On Evasion Games on Graphs

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Abstract. We consider an evasion game on a connected simple graph. We first show that the pursuit number of a graph G, the smallest k such that k pursuers win the game, is bounded above by the pathwidth of G. We next show that the pursuit number of G is two if and only if the pathwidth of G is one. We also show that for any integer $w \geq 2$, there exists a tree T such that the pursuit number of T is three and the pathwidth of T is w.

1 Introduction

In an evasion game on a connected simple graph, we have k pursuers and an evader. The evader moves invisibly along the edges of the graph. The pursuers must guess the position of the evader. At each round, k pursuers guess at most k vertices. The pursuers win if the current vertex of the evader is contained in the guessed vertices. Otherwise, the evader either stays at its vertex or moves to one of its neighbors. The pursuit number of a graph G, denoted by $\rho(G)$, is the minimum number k such that we have a winning strategy on G for k pursuers. In the active version, the evader is required to move at each round. We denote by $\rho^*(G)$ the pursuit number of a graph G for the active evasion game. We have $\rho(G) \geq \rho^*(G)$ by definition.

We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. Let $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ be a sequence of subsets of $V(G)$. The *width* of $\mathcal{X}$ is $\max_{1 \leq i \leq r} |X_i| - 1$. $\mathcal{X}$ is called a *path-decomposition* of G if the following conditions are satisfied:

(i) $\bigcup_{1 \leq i \leq r} X_i = V(G)$;
(ii) for any edge $(u, v) \in E(G)$, there exists an i such that $u, v \in X_i$;
(iii) for all l, m, and n with $1 \leq l \leq m \leq n \leq r$, $X_l \cap X_n \subseteq X_m$.

The *pathwidth* of G, denoted by $\mathrm{pw}(G)$, is the minimum width over all path-decompositions of G [5].

A connected graph of pathwidth one is called a *caterpillar*, which is a nontrivial tree that contains no 2-claw as a subtree, where a *k-claw* is a tree obtained from a complete bipartite graph $K_{1,3}$ by replacing each edge with a path of length k. A 2-*directional orthogonal ray tree (2DORT)* is a tree that contains no 3-claw as a subtree [7]. It is easy to see that the pathwidth of a 2DORT is at most 2. A caterpillar is a 2DORT by definition.

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It has been known that $\rho^*(G) = 1$ if and only if G is a 2DORT [2,4]. It was recently shown that $\rho^*(G) \leq \mathrm{pw}(G)+1$ for any graph G, and that for any integer $k \geq 2$, there exists a graph G such that $\mathrm{pw}(G) = k$ and $\rho^*(G) = k + 1$ [1]. It follows that $\rho^*(T) = O(\log n)$ for any n-vertex tree T, since $\mathrm{pw}(T) = O(\log n)$ for any n-vertex tree T [6]. Very recently, it was shown in [3] that there exists an n-vertex tree T such that $\rho^*(T) = \Omega(\log n)$.

We show the following four theorems.

Theorem 1. *For any graph G, $\rho(G) \leq \mathrm{pw}(G) + 1$. In particular, $\rho(T) = O(\log n)$ for any n-vertex tree T.*

It should be noted that for any integer $k \geq 0$, there exists a graph G such that $\mathrm{pw}(G) = k$ and $\rho(G) = k + 1$, since $\rho(G) \geq \rho^*(G)$ for any graph G. Notice also that there exists an n-vertex tree T such that $\rho(T) = \Omega(\log n)$, immediate from a result of [3] mentioned above.

Theorem 2. *$\rho(G) = 2$ if and only if $\mathrm{pw}(G) = 1$.*

It should be noted that $\rho(G) = 1$ if and only if $\mathrm{pw}(G) = 0$ (G has just one vertex), as mentioned in [4].

Theorem 3. *If $\mathrm{pw}(G) = 2$ then $\rho(G) = 3$.*

However, the converse of Theorem 3 does not hold as shown in the following.

Theorem 4. *For any integer $k \geq 2$, there exists a tree T such that $\rho(T) = 3$ and $\mathrm{pw}(T) = k$.*

2 Preliminaries

For a graph G, a k *pursuers' strategy* is a sequence $\mathcal{P} = (P_1, P_2, \ldots, P_r)$ of guessed vertices, where $P_i \subseteq V(G)$ and $|P_i| \leq k$ for any $i \in [r]$, where $[n] = \{1, 2, \ldots, n\}$ for a positive integer n; the pursuers guess the vertices in P_i at the i-th round of the game.

An *evader's strategy* is a sequence $\mathcal{M} = (m_0, m_1, \ldots, m_r)$ of vertices of G such that $m_i = m_{i-1}$ or m_i is adjacent to m_{i-1} for any $i \geq 1$; vertex m_0 is an initial position of the evader, and the evader stays at vertex m_i in the i-th round of the game.

A pursuers' strategy $\mathcal{P} = (P_1, P_2, \ldots, P_r)$ is a *winning strategy* if for any evader's strategy $\mathcal{M} = (m_0, m_1, \ldots, m_r)$, there exists an $i \geq 1$ such that $m_i \in P_i$. The *pursuit number* $\rho(G)$ of G is the minimum k such that there exists a winning strategy for k pursuers on G.

For a pursuer's strategy $\mathcal{P} = (P_1, P_2, \ldots, P_r)$, a vertex $v \in V(G)$ is said to be *contaminated* at the i-th round if there exists an evader's strategy $\mathcal{M} = (m_0, m_1, \ldots, m_r)$ such that $v = m_i$ and $m_j \notin P_j$ for any $j \in [i]$. Otherwise, v is said to be *clear* at the i-th round.

For a vertex $u \in V(G)$, let $N(u) = \{v \mid v \in V(G), (u,v) \in E(G)\} \cup \{u\}$, and for a vertex set $U \subseteq V(G)$, define that $N(U) = \bigcup_{u \in U} N(u)$. For a pursuers' strategy $\mathcal{P} = (P_1, P_2, \ldots, P_r)$, let $\mathcal{D}(\mathcal{P}) = (D_0, D_1, \ldots, D_r)$ be the sequence of *contaminated sets* of vertices for $\mathcal{P}$; D_i is the set of contaminated vertices at the i-th round, where $D_0 = V(G)$. It should be noted that

$$D_i = N(D_{i-1}) - P_i \tag{1}$$

for any $i \in [r]$, and $\mathcal{P}$ is a winning strategy if and only if $D_i = \emptyset$ for some $i \in [r]$.

3 Proof of Theorem 1

We can show that a path-decomposition $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ of G with width k is a winning strategy for $k+1$ pursuers on G by the same arguments as the proof of $\rho^*(G) \leq \mathrm{pw}(G) + 1$ in [1].

It is shown in [6] that $\mathrm{pw}(T) \leq \log_3(2n+1)$ for any n-vertex tree T. Thus, we have $\rho(T) = O(\log n)$ for any n-vertex tree T.

4 Proof of Theorems 2 and 3

Lemma 1. *If G contains a cycle then $\rho(G) \geq 3$.*

Proof. It suffices to show that $\rho(C) \geq 3$ for any cycle C. Let $\mathcal{P} = (P_1, P_2, \ldots, P_r)$ be a strategy for two pursuers on C, where $|P_i| \leq 2$ for any $i \in [r]$. We define an evader's strategy $\mathcal{M} = (m_0, m_1, \ldots, m_r)$ as follows. Let m_0 be any vertex in $V(C)$. We recursively define m_i as a vertex in $N(m_{i-1}) - P_i$ for any $i \in [r]$. Notice that $N(m_{i-1}) - P_i \neq \emptyset$, since $|N(m_{i-1})| = 3$ and $|P_i| \leq 2$. Thus, $\mathcal{P}$ is not a winning strategy, since $m_i \notin P_i$ for any $i \in [r]$, and we conclude that $\rho(C) \geq 3$.

Lemma 2. *If G is a 2-claw then $\rho(G) \geq 3$.*

Proof. Let T_2 be a 2-claw shown in Fig. 1. Define that

$$F_j = \{a_j, b_1, b_2, b_3, c\}, \text{ for any } j \in [3], \text{ and}$$
$$F_{j,j'} = \{a_j, a_{j'}, b_j, b_{j'}, c\}, \text{ for any } j \neq j' \in [3].$$

Let $\mathcal{P} = (P_1, P_2, \ldots, P_r)$ be a strategy for two pursuers on T_2, where $|P_i| \leq 2$ for any $i \in [r]$, and $\mathcal{D}(\mathcal{P}) = (D_0, D_1, \ldots, D_r)$ be the sequence of contaminated sets of vertices for $\mathcal{P}$, where $D_0 = V(T_2) = \{a_1, a_2, a_3, b_1, b_2, b_3, c\}$

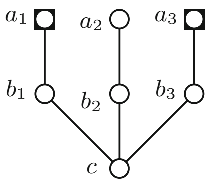

Fig. 1. 2-claw T_2.

Claim 1. *For any $i \in [r]$, $F_j \subseteq N(D_i)$ for some $j \in [3]$ or $F_{j,j'} \subseteq N(D_i)$ for some distinct $j, j' \in [3]$.*

Proof of Claim 1. The proof is by induction on i. We first show that $N(D_1)$ satisfies the claim. Since $D_0 = V(T_2)$, $D_1 = N(D_0) - P_1 = V(T_2) - P_1$ by (1). Let $P_1 = \{u, v\}$. If $(u, v) \notin E(T_2)$ then $N(u) \cap D_1 \neq \emptyset$ and $N(v) \cap D_1 \neq \emptyset$. Thus, $P_1 \subseteq N(D_1)$, and we have $N(D_1) = V(T_2)$, which satisfies the claim. If $(u, v) \in E(T_2)$ then P_1 is $\{a_j, b_j\}$ or $\{b_j, c\}$ for some $j \in [3]$. If $P_1 = \{a_j, b_j\}$ then $c \in D_1$, and so $N(D_1) = V(T_2) - \{a_j\}$, which contains $F_{j',j''}$ for $j' \neq j'' \in [3] - \{j\}$, and we are done. If $P_1 = \{b_j, c\}$ then $P_1 \subseteq N(D_1)$, and we have $N(D_1) = V(T_2)$, which satisfies the claim.

Suppose that Claim 1 holds for $i-1 \in [r-1]$, that is, $F_j \subseteq N(D_{i-1})$ for some $j \in [3]$ or $F_{j,j'} \subseteq N(D_{i-1})$ for some distinct $j, j' \in [3]$. We show that Claim 1 also holds for i. We distinguish two cases.

Case 1 $F_j \subseteq N(D_{i-1})$ for some $j \in [3]$: If $|F_j \cap P_i| \leq 1$, then $F_j \subseteq N(F_j - P_i)$, since any vertex of F_j is adjacent with another vertex of F_j. Therefore, $F_j \subseteq N(F_j - P_i) \subseteq N(N(D_{i-1}) - P_i) = N(D_i)$ by (1), and we are done. Thus, we assume in the following that $|F_j \cap P_i| = 2$. Without loss of generality, we assume that $F_1 = \{a_1, b_1, b_2, b_3, c\} \subseteq N(D_{i-1})$. We further distinguish three cases.

Case 1-1 $a_1 \in P_i$: In this case, P_i is $\{a_1, c\}$, $\{a_1, b_1\}$, $\{a_1, b_2\}$, or $\{a_1, b_3\}$. If $P_i = \{a_1, c\}$ then $\{b_1, b_2, b_3\} \subseteq D_i$ by (1). Thus, $N(D_i) = V(T_2)$, which satisfies the claim. If $P_i = \{a_1, b_1\}$ then $\{b_2, b_3, c\} \subseteq D_i$. Thus, $F_{2,3} = \{a_2, b_2, a_3, b_3, c\} \subseteq N(D_i)$, and we are done. If $P_i = \{a_1, b_2\}$ then $\{b_1, b_3, c\} \subseteq D_i$. Thus, $F_{1,3} = \{a_1, b_1, b_2, a_3, b_3, c\} \subseteq N(D_i)$, and we are done. If $P_i = \{a_1, b_3\}$ then $\{b_1, b_2, c\} \subseteq D_i$. Thus, $F_{1,2} = \{a_1, b_1, a_2, b_2, b_3, c\} \subseteq N(D_i)$, and we are done.

Case 1-2 $a_1 \notin P_i$ and $b_1 \in P_i$: In this case, P_i is $\{b_1, b_2\}$, $\{b_1, b_3\}$, or $\{b_1, c\}$. If $P_i = \{b_1, b_2\}$ then $\{a_1, b_3, c\} \subseteq D_i$ and $\{a_1, b_1, b_2, a_3, b_3, c\} \subseteq N(D_i)$ by (1). Thus, we have $F_{1,3} \subseteq N(D_i)$, and we are done. If $P_i = \{b_1, b_3\}$ then $\{a_1, b_2, c\} \subseteq D_i$ and $\{a_1, b_1, a_2, b_2, b_3, c\} \subseteq N(D_i)$. Thus, we have $F_{1,2} \subseteq N(D_i)$, and we are done. If $P_i = \{b_1, c\}$ then $\{a_1, b_2, b_3\} \subseteq D_i$ and $N(D_i) = V(T_2)$, which satisfies the claim.

Case 1-3 $a_1, b_1 \notin P_i$: In this case, P_i is $\{b_2, b_3\}$, $\{b_2, c\}$, or $\{b_3, c\}$. If $P_i = \{b_2, b_3\}$ then $\{a_1, b_1, c\} \subseteq D_i$ by (1) and $\{a_1, b_1, b_2, b_3, c\} \subseteq N(D_i)$. Thus, $F_1 \subseteq N(D_i)$, and we are done. If $P_i = \{b_2, c\}$ then $\{a_1, b_1, b_3\} \subseteq D_i$ and $\{a_1, b_1, b_2, a_3, b_3, c\} \subseteq N(D_i)$. Thus, $F_1 \subseteq N(D_i)$, and we are done. If $P_i = \{b_3, c\}$ then $\{a_1, b_1, b_2\} \subseteq D_i$ and $\{a_1, b_1, a_2, b_2, b_3, c\} \subseteq N(D_i)$. Thus, $F_1 \subseteq N(D_i)$, and we are done.

Case 2 $F_{j,j'} \subseteq N(D_{i-1})$ for some distinct $j, j' \in [3]$: If $|F_{j,j'} \cap P_i| \leq 1$, then $F_{j,j'} \subseteq N(F_{j,j'} - P_i)$, since any vertex of $F_{j,j'}$ is adjacent with another vertex of $F_{j,j'}$. Therefore, $F_{j,j'} \subseteq N(F_{j,j'} - P_i) \subseteq N(N(D_{i-1}) - P_i) = N(D_i)$ by (1), and we are done. Thus, we assume in the following that $|F_{j,j'} \cap P_i| = 2$. Without loss of generality, we assume that $F_{1,2} = \{a_1, a_2, b_1, b_2, c\} \subseteq N(D_{i-1})$. We further distinguish four cases.

Case 2-1 $P_i = \{a_1, a_2\}$: In this case, $\{b_1, b_2, c\} \subseteq D_i$ by (1) and $\{a_1, b_1, a_2, b_2, b_3, c\} \subseteq N(D_i)$. Thus, $F_{1,2} \subseteq N(D_i)$, and we are done.

Case 2-2 $a_1 \in P_i$ and $a_2 \notin P_i$: In this case, P_i is $\{a_1, b_1\}$, $\{a_1, b_2\}$, or $\{a_1, c\}$. If $P_i = \{a_1, b_1\}$ then $\{a_2, b_2, c\} \subseteq D_i$ and $\{b_1, a_2, b_2, b_3, c\} \subseteq N(D_i)$. Thus, $F_2 \subseteq N(D_i)$, and we are done. If $P_i = \{a_1, b_2\}$ then $\{b_1, a_2, c\} \subseteq D_i$ and $\{a_1, b_1, a_2, b_2, b_3, c\} \subseteq N(D_i)$. Thus, $F_{1,2} \subseteq N(D_i)$, and we are done. If $P_i = \{a_1, c\}$ then $\{b_1, a_2, b_2\} \subseteq D_i$ and $\{a_1, b_1, a_2, b_2, c\} \subseteq N(D_i)$. Thus, $F_{1,2} \subseteq N(D_i)$, and we are done.

Case 2-3 $a_1 \notin P_i$ and $a_2 \in P_i$: The proof is similar to the proof of Case 2-2, and omitted.

Case 2-4 $a_1, a_2 \notin P_i$, i.e., $P_i \subseteq \{b_1, b_2, c\}$: In this case, P_i is $\{b_1, b_2\}$, $\{b_1, c\}$ or $\{b_2, c\}$. If $P_i = \{b_1, b_2\}$, then $\{a_1, a_2, c\} \subseteq D_i$ by (1) and $\{a_1, b_1, a_2, b_2, b_3, c\} \subseteq N(D_i)$. Thus, $F_{1,2} \subseteq N(D_i)$, and we are done. If $P_i = \{b_1, c\}$ then $\{a_1, a_2, b_2\} \subseteq D_i$ and $\{a_1, b_1, a_2, b_2, c\} \subseteq N(D_i)$. Thus, $F_{1,2} \subseteq N(D_i)$, and we are done. If $P_i = \{b_2, c\}$ then $\{a_1, b_1, a_2\} \subseteq D_i$ and $\{a_1, b_1, a_2, b_2, c\} \subseteq N(D_i)$. Thus, $F_{1,2} \subseteq N(D_i)$, and we are done.

This completes the proof of Claim 1. □

Claim 2. *$\mathcal{P}$ is not a winning strategy.*

Proof of Claim 2. From Claim 1, $N(D_i) \geq 5$ for any $i \in [r]$. Therefore, $|D_i| = |N(D_{i-1}) - P_i| \geq 5 - 2$, which means that $D_i \neq \emptyset$ for any $i \in [r]$. Thus, we have the claim. □

From Claim 2, we conclude that $\rho(T_2) \geq 3$, and we have Lemma 2. □

If $\rho(G) \leq 2$ then G is a tree by Lemma 1, and G contains no 2-claw by Lemma 2, that is, G is a caterpillar. Thus, if $\rho(G) \leq 2$ then $\mathrm{pw}(G) \leq 1$. On the other hand, if $\mathrm{pw}(G) \leq 1$ then $\rho(G) \leq 2$ by Theorem 1. Thus, we have the following.

Lemma 3. *$\rho(G) \leq 2$ if and only if $\mathrm{pw}(G) \leq 1$.*

Since $\rho(G) = 1$ if and only if $\mathrm{pw}(G) = 0$, it follows from Lemma 3 that $\rho(G) = 2$ if and only if $\mathrm{pw}(G) = 1$, and we obtain Theorem 2.

If $\mathrm{pw}(G) = 2$ then $\rho(G) \geq 3$ by Lemma 3, and $\rho(G) \leq 3$ by Theorem 1. Thus, $\rho(G) = 3$ if $\mathrm{pw}(G) = 2$, and we obtain Theorem 3.

5 Proof of Theorem 4

We need some preliminaries. For a graph G and $U \subseteq V(G)$, we denote by $G - U$ the graph obtained from G by deleting all vertices of U and all edges incident to a vertex of U. The following is shown in [8].

Theorem 5. *Let G be a connected graph and $k \geq 1$ be an integer. If G has a vertex v such that $G - \{v\}$ has at least three connected components with pathwidth $k - 1$ or more, then $\mathrm{pw}(G) \geq k$.*

Lemma 4. *The pathwidth of a tree T is at most $k \geq 1$ if and only if there exists a path Q in T such that the pathwidth of every connected component of $T - V(Q)$ is at most $k - 1$.*

Proof. We first show the necessity. Let T be a tree of pathwidth at most k, and $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ be a path-decomposition of T with width at most k. Let $u \in X_1$ and $v \in X_r$ be any vertices, and Q be the path connecting u and v in T.

We show that

$$V(Q) \cap X_i \neq \emptyset \tag{2}$$

for any $i \in [r]$. Assume to the contrary that there exists X_j $(1 < j < r)$ with $X_j \cap V(Q) = \emptyset$. Define that $U_1 = (\bigcup_{i \le j-1} X_i) \cap V(Q)$ and $U_2 = (\bigcup_{i \ge j+1} X_i) \cap V(Q)$. From (iii), we have $U_1 \cap U_2 \subseteq X_j \cap V(Q) = \emptyset$, that is, $U_1 \cap U_2 = \emptyset$. Since Q is a path connecting u and v, Q contains an edge (x, y) with $x \in U_1$ and $y \in U_2$. However, we have no X_i such that $x, y \in X_i$, contradicting to (ii).

If we define $X'_i = X_i - V(Q)$ for any $i \in [r]$, $\mathcal{X}' = (X'_1, X'_2, \ldots, X'_r)$ is a path-decomposition of $T - V(Q)$. Since the width of $\mathcal{X}'$ is at most $k-1$ by (2), we conclude that every connected component of $T - V(Q)$ has pathwidth at most $k-1$. This completes the proof of the necessity.

We next show the sufficiency. Let Q be a path in T such that every connected component of $T - V(Q)$ has pathwidth at most $k-1$. Let $V(Q) = \{v_1, v_2, \ldots, v_p\}$ and $E(Q) = \{(v_i, v_{i+1}) \mid i \in [p-1]\}$. Let $C_1, C_2, \ldots C_q$ be the connected components of $T - V(Q)$ such that if C_i contains a vertex adjacent to v_j then C_{i+1} contains a vertex adjacent to $v_{j'}$ for some $j' \ge j$. For any $i \in [q]$, let $\mathcal{X}^i = (X^i_1, X^i_2, \ldots, X^i_{r_i})$ be a path-decomposition of C_i of width at most $k-1$, that is, $|X^i_j| \le k$ for any $j \in [r_i]$. Let $J(i)$ be an integer such that C_i contains a vertex adjacent to $v_{J(i)} \in V(Q)$. Since T is a tree, $J(i)$ is uniquely determined.

A path-decomposition of T of width at most k is constructed as follows. For any $i \in [q]$ and $j \in [r_i]$, let $Y^i_j = X^i_j \cup \{v_{J(i)}\}$ and $\mathcal{Y}^i = (Y^i_1, Y^i_2, \ldots, Y^i_{r_i})$. If $J(1) \ge 2$, $\mathcal{Z}^0$ is defined as an empty sequence. Otherwise, we define $\mathcal{Z}^0 = (Z^0_1, Z^0_2, \ldots, Z^0_{J(1)-1})$, where $Z^0_j = \{v_j, v_{j+1}\}$ for any $j \in [J(1)-1]$. For any $i \in [q-1]$ and $l \in [J(i+1) - J(i)]$, define that $Z^i_l = \{v_{J(i)+l-1}, v_{J(i)+l}\}$ and $\mathcal{Z}^i = (Z^i_1, Z^i_2, \ldots, Z^i_{J(i+1)-J(i)})$, where $\mathcal{Z}^i$ is an empty sequence if $J(i+1) = J(i)$. If $J(q) = p$ then $\mathcal{Z}^q$ is defined as an empty sequence. Otherwise, define that $Z^q_j = \{v_{J(q)+j-1}, v_{J(q)+j}\}$ for any $j \in [p - J(q)]$ and $\mathcal{Z}^q = (Z^q_1, Z^q_2, \ldots, Z^q_{p-J(q)})$. Define that $\mathcal{X}' = (\mathcal{Z}^0, \mathcal{Y}^1, \mathcal{Z}^1, \mathcal{Y}^2, \ldots, \mathcal{Y}^q, \mathcal{Z}^q)$. We denote $\mathcal{X}'$ by $(X'_1, X'_2, \ldots, X'_{r'})$.

We show that $\mathcal{X}'$ is a path-decomposition for T of width at most k. We first show that $\mathcal{X}'$ satisfies (i). Any vertex of Q is contained in some Z^i_j by definition. Since $\mathcal{X}^i$ is a path-decomposition of C_i, we have $V(C_i) = \bigcup_{j \in [r_i]} X^i_j \subseteq \bigcup_{j \in [r_i]} Y^i_j$. Thus, we conclude that $V(T) = \bigcup_{i \in [r']} X'_i$. Thus, $\mathcal{X}'$ satisfies (i).

We next show that $\mathcal{X}'$ satisfies (ii). We distinguish 3 cases. 1) $(u, v) \in E(C_i)$ for some $i \in [q]$: Since $\mathcal{X}^i$ is a path-decomposition of C_i, $u, v \in X^i_j$ for some $j \in [r_i]$. Thus, we conclude that $u, v \in X^i_j \subset Y^i_j = X'_l$ for some l. 2) $(u, v) \in E(Q)$: Since $u, v \in Z^i_j$ for some i, j, we conclude that $u, v \in X'_l$ for some l. 3) $(u, v) \in E(T)$ such that $u \in V(C_i)$ for some $i \in [q]$ and $v = v_{J(i)} \in V(Q)$: We have $u, v \in Y^i_j = X'_l$ for some l by definition. Thus, $\mathcal{X}'$ satisfies (ii).

We now show that $\mathcal{X}'$ satisfies (iii). Let l, m, and n be arbitrary integers with $1 \le l \le m \le n \le r'$. If $l = n$, $X'_m = X'_l = X'_n$, and we are done. Assume that $l \leq n-1$. Let $y(i)$ be an integer such that $X'_{y(i)+1} = Y^i_1$. If $y(i)+1 \leq l < n \leq y(i)+r_i$ for some $i \in [p]$, $X'_l \cap X'_n = Y^i_{l-y(i)} \cap Y^i_{n-y(i)} = (X^i_{l-y(i)} \cup \{v_{J(i)}\}) \cap (X^i_{n-y(i)} \cup \{v_{J(i)}\}) = (X^i_{l-y(i)} \cap X^i_{n-y(i)}) \cup \{v_{J(i)}\} \subseteq X^i_{m-y(i)} \cup \{v_{J(i)}\} = X'_m$, since $\mathcal{X}^i = (X^i_1, X^i_2, \dots, X^i_{r_i})$ is a path-decomposition of C_i. Thus, we have $X'_l \cap X'_n \subseteq X'_m$. Otherwise, $X'_l \cap X'_n$ contains only vertices of Q. Since any vertex in Q appears only in consecutive subsets in $\mathcal{X}'$, we have $X'_l \cap X'_n \subseteq X'_m$. Therefore, $\mathcal{X}'$ satisfies (iii).

Thus, $\mathcal{X}'$ is a path-decomposition of T. Since $|X'_i| \leq k+1$ for any $i \in [r']$ by definition, the width of $\mathcal{X}'$ is at most k, and we conclude that $\mathrm{pw}(T) \leq k$. This completes the proof of the sufficiency. □

Now, we are ready to prove Theorem 4. We prove the theorem by induction on k. The following lemma is the basis of the induction. For a graph G and $x, y \in V(G)$, a winning strategy $\mathcal{P} = (P_1, P_2, \dots, P_r)$ on G is called an (x, y)-*winning strategy* if the following conditions are satisfied:

- $x \in P_i$ if and only if $i = 1$, and
- $y \in P_i$ if and only if $i = r$.

Lemma 5. *For the* 2*-claw* T_2 *shown in Fig. 1,* $\rho(T_2) = 3$ *and* $\mathrm{pw}(T_2) = 2$. *Moreover, there exists an* (x, y)*-winning strategy for three pursuers on* T_2 *for some* $x, y \in V(T_2)$.

Proof. By Lemma 3, $\rho(T_2) \geq 3$. We show that $\mathcal{P} = (\{a_1, b_1, c\}, \{a_2, b_2, c\}, \{a_3, b_3, c\})$ is an (a_1, a_3)-winning strategy for three pursuers on T_2. Let $\mathcal{D}(\mathcal{P}) = (D_0, D_1, D_2, D_3)$ be the sequence of contaminated sets of vertices for $\mathcal{P}$, where $D_0 = V(T_2)$. By (1), $D_1 = N(D_0) - P_1 = \{a_2, b_2, a_3, b_3\}$, $D_2 = N(D_1) - P_2 = \{a_3, b_3\}$, and $D_3 = N(D_2) - P_3 = \emptyset$. Thus, $\mathcal{P}$ is an (a_1, a_3)-winning strategy for three pursuers on T_2, and we conclude that $\rho(T_2) = 3$.

From Theorem 5 and Lemma 4, we have $\mathrm{pw}(T_2) = 2$, since the pathwidth of every connected components of $T_2 - \{c\}$ is 1. This completes the proof of the lemma. □

We need some more preliminaries to show the induction step. Let G be a graph, $\mathcal{P} = (P_1, P_2, \dots, P_r)$ be a pursuers' strategy on G, and $\mathcal{D}(\mathcal{P}) = (D_0, D_1, \dots, D_r)$ be a sequence of contaminated sets of vertices for $\mathcal{P}$. From (1), we have the following.

Lemma 6. *For any* $i \in [r-1]$*, if* $N(D_i) - D_i \subseteq P_{i+1}$*, then* $D_{i+1} = D_i - P_{i+1}$.

For any $U \subseteq V(G)$, define that $N^1(U) = N(U)$, and $N^{i+1}(U) = N(N^i(U))$ for any $i \geq 1$. From (1) we have the following.

$$D_{i+k} \subseteq N^k(D_i) - P_{i+k} \tag{3}$$

for any $i \in [r-1]$ and $k \in [r-i]$. For any two vertices $u, v \in V(G)$, we denote by $\mathrm{dist}_G(u, v)$ the distance between vertices u and v in G. From (3), we have the following.

Lemma 7. *If* $\mathrm{dist}_G(u,v) \geq k+1$ *for any* $u \in U$ *and* $v \in D_i$*, then* $U \cap D_{i+k} = \emptyset$*.*

For a sequence $\mathcal{X} = (X_1, X_2, \ldots, X_r)$, r is called the *length* of $\mathcal{X}$ and denoted by $|\mathcal{X}|$. For sequences $\mathcal{X}^i = (X^i_1, X^i_2, \ldots, X^i_{r_i})$ for $i \in [n]$, $(\mathcal{X}^1, \mathcal{X}^2, \ldots, \mathcal{X}^n)$ is a sequence obtained by concatenating $\mathcal{X}^1, \mathcal{X}^2, \ldots, \mathcal{X}^n$, that is, $(\mathcal{X}^1, \mathcal{X}^2, \ldots, \mathcal{X}^n) = (X^1_1, X^1_2, \ldots, X^1_{r_1}, X^2_1, X^2_2, \ldots, X^2_{r_2}, \ldots, X^n_1, X^n_2, \ldots, X^n_{r_n})$. Notice that

$$|(\mathcal{X}^1, \mathcal{X}^2, \ldots, \mathcal{X}^n)| = \sum_{i=1}^{n} |\mathcal{X}^i| = \sum_{i=1}^{n} r_i.$$

Now, we are ready to show the induction step.

Lemma 8. *Let* T_{k-1} *(*$k \geq 3$*) be a tree with* $\rho(T_{k-1}) = 3$ *and* $\mathrm{pw}(T_{k-1}) = k-1$*. Assume that there exists an* (x,y)*-winning strategy for three pursuers on* T_{k-1} *for some* $x, y \in V(T_{k-1})$*. Then, we can construct from three copies of* T_{k-1} *a tree* T_k *with* $\rho(T_k) = 3$ *and* $\mathrm{pw}(T_k) = k$*. Moreover, there exists an* (x,y)*-winning strategy for three pursuers on* T_k *for some* $x, y \in V(T_k)$*.*

Proof. Let T_{k-1} be a tree with $\rho(T_{k-1}) = 3$ and $\mathrm{pw}(T_{k-1}) = k-1$, and $\mathcal{P} = (P_1, P_2, \ldots, P_r)$ be an (x,y)-winning strategy for three pursuers on T_{k-1}. Let T^i_{k-1} be a copy of T_{k-1} for $i \in [3]$, and $v^i \in V(T^i_{k-1})$ be the vertex corresponding to a vertex $v \in V(T_{k-1})$. Let $\mathcal{P}^i = (P^i_1, P^i_2, \ldots, P^i_r)$ be an (x^i, y^i)-winning strategy corresponding to $\mathcal{P}$ for $i \in [3]$. Let Q be a path with $V(Q) = \{q_i \mid i \in [r]\}$ and $E(Q) = \{(q_i, q_{i+1}) \mid i \in [r-1]\}$.

Define that T_k is a tree obtained from T^1_{k-1}, T^2_{k-1}, T^3_{k-1}, and Q by adding three edges (q_1, y^1), (q_r, y^2), and (q_r, x^3) (See Fig. 2).

Since the pathwidth of any connected component of $T_k - \{q_r\}$ is at least $\mathrm{pw}(T_{k-1})$, we have $\mathrm{pw}(T_k) \geq \mathrm{pw}(T_{k-1}) + 1 = k$ by Theorem 5. On the other hand, since T^1_{k-1}, T^2_{k-1}, and T^3_{k-1} are the connected components of $T_k - V(Q)$, we have $\mathrm{pw}(T_k) \leq \mathrm{pw}(T_{k-1}) + 1 = k$ by Lemma 4. Thus, we have $\mathrm{pw}(T_k) = k$.

We have $\rho(T_k) \geq 3$, since $\rho(T_{k-1}) = 3$. We will show an (x^1, y^3)-winning strategy for three pursuers on T_k, which means that $\rho(T_k) = 3$. Let $h = \lceil r/2 \rceil$,

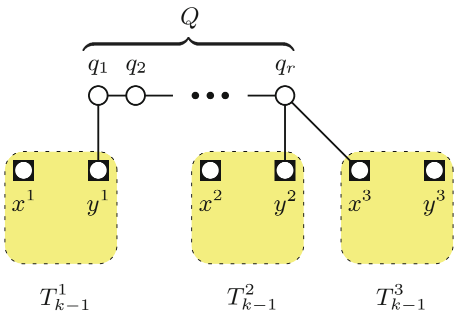

Fig. 2. Tree T_k.

and $\mathcal{R} = (R_1, R_2, \ldots, R_h)$ and $\mathcal{S} = (S_1, S_2, \ldots, S_{r+1})$ be sequences of subsets of $V(T_k)$ defined as follows.

$$R_j = \begin{cases} \{y_1, q_1, q_2\} & \text{if } j = 1, \\ \{q_{2j-2}, q_{2j-1}, q_{2j}\} & \text{if } 2 \leq j \leq h-1, \\ \{q_{r-2}, q_{r-1}, q_r\} & \text{if } j = h, \end{cases} \tag{4}$$

$$S_j = \begin{cases} \{y^1, y^2, q_1\} & \text{if } j = 1, \\ \{y^2, q_{j-1}, q_j\} & \text{if } 2 \leq j \leq r, \\ \{q_{r-1}, q_r, x^3\} & \text{if } j = r+1. \end{cases} \tag{5}$$

Define that $\mathcal{P}' = (\mathcal{P}^1, \mathcal{R}, \mathcal{P}^2, \mathcal{S}, \mathcal{P}^3)$. Notice that $|\mathcal{P}'| = |\mathcal{P}^1| + |\mathcal{R}| + |\mathcal{P}^2| + |\mathcal{S}| + |\mathcal{P}^3| = 4r + h + 1$. We now show the following.

Claim 3. *$\mathcal{P}'$ is an (x^1, y^3)-winning strategy for three pursuers on T_k.*

Proof of Claim 3. Let $r' = |\mathcal{P}'| = 4r + h + 1$ and $\mathcal{D}(\mathcal{P}') = (D_0, D_1, \ldots, D_{r'})$ be the sequence of contaminated sets of vertices for $\mathcal{P}'$. Since $\mathcal{P}^1$ is an (x^1, y^1)-winning strategy on T^1_{k-1} and $T^1_{k-1} - \{y^1\}$ is a connected component of $T_k - \{y^1\}$, we have

$$D_r = V(T_k) - V(T^1_{k-1}). \tag{6}$$

Similarly, by noting $|\mathcal{P}^1| + |\mathcal{R}| + |\mathcal{P}^2| = 2r + h$, we also have

$$D_{2r+h} \cap V(T^2_{k-1}) = \emptyset, \tag{7}$$

since $\mathcal{P}^2$ is an (x^2, y^2)-winning strategy on T^2_{k-1} and $T^2_{k-1} - \{y^2\}$ is a connected component of $T_k - \{y^2\}$.

(I). $D_{r+i} = V(T_k) - V(T^1_{k-1}) - \{q_j \mid j \in [2i]\}$ for any i with $0 \leq i \leq h-1$.

Proof of (I). We show (I) by induction on i. From (6), (I) holds for $i = 0$, since $\{q_j \mid j \in [2i]\} = \emptyset$ if $i = 0$. Let $i \geq 1$ and assume that (I) holds for $i-1$, that is, $D_{r+i-1} = V(T_k) - V(T^1_{k-1}) - \{q_1, q_2, \ldots, q_{2i-2}\}$. It follows that $N(D_{r+i-1}) - D_{r+i-1} = \{q_{2i-2}\}$, where we assume that $q_{2i-2} = y^1$ if $i = 1$. Therefore, $N(D_{r+i-1}) - D_{r+i-1} \subseteq R_i$ by (4). Thus from Lemma 6, $D_{r+i} = D_{r+i-1} - R_i = V(T_k) - V(T^1_{k-1}) - \{q_j \mid j \in [2i]\}$, and (I) holds for i. $\square$

From (I), we have $D_{r+h-1} = V(T_k) - V(T^1_{k-1}) - \{q_i \mid i \in [2h-2]\}$. Therefore, $N(D_{r+h-1}) - D_{r+h-1} = \{q_{2h-2}\} \subseteq R_h$, and we have

$$D_{r+h} = V(T_k) - V(T^1_{k-1}) - V(Q) \tag{8}$$

by Lemma 6. From Lemma 7 and (8), $D_{2r+h} \cap V(T^1_{k-1}) = \emptyset$, i.e., $D_{2r+h} \subseteq V(T_k) - V(T^1_{k-1})$. Thus from (7), we have

$$D_{2r+h} \subseteq V(T_k) - (V(T^1_{k-1}) \cup V(T^2_{k-1})). \tag{9}$$

Let $\mathcal{P}' = (P'_1, P'_2, \ldots, P'_{r'})$, and $\mathcal{M} = (m_0, m_1, \ldots, m_{r'})$ be an evader's strategy such that $m_i = x^3$ for $i \leq r+h$, and $m_i = q_{2r+h+1-i}$ for $r+h+1 \leq i \leq 2r+h$. Then, $m_i \notin P'_i$ for any $i \in [2r+h]$. Therefore, $q_1 \in D_{2r+h}$. Similarly, we can prove that $q_i \in D_{2r+h}$ for any $i \in [2r+h]$, and thus, $V(Q) \subseteq D_{2r+h}$. Since $V(T^3_{k-1}) \subseteq D_{2r+h}$, we have

$$\begin{aligned} V(T_k) - (V(T^1_{k-1}) \cup V(T^2_{k-1})) &= V(Q) \cup V(T^3_{k-1}) \\ &\subseteq D_{2r+h}. \end{aligned} \tag{10}$$

Thus, from (9) and (10), we have

$$D_{2r+h} = V(T_k) - (V(T^1_{k-1}) \cup V(T^2_{k-1})). \tag{11}$$

(II). $D_{2r+h+i} = V(T_k) - V(T^1_{k-1}) - V(T^2_{k-1}) - \{q_j \mid j \in [i]\}$ for any $i \in [r]$.

Proof of (II). From (11), $N(D_{2r+h}) \cap (V(T^1_{k-1}) \cup V(T^2_{k-1})) = \{y^1, y^2\}$. Thus from Lemma 6 and (5), we have

$$D_{2r+h+1} = V(T_k) - V(T^1_{k-1}) - V(T^2_{k-1}) - \{q_1\}. \tag{12}$$

We now show that

$$D_{2r+h+i} = V(T_k) - V(T^1_{k-1}) - V(T^2_{k-1}) - \{q_j \mid j \in [i]\} \tag{13}$$

by induction on i. Clearly, (13) holds for $i = 1$ by (12). Assume that (13) holds for $i-1$ with $i \geq 2$, that is, $D_{2r+h+i-1} = V(T_k) - V(T^1_{k-1}) - V(T^2_{k-1}) - \{q_j \mid j \in [i-1]\}$, and we will show that (13) also holds for i. By induction hypothesis, $N(D_{2r+h+i-1}) - D_{2r+h+i-1} \subseteq \{q_{i-1}, y^2\}$. Thus from Lemma 6 and (5), (13) holds for i. This completes the proof of (II). □

From (5) and (II), we have

$$\begin{aligned} D_{3r+h+1} &= V(T_k) - V(T^1_{k-1}) - V(T^2_{k-1}) - V(Q) - \{x^3\} \\ &= V(T^3_{k-1}) - \{x^3\}. \end{aligned} \tag{14}$$

Therefore, we have $D_{4r+h+1} = \emptyset$, since $\mathcal{P}^3$ is an (x^3, y^3)-winning strategy on T^3_{k-1} and $T^3_{k-1} - \{x^3\}$ is a connected component of $T_k - \{x^3\}$. Since $x^1 \in P'_i$ if and only if $i = 1$, and $y^3 \in P'_i$ if and only if $i = r'$, we conclude that $\mathcal{P}'$ is an (x^1, y^3)-winning strategy on T_k, and we have Claim 3. □

This completes the proof of the lemma. □

From Lemmas 5 and 8, we have Theorem 4.

6 Active Pursuit Number of T_k

We show the following for tree T_k defined in the previous section.

Theorem 6. *For any $k \geq 3$, $\rho^*(T_k) = 2$.*

Proof. For a bipartite graph G with a bipartition (B_0, B_1) and $P \subseteq V(G)$, define that $\mathcal{B}_G(P) = \max\{|P \cap B_0|, |P \cap B_1|\}$. For a pursuers strategy $\mathcal{P} = (P_1, P_2, \ldots, P_r)$ on G, $\mathcal{B}_G(\mathcal{P}) = \max\{\mathcal{B}_G(P_i) \mid i \in [r]\}$.

Lemma 9. *For a bipartite graph G, if there exists a winning strategy $\mathcal{P} = (P_1, P_2, \ldots, P_r)$ for the general evasion game on G with $\mathcal{B}_G(\mathcal{P}) \leq l$, then $\rho^*(G) \leq l$.*

Proof of Lemma 9. Let $\mathcal{P} = (P_1, P_2, \ldots, P_r)$ be a winning strategy for the general evasion game on G satisfying $\mathcal{B}_G(\mathcal{P}) \leq l$. Without loss of generality, we assume that r is odd, since otherwise, $\mathcal{P}' = (P_1, P_2, \ldots, P_r, \emptyset)$ is also a winning strategy of odd length on G satisfying $\mathcal{B}_G(\mathcal{P}') \leq l$.

Let (B_0, B_1) be a bipartition of G. Define that $W_i = P_i \cap B_{i \bmod 2}$, i.e.,

$$W_i = \begin{cases} P_i \cap B_0 & \text{if } i \text{ is even, and} \\ P_i \cap B_1 & \text{if } i \text{ is odd,} \end{cases}$$

for any $i \in [r]$. Define also that $W_i = W_{i-r}$ for $r + 1 \leq i \leq 2r$, and $\mathcal{W}^* = (W_1, W_2, \ldots, W_{2r})$. We will show that pursuers' strategy $\mathcal{W}^*$ is a winning strategy on G for the active evasion game.

Let $\mathcal{M}^* = (m_0, m_1, \ldots, m_{2r})$ be any evader's strategy on G for the active evasion game. From the definition of the active evasion game, the evader must move at each round and we have

$$m_i \in B_0 \Leftrightarrow m_{i-1} \in B_1 \text{ for any } i \in [2r]. \tag{15}$$

Since r is odd, we also have

$$m_0 \in B_0 \Leftrightarrow m_r \in B_1. \tag{16}$$

Define that

$$\begin{aligned} \mathcal{M}^L &= (m_0, m_1, \ldots, m_r) \text{ and} \\ \mathcal{M}^R &= (m_r, m_{r+1} \ldots, m_{2r}). \end{aligned}$$

It should be noted that $\mathcal{M}^L$ and $\mathcal{M}^R$ both can be considered as evader's strategies of r rounds for the general evasion game on G. Since $\mathcal{P}$ is a winning strategy on G for the general evasion game, there exist integers α and β with $1 \leq \alpha \leq r < \beta \leq 2r$ such that

$$m_\alpha \in P_\alpha, \text{ and} \tag{17}$$
$$m_\beta \in P_{\beta-r}. \tag{18}$$

We now show that there exists an integer $i \in [2r]$ such that $m_i \in W_i$. We distinguish two cases.

Case 1 $m_0 \in B_0$: From (15) and $m_0 \in B_0$, we have

$$m_i \in B_{i \bmod 2} \tag{19}$$

for any $i \in [r]$. Thus from (17) and (19), we have $m_\alpha \in P_\alpha \cap B_{\alpha \bmod 2}$, i.e., $m_\alpha \in W_\alpha$.

Case 2 $m_0 \in B_1$: From (15) and (16),

$$m_{r+i} \in B_{i \bmod 2} \tag{20}$$

for any $i \in [r]$. Let $\beta' = \beta - r$. From (18) and (20), we have $m_\beta \in P_{\beta'} \cap B_{\beta' \bmod 2} = W_{\beta'}$.

Thus, $\mathcal{W}^*$ is a winning strategy for the active evasion game on G. Since $\mathcal{B}_G(\mathcal{P}) \leq l$, we have $|W_i| \leq l$ for any $i \in [2r]$. Thus, $\rho^*(G) \leq l$, and we have the lemma. □

If $\mathcal{P} = (P_1, P_2, \ldots, P_r)$ is the (x, y)-winning strategy for three pursuers on T_k defined in the previous section then $\mathcal{B}_G(\mathcal{P}) \leq 2$, since $|P_i| = 3$ and every P_i contains a pair of adjacent vertices for any $i \in [r]$. Thus, we have $\rho^*(T_k) \leq 2$ for any $k \geq 2$ by Lemma 9. Since T_k contains a 3-claw if $k \geq 3$, we have $\rho^*(T_k) \geq 2$ if $k \geq 3$, and we conclude that $\rho^*(T_k) = 2$ for any $k \geq 3$. (Notice that $\rho^*(T_2) = 1$, since T_2 is a 2DORT.) This completes the proof of the theorem. □

7 Concluding Remarks

Since it is well-known that the longest path in a tree can be found in linear time, caterpillars and 2DORTs can be recognized in linear time [7]. Therefore, we can decide in linear time whether $\rho^*(G) = 1$ and $\rho(G) = 2$. The complexity of computing $\rho(G)$ and $\rho^*(G)$ is open.

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