On Fuzzy RDM-Arithmetic

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Abstract. The paper presents notion of horizontal membership function (HMF) and based on it fuzzy, relative distance measure (fuzzy RDM) arithmetic that is compared with standard fuzzy arithmetic (SF arithmetic). Fuzzy RDM-arithmetic possess such mathematical properties which allow for achieving complete fuzzy solution sets of problems, whereas SF-arithmetic, in general, delivers only approximate, partial solutions and sometimes no solutions of problems. The paper explains how to realize arithmetic operations with fuzzy RDM-arithmetic and shows examples of its application.

Keywords: Fuzzy arithmetic · Granular computing · Fuzzy RDM arithmetic · Horizontal membership function · Fuzzy HMF arithmetic · Multidimensional fuzzy arithmetic

1 Introduction

Operations of SF-arithmetic mostly are realized on fuzzy numbers (F-numbers) and on fuzzy intervals (F-intervals), [2,3,5–8]. Any trapezoidal F-interval A is fully characterized by the quadruple (a, b, c, d) of real numbers occuring in the special canonical form (1),

$$A(x) = \begin{cases} (x-a)/(b-a) & \text{for } x \in [a, b) \\ 1 & \text{for } x \in [b, c] \\ (d-x)/(d-c) & \text{for } x \in (c, d] \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Let $A = (a, b, c, d)$ be used as a shorthand notation of trapezoidal fuzzy intervals. When $b = c$ in (1), A is usually called triangular fuzzy number (TF-number). F-number is a special case of F-interval. Conventional interval can be defined as follows [10]: closed interval denoted by $[a, b]$ is the set of real numbers given by (2),

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\} \tag{2}$$

Figure 1 shows conventional interval, fuzzy interval, and fuzzy number.

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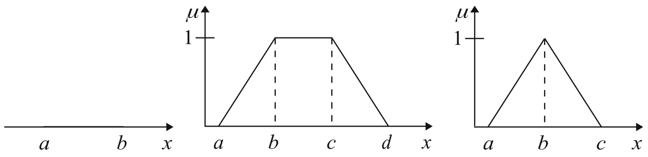

Fig. 1. Conventional interval (a, b), fuzzy interval (a, b, c, d), fuzzy number (a, b, c).

Fuzzy set A can be defined as sum of its μ-cuts, also called α-cuts. Definition of μ-cut is as follows [7]: Given a fuzzy set A defined on $\mathbb{R}$, and a real number $\mu \in [0, 1]$, the crisp set $A_\mu = \{x \in \mathbb{R} : A(x) \geq \mu\}$ is called μ-cut of A. The crisp set $S(A) = \{x \in \mathbb{R} : A(x) > 0\}$ is called the support of A. When $\max_{x \in \mathbb{R}} A(X) = 1$, A is called a normal fuzzy set.

Figure 2 shows μ-cut of F-interval on the level $\mu = 0.5$.

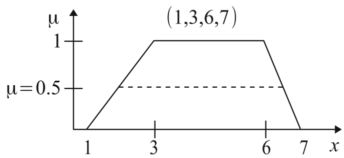

Fig. 2. Fuzzy interval about $(3, 6)$ characterized by quadruple $(1, 3, 6, 7)$ and its μ-cut $A_{0.5} = \{x \in \mathbb{R} : A(x) \geq 0.5\}$.

Because fuzzy set A can be defined as set of its μ-cuts A_μ then arithmetic operations $(+, -, \cdot, /)$ realized on fuzzy sets A and B can be understood as arithmetic operations on intervals. Hence, interval arithmetic (I-arithmetic) can be basis for F-arithmetic. In the practice mostly used I-arithmetic is Moore's arithmetic called also standard interval arithmetic (SI-arithmetic), [10]. Further on realization of basic operations of this arithmetic will be presented. If we have two intervals $[a_1, a_2]$ and $[b_1, b_2]$ then basic operations are realized according to (3) and (4).

$$[a_1, a_2] \oplus [b_1, b_2] = [a_1 \oplus b_1, a_2 \oplus b_2] \tag{3}$$

$$\begin{aligned}[a_1, a_2] \otimes [b_1, b_2] = [&\min(a_1 \otimes b_1, a_1 \otimes b_2, a_2 \otimes b_1, a_2 \otimes b_2),\\ &\max(a_1 \otimes b_1, a_1 \otimes b_2, a_2 \otimes b_1, a_2 \otimes b_2)]\end{aligned} \tag{4}$$

where: $\oplus \in \{+, -\}$, $\otimes \in \{\cdot, /\}$ and $0 \notin [b_1, b_2]$ if $\otimes = /$.

If an arithmetic operation on F-intervals A and B is to be realized then operations defined by (3) and (4) have to be performed for each μ-cut, $\mu \in [0, 1]$. Figure 3 shows subtraction of two identical F-intervals $X - X$ where $X = (1, 2, 4, 5)$.

In the case of SF-arithmetic subtraction of two identical F-intervals does not result in crisp zero but in fuzzy zero $(\tilde{0})$.

$$X - X \neq 0,\ X - X = \tilde{0} = (-4, -2, 2, 4) \tag{5}$$

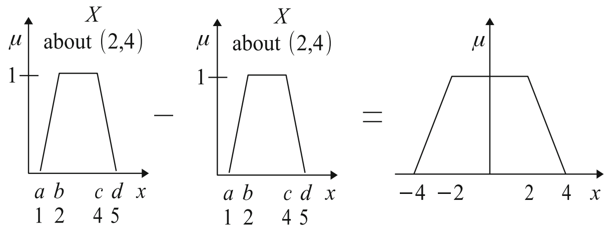

Fig. 3. Visualization of subtraction of two identical intervals $X - X$ determined by quadruple $(1, 2, 4, 5)$.

This result seems rather illogical, because F-interval X represents only one true value x that really occurred in a system. However, it is not known precisely but only approximately as $(1, 2, 4, 5)$. Hence, the difference should be equal to crisp zero. Let us consider now fuzzy equation $A - X = C$, where $A = (1, 2, 4, 5)$ and $C = (7, 8, 9, 10)$. Solving this equation with SF-arithmetic gives strange result $X = (-5, -, 5, -6, -6)$ shown in Fig. 4 in which $x_{\min} > x_{\max}$.

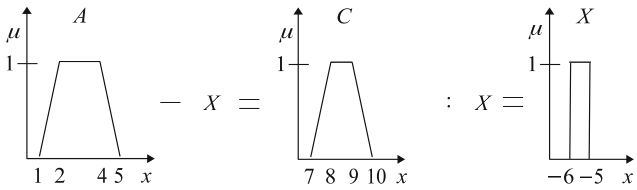

Fig. 4. Paradoxical and incomplete solution $X = [-6, -6, -5, -5]$ of fuzzy equation $A - X = C$ with $A = (1, 2, 4, 5]$ and $C = (7, 8, 9, 10)$ in which $x_{\min} > x_{\max}$.

One can also try to solve the equation $A - X = C$ using other forms of it, e.g.: $A - X - C = 0$, $A = C + X$, $A - C = X$. The form $A - X - C = 0$ gives paradoxical result $X = (-2, -4, -7, -9)$ in which $x_{\min} > x_{\max}$. The form $A = C+X$ gives result $X = (-6, -6, -5, -5)$ which is not fuzzy but conventional interval. The form $A - C = X$ gives result $X = (-9, -7, -4, -2)$ being inverse form of the result $X = (-2, -4, -7, -9)$ delivered by the form $A - X - C = 0$. This paradoxical phenomenon of different results achieved from different forms of one and the same equation has been described in many publications, e.g. in [4]. To "enable" solving of equations of type $A-X = C$ apart of the usual calculation way of the difference $A - B = X$ a second way called Hukuhara difference (H-difference) has been in SF-arithmetic introduced. It is calculated from equation $A = B + X_H$ [11]. Thus, officially two ways of difference calculation exist in SF-arithmetic. In SF-arithmetic many next paradoxes exist which are reason of its limited application. SF-arithmetic can solve only part of real problems but the rest lies outside its possibilities. What are reasons of this situation?

Reason 1 is assumption and conviction that result X of arithmetic operation on two 2D fuzzy intervals A and B also is a 2D fuzzy-interval. However, this is not true. Result X of such operation exist not in 2D-space but in 3D-space what will be shown further on. Each next F-interval added to the operation increases dimension of the result. This state of matter is diametrically different from the state in crisp, conventional arithmetic.

Reason 2. In SF-arithmetic and SI-arithmetic calculations are performed only with interval borders. Interiors of intervals do not take part in calculations. And after all calculations should be made on full and complete sets and only on their borders.

Reason 3. In SF-arithmetic as calculation result is accepted not complete solution set but partial, incomplete solution set. In interval arithmetic being basis of SF-arithmetic S.P. Shary [16] introduced 3 different notions of solutions of linear equation systems: the united solution set, the tolerable solution set, the controlled solution set. There also exists notion of the interval algebraic solution [9]. Basing on [9] definitions of the above solution sets for the case of equation $A - X = C$ are as below.

The united solution set $\sum_{\exists\exists}(A, C)$ is the set of solutions of real systems $a - x = c$ with $a \in A$ and $c \in C$, i.e.,

$$\sum\nolimits_{\exists\exists}(A, C) = \{x \in \mathbb{R} | \exists_{a\in A} \exists_{c\in C}(a - x = c)\} \tag{6}$$

The tolerable solution set $\sum_{\forall\exists}(A, C)$ is the set of all real values x such that for every real value $a \in A$ the real difference $a - x$ is contained in the interval vector C, that is;

$$\sum\nolimits_{\forall\exists}(A, C) = \{x \in \mathbb{R} | \forall_{a\in A} \exists_{c\in C}(a - x = c)\} \tag{7}$$

The controlled solution set $\sum_{\exists\forall}(A, C)$ of all real values $x \in \mathbb{R}$, such that for any $c \in C$ we can find the corresponding value $a \in A$ satisfying $a - x$;

$$\sum\nolimits_{\exists\forall}(A, C) = \{x \in \mathbb{R} | \exists_{a\in A} \forall_{c\in C}(a - x = c)\} \tag{8}$$

The united, tolerable and controlled solution sets are not complete algebraic solutions of the equation $A - X = C$ because there exists also point solutions being outside these sets. The full solution set was called "interval algebraic solution". Notion of it [9] adapted for the equation $A - X = C$ is as below.

The interval algebraic solution of interval equation $A - X = C$ is an interval X, which substituted in the equation $A - X$, using interval arithmetic, results in C, that is (9).

$$A - X = C \tag{9}$$

According to [9] "the interval algebraic solutions do not exist in general in the ordinary intervals space (the space without improper intervals)". However, let us remark that in definition of "the interval algebraic solution" as solution "an interval X" is understood, i.e. the same mathematical object as intervals A and C occurring in the equation $A - X = C$. As will be shown further on algebraic solution of expression $A - X = C$ exists and it can be called complete solution. It can be achieved with use of RDM fuzzy arithmetic what will be shown further on. However, this solution is not an interval but a multidimensional granule existing in 3D-space.

Reason 4. SF-arithmetic does not possess certain important mathematical properties which are necessary in solving more complicated problems as fuzzy equation systems. In particular, SF-arithmetic does not possess the inverse element $(-X)$ of addition and $(1/X)$ of multiplication. Hence properties (10) are true (except for degenerate intervals), 0 means crisp zero and 1 means crisp 1.

$$X + (-X) \neq 0,\ X \cdot (1/X) \neq 1 \tag{10}$$

In SI- and SF-arithmetic also subdistributivity law and cancellation law for multiplication does not hold in general, (11) and (12).

$$X(Y + Z) \neq XY + XZ \tag{11}$$

$$XZ = YZ \nRightarrow X = Y \tag{12}$$

2 RDM Variables and Horizontal Membership Functions

Figure 1 shows an interval. It is a model of x-value that is not precisely but only approximately known and the knowledge about it is expressed in the form $x \in [a, b]$. If such a model is used in calculations of SI-arithmetic then only the interval borders a and b take part in the calculations. The whole interior does not take part. The Relative-Distance-Measure (RDM) allows for participation also the interval interior in calculations. The RDM model of interval is given by [10] and shown in Fig. 5.

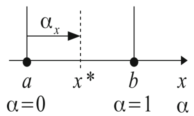

Fig. 5. Visualization of the RDM interval model.

In Fig. 5 x^* means the true precise value of variable x that has occurred in a real system but that is not precisely known. This true value can be expressed by (13).

$$x^* = a + (b - a)\alpha_x,\ \alpha_x \in [0, 1] \tag{13}$$

However, for simplicity notation (14) will be used.

$$x = a + (b - a)\alpha_x \tag{14}$$

The RDM-variable α_x has here meaning of the relative distance of the true value x^* from the interval beginning a. Thus, it can be interpreted as a local coordinate. A fuzzy interval (a, b, c, d) is shown in Fig. 1. For such intervals vertical MFs can be used. Vertical models express the vertical dependence $\mu = f(x)$. Vertical MF of fuzzy interval (a, b, c, d) is given by (15).

$$\mu(x) = \begin{cases} (x-a)/(b-a) & \text{for } x \in [a, b) \\ 1 & \text{for } x \in [b, c] \\ (d-x)/(d-c) & \text{for } x \in (c, d] \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

The model (15) is a model of the fuzzy interval borders only. It does not model the interval interior. This fact limits usefulness of the vertical model in calculations. A question can be asked: would it be possible to create a horizontal model of fuzzy interval in the form $x = f^{-1}(\mu)$? At first glance it seems impossible because such dependence would be ambiguous and hence would not be function. However, let us consider a horizontal cut of the MF shown in Fig. 6 on level μ. The cut is a usual 1D interval and has two boundaries $x_L(\mu)$ and $x_R(\mu)$ that are expressed by (16).

$$x_L(\mu) = a + (b-a)\mu, \; x_R(\mu) = d - (d-c)\mu \tag{16}$$

The RDM variable α_x with its increase transforms the left boundary $x_L(\mu)$ in the right one $x_R(\mu)$, Fig. 6.

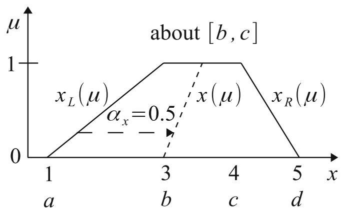

Fig. 6. Visualization of the horizontal approach to fuzzy membership functions.

The contour line $x(\mu, \alpha_x)$ of constant α_x-values in the interior of the MF (Fig. 6) is expressed by (17).

$$x(\mu, \alpha_x) = x_L(\mu) + [x_R(\mu) - x_L(\mu)]\alpha_x, \; \mu, \alpha_x \in [0, 1] \tag{17}$$

The contour line $x(\mu, \alpha_x)$ is set of points lying at equal relative distance α_x from the left boundary $x_L(\mu)$ of the MF in Fig. 6. A more precise form (18) of (17) can be called horizontal MF (HMF).

$$x = [a + (b-a)\mu] + [(d-a) - (d-c+b-a)\mu]\alpha_x, \; \mu, \alpha_x \in [0, 1] \tag{18}$$

The horizontal MF $x = f(\mu, \alpha_x)$ is function of two variables and exists in 3D-space, Fig. 7. It is unique and expresses the 2D MF with its interior shown

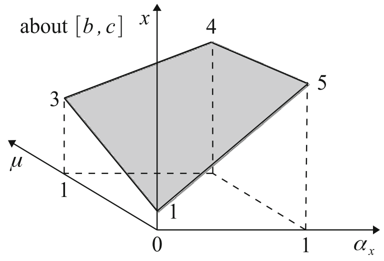

Fig. 7. The horizontal membership function $x = (1 + 2\mu) + (4 - 3\mu)\alpha_x$, $\mu, \alpha_x \in [0, 1]$, corresponding to vertical function shown in Fig. 6.

in Fig. 1. The HMF defines a 3D information granule, Fig. 7, and hence can be denoted as x^{gr}.

Formula (18) describes the trapezoidal MF. However, it can be adapted to triangular MF by setting $a = b$ and to rectangular MF by $a = b$ and $c = d$. Boundaries of these functions are here linear. To derive formulas for nonlinear boundaries, e.g. of Gauss type, formulas for the left $x_L(\mu, \alpha_x)$ and for the right boundary $x_R(\mu, \alpha_x)$ should be determined and set in (17). Concept of the horizontal MF was elaborated by A. Piegat [12–15, 17].

3 RDM Fuzzy Arithmetic with Horizontal Membership Functions

Let $x^{gr} = f(\mu, \alpha_x)$ be a horizontal MF representing a fuzzy interval X (19) and $y^{gr} = f(\mu, \alpha_y)$ be a horizontal MF representing a fuzzy interval Y (20).

$$X : x^{gr} = [a_x + (b_x - a_x)\mu] + [(d_x - a_x) - (d_x - c_x + b_x - a_x)\mu]\alpha_x, \ \mu, \alpha_x \in [0, 1] \quad (19)$$

$$Y : y^{gr} = [a_y + (b_y - a_y)\mu] + [(d_y - a_y) - (d_y - c_y + b_y - a_y)\mu]\alpha_y, \ \mu, \alpha_y \in [0, 1] \quad (20)$$

Addition of two independent fuzzy intervals, (21).

$$X + Y = Z : x^{gr}(\mu, \alpha_x) + y^{gr}(\mu, \alpha_y) = z^{gr}(\mu, \alpha_x, \alpha_y), \ \mu, \alpha_x, \alpha_y \in [0, 1] \quad (21)$$

For example, if X is trapezoidal MF $(1, 3, 4, 5)$, (22),

$$x^{gr}(\mu, \alpha_x) = (1 + 2\mu) + (4 - 3\mu)\alpha_x \quad (22)$$

and Y is trapezoidal MF $(1, 2, 3, 4)$, (23),

$$y^{gr}(\mu, \alpha_y) = (1 + \mu) + (3 - 2\mu)\alpha_y \quad (23)$$

then $z^{gr}(\mu, \alpha_x, \alpha_y)$ is given by (24),

$$z^{gr}(\mu, \alpha_x, \alpha_y) = (2 + 3\mu) + (4 - 3\mu)\alpha_x + (3 - 2\mu)\alpha_y, \ \mu, \alpha_x, \alpha_y \in [0, 1] \quad (24)$$

The 4D-solution (24) exists in the space which cannot be seen. Therefore we can be interested in its low dimensional representations. Frequently, the 2D-representation in the form of span $s(z^{gr})$ is determined. It can be found with known methods of function examination (25).

$$s(z^{gr}) = [\min_{\alpha_x,\alpha_y} z^{gr}(\mu,\alpha_x,\alpha_y), \max_{\alpha_x,\alpha_y} z^{gr}(\mu,\alpha_x,\alpha_y)] \tag{25}$$

In the case of discussed example, extrema of (25) lie not inside but on boundaries of the result domain. The minimum corresponds to $\alpha_x = \alpha_y = 0$ and the maximum to $\alpha_x = \alpha_y = 1$. Span of the 4D-result granule (24) is given by (26).

$$s(z^{gr}) = [2 + 3\mu, 9 - 2\mu],\ \mu \in [0,1] \tag{26}$$

The span (26) is not the addition result. The addition result has the form of 4D-function (24). The span is only a 2Dinformation about the maximal uncertainty of the result.

Subtraction of two independent fuzzy intervals, (27).

$$X - Y = Z : x^{gr}(\mu,\alpha_x) - y^{gr}(\mu,\alpha_y) = z^{gr}(\mu,\alpha_x,\alpha_y),\ \mu,\alpha_x,\alpha_y \in [0,1] \tag{27}$$

For example, if X and Y are trapezoidal MF (22) and (23) then the result is given by (28).

$$z^{gr}(\mu,\alpha_x,\alpha_y) = \mu + (4 - 3\mu)\alpha_x - (3 - 2\mu)\alpha_y,\ \mu,\alpha_x,\alpha_y \in [0,1] \tag{28}$$

If we are interested in the span representation $s(z^{gr})$ of the 4D-subtraction result, then it can be determined from (29).

$$s(z^{gr}) = [\min_{\alpha_x,\alpha_y} z^{gr}, \max_{\alpha_x,\alpha_y} z^{gr}] = [-3 + 3\mu, 4 - 2\mu],\ \mu \in [0,1] \tag{29}$$

The span (29) of z^{gr} (28) corresponds to $\alpha_x = 0$, $\alpha_y = 1$ for $\min z^{gr}$ and $\alpha_x = 1$, $\alpha_y = 0$ for $\max z^{gr}$.

Multiplication of two independent fuzzy intervals, (30).

$$X \cdot Y = Z : x^{gr}(\mu,\alpha_x) \cdot y^{gr}(\mu,\alpha_y) = z^{gr}(\mu,\alpha_x,\alpha_y),\ \mu,\alpha_x,\alpha_y \in [0,1] \tag{30}$$

For example, if X and Y are trapezoidal MF (22) and (23) then the multiplication result z^{gr} is expressed by (31).

$$\begin{aligned} &z^{gr}(\mu,\alpha_x,\alpha_y) = x^{gr} \cdot y^{gr} \\ &= (1 + 2\mu) + (4 - 3\mu)\alpha_x] \cdot [(1 + \mu) + (3 - 2\mu)\alpha_y],\ \mu,\alpha_x,\alpha_y \in [0,1] \end{aligned} \tag{31}$$

Formula (31) describes the full 4D-result of the multiplication. If we are interested in the 2D simplified representation of this result in the form of a span $s(z^{gr})$ then formula (32) should be used.

$$s(z^{gr}) = [\min_{\alpha_x,\alpha_y} z^{gr}, \max_{\alpha_x,\alpha_y} z^{gr}] = [(1 + 2\mu)(1 + \mu), (5 - \mu)(4 - \mu)],\ \mu \in [0,1] \tag{32}$$

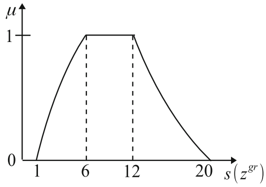

Fig. 8. MF of the span representation of the 4D multiplication result (31).

Figure 8 shows the MF of the span representation of the multiplication result.
Division X/Y of two independent fuzzy intervals, $0 \notin Y$, (33).

$$X/Y = Z : x^{gr}(\mu, \alpha_x)/y^{gr}(\mu, \alpha_y) = z^{gr}(\mu, \alpha_x, \alpha_y),\ \mu, \alpha_x, \alpha_y \in [0, 1] \tag{33}$$

For example, if X and Y are trapezoidal MF (22) and (23) then the division result z^{gr} is given by (34).

$$z^{gr}(\mu, \alpha_x, \alpha_y) = x^{gr}/y^{gr} = \frac{(1 + 2\mu) + (4 - 3\mu)\alpha_x}{(1 + \mu) + (3 - 2\mu)\alpha_y},\ \mu, \alpha_x, \alpha_y \in [0, 1] \tag{34}$$

The span representation $s(z^{gr})$ of the result (34) is expressed by (35) and is shown in Fig. 9.

$$s(z^{gr}) = [\min_{\alpha_x, \alpha_y} z^{gr}, \max_{\alpha_x, \alpha_y} z^{gr}] = \left[\frac{1 + 2\mu}{4 - \mu}, \frac{5 - \mu}{1 + \mu}\right],\ \mu \in [0, 1] \tag{35}$$

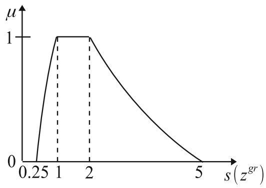

Fig. 9. Span representation of the 4D division result (34).

The solution granule of the division (34) is 4-dimensional, so it cannot be presented in its full space. However, it can be shown in a simplified way, in the $X \times Y \times Z$ 3D-space, without μ-coordinate. Figure 10 presents surfaces for constant $\mu = 0$ and $\mu = 1$ values.

As Fig. 10 shows, the solution granule (34) is uniform. This results from the fact that the divisor does not contain zero. Division results can be discontinuous and multigranular in more complicated cases.

What happens when uncertain denominator Y of division X/Y contains zero? Then the solution is multi-granular. Such situation does not occur in conventional crisp arithmetic.

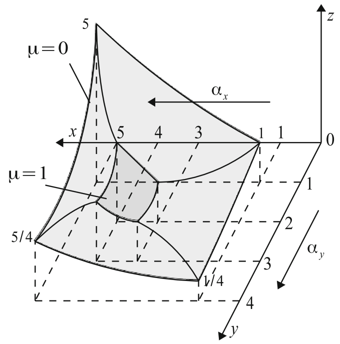

Fig. 10. Simplified view of the 4D-solution granule (34) $z^{gr}(x, y, z)$ in 3D-space $X \times Y \times Z$, without μ-coordinate

Let us now apply RDM fuzzy arithmetic to solve equation $A - X = C = (1, 2, 4, 5) - X = (7, 8, 9, 10)$ which previously has been "solved" with use of SF-arithmetic. Using Eq. (18) HMF of A is achieved in form of (36) and HMF of C in form of (37).

$$a^{gr} = (1 + \mu) + (4 - 2\mu)\alpha_a, \ \mu, \alpha_a \in [0, 1] \tag{36}$$

$$c^{gr} = (7 + \mu) + (3 - 2\mu)\alpha_c, \ \mu, \alpha_c \in [0, 1] \tag{37}$$

The solution x^{gr} can be found similarly as in crisp number arithmetic according to (38).

$$x^{gr} = a^{gr} - c^{gr} = [(1+\mu)+(4-2\mu)\alpha_a] - [(7+\mu)+(3-2\mu)\alpha_c], \ \mu, \alpha_a, \alpha_c \in [0, 1] \tag{38}$$

$$a^{gr} - x^{gr} = c^{gr} = a^{gr} - (a^{gr} - c^{gr}) = c^{gr} \tag{39}$$

One can easily check that after substitution of the solution x^{gr} in the solved equation $A - X = C$ (39) is achieved which gives the result $c^{gr} = c^{gr}$, which means that the solution (38) is the algebraic solution of the equation according to [9]. One can easily check that solution (38) is complete by substituting various possible triple combinations of variables $(\mu, \alpha_a, \alpha_c)$. E.g. for $\mu = 0$, $\alpha_a = 1/5$ and $\alpha_c = 2/5$ we have $a = 1.8$, $c = 8.2$ and $x = -6.4$. It is one of point solutions of equation $A - X = C$ because $a - x = 1.8 - (-6.4) = c = 8.2$. With various combinations $(\mu, \alpha_a, \alpha_c)$ one can generate each triple (a, x, c) satisfying equation $a - x = c$.

4 Mathematical Properties of Multidimensional RDM Fuzzy Arithmetic

Commutativity. For any fuzzy intervals X and Y, Eqs. (40) and (41) are true.

$$X + Y = Y + X \tag{40}$$

$$XY = YX \tag{41}$$

Associativity. For any fuzzy intervals X, Y and Z, Eqs. (42) and (43) are true.

$$X + (Y + Z) = (X + Y) + Z \tag{42}$$

$$X(YZ) = (XY)Z \tag{43}$$

Neutral element of addition and multiplication. In multidimensional RDM FA, there exist additive and multiplicative neutral elements such as the degenerate interval 0 and 1 for any interval X, Eqs. (44) and (45).

$$X + 0 = 0 + X = X \tag{44}$$

$$X \cdot 1 = 1 \cdot X = X \tag{45}$$

Inverse elements. In MD RDM FA, fuzzy interval $-X : -x^{gr} = -[a + (b - a)\mu] - [(d-a) - \mu(d-a+b-c)]\alpha_x$, $\mu, \alpha_x \in [0,1]$, is an additive inverse element of fuzzy interval $X : x^{gr} = [a+(b-a)\mu] + [(d-a) - \mu(d-a+b-c)]\alpha_x$, $\mu, \alpha_x \in [0,1]$.

If parameters of two fuzzy intervals X and Y are equal: $a_x = a_y$, $b_x = b_y$, $c_x = c_y$, $d_x = d_y$, then the interval $-Y$ is the additive inverse interval of X, when also inner RDM-variables are equal: $\alpha_x = \alpha_y$. It means full coupling (correlation) of both uncertain values x and y modelled by intervals.

Assuming that $0 \notin X$, a multiplicative inverse element of the fuzzy interval X is equal in MD RDM FA $\frac{1}{X} \big/ \frac{1}{x^{gr}} = \frac{1}{[a+(b-a)\mu]+[(d-a)-\mu(d-a+b-c)]\alpha_x}$, $\mu, \alpha_x \in [0,1]$.

If parameters of two fuzzy intervals X and Y are equal: $a_x = a_y$, $b_x = b_y$, $c_x = c_y$, $d_x = d_y$, then the interval $1/Y$ is the multiplicative inverse interval of X only when also inner RDM-variables are equal: $\alpha_x = \alpha_y$. It means full coupling (correlation) of both uncertain values x and y modelled by intervals. Such full or partial correlation of uncertain variables occurs in many real problems.

Subdistributivity law. The subdistributivity law holds in MD RDM FA (46).

$$X(Y + Z) = XY + XZ \tag{46}$$

The consequence of this law is a possibility of formulas transformations. They do not change the calculation result.

Cancellation law for addition and multiplication. Cancellation laws (47) and (48) hold in MD RDM FA:

$$X + Z = Y + Z \Rightarrow X = Y \tag{47}$$

$$XZ = YZ \Rightarrow X = Y \tag{48}$$

5 Application Example of RDM Fuzzy Arithmetic in Solving Differential Equation

Solving fuzzy differential equation (FD-equation) is difficult task that has been considered since many years and has not been finished until now. In the example both SF-arithmetic and RDM fuzzy arithmetic will be applied to solve a FD-equation (49). This equation is a type of benchmark because it has been discussed in few important papers on FD-equation solving methods, e.g. in [1].

Example. Find the solution of fuzzy differential equation (49) taken from [1].

$$\begin{cases} \dot{X}(t) = -X(t) + W\cos t \\ X(0) = (-1, 0, 1) \end{cases} \tag{49}$$

where $W = (-1, 0, 1)$.

The solution of Eq. (49) for $t \geq 0$ expressed in the form of μ-solution sets [1], for $\mu \in [0, 1]$, is given by (50).

$$x_\mu(t) = 0.5(\sin t + \cos t)[W]_\mu + ([X(0)]_\mu - 0.5[W]_\mu)\exp(-t) \tag{50}$$

The solution (50) obtained by standard fuzzy (SF-) arithmetic for $[W]_\mu = [\mu - 1, 1 - \mu]$ and $[X(0)]_\mu = [\mu - 1, 1 - \mu]$ is given by (51). This standard solution exists in 3D-space.

$$x_\mu^{SFA}(t) = 0.5(\sin t + \cos t)[\mu - 1, 1 - \mu] + ([\mu - 1, 1 - \mu] - 0.5[\mu - 1, 1 - \mu])\exp(-t) \tag{51}$$

In this case fuzzy numbers $[W]_\mu = [X(0)]_\mu = [\mu - 1, 1 - \mu]$ are equal. Figure 11(a) presents in 2D-space border values of the SFA solution (51).

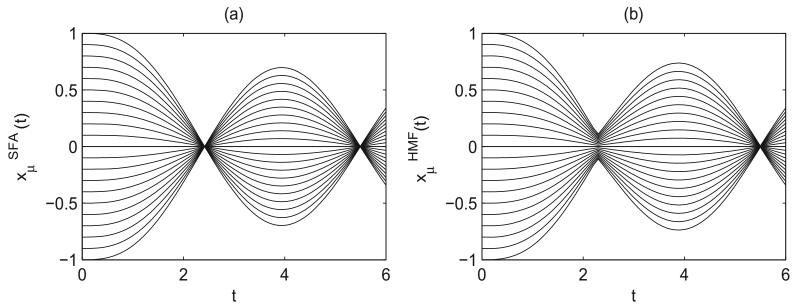

Fig. 11. 2D presentation of border values of the FD-equation (49) obtained with use of standard fuzzy arithmetic (a), and horizontal membership function FA (b), for $\mu \in [0 : 0.1 : 1]$, $t \in [0 : 0.1 : 6]$.

The fuzzy numbers in the form of horizontal membership functions are presented by (52) and (53), where α_W, α_X are RDM variables defined in 3D-space.

$$[W]_\mu = -1 + \mu + 2(1 - \mu)\alpha_W, \ \alpha_W \in [0, 1] \tag{52}$$

$$[X(0)]_\mu = -1 + \mu + 2(1-\mu)\alpha_X,\ \alpha_X \in [0,1] \tag{53}$$

In terms of SF-arithmetic both fuzzy numbers W and $X(0)$ are equal ($W = (-1,0,1)$ and $X(0) = (-1,0,1)$). But it does not mean that real values $w \in W$ and $x(0) \in X(0)$ that have occurred in the real system also have been equal. E.g., it is possible that $w = -0.5$ ($-0.5 \in (-1,0,1)$) and $x(0) = 0.3$ ($0.3 \in (-1.0.1)$). RDM fuzzy arithmetic thanks to RDM variables α_W and α_X enables modeling of such situations. If $\alpha_W = \alpha_X$ then $w = x(0)$. When $\alpha_W \neq \alpha_X$ then $w \neq x(0)$.

The μ-solution sets of FD-equation (49) using horizontal membership function is presented by (54). Let us notice that HMF-solution exists in 5D-space.

$$\begin{aligned} x_\mu^{HMF} &= 0.5(\sin t + \cos t)(-1 + \mu + 2(1-\mu)\alpha_W) \\ &\quad + [(-1 + \mu + 2(1-\mu)\alpha_X) - 0.5(-1 + \mu + 2(1-\mu)\alpha_W)] \exp(-t) \end{aligned} \tag{54}$$

where $\mu \in [0,1]$, $\alpha_W \in [0,1]$, $\alpha_X \in [0,1]$.

Figure 11(b) presents the 5D-solution (54) in 2D-space showing only border values of the HMF-solution.

To check that the solution achieved with use of HMF-functions is correct and complete let us take the testing point $x_\mu^0(t)$ from the HMF-solution (54). It can be proved that the testing point does not belong to the SF-arithmetic solution (51). Let us take the point $x_\mu^0(2.5) = 0.2245$, where $t = 2.5$, $\mu = 0$, $[W]_{\mu=0} = -1$, $[X(0)]_{\mu=0} = 1$, $\alpha_W = 0$ and $\alpha_X = 1$. The value $x_\mu^0(2.5) = 0.2245 \in x_\mu^{HMF}$ belongs to the HMF-solution but it does not belong to the SFA-solution $x_\mu^0(2.5) = 0.2245 \notin x_\mu^{SFA}$.

Solution of the Eq. (49) equals (50). Derivative of the solution (50) is (55).

$$[\dot{X}(t)]_\mu = 0.5(\cos t - \sin t)[W]_\mu - ([X(0)]_\mu - 0.5[W]_\mu) \exp(-t) \tag{55}$$

The left-part of Eq. (49) is the derivative (55). The derivative (55) for $t = 2.5$, $\mu = 0$, $[W]_{\mu=0} = -1$ and $[X(0)]_{\mu=0} = 1$ equals $[\dot{X}(2.5)]_{\mu=0} = 0.5767$. The right-part for $x_\mu^0(2.5) = 0.2245$, $t = 2.5$, $\mu = 0$, $[W]_{\mu=0} = -1$ also equals 0.5767, so the testing point $x_\mu^0(2.5) = 0.2245$ objectively belongs to the HMF-solution of Eq. (49).

In conclusion of the example one can say that standard fuzzy arithmetic can give only a part of full solution. The SF-arithmetic did not find the correct solution of Eq. (49) in the case $w \neq x(0)$. However, when using horizontal membership function with RDM variables α_W and α_X the fuzzy numbers W and $X(0)$ can be independent and w can be different from $x(0)$.

6 Conclusions

The paper has shown a new model of membership function called vertical MF and its application in RDM fuzzy arithmetic. RDM fuzzy arithmetic possess important mathematical properties which SF-arithmetic has not. These properties enable transformation of equations in the process of solving them.

Further on, they increase possibilities of fuzzy arithmetic in solving real problems. Examples of such problems were shown in the paper. Because uncertainty is prevalent in reality, RDM fuzzy arithmetic becomes important tool of solving real problems.

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