$
\|v(t,\cdot)\|_{L^2} \lesssim \|v_0\|_{L^2} + (1+t)\|v_1\|_{L^2}, \tag{6}
$$

$$
\|v_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-1}\|v_0\|_{H^\sigma} + \|v_1\|_{L^2}, \tag{7}
$$

$$
\||D|^\sigma v(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{\sigma}{2(\sigma-\delta)}} \|v_0\|_{H^\sigma} + (1+t)^{-\frac{\sigma-2\delta}{2(\sigma-\delta)}} \|v_1\|_{L^2}. \tag{8}
$$

The analogous results with the other range $\delta \in \big(\dfrac{\sigma}{2}, \sigma\big)$ were also obtained in [4]. We also note that the above estimates were obtained by using quite rough inequalities with the oscillating integrals in order to avoid extra condition on the dimension n. For optimal results, the authors would refer the reader to the papers [2, 3] and [1] where the more refined estimates were obtained successfully.

2 The External Damping Equations

Let us shift now our focus to one special case $\delta = 0$, that is the problem (equation in the abstract). We notice some minor difficulties that appear now. This can be seen from the characteristic equation $\lambda^2 + |\xi|^{2\delta}\lambda + |\xi|^{2\sigma} = 0$ which has the roots $\lambda_{1,2}(\xi) = \frac{1}{2}\Big(- |\xi|^{2\delta} \pm \sqrt{|\xi|^{4\delta} - 4|\xi|^{2\sigma}}\Big)$, that become $\lambda_{1,2}(\xi) = \frac{1}{2}\Big(- 1 \pm \sqrt{1 - 4|\xi|^{2\sigma}}\Big)$.

The absence of possible positive power of $|\xi|$ before and inside the square root sign for the trivial $\delta = 0$ seems to be difficult to apply the change of variables to estimate the Fourier integrals. The decaying rates such as $(1 + t)^{-\gamma}$ with a suitable $\gamma > 0$ are not too obvious. In order to deal at this point, the abstract settings for the diffusion problem introduced by Radu et al. [6] are helpful in order to obtain the desired estimates. We recall that for the problem

$$\begin{aligned} u_{tt} + Bu + u_t &= 0, \\ u(0, \cdot) &= u_0, \\ u_t(0, \cdot) &= u_1, \end{aligned} \tag{9}$$

some conditions can be applied on the operator B as such the Markovian or the ultra-contracting (see [6]) properties to study the diffusion phenomenon. By exploiting the refined Bessel functions techniques we can avoid the stochastic settings required on B and we can also see further that even in the case $s > 1$ some diffusion properties still hold for the $(-\Delta)^s$. This confirms the statement that diffusion phenomenon is closely related to the spectral asymptotic near 0 (the small frequencies $|\xi|$) for the operator B. Our results in this direction are the following.

Proposition 2.1 *The solution $v(t, x)$ of the linear Cauchy problem for external damping model*

$$\begin{aligned} v_{tt} + (-\Delta)^\sigma v + v_t &= 0, \\ v(0, x) &= v_0(x), \\ v_t(0, x) &= v_1(x), \end{aligned} \tag{10}$$

and its derivatives satisfy the following $(L^1 \cap L^2) - L^2$ estimates

$$\|v(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{n}{4\sigma}} \|v_0\|_{L^1 \cap L^2} + (1 + t)^{-\frac{n}{4\sigma}} \|v_1\|_{L^1 \cap H^{-\sigma}}, \tag{11}$$

$$\|v(t, \cdot)\|_{\dot{H}^\sigma} \lesssim (1 + t)^{-\frac{n}{4\sigma} - \frac{1}{2}} \|v_0\|_{L^1 \cap \dot{H}^\sigma} + (1 + t)^{-\frac{n}{4\sigma} - \frac{1}{2}} \|v_1\|_{L^1 \cap L^2}, \tag{12}$$

$$\|v_t(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{n}{4\sigma} - 1} \|v_0\|_{L^1 \cap \dot{H}^\sigma} + (1 + t)^{-\frac{n}{4\sigma} - 1} \|v_1\|_{L^1 \cap L^2}, \tag{13}$$

$$\|v(t, \cdot)\|_{\dot{H}^k} \lesssim (1 + t)^{-\frac{n}{4\sigma} - \frac{k}{2\sigma}} \|v_0\|_{L^1 \cap \dot{H}^k} + (1 + t)^{-\frac{n}{4\sigma} - \frac{k}{2\sigma}} \|v_1\|_{L^1 \cap \dot{H}^{k - \frac{1}{2}}} \tag{14}$$

for all $k \geq 0$, and the $L^2 - L^2$ estimates

$$\|v(t,.)\|_{L^2} \lesssim \|v_0\|_{L^2} + (1+t)\|v_1\|_{L^2}, \tag{15}$$

$$\|v_t(t,.)\|_{L^2} \lesssim (1+t)^{-1}\|v_0\|_{H^\sigma} + \|v_1\|_{L^2}, \tag{16}$$

$$\||D|^\sigma v(t,.)\|_{L^2} \lesssim (1+t)^{-\frac{1}{2}}\|v_0\|_{H^\sigma} + (1+t)^{-\frac{1}{2}}\|v_1\|_{L^2}. \tag{17}$$

As it was done with the classical damping model, we can also compare the large time behaviors of solutions $v(t,x)$ for the problem (10) and $w(t,x)$ for the problem

$$w_t + (-\Delta)^\sigma w = 0, \tag{18}$$

$$w(x,0) = v_0 + v_1. \tag{19}$$

Proposition 2.2 (The Asymptotic Profile of Solution) *The difference $v - w$ of solutions of these two problems with the initial data from the space $(L^1 \cap H^\sigma) \times (L^1 \cap L^2)$ satisfies the following decay rates*

$$\|v-w\|_2 \lesssim (t+1)^{-1-\frac{n}{4\sigma}}\big(\|v_0\|_{L_1\cap L_2} + \|v_1\|_{L_1\cap H^{-\sigma}}\big),$$

$$\|v-w\|_{\dot{H}^k} \lesssim (t+1)^{-1-\frac{n}{4\sigma}-k}\big(\|v_0\|_{L_1\cap\dot{H}^k} + \|v_1\|_{L_1} + \||D|^{2k\sigma}v_1\|_{H^{-\sigma}}\big).$$

These linear estimates allow us to obtain elementary results on the possible range of the admissible exponents p in some nonlinear problems.

Theorem 2.3 (Main Theorem) *Let us consider the following Cauchy problem*

$$u_{tt} + (-\Delta)^\sigma u + u_t = ||D|^a u|^p, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \tag{20}$$

with $\sigma > 0$, $a \in [0,\sigma)$, where the data are chosen from the space $A := (L^1 \cap H^\sigma) \times (L^1 \cap L^2)$. Assume that $\dfrac{n+2\sigma}{n+a} < p < \dfrac{n}{n+2(a-\sigma)}$ and $p \geq 2$. Then there exists the (global) solution $u(t,x) \in \mathcal{C}([0,\infty), H^\sigma) \cap \mathcal{C}^1([0,\infty), L^2)$ for any small data (u_0,u_1). Moreover, the estimates (11)–(13) are satified also for the solution $u(x,t)$, where the Cauchy data (v_0,v_1) of the linear problem in the right-hand sides are replaced by (u_0,u_1) in the non-linear case, respectively.

Remark 2.4 The interval $\big(\frac{n+2\sigma}{n+a}, \frac{n}{n+2(a-\sigma)}\big)$ can be empty in several cases. For instance, taking the limit $a \to \sigma^-$ one can see that the expression $\frac{n}{n+2(a-\sigma)}$ has the values closer to 1^+, which is contrary to the condition $p \geq 2$. In order to get a non-void interval for p, we may apply the conditions for a, σ more strictly as follows: $a < \frac{4\sigma^2}{n+4\sigma}$ and $\sigma > \frac{n+4a}{4}$.

Proof We introduce for all $t > 0$ the function spaces $X(t) := \mathcal{C}\big([0,t], H^\sigma\big) \cap \mathcal{C}^1\big([0,t], L^2\big)$ with the norm

$$\|u(\tau,\cdot)\|_{X(t)} = \sup_{0\le\tau\le t} \big(f_0(\tau)^{-1}\|u(\tau,\cdot)\|_{L^2} + f_\sigma(\tau)^{-1}\||D|^\sigma u(\tau,\cdot)\|_{L^2} + g(\tau)^{-1}\|u_t(\tau,\cdot)\|_{L^2}\big),$$

and the space $X_0(t) := \mathcal{C}\big([0,t], H^\sigma\big)$ with the norm

$$\|w(\tau,\cdot)\|_{X_0(t)} := \sup_{0\le\tau\le t} \Big(f_0(\tau)^{-1}\|w(\tau,\cdot)\|_{L^2} + f_\sigma(\tau)^{-1}\||D|^\sigma w(\tau,\cdot)\|_{L^2}\Big),$$

where from the estimates of Proposition 2.1 we choose $f_0(\tau) := (1+\tau)^{-\frac{n}{4\sigma}}$, $f_\sigma(\tau) := (1+\tau)^{-\frac{n}{4\sigma}-1/2}$, $g(\tau) := (1+\tau)^{-\frac{n}{4\sigma}-1}$.

We define the operator $N:\ u \in X(t) \to Nu \in X(t)$ by:

$$Nu(t,x) = G_0(t,x) *_x u_0(x) + G_1(t,x) *_x u_1(x) + \int_0^t G_1(t-\tau,x) *_x \big||D|^a u(\tau,x)\big|^p d\tau.$$

In order to prove the Theorem we will show that for the exponent p satisfying the given conditions the estimate

$$\|Nu(t,\cdot)\|_{X(t)} \lesssim \|(u_0,u_1)\|_A + \|u(t,\cdot)\|^p_{X_0(t)}. \tag{21}$$

and the Lipschitz property

$$\|Nu(t,\cdot) - Nv(t,\cdot)\|_{X(t)} \lesssim \big\|u(t,\cdot) - v(t,\cdot)\big\|_{X_0(t)}\Big(\big\|u(t,\cdot)\big\|^{p-1}_{X_0(t)} + \big\|v(t,\cdot)\big\|^{p-1}_{X_0(t)}\Big) \tag{22}$$

must hold. We use the $L^1 \cap L^2 - L^2$ estimates if $\tau \in [0,t/2]$ and the $L^2 - L^2$ estimates if $\tau \in [t/2,t]$. Then

$$\begin{aligned}\|\partial_t^j |D|^{k\sigma} Nu(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{n}{4\sigma}-(k/2+j)}\|(u_0,u_1)\|_{(L^1\cap H^{\sigma(k+j)})\times(L^1\cap H^{\sigma(k+j-1)})}\\ &\quad + \int_0^{t/2}(1+t-\tau)^{-\frac{n}{4\sigma}-(k/2+j)}\big\|\big||D|^a u(\tau,\cdot)\big|^p\big\|_{L^1\cap L^2}d\tau\\ &\quad + \int_{t/2}^t (1+t-\tau)^{1-(3k/2+j)}\big\|\big||D|^a u(\tau,\cdot)\big|^p\big\|_{L^2}d\tau,\end{aligned}$$

where $j,k = 0,1$ and $(j,k) \neq (1,1)$. We will estimate $\big||D|^a u(\tau,\cdot)\big|^p$ in $L^1 \cap L^2$ and in L^2. Obviously $\big\|\big||D|^a u(\tau,\cdot)\big|^p\big\|_{L^1\cap L^2} \lesssim \big\||D|^a u(\tau,\cdot)\big\|^p_{L^p} + \big\|u(\tau,\cdot)\big\|^p_{L^{2p}}$, and $\big\|\big||D|^a u(\tau,\cdot)\big|^p\big\|_{L^2} = \big\||D|^a u(\tau,\cdot)\big\|^p_{L^{2p}}$.

We apply the fractional Gagliardo–Nirenberg inequality (see [3, 5] and [4] for the formulation, proof, and notations) with the interpolation exponents $\theta_{a,\sigma}(p,2)$ and $\theta_{a,\sigma}(2p,2)$ from the interval $[\frac{a}{\sigma},1)$. This gives the condition $2 \le p < \dfrac{n}{n+2(a-\sigma)}$. Accordingly:

$$\big\| \big| |D|^a u(\tau,\cdot)\big|^p \big\|_{L^1\cap L^2} \lesssim (1+\tau)^{\frac{-p(n+a)+n}{2\sigma}} \|u(\tau,\cdot)\|^p_{X_0(\tau)},$$

because of $\theta_{a,\sigma}(p,2) < \theta_{a,\sigma}(2p,2)$, meanwhile

$$\begin{aligned}\big\| \big| |D|^a u(\tau,\cdot)\big|^p \big\|_{L^2} &\lesssim (1+\tau)^{p\left(-\frac{n}{4\sigma}-\frac{\theta_{a,\sigma}(2p,2)}{2}\right)} \|u(\tau,\cdot)\|^p_{X_0(\tau)}\\ &= (1+\tau)^{-\frac{p(n+a)-n/2}{2\sigma}} \|u(\tau,\cdot)\|^p_{X_0(\tau)}.\end{aligned}$$

Combining the last estimates we conclude

$$\begin{aligned}\|\partial_t^j |D|^{k\sigma} Nu(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{n}{4\sigma}-(k/2+j)} \|(u_0,u_1)\|_{(L^1\cap H^{\sigma(k+j)})\times(L^1\cap H^{\sigma(k+j-1)})}\\ &\quad + (1+t)^{-\frac{n}{4\sigma}-(k/2+j)} \|u\|^p_{X_0(t)} \int_0^{t/2} (1+\tau)^{\frac{-p(n+a)+n}{2\sigma}}\, d\tau\\ &\quad + (1+t)^{-\frac{p(n+a)-n/2}{2\sigma}} \|u\|^p_{X_0(t)} \int_{t/2}^{t} (1+t-\tau)^{1-(3k/2+j)}\, d\tau.\end{aligned}$$

If $p > \dfrac{n+2\sigma}{n+a}$, then the term $(1+\tau)^{\frac{-p(n+a)+n}{2\sigma}}$ is integrable. Moreover, we have

$$\begin{aligned}&(1+t)^{-\frac{p(n+a)-n/2}{2\sigma}} \|u\|^p_{X_0(t)} \int_{t/2}^{t} (1+t-\tau)^{1-(3k/2+j)}\, d\tau\\ &\quad = (1+t)^{-\frac{p(n+a)-n/2}{2\sigma}} \|u\|^p_{X_0(t)} \int_0^{t/2} (1+\tau)^{1-(3k/2+j)}\, d\tau\\ &\quad \lesssim (1+t)^{-\frac{n}{4\sigma}-(k/2+j)} \|u\|^p_{X_0(t)}.\end{aligned}$$

We can follow the same approach to prove other inequalities in (21)–(22). This completes the proof. □

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A Note on the Blow-Up of Solutions to Nakao's Problem

Yuta Wakasugi

Abstract We consider the critical exponent problem for the Cauchy problem of the system of semilinear damped wave and wave equations. This problem is proposed by Professor Mitsuhiro Nakao. In this note, we prove some blow-up results, which give the answer to the one-dimensional case and a partial answer to higher dimensional cases.

Keywords Blow-up of solutions • Nakao's problem

Mathematics Subject Classification (2010) Primary 35L52; Secondary 35B44

1 Introduction

In this note, we consider the system of semilinear damped wave and wave equations

$$\begin{cases} u_{tt} - \Delta u + u_t = |v|^p, \\ v_{tt} - \Delta v = |u|^q, \end{cases} \qquad t > 0, x \in \mathbb{R}^N \tag{1}$$

with initial data

$$(u, u_t, v, v_t)(0, x) = (u_0, u_1, v_0, v_1)(x). \tag{2}$$

Professor Mitsuhiro Nakao, Emeritus of Kyushu University, proposed the critical exponent problem to (1)–(2) (see Sect. 4 of Nishihara and Wakasugi [4]). Here the word *critical exponent* means the threshold condition of the exponents p, q for global existence and blow-up of solutions with small initial data. In this note, we give the answer to the one-dimensional case and a partial answer to higher dimensional cases.

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We assume that the exponents p, q satisfy $1 < p, q < \infty$ when $N = 1, 2$, and $1 < p, q \leq N/(N-2)$ when $N \geq 3$, and the initial data $(u_0, u_1), (v_0, v_1)$ belong to $H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ with compact support. Then, the local existence of weak solutions is proved in a standard way (see, for example, [5]). Moreover, there exists a maximal existence time $T_* \in (0, \infty]$ such that the local weak solution (u, v) belong to $[C([0, T_*); H^1(\mathbb{R}^N)) \cap C^1([0, T_*); L^2(\mathbb{R}^N))]^2$ and if $T_* < +\infty$, then the $H^1 \times L^2$-norm of the solution blows up at the time T_*.

For the system of semilinear wave equation

$$\begin{cases} u_{tt} - \Delta u = |v|^p, \\ v_{tt} - \Delta v = |u|^q, \end{cases} \tag{3}$$

it is known that if the number

$$\max\left\{\frac{q+2+p^{-1}}{pq-1}, \frac{p+2+q^{-1}}{pq-1}\right\} - \frac{N-1}{2} \tag{4}$$

is negative, then the global solution uniquely exists for small initial data, and if the number (4) is nonnegative and the initial data satisfies some positivity condition, then the solution must blow up in finite time (see Del Santo et al. [1] and Kurokawa et al. [2] and the references therein).

On the other hand, for the system of semilinear damped wave equation

$$\begin{cases} u_{tt} - \Delta u + u_t = |v|^p, \\ v_{tt} - \Delta v + v_t = |u|^q, \end{cases} \tag{5}$$

the critical exponent is given by

$$\max\left\{\frac{q+1}{pq-1}, \frac{p+1}{pq-1}\right\} - \frac{N}{2}, \tag{6}$$

that is, if the number (6) is negative, then there exists a unique global solution for small initial data, while the local solution with some positive data blows up in finite time if the number (6) is nonnegative (see Sun and Wang [6] and [3] and the references therein). We note that the number (6) is less than the number (4).

In view of the above results, we expect that the system (1) has another critical exponent, which will be between the numbers (6) and (4). However, there are no results about both global existence and blow-up of solutions as far as the author knows. In this note, we give a sufficient condition for the blow-up of solutions to (1). To state our results, we give the definition of weak solutions to (1):

Definition 1.1 Let $T \in (0, \infty]$ and let $(u_0, u_1, v_0, v_1) \in [L^1_{loc}(\mathbb{R}^N)]^4$. We say that a pair of function $(u, v) \in [L^1_{loc}([0, T) \times \mathbb{R}^N)]^2$ is a weak solution to (1) on the time interval $[0, T)$ if the identities

$$\int_{[0,T)\times\mathbb{R}^N} u\,(\psi_{tt} - \Delta\psi - \psi_t)\,dxdt = \int_{\mathbb{R}^N} [(u_0 + u_1)\psi(0,x) - u_0\psi_t(0,x)]\,dx + \int_{[0,T)\times\mathbb{R}^N} |v|^p \psi\,dxdt,$$

$$\int_{[0,T)\times\mathbb{R}^N} v\,(\psi_{tt} - \Delta\psi)\,dxdt = \int_{\mathbb{R}^N} [v_1\psi(0,x) - v_0\psi_t(0,x)]\,dx + \int_{[0,T)\times\mathbb{R}^N} |u|^q \psi\,dxdt$$

hold for any $\psi \in C_0^\infty([0,T) \times \mathbb{R}^N)$. If we can take T arbitrarily large, we call (u, v) a global-in-time weak solution.

Our main result is the following:

Theorem 1.2 *Let $N \geq 1$ and let $1 < p, q < \infty$ satisfy*

$$\max\left\{\frac{q+2}{pq-1} + 1,\ \frac{2(q+1)}{pq-1},\ \frac{2(p+1)}{pq-1}\right\} - N \geq 0. \tag{7}$$

Moreover, we assume that the initial data $(u_0, u_1, v_0, v_1) \in [L^1_{loc}(\mathbb{R}^N)]^4$ satisfy

$$\liminf_{R\to\infty} \int_{|x|<R} (u_0 + u_1)(x)dx > 0, \quad \liminf_{R\to\infty} \int_{|x|<R} v_1(x)dx > 0. \tag{8}$$

Then, there is no global weak solution to (1)–(2).

Corollary 1.3 *Let $1 < p, q < \infty$ $(N = 1, 2)$, $1 < p, q \leq N/(N-2)$ $(N \geq 3)$ satisfy (7) and assume that $(u_0, u_1), (v_0, v_1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ have compact support and satisfy (8). Then the maximal existence time T_* of the local weak solution $(u, v) \in [C([0, T_*); H^1(\mathbb{R}^N)) \cap C^1([0, T_*); L^2(\mathbb{R}^N))]^2$ is finite and we have*

$$\lim_{t\to T_*-} (\|(u, u_t)(t)\|_{H^1\times L^2} + \|(v, v_t)(t)\|_{H^1\times L^2}) = +\infty.$$

Remark 1.4 When $N = 1$, the condition (7) is always valid and there is no restriction on the exponents p, q. When $N \geq 2$, the number in the left-hand side of (7) is between (6) and (4). However, the number in (7) seems not to be optimal.

Remark 1.5 Corollary 1.3 follows from Theorem 1.2 and a contradiction argument. In fact, first, the local existence result (see, for example, [5]) shows that if the initial data belong to $H^1 \times L^2$ and have compact support, then there exists a time $T > 0$, which depends only on the $H^1 \times L^2$ norm of the data, such that the weak solution $(u, v) \in [C([0, T); H^1(\mathbb{R}^N)) \cap C^1([0, T); L^2(\mathbb{R}^N))]^2$ uniquely exists on the interval $[0, T]$. Furthermore, for each $t \in [0, T)$, the solution $(u(t), v(t))$ have compact support. Let T_* be the maximal existence time of such a weak solution. Then, Theorem 1.2 implies that T_* must be finite. Let us suppose that the local solution satisfies

$$\liminf_{t\to T_*-} (\|(u,u_t)(t)\|_{H^1\times L^2} + \|(v,v_t)(t)\|_{H^1\times L^2}) \le M$$

with some $M \ge 0$. Then, there exists a sequence of time $\{t_j\}_{j=1}^{\infty} \subset [0,T_*)$ such that $\{t_j\}$ tends to T_* and

$$\left(\|(u,u_t)(t_j)\|_{H^1\times L^2} + \|(v,v_t)(t_j)\|_{H^1\times L^2}\right) \le M+1$$

for any $j \ge 1$. Moreover, by the local existence result, there exists a time T, which depends only on M, such that the solution $(u,v)(t_j)$ can be uniquely extended to the time interval $[t_j, t_j+T]$. However, for sufficiently large j, $t_j + T > T_*$ holds and this contradicts the maximality of T_*.

2 Proof of the Blow-Up by a Test Function Method

In this section, we give a proof of Theorem 1.2. The proof is based on a test function method developed by Q.S. Zhang [7]. We suppose that there exists a global-in-time weak solution (u,v). Let $\tau \in [\tau_0,\infty)$ be a parameter, where $\tau_0 \ge 1$ is determined later. We define a test function $\psi_\tau(t,x)$ by $\psi_\tau(t,x) = \eta_\tau(t)\phi_\tau(x) = \eta(t/\tau)\phi(x/\tau^\alpha)$, where $\alpha \in [1/2,1]$, $\phi(x) = \eta(|x|)$ and $\eta \in C_0^\infty([0,\infty))$, $0 \le \eta \le 1$, $\eta(t) = 1$ $(0 \le t \le 1/2)$, $\eta(t) = 0$ $(t \ge 1)$. It is easy to see that

$$|\eta'(t)| \le C\eta(t)^{1/r}, \quad |\eta''(t)| \le C\eta(t)^{1/r}, \quad |\Delta\phi(x)| \le C\phi(x)^{1/r}$$

for any $r > 1$, hereafter C denotes generic constants which may change from line to line. We define

$$V_\tau = \int_0^\infty \int_{\mathbb{R}^N} |v|^p \psi_\tau dxdt, \quad U_\tau = \int_0^\infty \int_{\mathbb{R}^N} |u|^q \psi_\tau dxdt$$

and

$$S_\tau = \int_{\mathbb{R}^N} v_1 \phi_\tau dx, \quad T_\tau = \int_{\mathbb{R}^N} (u_0 + u_1)\phi_\tau dx.$$

By the assumption (8), there is some $\tau_0 \ge 1$ such that for any $\tau \ge \tau_0$ we have $S_\tau, T_\tau \ge 0$. By noting this and $\partial_t \psi_\tau(0,x) = 0$ we deduce from Definition 1.1 that

$$V_\tau \le \int_0^\infty \int_{\mathbb{R}^N} |u| \left|(\partial_t^2 - \Delta - \partial_t)\psi_\tau\right| dxdt =: K_1 + K_2 + K_3.$$

The Hölder inequality yields

$$\begin{aligned} K_1 &\le C\tau^{-2}\int_0^\infty \int_{\mathbb{R}^N} |u||\eta''(t/\tau)|\phi_\tau(x)dxdt \\ &\le C\tau^{-2+(1+\alpha N)/q'}\tilde{U}_\tau^{1/q}, \end{aligned}$$

where q' stands for the conjugate of q and

$$\tilde{U}_\tau = \int_{\tau/2}^{\tau}\int_{\mathbb{R}^N} |u|^q\psi_\tau dxdt.$$

Similarly, we see that

$$K_2 \le C\tau^{-2\alpha+(1+\alpha N)/q'}\hat{U}_\tau^{1/q}, \quad K_3 \le C\tau^{-1+(1+\alpha N)/q'}\tilde{U}_\tau^{1/q},$$

where

$$\hat{U}_\tau = \int_0^\tau \int_{\tau^\alpha/2\le|x|\le\tau^\alpha} |u|^q\psi_\tau dxdt.$$

Therefore, we obtain

$$V_\tau \le C\tau^{-1+(1+\alpha N)/q'}\left(\tilde{U}_\tau^{1/q} + \hat{U}_\tau^{1/q}\right). \tag{9}$$

Here we used that $-2 + (1+\alpha N)/q' \le -1 + (1+\alpha N)/q'$ and $\alpha \in [1/2, 1]$.

By the same argument and noting that there is no damping term in the equation of v, we also have

$$U_\tau \le C\tau^{-2\alpha+(1+\alpha N)/p'}\left(\tilde{V}_\tau^{1/p} + \hat{V}_\tau^{1/p}\right), \tag{10}$$

where $\tilde{V}_\tau, \hat{V}_\tau$ are defined by the same way as $\tilde{U}_\tau, \hat{U}_\tau$. From the inequalities $\tilde{U}_\tau, \hat{U}_\tau \le U_\tau$ and $\tilde{V}_\tau, \hat{V}_\tau \le V_\tau$, we also obtain the following two estimates:

$$U_\tau \le C\tau^{a(\alpha)}\left(\tilde{U}_\tau + \hat{U}_\tau\right)^{1/(pq)}, \quad V_\tau \le C\tau^{b(\alpha)}\left(\tilde{V}_\tau + \hat{V}_\tau\right)^{1/(pq)}, \tag{11}$$

where

$$\begin{aligned} a(\alpha) &= (1+\alpha N)\left(1 - \frac{1}{pq}\right) - 2\alpha - \frac{1}{p}, \\ b(\alpha) &= (1+\alpha N)\left(1 - \frac{1}{pq}\right) - 1 - \frac{2\alpha}{q}. \end{aligned}$$

Now, once we assume that $a(\alpha) \leq 0$ or $b(\alpha) \leq 0$ holds for some $\alpha \in [1/2, 1]$, then, it follows from (11) that either U_τ or V_τ, let us say U_τ, is bounded uniformly in $\tau \geq \tau_0$. By the definition of U_τ, letting $\tau \to \infty$ leads to $u \in L^q([0,\infty) \times \mathbb{R}^N)$. However, this implies $\tilde{U}_\tau, \hat{U}_\tau \to 0$ as $\tau \to \infty$ and we obtain from (11) that $U_\tau \to 0$, which means u is identically zero. This and the definition of the weak solution imply that $u_0 + u_1$ is also identically zero, which contradicts the assumption (8).

Therefore, it suffices to find the condition that for fixed p, q, there exists $\alpha \in [1/2, 1]$ such that $a(\alpha) \leq 0$ or $b(\alpha) \leq 0$ holds. To do this, for fixed p, q, we consider the minimum of the functions $a(\alpha)$ and $b(\alpha)$ under $\alpha \in [1/2, 1]$. By noting that the functions $a(\alpha)$ and $b(\alpha)$ are linear with respect to α, the minimums must be attained at the endpoints $\alpha = 1/2$ or $\alpha = 1$. Hence, the condition we need is

$$\min \{a(1/2), a(1), b(1/2), b(1)\} \leq 0.$$

For example, $a(1)$ is computed as

$$\begin{aligned} a(1) &= (1+N)\left(1 - \frac{1}{pq}\right) - 2 - \frac{1}{p} \\ &= \frac{1}{pq}\left[(1+N)(pq-1) - 2pq - q\right] \\ &= \frac{1}{pq}\left[(N-1)(pq-1) - (q+2)\right] \\ &= \frac{pq-1}{pq}\left(N - 1 - \frac{q+2}{pq-1}\right), \end{aligned}$$

and hence, the condition $a(1) \leq 0$ is written as

$$\frac{q+2}{pq-1} + 1 - N \geq 0.$$

In the same way, we see that the conditions $a(1/2) \leq 0, b(1/2) \leq 0, b(1) \leq 0$ are equivalent to

$$\frac{2(q+1)}{pq-1} - N \geq 0, \quad \frac{2(p+1)}{pq-1} - N \geq 0, \quad \frac{2p+1}{pq-1} - N \geq 0,$$

respectively. Finally, noting that the third condition in the above is stronger than the second one, we omit the third one and reach the condition (7).

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Regular Singular Problems for Hyperbolic Systems and Their Asymptotic Integration

Jens Wirth

Abstract In this short note we discuss Cauchy problems for t-dependent hyperbolic systems with lower order terms becoming singular at the final time, but in such a way that a controlled loss of Sobolev regularity appears. Our aim is to describe this loss in terms of the full symbol of the operator.

Keywords Asymptotic integration • Hyperbolic systems • Singular problem

1 Introduction

We consider Cauchy problems for hyperbolic equations on a finite time-strip $[0, T)$, where coefficients are allowed to become singular at the final time $t = T$. Typical examples include the wave equation with mass and dissipation

$$u_{tt} - \Delta u + b(t)u_t + m(t)u = 0, \qquad u(0,\cdot) = u_0, \quad u_t(0,\cdot) = u_1, \tag{1}$$

where $b(t) \sim b_0(T-t)^{-1}$ and $m(t) \sim m_0(T-t)^{-2}$, both as $t \to T$, or $d\times d$ hyperbolic systems

$$\mathrm{D}_t U = \sum_{j=1}^{n} A_j(t)\mathrm{D}_{x_j} U + B(t)U, \qquad U(0,\cdot) = U_0 \tag{2}$$

with bounded coefficients A_j and where $B(t) \sim B_0(T-t)^{-1}$ as $t \to T$. In both cases the problems are H^s-well-posed locally in $[0, T)$ and it is interesting to ask for the behaviour of H^s-norms as t approaches T. The asymptotic bounds on the lower order terms imply a controlled loss (or gain) of a finite amount of Sobolev regularity

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and they are related to so-called Levi conditions for weakly hyperbolic equations. Results of this paper generalize some results of Del Santo, Kinoshita, and Reissig [1] obtained for (1) with $b(t) = 0$ and scaling-critical $m(t)$.

This situation is dual to the one considered by the author and Nunes in [2] or by the author in [6], where lower order terms decay exactly at the scaling powers in order to influence the large-time asymptotics significantly (without changing the overall asymptotic type of the equation).

The basic strategy of our approach is to apply a partial Fourier transform with respect to the spatial variables, which leads to a ξ-dependent family of ordinary differential equations and it remains to solve these equations and to give sharp asymptotic bounds on its fundamental solution. Away from the singularity this will be done by standard hyperbolic theory using a factorization/diagonalization of the full symbol, while close to the singularity we will use the Fuchs type nature of the problem and apply asymptotic integration arguments. The approach follows [4–6] and is inspired by [3].

2 Results

2.1 Notation, Model Problem, and Main Assumptions

In order to formulate precise assumptions for the aforementioned examples we introduce the spaces

$$\mathcal{T}\{m\} = \{f \in \mathrm{C}^\infty([0,T)) : |\mathrm{D}_t^k f(t)| \le C_k (T-t)^{-m-k}\} \tag{3}$$

of admissible coefficient functions. The parameter m describes an order and is related to the place of the coefficient in our problem. In the model (1) we require that $b \in \mathcal{T}\{1\}$ and $m \in \mathcal{T}\{2\}$, where in (2) the precise requirement will be that $A \in \mathcal{T}\{0\} \otimes \mathbb{C}^{d\times d}$ and $B \in \mathcal{T}\{1\} \otimes \mathbb{C}^{d\times d}$. The second model has to be accompanied by an assumption of uniform strict hyperbolicity.

In order to separate the influence of the regular and the singular part of the problem we introduce the zones

$$\begin{aligned} \mathcal{Z}_{\mathrm{sing}}(N) &= \{(t,\xi) : |\xi|(T-t) \le N\} \subset [0,T) \times \mathbb{R}^n, \\ \mathcal{Z}_{\mathrm{reg}}(N) &= \{(t,\xi) : |\xi|(T-t) \ge N\} \subset [0,T) \times \mathbb{R}^n, \end{aligned} \tag{4}$$

in the extended phase space and solve problems locally in them. For this we will make use of the following symbol classes $\mathcal{S}\{m_1, m_2\}$ defined as

$$\bigcup_N \left\{ a \in \mathrm{C}^\infty(\mathcal{Z}_{\mathrm{reg}}(N)) \;:\; |\mathrm{D}_t^k \mathrm{D}_\xi^\alpha a(t,\xi)| \le C_{k,\alpha} |\xi|^{m_1 - |\alpha|} (T-t)^{-m_2-k} \right\}, \tag{5}$$

already employed in earlier works, see, e.g., [5]. We further denote by $a(t, \mathrm{D})$ the Fourier multiplier associated with a symbol $a(t, \xi)$ (of sufficient regularity) and consider in the following the model problem

$$\mathrm{D}_t U = A(t, \mathrm{D})U, \qquad U(0, \cdot) = U_0, \tag{6}$$

for a matrix-valued symbol $A(t, \xi)$ satisfying the main assumptions:

(A1). We assume $A \in \mathcal{S}\{1, 0\} \otimes \mathbb{C}^{d\times d}$. We assume further that there exists a homogeneous symbol $A_1(t, \xi)$ satisfying $A_1(t, \rho\xi) = \rho A_1(t, \xi)$ for all $\rho > 1$ such that $A - A_1 \in \mathcal{S}\{0, 1\} \otimes \mathbb{C}^{d\times d}$.

(A2). The eigenvalues of $A_1(t, \xi)$ are real and distinct. We denote them by $\lambda_1(t, \xi) < \lambda_2(t, \xi) < \cdots < \lambda_d(t, \xi)$ and assume further that they are uniformly distinct in the sense that

$$\inf_{t\in[0,T)} \inf_{\xi\neq 0} |\xi|^{-1} |\lambda_i(t, \xi) - \lambda_j(t, \xi)| \geq c_0 > 0. \tag{7}$$

(A3). There exists a matrix $A_\star \in \mathbb{C}^{d\times d}$, such that

$$\int_{t_\xi}^{T} \|(T - t)A(t, \xi) - A_\star\|^\sigma \frac{\mathrm{d}t}{T - t} < \infty \tag{8}$$

with some constant $\sigma \geq 1$. Here and later on t_ξ denotes the implicit function defined by $(T - t_\xi)|\xi| = N$. We further assume that the eigenvalue of $A_\star$ with lowest imaginary part is simple.

We will comment on each of the assumptions and its implications in due course. The first two describe the regular nature within $\mathcal{Z}_{\mathrm{reg}}(N)$ with $A_1(t, \xi)$ the hyperbolic principal part of the problem, while the last assumptions concern the singular part of the problem with $A_\star$ playing the role of a second principal symbol at the singularity. A spatial Fourier transform rewrites (6) as

$$\mathrm{D}_t \widehat{U}(t, \xi) = A(t, \xi)\widehat{U}(t, \xi), \qquad \widehat{U}(0, \cdot) = \widehat{U}_0, \tag{9}$$

and our aim is to derive asymptotically sharp results on the corresponding fundamental solution $\mathcal{E}(t, 0, \xi)$, i.e., the matrix-valued solution to

$$\mathrm{D}_t \mathcal{E}(t, s, \xi) = A(t, \xi)\mathcal{E}(t, s, \xi), \qquad \mathcal{E}(s, s, \xi) = \mathrm{I} \in \mathbb{C}^{d\times d}. \tag{10}$$

2.2 Treatment in the Regular Zone

The consideration in the regular zone is rather standard and will be discussed only briefly. By the uniform strict hyperbolicity due to (A2) we infer that there exists a diagonalizer $M \in \mathcal{S}\{0,0\}$ of A_1 such that $M^{-1} \in \mathcal{S}\{0,0\}$ and

$$M^{-1}(t,\xi)A_1(t,\xi)M(t,\xi) = \mathcal{D}(t,\xi) = \operatorname{diag}\big(\lambda_1(t,\xi),\dots,\lambda_d(t,\xi)\big) \tag{11}$$

holds true for all $\xi \neq 0$. Therefore, $U^{(0)}(t,\xi) = M^{-1}(t,\xi)\widehat{U}(t,\xi)$ solves

$$\mathrm{D}_t U^{(0)}(t,\xi) = \big(\mathcal{D}(t,\xi) + R_0(t,\xi)\big)U^{(0)}(t,\xi), \tag{12}$$

where the remainder term R_0 is given by

$$R_0 = M^{-1}(A - A_1)M + \big(\mathrm{D}_t M^{-1}\big)M \in \mathcal{S}\{0,1\}. \tag{13}$$

The strategy within the regular zone lies in improvements of remainders by successive diagonalization steps. We only recall the result, the proofs are analogous to the ones given in [4, 5] for the large-time situation.

Proposition 2.1 *Assume (A1) and (A2). Then for each number k there exists a zone-constant N and matrix-valued symbols $N_k \in \mathcal{S}\{0,0\}$, invertible within $\mathcal{Z}_{\mathrm{reg}}(N)$ and with $N_k^{-1} \in \mathcal{S}\{0,0\}$, such that the operator identity*

$$\big(\mathrm{D}_t - \mathcal{D} - R_0\big)N_k = N_k\big(\mathrm{D}_t - \mathcal{D} - F_{k-1} - R_k\big) \tag{14}$$

holds true with symbols $R_k \in \mathcal{S}\{-k,k+1\}$, diagonal symbols $F_{k-1} \in \mathcal{S}\{0,1\}$ and in such a way that $N_k - N_{k-1} \in \mathcal{S}\{-k,k\}$ and $F_{k-1} - F_{k-2} \in \mathcal{S}\{1-k,k\}$.

We comment on one particular term which is needed later on. The hyperbolic sub-principal part F_0 is given by

$$F_0 = \operatorname{diag} R_0 = \operatorname{diag}\big(M^{-1}(A - A_1)M + \big(\mathrm{D}_t M^{-1}\big)M\big) \tag{15}$$

modulo $\mathcal{S}\{-1,2\}$. Furthermore, as $F_0 \in \mathcal{S}\{0,1\}$, we find constants $\kappa_\pm \in \mathbb{R}$ such that

$$C_-\left(\frac{T-t}{T-s}\right)^{\kappa_-} \le \left\|\exp\left(\mathrm{i}\int_s^t F_0(\tau,\xi)\,\mathrm{d}\tau\right)\right\| \le C_+\left(\frac{T-t}{T-s}\right)^{\kappa_+} \tag{16}$$

holds true. We denote by $\mathcal{E}_k(t,s,\xi)$ the fundamental solution of the transformed system within the regular zone. Choosing $k > \kappa_+ - \kappa_-$ allows for good estimates for this fundamental solution.

Lemma 2.2 *Let (A1) and (A2) be satisfied and assume $k \ge \kappa_+ - \kappa_- + 1$. Then the fundamental solution $\mathcal{E}_k(t,0,\xi)$ can be represented as*

$$\mathcal{E}_k(t,0,\xi) = \exp\left(\mathrm{i}\int_0^t \big(\mathcal{D}(\tau,\xi) + F_{k-1}(\tau,\xi)\big)\,\mathrm{d}\tau\right)\mathcal{Q}_k(t,0,\xi), \tag{17}$$

where $\mathcal{Q}_k(t,0,\xi)$ is uniformly bounded in t a symbol of order 0 with respect to ξ, i.e.,

$$\|\mathrm{D}_\xi^\alpha \mathcal{Q}_k(t,0,\xi)\| \le C_\alpha |\xi|^{-|\alpha|}, \qquad (t,\xi) \in \mathcal{Z}_{\mathrm{reg}}(N), \tag{18}$$

for $|\alpha| < k - (\kappa_+ - \kappa_-)$. In particular the estimate

$$\|\mathcal{E}(t,0,\xi)\| \le C\|\mathcal{E}_k(t,0,\xi)\| \le C(T-t)^{\kappa_+} \tag{19}$$

holds true.

Proof The fundamental solution of the diagonal part $\mathrm{D}_t - \mathcal{D}(t,\xi) - F_{k-1}(t,\xi)$ is given by the exponential term in (17). As the entries of $\mathcal{D}(t,\xi)$ are real and entries of $F_{k-1} - F_0 \in \mathcal{S}\{-1,2\}$ are uniformly integrable over $\mathcal{Z}_{\mathrm{reg}}(N)$, the behaviour of its norm depends only on the entries of $F_0(t,\xi)$ and hence on the constants $\kappa_\pm$ from (16). Using (17) as ansatz for the unknown matrix $\mathcal{Q}_k$, we obtain a system

$$\mathrm{D}_t \mathcal{Q}_k(t,0,\xi) = \mathcal{R}_k(t,0,\xi)\mathcal{Q}_k(t,0,\xi), \qquad \mathcal{Q}_k(0,0,\xi) = \mathrm{I}, \tag{20}$$

where $\mathcal{R}_k(t,0,\xi)$ is obtained from $R_k(t,\xi)$ by conjugation with the fundamental solution of the diagonal part. This yields

$$\|\mathcal{R}_k(t,0,\xi)\| \le C(T-t)^{\kappa_+-\kappa_-}\|R_k(t,\xi)\| \le C|\xi|^{-k}(T-t)^{\kappa_+-\kappa_--k-1} \tag{21}$$

together with

$$\begin{aligned}\|\mathrm{D}_\xi^\alpha R_k(t,0,\xi)\| &\le C(T-t)^{\kappa_+-\kappa_--k-1}\big(|\xi|^{-k-|\alpha|} + (T-t)^{|\alpha|}|\xi|^{-k}\big)\\ &\le C_\epsilon (T-t)^{-\varepsilon-1}|\xi|^{-\varepsilon-|\alpha|}\end{aligned} \tag{22}$$

for $|\alpha| < k - (\kappa_+ - \kappa_-)$ and with sufficiently small $\varepsilon > 0$. The last estimate makes use of the definition of the regular zone. Finally, representing $\mathcal{Q}_k(t,0,\xi)$ as Peano–Baker series

$$\mathcal{Q}_k(t,0,\xi) = \mathrm{I} + \sum_{k=1}^\infty \mathrm{i}^k \int_0^t \mathcal{R}_k(t_1,0,\xi)\cdots\int_0^{t_k-1}\mathcal{R}_k(t_k,0,\xi)\,\mathrm{d}t_k\cdots\mathrm{d}t_1 \tag{23}$$

and estimating it term by term yields

$$\begin{aligned}\|\mathcal{Q}_k(t,0,\xi)\| &\le \exp\left(C|\xi|^{-k}\int_0^t \frac{\mathrm{d}\tau}{(T-\tau)^{1-\kappa_++\kappa_-+k}}\right)\\ &\le C\exp\big(C|\xi|^{-k}(T-t)^{\kappa_+-\kappa_--k}\big) \le C\end{aligned} \tag{24}$$

and similarly for derivatives

$$\|D_\xi^\alpha \mathcal{Q}_k(t,0,\xi)\| \;\le\; |\xi|^{-|\alpha|}\exp\left(C_\varepsilon|\xi|^{-\varepsilon}\int_0^t \frac{\mathrm{d}\tau}{(T-\tau)^{1+\varepsilon}}\right) \;\le\; C|\xi|^{-|\alpha|}. \tag{25}$$

As the diagonalizers $M(t,\xi)$ and $N_k(t,\xi)$ are uniformly bounded and invertible over $\mathcal{Z}_{\mathrm{reg}}(N)$ the estimate (19) follows. □

One particular case of the last estimate will be needed later on. It follows that

$$\|\mathcal{E}(t_\xi,0,\xi)\| \le C(T-t_\xi)^{\kappa_+} \le C|\xi|^{-\kappa_+} \tag{26}$$

and the regular zone will in general contribute some loss of regularity in the end. Note that this is due to the presence of the lower order terms and already known for the case of wave equations with dissipation. For Klein–Gordon equations with a singular mass as treated in [1] one obtains $\kappa_+ = \kappa_- = 0$ and the mass term is not felt in the regular zone.

2.3 Treatment in the Singular Zone

In this part of the extended phase space we rewrite the system for $\widehat{U}(t,\xi)$ as system of Fuchs type

$$(T-t)\mathrm{D}_t\widehat{U} = (T-t)A(t,\xi)\widehat{U} \tag{27}$$

and make use of assumption (A3) in combination with the Levinson theorem [2, Theorem A.1] and Hartmann–Wintner theorem [2, Theorem A.2] for Fuchs type equations. This allows to reduce the above equation to the explicitly solvable model equation $(T-t)\mathrm{D}_t\widehat{V} = A_\star\widehat{V}$ and yields for the original fundamental solution the following estimate. For details on the proof, see [2] and [6].

Lemma 2.3 *Assume (A3) and let* $\mu = \inf \operatorname{Im}\operatorname{spec} A_\star$. *Then the fundamental solution* $\mathcal{E}$ *to (9) satisfies*

$$\|\mathcal{E}(t,t_\xi,\xi)\| \le \begin{cases} C\left(\frac{T-t}{T-t_\xi}\right)^{-\mu}, & \sigma = 1,\\ C_\epsilon\left(\frac{T-t}{T-t_\xi}\right)^{-\mu-\epsilon}, & \sigma > 1 \text{ for } \epsilon > 0 \text{ arbitrary.}\end{cases} \tag{28}$$

2.4 Combination of Estimates and Main Result

We consider estimates in the H^s-Sobolev scale. For such estimates only uniform bounds of the fundamental solution $\mathcal{E}(t,0,\xi)$ are needed and they are encoded in the numbers $\kappa_\pm$ arising from the main lower order part F_0 in the regular zone and $\mu = \inf \operatorname{Im} \operatorname{spec} A_\star$ in the singular zone. We distinguish two cases and assume first that $\mu + \kappa_+ \le 0$. Then Lemmata 2.2 and 2.3 (for simplicity with $\sigma = 1$) yield

$$\begin{aligned}\|\mathcal{E}(t,0,\xi)\| &\le C(T-t)^{-\mu}\begin{cases}1 & |\xi|\le N,\\ (T-t_\xi)^{\mu+\kappa_+}, & |\xi|\ge N, \quad t\ge t_\xi,\\ (T-t)^{\mu+\kappa_+}, & |\xi|\ge N, \quad t\le t_\xi,\end{cases}\\ &\le C(T-t)^{-\mu}\langle\xi\rangle^{-\mu-\kappa_+}\end{aligned} \tag{29}$$

based on the definition of the regular zone. On the other hand, if $\mu + \kappa_+ > 0$

$$\begin{aligned}\|\mathcal{E}(t,0,\xi)\| &\le C\begin{cases}(T-t)^{-\mu} & |\xi|\le N,\\ (T-t)^{-\mu}(T-t_\xi)^{\mu+\kappa_+}, & |\xi|\ge N, \quad t\ge t_\xi,\\ (T-t)^{\kappa_+}, & |\xi|\ge N, \quad t\le t_\xi,\end{cases}\\ &\le C(T-t)^{-\mu}\end{aligned} \tag{30}$$

uniform with respect to $\xi \in \mathbb{R}^n$. In the first case we observe a loss of Sobolev regularity, while in the second case only the norm has a non-trivial behaviour.

Theorem 2.4 *Assume (A1)–(A3) with* $\sigma = 1$ *and that the initial data satisfy* $U_0 \in \mathrm{H}^s(\mathbb{R}^n;\mathbb{C}^d)$. *Then there exists a unique solution* $U \in \mathrm{C}([0,T);\mathrm{H}^s(\mathbb{R}^n;\mathbb{C}^d))$ *such that*

$$(T-t)^{\mu}\|U(t,\cdot)\|_{\mathrm{H}^s} \le C\begin{cases}\|U_0\|_{\mathrm{H}^s}, & \mu+\kappa_+\ge 0,\\ \|U_0\|_{\mathrm{H}^{s-\mu-\kappa_+}}, & \mu+\kappa_+ < 0\end{cases} \tag{31}$$

holds true uniformly in t.

If $\sigma > 1$, a further ϵ (or better logarithmic) loss appears. We omit the details.

3 Particular Cases of Interest

Finally we want to come back to our initial example. We consider a wave equation with singular mass and dissipation (1) with $b \in \mathcal{T}\{1\}$ and $m \in \mathcal{T}\{2\}$. In order to transform it into a first order system we use a symbol $h \in \mathcal{S}\{1,0\}$ satisfying $h(t,\xi) = |\xi|$ for $(T-t)|\xi| > 2$ and $h(t,\xi) = (T-t)^{-1}$ for $(T-t)|\xi| < 1$ and consider the new unknown $U = \big(h(t,\mathrm{D})u, \mathrm{D}_t u\big)^\top$. The corresponding system of first order satisfies (A1) and (A2). The main terms are given by

$$\mathcal{D}(t,\xi) = \begin{pmatrix} |\xi| & \\ & -|\xi| \end{pmatrix}, \qquad F_0(t,\xi) = \frac{ib(t)}{2}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \tag{32}$$

in the regular part and

$$A_\star = \begin{pmatrix} ib_0 & m_0 \\ 1 & \mathrm{i} \end{pmatrix} \tag{33}$$

near the singularity, if we require (A3) in the form of

$$\int_0^T |(T-t)b(t) - b_0|^\sigma \frac{\mathrm{d}t}{T-t} + \int_0^T |(T-t)^2 m(t) - m_0|^\sigma \frac{\mathrm{d}t}{T-t} < \infty. \tag{34}$$

For simplicity we restrict our consideration to $\sigma = 1$. Then $\kappa_+ = \kappa_- = -b_0 \pm 0$ and $\operatorname{spec} A_\star = \{\mathrm{i}(b_0+1)/2 \pm \sqrt{m_0 - (b_0-1)^2/4}\}$ our main result reads for this particular model case read as in the following table:

Case	Behaviour of H^s norm	Required $H^{s+\delta}$-bound on data
$4m_0 > (b_0-1)^2$	$(T-t)^{-\frac{b_0+1}{2}}$	$\delta = 0$
$b_0(b_0-2) \le 4m_0 < (b_0-1)^2$	$(T-t)^{-\frac{b_0+1-((b_0-1)^2-4m_0)^{1/2}}{2}}$	$\delta = 0$
$4m_0 < b_0(b_0-2)$	$(T-t)^{-\frac{b_0+1-((b_0-1)^2-4m_0)^{1/2}}{2}}$	$\delta = \frac{((b_0-1)^2-4m_0)^{1/2}-1}{2}$

They generalize [1]. If $b(t) = 0$ we obtain $\kappa_\pm = 0$ and $\mu = 1/2$ for $m_0 > 1/4$ and $\mu = 1/2 - \sqrt{1/4 - m_0}$ if $m_0 < 1/4$.

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Part VIII
Wavelet Theory and Image Processing

Qiuhui Chen and Keiko Fujita

A Generalization of Average Interpolating Wavelets

Kensuke Fujinoki

Abstract We consider two-dimensional average interpolating wavelets, which are generated from the average interpolating lifting scheme on a two-dimensional triangular lattice. The resulting set of biorthogonal functions is the generalization of the one-dimensional Cohen–Daubechies–Feauveau $(1, N)$ biorthogonal wavelet whose scaling function is an average interpolating function of the order N. Some properties of the biorthogonal bases and associated filters, such as the order of zeros, regularity, and decay will be described.

Keywords Average interpolation • Lifting • Triangular lattice • Wavelet

Mathematics Subject Classification (2010) Primary 65T60; Secondary 41A05

1 Introduction

An interpolating scaling function $\phi \in L^2(\mathbb{R})$ described in [1] is an interpolation function in the sense that

$$\phi(k) = \begin{cases} 0 & k \in \mathbb{Z} \text{ and } k \neq 0 \\ 1 & k = 0. \end{cases} \tag{1}$$

Let V_0 be a space generated by the linear combination of $\{\phi(t-k)\}_{k\in\mathbb{Z}}$. This interpolating scaling function ϕ recovers any function $f \in V_0$ by interpolating its discrete samples: $f(t) = \sum_{k\in\mathbb{Z}} f(k)\phi(t-k)$. We denote the Fourier transform of $f(t)$ by $\widehat{f}(\omega) = \int_{\mathbb{R}} f(t)\, e^{-i\omega t} dt, \omega \in \mathbb{R}$, and a discrete sequence $\{f[k]\}_{k\in\mathbb{Z}}$ by $\hat{f}(\omega) = \sum_{k\in\mathbb{Z}} f[k]e^{-i\omega k}$. If $\widehat{\phi}$ is defined by an infinite product of the Fourier

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transform of a discrete low-pass (LP) filter $\{h[k]\}_{k\in\mathbb{Z}}$

$$\widehat{\phi}(\omega) = \prod_{j=1}^{\infty} \frac{1}{\sqrt{2}} \hat{h}\left(\frac{\omega}{2^j}\right),$$

the interpolating property is also expressed as

$$\hat{h}(\omega) + \hat{h}(\omega + \pi) = \sqrt{2}. \tag{2}$$

Examples of such an interpolating function are spline functions and Deslauriers–Dubuc functions.

In this paper, we consider the average interpolating scaling function introduced in [2], which is defined by the dilation equation

$$\phi(t) = \sum_{k=-N+1}^{N} h[k]\phi(2t-k),$$

where $\operatorname{supp}\phi = [-N+1, N]$. Since ϕ is an average interpolating function, it satisfies

$$\int_k^{k+1} \phi(t)\, dt = \begin{cases} 0 & k \neq 0 \\ 1 & k = 0. \end{cases}$$

As for the ordinary interpolating scaling function, the average interpolating function reproduces polynomials up to degree $N-1$. The corresponding wavelet is defined by $\psi(t) = \phi(2t) - \phi(2t-1)$, which satisfies $\widehat{\psi}(0) = \int_{\mathbb{R}} \psi(t)dt = 0$.

Such functions comprise the family of the Cohen–Daubechies–Feauveau (CDF) $(1, N)$ biorthogonal wavelet (see [3]). We denote by N the order of the interpolation. When $N = 1$, it is identical to the Haar wavelet. A dual scaling function $\widetilde{\phi}$ of the CDF$(1, N)$ wavelet is always the Haar scaling function.

2 Lifting on Lattice

To generalize the average interpolating scaling function ϕ as well as CDF $(1, N)$ wavelet to a two-dimensional lattice, we first define two vectors $\boldsymbol{t}_1, \boldsymbol{t}_2 \in \mathbb{R}^2$ that generate a lattice

$$\Lambda = \{\boldsymbol{t} = n_1\boldsymbol{t}_1 + n_2\boldsymbol{t}_2 |\, (n_1, n_2) \in \mathbb{Z}^2\}.$$

Wavelet functions and filters are defined on the lattice Λ. The Fourier domain of the lattice is the reciprocal lattice

$$\widehat{\Lambda} = \{2\pi(\boldsymbol{\lambda} = m_1\boldsymbol{\lambda}_1 + m_2\boldsymbol{\lambda}_2) | \, (m_1, m_2) \in \mathbb{Z}^2\},$$

which is given by the vectors $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in \mathbb{R}^2$ that satisfy $\boldsymbol{\lambda}_k \cdot \boldsymbol{t}_k = \mathbf{0}$ for $k = 1, 2$.

For the triangular lattice, we define the vectors $\boldsymbol{t}_1 = \begin{pmatrix}1 & 0\end{pmatrix}^T$, $\boldsymbol{t}_2 = \begin{pmatrix}-1/2 & \sqrt{3}/2\end{pmatrix}^T$ and $\boldsymbol{\lambda}_1 = \begin{pmatrix}0 & 2/\sqrt{3}\end{pmatrix}^T$, $\boldsymbol{\lambda}_2 = \begin{pmatrix}1 & 1/\sqrt{3}\end{pmatrix}^T$. Additionally, $\boldsymbol{\lambda}_3 = \boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2$ and $\boldsymbol{t}_3 = -\boldsymbol{t}_1 - \boldsymbol{t}_2$ are also defined.

A two-dimensional LP filter $\{h[\boldsymbol{t}]\}_{\boldsymbol{t}\in\Lambda}$ is assumed to be given on the lattice and its Fourier transform is defined by

$$\hat{h}(\boldsymbol{\omega}) = \sum_{\boldsymbol{t}\in\Lambda} h[\boldsymbol{t}] e^{-i\boldsymbol{\omega}\cdot\boldsymbol{t}}, \quad \boldsymbol{\omega} \in \mathbb{R}^2.$$

Here, $\hat{h}(\boldsymbol{\omega})$ is a periodic function that has double periodicity $\hat{h}(\boldsymbol{\omega}) = \hat{h}(\boldsymbol{\omega} + 2\pi\boldsymbol{\lambda})$, $\boldsymbol{\lambda} \in \widetilde{\Lambda}$, which implies that we have two alias points $\boldsymbol{\omega} = \pi\boldsymbol{\lambda}_1$ and $\boldsymbol{\omega} = \pi\boldsymbol{\lambda}_2$. The simplest set of biorthogonal filters $\{h[\boldsymbol{t}], g_k[\boldsymbol{t}], \tilde{h}[\boldsymbol{t}], \tilde{g}_k[\boldsymbol{t}]\}_{\boldsymbol{t}\in\Lambda, k=1,2,3}$ defined on the triangular lattice Λ is the biorthogonal Haar filters generalized to two-dimension which is introduced in [4].

The lifting scheme proposed by Sweldens [5] corresponds to modify biorthogonal filters without losing the biorthogonality. With the lifting on the Fourier domain, a biorthogonal set of filters $\{\hat{h}, \hat{g}_k, \hat{\hat{h}}, \hat{\hat{g}}_k\}$ is modified by the predictors $\widehat{\mathcal{P}}_k$ and the updaters $\widehat{\mathcal{U}}_k$ as $\{\hat{h}^{\mathcal{U}}, \hat{g}_k^{\mathcal{P}}, \hat{\hat{h}}^{\mathcal{P}}, \hat{\hat{g}}_k^{\mathcal{U}}\}$, which may be written as

$$\hat{h}^{\mathcal{U}}(\boldsymbol{\omega}) = \hat{h}(\boldsymbol{\omega}) + \sum_{k=1}^{3} \hat{g}_k(\boldsymbol{\omega}) \overline{\widehat{\mathcal{U}}_k(2\boldsymbol{\omega})}, \quad \hat{g}_k^{\mathcal{P}}(\boldsymbol{\omega}) = \hat{g}_k(\boldsymbol{\omega}) - \hat{h}(\boldsymbol{\omega}) \overline{\widehat{\mathcal{P}}_k(2\boldsymbol{\omega})},$$

$$\hat{\hat{h}}^{\mathcal{P}}(\boldsymbol{\omega}) = \hat{\hat{h}}(\boldsymbol{\omega}) + \sum_{k=1}^{3} \hat{\hat{g}}_k(\boldsymbol{\omega}) \widehat{\mathcal{P}}_k(2\boldsymbol{\omega}), \quad \hat{\hat{g}}_k^{\mathcal{U}}(\boldsymbol{\omega}) = \hat{\hat{g}}_k(\boldsymbol{\omega}) - \hat{\hat{h}}(\boldsymbol{\omega}) \widehat{\mathcal{U}}_k(2\boldsymbol{\omega}).$$

An arbitrary filter set is found from applying the lifting to the lazy wavelet filters. The lazy wavelet filters are given by $\hat{h}(\boldsymbol{\omega}) = \hat{\hat{h}}(\boldsymbol{\omega}) = 1$ and $\hat{g}_k(\boldsymbol{\omega}) = \hat{\hat{g}}_k(\boldsymbol{\omega}) = e^{-i\boldsymbol{\omega}\cdot\boldsymbol{t}_k}$, $k = 1, 2, 3$. Here, letting $\widehat{\mathcal{P}}_k(\boldsymbol{\omega}) = 1$, $\widehat{\mathcal{U}}_k(\boldsymbol{\omega}) = 1/4$, for $k = 1, 2, 3$, gives the order $N = 1$ case, which is the biorthogonal Haar wavelet filters on Λ described above. In this choice, the predictors need to be applied first to modify $\hat{g}_k$, and then $\hat{h}^{\mathcal{U}}$ is calculated.

3 Generalization of Average Interpolating Filters

Here we generalize the CDF $(1, N)$ filter to the lattice Λ. If we set

$$\widehat{\mathcal{U}}_k(\boldsymbol{\omega}) = \frac{1}{32}\left(-e^{-i\boldsymbol{\omega}\cdot\boldsymbol{t}_k} + 8 + e^{i\boldsymbol{\omega}\cdot\boldsymbol{t}_k}\right), \tag{3}$$

holding $\widehat{\mathcal{P}}_k(\boldsymbol{\omega}) = 1$, we obtain the system of the generalized CDF $(1, N)$ wavelet of order $N = 3$. In this case, ϕ is an average interpolating scaling function of $N = 3$. The associated LP filters satisfy $\hat{\hat{h}}(\omega\boldsymbol{\lambda}_k)\,\overline{\hat{h}(\omega\boldsymbol{\lambda}_k)} \propto (\omega - \pi)^3, k = 1, 2, 3$. If we change the updaters as

$$\widehat{\mathcal{U}}_k(\boldsymbol{\omega}) = \frac{1}{512}\left(3e^{-4i\boldsymbol{\omega}\cdot\boldsymbol{t}_k} - 22e^{-2i\boldsymbol{\omega}\cdot\boldsymbol{t}_k} + 128 + 22e^{2i\boldsymbol{\omega}\cdot\boldsymbol{t}_k} - 3e^{4i\boldsymbol{\omega}\cdot\boldsymbol{t}_k}\right), \tag{4}$$

then we obtain the $N = 5$ case.

As we have only changed $\mathcal{U}_k$, the resulting primal HP filters $g_k[\boldsymbol{t}]$ are still the same of those of the Haar filters. More precisely, in this choice, $g_k[\boldsymbol{t}]$ and $\tilde{h}[\boldsymbol{t}]$ do not depend on the order N. However, $\tilde{g}_k[\boldsymbol{t}]$ and $h[\boldsymbol{t}]$ are significantly changed when the order of the interpolation N is increased. Thus, the generalized $(1, N)$ family obtained by the lifting always has the same filters $g_k[\boldsymbol{t}]$ and $\tilde{h}[\boldsymbol{t}]$, but the other filters $\tilde{g}_k[\boldsymbol{t}]$ and $h[\boldsymbol{t}]$ depend on the order N. In general, the frequency responses of a filter is improved when we increase the interpolation order N. The filters $\{\hat{h}, \hat{g}_k, \hat{\hat{h}}, \hat{\hat{g}}_k\}$ satisfy $\hat{h}(\boldsymbol{0}) = \hat{g}_k(\pi\boldsymbol{\lambda}_k) = 2$, $\hat{h}(\pi\boldsymbol{\lambda}_k) = \hat{g}_k(\boldsymbol{0}) = 0$, and similarly for the dual filters. The dual LP filter $\hat{\hat{h}}(\boldsymbol{\omega})$ is always the Haar filter and thus an interpolating in the sense that

$$\hat{\hat{h}}(\boldsymbol{\omega}) + \sum_{k=1}^{3} \hat{\hat{h}}(\boldsymbol{\omega} + \pi\boldsymbol{\lambda}_k) = 2,$$

which corresponds to (2).

4 Scaling Function and Wavelets

Once the filters are found, the two-dimensional scaling function and wavelets are given by

$$\widehat{\phi}(\boldsymbol{\omega}) = \prod_{j=1}^{\infty} \frac{1}{2}\hat{h}\left(\frac{\boldsymbol{\omega}}{2^j}\right), \qquad \widehat{\psi}_k(\boldsymbol{\omega}) = \frac{1}{2}\hat{g}_k\left(\frac{\boldsymbol{\omega}}{2}\right)\prod_{j=2}^{\infty} \frac{1}{2}\hat{h}\left(\frac{\boldsymbol{\omega}}{2^j}\right), \tag{5}$$

which satisfy

$$\widehat{\phi}(\boldsymbol{\omega}) = \frac{1}{2}\hat{h}\left(\frac{\boldsymbol{\omega}}{2}\right)\widehat{\phi}\left(\frac{\boldsymbol{\omega}}{2}\right), \qquad \widehat{\psi}_k(\boldsymbol{\omega}) = \frac{1}{2}\hat{g}_k\left(\frac{\boldsymbol{\omega}}{2}\right)\widehat{\phi}\left(\frac{\boldsymbol{\omega}}{2}\right),$$

where

$$\widehat{\phi}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} \phi(\boldsymbol{r})\, e^{-i\boldsymbol{\omega}\cdot\boldsymbol{r}} d\boldsymbol{r}, \qquad \boldsymbol{r} \in \mathbb{R}^2.$$

Dual functions $\widehat{\widetilde{\phi}}$ and $\widehat{\widetilde{\psi}}_k$ are defined in a similar way with $\tilde{h}$ and $\tilde{g}_k$. They are normalized as $\widehat{\phi}(\boldsymbol{0}) = \widehat{\widetilde{\phi}}(\boldsymbol{0}) = 1$ and $\widehat{\psi}_k(\boldsymbol{0}) = \widehat{\widetilde{\psi}}_k(\boldsymbol{0}) = 0$.

On the Bravais lattice Λ, the two-dimensional scaling function and wavelets $\{\phi(\boldsymbol{r}), \psi_k(\boldsymbol{r}), \widetilde{\phi}(\boldsymbol{r}), \widetilde{\psi}_k(\boldsymbol{r})\}_{\boldsymbol{r}\in\mathbb{R}^2}$ are defined by

$$\phi(\boldsymbol{r}) = \sum_{\boldsymbol{t}\in\Lambda} 2\, h[\boldsymbol{t}]\phi(2\boldsymbol{r}-\boldsymbol{t}), \quad \psi_k(\boldsymbol{r}) = \sum_{\boldsymbol{t}\in\Lambda} 2\, g_k[\boldsymbol{t}]\phi(2\boldsymbol{r}-\boldsymbol{t}),$$
$$\widetilde{\phi}(\boldsymbol{r}) = \sum_{\boldsymbol{t}\in\Lambda} 2\, \tilde{h}[\boldsymbol{t}]\widetilde{\phi}(2\boldsymbol{r}-\boldsymbol{t}), \quad \widetilde{\psi}_k(\boldsymbol{r}) = \sum_{\boldsymbol{t}\in\Lambda} 2\, \tilde{g}_k[\boldsymbol{t}]\widetilde{\phi}(2\boldsymbol{r}-\boldsymbol{t}).$$

By the lifting, the prediction of $\{h, g_k, \tilde{h}, \tilde{g}_k\}$ affects $\{\tilde{h}, g_k\}$. This modifies the new biorthogonal system $\{\phi, \psi_k, \widetilde{\phi}, \widetilde{\psi}_k\}$ to $\{\phi, \psi_k^{\mathcal{P}}, \widetilde{\phi}^{\mathcal{P}}, \widetilde{\psi}_k^{\mathcal{P}}\}$ as

$$\psi_k^{\mathcal{P}}(\boldsymbol{r}) = \psi_k(\boldsymbol{r}) - \sum_{\boldsymbol{t}\in\Lambda} \mathcal{P}_k[-\boldsymbol{t}]\, \phi(\boldsymbol{r}-\boldsymbol{t}),$$

$$\widetilde{\phi}^{\mathcal{P}}(\boldsymbol{r}) = 2\sum_{\boldsymbol{t}\in\Lambda} \tilde{h}[\boldsymbol{t}]\, \widetilde{\phi}^{\mathcal{P}}(2\boldsymbol{r}-\boldsymbol{t}) + \sum_{k=1}^{3}\sum_{\boldsymbol{t}\in\Lambda} \mathcal{P}_k[\boldsymbol{t}]\, \widetilde{\psi}_k^{\mathcal{P}}(\boldsymbol{r}-\boldsymbol{t}),$$
$$\widetilde{\psi}_k^{\mathcal{P}}(\boldsymbol{r}) = 2\sum_{\boldsymbol{t}\in\Lambda} \tilde{g}_k[\boldsymbol{t}]\, \widetilde{\phi}^{\mathcal{P}}(2\boldsymbol{r}-\boldsymbol{t}),$$

and similarly the update of $\{\phi, \psi_k, \widetilde{\phi}, \widetilde{\psi}_k\}$ would result in $\{\phi^{\mathcal{U}}, \psi_k^{\mathcal{U}}, \widetilde{\phi}, \widetilde{\psi}_k^{\mathcal{U}}\}$ as

$$\phi^{\mathcal{U}}(\boldsymbol{r}) = 2\sum_{\boldsymbol{t}\in\Lambda} h[\boldsymbol{t}]\, \phi^{\mathcal{U}}(2\boldsymbol{r}-\boldsymbol{t}) + \sum_{k=1}^{3}\sum_{\boldsymbol{t}\in\Lambda} \mathcal{U}_k[\boldsymbol{t}]\, \psi_k^{\mathcal{U}}(\boldsymbol{r}-\boldsymbol{t}),$$
$$\psi_k^{\mathcal{U}}(\boldsymbol{r}) = 2\sum_{\boldsymbol{t}\in\Lambda} g_k[\boldsymbol{t}]\, \phi^{\mathcal{U}}(2\boldsymbol{r}-\boldsymbol{t}),$$
$$\widetilde{\psi}_k^{\mathcal{U}}(\boldsymbol{r}) = \widetilde{\psi}_k(\boldsymbol{r}) - \sum_{\boldsymbol{t}\in\Lambda} \mathcal{U}_k[-\boldsymbol{t}]\, \widetilde{\phi}(\boldsymbol{r}-\boldsymbol{t}).$$

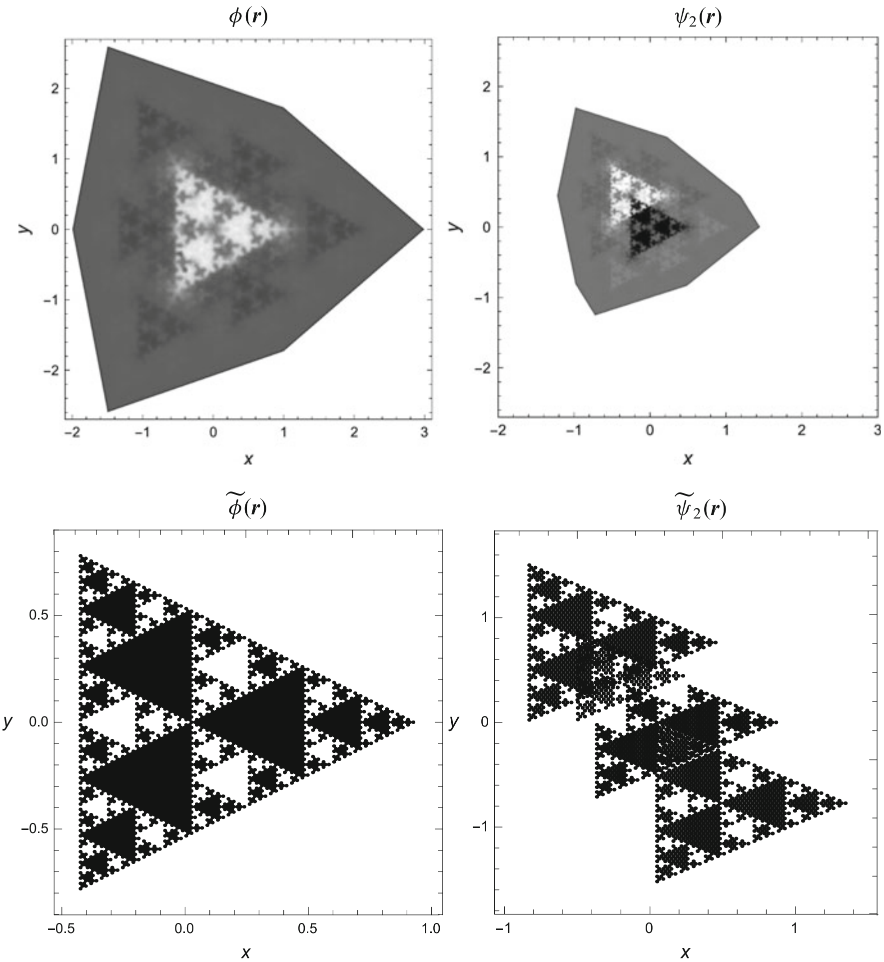

Fig. 1 Scaling functions $\{\phi, \widetilde{\phi}\}$ and wavelets $\{\psi_2, \widetilde{\psi}_2\}$ of $N = 3$

If $\{h, g_k, \tilde{h}, \tilde{g}_k\}$ and $\{\mathcal{P}_k, \mathcal{U}_k\}$ are finite sequences, the resulting system of functions has a compact support on the lattice Λ.

Some of the scaling functions and the wavelets with their dual functions are shown in Fig. 1. Note that only the $k = 2$ case for $\{\psi_k, \widetilde{\psi}_k\}$ is shown because the other wavelets of $k = 1, 3$ are defined by rotating them by $\pm 2\pi/3$ on the lattice. The primal HP filters $g_k[\boldsymbol{t}]$ have not changed by setting the updater defined by (3) because these operations would affect the dual HP filters $\tilde{g}_k[\boldsymbol{t}]$. However, the associate primal wavelets $\psi_k(\boldsymbol{r})$ are changing because $h[\boldsymbol{t}]$ and $\phi(\boldsymbol{r})$ have already changed. Only $\widetilde{\phi}(\boldsymbol{r})$ is the same as that of the Haar whereas $\phi(\boldsymbol{r})$ with the order N is an average interpolating scaling function.

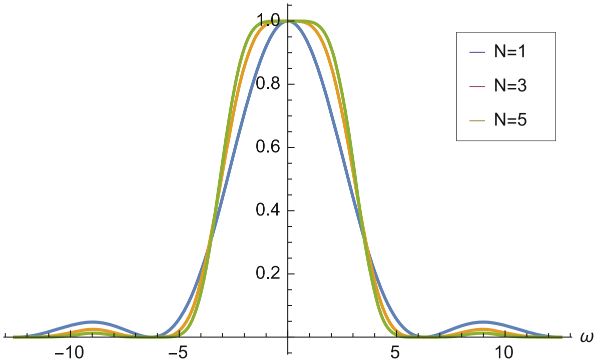

Fig. 2 $|\widehat{\phi}(\omega \boldsymbol{\lambda}_k)|$ for $k = 1, 2, 3$ of the order N

While $\widetilde{\phi}$ turns out to be the Haar wavelet, a jaggy function that has fractal shape, $\widetilde{\psi}_k$ is similar to $\widetilde{\phi}$ because $\widetilde{\psi}_k$ is based on a linear combination of shifted ϕ. This is true for ϕ and ψ_k, but they have a much larger support on Λ, and the regularity seems to be slightly improved. The lifting (3) and (4) do not only improve $\widetilde{\psi}_k$, they improve ϕ and ψ_k. Figure 2 illustrates the frequency decay of the scaling function $\widehat{\phi}$ for $N = 1, 3, 5$. To see a more clear structure of the decay, we only show the one-dimensional response $|\widehat{\phi}(\omega \boldsymbol{\lambda}_k)|, k = 1, 2, 3$ after iterating (5) until $j = 5$. We observe that it provides a fast decay in ω space as the order N increases.

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Gabor Transformation on the Sphere and Its Inverse Transformation

Keiko Fujita

Abstract We will consider the Gabor transformation for the square integrable function on the two-dimensional sphere and its inverse transformation. By using an integral over $\mathbf{R}^3$ we will give the inverse Gabor transformation concretely.

Keywords Expansion formula • Gabor transformation • Sphere

Mathematics Subject Classification (2010) Primary 42C40; Secondary 33C50

1 Introduction

We have studied the windowed Fourier transform, whose windows function is the Gaussian function, of an analytic functional on the n-dimensional sphere and the Gabor transform of the square integrable function on the n-dimensional sphere in [3]. We expressed their transforms in a series expansion by means of the Bessel function. Then in case of the circle, we expressed the results more explicitly and considered their inverse transformations in [2].

In this paper, we will consider the Gabor transformation for the square integrable function on the two-dimensional sphere and will give the inverse transformation concretely by using an integral transformation.

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2 Gabor Transformation on the Sphere

2.1 Fourier Transformation and Windowed Fourier Transformation

Let S_r^2 be the sphere with radius $r > 0$ in $\mathbf{R}^3$; that is,

$$S_r^2 = \{(x_1, x_2, x_3) \in \mathbf{R}^3; x_1^2 + x_2^2 + x_3^2 = r^2\}.$$

For $z = (z_1, z_2, z_3)$ and $w = (w_1, w_2, w_3) \in \mathbf{C}^3$, we set

$$z \cdot w = z_1 w_1 + z_2 w_2 + z_3 w_3, \quad z^2 = z \cdot z.$$

For an integrable function f on S_r^2, define the Fourier transform of f by

$$\mathcal{F}f(\omega) = \int_{S_r^2} e^{-ix\cdot\omega} \overline{f(x)} d\Omega_r(x), \tag{1}$$

where $d\Omega_r$ is the normalized invariant measure on S_r^2. We call the mapping $f \mapsto \mathcal{F}f$ the Fourier transformation. Note that the volume of S_r^2 is $\mathrm{vol}(S_r^2) = 4\pi r^2$. Let $L^2(S_r^2)$ be the space of square integrable functions on S_r^2. For $f, g \in L^2(S_r^2)$, we define a sesquilinear form $(f,\ g)_{S_r^2}$ by

$$(f,\ g)_{S_r^2} \equiv \int_{S_r^2} f(\omega)\overline{g(\omega)} d\Omega_r(\omega).$$

Then $(f,\ g)_{S_r^2}$ gives an inner product on $L^2(S_r^2)$ and $\|f\|_{S_r^2} = \sqrt{(f, f)_{S_r^2}}$ gives a norm on $L^2(S_r^2)$. From now on we consider $L^2(S_r^2)$ the space of square integrable functions on S_r^2 with the inner product $(\ ,\ )_{S_r^2}$.

For $f \in L^2(S_r^2)$ and $\omega, \zeta \in \mathbf{C}^3$, we define the windowed Fourier transformation $\mathcal{WF}$ with the window function $w(x) = \exp(-x^2/2)$ by

$$\begin{aligned} \mathcal{WF} : f \mapsto \mathcal{WF}f(\zeta, \omega) &= \int_{S_r^2} e^{-ix\cdot\omega} e^{-\frac{(x-\zeta)^2}{2}} \overline{f(x)} d\Omega_r(x) && (2) \\ &= e^{\frac{-r^2-\zeta^2}{2}} \int_{S_r^2} e^{-ix\cdot(\omega+i\zeta)} \overline{f(x)} d\Omega_r(x) \\ &= e^{\frac{-r^2-\zeta^2}{2}} \mathcal{F}f(\omega + i\zeta). && (3) \end{aligned}$$

2.2 Gabor Transformation on the Sphere

Let $\omega_0 \in \mathbf{R}^3$ be fixed. Put $G_{\omega_0}(x) = e^{-x^2/2}e^{-ix\cdot\omega_0}$. For $f \in L^2(S_r^2)$ and $a \in \mathbf{R}_+ = \{x : x > 0\}$, we define the Gabor transformation $\mathcal{G}_{\omega_0}$ by

$$\mathcal{G}_{\omega_0} : f \mapsto \mathcal{G}_{\omega_0} f(\zeta, a) = a^{-\frac{3}{2}} \int_{S_r^2} G_{\omega_0}\left(\frac{x-\zeta}{a}\right)\overline{f(x)}d\Omega_r(x) \tag{4}$$

$$= a^{-\frac{3}{2}} \int_{S_r^2} e^{-\frac{1}{2}(\frac{x-\zeta}{a})^2} e^{-i\frac{x-\zeta}{a}\cdot\omega_0}\overline{f(x)}d\Omega_r(x)$$

$$= a^{-\frac{3}{2}} e^{i\frac{\zeta\cdot\omega_0}{a}} e^{-\frac{r^2+\zeta^2}{2a^2}} \int_{S_r^2} e^{-i\frac{x}{a}\cdot(\omega_0+i\frac{\zeta}{a})}\overline{f(x)}d\Omega_r(x)$$

$$= a^{-\frac{3}{2}} e^{i\frac{\zeta\cdot\omega_0}{a}} e^{-\frac{r^2+\zeta^2}{2a^2}} \mathcal{F}f\left(\frac{a\omega_0 + i\zeta}{a^2}\right). \tag{5}$$

3 Expansion Formula

Let $P_{k,2}(t)$ be the Legendre polynomial of degree k:

$$P_{k,2}(t) = \sum_{l=0}^{[k/2]}(-1)^l \frac{\Gamma(k-l+1/2)}{l!(k-2l)!\sqrt{\pi}}(2t)^{k-2l},$$

where $\Gamma(\cdot)$ is the Gamma function. We define the extended Legendre polynomial by

$$P_{k,2}(z,w) = (\sqrt{z^2})^k(\sqrt{w^2})^k P_{k,n}\left(\frac{z}{\sqrt{z^2}}\cdot\frac{w}{\sqrt{w^2}}\right), \quad z, w \in \mathbf{C}^3.$$

Then $P_{k,2}(z,w)$ is a homogeneous harmonic polynomial of degree k in z and in w; that is, $P_{k,2}(z,w) = P_{k,2}(w,z)$ and $\left(\frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2} + \frac{\partial^2}{\partial z_3^2}\right)P_{k,2}(z,w) = 0$. $P_{k,2}(z,\cdot)$ is the orthogonal polynomial with respect to the measure $d\Omega_r$;

$$\frac{N(k,2)}{r^{2k}}\int_{S_r^2} P_{k,2}(z,\omega)P_{j,2}(\omega,w)d\Omega_r(\omega) = \delta_{kj}P_{k,2}(z,w). \tag{6}$$

The dimension $N(k,2)$ of the space of homogeneous harmonic polynomials of degree k is given by

$$N(k,2) = 2k+1.$$

For $f \in L^2(S_r^2)$, define

$$f_k(x) = \frac{N(k,2)}{r^{2k}} \int_{S_r^2} f(\omega) P_{k,2}(x,\omega) d\Omega_r(\omega). \tag{7}$$

Then we have

$$f(x) = \sum_{k=0}^{\infty} f_k(x) \tag{8}$$

in the sense of $L^2(S_r^2)$ and we have the following Poisson integral formula:

$$\begin{aligned} f(x) &= \lim_{t \uparrow 1} \int_{S_r^2} f(y) \sum_{k=0}^{\infty} \frac{N(k,2)}{r^{2k}} P_{k,2}(tx, y) d\Omega_r(y), \qquad x \in S_r^2, \\ &= \lim_{t \uparrow 1} \int_{S_r^2} f(y) \frac{r^4 - t^2 x^2 y^2}{(r^4 - 2r^2 tx \cdot y + t^2 x^2 y^2)^{3/2}} d\Omega_r(y), \quad x \in S_r^2. \end{aligned}$$

3.1 Expansion Formula of the Exponential Function

For $\nu \neq -1, -2, \ldots$, we define the Bessel function of order ν by

$$J_\nu(t) = \left(\frac{t}{2}\right)^{\nu} \sum_{l=0}^{\infty} \frac{1}{l!\Gamma(\nu + l + 1)} \left(\frac{it}{2}\right)^{2l}.$$

We put

$$\tilde{j}_k(t) = \Gamma(k + 3/2) \left(\frac{2}{t}\right)^{k+1/2} J_{k+1/2}(t) = \sum_{l=0}^{\infty} \frac{\Gamma(k+3/2)}{l!\Gamma(k+l+3/2)} \left(\frac{it}{2}\right)^{2l}. \tag{9}$$

Note that $\tilde{j}_k(-t) = \tilde{j}_k(t)$ and $\tilde{j}_k(0) = 1$. By using this notation we have

$$e^{z \cdot w} = \sum_{k=0}^{\infty} \frac{\sqrt{\pi} N(k,2)}{2^{k+1}\Gamma(k+\frac{3}{2})} \tilde{j}_k(i\sqrt{z^2}\sqrt{w^2}) P_{k,2}(z,w). \tag{10}$$

See [4] for example.

By (1), (6), (7), (8) and (10), for $f \in L^2(S_r^2)$, the Fourier transform $\mathcal{F}f(w)$ is given by

$$\mathcal{F}f(w) = \sum_{k=0}^{\infty} \frac{\sqrt{\pi}(-i)^k r^{2k}}{2^{k+1}\Gamma(k+\frac{3}{2})} \tilde{j}_k(r\sqrt{w^2}) \overline{f_k(w)}.$$

When $f(x) = G_{\omega_0}(x) = e^{-x^2/2}e^{-ix\cdot\omega_0}$, by (7) and (10)

$$f_k(\omega) = e^{-r^2/2}\frac{\sqrt{\pi}N(k,2)(-i)^k}{2^{k+1}\Gamma(k+3/2)}\tilde{j}_k(r\sqrt{\omega_0^2})P_{k,2}(\omega,\omega_0)$$

and for $\omega \in \mathbf{R}^3$, we have

$$\mathcal{F}G_{\omega_0}(\omega) = e^{-r^2/2}\sum_{k=0}^{\infty}\frac{N(k,2)\pi r^{2k}}{(2^{k+1}\Gamma(k+\frac{3}{2}))^2}\tilde{j}_k(r\sqrt{\omega^2})\tilde{j}_k(r\sqrt{\omega_0^2})P_{k,2}(\omega,\omega_0). \tag{11}$$

On the other hand, by (6), (9) and (10),

$$\begin{aligned}\mathcal{F}G_{\omega_0}(\omega) &= \int_{S_r^2} e^{-ix\cdot\omega}\overline{e^{-\frac{x^2}{2}}e^{-ix\cdot\omega_0}}d\Omega_r(x) = e^{-\frac{r^2}{2}}\int_{S_r^2} e^{-ix\cdot(\omega-\omega_0)}d\Omega_r(x)\\ &= e^{-r^2/2}\tilde{j}_0\left(r\sqrt{(\omega-\omega_0)^2}\right). \end{aligned}\tag{12}$$

Therefore by (9), (11) and (12), we have the following Proposition:

Proposition 3.1 *We have the following formula:*

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{\sqrt{\pi}(-1)^k r^{2k}}{2^{2k+1}k!\Gamma(k+3/2)}(\omega-\omega_0)^{2k}\\ &= \tilde{j}_0\left(r\sqrt{(\omega-\omega_0)^2}\right) = \sum_{k=0}^{\infty}\frac{N(k,2)\pi r^{2k}}{(2^{k+1}\Gamma(k+\frac{3}{2}))^2}\tilde{j}_k(r\sqrt{\omega^2})\tilde{j}_k(r\sqrt{\omega_0^2})P_{k,2}(\omega,\omega_0).\end{aligned}$$

3.2 Expansion Formula of Windowed Fourier Transform

By (10), we have

$$\begin{aligned}&\int_{S_r^2}\exp\left(-ix\cdot(\omega+i\zeta)\right)\overline{f(x)}d\Omega_r(x)\\ &= \int_{S_r^2}\sum_{k=0}^{\infty}\frac{\sqrt{\pi}N(k,2)(-i)^k}{2^{k+1}\Gamma(k+\frac{3}{2})}\tilde{j}_k\left(r\sqrt{(\omega+i\zeta)^2}\right)P_{k,2}(x,\omega+i\zeta)\overline{f(x)}d\Omega_r(x)\\ &= \sum_{k=0}^{\infty}\frac{\sqrt{\pi}(-i)^k r^{2k}}{2^{k+1}\Gamma(k+\frac{3}{2})}\tilde{j}_k(r\sqrt{(\omega+i\zeta)^2})\frac{N(k,2)}{r^{2k}}\int_{S_r^2}P_{k,2}(x,\omega+i\zeta)\overline{f(x)}d\Omega_r(x)\\ &= \sum_{k=0}^{\infty}\frac{\sqrt{\pi}(-i)^k r^{2k}}{2^{k+1}\Gamma(k+\frac{3}{2})}\tilde{j}_k\left(r\sqrt{(\omega+i\zeta)^2}\right)\overline{f_k(\omega+i\zeta)}.\end{aligned}$$

Thus by (3), for $f \in L^2(S_r^2)$ we have

$$\mathcal{W}\mathcal{F}f(\zeta,\omega) = e^{(-r^2-\zeta^2)/2}\sum_{k=0}^{\infty}\frac{\sqrt{\pi}(-i)^k r^{2k}}{2^{k+1}\Gamma(k+\frac{3}{2})}\tilde{j}_k\left(r\sqrt{(\omega+i\zeta)^2}\right)\overline{f_k(\omega+i\zeta)}.$$

3.3 Expansion Formula of Gabor Transform

By (6), (7) and (10), we have

$$\int_{S_r^2} e^{-i\frac{x}{a}\cdot(\frac{a\omega_0+i\zeta}{a})}\overline{f(x)}d\Omega_r(x) = \sum_{k=0}^{\infty}\frac{\sqrt{\pi}r^{2k}(-i)^k\tilde{j}_k\left(r\frac{\sqrt{(a\omega_0+i\zeta)^2}}{a^2}\right)}{2(2a^2)^k\Gamma(k+\frac{3}{2})}\overline{f_k(a\omega_0+i\zeta)}.$$

Therefore by (5) we have

$$\begin{aligned}&\mathcal{G}_{\omega_0}f(\zeta,a)\\ &= a^{-\frac{3}{2}}e^{i\frac{\zeta\cdot\omega_0}{a}}e^{-\frac{r^2+\zeta^2}{2a^2}}\sum_{k=0}^{\infty}\frac{r^{2k}\sqrt{\pi}(-i)^k}{2(2a^2)^k\Gamma(k+\frac{3}{2})}\tilde{j}_k\left(r\frac{\sqrt{(a\omega_0+i\zeta)^2}}{a^2}\right)\overline{f_k(a\omega_0+i\zeta)}.\end{aligned}$$

4 Inverse Transformation

Put

$$E(z,w) = \sum_{k=0}^{\infty}\frac{(-i)^k}{r^{2k}2^k k!\tilde{j}_k(\sqrt{z^2}\sqrt{w^2})}P_{k,2}(z,w). \tag{13}$$

Let

$$K_\nu(s) = K_{-\nu}(s) = \int_0^\infty \exp(-s\cosh t)\cosh\nu t dt,\quad 0 < s < \infty$$

be the modified Bessel function. Put

$$\rho(s) = a_0 s^{1/2}K_{-1/2}(s) + a_1 s^{3/2}K_{1/2}(s) = (a_0 + a_1 s)s^{1/2}K_{1/2}(s),$$

where the constants a_0, a_1 are defined by

$$\int_0^\infty s^{2k+1}\rho(s)ds = \frac{N(k,2)k!\Gamma(k+3/2)2^{2k+1}}{\sqrt{\pi}}.$$

Define a measure $d\mu$ on $\mathbf{R}^3$ by

$$\int_{\mathbf{R}^3} f(x)d\mu(x) = \int_0^\infty \int_{S_1^2} f(s\omega)d\Omega_1(\omega)s\rho(s)ds.$$

For the measure $d\mu$ see [1] for example. Put

$$F(w) = \sum_{k=0}^{\infty} \frac{\sqrt{\pi}(-i)^k r^{2k}}{2^{k+1}\Gamma(k+\frac{3}{2})}\tilde{j}_k(r\sqrt{w^2})\overline{f_k(w)}. \tag{14}$$

For $z \in S_r^2$, by (13) and (14), we have

$$\begin{aligned}\int_{\mathbf{R}^3} \overline{F(w)}E(z,w)d\mu(w) &= \int_0^\infty \int_{S_1^2} \overline{F(r\omega)}E(z,s\omega)d\Omega_1(\omega)s\rho(s)ds \\ &= \int_0^\infty \sum_{k=0}^{\infty} \frac{\Gamma(3/2)s^{2k}}{k!\Gamma(k+3/2)2^{2k}} f_k(z)s\rho(s)ds \\ &= \sum_{k=0}^{\infty} f_k(z).\end{aligned}$$

Therefore the mapping

$$f \mapsto \int_{\mathbf{R}^3} \overline{f(x)}E(z,x)d\mu(x)$$

gives the inverse mapping of the Fourier transformation. For $\mathcal{WF}f(\zeta,\omega)$, by (3), we have $e^{(r^2+\zeta^2)/2}\mathcal{WF}f(\zeta,\omega) = \mathcal{F}f(\omega+i\zeta)$. Since $\omega, \zeta \in \mathbf{C}^3$, $y_1 = \omega + i\zeta \in \mathbf{C}^3$. Thus we have

$$\int_{\mathbf{R}^3} \overline{\mathcal{F}f(y_1)}E(z,y_1)d\mu(y_1) = f(z), \quad z \in S_r^2.$$

Similarly for $\mathcal{G}_{\omega_0}f(\zeta,a)$, by (5), we have

$$a^{3/2}e^{-i\zeta\cdot\omega_0/a}e^{(r^2+\zeta^2)/(2a^2)}\mathcal{G}_{\omega_0}f(\zeta,a) = \mathcal{F}f\left((a\omega_0+i\zeta)/a^2\right).$$

Since $y_2 = (a\omega_0 + i\zeta)/a^2 \in \mathbf{C}^3$, we can integrate $\mathcal{F}f(y_2)$ over $\mathbf{R}^3$ and we have

$$\int_{\mathbf{R}^3} \overline{\mathcal{F}f(y_2)}E(z,y_2)d\mu(y_2) = f(z), \quad z \in S_r^2.$$

Thus we have the following theorem:

Theorem 4.1 *Let* $\omega_0 \in \mathbf{R}^3$ *is fixed. For* $\omega, \zeta \in \mathbf{C}^3$ *and* $a \in \mathbf{R}^+$*, put* $y_1 = \omega + i\zeta$ *and* $y_2 = (a\omega_0 + i\zeta)/a^2$*. Then the mapping*

$$F(\zeta, \omega) \mapsto \int_{\mathbf{R}^3} \overline{e^{(r^2+\zeta^2)/2} F(\zeta, \omega)} E(z, y_1) d\mu(y_1), \quad z \in S_r^2,$$

gives an inverse mapping of $\mathcal{WF}$ *defined by* (2) *and the mapping*

$$G(\zeta, a) \mapsto \int_{\mathbf{R}^3} \overline{a^{3/2} e^{-i\zeta\cdot\omega_0/a} e^{(r^2+\zeta^2)/(2a^2)} G(\zeta, a)} E(z, y_2) d\mu(y_2), \quad z \in S_r^2,$$

gives an inverse mapping of the Gabor transformation defined by (4).

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A Model of Relationship Between Waveform-Averaging and Slow Auditory Brainstem Response by Using Discrete Stationary Wavelet Analysis

Nobuko Ikawa, Akira Morimoto, and Ryuichi Ashino

Abstract The relationship between the slow component of auditory brainstem response (ABR) and the number of averaging is investigated using the discrete stationary wavelet analysis (SWT). A new model to analyze the phase shifts of the spontaneous electroencephalogram (EEG) is presented.

Keywords Auditory brainstem response (ABR) • Discrete stationary wavelet transform (SWT) • Electroencephalogram (EEG)

Mathematics Subject Classification (2010) Primary 42C40; Secondary 42C99

1 Introduction

According to an increase of the aging people, the prevention and the treatment of dementia are required. The particularly hearing ability measurement is necessary for the maintenance of the normal brain function. For anti-aging, our ultimate aim is the development of the portable objective audiometry device.

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In this paper we propose a model of auditory brainstem response (ABR) using a discrete stationary wavelet analysis (SWT). The ABR is the representation of electrical activity generated by the eighth cranial nerve and brainstem in response to auditory stimulation. In physiology and anatomy, it is well known that the ABR waveform has seven peaks (see [1]) and that the time (latency) and amplitude analyses of these peaks supply the response according to peripheral hearing status and each location of the human brainstem pathway. Therefore it is also well known that the ABR is useful to the objective hearing test assistant. Electrodes are placed on the scalp and coupled via leads to an amplifier and a signal averager. Spontaneous electroencephalogram (EEG) including the auditory evoked potentials (AEP) from the scalp is recorded while the ears are stimulated via earphones by brief clicks or tones. A series of waveforms unique to the auditory neural structures is viewed after time-locking the EEG recording to each auditory stimulus and averaging several thousand recordings.

2 Conventional Waveform-Averaging Method

The ABR used in this paper was recorded in an acoustically quiet room with subjects either reclined in a comfortable chair or lying on a bed with electrodes placed on the scalp. The electrodes were placed high on each subject's vertex, on the earlobes of both ears, and on the forehead(ground). The subjects are healthy 20-year-old male adults.

Making Epochs

1. As input stimuli, we made several Dirac combs (impulse trains), more precisely, series of periodic acoustic stimuli composed of clicks with intensities of 30, 40, 50, 60, 70, 80 dB nHL, a duration of 0.1 ms, and a frequency of 20 Hz.
2. We presented the input stimuli to both ears using a sensory stimulator.
3. We recorded the electric potentials.
4. We converted the recorded electric potentials to digital data with a sampling frequency 50,000 Hz.
5. We applied a 100–1500 Hz band-pass filter to digital data.
6. We cut the digital data into 512-points data, which are called epochs. The duration of epoch is 10.24 ms.

Averaging Epochs Denote by Epoch_k, the k-th epoch. Define

$$\mathrm{ABR}_N = \frac{1}{N}\sum_{k=1}^{N} \mathrm{Epoch}_k.$$

We call ABR_N by N-average ABR. Usually, 2000-average ABR is simply called ABR. We study the dependency of ABR_N with respect to N for $N = 10, 20, 30, 40, 100, 200, 300, 1000, 1500$, and 2000.

3 Wavelet Analysis of ABR

To represent ABR waveform components, we not only analyze the frequency characteristics of ABR, but also represent both the time (latency) and frequency characteristics of each component of ABR. Using the SWT in this latency-frequency analysis, we describe an estimation method of reproducing ABR signals from the observed values obtained with a number of averaging procedures, as following algorithm:

Wavelet Analysis

1. We used MATLAB2015b and the wavelet toolbox.
2. We set the decomposition level to 8.
3. We used the bi-orthogonal 5.5 wavelet functions for the SWT.
4. Relationship between details and approximation and frequency ranges used in our analysis is given in Table 1.
5. We presented results of reconstructed waveforms of SWT in the case of 70 dB nHL (shown in Fig. 1).
6. We could estimate the peak latencies by the details $D5_N$ to $D8_N$ and $A8_N$ in each case of averaging.

Table 1 Relationship between details and approximation and frequency ranges

Details and approximation	Frequency band (Hz)
D1	12,500–25,000
D2	6250–12,500
D3	3125–6250
D4	1562–3125
D5	781–1562
D6	390–781
D7	195–390
D8	97–195
A8	0–97

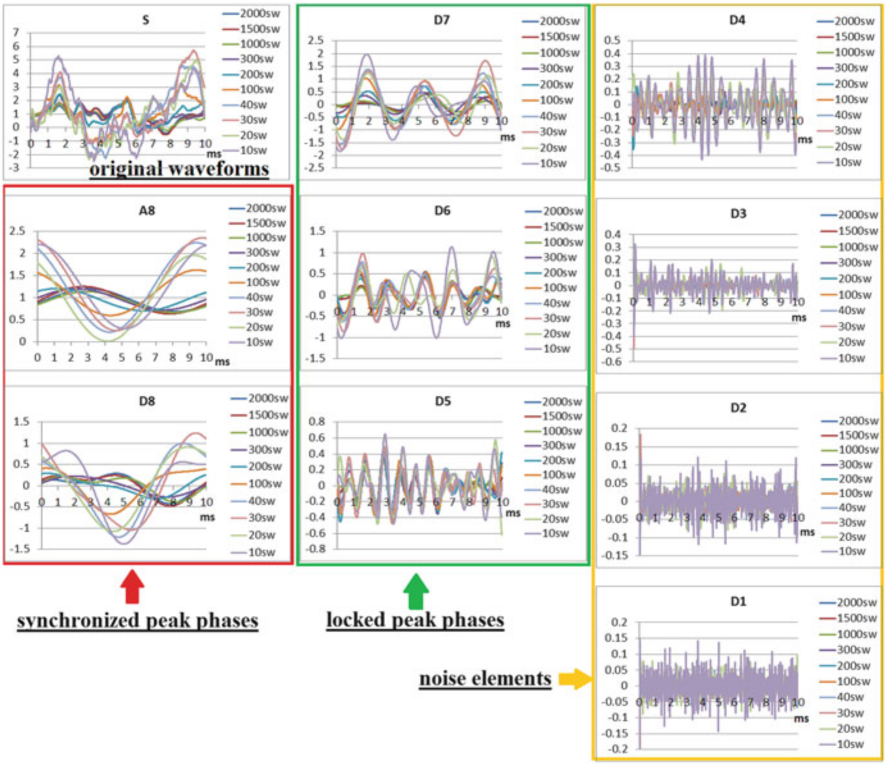

Fig. 1 An example of graphs. The averaging waveforms (S) and their reconstructed waveforms of details $D1_N$ to $D8_N$ and approximations $A8_N$ for $N = 10, 20, 30, 40, 100, 200, 300, 1000, 1500,$ and 2000

4 Proposed Model

In [2] we reported that we applied the polynomial fittings to $A8_N$. In this paper we also apply the polynomial fittings to $D8_N$.

4.1 Dependency of the $A8_N$ *and* $D8_N$ *with Respect to N*

Denote by $A8_N$ and $D8_N$, the approximation A8 and the detail D8 of ABR_N, respectively. Denote by t_N, the time coordinate of the local maximum of $A8_N$ and $D8_N$. We have the following Observations 4.1 and 4.2.

Observation 4.1 *The local maximums of* $A8_N$ *and* $D8_N$ *are decreasing as N increases.*

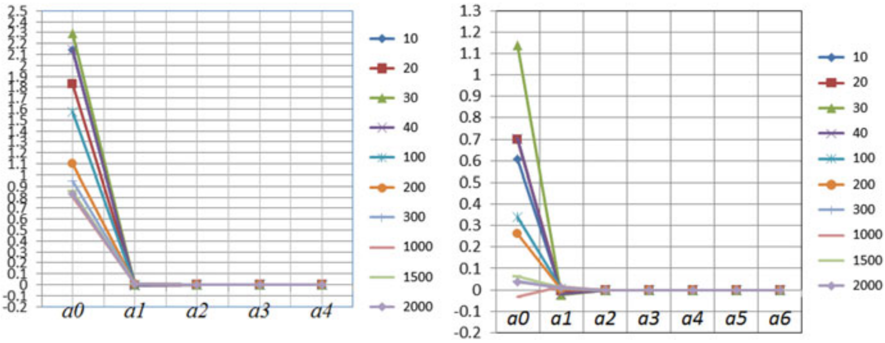

Fig. 2 Coefficients of polynomial fittings of A8 (*left side*) and D8 (*right side*)

Observation 4.2 *For* $N = 200, 300, 1000, 1500, 2000$, *each of* $\mathrm{D8}_N$ *has only one local maximum in* $[1, 6]$, *the coordinate of which is denoted by* $t_{\mathrm{D8},N}$. *Then, we have*

$$1 \le t_{\mathrm{D8},200} \le t_{\mathrm{D8},300} \le t_{\mathrm{D8},1000} \le t_{\mathrm{D8},1500} \le t_{\mathrm{D8},2000} \le 6.$$

We have the similar result for $\mathrm{A8}_N$.

For each $N = 10, 20, \ldots, 2000$, to estimate local maximums of $\mathrm{A8}_N$ and $\mathrm{D8}_N$ and their coordinates $t_{\mathrm{A8},N}$ and $t_{\mathrm{D8},N}$, we use the K-th degree polynomial fitting:

$$y = \sum_{k=0}^{K} a_k(N)\, t^k, \quad (K = 4 \text{ when } A8_N,\ K = 6 \text{ when } D8_N). \tag{1}$$

For a data $\{f_n\}$, we denote by polyfit($\{f_n\}$), the polynomial of degree 4 or 6 fitted to $\{f_n\}$. Here, the coefficients $\{a_k(N)\}_{N=10,20,\ldots,2000}$ are the best coefficients of the polynomial fittings model derived from the experiment data using the least squares method. We apply the polynomial fitting to $\mathrm{A8}_N$ and $\mathrm{D8}_N$.

Results

1. For the coefficients $\{a_k(N)\}_{N=10,20,\ldots,2000}$ we can see the remarkable dependency of $a_0(N)$ on N and the dependencies of the other coefficients are smaller than $a_0(N)$. See Fig. 2.
2. The polynomial fittings are very accurate because the least squares errors are less than 0.0011.
3. We can estimate the coordinates $t_{\mathrm{A8},N}$ and $t_{\mathrm{D8},N}$ of the local maximums.

4.2 A New Slow ABR Model

It is important to analyze the dependency of slow components of ABR_N on N. In particular, the time latency around 5 ms is the most important for the auditory brainstem response. To analyze the time latency, we will search the local maximum, denoted by $\delta(N)$, closest to the time latency. At the same time we independently express $a_0(N)$ because when $t = 0$ is $\delta(N) = 0$. Let us propose a new slow ABR model, which is an improved version of (1), as follows.

$$y = a_0(N) + \sum_{k=1}^{K} a_k(N)\,(t - \delta(N))^k. \tag{2}$$

5 Conclusions

The ABR, usually defined by the average of 2000 epochs, is widely used as an index to assist hearing and brain function diagnoses. We are interested in the dependency of the ABR_N on the number N of averaging. It is believed that the spontaneous EEG should synchronize with the Dirac combs. To show this synchronization, we have studied the slow component of ABR_N using $\mathrm{A8}_N$ (see [2]) and $\mathrm{D8}_N$ because the spontaneous EEG and $\mathrm{A8}_N$ share the same frequency band. Especially the frequency band of $\mathrm{D8}_N$ is contained in the frequency band of the slow component of ABR_N. Therefore we propose the new model (2) in this paper. Our main results are the followings.

1. The ABR peak latencies observed in the $\mathrm{D5}_N$, $\mathrm{D6}_N$, and $\mathrm{D7}_N$ synchronize each other. (Shown in the center graphs of Fig. 1.) In the ten graphs of $\mathrm{D5}_N$, which are the main components of fast ABR_N, we observed the peak latencies of the $D5_N$ are constants. Hence the peak latencies are independent of N.
2. The local maximums of $\mathrm{A8}_N$ and $\mathrm{D8}_N$ are decreasing as N increases.
3. For $N = 200, 300, 1000, 1500, 2000$, the coordinates $t_{\mathrm{A8},N}$ and $t_{\mathrm{D8},N}$, which correspond to the time coordinate of the local maximum of $\mathrm{A8}_N$ and $\mathrm{D8}_N$, increase monotonically with

$$1 \le t_{\mathrm{A8},N} \le 6, \quad 1 \le t_{\mathrm{D8},N} \le 6.$$

Future Study In this paper, we have used the polynomial fitting. Using the polynomial fitting in the form (2), we may have some information on the time-phase shifts. Furthermore, we will investigate dependencies of details $\mathrm{D8}_N$ and approximations $\mathrm{A8}_N$ on N because details $\mathrm{D8}_N$ are also included in the slow components of ABR_N.

Acknowledgements The investigation reported in this paper was supported by CFME of Chiba University. This work was partially supported by JSPS.KAKENHI (C)26400199, (C)25400202 of Japan.

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The Analysis of Big Data by Wavelets

Kiyoshi Mizohata

Abstract The amount of social media data is now growing exponentially. Such data is now called Big Data. In this paper, we shall show several interesting results obtained by the wavelet analysis of Nico Nico Douga (famous social media in Japan) which is a typical example of Big Data, using Hadoop distributed file system.

Keywords Big Data • Wavelets

1 Introduction

We live in the world where the amount of the data set is increasing exponentially. Such big data set is now called Big Data. In this paper we first show how to deal with Big Data written in Japanese. In general, in order to deal with Big Data, we must use Hadoop system. But in this case, to deal with data written in Japanese, we must be careful. Next, we shall explain several interesting results obtained by the wavelet analysis.

2 Pre-processing of Japanese Big Data

In this investigation, we analyze comments of Nico Nico Douga (famous social media in Japan). Nico Nico Douga is famous video sharing website in Japan managed by Dwango. Users can upload video clips. Many comments can be overlaid directly onto the video by many viewers. So comments of video clips are one of the most famous Big Data in Japan.

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To investigate this Big Data (300 GB) by Hadoop system, pre-processing of Japanese data is required since Hadoop system cannot analyze Japanese words. Japanese words must be reformed by Mecab, a famous open source morphological analyzer for Japanese nouns, verbs, and adjectives. By using Mecab, we can do pre-processing of Japanese data by Hadoop system.

3 Wavelet Analysis of Comments

Investigations of the number of comments of this Big Data lead us to very interesting results. In this paper we show one typical example, concerning to the musician A. A is now one of the most famous musicians in Japan. (A is, of course, pseudonym) We want to know a turning point of A's life by the comments of Nico Nico Douga.

Let us find the number of comments related to the musician A. By counting comments with Hadoop system [1] after pre-processing, we obtain the following comment data. See Fig. 1.

We decompose this data using D_2 wavelets [2]. Denote by H_1 the high frequency part of the data and by L_1 the low frequency part of the data.

The lowest value of H_1 data corresponds to a turning point. See a small circle in Fig. 2. In Fig. 3, we also put a circle to a turning point. More interesting results can be found by decomposing L_1 to H_2 and L_2 (Fig. 4).

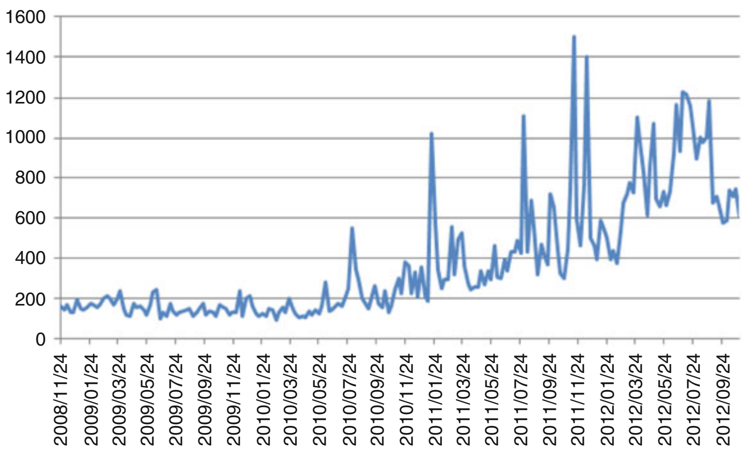

Fig. 1 The number of comments by week

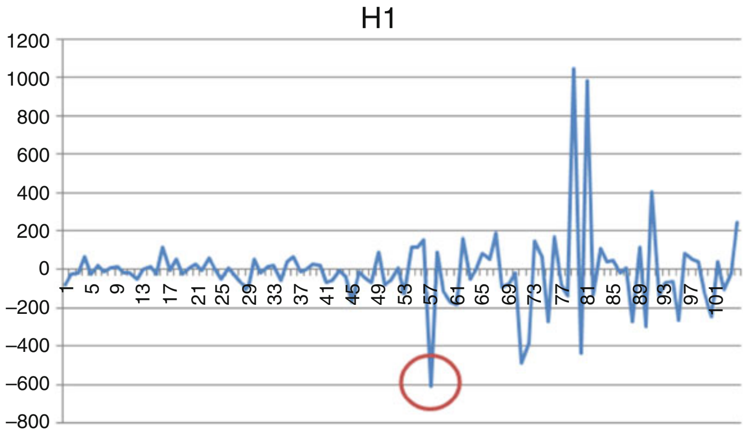

Fig. 2 H_1 data

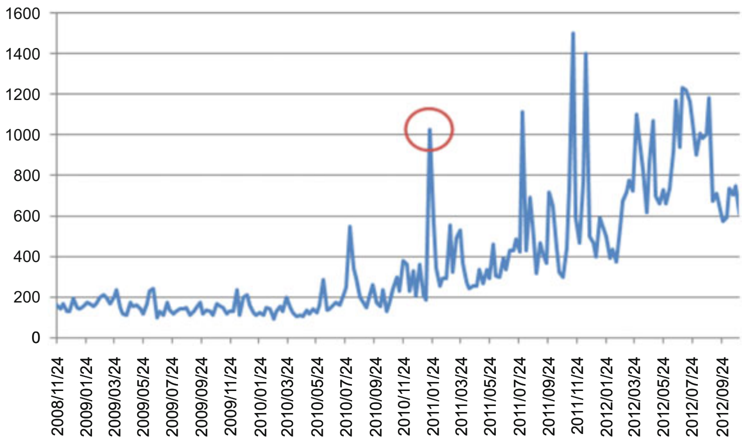

Fig. 3 Turning point of data by H_1

It is obvious that a turning point of data by H_2 data (circle in Fig. 5) is important. This is a turning point of A's life. By analyzing A's turning point more precisely, and also other famous people's data, we can find more interesting results.

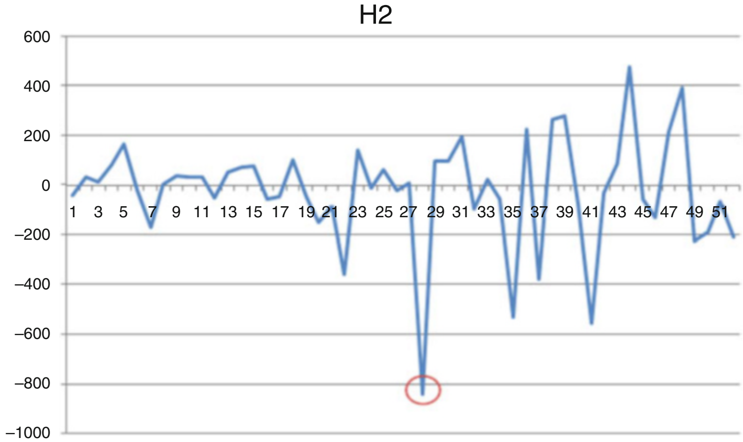

Fig. 4 H_2 data

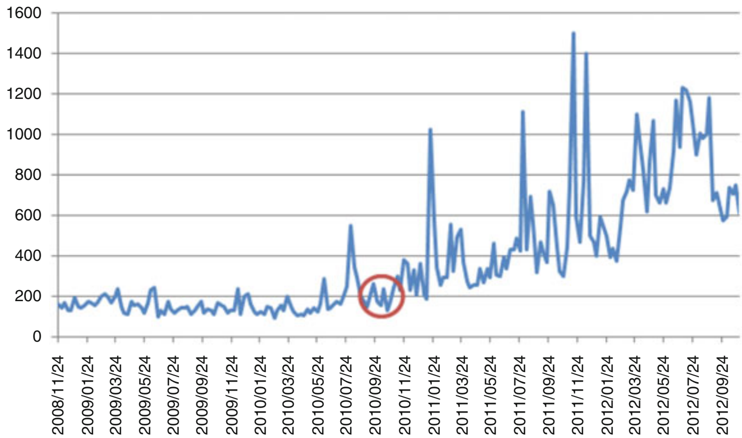

Fig. 5 Turning point of data by H_2

4 Conclusions

These results show that wavelets are strong tools to analyze Big Data. Using wavelets, we can detect important edges of data, and also turning points of a person's life. But, on the other hand, there is a difficult problem. It spends a lot of time to analyze Big Data by Hadoop system. More efficient algorithm must be found for dealing with Big Data.

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Image Source Separation Based on N-tree Discrete Wavelet Transforms

Ryuichi Ashino, Takeshi Mandai, and Akira Morimoto

Abstract An image source separation method using N-tree discrete wavelet transforms is proposed. Key ideas of solving an image separation problem are sketched. The results of numerical experiments show the validity of the proposed method.

Keywords Image source separation • N-tree discrete wavelet transform • Shift parameter

Mathematics Subject Classification (2010) Primary 42C40; Secondary 65T60

1 Introduction

We have studied blind source separation problems in [1], based on wavelet analysis in [2]. We consider an image separation problem whose mixing model superposes shifted source images as shown in Fig. 1. In [3], we treated this problem using continuous multiwavelet transforms proposed in [4]. In this paper, we sketch the key ideas of solving the problem using N-tree discrete wavelet transforms.

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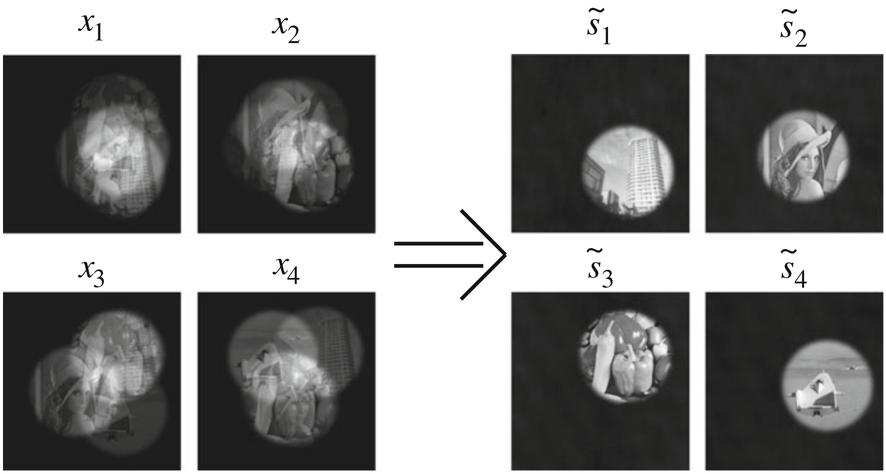

Fig. 1 Example of image separation. *Left*: observed images. *Right*: separated source images

2 Image Separation Problem

We consider an image as a periodic extension of a real matrix $s \in \mathbb{R}^{P\times Q}$, where P, $Q \in \mathbb{N}$. Let $s_m \in \mathbb{R}^{P\times Q}$, $m = 1, \ldots, M$, be source images. Assume that we get observed images x_j, $j = 1, \ldots, J$, by the following mixing model

$$x_j(p,q) = \sum_{m=1}^{M} a_{j,m}\, s_m(p - c^1_{j,m}, q - c^2_{j,m}), \tag{1}$$

where $a_{j,m} \in \mathbb{R}$ are mixing coefficients and $c_{j,m} = (c^1_{j,m}, c^2_{j,m}) \in \mathbb{Z}^2$ are shift parameters. As in Fig. 1, for given observed images, our purposes are to estimate model parameters M, $a_{j,m}$, $c_{j,m}$ and to separate source images. This problem is called an *image separation*. We assume the number of observed images J is larger than or equal to the number of source images M. Under this assumption, if we estimate all parameters, then we can separate source images using the Fourier transform of the mixing model.

3 Key Ideas

Let us sketch the key ideas of solving our image separation problem. The first idea is how to estimate the number M of source images and shift parameters $c_{j,m}$. The second idea is how to estimate mixing coefficients $a_{j,m}$. For these purposes, we need a linear shift-invariant edge extraction method. For details, see [3, Algorithm 3.1] and [5, Algorithm 3].

3.1 Estimation of Shift Parameters

We apply a shift-invariant edge extraction algorithm to two observed images x_1 and x_2, and make edge images e_1 and e_2. See Fig. 2. Let us consider a variant of correlation $R_{1,2}$ between e_1 and e_2:

$$R_{1,2}(c^1, c^2) = \sum_{p,q} e_1(p,q)\, e_2(p+c^1, q+c^2).$$

Here we avoid the normalization because we sum up these variants of correlations for several types of edge images. If the part of the first edge image corresponding to the source image s_m and the same part of the shifted second edge image overlap as in Fig. 2 bottom right, then the absolute value of the inner product is large.

Figure 3 shows an example of the correlation $R_{1,2}$ between the observed images x_1 and x_2 illustrated in Fig. 1. We estimate the number M of source images by the number of peaks of the correlation $R_{1,2}$. Since Fig. 3 has four peaks, the number of source images in Fig. 1 is four. The coordinates (c^1, c^2), which attain the peaks of $R_{1,2}(c^1, c^2)$, correspond to relative shift parameters $c_{2,m} - c_{1,m} = (c^1_{2,m} - c^1_{1,m}, c^2_{2,m} - c^2_{1,m})$.

3.2 Estimation of Mixing Coefficients

If we use a linear shift-invariant edge extraction method, then edge images follow the mixing model (1). We choose a relative shift parameter by the above-mentioned method. The edge image e_1 and the shifted edge image e_2, which is shifted by the relative shift parameter, overlap at the source image s_m as in Fig. 4 left. We consider ratios of the shifted e_2 to e_1. At the coordinates where e_1 and the shifted e_2 overlap, ratios take the same value $a_{2,m}/a_{1,m}$. See Fig. 4 right. We draw a histogram of ratios and select a coordinate which attains the largest peak. This coordinate corresponds to the ratio of mixing coefficients $a_{2,m}$ to $a_{1,m}$.

4 N-tree Discrete Wavelet Transform

We propose to use details of N-tree discrete wavelet transform (N-tree DWT) as a linear shift-invariant edge extraction method. Selesnick and others proposed the dual-tree complex wavelet transform in [6]. We extended it to N-tree version in [7–9]. We applied it to watermarking in [10, 11] and blind source separation problems in [5].

The merits of N-tree DWT are that the calculation speed is faster than a continuous wavelet transform and that approximations and details are almost shift-

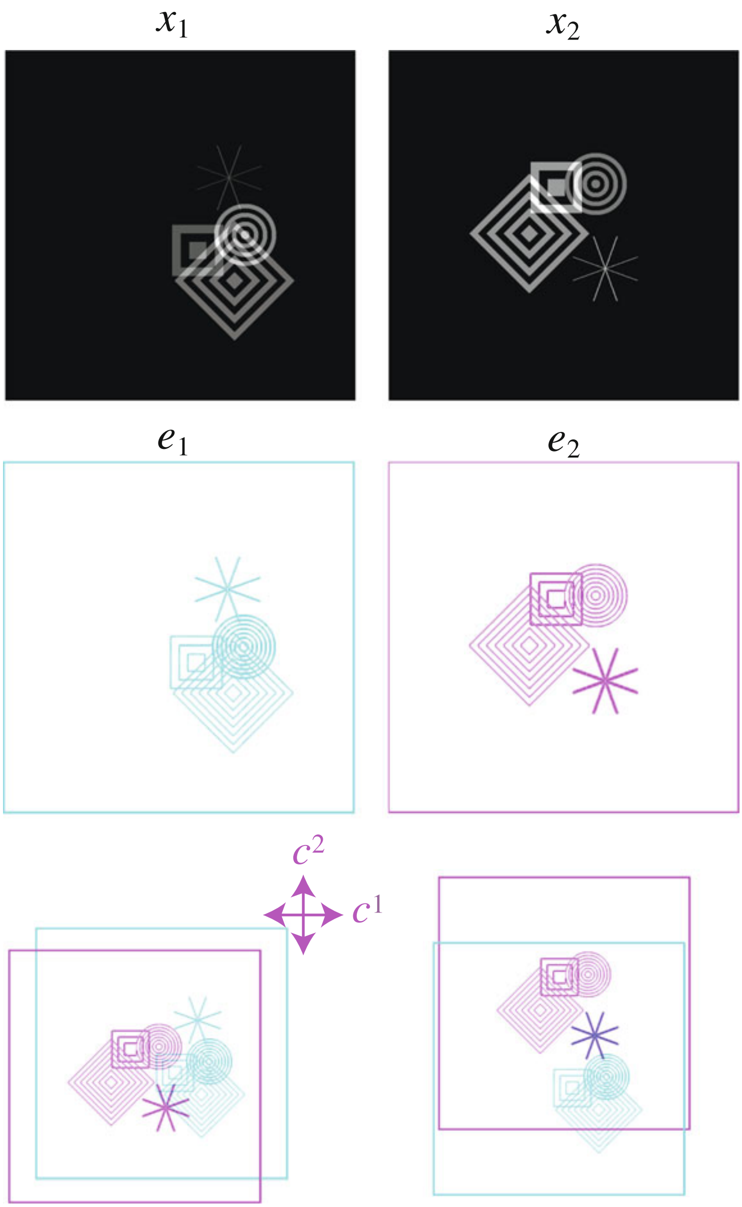

Fig. 2 *Upper*: observed images. *Middle*: edge images. *Bottom left*: fix the first edge image e_1 and move the second edge image e_2. *Bottom right*: search shift parameters where edge images overlap

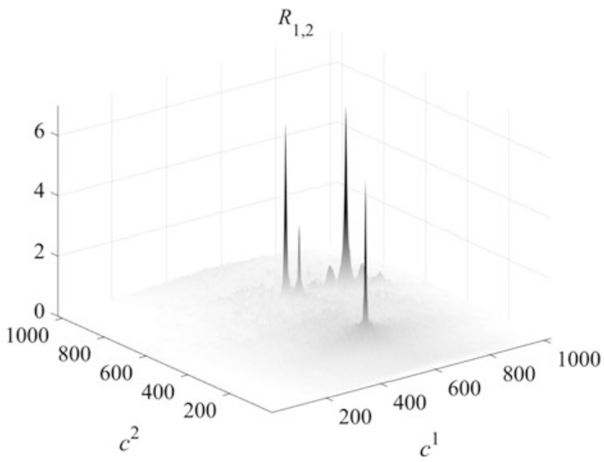

Fig. 3 Correlation $R_{1,2}(c^1, c^2)$ between x_1 and x_2 illustrated in Fig. 1

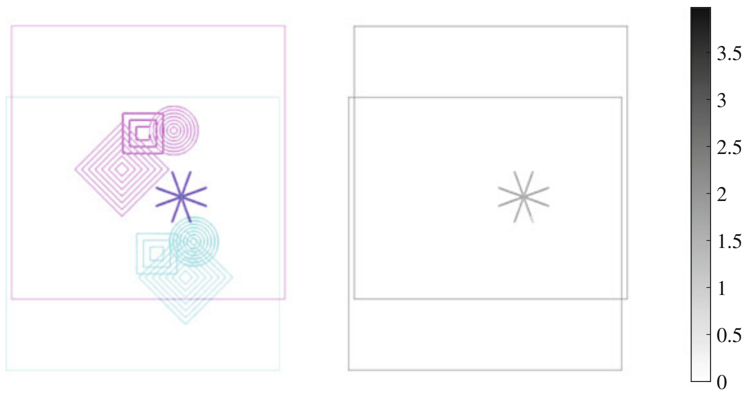

Fig. 4 *Left*: e_1 and the shifted e_2 overlap at the source image. *Right*: the intensity map of ratios of the shifted e_2 to e_1, where the color bar represents the intensity

invariant if we use good wavelet functions explained in [12]. Figure 5 shows details with level three of five shifted impulses. We use Cohen-Daubechies-Feauveau's biorthogonal wavelets (CDF) in [13]. Details using CDF with 7/9 taps are not shift-invariant as in Fig. 5 left, but details using CDF with 39/41 taps are almost shift-invariant as in Fig. 5 right.

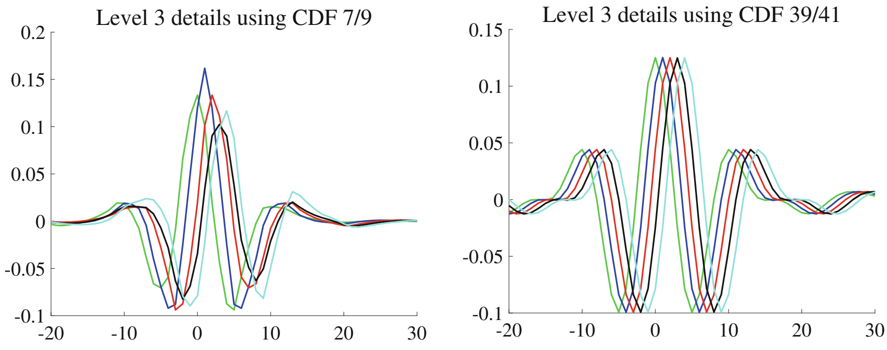

Fig. 5 Details with level three of five shifted impulses. *Left*: CDF 7/9 taps. *Right*: CDF 39/41 taps

5 Numerical Results

We prepared 40 patterns of observed images under the following conditions. We use $J = M = 4$, 512×512 four standard images, uniformly random mixing coefficients on $[0.2, 0.8]$ and uniformly random shift parameters on $[-200, 200]$. All the cases can be separated by our proposed method using N-tree DWT, where $N = 2$.

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Touch-Less Personal Verification Using Palm and Fingers Movements Tracking

Marcin Piekarczyk and Marek R. Ogiela

Abstract In this paper the approach to personal authentication based on analysis of biomechanical characteristics related to palm movements is considered. The basic concept discussed in this research assumes that the hand motion dynamics, treated as a biometrics, can be a sufficient base for efficient user identification. As an input pattern for recognition system the natural finger-based gestures are investigated. The appropriate data is gathered from touch-less sensor device in the form of time-ordered data series related to spatial coordinates of fingertips positions and its velocities. The proposed matching scheme exploits data series analysis in joint with feature-based classification. The research is also focused on the analysis of such type of natural gestures which can be performed with as little awareness as possible. The possibility of using gestures performed in a high degree automatically and non-consciously can be considered as significant advantage in practical applications.

Keywords Behavioral biometrics • Biometrics • Finger tracking • Gesture-based identification • Natural gestures • Palm movements

Mathematics Subject Classification (2010) Primary 68T10; Secondary 92C55

1 Introduction

Nowadays, the utilization of the gestures for control, guidance, or authentication purposes is widely considered. Gesture-oriented interfaces are implemented in various computer-based systems to support remote control (medical visualization

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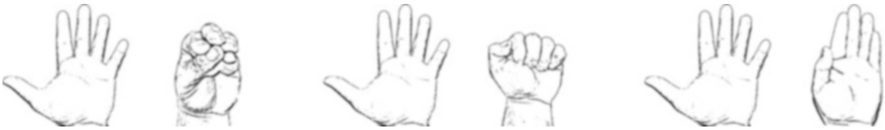

Fig. 1 Examples of natural gestures

systems, virtual reality navigation, computer games) or direct control (touch-screens in standalone and mobile devices). The recognition systems are able to gather information about user's performed gestures from different sources like vision [1], touch-sensitive surfaces [9], accelerometers [10], gyroscopes, or even magnetic field sensors. The last three elements are often integrated as MEMS inside mobile devices. Also, in the field of security, the gestures are perceived as useful data source for authentication purposes. Following the main idea of using gestures instead of standard text password different research and implementations have been developed involving accelerometer-based recognition [10], touch-based drawing gestures [9], or remote palm-based gestures [6, 7].

In this paper we discuss the authentication system where multi-fingers gestures are considered. It extends the previously proposed algorithms [6, 7] where only single finger or writing device is observed and analyzed. Especially, we focus on the so-called natural type of gestures due to their usability in practical applications. Even though it may be classified as behavioral biometrics like handwritten signatures [4, 5] this type of gestures inherits also some advantages over physiological biometrics.

In the context of this paper, we refer the term natural gestures to such type of the gestures which can be performed with as little awareness as possible like instinctive (habit-based) activity i.e without much thinking, almost automatically and where motion pattern exploits natural limitations of musculoskeletal system of the hand to define the range of movement in a easy and deterministic way. Examples of such type of gestures are presented in Fig. 1. The possibility of using gestures performed in a high degree automatically and non-consciously can be considered as significant advantage in practical implementations where exactness of the movement and repeatability is crucial.

2 Recognition and Verification Scheme

2.1 Data Acquisition

We assume the working environment is organized in similar way to systems proposed in [6, 7], where the motion detector is placed vertically and is able to trace palm and fingers shifted above its active surface. We assume that all five fingers are traced and information about the fingertips movements is described as time series in the form of $p^{finger}(t), v^{finger}(t)$ where $finger \in [thumb, index, middle, ring, pinky]$.

The series $p(t)$ means position coordinates data and $v(t)$ means velocity data. Both data series are expressed in 3D Cartesian coordinate system where reference frame is associated with sensor device. In the proposed approach we didn't take into consideration the information about velocity because we want to obtain the authentication scheme independent of the gesture execution speed. Finally, we transform data into one dimensional space using standard Euclidean norm and calculating the magnitude for each component of the signal [Eq. (1)].

$$s = \begin{bmatrix} p^{thumb}(t) \\ p^{index}(t) \\ p^{middle}(t) \\ p^{ring}(t) \\ p^{pinky}(t) \end{bmatrix} = \begin{bmatrix} s_0(t) \\ s_1(t) \\ s_2(t) \\ s_3(t) \\ s_4(t) \end{bmatrix} = \begin{bmatrix} \| s_0(t) \| \\ \| s_1(t) \| \\ \| s_2(t) \| \\ \| s_3(t) \| \\ \| s_4(t) \| \end{bmatrix} \tag{1}$$

2.2 Preprocessing

During the preprocessing phase the following main operations are executed in sequence to obtain the usable signal range:

- dropping off insignificant parts of the signal from the beginning and the end,
- narrowing the signal data to closest left/right minima.

The first step is simply calculated based on lack of the activity. The preprocessing steps are primarily performed for index fingertip signal which is treated as main reference data in this context. Subsequently, the information about usable signal range is applied to the other fingertip signals. Finally, the normalization to the value range [0,1] is calculated (Fig. 2).

2.3 Hand Geometry

Apart from information about finger movements we also try to estimate some coefficients related to individual hand geometry. It is biometric-related information unique for different persons. We calculate and store these coefficients in the form of reference weights $r = [r_1, r_2, r_3, r_4, r_5]$ as it is illustrated in Fig. 3.

2.4 Feature-Based Classification

To make a classification process we assume that all data series are transformed using Discrete Cosine Transform (DCT) [2] in accordance with DCT-II variant defined in

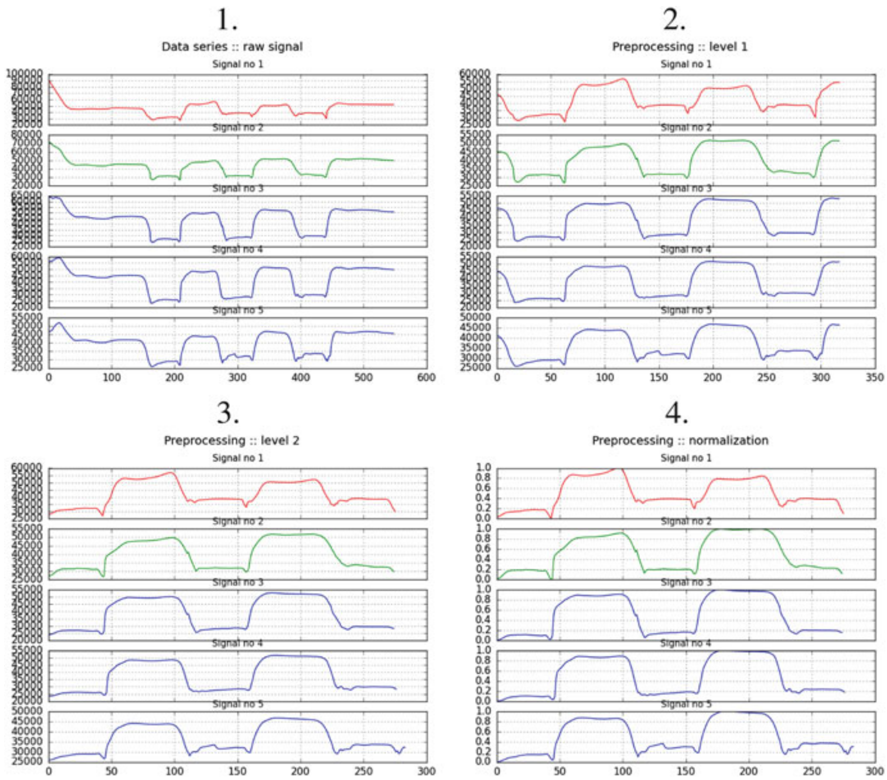

Fig. 2 Preprocessing steps: input raw data (1), cutting the insignificant sections (2), narrowing down to closest minima (3), normalization (4)

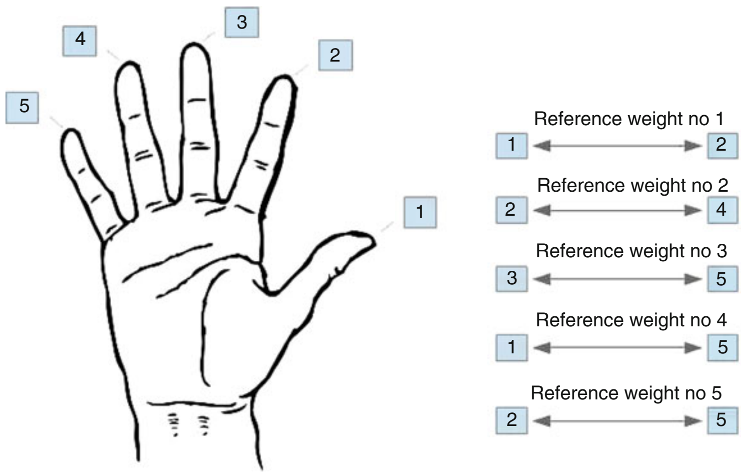

Fig. 3 Structure of the reference weights calculated as distances between fingertips

Eqs. (2) and (3).

$$y[k] = 2f \sum_{n=0}^{N-1} x[n] \cos \frac{\pi}{N} \left(\frac{2n+1}{2} \right) k \qquad \text{where } k = 0, \ldots, N-1 \tag{2}$$

$$f = \frac{1}{\sqrt{4N}} \text{ if } k = 0 \qquad \text{or} \qquad f = \frac{1}{\sqrt{2N}} \text{ if } k \in [1, N-1] \tag{3}$$

In result we obtain new data series composed of DCT factors [Eq. (4)]. DCT transformation has a interesting property that the most energy (highest values) and the most information about original signal is accumulated in finite number of its first coefficients. Due to this (in practice) very often the limited subset of the first DCT factors is applied instead of the full representation. In this paper the limited subset of DCT coefficients is denoted as DCT-m where m parameter indicates the number of DCT coefficients taken into calculations, i.e. DCT-50 means that the first 50 factors from original DCT chain is taken into consideration. Such full or limited collection (subset) of DCT factors is treated as global features describing the examined signal.

$$DCT(s) = \begin{bmatrix} DCT(s_0(t)) \\ DCT(s_1(t)) \\ DCT(s_2(t)) \\ DCT(s_3(t)) \\ DCT(s_4(t)) \end{bmatrix} = \begin{bmatrix} dct_0(s) \\ dct_1(s) \\ dct_2(s) \\ dct_3(s) \\ dct_4(s) \end{bmatrix} \tag{4}$$

Subsequently, it is calculated the DTW-based distance between examined signal (CS) and all patterns stored in memory (S_j) according to Eq. (5). In these calculations standard DTW algorithm [3] with the Manhattan distance as a local cost measure has been used.

$$DTW(CS, S^j) = \begin{bmatrix} DTW(dct_0(CS), dct_0(S^j)) \\ DTW(dct_1(CS), dct_1(S^j)) \\ DTW(dct_2(CS), dct_2(S^j)) \\ DTW(dct_3(CS), dct_3(S^j)) \\ DTW(dct_4(CS), dct_4(S^j)) \end{bmatrix} = \begin{bmatrix} dtw_0 \\ dtw_1 \\ dtw_2 \\ dtw_3 \\ dtw_4 \end{bmatrix} \tag{5}$$

Final classification is made by using k-NN classifier based on weighted Euclidian-like distance [Eq. (6)] where $\Delta r = |r^{CS} - r^{S_j}|$ and vector $w = [1, 5, 5, 1, 5]$ is responsible for reducing the impact of thumb-related distances.

$$d = \sqrt{\sum_i w_i \cdot (\Delta r_i \cdot dtw_i)^2} \tag{6}$$

The brief scheme of the proposed recognition system operating in two phases indicated as offline (collecting patterns for learning set) and online (real-time classification) is presented in Fig. 4.

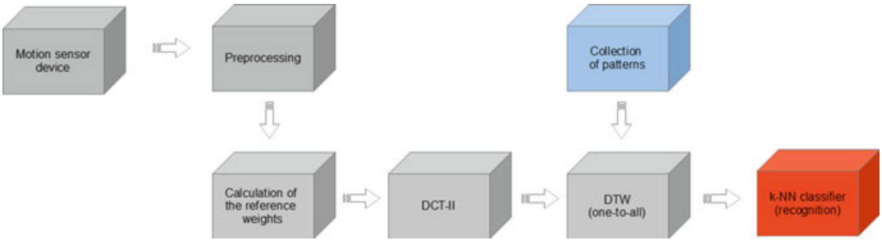

Fig. 4 The main components of the verification scheme

Table 1 The results of 3-NN classification for different number of DCT factors

Errors	DCT-5	DCT-10	DCT-15	DCT-20
FRR	12.5	8.3	12.5	16.7
FAR	4.2	2.8	4.2	5.6

3 Experimental Results

The initial accuracy of the authentication scheme proposed in the paper has been verified using limited set of the examined gestures (4 persons, 10 gestures each). Three randomly chosen realizations have been selected as representative templates and the remaining ones as learning set. The 3-NN classifier was considered. The preliminary results received during the tests are presented in Table 1.

4 Conclusions

In this paper the idea of utilization of hand-based motion characteristics for the verification purposes has been discussed. We have proposed the automatic recognition scheme based on natural gestures where multi-fingers dynamics is observed. Preliminary tests provide promising results. In future research it is necessary to focus on ramarkably improving the effectiveness and investigate the applicability of the model in the field of cryptography (i.e., gesture-based fuzzy vault scheme [8]).

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