Chapter X
Brownian Motion: The LIL and Some Fine-Scale Properties

In this chapter, we analyze the *growth* of the Brownian paths $t \mapsto B_t$ as $t \to \infty$. We will see by a property of "time inversion" of Brownian motion that this leads to small-scale properties as well. First, however, let us record some basic properties of the Brownian motion that follow somewhat directly from its definition.

Theorem 10.1 Let $B = \{B_t : t \geq 0\}$ be a standard one-dimensional Brownian motion starting at 0. Then

1. *(Symmetry)* $W_t := -B_t, t \geq 0$, is a standard Brownian motion starting at 0.
2. *(Homogeneity and Independent Increments)* $\{B_{t+s} - B_s : t \geq 0\}$ is a standard Brownian motion independent of $\{B_u : 0 \leq u \leq s\}$, for every $s \geq 0$.
3. *(Scale-Change Invariance).* For every $\lambda > 0$, $\{B_t^{(\lambda)} := \lambda^{-\frac{1}{2}} B_{\lambda t} : t \geq 0\}$ is a standard Brownian motion starting at 0.
4. *(Time-Inversion Invariance)* $W_t := t B_{1/t}, t > 0$, $W_0 = 0$, is a standard Brownian motion starting at 0.

Proof Each of these is obtained by showing that the conditions defining a Brownian motion are satisfied. In the case of the time-inversion property, one may apply the strong law of large numbers to obtain continuity at $t = 0$. That is, if $0 < t_n \to 0$ then write $s_n = 1/t_n \to \infty$ and $N_n := [s_n]$, where $[\cdot]$ denotes the greatest integer function, so that by the strong law of large numbers, with probability one

$$W_{t_n} = \frac{1}{s_n} B_{s_n} = \frac{N_n}{s_n} \frac{1}{N_n} \sum_{j=1}^{N_n} (B_i - B_{i-1}) + \frac{1}{s_n}(B_{s_n} - B_{N_n}) \to 0,$$

since $B_i - B_{i-1}$, $i \geq 1$, is an i.i.d. mean-zero sequence, $N_n/s_n \to 1$, and $(B_{s_n} - B_{N_n})/s_n \to 0$ a.s. as $n \to \infty$ (see Exercise 2). ■

Although the Brownian motion paths cannot be differentiable, it is possible to determine an *order of continuity* using the next general theorem.

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Definition 10.1 A stochastic process (or random field) $Y = \{Y_u : u \in \Lambda\}$ is a *version* of $X = \{X_u : u \in \Lambda\}$ taking values in a metric space if Y has the same finite dimensional distributions as X.

Theorem 10.2 *(Kolmogorov-Chentsov Theorem)* Suppose $X = \{X_u : u \in \Lambda\}$ is a stochastic process (or random field) with values in a complete metric space (S, ρ), indexed by a bounded rectangle $\Lambda \subset \mathbb{R}^k$ and satisfying

$$\mathbb{E}\rho^{\alpha}(X_u, X_v) \leq c|u - v|^{k+\beta}, \quad \text{for all } u, v \in \Lambda,$$

where c, α, β are positive numbers. Then there is a version $Y = \{Y_u : u \in \Lambda\}$ of X which is a.s. Hölder continuous of any exponent γ such that $0 < \gamma < \frac{\beta}{\alpha}$.

Proof Without essential loss of generality we take $\Lambda = [0, 1]^k$ and the norm $|\cdot|$ to be the *maximum norm* given by $|u| = \max\{|u_i| : 1 \leq i \leq k\}$, $u = (u_1, \ldots, u_k)$. For each $N = 1, 2, \ldots$, let L_N be the finite lattice $\{j2^{-N} : j = 0, 1, \ldots 2^N\}^k$. Write $L = \cup_{N=1}^{\infty} L_N$. Define $M_N = \max\{\rho(X_u, X_v) : (u, v) \in L_N^2, |u - v| \leq 2^{-N}\}$. Since (i) for a given $u \in L_N$ there are no more than 3^k points in L_N such that $|u - v| \leq 2^{-N}$, (ii) there are $(2^N + 1)^k$ points in L_N, and (iii) for every given pair (u, v), the condition of the theorem holds, one has by Chebyshev's inequality that

$$P(M_N > 2^{-\gamma N}) \leq c3^k(2^N + 1)^k(\frac{2^{-N(k+\beta)}}{2^{-\alpha\gamma N}}). \tag{10.1}$$

In particular, since $\gamma < \beta/\alpha$,

$$\sum_{N=1}^{\infty} P(M_N > 2^{-\gamma N}) < \infty. \tag{10.2}$$

Thus there is a random positive integer $N^* \equiv N^*(\omega)$ and a set Ω^* with $P(\Omega^*) = 1$, such that

$$M_N(\omega) \leq 2^{-\gamma N} \quad \text{for all } N \geq N^*(\omega), \omega \in \Omega^*. \tag{10.3}$$

Fix $\omega \in \Omega^*$ and let $N \geq N^*(\omega)$. By exactly the same induction argument as used for the proof of Lemma 3 in Chapter VII, one has for all $m \geq N + 1$,

$$\rho(X_u, X_v) \leq 2\sum_{j=N}^{m} 2^{-\gamma j}, \quad \text{for all } u, v \in L_m, |u - v| \leq 2^{-N}. \tag{10.4}$$

Since $2\sum_{\nu=N}^{\infty} 2^{-\gamma\nu} = 2^{-\gamma N+1}(1 - 2^{-\gamma})^{-1}$, and $L = \cup_{m=N+1}^{\infty} L_m$ for all $N \geq N^*(\omega)$, it follows that

$$\begin{aligned}&\sup\{\rho(X_u, X_v) : u, v \in L, |u-v| \le 2^{-N}\}\\ &= \sup\{\rho(X_u, X_v) : u, v \in \cup_{m=N+1}^{\infty} L_m, |u-v| \le 2^{-N}\}\\ &\le 2^{-\gamma N+1}(1-2^{-\gamma})^{-1}, \quad N \ge N^*(\omega), \omega \in \Omega^*. \end{aligned} \tag{10.5}$$

This proves that on Ω^*, $u \to X_u$ is uniformly continuous (from L into (S, ρ)), and is Hölder continuous with exponent γ. Now define $Y_u := X_u$ if $u \in L$ and otherwise $Y_u := \lim X_{u_N}$, with $u_N \in L$ and $u_N \to u$, if $u \notin L$. Because of uniform continuity of $u \to X_u$ on L (for $\omega \in \Omega^*$), and completeness of (S, ρ), the last limit is well-defined. For all $\omega \notin \Omega^*$, let Y_u be a fixed element of S for all $u \in [0, 1]^k$. Finally, letting $\gamma_j \uparrow \beta/\alpha$, $\gamma_j < \beta/\alpha$, $j \ge 1$, and denoting the exceptional set above as Ω_j^*, one has the Hölder continuity of Y for every $\gamma < \beta/\alpha$ on $\Omega^{**} := \cap_{j=1}^{\infty} \Omega_j^*$ with $P(\Omega^{**}) = 1$.

That Y is a version of X may be seen as follows. For any $r \ge 1$ and r vectors $u_1, \ldots, u_r \in [0, 1]^k$, there exist $u_{jN} \in L$, $u_{jN} \to u_j$ as $N \to \infty$ $(1 \le j \le r)$. Then $(X_{u_{1N}}, \ldots, X_{u_{rN}}) = (Y_{u_{1N}}, \ldots, Y_{u_{rN}})$ a.s., and $(X_{u_{1N}}, \ldots, X_{u_{rN}}) \to (X_{u_1}, \ldots, X_{u_r})$ in probability, $(Y_{u_{1N}}, \ldots, Y_{u_{rN}}) \to (Y_{u_1}, \ldots, Y_{u_r})$ almost surely. ■

Corollary 10.3 *(Brownian Motion)* Let $X = \{X_t : t \ge 0\}$ be a real-valued Gaussian process defined on $(\Omega, \mathcal{F}, P)$, with $X_0 = 0$, $\mathbb{E}X_t = 0$, and $\mathrm{Cov}(X_s, X_t) = s \wedge t$, for all $s, t \ge 0$. Then X has a version $B = \{B_t : t \ge 0\}$ with continuous sample paths, which are Hölder continuous on every bounded interval with exponent γ for every $\gamma \in (0, \frac{1}{2})$.

Proof Since $\mathbb{E}|X_t - X_s|^{2m} = c(m)(t-s)^m$, $0 \le s \le t$, for some constant $c(m)$, for every $m > 0$, the Kolmogorov–Chentsov Theorem 10.2 implies the existence of a version $B^{(0)} = \{B_t^{(0)} : 0 \le t \le 1\}$ with the desired properties on $[0, 1]$. Let $B^{(n)}$, $n \ge 1$, be independent copies of $B^{(0)}$, indedpendent of $B^{(0)}$. Define $B_t = B_t^{(0)}$, $0 \le t \le 1$, and $B_t = B_1^{(0)} + \cdots + B_1^{(n-1)} + B_{t-[t]}^{(n)}$, for $t \in [n, n+1)$, $n = 1, 2, \ldots$. ■

Corollary 10.4 *(Brownian Sheet)* Let $X = \{X_u : u \in [0, \infty)^2\}$ be a real-valued Guassian random field satisfying $\mathbb{E}X_u = 0$, $\mathrm{Cov}(X_u, X_v) = (u_1 \wedge v_1)(u_2 \wedge v_2)$ for all $u = (u_1, u_2)$, $v = (v_1, v_2)$. Then X has a continuous version on $[0, \infty)^2$, which is Hölder continuous on every bounded rectangle contained in $[0, \infty)^2$ with exponent γ for every $\gamma \in (0, \frac{1}{2})$.

Proof First let us note that on every compact rectangle $[0, M]^2$, $\mathbb{E}|X_u - X_v|^{2m} \le c(M)|u-v|^m$, for all $m = 1, 2, \ldots$. For this it is enough to check that on each horizontal line $u = (u_1, c)$, $0 \le u_1 < \infty$, X_u is a one-dimensional Brownian motion with mean zero and variance parameter $\sigma^2 = c$ for $c \ge 0$. The same holds on vertical lines. Hence $\mathbb{E}|X_{(u_1,u_2)} - X_{(v_1,v_2)}|^{2m} \le 2^{2m-1}\big(\mathbb{E}|X_{(u_1,u_2)} - X_{(v_1,u_2)}|^{2m} + \mathbb{E}|X_{(v_1,u_2)} - X_{(v_1,v_2)}|^{2m}\big) \le 2^{m-1}c(m)\big(u_2^m|u_1 - v_1|^m + v_1^m|u_2 - v_2|^m\big) \le 2^{m-1}c(m) M^m 2|u-v|^m$, where $u = (u_1, u_2)$, $v = (v_1, v_2)$. ■

Remark 10.1 One may define the *Brownian sheet* on the index set $\Lambda_{\mathcal{R}}$ of all rectangles $R = [u, v)$, with $u = (u_1, u_2)$, $v = (v_1, v_2)$, $0 \le u_i \le v_i < \infty$ $(i = 1, 2)$, by setting

$$X_R \equiv X_{[u,v)} := X_{(v_1,v_2)} - X_{(v_1,u_2)} - X_{(u_1,v_2)} + X_{(u_1,u_2)}. \tag{10.6}$$

Then X_R is Gaussian with mean zero and variance $|R|$, the area of R. Moreover, if R_1 and R_2 are nonoverlapping rectangles, then X_{R_1} and X_{R_2} are independent. More generally, $\mathrm{Cov}(X_{R_1}, X_{R_2}) = |R_1 \cap R_2|$. Conversely, given a Gaussian family $\{X_R : R \in \Lambda_{\mathcal{R}}\}$ with these properties, one can restrict it to the class of rectangles $\{R = [0, u) : u = (u_1, u_2) \in [0, \infty)^2\}$ and identify this with the Brownian sheet in Corollary 10.4. It is simple to check that for all n-tuples of rectangles $R_1, R_2, \ldots, R_n \subset [0, \infty)^2$, the matrix $((|R_i - R_j|))_{1\le i,j\le n}$ is symmetric and non-negative definite. So the finite dimensional distributions of $\{X_R : R \in \Lambda_{\mathcal{R}}\}$ satisfy Kolmogorov's consistency condition.

In order to prove our main result of this section, we will make use of the following important inequality due to Paul Lévy.

Proposition 10.5 *(Lévy's Inequality)* Let X_j, $j = 1, \ldots, N$, be independent and symmetrically distributed (about zero) random variables. Write $S_j = \sum_{i=1}^{j} X_i$, $1 \le j \le N$. Then, for every $y > 0$,

$$P\left(\max_{1\le j\le N} S_j \ge y\right) \le 2P(S_N \ge y) - P(S_N = y) \le 2P(S_N \ge y).$$

Proof Write $A_j = [S_1 < y, \ldots, S_{j-1} < y,\ S_j \ge y]$, for $1 \le j \le N$. The events $[S_N - S_j < 0]$ and $[S_N - S_j > 0]$ have the same probability and are independent of A_j. Therefore

$$\begin{aligned}
P\left(\max_{1\le j\le N} S_j \ge y\right) &= P(S_N \ge y) + \sum_{j=1}^{N-1} P(A_j \cap [S_N < y]) \\
&\le P(S_N \ge y) + \sum_{j=1}^{N-1} P(A_j \cap [S_N - S_j < 0]) \\
&= P(S_N \ge y) + \sum_{j=1}^{N-1} P(A_j)P([S_N - S_j < 0]) \\
&= P(S_N \ge y) + \sum_{j=1}^{N-1} P(A_j \cap [S_N - S_j > 0]) \\
&\le P(S_N \ge y) + \sum_{j=1}^{N-1} P(A_j \cap [S_N > y]) \\
&\le P(S_N \ge y) + P(S_N > y) \\
&= 2P(S_N \ge y) - P(S_N = y).
\end{aligned} \tag{10.7}$$

This establishes the basic inequality. ■

Corollary 10.6 For every $y > 0$ one has for any $t > 0$,

$$P\left(\max_{0\le s\le t} B_s \ge y\right) \le 2P(B_t \ge y).$$

Proof Partition $[0, t]$ by equidistant points $0 < u_1 < u_2 < \cdots < u_N = t$, and let $X_1 = B_{u_1}, X_{j+1} = B_{u_{j+1}} - B_{u_j}, 1 \le j \le N-1$, in the proposition. Now let $N \to \infty$, and use the continuity of Brownian motion. ■

In fact one may use a reflection principle argument (strong Markov property) to see that this inequality is sharp for Brownian motion

$$P(\max_{0\le s\le t} B_s \ge y) = 2P(B_t \ge y). \tag{10.8}$$

Alternatively, the following proposition concerns the **simple symmetric random walk** defined by $S_0 = 0, S_j = X_1 + \cdots + X_j, j \ge 1$, with $X_1, X_2, \ldots$ i.i.d. ±1-valued with equal probabilities. It also demonstrates the remarkable strength of the **reflection method**, allowing one in particular to compute the distribution of the maximum of a random walk over a finite time. The above-indicated equality (10.8) then becomes a consequence of the functional central limit theorem proved in Section 1.8, (Theorem 7.15); especially see (9.27).

Proposition 10.7 For the simple symmetric random walk one has for every positive integer y,

$$P\left(\max_{0\le j\le N} S_j \ge y\right) = 2P(S_N \ge y) - P(S_N = y).$$

Proof In the notation of Lévy's inequality given in Proposition 10.5 one has, for the present case of the random walk moving by ±1 units at a time, that $A_j = [S_1 < y, \ldots, S_{j-1} < y, S_j = y]$, $1 \le j \le N$. Then in (10.7) the probability inequalities are all equalities for this special case. ■

Corollary 10.8 Equation (10.8) holds for every $y > 0, t > 0$.

Theorem 10.9 *(Law of the Iterated Logarithm (LIL) for Brownian Motion)* Each of the following holds with probability one:

$$\overline{\lim}_{t\to\infty}\frac{B_t}{\sqrt{2t\log\log t}} = 1, \qquad \underline{\lim}_{t\to\infty}\frac{B_t}{\sqrt{2t\log\log t}} = -1.$$

Proof Let $\varphi(t) := \sqrt{2t\log\log t}$, $t > 0$. Let us first show that for any $0 < \delta < 1$, one has with probability one that

$$\overline{\lim}_{t\to\infty}\frac{B_t}{\varphi(t)} \le 1 + \delta. \tag{10.9}$$

For arbitrary $\alpha > 1$, partition the time interval $[0, \infty)$ into subintervals of exponentially growing lengths $t_{n+1} - t_n$, where $t_n = \alpha^n$, and consider the event

$$E_n := \left[\max_{t_n \le t \le t_{n+1}} \frac{B_t}{(1+\delta)\varphi(t)} > 1\right].$$

Since $\varphi(t)$ is a nondecreasing function, one has, using Corollary 10.6, a scaling property, and Lemma 2 from Chapter IV, that

$$\begin{aligned} P(E_n) &\le P\left(\max_{0\le t\le t_{n+1}} B_t > (1+\delta)\varphi(t_n)\right) \\ &= 2P\left(B_1 > \frac{(1+\delta)\varphi(t_n)}{\sqrt{t_{n+1}}}\right) \\ &\le \sqrt{\frac{2}{\pi}}\frac{\sqrt{t_{n+1}}}{(1+\delta)\varphi(t_n)} e^{-\frac{(1+\delta)^2\varphi^2(t_n)}{2t_{n+1}}} \le c\frac{1}{n^{(1+\delta)^2/\alpha}} \end{aligned} \tag{10.10}$$

for a constant $c > 0$ and all $n > \frac{1}{log\alpha}$. For a given $\delta > 0$ one may select $1 < \alpha < (1+\delta)^2$ to obtain $P(E_n\ i.o.) = 0$ from the Borel–Cantelli lemma (Part I). Thus we have (10.9). Since $\delta > 0$ is arbitrary we have with probability one that

$$\overline{\lim}_{t\to\infty} \frac{B_t}{\varphi(t)} \le 1. \tag{10.11}$$

Next let us show that with probability one,

$$\overline{\lim}_{t\to\infty} \frac{B_t}{\varphi(t)} \ge 1. \tag{10.12}$$

For this consider the independent increments $B_{t_{n+1}} - B_{t_n}$, $n \ge 1$. For $\theta = \frac{t_{n+1}-t_n}{t_{n+1}} = \frac{\alpha-1}{\alpha} < 1$, using Feller's tail probability estimate (Lemma 2, Chapter IV) and Brownian scale change,

$$\begin{aligned} P\left(B_{t_{n+1}} - B_{t_n} > \theta\varphi(t_{n+1})\right) &= P\left(B_1 > \sqrt{\frac{\theta}{t_{n+1}}}\varphi(t_{n+1})\right) \\ &\ge \frac{c'}{\sqrt{2\theta\log\log t_{n+1}}} e^{-\theta\log\log t_{n+1}} \\ &\ge \frac{c}{\sqrt{\log n}} n^{-\theta} \end{aligned} \tag{10.13}$$

for suitable positive constants c, c' depending on α and for all $n > \frac{1}{log\alpha}$. It follows from the Borel–Cantelli Lemma (Part II) that with probability one,

$$B_{t_{n+1}} - B_{t_n} > \theta\varphi(t_{n+1}) \ i.o. \tag{10.14}$$

Also, by (10.11) and replacing $\{B_t : t \geq 0\}$ by the standard Brownian motion $\{-B_t : t \geq 0\}$,

$$\underline{\lim}_{t\to\infty} \frac{B_t}{\varphi(t)} \geq -1, \ a.s. \tag{10.15}$$

Since $t_{n+1} = \alpha t_n > t_n$, we have

$$\frac{B_{t_{n+1}}}{\sqrt{2t_{n+1}\log\log t_{n+1}}} = \frac{B_{t_{n+1}} - B_{t_n}}{\sqrt{2t_{n+1}\log\log t_{n+1}}} + \frac{1}{\sqrt{\alpha}} \frac{B_{t_n}}{\sqrt{2t_n(\log\log t_n + \log\log\alpha)}}. \tag{10.16}$$

Now, using (10.14) and (10.15), it follows that with probability one,

$$\overline{\lim}_{n\to\infty} \frac{B_{t_{n+1}}}{\varphi(t_{n+1})} \geq \theta - \frac{1}{\sqrt{\alpha}} = \frac{\alpha-1}{\alpha} - \frac{1}{\sqrt{\alpha}}. \tag{10.17}$$

Since $\alpha > 1$ may be selected arbitrarily large, one has with probability one that

$$\overline{\lim}_{t\to\infty} \frac{B_t}{\varphi(t)} \geq \overline{\lim}_{n\to\infty} \frac{B_{t_{n+1}}}{\varphi(t_{n+1})} \geq 1. \tag{10.18}$$

This completes the computation of the limit superior. To get the limit inferior simply replace $\{B_t : t \geq 0\}$ by $\{-B_t : t \geq 0\}$. ■

The time inversion property for Brownian motion turns the law of the iterated logarithm (LIL) into a statement concerning the degree (or lack) of *local smoothness.* (Also see Exercise 7).

Corollary 10.10 Each of the following holds with probability one:

$$\overline{\lim}_{t\to 0} \frac{B_t}{\sqrt{2t\log\log\frac{1}{t}}} = 1, \qquad \underline{\lim}_{t\to 0} \frac{B_t}{\sqrt{2t\log\log\frac{1}{t}}} = -1.$$

Exercise Set X

1. (*Ornstein–Uhlenbeck Process*) Fix parameters $\gamma > 0, \sigma > 0, x \in \mathbb{R}$. Use the Kolmogorov–Chentsov theorem to obtain the existence of a continuous Gaussian process $X = \{X_t : t \geq 0\}$ starting at $X_0 = x$ with $\mathbb{E}X_t = xe^{-\gamma t}$, and $\mathrm{Cov}(X_s, X_t) = \frac{\sigma^2}{\gamma} e^{-\gamma t}\sinh(\gamma s), 0 < s \leq t$.

2. (i) Use Feller's tail estimate (Lemma 2, Chapter IV). to prove that $\max\{|B_i - B_{i-1}| : i = 1, 2, \ldots, N+1\}/N \to 0$ a.s. as $N \to \infty$.
   (ii) Without using the law of the iterated logarithm for standard Brownian motion B, show directly that $\limsup_{n\to\infty} \frac{B_n}{\sqrt{2n\log n}} \leq 1$ almost surely.

3. Show that with probability one, standard Brownian motion has arbitrarily large zeros. [*Hint*: Apply the LIL.]
4. Fix $t \geq 0$ and use the law of the iterated logarithm to show that $\lim_{h\to 0} \frac{B_{t+h}-B_t}{h}$ exists with probability zero. [*Hint*: Check that $Y_h := B_{t+h} - B_t, h \geq 0$, is distributed as standard Brownian motion starting at 0. Consider $\frac{1}{h}Y_h = \frac{Y_h}{\sqrt{2h\log\log(1/h)}} \frac{\sqrt{2h\log\log(1/h)}}{h}$.]
5. For the simple symmetric random walk, find the distributions of the extremes: (a) $M_N = \max\{S_j : j = 0, \ldots, N\}$, and (b) $m_N = \min\{S_j : 0 \leq j \leq N\}$.
6. Consider the simple symmetric random walk $S_0 = 0, S_n = X_1 + \cdots + X_n, n \geq 1$, where $X_k, k \geq 1$, are iid symmetric Bernoulli ± 1 valued random variables. Denote the range by $R_n = \max_{m\leq n} S_m - \min_{m\leq n} S_m, n \geq 1$. Show that $\frac{R_n}{\sqrt{n}}$ converges in distribution to a nonnegative random variable as $n \to \infty$.
7. (*Lévy Modulus of Continuity*[1]) Use the wavelet construction $B_t := \sum_{n,k} Z_{n,k} S_{n,k}(t), 0 \leq t \leq 1$, of standard Brownian motion to establish the following fine-scale properties.

   (i) Let $0 < \delta < \frac{1}{2}$. With probability one there is a random constant K such that if $|t-s| \leq \delta$ then $|B_t - B_s| \leq K\sqrt{\delta \log \frac{1}{\delta}}$. [*Hint*: Fix N and write the increment as a sum of three terms: $B_t - B_s = Z_{00}(t-s) + \sum_{n=0}^{N} \sum_{k=2^n}^{2^{n+1}-1} Z_{n,k} \int_s^t H_{n,k}(u)du + \sum_{n=N+1}^{\infty} \sum_{k=2^n}^{2^{n+1}-1} Z_{n,k} \int_s^t H_{n,k}(u)du = a + b + c$. Check that for a suitable (random) constant K' one has $|b| \leq |t-s|K' \sum_{n=0}^{N} n^{\frac{1}{2}} 2^{\frac{n}{2}} \leq |t-s|K' \frac{\sqrt{2}}{\sqrt{2}-1}\sqrt{N}2^{\frac{N}{2}}$, and $|c| \leq K' \sum_{n=N+1}^{\infty} n^{\frac{1}{2}} 2^{-\frac{n}{2}} \leq K' \frac{\sqrt{2}}{\sqrt{2}-1}\sqrt{N}2^{-\frac{N}{2}}$. Use these estimates, taking $N = [-\log_2(\delta)]$ such that $\delta 2^N \sim 1$, to obtain the bound $|B_t - B_s| \leq |Z_{00}|\delta + 2K'\sqrt{-\delta\log_2(\delta)}$. This is sufficient since $\delta < \sqrt{\delta}$.]

   (ii) The modulus of continuity is sharp in the sense that with probability one, there is a sequence of intervals $(s_n, t_n), n \geq 1$, of respective lengths $t_n - s_n \to 0$ as $n \to \infty$ such that the ratio $\frac{B_{t_n} - B_{s_n}}{\sqrt{-(t_n-s_n)\log(t_n-s_n)}}$ is bounded below by a positive constant. [*Hint*: Use Borel–Cantelli I together with Feller's tail probability estimate for the Gaussian distribution to show that $P(A_n\ i.o.) = 0$, where $A_n := [|B_{k2^{-n}} - B_{(k-1)2^{-n}}| \leq c\sqrt{n2^{-n}}, k = 1, \ldots, 2^n]$ and c is fixed in $(0, \sqrt{2\log 2})$. Interpret this in terms of the certain occurrence of the complimentary event $[A_n\ i.o.]^c$.]

   (iii) The paths of Brownian motion are a.s. nowhere differentiable.

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[1] The calculation of the modulus of continuity for Brownian motion is due to Lévy, P. (1937). However this exercise follows Pinsky, M. (1999): Brownian continuity modulus via series expansions, *J. Theor. Probab.* **14** (1), 261–266.