2

LAMINAR FLOW AND THE MOMENTUM EQUATION

In discussing Newton's law of viscosity, we have described fluid motion as flowing parallel layers which, because of viscosity, establish a velocity gradient dependent upon the shear stress applied to the fluid. This velocity gradient has been regarded as a potential or a "reason" for momentum transport from layer to layer.

In this chapter, we shall first derive simple differential equations of momentum for special cases of flow, for example, flow of a falling film, flow between two parallel plates, and flow through tubes. To make it possible for the student to participate in developing complex formulas, these derivations make use of the concept of a momentum balance and the definition of viscosity. These classic examples of viscous flow patterns certainly apply to rather simplified and idealized conditions. You may be tempted, therefore, to disregard the importance of thoroughly understanding these examples; however, we should point out that despite the simplicity of the following calculations, you will gain an appreciation of the variables involved. Also, you will obtain a basic tool for analyzing engineering problems: the ability to arrive at pertinent differential equations.

2.1 MOMENTUM BALANCE*

A momentum balance is applied to a small control volume of fluid to develop a differential equation. The differential equations, when their solutions comply with the physical restrictions (boundary conditions), yield the algebraic relationships which can be used to determine the engineering characteristics of the system. The solutions give the velocity distributions from which other characteristics, including the shear stress at the fluid-solid interface, are developed. As we shall see in Chapter 3, the shear stress at the fluid-solid

*The general aspects of the developments in Sections 2.1-2.6 are similar to those found in Chapters 2-3 in R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, Wiley, New York, 1960.

interface is very important in analyzing the disposition of energy flowing through a system. For steady state flow, the momentum balance is

$$\begin{pmatrix} \text{rate of} \\ \text{momentum in} \end{pmatrix} - \begin{pmatrix} \text{rate of} \\ \text{momentum out} \end{pmatrix} + \begin{pmatrix} \text{sum of forces} \\ \text{acting on system} \end{pmatrix} = 0. \tag{2.1}$$

Momentum in (or out) may enter a system by momentum transfer according to Newton's equation of viscosity (if the fluid is Newtonian; otherwise various equations for non-Newtonian fluids are used), or it may enter due to the overall fluid motion. The forces applied to the balance are pressure forces and/or gravity forces.

The momentum balance is actually a force balance because we are concerned with the *rate of momentum* that enters and leaves the unit volume. Units of momentum are MLT^{-1} (M = mass, L = length, T = time), whereas a rate of momentum is MLT^{-2}. Classical physics states clearly that forces ($F = ma$) are involved when we consider momentum rates. Thus, if the term momentum balance confuses the reader, he or she is reassured that a force balance is being applied.

2.2 FLOW OF A FALLING FILM

Consider the flow of a liquid at steady state along an inclined plane (Fig. 2.1). The liquid is at a constant temperature, and therefore its density and viscosity are constant. Furthermore, we consider only that portion of the plane where the entrance and exit of the liquid to the plane are sufficiently remote so as not to influence the velocity v_z. In this situation, v_z is not a function of z but obviously a function of x.* Figure 2.1 also depicts the unit volume as a "shell" with a thickness Δx and length L; the width of the shell extends a distance W, perpendicular to the page. The terms used in Eq. (2.1) are as follows:

Rate of momentum in across surface at x (moment transport due to viscosity)	$(LW)(\tau_{xz})\vert_x$
Rate of momentum out across surface at $x + \Delta x$ (due to viscosity)	$(LW)(\tau_{xz})\vert_{x+\Delta x}$
Rate of momentum in across surface at $z = 0$ (due to fluid motion)	$(W\Delta x v_z)(\rho v_z)\vert_{z=0}$
Rate of momentum out across surface at $z = L$ (due to fluid motion)	$(W\Delta x v_z)(\rho v_z)\vert_{z=L}$
Gravity force acting on fluid	$(LW\Delta x)(\rho g \cos \beta)$

In this particular problem, the pressure forces are irrelevant because the pressure does not vary with z. Also note that all terms in the list, including the first two, are z-directed forces. Figure 2.1 shows that momentum in by viscous transport is x-directed, but if we think of

*In the region where $v_z = f(x)$ and $v_z \neq f(z)$, we say that the flow is ***fully developed***.

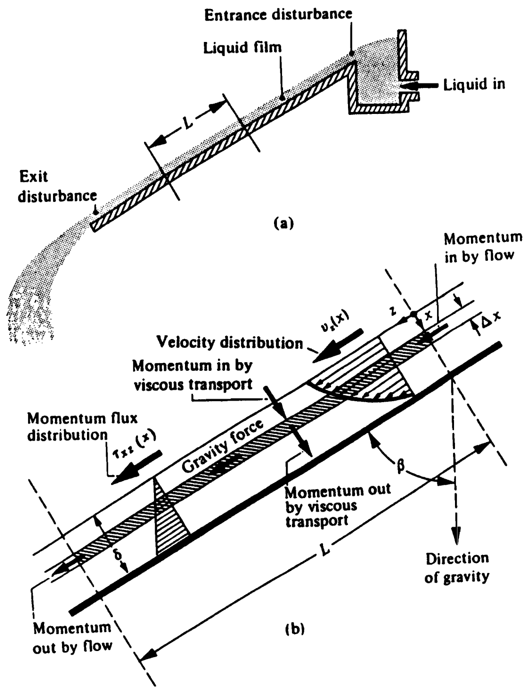


Fig. 2.1 Flow of a falling film.

interpreting τ_{xz} in an alternative way—namely, as a shear stress—we certainly realize that we are dealing with a z-directed force.

When all these terms are substituted into the momentum balance, we get

$$LW\tau_{xz}|_x - LW\tau_{xz}|_{x+\Delta x} + W\Delta x \rho v_z^2|_{z=0} - W\Delta x \rho v_z^2|_{z=L} + LW\Delta x \rho g \cos\beta = 0. \quad (2.2)$$

Because we are restricted to that part of the inclined plane which does not feel the effects of the exit and entrance, v_z is independent of z. Therefore, the third and fourth terms cancel one another out. Equation (2.2) is now divided by $LW\Delta x$ and, if Δx is allowed to be infinitely small, we obtain

$$\lim_{\Delta x \to 0} \frac{\tau_{xz}|_{x+\Delta x} - \tau_{xz}|_x}{\Delta x} = \rho g \cos\beta. \quad (2.3)$$

We have now recognized the definition of the first derivative of τ_{xz} with respect to x, and have thus developed the differential equation pertinent to our system:

$$\frac{d\tau_{xz}}{dx} = \rho g \cos\beta. \quad (2.4)$$

This equation is integrated to yield

$$\tau_{xz} = \rho g x \cos\beta + C_1. \quad (2.5)$$

Equation (2.5) describes the momentum flux (or alternatively the shear-stress distribution), but contains an integration constant C_1. This constant is evaluated by recognizing that the shear stress in the liquid is very nearly zero at a liquid-gas interface. In other words, the gas phase, in this instance, offers little resistance to liquid flow, which results in a realistic *boundary condition*:

$$\text{B.C.1} \qquad \text{at } x = 0, \qquad \tau_{xz} = 0. \tag{2.6}$$

Substitution of this boundary condition into Eq. (2.5) requires that $C_1 = 0$; hence the momentum flux is

$$\tau_{xz} = \rho g x \cos \beta. \tag{2.7}$$

If the fluid is Newtonian, then we know that the momentum flux is related to the velocity gradient according to

$$\tau_{xz} = -\eta \frac{dv_z}{dx}. \tag{2.8}$$

Substituting this expression for τ_{xz} in Eq. (2.7) gives the distribution of the velocity gradient:

$$\frac{dv_z}{dx} = -\frac{\rho g \cos \beta}{\eta} x. \tag{2.9}$$

Integrating Eq. (2.9), we have

$$v_z = -\left[\frac{\rho g \cos \beta}{2\eta}\right] x^2 + C_2. \tag{2.10}$$

Another integration constant has evolved which is evaluated by examining the other boundary condition, namely, that at the fluid-solid interface the fluid clings to the wall; that is,

$$\text{B.C.2} \qquad \text{at } x = \delta, \qquad v_z = 0. \tag{2.11}$$

Substituting this into Eq. (2.10), we determine the constant of integration; $C_2 = (\rho g \cos \beta / 2\eta)\delta^2$. Therefore the velocity distribution is

$$v_z = \frac{\rho g \delta^2 \cos \beta}{2\eta}\left[1 - \left[\frac{x}{\delta}\right]^2\right]. \tag{2.12}$$

and is parabolic. Once the velocity profile has been found, a number of quantities may be calculated.

i) The *maximum velocity*, V_z^{max}, is that velocity at $x = 0$:

$$V_z^{max} = \frac{\rho g \delta^2 \cos \beta}{2\eta}. \tag{2.13}$$

ii) The *average velocity*, $\bar{v}_z$, is simply

$$\bar{v}_z = \frac{1}{\delta}\int_0^\delta v_z\, dx = \frac{\rho g \delta \cos \beta}{2\eta}\int_0^\delta \left[1 - \left[\frac{x}{\delta}\right]^2\right] dx = \frac{\rho g \delta^2 \cos \beta}{3\eta}. \tag{2.14}$$

iii) The *volume flow rate*, Q, is given by the product of the average velocity and the cross-section of flow:

$$Q = \bar{v}_z(W\delta) = \frac{\rho g W\delta^3 \cos\beta}{3\eta}. \tag{2.15}$$

The foregoing analytical results are valid only when the film is falling in laminar flow (straight streamlines). This condition can easily be satisfied for the slow flow of viscous films, but experimentally, it has been found that as the film velocity increases, the film thickness increases (according to Eq. (2.14)) to a critical value, depending on the liquid's kinematic viscosity, where turbulence replaces laminar flow. Of course, when turbulent flow develops, Eqs. (2.12)-(2.15) are no longer valid.

Example 2.1 A viscous molten glass covers a molten metal, and together they flow slowly down an inclined plane that makes an angle β with the vertical. The thickness of the glass is δ_1, and the combined thickness of both layers is δ_2. Of course, each layer has its own viscosity. For a plane of length L, assume laminar flow that is fully developed and derive an equation for the velocity distribution in each layer.

Solution. We should recognize that Eq. (2.4) applies to both layers. In the glass (i.e., the top layer), $\tau_{xz} = 0$ at $x = 0$. Therefore, after integrating Eq. (2.4) the shear stress in the glass is

$$\tau_{xz} = \rho_g \, x g \cos\beta$$

or

$$\frac{dv_z}{dx} = -\frac{x g \cos\beta}{\nu_g}$$

where the subscript g is for glass. Another integration gives:

$$v_z = -\frac{x^2 g \cos\beta}{2\nu_g} + c_1, \qquad \delta_1 \le x \le 0.$$

In the metal, we rewrite Eq. (2.4) as

$$\frac{d^2 v_z}{dx^2} = -\frac{g \cos\beta}{\nu_m}$$

and integrate twice. The result is

$$v_z = -\frac{x^2 g \cos\beta}{2\nu_m} - c_2 x + c_3, \qquad \delta_1 \le x \le \delta_2$$

where the subscript m is for metal.

In order to evaluate the three constants, c_1, c_2 and c_3, we apply the following conditions:

B.C. 1: $v_z = 0$ at $x = \delta_2$,

B.C. 2: v_z (metal) $= v_z$ (glass) at $x = \delta_1$,

B.C. 3: τ_{xz} (metal) $= \tau_{xz}$ (glass) at $x = \delta_1$.

From these three conditions (and lines and lines of algebra), the three constants can be determined so that the velocity distribution in both layers can be found. The results are

$$c_2 = \frac{(\rho_g - \rho_m)\,\delta_1\, g \cos\beta}{\eta_m},$$

$$c_3 = \frac{\delta_2^2\, g \cos\beta}{2\nu_m} + c_2\,\delta_2,$$

and

$$c_1 = \left[\frac{\delta_2^2 - \delta_1^2}{\nu_m} + \frac{\delta_1^2}{\nu_g}\right]\frac{g\cos\beta}{2} + c_2(\delta_2 - \delta_1).$$

2.3 FULLY DEVELOPED FLOW BETWEEN PARALLEL PLATES

Consider the flow of fluid between parallel plates in Fig. 2.2. The velocity at the entrance is uniform and, as the flow progresses, velocity gradients must form because the fluid clings to the wall. At some distance downstream from the entrance, the velocity profile becomes independent of the distance from the entrance, and the flow is then fully developed. Let this region of fully developed flow start at $x = 0$ and consider the unit volume in Fig. 2.2 with a thickness Δy, width W, and length L.

Rate of momentum in across surface at y (momentum transport due to viscosity)	$(LW)(\tau_{yx})\|_y$
Rate of momentum out across surface at $y + \Delta y$ (due to viscosity)	$(LW)(\tau_{yx})\|_{y+\Delta y}$
Rate of momentum in across surface at $x = 0$ (due to fluid motion)	$(W\Delta y v_x)(\rho v_x)\|_{x=0}$
Rate of momentum out across surface at $x = L$ (due to fluid motion)	$(W\Delta y v_x)(\rho v_x)\|_{x=L}$
Pressure force on liquid at $x = 0$	$\Delta y W[P(x = 0)] = P_0 \Delta y W$
Pressure force on liquid at $x = L$	$-\Delta y W[P(x = L)] = -P_L \Delta y W$

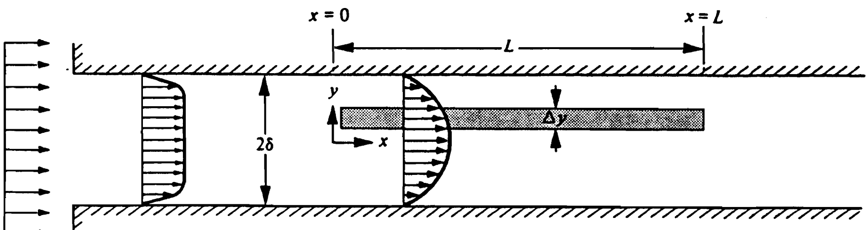


Fig. 2.2 Flow between parallel plates.

Again, the momentum in and out of the system due to the fluid motion are equal. We are left with

$$(LW)(\tau_{yx})\big|_y - (LW)(\tau_{yx})\big|_{y+\Delta y} + (P_0 - P_L)\Delta y W = 0. \tag{2.16}$$

Dividing through by $LW\Delta y$ and letting Δy approach zero, we develop the differential equation

$$\frac{d\tau_{yx}}{dy} = \frac{(P_0 - P_L)}{L}. \tag{2.17}$$

The boundary conditions are described at the centerline ($y = 0$) and at the solid wall ($y = \delta$) as follows:

$$\text{B.C. 1} \qquad \text{at } y = 0, \qquad \tau_{yx} = 0; \tag{2.18}$$

$$\text{B.C. 2} \qquad \text{at } y = \delta, \qquad v_x = 0.$$

It is left as an exercise for the reader to show that the shear stress distribution is given by

$$\tau_{yx} = \frac{(P_0 - P_L)}{L} y, \tag{2.19}$$

and the velocity distribution (for a Newtonian fluid) by

$$v_x = \frac{1}{2\eta}\left(\delta^2 - y^2\right)\frac{(P_0 - P_L)}{L}. \tag{2.20}$$

We determine other characteristics of the system by the method shown in Section 2.2. These are:

i) The maximum velocity

$$V_x^{\max} = \frac{1}{2\eta}\delta^2 \frac{(P_0 - P_L)}{L}. \tag{2.21}$$

ii) The average velocity

$$\bar{v}_x = \frac{1}{\delta}\int_0^\delta v_x\, dy = \frac{\delta^2}{3\eta}\frac{(P_0 - P_L)}{L}. \tag{2.22}$$

iii) The volume flow rate

$$Q = \frac{2}{3}\frac{W\delta^3}{\eta}\frac{(P_0 - P_L)}{L}. \tag{2.23}$$

On looking back through this example, we note that in this instance the fluid flows because of the pressure drop (P_0 - P_L). For horizontal flow, such a pressure drop would be necessary to make the fluid flow, in contrast to the flow down an inclined plane (Section 2.2) on which gravity exerts the necessary force for fluid motion.

2.4 FULLY DEVELOPED FLOW THROUGH A CIRCULAR TUBE

In this section we derive the momentum balance for steady flow through a long cylindrical tube for a Newtonian fluid and then for a non-Newtonian fluid, using an empirical equation that is often applied to polymeric melts.

2.4.1 Newtonian Fluids

Consider the fully developed flow of a fluid in a long tube of length L and radius R; we specify fully developed flow so that end effects are negligible. Since we are dealing with a pipe, it is convenient to work with cylindrical coordinates. Therefore the shell in Fig. 2.3 is cylindrical, of thickness Δr and length L.

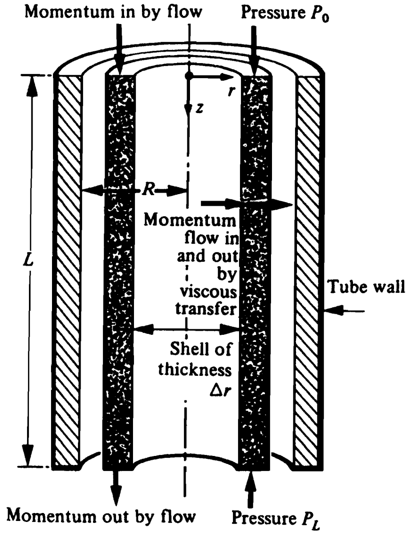


Fig. 2.3 Cylindrical shell chosen for momentum balance in tubes.

Rate of momentum in
across surface at r
(due to viscosity) $$(2\pi r L \tau_{rz})|_r$$

Note that here we include the area factor $(2\pi r L)$ in parentheses. This is because the area as well as the shear stress is a function of r.

Rate of momentum out
across surface at $r + \Delta r$
(due to viscosity) $$(2\pi r L \tau_{rz})|_{r+\Delta r}$$

Since we are considering fully developed flow, the momentum fluxes due to flow are equal; hence these terms are omitted.

Gravity force acting on the cylindrical shell $$(2\pi r \Delta r L)\rho g$$

Pressure force acting on surface at $z = 0$ $(2\pi r\Delta r)P_0$

Pressure force acting on surface at $z = L$ $-(2\pi r\Delta r)P_L$

We now add up the contributions to the momentum balance:

$$\left(2\pi rL\tau_{rz}\right)\big|_r - \left(2\pi rL\tau_{rz}\right)\big|_{r+\Delta r} + 2\pi r\Delta rL\rho g + 2\pi r\Delta r\left(P_0 - P_L\right) = 0. \tag{2.24}$$

Note that all terms contain the factor r; however, since r is a variable, it should not be used as a common divisor. By dividing through by $2\pi L\Delta r$ and taking the limit as Δr goes to zero, we develop the differential equation

$$\frac{d}{dr}\left(r\tau_{rz}\right) = \left[\frac{P_0 - P_L}{L} + \rho g\right] r. \tag{2.25}$$

Integration yields

$$\tau_{rz} = \left[\frac{P_0 - P_L}{L} + \rho g\right]\frac{r}{2} + \frac{C_1}{r}. \tag{2.26}$$

At $r = 0$, the velocity gradient (hence, the shear stress) equals zero; this can be realized because of the symmetry of flow.

Thus for this case,

$$\text{B.C. 1} \qquad \text{at } r = 0, \qquad \tau_{rz} = 0. \tag{2.27}$$

Therefore, $C_1 = 0$, and the momentum flux is given by

$$\tau_{rz} = \left[\frac{P_0 - P_L}{L} + \rho g\right]\frac{r}{2}. \tag{2.28}$$

Substituting Newton's law of viscosity

$$\tau_{rz} = -\eta\frac{dv_z}{dr}, \tag{2.29}$$

and noting

$$\text{B.C. 2} \qquad \text{at } r = R, \qquad v_z = 0, \tag{2.30}$$

we obtain the solution for the velocity distribution:

$$v_z = \left[\frac{P_0 - P_L}{L} + \rho g\right]\left[\frac{R^2}{4\eta}\right]\left[1 - \left[\frac{r}{R}\right]^2\right]. \tag{2.31}$$

As before:

i) The maximum velocity is at $r = 0$, and is given by

$$V_z^{\max} = \left[\frac{P_0 - P_L}{L} + \rho g\right]\frac{R^2}{4\eta}. \tag{2.32}$$

ii) The average velocity is

$$\bar{v}_z = \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R v_z r \, dr \, d\theta = \left[\frac{P_0 - P_L}{L} + \rho g \right] \frac{R^2}{8\eta}. \tag{2.33}$$

iii) The volume flow rate is

$$Q = \left[\frac{P_0 - P_L}{L} + \rho g \right] \left[\frac{\pi R^4}{8\eta} \right]. \tag{2.34}$$

This latter result, which is commonly referred to as the *Hagen-Poiseuille law*, is valid for laminar steady-state flow of incompressible fluids in tubes having sufficient length to make end effects negligible. An entrance length given by $L_e = 0.035\ D\text{Re}$ is required before we can establish fully developed parabolic velocity distribution.

Example 2.2 Water at 290 K flows through a horizontal tube of diameter 1.6 mm with a pressure drop of 900 N m^{-3}. Find the mass flow rate through the tube.

Solution. In this situation, the force of gravity does not act on the fluid in the direction of flow, so according to Eq. (2.34) the volume flow rate is

$$Q = \left[\frac{P_0 - P_L}{L} \right] \frac{\pi R^4}{8\eta}.$$

The viscosity and density of water at 290 K are 1.080×10^{-3} N s m^{-2} and 10^3 kg m^{-3}, respectively. Substituting in values, we obtain:

$$Q = \frac{900\ \text{N}}{\text{m}^3} \left| \frac{\pi}{8} \right| \frac{(0.8 \times 10^{-3})^4\ \text{m}^4}{} \left| \frac{\text{m}^2}{1.080 \times 10^{-3}\ \text{N s}} \right. = 1.34 \times 10^{-7}\ \text{m}^3\ \text{s}^{-1}$$

Thus we see that the mass flow rate is

$$\rho Q = (10^3)(1.34 \times 10^{-7}) = 1.34 \times 10^{-4}\ \text{kg s}^{-1}$$

We should then check if the flow is laminar, by evaluating the Reynolds number. As mentioned in Chapter 1, the criterion is Re < 2100:

$$\text{Re} = \frac{D\bar{V}}{\nu} = \frac{D\bar{V}\rho}{\eta}.$$

Also

$$\rho\bar{V} = \frac{\rho Q}{\pi D^2/4}.$$

so that the Reynolds number may be written in the alternative form

$$\mathrm{Re} = \frac{D\rho Q}{(\pi D^2/4)\eta} = \frac{4\rho Q}{\pi D\eta}.$$

Therefore,

$$\mathrm{Re} = \frac{4}{\pi}\left|\frac{1.34\times10^{-4}\ \mathrm{kg}}{\mathrm{s}}\right|\frac{}{1.6\times10^{-3}\ \mathrm{m}}\left|\frac{\mathrm{m}^2}{1.080\times10^{-3}\ \mathrm{N\,s}}\right|\frac{1\ \mathrm{N\,s}^2}{1\ \mathrm{kg\,m}}$$

$$= 77.6$$

Because Re < 2100, the flow is laminar.

2.4.2 Power law non-Newtonian fluids

Now we consider the isothermal flow of a non-Newtonian fluid through the circular tube of Fig. 2.3. The momentum balance is precisely the same as in Section 2.4.1 up to Eq. (2.28). We assume that the relationship between the shear stress and the shear rate is given by a power law, Eq. (1.32), as is often used for polymeric melts. Then

$$\tau_{rz} = -\eta\frac{\partial v_z}{\partial r} = -\eta_0\left[\frac{\partial v_z}{\partial r}\right]^n \tag{2.35}$$

and by combining Eqs. (2.28) and (2.35), we obtain

$$\left[\frac{\partial v_z}{\partial r}\right] = -\left[\frac{1}{2\eta_0}\left[\frac{P_0 - P_L}{L} + \rho g\right]\right]^{1/n} r^{1/n}. \tag{2.36}$$

Integrating with $v_z = 0$ at $r = R$, the velocity distribution is obtained:

$$v_z = \left[\frac{1}{2\eta_0}\left[\frac{P_0 - P_L}{L} + \rho g\right]\right]^{1/n}\left[\frac{n}{n+1}\right]\left[R^{(n+1)/n} - r^{(n+1)/n}\right]. \tag{2.37}$$

The maximum velocity is at $r = 0$ and is given by

$$V_z^{\max} = \left[\frac{1}{2\eta_0}\left[\frac{P_0 - P_L}{L} + \rho g\right]\right]^{1/n}\left[\frac{n}{n+1}\right] R^{(n+1)/n} \tag{2.38}$$

so that Eq. (2.37) can be written in a simpler form:

$$v_z = V_z^{\max}\left[1 - \left[\frac{r}{R}\right]^{(n+1)/n}\right]. \tag{2.39}$$

The volume flow rate is

$$Q = \int_0^R 2\pi\, r\, v_z\, dr = 2\pi\, V_z^{\max} \int_0^R r \left[1 - \left[\frac{r}{R} \right]^{(n+1)/n} \right] dr$$

or

$$Q = \left[\frac{n+1}{3n+1} \right] \pi R^2 V_z^{\max} . \tag{2.40}$$

The average velocity is simply

$$\bar{v}_z = \frac{Q}{\pi R^2}$$

or

$$\bar{v}_z = \left[\frac{n+1}{3n+1} \right] V_z^{\max} . \tag{2.41}$$

2.5 EQUATION OF CONTINUITY AND THE MOMENTUM EQUATION

In the previous sections of this chapter, we determined velocity distributions for some simple flow systems by applying differential momentum balances. The balances for these systems served to illustrate the application of the momentum equation. In general, when dealing with isothermal fluid systems which do not involve changes in compositions, we can solve problems by starting with general expressions. This method is better than developing formulations peculiar to the specific problem at hand. The general momentum equation is also called the *equation of motion* or the *Navier-Stokes' equation*; in addition the *equation of continuity* is frequently used in conjunction with the momentum equation.

The continuity equation is developed simply by applying the law of conservation of mass to a small volume element within a flowing fluid. The principle of conservation of mass is quite simple to apply and we assume that the reader has used it in developing material balances. We develop the momentum equation by applying the momentum balance which, in its general form, is an extension of Eq. (2.1). With the aid of these two equations, we can mathematically describe the problems encountered in the previous section, as well as more complicated problems. However, as we shall see, these expressions are rather cumbersome, and exact solutions can be found only in very limited cases. Hence these equations are used primarily as starting points for solving problems. The equations of continuity and motion are simplified to fit the problem at hand. Although theoretically these equations are valid for both laminar and turbulent flows, in practice they are applied only to laminar flow.

2.5.1 Equation of continuity

Consider the stationary volume element within a fluid moving with a velocity having the components v_x, v_y, and v_z, as shown in Fig. 2.4. We begin with the basic representation of the conservation of mass:

$$\begin{pmatrix} \text{rate of mass} \\ \text{accumulation} \end{pmatrix} = \begin{pmatrix} \text{rate of} \\ \text{mass in} \end{pmatrix} - \begin{pmatrix} \text{rate of} \\ \text{mass out} \end{pmatrix} . \tag{2.42}$$

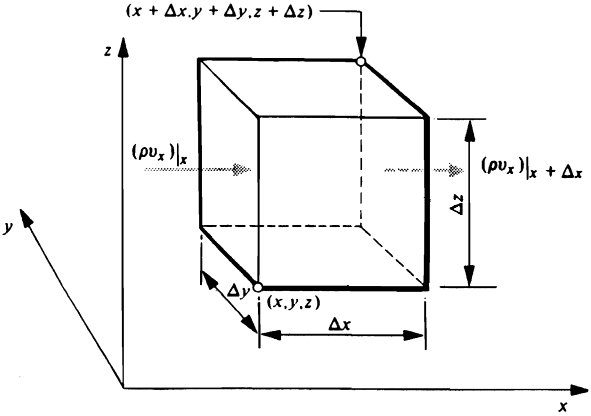


Fig. 2.4 Volume element fixed in space with fluid flowing through it.

First, look at the faces perpendicular to the x-axis. The volume flow rate of fluid in across the face at x is simply the product of the velocity (x-component) and the cross-sectional area, yielding $\Delta y \Delta z v_x|_x$. The rate of mass in through the face at x is then $\Delta y \Delta z(\rho v_x)|_x$. Similarly, the rate of mass out through the face at $x + \Delta x$ is $\Delta y \Delta z(\rho v_x)|_{x+\Delta x}$. We may write analogous expressions for the other two pairs of faces, and then enter all the terms that account for the fluid entering and leaving the system into the mass balance, and leave the accumulation term to be developed.

The *accumulation* is the rate of change of mass within the control volume

$$\Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t}.$$

The mass balance thus becomes

$$\Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t} = \Delta y \Delta z\left[\rho v_x|_x - \rho v_x|_{x+\Delta x}\right] + \Delta x \Delta z\left[\rho v_y|_y - \rho v_y|_{y+\Delta y}\right]$$

$$+ \Delta x \Delta y\left[\rho v_z|_z - \rho v_z|_{z+\Delta z}\right]. \qquad (2.43)$$

Then, dividing through by $\Delta x \Delta y \Delta z$, and taking the limit as these dimensions approach zero, we get the *equation of continuity*:

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial}{\partial x}\rho v_x + \frac{\partial}{\partial y}\rho v_y + \frac{\partial}{\partial z}\rho v_z\right]. \qquad (2.44)$$

A very important form of Eq. (2.44) is the form that applies to a fluid of constant density. For this case, which frequently occurs in engineering problems, the continuity equation reduces to

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

or in vector notation

$$\nabla \cdot \boldsymbol{v} = 0. \qquad (2.45)$$

2.5.2 The momentum equation

When Eq. (2.1) is extended to include unsteady-state systems, the momentum balance takes the form:

$$\begin{bmatrix}\text{rate of}\\ \text{momentum}\\ \text{accumulation}\end{bmatrix} = \begin{bmatrix}\text{rate of}\\ \text{momentum}\\ \text{in}\end{bmatrix} - \begin{bmatrix}\text{rate of}\\ \text{momentum}\\ \text{out}\end{bmatrix} + \begin{bmatrix}\text{sum of}\\ \text{forces acting}\\ \text{on system}\end{bmatrix}. \tag{2.46}$$

For simplicity, we begin by considering only the x-component of each term in Eq. (2.46); the y- and z-components may be handled in the same manner.

Figure 2.5(a) shows the x-components of τ as if they were made up of viscous momentum fluxes rather than shear stresses. On the other hand Fig. 2.5(b) shows the x-component of τ as stresses. Note the appearance of τ_{xx}, which by the scheme of subscripts represents the transport of x-momentum in the x-direction. Alternatively, we view τ_{xx} as the x-directed *normal* stress on the x-face, in contrast to τ_{yx} which we view as the x-directed *shear* stress on the y-face.

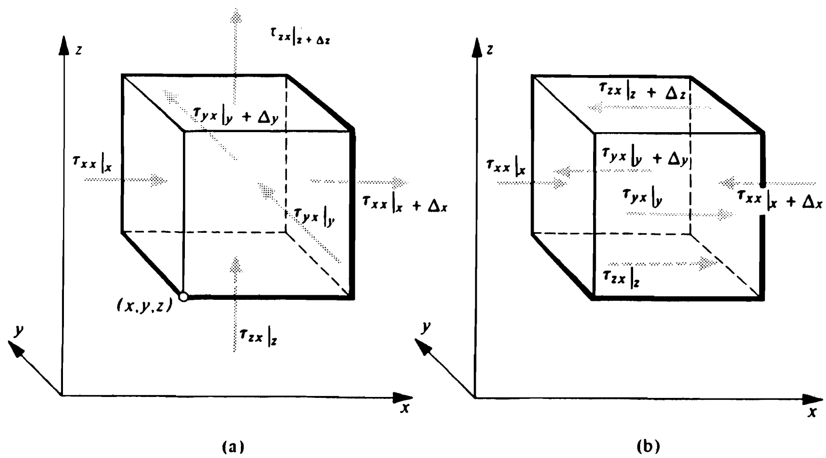


Fig. 2.5 Momentum transport (x-component) due to viscosity into the volume element. (a) Directions of viscous momentum transport. (b) Directions of stresses.

Let us now develop the terms that enter into Eq. (2.46). First, the net rate at which the x-component of the *convective* momentum enters the unit volume, is

$$\Delta y \Delta z\left(\rho v_x v_x|_x - \rho v_x v_x|_{x+\Delta x}\right) + \Delta x \Delta z\left(\rho v_y v_x|_y - \rho v_y v_x|_{y+\Delta y}\right)$$
$$+ \Delta x \Delta y\left(\rho v_z v_x|_z - \rho v_z v_x|_{z+\Delta z}\right). \tag{2.47}$$

Similarly, the net rate of *viscous* momentum flow into the unit volume across the six faces is

$$\Delta y \Delta z \left(\tau_{xx}\big|_x - \tau_{xx}\big|_{x+\Delta x}\right) + \Delta x \Delta z \left(\tau_{yx}\big|_y - \tau_{yx}\big|_{y+\Delta y}\right) + \Delta x \Delta y \left(\tau_{zx}\big|_z - \tau_{zx}\big|_{z+\Delta z}\right). \tag{2.48}$$

The reader who has not come in contact with this development before might find a brief explanation of the meaning of $\rho v_y v_x$ and $\rho v_z v_x$ useful. Remember that we are applying the law of conservation of momentum to the x-component of momentum. Thus v_x represents the x-velocity, and the rate at which mass enters the system through the y-face is given by $\Delta x \Delta z \rho v_y|_y$. Hence the rate at which x-momentum enters through the y-face is simply the product of mass-flow rate and velocity:

$$\Delta x \Delta z \rho v_y v_x|_y.$$

In most cases, the forces acting on the system are those arising from the pressure P and the gravitational force per unit mass $\boldsymbol{g}$. In the x-direction, these forces are

$$\Delta y \Delta z \left(P|_x - P|_{x+\Delta x}\right), \tag{2.49}$$

and

$$\rho g_x \Delta x \Delta y \Delta z, \tag{2.50}$$

respectively. Here g_x is the x-component of $\boldsymbol{g}$. Finally, the rate of accumulation of x-momentum within the element is

$$\Delta x \Delta y \Delta z \left[\frac{\partial}{\partial t} \rho v_x\right]. \tag{2.51}$$

Entering Eqs. (2.47)-(2.51) into the momentum balance, dividing through by $\Delta x \Delta y \Delta z$, and taking the limit as all three approach zero, we obtain the x-component of the momentum equation:

$$\frac{\partial}{\partial t} \rho v_x = -\left[\frac{\partial}{\partial x} \rho v_x v_x + \frac{\partial}{\partial y} \rho v_y v_x + \frac{\partial}{\partial z} \rho v_z v_x\right] - \left[\frac{\partial}{\partial x} \tau_{xx} + \frac{\partial}{\partial y} \tau_{yx} + \frac{\partial}{\partial z} \tau_{zx}\right] - \frac{\partial P}{\partial x} + \rho g_x. \tag{2.52}$$

The y- and z-components, which we obtain in a similar manner, are

$$\frac{\partial}{\partial t} \rho v_y = -\left[\frac{\partial}{\partial x} \rho v_x v_y + \frac{\partial}{\partial y} \rho v_y v_y + \frac{\partial}{\partial z} \rho v_z v_y\right] - \left[\frac{\partial}{\partial x} \tau_{xy} + \frac{\partial}{\partial y} \tau_{yy} + \frac{\partial}{\partial z} \tau_{zy}\right] - \frac{\partial P}{\partial y} + \rho g_y, \tag{2.53}$$

and

$$\frac{\partial}{\partial t}\rho v_z = -\left[\frac{\partial}{\partial x}\rho v_x v_z + \frac{\partial}{\partial y}\rho v_y v_z + \frac{\partial}{\partial z}\rho v_z v_z\right] - \left[\frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial y}\tau_{yz} + \frac{\partial}{\partial z}\tau_{zz}\right] - \frac{\partial P}{\partial z} + \rho g_z . \tag{2.54}$$

To describe the general case, all three Equations (2.52), (2.53), and (2.54) are needed. Vector notation can reduce these to one equation which is just as meaningful as all three. The quantities ρv_x, ρv_y, and ρv_z are the components of the mass velocity $\rho \boldsymbol{v}$; similarly g_x, g_y, and g_z are the components of $\boldsymbol{g}$. Vectorial representation of a velocity and an acceleration is familiar to most readers. However, the terms $\partial P/\partial x$, $\partial P/\partial y$, and $\partial P/\partial z$ all represent pressure *gradients*. By itself, pressure is a scalar quantity, but the gradient of pressure is a vector denoted, in general, by ∇P (sometimes written grad P). Therefore

$$\nabla P = \frac{\partial}{\partial x}P + \frac{\partial}{\partial y}P + \frac{\partial}{\partial z}P,$$

and ∇ can be thought to be an operator, such that

$$\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

The terms $\rho v_x v_x$, $\rho v_x v_y$, $\rho v_x v_z$, $\rho v_y v_z$, etc., are the nine components of the convective momentum flux $\rho \boldsymbol{v}\boldsymbol{v}$, which is the *dyadic product of* $\rho \boldsymbol{v}$ and $\boldsymbol{v}$. Also, τ_{xx}, τ_{xy}, etc., are the nine components of $\boldsymbol{\tau}$.

The vector equation representing Eqs. (2.52)-(2.54) is finally written:

$$\frac{\partial}{\partial t}\rho \boldsymbol{v} = -[\nabla \cdot \rho \boldsymbol{v}\boldsymbol{v}] - \nabla P - [\nabla \cdot \boldsymbol{\tau}] + \rho \boldsymbol{g}. \tag{2.55}$$

To interpret the mathematical nature of $\nabla \cdot \rho \boldsymbol{v}\boldsymbol{v}$ and $\nabla \cdot \boldsymbol{\tau}$ in physical terms is difficult. However, for sufficient understanding of this text it is enough if the reader accepts them as mathematical shorthands of the appropriate terms in Eqs. (2.52)-(2.54).

So far we have developed a general expression, namely, Eq. (2.55) for the law of conservation of momentum. However, in order to use this equation for the determination of velocity distributions, it is necessary to insert expressions for the various stresses in terms of velocity gradients and fluid properties. The following equations are presented without proof because the arguments involved are quite lengthy. For *Newtonian fluids*, the nine components of $\boldsymbol{\tau}$ are written as follows.[1]

Normal stresses:

$$\tau_{xx} = -2\eta \frac{\partial v_x}{\partial x} + \frac{2}{3}\eta(\nabla \cdot \boldsymbol{v}) \tag{2.56}$$

$$\tau_{yy} = -2\eta \frac{\partial v_y}{\partial y} + \frac{2}{3}\eta(\nabla \cdot \boldsymbol{v}) \tag{2.57}$$

$$\tau_{zz} = -2\eta \frac{\partial v_z}{\partial z} + \frac{2}{3}\eta(\nabla \cdot \boldsymbol{v}) \tag{2.58}$$

[1] V. L. Streeter, *Fluid Dynamics*, McGraw-Hill, New York, 1948, Chapter 10.

Shear stresses

$$\tau_{xy} = \tau_{yx} = -\eta \left[\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right] \tag{2.59}$$

$$\tau_{yz} = \tau_{zy} = -\eta \left[\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right] \tag{2.60}$$

$$\tau_{zx} = \tau_{xz} = -\eta \left[\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right] \tag{2.61}$$

These equations constitute a more general statement of Newton's law of viscosity than that given in Eq. (1.2), and apply to complex flow situations. When the fluid flows between two parallel plates in the x-direction so that v_x is a function of y alone, where the y-direction is perpendicular to the plates' surfaces (Fig. 1.4), then Eqs.(2.56)-(2.61) yield

$$\tau_{xx} = \tau_{yy} = \tau_{zz} = \tau_{yz} = \tau_{xz} = 0 \qquad \text{and} \qquad \tau_{yx} = -\eta \left[\frac{\partial v_x}{\partial y} \right],$$

which is the same as the simple relationship previously used to describe Newton's law of viscosity. Also in many other problems of physical significance in which v_x is recognized as a function of both y and x, we find that $\partial v_x/\partial y \gg \partial v_x/\partial x$, and the simple rate Eq. (1.2) can be used for τ_{yx} as an example with a high degree of accuracy rather than Eq. (2.59).

2.5.3 Navier-Stokes' equation, constant ρ and η

The continuity equation for constant density is given by Eq. (2.45) or in vector notation,

$$\nabla \cdot \boldsymbol{v} = 0. \tag{2.62}$$

Regarding the momentum equation, we can write Eqs. (2.52)-(2.54) with constant ρ and η:*

$$\rho \left[\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right] = -\frac{\partial P}{\partial x} + \eta \left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right] + \rho g_x, \tag{2.63}$$

$$\rho \left[\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right] = -\frac{\partial P}{\partial y} + \eta \left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right] + \rho g_y, \tag{2.64}$$

$$\rho \left[\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right] = -\frac{\partial P}{\partial z} + \eta \left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right] + \rho g_z. \tag{2.65}$$

*This development is the subject of Problem 2.11.

The bracketed terms on the left side of these equations merit attention. Consider a control volume of fluid moving in space with no mass flow across its surface. The change in the x-component of its velocity with time and position is given by

$$\Delta v_x = \frac{\partial v_x}{\partial t} \Delta t + \frac{\partial v_x}{\partial x} \Delta x + \frac{\partial v_x}{\partial y} \Delta y + \frac{\partial v_x}{\partial z} \Delta z, \tag{2.66}$$

and since the x-component of acceleration is defined as

$$a_x = \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t} = \lim_{\Delta t \to 0} \left\{ \frac{\partial v_x}{\partial t} \frac{\Delta t}{\Delta t} + \frac{\partial v_x}{\partial x} \frac{\Delta x}{\Delta t} + \frac{\partial v_x}{\partial y} \frac{\Delta y}{\Delta t} + \frac{\partial v_x}{\partial z} \frac{\Delta z}{\Delta t} \right\}, \tag{2.67}$$

then we obtain

$$a_x = \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} = \frac{Dv_x}{Dt}. \tag{2.68}$$

This is the acceleration one would feel if riding with the control volume of fluid. We also refer to this time derivative of velocity, Dv_x/Dt, as the *substantial* derivative. Analogous expressions exist for the y- and z-directions. In general, one notation can represent all three substantial derivatives, so that Eqs. (2.63)-(2.65) become

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla P + \eta \nabla^2 \mathbf{v} + \rho \mathbf{g}. \tag{2.69}$$

Equation (2.69), or Eqs. (2.63)-(2.65) which taken together represent the expansion of Eq. (2.69), is often referred to as the *Navier-Stokes' equation*. In the form of Eq. (2.69), we can recognize it as a statement of Newton's law in the form *mass* (ρ) $\times$ *acceleration* ($D\mathbf{v}/Dt$) equals the *sum of forces*, namely, the pressure force ($-\nabla P$), the viscous force ($\eta \nabla^2 \mathbf{v}$), and the gravity or body force $\rho \mathbf{g}$.

2.6 THE MOMENTUM EQUATION IN RECTANGULAR AND CURVILINEAR COORDINATES

In many instances rectangular coordinates are not useful for analyzing problems. For example, in the Hagen-Poiseuille problem discussed in Section 2.4, we described the axial velocity v_z as a function of only a single variable r by employing cylindrical coordinates. If rectangular coordinates had been used instead, v_z would have been a very complicated function of x and y. Similarly, it would have been difficult to describe and apply the boundary condition at the tube wall.

The equations of continuity and motion in Section 2.5 were given in rectangular coordinates and are repeated here, along with spherical or cylindrical coordinates in Tables 2.1-2.7.

Table 2.1 The continuity equation in different coordinates systems

Rectangular coordinates (x, y, z):

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0 \tag{A}$$

Cylindrical coordinates (r, θ, z):

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0 \tag{B}$$

Spherical coordinates (r, θ, ϕ):

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 v_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\rho v_\theta \sin\theta) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \phi}(\rho v_\phi) = 0 \tag{C}$$

*Tables 2.1-2.7 are from R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, Wiley, New York, 1960, pages 83-91. Reprinted by permission.

Table 2.2 The momentum equation in rectangular coordinates (x, y, z)

In terms of τ:

x-component
$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial P}{\partial x} - \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) + \rho g_x \quad \text{(A)}$$

y-component
$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial P}{\partial y} - \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) + \rho g_y \quad \text{(B)}$$

z-component
$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} - \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) + \rho g_z \quad \text{(C)}$$

In terms of velocity gradients for a Newtonian fluid with constant ρ and η:

x-component
$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial P}{\partial x} + \eta\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \rho g_x \quad \text{(D)}$$

y-component
$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial P}{\partial y} + \eta\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right) + \rho g_y \quad \text{(E)}$$

z-component
$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \eta\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho g_z \quad \text{(F)}$$

Table 2.3 The momentum equation in cylindrical coordinates (r, θ, z)

In terms of τ:

r-component*
$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial P}{\partial r} - \left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{rz}}{\partial z}\right) + \rho g_r \quad \text{(A)}$$

θ-component
$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} - \left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z}\right) + \rho g_\theta \quad \text{(B)}$$

z-component
$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} - \left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z}\right) + \rho g_z \quad \text{(C)}$$

In terms of velocity gradients for a Newtonian fluid with constant ρ and η:

r-component*
$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial P}{\partial r} + \eta\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right] + \rho g_r \quad \text{(D)}$$

θ-component
$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \eta\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right] + \rho g_\theta \quad \text{(E)}$$

z-component
$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \eta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z \quad \text{(F)}$$

*The term $\rho v_\theta^2\, r$ is the *centrifugal force*. It gives the effective force in the r-direction resulting from fluid motion in the θ-direction. This term arises automatically on transformation from rectangular to cylindrical coordinates; it does not have to be added on physical grounds.

Table 2.4 The momentum equation in spherical coordinates (r, θ, ϕ)

In terms of τ:

r-component
$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right)$$
$$= -\frac{\partial P}{\partial r} - \left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{rr}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\tau_{r\theta}\sin\theta)\right.$$
$$\left. + \frac{1}{r\sin\theta}\frac{\partial \tau_{r\phi}}{\partial\phi} - \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r}\right) + \rho g_r \qquad \text{(A)}$$

θ-component
$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2\cot\theta}{r}\right)$$
$$= -\frac{1}{r}\frac{\partial P}{\partial \theta} - \left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\tau_{\theta\theta}\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial \tau_{\theta\phi}}{\partial\phi}\right.$$
$$\left. + \frac{\tau_{r\theta}}{r} - \frac{\cot\theta}{r}\tau_{\phi\phi}\right) + \rho g_\theta \qquad \text{(B)}$$

ϕ-component
$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r}{r} + \frac{v_\theta v_\phi}{r}\cot\theta\right)$$
$$= -\frac{1}{r\sin\theta}\frac{\partial P}{\partial \phi} - \left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\phi}) + \frac{1}{r}\frac{\partial \tau_{\theta\phi}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial \tau_{\phi\phi}}{\partial\phi}\right.$$
$$\left. + \frac{\tau_{r\phi}}{r} + \frac{2\cot\theta}{r}\tau_{\theta\phi}\right) + \rho g_\phi \qquad \text{(C)}$$

In terms of velocity gradients for a Newtonian fluid with constant ρ and η:

r-component
$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right)$$
$$= -\frac{\partial P}{\partial r} + \eta\left(\nabla^2 v_r - \frac{2}{r^2}v_r - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta} - \frac{2}{r^2}v_\theta\cot\theta\right.$$
$$\left. - \frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right) + \rho g_r \qquad \text{(D)}$$

θ-component
$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2\cot\theta}{r}\right)$$
$$= -\frac{1}{r}\frac{\partial P}{\partial \theta} + \eta\left(\nabla^2 v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial\phi}\right) + \rho g_\theta \qquad \text{(E)}$$

ϕ-component
$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r}{r} + \frac{v_\theta v_\phi}{r}\cot\theta\right)$$
$$= -\frac{1}{r\sin\theta}\frac{\partial P}{\partial \phi} + \eta\left(\nabla^2 v_\phi - \frac{v_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\phi}\right.$$
$$\left. + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial\phi}\right) + \rho g_\phi \qquad \text{(F)}$$

Table 2.5 Components of the stress tensor in rectangular coordinates (x, y, z)

$$\tau_{xx} = -\eta\left[2\frac{\partial v_x}{\partial x} - \tfrac{2}{3}(\nabla \cdot v)\right] \quad \text{(A)}$$

$$\tau_{yy} = -\eta\left[2\frac{\partial v_y}{\partial y} - \tfrac{2}{3}(\nabla \cdot v)\right] \quad \text{(B)}$$

$$\tau_{zz} = -\eta\left[2\frac{\partial v_z}{\partial z} - \tfrac{2}{3}(\nabla \cdot v)\right] \quad \text{(C)}$$

$$\tau_{xy} = \tau_{yx} = -\eta\left[\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right] \quad \text{(D)}$$

$$\tau_{yz} = \tau_{zy} = -\eta\left[\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right] \quad \text{(E)}$$

$$\tau_{zx} = \tau_{xz} = -\eta\left[\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right] \quad \text{(F)}$$

$$(\nabla \cdot v) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \quad \text{(G)}$$

Table 2.6 Components of the stress tensor in cylindrical coordinates (r, θ, z)

$$\tau_{rr} = -\eta\left[2\frac{\partial v_r}{\partial r} - \tfrac{2}{3}(\nabla \cdot v)\right] \quad \text{(A)}$$

$$\tau_{\theta\theta} = -\eta\left[2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) - \tfrac{2}{3}(\nabla \cdot v)\right] \quad \text{(B)}$$

$$\tau_{zz} = -\eta\left[2\frac{\partial v_z}{\partial z} - \tfrac{2}{3}(\nabla \cdot v)\right] \quad \text{(C)}$$

$$\tau_{r\theta} = \tau_{\theta r} = -\eta\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right] \quad \text{(D)}$$

$$\tau_{\theta z} = \tau_{z\theta} = -\eta\left[\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right] \quad \text{(E)}$$

$$\tau_{zr} = \tau_{rz} = -\eta\left[\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right] \quad \text{(F)}$$

$$(\nabla \cdot v) = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \quad \text{(G)}$$

Table 2.7 Components of the stress tensor in spherical coordinates (r, θ, ϕ)

$$\tau_{rr} = -\eta\left[2\frac{\partial v_r}{\partial r} - \tfrac{2}{3}(\nabla \cdot v)\right] \quad \text{(A)}$$

$$\tau_{\theta\theta} = -\eta\left[2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) - \tfrac{2}{3}(\nabla \cdot v)\right] \quad \text{(B)}$$

$$\tau_{\phi\phi} = -\eta\left[2\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot\theta}{r}\right) - \tfrac{2}{3}(\nabla \cdot v)\right] \quad \text{(C)}$$

$$\tau_{r\theta} = \tau_{\theta r} = -\eta\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right] \quad \text{(D)}$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} = -\eta\left[\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right] \quad \text{(E)}$$

$$\tau_{\phi r} = \tau_{r\phi} = -\eta\left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right] \quad \text{(F)}$$

$$(\nabla \cdot v) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(v_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} \quad \text{(G)}$$

2.7 APPLICATION OF NAVIER-STOKES' EQUATION

In this section, we show how to set up problems of viscous flow, by selecting the appropriate equation of motion that applies to the problem at hand and by simplifying it to manageable proportions so that it still relates to the given problem, and yet is not oversimplified. We do so by discarding those terms which are zero, and then recognizing those terms which can be neglected. To decide this, is, to a certain extent, a matter of experience, but in most instances even the novice can make intelligent decisions by making an order-of-magnitude estimate. For this purpose, we shall discuss below an order-of-magnitude technique that can be used to arrive at a more simplified, but still relevant, equation of motion. We also introduce other topics, such as the *boundary layer* and *drag forces* exerted by fluids on solids.

2.7.1 Flow over a flat plate

Figure 2.6 depicts the velocity profile of a fluid flowing parallel to a flat plate. Before it meets the leading edge of the plate, we assume that the fluid has a uniform velocity V_∞. At any point x downstream from the leading edge of the plate, we observe that the velocity increases from zero at the wall to very near V_∞ at a very short distance δ from the plate. The loci of positions where $v_x/V_\infty = 0.99$, is δ, and it is defined as the *boundary layer*. At the leading edge of the plate ($x = 0$), δ is zero, and it progressively grows as flow proceeds down the plate.

Whenever problems of this type are encountered, namely, in the flow of fluid past a stationary solid, the viscous effects are felt only within the fluid near the solid, that is, $y < \delta$. Of course, this is exactly where the behavior of the fluid should be analyzed, because for $y > \delta$, the happenings from the point of view of this discussion are essentially uneventful,

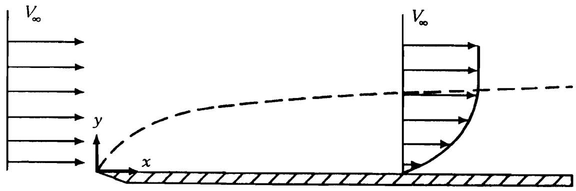


Fig. 2.6 Velocity profile and momentum boundary layer of flow parallel to a flat plate.

due to the fact that in this region v_x is essentially uniform and constant, being equal to V_∞. Since v_x is uniform and constant for $y > \delta$, Eq. (D) in Table 2.2 reveals that the pressure gradient $\partial P/\partial x$ is zero or, stated differently, pressure everywhere in the bulk stream is uniform. In turn, the pressure within the boundary layer is equal to the pressure in the bulk stream, so that $\partial P/\partial x$ is also zero within the boundary layer. We are now almost ready to pick out the appropriate equation of motion for the flow pattern in Fig. 2.6, but before we do so, let us examine which velocity components are relevant.

As discussed above, v_x is a function of y, and the determination of this functional relationship is indeed a major part of describing the flow and how the fluid and the solid surface interact. Also note that v_x depends on x. This results from the fact that, as the fluid progresses down the plate, it is retarded more and more by the drag at the plate's surface. Thus $\partial v_x/\partial x$ is not zero, and the equation of continuity for steady two-dimensional flow of fluid with constant ρ and η is

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0. \tag{2.70}$$

Thus v_y exists and we should consider both the x- and y-components in Table 2.2.

For the steady-state case with constant density and viscosity, Eqs. (D) and (E) in Table 2.2 reduce to

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right], \tag{2.71}$$

and

$$v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = \nu \left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} \right] + g_y. \tag{2.72}$$

When we remember that we are primarily interested in the region $y \leq \delta$, at this point it is convenient to define some dimensionless parameters:†

$$u^* = \frac{v_x}{V_\infty}, \quad x^* = \frac{x}{L}, \quad \delta^* = \frac{\delta}{L},$$

†The *Reynolds number* Re_L, which was briefly introduced in Chapter 1 reappears here again. Also the *Froude number* Fr_L, is introduced. These two dimensionless numbers, which so often occur in engineering studies, have been given names in honor of those two early workers in fluid mechanics.

$$v^* = \frac{v_y}{V_\infty}, \quad y^* = \frac{y}{L}, \quad \mathrm{Re}_L = \frac{V_\infty L}{\nu},$$

$$\mathrm{Fr}_L = \frac{V_\infty^2}{g_y L}.$$

Substituting these parameters into Eqs. (2.71), (2.72), and the continuity equation, we get

Continuity:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0. \tag{2.73}$$

Momentum:

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{1}{\mathrm{Re}_L}\left[\frac{\partial^2 u^*}{\partial (x^*)^2} + \frac{\partial^2 u^*}{\partial (y^*)^2}\right]; \tag{2.74}$$

$$u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = \frac{1}{\mathrm{Re}_L}\left[\frac{\partial^2 v^*}{\partial (x^*)^2} + \frac{\partial^2 v^*}{\partial (y^*)^2}\right] + \frac{1}{\mathrm{Fr}_L}. \tag{2.75}$$

The next step is to make order-of-magnitude estimates of the terms in Eqs. (2.74) and (2.75), realizing that we are interested only in the happenings within the boundary layer.

Order of magnitudes:

$$\Delta x^* \cong 0 \text{ to } 1. \tag{2.76}$$

$$\Delta y^* \cong \delta^*. \tag{2.77}$$

$$\Delta u^* \cong 1. \tag{2.78}$$

From (2.76) and (2.78), we get

$$\frac{\partial u^*}{\partial x^*} \cong \frac{1}{1} = 1, \tag{2.79}$$

and from continuity,

$$\frac{\partial v^*}{\partial y^*} \cong 1. \tag{2.80}$$

From (2.77) and (2.78), we obtain

$$\frac{\partial u^*}{\partial y^*} \cong \frac{1}{\delta^*}, \tag{2.81}$$

and from (2.79) and (2.80),

$$\frac{\partial v^*}{\partial x^*} \cong \frac{\delta^*}{1} = \delta^* . \tag{2.82}$$

Now we can estimate all the second derivatives:

$$\frac{\partial^2 u^*}{\partial (x^*)^2} = \frac{\partial}{\partial x^*}\left[\frac{\partial u^*}{\partial x^*}\right] \cong \frac{1}{1} = 1. \tag{2.83}$$

$$\frac{\partial^2 u^*}{\partial (y^*)^2} = \frac{\partial}{\partial y^*}\left[\frac{\partial u^*}{\partial y^*}\right] \cong \frac{1}{\delta^*}\left[\frac{1}{\delta^*}\right] = \frac{1}{\delta^{*2}}. \tag{2.84}$$

$$\frac{\partial^2 v^*}{\partial (x^*)^2} \cong \delta^* . \tag{2.85}$$

$$\frac{\partial^2 v^*}{\partial (y^*)^2} \cong \frac{1}{\delta^*}. \tag{2.86}$$

Insertion of the various magnitudes into Eqs. (2.74) and (2.75) reveals two important facts: $\partial^2 u^*/\partial(y^*)^2 \gg \partial^2 u^*/\partial(x^*)^2$, and the equation involving the x-component of the velocity has much larger terms than that for v_y. Hence we deal only with

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2}, \tag{2.87}$$

which is the *boundary layer* equation for a flat plate with zero pressure gradient. We now proceed to solve Eq. (2.87) for the boundary conditions

$$\text{B.C. 1} \quad \text{at } y = 0, \quad v_x = 0, \quad v_y = 0; \tag{2.88}$$

$$\text{B.C. 2} \quad \text{at } y = \infty, \quad v_x = V_\infty. \tag{2.89}$$

In order to simplify Eq. (2.87), we define the *stream function* ψ as

$$v_x \equiv \frac{\partial \psi}{\partial y} \quad \text{and} \quad v_y \equiv -\frac{\partial \psi}{\partial x}. \tag{2.90}$$

The use of the stream functions simplifies Eq. (2.87) and automatically satisfies continuity (Eq. (2.70)). Substituting Eq. (2.90) into Eq. (2.87) yields:

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \nu \frac{\partial^3 \psi}{\partial y^3}. \tag{2.91}$$

A *similarity argument*‡ shows that the stream function may be expressed as

$$\psi \equiv \sqrt{V_\infty \nu x}\, f(\beta), \tag{2.92}$$

‡Blasius showed that Eq. (2.91) could be solved in this manner. (See H. Blasius, *NACA Tech. Mem.*, 1949, page 1217.)

where

$$\beta = \frac{y}{\sqrt{x}} \left(V_\infty / \nu\right)^{1/2}, \tag{2.93}$$

From Eqs. (2.90), (2.92), and (2.93),

$$v_x = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \beta} \frac{\partial \beta}{\partial y} = V_\infty \frac{df}{d\beta}, \tag{2.94}$$

$$v_y = -\frac{\partial \psi}{\partial x} = \frac{1}{2} \left[\frac{\nu V_\infty}{x}\right]^{1/2} \left[\beta \frac{df}{d\beta} - f\right]. \tag{2.95}$$

Then Eq. (2.91) becomes

$$2 \frac{d^3 f}{d\beta^3} + f \frac{d^2 f}{d\beta^2} = 0. \tag{2.96}$$

Mathematically, the use of ψ and β has reduced a partial differential equation to an ordinary differential equation with the boundary conditions also taking equivalent forms:

$$\text{B.C. 1} \quad \text{at } \beta = 0, \quad f = 0, \quad \frac{df}{d\beta} = 0; \tag{2.97}$$

$$\text{B.C. 2} \quad \text{at } \beta = \infty, \quad \frac{df}{d\beta} = 1. \tag{2.98}$$

Equation (2.96) may be solved by expressing $f(\beta)$ in a power series, that is, $f = \sum_0^\infty a_k \beta^k$. The technique is too involved to develop here, but the solution conforming to the boundary conditions becomes

$$f = \frac{\alpha\beta^2}{2!} - \frac{1}{2}\frac{\alpha^2\beta^5}{5!} + \frac{11}{4}\frac{\alpha^3\beta^8}{8!} + \dots. \tag{2.99}$$

where $\alpha = 0.332$. Then Eqs. (2.94) and (2.95) give expressions for v_x and v_y; the solution for v_x is shown graphically in Fig. 2.7. The position, where $v_x/V_\infty = 0.99$, is located at $\beta = 5.0$; thus the boundary-layer thickness δ is

$$\delta = 5.0 \left[\frac{\nu x}{V_\infty}\right]^{1/2}. \tag{2.100}$$

Note that if we divide Eq. (2.100) by x, both sides become dimensionless:

$$\frac{\delta}{x} = 5.0 \left[\frac{\nu}{x V_\infty}\right]^{1/2} = \frac{5.0}{\sqrt{\text{Re}_x}}. \tag{2.101}$$

Note also that as a result of the analysis, the Reynolds number ($\text{Re}_x = xV_\infty/\nu$) has evolved; in this instance we give Re the subscript x in order to emphasize that it is a local value with

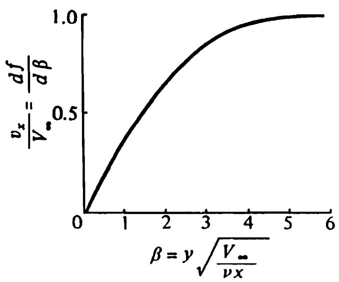


Fig. 2.7 Solution for the velocity distribution in the boundary layer over a flat plate. (From L. Howarth, *Proc. Roy. Soc.*, London **A164**, 547 (1938).)

the *characteristic dimension x*. We can also calculate the *drag force*, which is exerted by the fluid on the plate's surface. If the plate has a length L and width W, the drag force F_K is

$$F_K = \int_0^W \int_0^L \left[\eta \frac{\partial v_x}{\partial y}\right]_{y=0} dxdz. \tag{2.102}$$

In other words, the shear stress at the solid surface is integrated over the entire surface. From Fig. 2.7 we find that

$$\left[\frac{\partial(v_x/V_\infty)}{\partial\beta}\right]_{\beta=0} = 0.332. \tag{2.103}$$

Knowing the integrand in Eq. (2.102), we can now perform the integration. The result is

$$F_K = 0.664\sqrt{\rho\eta L W^2 V_\infty^3}. \tag{2.104}$$

This is the drag force exerted by the fluid on one surface only.

2.7.2 Flow in inlet of circular tubes

In Section 2.4, we considered the flow of fluid in a long tube so that end effects were negligible. Now we wish to study the flow conditions at the inlet where the flow is not fully developed. The fluid enters the tube with a uniform velocity V_0 in the z-direction. The important component for this system is the z-component, just as the x-component is most important for the flow past a flat plate. According to Eq. (F) in Table 2.3, the momentum equation with $v_\theta = 0$, $\partial v_z/\partial t = 0$, and $g_z = 0$, reduces to

$$v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial v_z}{\partial r}\right] + \frac{\partial^2 v_z}{\partial z^2}\right]$$

Again, the viscous effect in the direction parallel to flow is negligible, so that $\partial^2 v_z/\partial z^2 \cong 0$, and we are left with

$$v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \frac{\nu}{r}\frac{\partial}{\partial r}\left[r \frac{\partial v_z}{\partial r}\right] \tag{2.105}$$

We can deduce the equation of continuity from Eq. (B) in Table 2.1:

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{\partial v_z}{\partial z} = 0. \tag{2.106}$$

The method for the solution of Eq. (2.105) is not given here, but one should understand why Eq. (2.105) is the starting point. Langhaar[2] has developed the solution for this problem, as described in Fig. 2.8. His analysis shows that a fully developed flow is not established until $\nu z/R^2 V_0 \cong 0.07$. Thus an entrance length ($z = L_e$) of approximately 0.035 $(D^2 V_0 \rho)/\eta$ is required for buildup to the parabolic profile of the fully developed flow.

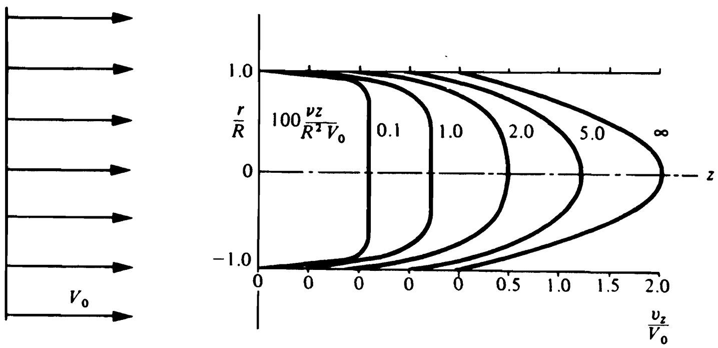


Fig. 2.8 Velocity distribution for laminar flow in the inlet section of a tube.

2.7.3 Creeping flow around a solid sphere

Consider the flow of an incompressible fluid about a solid sphere (Fig. 2.9). The fluid approaches the sphere upward along the z-axis with a uniform velocity V_∞. Clearly, the momentum equation for this situation does not involve the ϕ-component. In addition, if the flow is slow enough, the acceleration terms in Navier-Stokes' equation can be ignored. Therefore, in spherical coordinates, from Eqs. (D) and (E) in Table 2.4, we obtain for the r-component:

$$-\frac{\partial P}{\partial r} + \eta\left[\nabla^2 v_r - \frac{2}{r^2} v_r - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{2}{r^2} v_\theta \cot\theta\right] + \rho g_r = 0, \tag{2.107}$$

[2]H. L. Langhaar, *J. Appl. Mech.* **9**, A55-58 (1942).

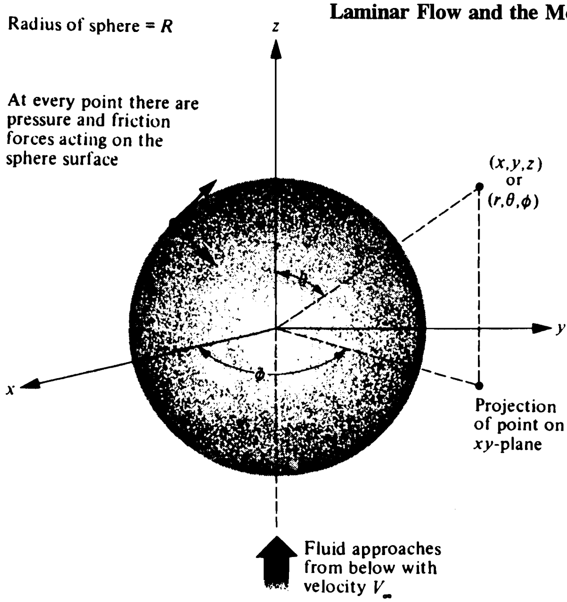


Fig. 2.9 Coordinate system used in describing the flow of a fluid about a rigid sphere.

and for the θ-component:

$$-\frac{1}{r}\frac{\partial P}{\partial \theta} + \eta\left[\nabla^2 v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2 \sin^2\theta}\right] + \rho g_\theta = 0. \quad (2.108)$$

The continuity equation (Eq. (C) in Table 2.1) is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(v_\theta \sin\theta\right) = 0. \quad (2.109)$$

The momentum flux-distribution, pressure distribution, and velocity components have been found analytically:[3]

$$\tau_{r\theta} = \frac{3}{2}\frac{\eta V_\infty}{R}\left[\frac{R}{r}\right]^4 \sin\theta, \quad (2.110)$$

$$P = P_0 - \rho g z - \frac{3}{2}\frac{\eta V_\infty}{R}\left[\frac{R}{r}\right]^2 \cos\theta, \quad (2.111)$$

$$v_r = V_\infty\left[1 - \frac{3}{2}\left[\frac{R}{r}\right] + \frac{1}{2}\left[\frac{R}{r}\right]^3\right]\cos\theta. \quad (2.112)$$

[3]V. L. Streeter, *Fluid Dynamics*, McGraw-Hill, New York, 1948, pages 235-240.

$$v_\theta = -V_\infty \left[1 - \frac{3}{4}\left[\frac{R}{r}\right] - \frac{1}{4}\left[\frac{R}{r}\right]^3\right] \sin\theta. \tag{2.113}$$

We can check the validity of the results by showing that Eqs. (2.107)-(2.109) and the following conditions are satisfied:

B.C. 1 at $r = R$, $v_r = 0 = v_\theta$;

B.C. 2 at $r = \infty$, $v_z = V_\infty$.

In Eq. (2.111), the quantity P_0 is the pressure in the plane $z = 0$ far away from the sphere, $-\rho g z$ is simply the hydrostatic effect, and the term containing V_∞ results from the fluid flow around the sphere. These equations are valid for a Reynolds number (DV_∞/ν) less than approximately unity.

With these results, we can calculate the net force which is exerted by the fluid on the sphere. This force is computed by integrating the normal force and tangential force over the sphere surface as follows.

The *normal force* acting on the solid surface is due to the pressure given by Eq. (2.111) with $r = R$ and $z = R\cos\theta$. Thus

$$P(r = R) = P_0 - \rho g R\cos\theta - \frac{3}{2}\frac{\eta V_\infty}{R}\cos\theta.$$

The z-component of this pressure multiplied by the surface area on which it acts, $R^2\sin\theta\, d\theta d\phi$, is integrated over the surface of the sphere to yield the net force due to the pressure difference:

$$F_n = \int_0^{2\pi}\int_0^{\pi}\left[P_0 - \rho g R\cos\theta - \frac{3}{2}\frac{\eta V_\infty}{R}\cos\theta\right] R^2\sin\theta\, d\theta d\phi. \tag{2.114}$$

Equation (2.114), integrated, yields two terms:

$$F_n = \frac{4}{3}\pi R^3\rho g + 2\pi\eta R V_\infty, \tag{2.115}$$

the *buoyant force* and *form drag*, respectively.

At each point on the surface, there is also a shear stress acting tangentially, $-\tau_{r\theta}$, which is the force acting on a unit surface area. The z-component of this force is $(-\tau_{r\theta})(-\sin\theta)R^2\sin\theta\, d\theta d\phi$; again, integration over the sphere's surface yields

$$F_t = \int_0^{2\pi}\int_0^{\pi}\left(\tau_{r\theta}\big|_{r=R}\sin\theta\right)R^2\sin\theta\, d\theta d\phi.$$

From Eq. (2.110), we get

$$\tau_{r\theta}\big|_{r=R} = \frac{3}{2}\frac{\eta V_\infty}{R}\sin\theta,$$

so that the *friction drag* results:

$$F_t = 4\pi\eta R V_\infty. \tag{2.116}$$

The total force F of the fluid on the sphere is given by the sum of Eqs. (2.115) and (2.116):

$$F = \frac{4}{3}\pi R^3 \rho g + 6\pi\eta R V_\infty . \tag{2.117}$$

These two terms are designated as F_s (the force exerted even if the fluid is stationary) and F_K (the force associated with fluid movement). Thus these forces are

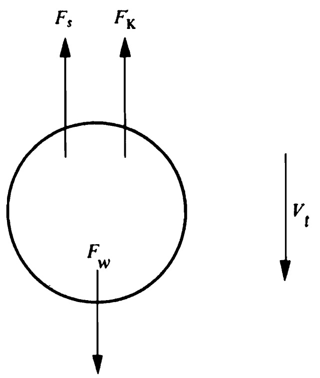
$$F_s = \frac{4}{3}\pi R^3 \rho g, \tag{2.118}$$

$$F_K = 6\pi\eta R V_\infty . \tag{2.119}$$

We use Eq. (2.119), known as *Stokes' law*, primarily for determining the terminal velocity, V_t of small spherical particles moving through fluid media. The fluid media are stagnant; the spherical particle moves through the fluid, and V_∞ is viewed as V_t. With this in mind, we may use Stokes' law as the basis of a *falling-sphere* viscometer, in which the liquid is placed in a tall transparent cylinder and a sphere of known mass and diameter is dropped into it. The terminal velocity of the falling sphere can be measured, and this in turn relates to the fluid's viscosity.

Example 2.3 Apply Stokes' law to the falling sphere viscometer and write an expression for the viscosity of the liquid in the viscometer.

Solution. A force balance on the sphere, as it falls through the liquid, is made according to the diagram:



Here F_s is the buoyant force exerted by the liquid and is therefore directed upward; F_K is often called the *drag force* and as such always acts in the opposite direction to that of motion and is therefore directed upward. The only force in the downward direction is the weight of the sphere. The terminal velocity is reached when the force system is in equilibrium. Therefore $F_s + F_K = F_w$, and by substituting in expressions for each of these forces, we have

$$\frac{4}{3}\pi R^3 \rho g + 6\pi\eta R V_t = \frac{4}{3}\pi R^3 \rho_s g. \tag{2.120}$$

where ρ_s is the sphere's density.

By solving Eq. (2.120) for η, we arrive at

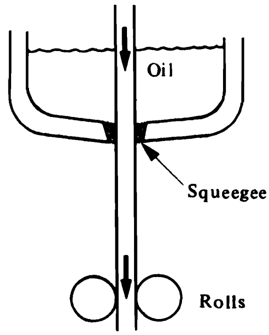
$$\eta = \frac{2R^2(\rho_s - \rho)g}{9V_t}. \tag{2.121}$$

The result is valid only if $2RV_t/v$ is less than approximately unity.

PROBLEMS

2.1 Refer to the results of Example 2.1. The viscosity of the glass is 1 N s m^{-2}, and the viscosity of the metal is 3 × 10^{-3} N s m^{-2}. The densities of the glass and the metal are 3.2 kg m^{-3} and 7.0 kg m^{-3}, respectively. For $\beta = \pi/8$ and $\delta_1 = 1$ mm and $\delta_2 = 2$ mm, calculate the maximum velocities and average velocities of the glass and the metal.

2.2 A continuous sheet of metal is cold-rolled by passing vertically between rolls. Before entering the rolls, the sheet passes through a tank of lubricating oil equipped with a squeegee device that coats both sides of the sheet uniformly as it exits. The amount of oil that is carried through can be controlled by adjusting the squeegee device. Prepare a control chart that can be used to determine the thickness of oil (in mm) on the plate just before it enters the roll as a function of the mass rate of oil (in kg per hour). Values of interest for the thickness of the oil film range from 0-0.6 mm. *Data*: Oil density, 962 kg m^{-3}; oil viscosity, 4.1 × 10^{-3} N s m^{-2}; width of sheet, 1.5 m; velocity of sheet, 0.3 m s^{-1}.



2.3 A Newtonian liquid flows simultaneously through two parallel and vertical channels of different geometries. Channel "A" is circular with a radius R, and "B" is a slit of thickness 2δ and width W; $2\delta \ll W$. Assume fully developed flow in both channels and derive an equation which gives the ratio of the volume flow rate through A to that through B.

2.4 Develop expressions for the flow of a fluid between vertical parallel plates. The plates are separated by a distance of 2δ. Consider fully developed flow and determine
a) the velocity distribution,
b) the volume flow rate.
Compare your expressions with Eqs. (2.20) and (2.23).

2.5 Repeat Problem 2.4 but now orient the plates at an angle β to the direction of gravity and obtain expressions for
a) the velocity distribution,
b) the volume flow rate.
Compare your expressions with the results of Problem 2.4 and Eqs. (2.20) and (2.23).

2.6 A liquid is flowing through a vertical tube 0.3 m long and 2.5 mm in I.D. The density of the liquid is 1260 kg m^{-3} and the mass flow rate is 3.8 × 10^{-5} kg s^{-1}.
a) What is the viscosity in N s m^{-2}?
b) Check on the validity of your results.

2.7 Water (viscosity 10^{-3} N s m^{-2}) flows parallel to a flat horizontal surface. The velocity profile at $x = x_1$ is given by

$$v_x = 6 \sin \left[\frac{\pi}{2} \right] y$$

with v_x in m s^{-1} and y as distance from surface in mm.

a) Find the shear stress at the wall at x_1. Express results in N m^{-2}.
b) Farther downstream, at $x = x_2$, the velocity profile is given by

$$v_x = 4 \sin \left[\frac{\pi}{2} \right] y$$

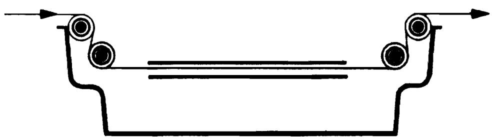
Is the flow "fully developed"? Explain.
c) Is there a y-component to flow (i.e., v_y)? Explain with the aid of the continuity equation.

2.8 For a polymeric melt that follows a power law for shear stress versus shear strain rate, derive an equation for the velocity profile and volume flow rate for flow between parallel plates.

2.9 The power law polymer of Problem 2.8 has constants $\eta_0 = 1.2 \times 10^4$ N s m^{-2} and $n = 0.35$. It is injected through a gate into a thin cavity, which has a thickness of 2 mm, a width of 10 mm and a length of 20 mm. If the injection rate is constant at 200 mm^3 s^{-1}, estimate the time to fill the cavity and the injection pressure at the gate.

2.10 A wire is cooled after a heat treating operation by being pulled through the center of an open-ended, oil-filled tube which is immersed in a tank. In a region in the tube where end effects are negligible, obtain an expression for the velocity profile assuming steady state and all physical properties constant.

Tube inner radius: R
Wire radius: KR
Wire velocity: U



2.11 Starting with the x-component of the momentum equation (Eq. (2.52)), develop the x-component for the Navier-Stokes' equation (constant ρ and η, (Eq. 2.63)).

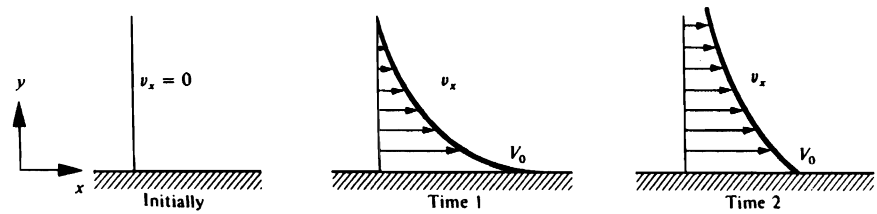
2.12 Air at 289 K flows over a flat plate with a velocity of 9.75 m s^{-1}. Assume laminar flow and a) calculate the boundary-layer thickness 50 mm from the leading edge; b) calculate the rate of growth of the boundary layer at that point; i.e., what is $d\delta/dx$ at that point? *Properties of air at 289 K*: density: 1.22 kg m^{-3}; viscosity: 1.78×10^{-5} N s m^{-2}.

2.13 A fluid flows upward through a vertical cylindrical annulus of length L. Assume that the flow is fully developed. The inner radius of the annulus is κR, and the outer radius is R. a) Write the momentum equation in terms of velocity. b) Solve for the velocity profile. c) Solve for the maximum velocity.

2.14 In steelmaking, deoxidation of the melt is accomplished by the addition of aluminum, which combines with the free oxygen to form alumina, Al_2O_3. It is then hoped that most of these alumina particles will float up to the slag layer for easy removal from the process, because their presence in steel can be detrimental to mechanical properties. Determine the size of the smallest alumina particles that will reach the slag layer from the bottom of the steel two minutes after the steel is deoxidized. It may be assumed that the alumina particles are spherical in nature. For the purpose of estimating the steel's viscosity use the data for Fe-0.5 wt pct C in Fig. 1.11. *Data*: Temperature of steel melt: 1873 K; steel melt depth: 1.5 m; density of steel: 7600 kg m^{-3}; density of alumina: 3320 kg m^{-3}.

2.15

a) Consider a very large flat plate bounding a liquid that extends to $y = +\infty$. Initially, the liquid and the plate are at rest; then suddenly the plate is set into motion with velocity V_0 as shown in the figure below. Write (1) the pertinent differential equation in terms of velocity, for constant properties, that applies from the instant the plate moves, and (2) the appropriate boundary and initial conditions. The solution to these equations will be discussed in Chapter 9.



b) A liquid flows upward through a long vertical conduit with a square cross section. With the aid of a clearly labeled sketch, write (1) a pertinent differential equation that describes the flow for constant properties, and (2) the appropriate boundary conditions. Consider only that portion of the conduit where flow is fully developed and be sure that your sketch and equations correspond to one another.

2.16 Molten aluminum is degassed by gently bubbling a 75%N_2-25%Cl_2 gas through the melt. The gas passes through a graphite tube at a volumetric flow rate of 6.6×10^{-5} m^3 s^{-1}. Calculate the pressure that should be maintained at the tube entrance if the pressure over the bath is 1.014×10^5 N m^{-2} (1 atm). *Data*: Tube dimensions: $L = 0.9$ m; inside diameter = 2 mm. Temperature of aluminum melt is 973 K; density of aluminum is 2500 kg m^{-3}.

2.17 Glass flows through a small orifice by gravity to form a fiber. The free-falling fiber does not have a uniform diameter; furthermore as it falls through the air it cools so that its viscosity changes. a) Write the momentum equation for this situation. b) Write appropriate boundary conditions.

2.18 A liquid flows upward through a tube, overflows, and then flows downward as a film on the outside.

a) Develop the pertinent momentum balance that applies to the falling film, for steady-state laminar flow, neglecting end effects.

b) Develop an expression for the velocity distribution.