Total Dual Integrality of Triangle Covering

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Abstract. This paper concerns weighted triangle covering in undirected graph $G = (V, E)$, where a nonnegative integral vector $\mathbf{w} = (w(e) : e \in E)^T$ gives weights of edges. A subset S of E is a *triangle cover* in G if S intersects every triangle of G. The weight of a triangle cover is the sum of $w(e)$ over all edges e in it. The characteristic vector $\mathbf{x}$ of each triangle cover in G is an integral solution of the linear system

$$\pi : A\mathbf{x} \geq \mathbf{1}, \mathbf{x} \geq \mathbf{0},$$

where A is the triangle-edge incidence matrix of G. System π is *totally dual integral* if $\max\{\mathbf{1}^T\mathbf{y} : A^T\mathbf{y} \leq \mathbf{w}, \mathbf{y} \geq \mathbf{0}\}$ has an integral optimum solution $\mathbf{y}$ for each integral vector $\mathbf{w} \in \mathbb{Z}_+^E$ for which the maximum is finite. The total dual integrality of π implies the nice combinatorial min-max relation that the minimum weight of a triangle cover equals the maximize size of a triangle packing, i.e., a collection of triangles in G (repetitions allowed) such that each edge e is contained in at most $w(e)$ of them. In this paper, we obtain graphical properties that are necessary for the total dual integrality of system π, as well as those for the (stronger) total unimodularity of matrix A and the (weaker) integrality of polyhedron $\{\mathbf{x} : A\mathbf{x} \geq \mathbf{1}, \mathbf{x} \geq \mathbf{0}\}$. These necessary conditions are shown to be sufficient when restricted to planar graphs. We prove that the three notions of integrality coincide, and are commonly characterized by excluding odd pseudo-wheels from the planar graphs.

Keywords: Triangle packing and covering · Totally dual integral system · Totally unimodular matrix · Integral polyhedron · Planar graph · Hypergraph

1 Introduction

Covering and packing triangles in graphs has been extensively studied for decades in graph theory [6,7,14] and optimization theory [2,9]. In this paper, we study the problem from both a polyhedral perspective and a graphical persective – characterizing polyhedral integralities of triangle covering and packing with graphical structures.

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Graphs considered in this paper are undirected, simple and finite. A weighted graph $(G, \mathbf{w})$ consists of a graph G (with vertex set $V(G)$ and edge set $E(G)$) and an edge weight (function) $\mathbf{w} \in \mathbb{Z}_+^{E(G)}$. The weight of any edge subset S is $w(S) = \sum_{e \in S} w(e)$. By a *triangle cover* of G we mean an edge subset S ($\subseteq E(G)$) whose removal from G leaves a triangle-free graph. Let $\tau_w(G)$ denote the minimum weight of a triangle cover of $(G, \mathbf{w})$. By a *triangle packing* of $(G, \mathbf{w})$ we mean a collection of triangles in G (repetition allowed) such that each edge $e \in E(G)$ is contained in at most $w(e)$ of them. Let $\nu_w(G)$ denote the maximum size of a triangle packing of $(G, \mathbf{w})$. In case of $\mathbf{w} = \mathbf{1}$, we write $\tau_w(G)$ and $\nu_w(G)$ as $\tau(G)$ and $\nu(G)$, respectively.

Tuza's Conjecture and Variants. A vast literature on triangle covering and packing concerns Tuza's conjecture [14] that $\tau(G) \leq 2\nu(G)$ for all graphs G and its weighted version [2] that $\tau_w(G) \leq 2\nu_w(G)$ for all graphs G and all $\mathbf{w} \in \mathbb{Z}_+^{E(G)}$. Both conjectures remain wide open. The best known general results $\tau(G) \leq 2.87\nu(G)$ and $\tau_w(G) \leq 2.92\nu_w(G)$ are due to Haxell [7] and Chapuy et al. [2], respectively. Many researchers have pursued the conjectures by showing the conjectured inequalities hold for certain special class of graphs. In particular, Tuza [15] and Chapuy et al. [2] confirmed their own conjectures for planar graphs. Haxell et al. [6] proved the stronger inequality $\tau(G) \leq 1.5\nu(G)$ if G is planar and K_4-free, where K_4 denotes the complete graph on 4 vertices.

Along a different line, Lakshmanan et al. [10] proved that the equation $\tau(G) = \nu(G)$ holds whenever G is (K_4, gem)-free or G's triangle graph is odd-hole-free. A natural question arises for the weighted version: When does $\tau_w(G) = \nu_w(G)$ hold? This question is closely related to the notion of total dual integrality from the theory of polyhedral combinatorics.

Total Dual Integrality. A rational system $\{A\mathbf{x} \geq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is called *totally dual integral* (TDI) if the maximum in the LP duality equation

$$\min\{\mathbf{c}^T\mathbf{x} : A\mathbf{x} \geq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\} = \max\{\mathbf{b}^T\mathbf{y} : A^T\mathbf{y} \leq \mathbf{c}, \mathbf{y} \geq \mathbf{0}\}$$

has an integral optimum solution $\mathbf{y}$ for each integral vector $\mathbf{c}$ for which the maximum is finite. The model of TDI systems introduced by Edmonds and Galies [5] plays a crucial role in combinatorial optimization and serves as a general framework for establishing many important combinatorial min-max relations [3,4,11,12]. Schrijver and Seymour [13] derived the following useful tool for proving total dual integrality.

Theorem 1 [13]**.** *The rational system $A\mathbf{x} \geq \mathbf{b}, \mathbf{x} \geq \mathbf{0}$ is TDI, if and only if*

$$\max\{\mathbf{b}^T\mathbf{y} : A^T\mathbf{y} \leq \mathbf{c}, \mathbf{y} \geq \mathbf{0}, 2\mathbf{y}\, is\, integral\}$$

has an integral optimum solution $\mathbf{y}$ *for each integral vector* $\mathbf{c}$ *for which the maximum is finite.*

Edmonds and Giles [5] showed that total dual integrality implies primal integrality as specified by the following theorem.

Theorem 2 [5]**.** *If rational system* $A\mathbf{x} \geq \mathbf{b}, \mathbf{x} \geq \mathbf{0}$ *is TDI and* $\mathbf{b}$ *is integral, then the polyhedron* $\{\mathbf{x} : A\mathbf{x} \geq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ *is integral, i.e.,* $\min\{\mathbf{c}^T\mathbf{x} : A\mathbf{x} \geq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ *is attained by an integral vector for each integral vector* $\mathbf{c}$ *for which the minimum is finite.*

Given a weighted graph $(G, \mathbf{w})$, let $\Lambda(G)$ denote the set of triangles in G. To see the relation between the equation $\tau_w(G) = \nu_w(G)$ and TDI systems, let us consider the hypergraph $\mathcal{H}_G = (E(G), \Lambda(G))$ of triangles in G. We assume $\Lambda(G) \neq \emptyset$ to avoid triviality. The edge-vertex incidence matrix A_G of $\mathcal{H}_G$ is exactly the triangle-edge incidence matrix of G, whose rows and columns are indexed by triangles and edges of G, respectively, such that for any $\triangle \in \Lambda(G)$ and $e \in E(G)$, $A_{\triangle,e} = 1$ if $e \in \triangle$ and $A_{\triangle,e} = 0$ otherwise. In standard terminologies from the theory of packing and covering [4,12], we write

$$\tau_w(\mathcal{H}_G) = \min\{\mathbf{w}^T\mathbf{x} : A_G\mathbf{x} \geq \mathbf{1}, \mathbf{x} \in \mathbb{Z}_+^{E(G)}\}, \quad (1.1)$$

$$\nu_w(\mathcal{H}_G) = \max\{\mathbf{1}^T\mathbf{y} : A_G^T\mathbf{y} \leq \mathbf{w}, \mathbf{y} \in \mathbb{Z}_+^{\Lambda(G)}\}, \quad (1.2)$$

$$\tau_w^*(\mathcal{H}_G) = \min\{\mathbf{w}^T\mathbf{x} : A_G\mathbf{x} \geq \mathbf{1}, \mathbf{x} \geq \mathbf{0}\}, \quad (1.3)$$

$$\nu_w^*(\mathcal{H}_G) = \max\{\mathbf{1}^T\mathbf{y} : A_G^T\mathbf{y} \leq \mathbf{w}, \mathbf{y} \geq \mathbf{0}\}. \quad (1.4)$$

Combinatorially, each feasible 0–1 solution $\mathbf{x}$ of (1.1) is the characteristic vector of a triangle cover of G, and vice versa. Thus such an $\mathbf{x}$ is also referred to as a triangle cover (or an integral triangle cover to emphasis the integrality) of G. Moreover the minimality of $\tau_w(\mathcal{H}_G)$ implies that

$$\tau_w(\mathcal{H}_G) = \tau_w(G).$$

Similarly, each feasible solution $\mathbf{y}$ of (1.2) is regarded as a triangle packing (or an integral triangle packing) which contains, for each $\triangle \in \Lambda(G)$, exactly $y(\triangle)$ copies of $\triangle$. In particular,

$$\nu_w(\mathcal{H}_G) = \nu_w(G).$$

Usually, feasible solutions of (1.3) and (1.4) are called *fractional triangle covers* and *fractional triangle packings* of G, respectively. Writing $\tau_w^*(G) = \tau_w^*(\mathcal{H}_G)$ and $\nu_w^*(G) = \nu_w^*(\mathcal{H}_G)$, the LP-duality theorem gives

$$\tau_w(G) \geq \tau_w^*(G) = \nu_w^*(G) \geq \nu_w(G).$$

It is well known (see e.g., page 1397 of [12]) that

$$\tau_w(G) = \nu_w(G) \text{ holds for each } \mathbf{w} \in \mathbb{Z}_+^{E(G)} \text{ if and only if } A_G\mathbf{x} \geq \mathbf{1}, \mathbf{x} \geq \mathbf{0} \text{ is TDI.}$$

Total Unimodularity. A matrix A is *totally unimodular* (TUM) if each subdeterminant of A is 0, 1 or -1. Total unimodular matrices often imply stronger integrality than TDI systems (see e.g., [8]).

Theorem 3. *An integral matrix* A *is totally unimodular if and only if the system* $A\mathbf{x} \geq \mathbf{b}, \mathbf{x} \geq \mathbf{0}$ *is TDI for each vector* $\mathbf{b}$.

The 0–1 TUM matrices are connected to balanced hypergraphs. Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. Let $k \geq 2$ be an integer. In $\mathcal{H}$, a *cycle of length* k is a sequence $v_1 e_1 v_2 e_2 \ldots v_k e_k v_1$ such that $v_1, \ldots, v_k \in \mathcal{V}$ are distinct, $e_1, \ldots, e_k \in \mathcal{E}$ are distinct, and $\{v_i, v_{i+1}\} \subseteq e_i$ for each $i = 1, \ldots, k$, where $v_{k+1} = v_1$. Hypergraph $\mathcal{H}$ is called *balanced* if every odd cycle, i.e., cycle of odd length, has an edge that contains at least three vertices of the cycle.

Theorem 4 (Berge [1]). *Let $\mathcal{H}$ be a hypergraph such that every edge consists of at most three vertices. Then the vertex-edge incidence matrix of $\mathcal{H}$ is TUM if and only if $\mathcal{H}$ is balanced.*

Our Results. Let $\mathfrak{B}$, $\mathfrak{M}$, and $\mathfrak{I}$ be the sets of graphs G such that the triangle-edge incidence matrices A_G are TUM, systems $A_G\mathbf{x} \geq \mathbf{1}, \mathbf{x} \geq \mathbf{0}$ are TDI, and polyhedra $\{\mathbf{x} | A_G\mathbf{x} \geq \mathbf{1}, \mathbf{x} \geq \mathbf{0}\}$ are integral, respectively. In terminologies of hypergraph theory (see e.g., Part VIII of [12]),

$G \in \mathfrak{B} \Leftrightarrow \mathcal{H}_G$ is balanced (by Theorem 4 because $\mathcal{H}_G$ is 3-uniform).

$G \in \mathfrak{M} \Leftrightarrow \mathcal{H}_G$ is Mengerian, i.e., $\tau_w(G) = \nu_w(G)$ holds for each $\mathbf{w} \in \mathbb{Z}_+^{E(G)}$.

$G \in \mathfrak{I} \Leftrightarrow \mathcal{H}_G$ is ideal, i.e., $\tau_w(G) = \tau_w^*(G)$ holds for each $\mathbf{w} \in \mathbb{Z}_+^{E(G)}$.

Recalling Theorems 2 and 3, given any graph G, the total modularity (balancedness): $G \in \mathfrak{B}$ implies the total dual integrality (Mengerian property): $G \in \mathfrak{M}$, while $G \in \mathfrak{M}$ implies primal integrality: $G \in \mathfrak{I}$. It follows that

$$\mathfrak{B} \subseteq \mathfrak{M} \subseteq \mathfrak{I}. \tag{1.5}$$

In Sect. 2, first we strengthen (1.5) to $\mathfrak{B} \subsetneq \mathfrak{M} \subsetneq \mathfrak{I}$ (Theorem 5). Then we obtain necessary conditions for a graph to be a member of $\mathfrak{I}$ (Lemma 4) or a minimal graph outside $\mathfrak{B}$ (Theorem 6 and its corollaries) in terms of the pattern of the so-called odd triangle-cycles (Definition 1). Building on these conditions, we establish in Sect. 3 the following characterization for total dual integrality of covering triangle in planar graphs G (Theorem 9):

$$G \in \mathfrak{M} \Leftrightarrow G \in \mathfrak{B} \Leftrightarrow G \in \mathfrak{I} \text{ is } K_4\text{-free} \Leftrightarrow G \text{ is } K_4\text{-free} \,\&\, \text{odd pseudo-wheel-free},$$

where odd pseudo-wheels correspond to odd induced cycles in the triangle graph of G (Definition 2). We conclude in Sect. 4 with remarks on characterizing general graphs $G \in \mathfrak{M}$ and general graphs $G \in \mathfrak{I}$. For easy reference, Appendix gives a list of mathematical symbols used in the paper.

2 General Graphs

In this section, we study TUM, TDI and integral properties for covering and packing triangle in general graphs. We often identify a graph G with its edge set $E(G)$. The following definition is crucial to our discussions.

Definition 1. A *triangle-cycle* in G is a sequence $C = e_1\triangle_1 e_2 \cdots e_k \triangle_k e_1$ with $k \geq 3$ such that $e_1, \cdots, e_k$ are distinct edges, $\triangle_1, \cdots, \triangle_k$ are distinct triangles, and $\{e_i, e_{i+1}\} \subseteq \triangle_i$ for each $i \in \{1, 2, \cdots, k\}$, where $e_{k+1} = e_1$. In $\cup_{i=1}^k \triangle_i$, the edges $e_1, e_2, \ldots, e_k$ are *join edges* and other edges are *non-join edges.*

Let $C = e_1\triangle_1 e_2 \cdots e_k \triangle_k e_1$ be a triangle-cycle. We call C *odd* if its *length* k is odd. By abusing notations, we identify C with the graph $\cup_{i=1}^k \triangle_i$, whose edge set we denote as $E(C)$. We write $J_C = \{e_1, \cdots, e_k\}$ for the set of join edges, and $N_C = E(C) \backslash J_C$ for the set of non-join edges. Let $\mathscr{T}_C$ denote the set of triangles in C. A triangle in $\mathscr{T}_C$ is *basic* if it belongs to $\mathscr{B}_C = \{\triangle_1, \cdots, \triangle_k\}$. Two basic triangles $\triangle_i$ and $\triangle_j$ are *consecutive* if $|i - j| \in \{1, k-1\}$. Triangles in $\mathscr{T}_C$ can be classified into four categories:

$$\mathscr{T}_{C,i} = \{\triangle \in \mathscr{T}_C : |\triangle \cap J_C| = i\}, \quad i = 0, 1, 2, 3.$$

It is clear from Definition 1 that $\mathscr{B}_C \subseteq \mathscr{T}_{C,2} \cup \mathscr{T}_{C,3}$. We will establish a strengthening $\mathfrak{B} \subsetneq \mathfrak{M} \subsetneq \mathfrak{I}$ of the inclusion relations (1.5). The proof needs the following equivalence implied by hypergraph theory.

Lemma 1. *Let G be a graph. Then $G \in \mathfrak{B}$ if and only if every odd triangle-cycle C in G (if any) contains a basic triangle that belongs to $\mathscr{T}_{C,3}$;*

Proof. Recall that $G \in \mathfrak{B}$ if and only if hypergraph $\mathcal{H}_G = (E(G), \Lambda(G))$ is balanced. By definition, the balance condition amounts to saying that every odd triangle-cycle C in G (if any) has a triangle $\triangle$ which contains at least 3 joins. It must be the case that $\triangle$ is formed by exactly 3 joins, giving $\triangle \in \mathscr{T}_{C,3}$. □

Observe that the balanced, Mengerian, and integral properties are all closed under taking subgraphs (see, e.g., Theorems 78.2 and 79.1 of [12]).

Lemma 2. *Let G be a graph and H a subgraph of G. If $G \in \mathfrak{X}$ for some $\mathfrak{X} \in \{\mathfrak{B}, \mathfrak{M}, \mathfrak{I}\}$, then $H \in \mathfrak{X}$.* □

Lemma 3. *$K_4 \in \mathfrak{I} \setminus \mathfrak{M}$.*

Proof. Note that $K_4 \notin \mathfrak{M}$ follows from the fact that $\tau(K_4) = 2$ and $\nu(K_4) = 1$. To see $K_4 = (V, E) \in \mathfrak{I}$, for any $\mathbf{x} \in \mathbb{Q}^E$, let $F(\mathbf{x}) = \{e \in E : 0 < x(e) < 1\}$ consist of "fractional" edges w.r.t $\mathbf{x}$. Taking arbitrary $\mathbf{w} \in \mathbb{Z}_+^E$, we consider an optimal fractional triangle cover $\mathbf{x}^*$ for $(K_4, \mathbf{w})$ such that

$$F(\mathbf{x}^*) \text{ is as small as possible.}$$

We are done by showing that $\mathbf{x}^*$ is integral. Suppose it were not the case. The optimality says that $\mathbf{w}^T\mathbf{x}^* = \tau_w^*(K_4)$ and $\mathbf{x}^* \leq \mathbf{1}$. Thus $F(\mathbf{x}^*) \neq \emptyset$.

If $x^*(e) = 1$ for some $e \in E$, then $\mathbf{x}^*|_{E\setminus\{e\}}$ is a fractional triangle cover for $K_4 \setminus e$ such that $(\mathbf{w}|_{E\setminus\{e\}})^T\mathbf{x}^*|_{E\setminus\{e\}} = \tau_w^*(K_4) - w(e)$. Since $K_4 \setminus e \in \mathfrak{B} \subseteq \mathfrak{I}$, there is a triangle cover S of $K_4 \setminus e$ with minimum weight $w(S) \leq \tau_w^*(K_4) - w(e)$. So $S \cup \{e\}$ is a triangle cover of K_4 with weight $w(S) + w(e) \leq \tau_w^*(K_4)$, and hence the incidence vector $\mathbf{x} \in \{0, 1\}^E$ of $S \cup \{e\}$ is an optimal fractional triangle cover for $(K_4, \mathbf{w})$ with $F(\mathbf{x}) = \emptyset \subsetneq F(\mathbf{x}^*)$ contradicting the minimality of $F(\mathbf{x}^*)$.

Therefore $x^*(e) < 1$ for all $e \in E$, and $A_{K_4}\mathbf{x}^* \geq \mathbf{1}$ enforces that every triangle of K_4 intersects $F(\mathbf{x}^*)$ with at least 2 edges. Thus $F(\mathbf{x}^*)$ contains four edges e_1, e_2, e_3, e_4 that induce a cycle of K_4, where $\{e_1, e_3\}$ and $\{e_2, e_4\}$ are two matchings of K_4. Without loss of generality we may assume that $x^*(e_1) = \min_{i=1}^4 x^*(e_i)$. Let $\mathbf{x} \in \mathbb{Q}_+^E$ be defined by $x(e_i) = x^*(e_i) + (-1)^i x^*(e_1)$ for $i = 1, 2, 3, 4$ and $x(e) = x^*(e)$ for $e \in E \setminus \{e_1, e_2, e_3, e_4\}$. It is straightforward that

$$\mathbf{w}^T\mathbf{x} = \mathbf{w}^T\mathbf{x}^* \text{ and } F(\mathbf{x}) \subseteq F(\mathbf{x}^*) \setminus \{e_1\}.$$

Since every triangle of K_4 intersects each of $\{e_1, e_3\}$ and $\{e_2, e_4\}$ with exactly one edge, we have $A_{K_4}\mathbf{x} = A_{K_4}\mathbf{x}^* \geq \mathbf{1}$, which along with $\mathbf{w}^T\mathbf{x} = \mathbf{w}^T\mathbf{x}^*$ says that $\mathbf{x} \in \{0,1\}^E$ is an optimal fractional triangle cover for $(K_4, \mathbf{w})$. However, $F(\mathbf{x}) \subsetneq F(\mathbf{x}^*)$ gives a contradiction. □

Theorem 5. $\mathfrak{B} \subsetneq \mathfrak{M} \subsetneq \mathfrak{I}$.

Proof. In view of Lemma 3, it suffices to show that the graph $G = (V, E)$ depicted in Fig. 1 belongs to $\mathfrak{M} \setminus \mathfrak{B}$. Note that $G = e_1 \triangle_1 e_2 \cdots e_7 \triangle_7 e_1$ is an odd triangle-cycle of length 7, where $\mathscr{B}_G = \{\triangle_1, \triangle_2, \ldots, \triangle_7\}$ and $\Lambda = \Lambda(G) = \mathscr{T}_G = \{\triangle_1, \ldots, \triangle_7, \triangle_8\}$.

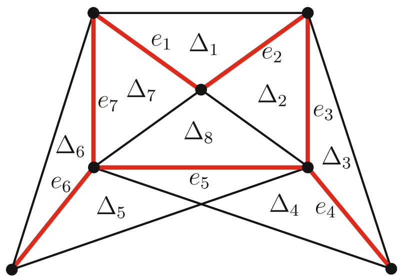

Fig. 1. Graph $G \in \mathfrak{M} \setminus \mathfrak{B}$.

It is routine to check that none of G's basic triangles $\triangle_1, \triangle_2, \ldots, \triangle_7$ belongs to $\mathscr{T}_{G,3}$. Hence Lemma 1 asserts that $G \notin \mathfrak{B}$. To prove $G \in \mathfrak{M}$, by Theorem 1, it suffices to prove that, for any $\mathbf{w} \in \mathbb{Z}_+^E$ and an optimal solution $\mathbf{y}^*$ of $\max\{\mathbf{1}^T\mathbf{y} : A_G^T\mathbf{y} \leq \mathbf{w}, \mathbf{y} \geq \mathbf{0}, 2\mathbf{y} \in \mathbb{Z}_+^\Lambda\}$, there is an integral triangle packing $\mathbf{z} \in \mathbb{Z}_+^\Lambda$ of $(G, \mathbf{w})$ such that $\mathbf{1}^T\mathbf{z} \geq \mathbf{1}^T\mathbf{y}^*$.

Let $\mathbf{y}' \in \{0, 1/2\}^\Lambda$ be defined by $y'(\triangle) = y^*(\triangle) - \lfloor y^*(\triangle) \rfloor$ for each $\triangle \in \Lambda$, and let $\mathbf{w}' \in \mathbb{Z}_+^E$ be defined by $w'(e) = w(e) - \sum_{\triangle \in \Lambda : e \in \triangle} \lfloor y(\triangle) \rfloor$ for each $e \in E$. Then $\mathbf{y}'$ is a fractional triangle packing of $(G, \mathbf{w}')$ such that

$$\mathbf{1}^T\mathbf{y}' = \mathbf{1}^T\mathbf{y}^* - \sum\nolimits_{\triangle \in \Lambda} y^*(\triangle).$$

If there is an integral packing $\mathbf{z}'$ of $(G, \mathbf{w}')$ such that $\mathbf{1}^T\mathbf{z}' \geq \mathbf{1}^T\mathbf{y}'$, then $\mathbf{z}$ with $z(\triangle) = \lfloor y^*(\triangle) \rfloor + z'(\triangle)$ for each $\triangle \in \Lambda$ is an integral packing of $(G, \mathbf{w})$ satisfying $\mathbf{1}^T\mathbf{z} \geq \sum_{\triangle \in \Lambda} y^*(\triangle) + \mathbf{1}^T\mathbf{y}' = \mathbf{1}^T\mathbf{y}^*$ as desired. We next show such a $\mathbf{z}'$ does exist by distinguishing two cases for integral weight $\mathbf{w}'$.

In case of $w'(e) \geq 1$ for each $e \in E$, we observe that $\mathbf{z}'$ with $z'(\triangle_i) = 1$ for $i = 1, 3, 6, 8$ and $z'(\triangle_i) = 0$ for $i = 2, 4, 5, 7$ is a triangle packing of $(G, \mathbf{w}')$ with $\mathbf{1}^T\mathbf{z}' = 4 = |\Lambda|/2 \geq \mathbf{1}^T\mathbf{y}'$.

In case of $w'(e) = 0$ for some $e \in G$, the restriction $\mathbf{y}''$ of $\mathbf{y}'$ to $\Lambda(G \setminus e)$ is a fractional triangle packing of $(G \setminus e, \mathbf{w}'|_{E\setminus e})$ with $\mathbf{1}^T\mathbf{y}'' = \mathbf{1}^T\mathbf{y}'$. Using Lemma 1, it is routine to check that $G \setminus e \in \mathfrak{B}$, which along with $\mathfrak{B} \subseteq \mathfrak{M}$ gives an integral triangle packing $\mathbf{z}''$ of $(G\setminus e, \mathbf{w}'|_{E\setminus e})$ with $\mathbf{1}^T\mathbf{z}'' \geq \mathbf{1}^T\mathbf{y}''$. For each triangle $\triangle \in \Lambda$, set $z'(\triangle)$ to 0 if $e \in \triangle$ and to $z''(\triangle)$ otherwise. It follows that $\mathbf{z}' \in \mathbb{Z}_+^{\Lambda}$ is an integral triangle packing of $(G, \mathbf{w}')$ with $\mathbf{1}^T\mathbf{z}' = \mathbf{1}^T\mathbf{z}'' \geq \mathbf{1}^T\mathbf{y}'$ as desired. □

Lemma 4. *If C is an odd triangle-cycle of graph $G \in \mathfrak{I}$, then C contains either a basic triangle belonging to $\mathscr{T}_{C,3}$ or a non-basic triangle belonging to $\mathscr{T}_{C,0} \cup \mathscr{T}_{C,1}$.*

Proof. By contradiction, suppose that graph $G \in \mathfrak{I}$ and its odd triangle-cycle C of length $2k+1$ form a counterexample, i.e., $\mathscr{B}_C \subseteq \mathscr{T}_{C,2}$ and $\mathscr{T}_C \setminus \mathscr{B}_C \subseteq \mathscr{T}_{C,2} \cup \mathscr{T}_{C,3}$. By Observation 2, we have $C \in \mathfrak{I}$. Let $\mathbf{w} \in \{1, \infty\}^{E(C)}$ be defined by $w(e) = 1$ for all $e \in J_C$ and $w(e) = \infty$ for all $e \in N_C$. On one hand, $\mathscr{B}_C \subseteq \mathscr{T}_{C,2}$ implies that each join edge of C exactly belongs to two basic triangles. To break all $2k+1$ basic triangles, we have to delete at least $k+1$ join edges unless we use some non-join edge (with infinity weight). Thus $\tau_w(C) \geq k+1$.

On the other hand, note that every triangle of C contains at least two join edges in J_C. Thus $\mathbf{x} \in \{1/2, 0\}^{E(C)}$ with $x(e) = 1/2$ if $e \in J_C$ and $x(e) = 0$ otherwise is a fractional triangle cover of C. This along with $|J_C| = 2k+1$ and $\mathbf{w}|_{J_C} = \mathbf{1}$ shows that $\tau_w^*(C) \leq |J_C|/2 = k + 1/2$. However, $\tau_w(C) > \tau_w^*(C)$ contradicts $C \in \mathfrak{I}$. □

The concept of triangle graph provides an efficient tool for studying triangle covering. Suppose that G is a graph with at least a triangle. Its *triangle graph*, denoted as $T(G)$, is a graph whose vertices are named as triangles of G such that $\triangle_i\triangle_j$ is an edge in $T(G)$ if and only if $\triangle_i$ and $\triangle_j$ are distinct triangles in G which share a common edge. For example, the graph G in Fig. 1 has its triangle graph as depicted in Fig. 2.

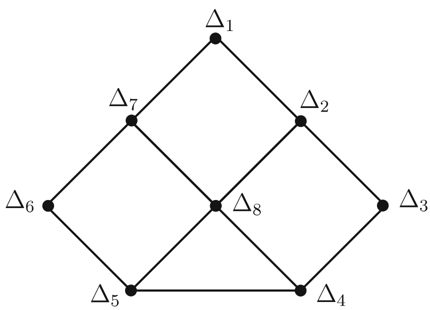

Fig. 2. The triangle graph $T(G)$ of G in Fig. 1.

A graph $G \notin \mathfrak{B}$ is called *minimal* if every proper subgraph H of G belongs to $\mathfrak{B}$. Let $\mathfrak{N}$ denote the set of these minimal graphs.

Theorem 6. *If $G \in \mathfrak{N}$, then G is either K_4 or an odd triangle-cycle with length at least 5 such that $\mathscr{B}_G \subseteq \mathscr{T}_{G,2}$ and $\mathscr{T}_G \setminus \mathscr{B}_G \subseteq \mathscr{T}_{G,1} \cup \mathscr{T}_{G,3}$.*

Proof. Clearly, $K_4 \in \mathfrak{N}$. So we consider $G \neq K_4$. Since $G \notin \mathfrak{B}$ is minimal, G is K_4-free, and by Lemma 1, $G = e_1\triangle_1 e_2 \cdots e_k \triangle_k e_1$ is an odd triangle-cycle such that $\mathscr{B}_G \subseteq \mathscr{T}_{C,2}$, where $k \geq 5$ is odd. Observe that triangle-cycle G corresponds to a cycle $\tilde{C} = \tilde{e}_1\triangle_1\tilde{e}_2 \cdots \tilde{e}_k\triangle_k\tilde{e}_1$ in triangle graph $T(G)$. We first present a series of useful properties.

Property 1. If $\triangle_i\triangle_j$ is a chord of $\tilde{C}$, then the common edge of $\triangle_i$ and $\triangle_j$ is an non-join edge.

Since $\{\triangle_i, \triangle_j\} \subseteq \mathscr{T}_{G,2}$ and they are not consecutive in G, $\triangle_i \cap J_G$ and $\triangle_j \cap J_G$ are disjoint. ■

Property 2. If both $\triangle_i\triangle_j$ and $\triangle_j\triangle_k$ are chords of $\tilde{C}$, then $\triangle_i, \triangle_j, \triangle_k$ share the same non-join edge in G, and $\triangle_i\triangle_k$ is a chord of $\tilde{C}$.

It follows from Property 1 that each of $\triangle_i, \triangle_j, \triangle_k$ has only one non-join edge.■

Property 3. If $\triangle_{i_1}, \triangle_{i_2}, \ldots, \triangle_{i_t}$ are all basic triangles in $\mathscr{B}_G$ that contain $e \in N_G$, where $t \geq 2$ and $i_1 < i_2 < \cdots < i_t$, then for each $j = 1, 2, \ldots, t$, $|\{\triangle_{i_j}, \triangle_{i_j+1} \ldots, \triangle_{i_{j+1}-1}, \triangle_{i_{j+1}}\}|$ is even (where $i_{t+1} = i_1$ in case of $j = t$).

Otherwise, $C_j = e\triangle_{i_j} e_{i_j+1}\triangle_{i_j+1} \cdots \triangle_{i_{j+1}-1} e_{i_{j+1}}\triangle_{i_{j+1}} e$ is an odd triangle-cycle of G for some $1 \leq j \leq t$. Observe that every basic triangle of C_j belongs to $\mathscr{T}_{C_j,2}$. Thus Lemma 1 says that $C_j \notin \mathfrak{B}$, which along with the minimality of $G \in \mathfrak{N}$ enforces that $C_j = G$. However this is absurd because C_j does not contain the join edge $e_{i_{j+2}} \in J_G$ of G. ■

Property 4. For each $e \in N_G$, there are exactly an odd number of basic triangles in $\mathscr{B}_G$ that contain e.

Since G is the union of its basic triangles, e is contained by some basic triangle of G. The property is instant from Property 3 and the odd length k of the triangle-cycle G. ■

We now proceed to prove $\mathscr{T}_G \setminus \mathscr{B}_G \subseteq \mathscr{T}_{G,1} \cup \mathscr{T}_{G,3}$. Suppose for a contradiction that there exists $\triangle \in \mathscr{T}_G \setminus \mathscr{B}_G$ with $\triangle \in \mathscr{T}_{G,0}$. Then $\triangle$ consists of three non-join edges $p, q, r \in N_G$. Let

$$\mathscr{B}_p = \{\triangle \in \mathscr{B}_G : p \in \triangle\}, \mathscr{B}_q = \{\triangle \in \mathscr{B}_G : q \in \triangle\}, \mathscr{B}_r = \{\triangle \in \mathscr{B}_G : r \in \triangle\}$$

denote the sets of basic triangles (of G) that contain p, q, r, respectively. Notice from Property 4 that

$$|\mathscr{B}_p|, |\mathscr{B}_q| \text{ and } |\mathscr{B}_r| \text{ are odd numbers.}$$

We distinguish between two cases depending on whether all of $\mathscr{B}_p, \mathscr{B}_q, \mathscr{B}_r$ are singletons or not.

Case 1. $|\mathscr{B}_p| = |\mathscr{B}_q| = |\mathscr{B}_r| = 1$. We may assume without loss of generality that $\mathscr{B}_j = \{\triangle_{i_j}\}$ for $j \in \{p, q, r\}$ and $i_p < i_q < i_r$. Note that

$$C_{pq} = p\triangle_{i_p} e_{i_p+1}\triangle_{i_p+1} \cdots e_{i_q}\triangle_{i_q} q\triangle p,$$
$$C_{qr} = q\triangle_{i_q} e_{i_q+1}\triangle_{i_q+1} \cdots e_{i_r}\triangle_{i_r} r\triangle q,$$
$$C_{rp} = r\triangle_{i_r} e_{i_r+1}\triangle_{i_r+1} \cdots e_{i_p}\triangle_{i_p} p\triangle r$$

are triangle-cycles of G whose basic triangles each contain exactly two join edges. Observe that the sum of lengths of C_{pq}, C_{qr}, C_{rp} equals $k+6$, which is odd. So at least one of C_{pq}, C_{qr}, C_{rp}, say C_{pq}, has an odd length. It follows from $\mathscr{B}_{C_{pq}} \subseteq \mathscr{T}_{C_{pq},2}$ and Lemma 1 that $C_{pq} \notin \mathfrak{B}$. Now the minimality of $G \in \mathfrak{N}$ enforces $C_{pq} = G$. Hence the join edge $e_{i_q+2} \in J_G$ must be one of $e_{i_p}, e_{i_p+1}, \ldots, e_{i_q-1}$, from which we deduce that $e_{i_q+2} = e_{i_p}$ (and $i_q + 1 = i_r$). As e_{i_q+2} has a common vertex with e_{i_q}, it follows that e_{i_p}, e_{i_q+1} and r form a triangle, and $p, q, r, e_{i_p}, e_{i_p+1}, e_{i_q+1}$ induce a K_4, contradicting the fact that G is K_4-free.

Case 2. $\max\{|\mathscr{B}_p|, |\mathscr{B}_q|, |\mathscr{B}_r|\} \geq 3$. Suppose without loss of generality that $\mathscr{B}_p = \{\triangle_{i_1}, \cdots, \triangle_{i_t}\}$ where $t \geq 3$ and $i_1 < i_2 \cdots < i_t$. Setting $i_{t+1} = i_1$, since $\mathscr{B}_p \cap \mathscr{B}_q = \emptyset$, we have $|\mathscr{B}_q| = \sum_{j=1}^{t} |\{\triangle_{i_j}, \triangle_{i_j+1} \cdots, \triangle_{i_{j+1}}\} \cap \mathscr{B}_q|$. Recall that $|\mathscr{B}_q|$ is odd. So there exists $j \in \{1, \ldots, t\}$ such that $\{\triangle_{i_j}, \triangle_{i_j+1} \cdots, \triangle_{i_{j+1}}\} \cap \mathscr{B}_q$ consists of

$$\text{an odd number } s \text{ of basic triangles } \triangle_{h_1}, \ldots, \triangle_{h_s},$$

where $i_j < h_1 < \cdots < h_s < i_{j+1}$. By Property 3, $|\{\triangle_{i_j}, \triangle_{i_j+1} \ldots, \triangle_{i_{j+1}}\}|$ is even, and $|\{\triangle_{h_\ell}, \triangle_{h_\ell+1}, \ldots, \triangle_{h_{\ell+1}}\}|$ is even for each $\ell \in \{1, \ldots, s-1\}$. Note that

$$\begin{aligned}
&|\{\triangle_{i_j}, \triangle_{i_j+1} \ldots, \triangle_{i_{j+1}}\}| \\
&= |\{\triangle_{i_j}, \triangle_{i_j+1} \ldots, \triangle_{h_1}\}| + \left(\sum_{\ell=1}^{s-1} |\{\triangle_{h_\ell}, \triangle_{h_\ell+1}, \ldots, \triangle_{h_{\ell+1}}\}|\right) \\
&\quad + |\{\triangle_{h_s}, \triangle_{h_s+1} \ldots, \triangle_{i_{j+1}}\}| - s \\
&\equiv (h_1 - i_j) + (i_{j+1} - h_s) - s \pmod 2
\end{aligned}$$

Since s is odd, either $h_1 - i_j$ or $i_{j+1} - h_s$ is odd. Suppose by symmetry that $h_1 - i_j$ is odd. It follows that $C = p\triangle_{i_j} e_{i_j+1}\triangle_{i_j+1} \cdots e_{h_1}\triangle_{h_1} q\triangle p$ is a triangle-cycle of G such that $\mathscr{B}_C \subseteq \mathscr{T}_{C,2}$. As the length $h_1 - i_j + 2$ is odd, we deduce from Lemma 1 that $C \notin \mathfrak{B}$. In turn $G \in \mathfrak{N}$ enforces $C = G$. Similar to Case 1, $e_{h_1+2} \in J_G \subseteq C$ implies that $\triangle_{h_1}, \triangle_{i_{j+1}}, \triangle$ form a K_4, a contradiction to the K_4-freeness of G. The contradiction shows that $(\mathscr{T}_G \setminus \mathscr{B}_G) \cap \mathscr{T}_{G,0} = \emptyset$.

It remains to prove $(\mathscr{T}_G \setminus \mathscr{B}_G) \cap \mathscr{T}_{G,2} = \emptyset$. Suppose on the contrary that there exists $\triangle \in \mathscr{T}_G \setminus \mathscr{B}_G$ which consists of two join edges $p, q \in J_G$ and one non-join edge $r \in N_G$. Again we set $\mathscr{B}_p = \{\triangle \in \mathscr{B}_G : p \in \triangle\}$, $\mathscr{B}_q = \{\triangle \in \mathscr{B}_G : q \in \triangle\}$ and $\mathscr{B}_r = \{\triangle \in \mathscr{B}_G : r \in \triangle\}$. Recalling $\mathscr{B}_G \subseteq \mathscr{T}_{G,2}$, we derive

$|\mathscr{B}_p| = |\mathscr{B}_q| = 2$. Suppose without loss of generality that $\mathscr{B}_p = \{\triangle_{i_p}, \triangle_{i_p+1}\}$, $\mathscr{B}_q = \{\triangle_{i_q}, \triangle_{i_q+1}\}$ and $i_p < i_p+1 < i_q < i_q+1$ (note $p = e_{i_p+1}, q = e_{i_q+1}$). Recall from Property 4 that $|\mathscr{B}_r|$ is an odd number. Observe that both $C = p\triangle_{i_p+1}e_{i_p+2}\triangle_{i_p+2}\cdots e_{i_q}\triangle_{i_q}q\triangle p$ and $C' = q\triangle_{i_q+1}e_{i_q+2}\triangle_{i_q+2}\cdots e_{i_p}\triangle_{i_p}p\triangle q$ are triangle-cycles whose basic triangles each contain exactly 2 join edges. Because the length of G is odd, exactly one of C and C', say C, whose length is odd. By Lemma 1(i), $C \notin \mathfrak{B}$. In turn $G \in \mathfrak{N}$ gives $C = G$. Since neither $\triangle_{i_p}$ nor $\triangle_{i_q+1}$ is a basic triangle of C and $\triangle_{i_p} \neq \triangle_{i_q+1}$, we derive that $e_{i_q+2} \in G \setminus C$, a contradiction to $C = G$. This completes the proof of Theorem 6. □

Let $\mathfrak{X} \in \{\mathfrak{M}, \mathfrak{I}\}$. If graph $G \in \mathfrak{X} \setminus \mathfrak{B}$ is *minimal* in the sense that every proper subgraph H of G is outside $\mathfrak{X} \setminus \mathfrak{B}$, then $H \in \mathfrak{X}$ (by Lemma 2) enforces $H \in \mathfrak{B}$. Hence $G \in \mathfrak{N}$. Conversely, if $G \in \mathfrak{X} \cap \mathfrak{N}$, then every subgraph H of G satisfies $H \in \mathfrak{B} \subseteq \mathfrak{X}$, giving $H \notin \mathfrak{X} \setminus \mathfrak{B}$. Thus the set of *minimal graphs* in $\mathfrak{X} \setminus \mathfrak{B}$ is

$$\{G \in \mathfrak{X} \setminus \mathfrak{B} : H \notin \mathfrak{X} \setminus \mathfrak{B} \text{ for every } H \subsetneq G\} = \mathfrak{N} \cap \mathfrak{X}, \text{ where } \mathfrak{X} \in \{\mathfrak{M}, \mathfrak{I}\}. \quad (2.1)$$

Corollary 1. *If $G \in \mathfrak{N} \cap \mathfrak{I}$ (i.e., $G \in \mathfrak{I} \setminus \mathfrak{B}$ is minimal) , then G is either K_4 or an odd triangle-cycle such that $\mathscr{B}_G \subseteq \mathscr{T}_{G,2}$, $\mathscr{T}_G \setminus \mathscr{B}_G \subseteq \mathscr{T}_{G,1} \cup \mathscr{T}_{G,3}$, and $\mathscr{T}_{G,1} = \mathscr{T}_{G,1} \setminus \mathscr{B}_G \neq \emptyset$.*

Proof. In view of Theorem 6, it suffices to consider G being an odd triangle-cycle such that $\mathscr{B}_G \subseteq \mathscr{T}_{G,2}$ and $\mathscr{T}_G \setminus \mathscr{B}_G \subseteq \mathscr{T}_{G,1} \cup \mathscr{T}_{G,3}$. In turn, Lemma 4 implies the existence of at least a non-basic triangle of G that belongs to $\mathscr{T}_{G,1}$. □

Corollary 2. *If $G \in \mathfrak{N} \cap \mathfrak{M}$ (i.e., $G \subseteq \mathfrak{M} \setminus \mathfrak{B}$ is minimal) , then G is an odd triangle-cycle such that $\mathscr{B}_G \subseteq \mathscr{T}_{G,2}$, $\mathscr{T}_G \setminus \mathscr{B}_G \subseteq \mathscr{T}_{G,1} \cup \mathscr{T}_{G,3}$, and $\mathscr{T}_{G,1} = \mathscr{T}_{G,1} \setminus \mathscr{B}_G \neq \emptyset$.*

Proof. Note from $G \in \mathfrak{M}$ that $G \neq K_4$. As $\mathfrak{M} \subseteq \mathfrak{I}$, the conclusion is immediate from Corollary 1. □

3 Planar Graphs

In this section, we study the planar case more closely, and characterize planar graphs in $\mathfrak{M}$ by excluding pseudo-wheels defined as follows.

Definition 2. A *triangle-cycle* C is a *pseudo-wheel* if it has length at least 4, $\mathscr{T}_C = \mathscr{B}_C$ and each pair of non-consecutive basic triangles of C is edge-disjoint.

It is easy to see that a triangle-cycle C is a pseudo-wheel if and only if its triangle graph $T(C)$ is an induced cycle with length at least 4. Thus every wheel other than K_4 is a pseudo-wheel. Two pseudo-wheels that are not wheels are shown in Fig. 3.

Lemma 5. *If C is an odd pseudo-wheel, then $C \notin \mathfrak{I}$.*

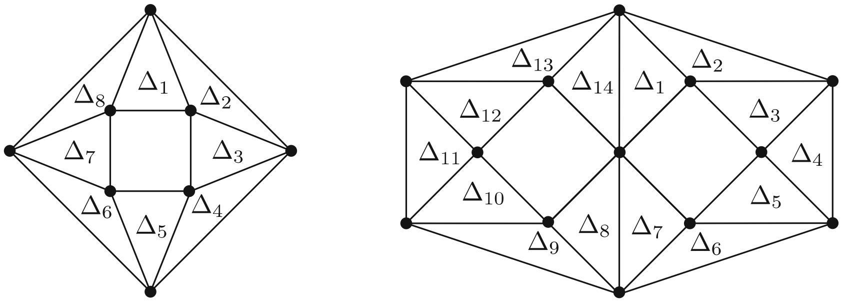

Fig. 3. Examples of pseudo-wheels.

Proof. Suppose the length of C is $2k+1$. Let $\mathbf{w} \in \mathbb{Z}_+^E(C)$ be defined by $w(e) = 1$ for all $e \in J_C$ and $w(e) = \infty$ for all $e \in N_C$. Then $\tau_w(C) = k + 1$. On the other hand $\mathbf{x} \in \{0, 1/2\}^{E(C)}$ with $x(e) = 1/2$ for all $e \in J_C$ and $x(e) = 0$ for all $e \in N_C$ is a fractional triangle cover of C, showing $\tau_w^*(C) \le \mathbf{w}^T\mathbf{x} = k + 1/2 < \tau_w(C)$. □

If $\triangle^i, \triangle^o, \triangle$ are distinct triangles of plane graph G such that $\triangle^i$ is inside $\triangle$ and $\triangle^o$ is outside $\triangle$, then we say that $\triangle$ is a *separating triangle* of $\triangle^i$ and $\triangle^o$, or $\triangle$ *separates* $\triangle^i$ from $\triangle^o$.

A *triangle-path* in graph G is a sequence $P = \triangle_1 e_1 \cdots e_k \triangle_{k+1}$ with $k \ge 1$ such that $e_1, \cdots, e_k$ are distinct edges, $\triangle_1, \cdots, \triangle_{k+1}$ are distinct triangles of G, and $\{e_1\} \subseteq \triangle_1, \{e_k\} \subseteq \triangle_{k+1}, \{e_i, e_{i+1}\} \subseteq \triangle_{i+1}$ for each $i \in [k-1]$. In $\cup_{i=1}^{k+1} \triangle_i$, the edges $e_1, e_2, \ldots, e_k$ are called *join edges* and other edges are called *non-join edges*. Let J_P denote the set of join edges of P. The length of P is defined as k. We often say that P is a triangle-path from $\triangle_1$ to $\triangle_{k+1}$.

Lemma 6. *Let G be a plane graph in which $\triangle$ is a separating triangle of triangles $\triangle^i$ and $\triangle^o$. Then $\triangle$ contains at least one join edge of every triangle-path from $\triangle^i$ to $\triangle^o$ in G.*

Proof. Consider an arbitrary triangle-path $P = \triangle_1 e_1 \cdots e_k \triangle_{k+1}$ in G from $\triangle_1 = \triangle^i$ to $\triangle_{k+1} = \triangle^o$. We prove $\triangle \cap \{e_1, \ldots, e_k\} \ne \emptyset$ by induction on k. The basic case of $k = 1$ is trivial. We consider $k \ge 2$ and assuming that the lemma holds when triangle-path involved has length at most $k - 1$. If $\triangle_2 = \triangle$, then we are done. If $\triangle_2 \ne \triangle$, then either $\triangle$ separates $\triangle_1$ from $\triangle_2$ or separates $\triangle_2$ and $\triangle_{k+1}$. Observe that $\triangle_1 e_1 \triangle_2$ is a triangle-path of length $1 < k$, and $\triangle_2 e_2 \ldots, e_k \triangle_{k+1}$ is a triangle-path of length $k-1$. From the induction hypothesis, we derive $e_1 \in \triangle$ in the former case, and $e_j \in \triangle$ for some $j = 2, \ldots, k$ in the latter case. □

Lemma 7. *Let $C = e_1 \triangle_1 e_2 \cdots e_k \triangle_k e_1$ with $k \ge 3$ be a triangle-cycle. If C is plane and $\mathscr{B}_C \subseteq \mathscr{T}_{C,2}$, then $\triangle_h$ does not separate $\triangle_i$ from $\triangle_j$ for any distinct $h, i, j \in \{1, \ldots, k\}$.*

Proof. Note that C contains a triangle-path P from $\triangle_i$ and $\triangle_j$ with $J_P \subseteq J_C \setminus \triangle_h$. The triangle-path P along with Lemma 6 implies the result. □

Theorem 7. *If C is a planar triangle-cycle such that $\mathscr{B}_C \subseteq \mathscr{T}_{C,2}$, then $\mathscr{T}_C \subseteq \mathscr{T}_{C,0} \cup \mathscr{T}_{C,2}$.*

Proof. Suppose that $C = e_1\triangle_1 e_2 \cdots e_k \triangle_k e_1$ with $k \geq 3$ is plane, and there exists $\triangle \in \mathscr{T}_C$ with $\triangle \in \mathscr{T}_{C,1} \cup \mathscr{T}_{C,3}$.

Case 1. $\triangle \in \mathscr{T}_{C,3}$ consists of three join edges e_h, e_i, e_j, where $1 \leq h < i < j \leq k$. The structure of the triangle graph $T(C)$ is illustrated in the left part of Fig. 4.

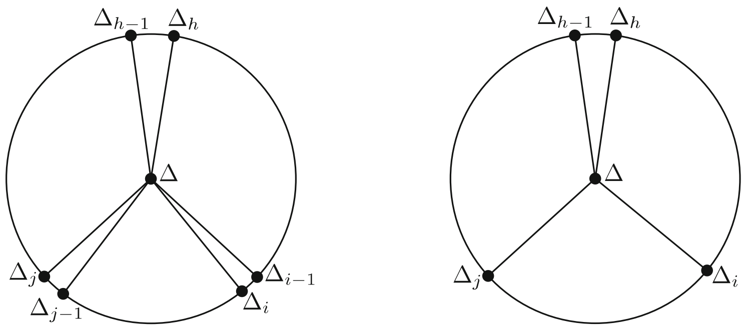

Fig. 4. The triangle graph $T(C)$ in the two cases of the proof for Theorem 7.

For each pair $(s,t) \in \{(h, i-1), (i, j-1), (j, h-1)\}$, there is a triangle-path in C from $\triangle_s$ to $\triangle_t$ whose set of join edges is disjoint from $\{e_h, e_i, e_j\} = \triangle$. It follows from Lemma 6 that

$$\begin{aligned} &\triangle \text{ does not separate } \triangle_s \text{ from } \triangle_t \text{ for each} \\ &(s,t) \in \{(h, i-1), (i, j-1), (j, h-1)\}. \end{aligned} \tag{3.1}$$

Suppose that $\triangle$ separates $\triangle_{h-1}$ from $\triangle_h$, and separates $\triangle_{i-1}$ from $\triangle_i$. Without loss of generality let $\triangle_{h-1}$ and $\triangle_h$ sit inside and outside $\triangle$, respectively. Then (3.1) implies that $\triangle_j$ and $\triangle_{i-1}$ are inside and outside $\triangle$, respectively. In turn, i is inside $\triangle$, and (3.1) says that $\triangle_{j-1}$ is inside $\triangle$. Now $\triangle_{j-1}$ and $\triangle_j$ are both inside $\triangle$, i.e., $\triangle$ does not separate $\triangle_{j-1}$ from $\triangle_j$. Hence, by symmetry we

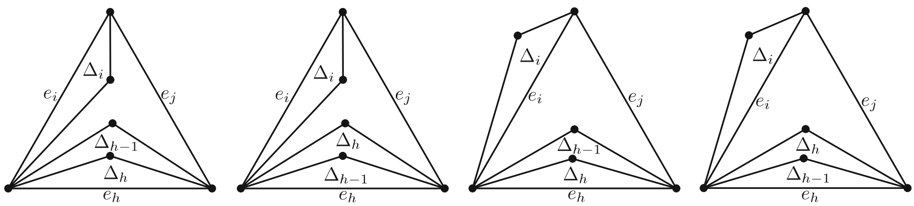

Fig. 5. $\triangle_{h-1}$ and $\triangle_h$ are both inside $\triangle$.

may assume that $\triangle$ does not separate $\triangle_{h-1}$ from $\triangle_h$, and further that $\triangle_{h-1}$ and $\triangle_h$ are both inside $\triangle$. as illustrated in Fig. 5.

As $e_h \in \triangle_{h-1} \cap \triangle_h$, it is easy to see that either $\triangle_{h-1}$ separates $\triangle_h$ from $\triangle_i$ or $\triangle_h$ separates $\triangle_{h-1}$ from $\triangle_i$. The contradiction to Lemma 7 finishes our discussion on Case 1.

Case 2. $\triangle \in \mathscr{T}_{C,1}$ consist of join edge e_h of C (shared with $\triangle_{h-1}, \triangle_h$), non-join edge f (shared with $\triangle_i$) and non-join edge g (shared with $\triangle_j$), where h, i, j are distinct. See the right part of Fig. 4. Similar to Case 1, it can be derived from Lemma 6 that

$$\triangle \text{ does not separate} \triangle_s \text{ from } \triangle_t \text{ for each} (s,t) \in \{(h,i),(i,j),(j,h-1)\}.$$

Therefore $\triangle$ does not separate $\triangle_{h-1}$ and $\triangle_h$. Suppose without loss of generality that both $\triangle_{h-1}$ and $\triangle_h$ are inside $\triangle$. Then C has one of the structures as illustrated in Fig. 5 with f in place of e_i and g in place of e_j. Again, either $\triangle_{h-1}$ separating $\triangle_h$ from $\triangle_i$ or $\triangle_h$ separating $\triangle_{h-1}$ from $\triangle_i$ contradicts to Lemma 7. This completes the proof. □

Theorem 8. *Let G be a planar graph. Then $G \in \mathfrak{N}$ if and only if G is K_4 or an odd pseudo-wheel.*

Proof. Sufficiency: Clearly $K_4 \in \mathfrak{N}$. If G is an odd pseudo-wheel C, then G is an odd triangle-cycle such that $\mathscr{B}_G \subseteq \mathscr{T}_{G,2}$. By Lemma 1, $G \notin \mathfrak{B}$. Since the triangle graph $T(C)$ is an induced cycle, every proper subgraph of C is triangle-cycle-free, and hence belongs to $\mathfrak{B}$, giving $G \in \mathfrak{N}$.

Necessity: Suppose that $G \in \mathfrak{N}$ and $G \neq K_4$. By Theorem 6, an odd triangle-cycle with length at least 5 such that $\mathscr{B}_G \subseteq \mathscr{T}_{G,2}$ and $\mathscr{T}_G \setminus \mathscr{B}_G \subseteq \mathscr{T}_{G,1} \cup \mathscr{T}_{G,3}$. In turn, Theorem 7 enforces

$$\mathscr{T}_C = \mathscr{B}_C.$$

Suppose for a contradiction that there exists non-consecutive triangles $\triangle_i, \triangle_j \in \mathscr{B}_G$ that share a common non-join edge e, where $i < j-1$. Then G contains two triangle-cycles $C_1 = e\triangle_i e_{i+1}\triangle_{i+1} \cdots e_j \triangle_j e$ and $C_2 = e\triangle_j e_{j+1}\triangle_{j+1} \cdots e_i \triangle_i e$. Because G is odd, one of C_1 and C_2, say C_1, is odd. As C_1 is a proper subgraph of $G \in \mathfrak{N}$, we have $C_1 \in \mathfrak{B}$. By Lemma 1, there exists a basic triangle $\triangle_h$ in $\mathscr{B}_{C_1} \cap \mathscr{T}_{C_1,3}$. Because $\triangle_h \in \mathscr{T}_{G,2}$, it must be the case that $e \in \triangle_h$. Thus $\triangle_i, \triangle_j, \triangle_h$ share a common non-join edge e of G. However in any planar embedding for G, there is one triangle in $\{\triangle_i, \triangle_j, \triangle_h\}$, which is a separating triangle of the other two. This is a contradiction to Lemma 7. Thus each pair of non-consecutive basic triangles of G is edge-disjoint, and G is an odd pseudo-wheel. □

Theorem 9. *Let G be a planar graph, then the following are equivalent:*

(i) $G \in \mathfrak{B}$;
(ii) $G \in \mathfrak{M}$;

(iii) $G \in \mathfrak{I}$ *is* K_4*-free; and*
(iv) G *is* K_4*-free and odd pseudo-wheel free.*

Proof. Recalling (1.5) and Lemma 3, $\mathfrak{B} \subseteq \mathfrak{M} \subseteq \mathfrak{I}$ and $K_4 \in \mathfrak{I} \setminus \mathfrak{M}$ imply the relation $(i) \Rightarrow (ii) \Rightarrow (iii)$. If G contains an odd pseudo-wheel H, then $H \notin \mathfrak{I}$ by Lemma 5, which along with Lemma 2 would give $G \notin \mathfrak{I}$. So we have $(iii) \Rightarrow (iv)$.

It remains to prove $(iv) \Rightarrow (i)$. If $G \notin \mathfrak{B}$, we take $H \subseteq G$ to be minimal, i.e., $H \in \mathfrak{N}$. Theorem 8 says that H is K_4 or an odd pseudo-wheel, i.e., G is not K_4-free and G is not odd pseudo-wheel free. □

4 Remarks

Lemma 4 provides us a necessary condition for $G \in \mathfrak{I}$ as follows:

$$(\mathscr{B}_C \cap \mathscr{T}_{C,3}) \cup ((\mathscr{T}_{C,0} \cup \mathscr{T}_{C,1}) \setminus \mathscr{B}_C) \neq \emptyset \text{ for any odd triangle-cycle } C \text{ of } G. \quad (4.1)$$

It would be interesting to see if the condition is sufficient for $G \in \mathfrak{I}$. A supporting evidence is the following.

Remark 1. Condition (4.1) is a necessary and sufficient condition for K_4-free planar graph G to be a member of $\mathfrak{I}$.

Proof. By Theorem 9, a K_4-free planar graph $G \in \mathfrak{I}$ implies $G \in \mathfrak{B}$, and thus $\mathscr{B}_C \cap \mathscr{T}_{C,3} \neq \emptyset$ for every odd triangle-cycle C in G. On the other hand, given a K_4-free planar graph G satisfying (4.1), we see from Definition 2 that G does not contain any odd pseudo-wheel. It follows from Theorem 8 that G does not contain any subgraph in $\mathfrak{N}$, which implies $G \in \mathfrak{B} \subseteq \mathfrak{I}$. □

As $\mathfrak{M} \subseteq \mathfrak{I}$, condition (4.1) is also necessary for $G \in \mathfrak{M}$, but it is not sufficient for the total dual integrality. This can be seen from $K_4 \notin \mathfrak{M}$, which satisfies (4.1): K_4 has four odd triangle-cycles with length 3 each containing a triangle without any join edge, and for each odd triangle-cycle C, there is a triangle in $\mathscr{T}_C \setminus \mathscr{B}_C$ that belongs to $\mathscr{T}_{C,0}$. This motivates us to ask about the necessity and sufficiency of the following conditions for $G \in \mathfrak{M}$:

$$(\mathscr{B}_C \cap \mathscr{T}_{C,3}) \cup (\mathscr{T}_{C,1} \setminus \mathscr{B}_C) \neq \emptyset \text{ for any odd triangle-cycle } C \text{ of } G. \quad (4.2)$$

Note that condition (4.2) implies G contains neither K_4 nor odd pseudo-wheels. Similar to Remark 1, Theorems 8 and 9 provide the following fact.

Remark 2. Condition (4.2) is a necessary and sufficient condition for planar graph G to be a member of $\mathfrak{M}$. □

Appendix: A List of Mathematical Symbols

$(G, \mathbf{w})$	Weighted graph $G = (V(G), E(G))$ with $\mathbf{w} \in \mathbb{Z}_+^{E(G)}$
$\tau_w(G)$	The minimum weight of an integral triangle cover in $(G, \mathbf{w})$
$\nu_w(G)$	The maximum size of an integral triangle packing in $(G, \mathbf{w})$
$\tau_w^*(G)$	The minimum weight of a fractional triangle cover in $(G, \mathbf{w})$
$\nu_w^*(G)$	The maximum size of a fractional triangle packing in $(G, \mathbf{w})$
$\tau(G)$	$\tau_w(G)$ when $\mathbf{w} = \mathbf{1}$
$\nu(G)$	$\nu_w(G)$ when $\mathbf{w} = \mathbf{1}$
A_G	The triangle-edge incidence matrix of graph G
$\Lambda(G)$	The set of triangles in graph G
$\mathfrak{B}$	The set of graphs G such that A_G are TUM
$\mathfrak{M}$	The set of graphs G such that systems $A_G x \geq 1, x \geq 0$ are TDI
$\mathfrak{I}$	The set of graphs G such that $\{\mathbf{x} : A_G \mathbf{x} \geq 1, \mathbf{x} \geq \mathbf{0}\}$ are intergal
$\mathfrak{N}$	The set of minimal graphs not belonging to $\mathfrak{B}$
$\mathscr{T}_C$	The set of triangles in triangle-cycle $C = e_1 \triangle_1 e_2 \cdots e_k \triangle_k e_1 = \cup_{i=1}^k \triangle_i$
$\mathscr{B}_C$	The set of basic triangles in triangle-cycle C, i.e., $\{\triangle_1, \cdots, \triangle_k\}$
J_C	The set of join edges in triangle-cycle C, i.e., $\{e_1, \cdots, e_k\}$
N_C	The set of nonjoin edges in triangle-cycle C, i.e., $E(C) \backslash J_C$
$\mathscr{T}_{C,i}$	$\{\triangle \in \mathscr{T}_C : \lvert \triangle \cap J_C \rvert = i\}$, $i = 0, 1, 2, 3$

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