L-Fuzzy Bags

Fateme Kouchakinejad, Mashaallah Mashinchi and Radko Mesiar

Abstract This chapter studies L-fuzzy bags and some of its applications in which L is a complete lattice. Furthermore, the concepts of α-cuts, (L-fuzzy) bag relations and related theorems are given. The chapter ends with the characterization of the algebraic structure of bags and L-fuzzy bags.

1 Introduction

The theory of bags, an alternative name for multisets, as a natural extension of the set theory was introduced by Yager [19]. So far, bags have been employed in practice; for example, in flexible querying [16], representation of relational information [19], decision problem analysis [2], criminal career analysis [8], and in biology [13]. As another example, bags can play the role of primary data bases in the real world problems. As a matter of fact, all of information should be considered in the data mining tasks [6], and in particular in the fuzzy clustering where each data point has a membership degree in each cluster. So, from the mathematical point of view, each cluster should be considered as a fuzzy bag, see [18]. Some other applications can be found in [3, 11, 14–17]. However, due to some existing drawbacks in the first definition of bags [19], the necessity of a revision of this notion has grown. The definitions proposed by Delgado et al. [5] for bags and fuzzy bags have improved these drawbacks. As it is shown in [9], there is some incompatibility with the nature of fuzziness in the

F. Kouchakinejad (✉)
Department of Mathematics, Graduate University of Advanced Technology, Kerman, Iran
e-mail: kouchakinezhad@gmail.com

M. Mashinchi
Department of Statistics, Faculty of Mathematics and Computer Sciences, Shahid Bahonar University of Kerman, Kerman, Iran
e-mail: mashinchi@uk.ac.ir

R. Mesiar
Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, 810 05 Bratislava, Slovak Republic
e-mail: radko.mesiar@stuba.sk

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fuzzy bag's definition in [5]. The proposed definition for fuzzy bags in [10] resolved this problem. In this chapter, we summarize our recent results concerning bags and L-fuzzy bags from [9, 10] adding several examples and observations.

The chapter is structured as follows. In the next section, basic definitions and results concerning bags and L-fuzzy bags are reviewed. Section 3 deals with relations on bags and L-fuzzy bags. In Sect. 4, the α-cuts of L-fuzzy bags are studied. Section 5 brings the characterization of the algebraic structure of bags and L-fuzzy bags. Finally, some concluding remarks are added.

2 Definitions

It should be mentioned that, in general, non-empty sets P and O can be arbitrary (finite or infinite) but they are considered to be finite in this chapter. Throughout this chapter, $I_n = \{1, 2, \ldots, n\}$, where $n \in \mathcal{N}$ and $\mathcal{N}$ is the set of natural numbers. Also, P and O are two finite universes (sets) called "properties" and "objects", respectively. We have the following definitions.

Definition 1 ([5]) A (crisp) bag $\mathcal{B}^f$ is a pair (f, B^f), where $f : P \to \mathcal{P}(O)$ is a function and B^f is the following subset of $P \times \mathcal{N}_0$

$$B^f = \{(p, card(f(p)))|p \in P\}.$$

Here, $\mathcal{P}(O)$ is the power set of O, $\mathcal{N}_0 = \mathcal{N} \cup \{0\}$, $card(X)$ is the cardinality of set X.

We will use the convention that $card(\emptyset) = 0$ if necessary. Also, we will not distinguish $\{(p, card(f(p))), p \in P\}$ and $\{(p, card(f(p))), p \in P, f(p) \neq \emptyset\}$.

Note 1 For the sake of simplicity, whenever $f(p) = \emptyset$ we may not write $(p, 0)$ in the set B^f.

In this characterization, a bag $\mathcal{B}^f$ consists of two parts. The first one is the function f that can be seen as an information source about the relation between objects and properties. The second part B^f is a summary of the information in f obtained by means of the count operation $card(.)$. This summary corresponds to the classical view of bags in the sense of [19]. Observe that, up to trivial cases, the knowledge of B^f is not enough to recover the original information source f (this was the main drawback of the original approach to bags in [19]). Obviously, f determines $\mathcal{B}^f$ univocally. However, we prefer to keep the notation (f, B^f) for bags as proposed in [4, 5] due to the higher transparency and link to the original notion of bags given in [19].

Notation 1 $\mathbf{B}(P, O)$ is the set of all bags $\mathcal{B}^f = (f, B^f)$ defined in Definition 1.

Table 1 Several functions: age-people

p	17	21	27	35
$f_1(p)$	{Bill, Sue}	{John, Tom}	∅	∅
$f_2(p)$	{Bill, Sue}	{John, Tom, Stan}	∅	{Ben}
$f_3(p)$	∅	{Stan}	{Ana}	{Ben}
$f_4(p)$	{Bill}	{John, Stan}	∅	∅
$f_5(p)$	{John, Tom}	{Ana, Stan}	∅	∅

Definition 2 We have $\mathcal{B}^0 = (0, B^0)$ and $\mathcal{B}^1 = (1, B^1)$ where, $0(p) = \emptyset$, $1(p) = O$ for all $p \in P$, $B^0 = \{(p, 0), p \in P\}$ and $B^1 = \{(p, card(O)), p \in P\}$. Clearly, $\mathcal{B}^0, \mathcal{B}^1 \in \mathbf{B}(P, O)$.

Example 1 ([4]) Let $O = \{$John, Ana, Bill, Tom, Sue, Stan, Ben$\}$ and $P = \{17, 21, 27, 35\}$ be the set of objects and the set of properties, respectively. Let $f_1, f_2, f_3, f_4, f_5 : P \to \mathcal{P}(O)$ be the functions in Table 1 with $f_i(p) \subseteq O$ for all $p \in P$. So, we can define bags $\mathcal{B}^{f_i} = (f_i, B^{f_i})$, $1 \le i \le 5$. Where,
$B^{f_1} = \{(17, 2), (21, 2)\}$,
$B^{f_2} = \{(17, 2), (21, 3), (35, 1)\}$,
$B^{f_3} = \{(21, 1), (27, 1), (35, 1)\}$,
$B^{f_4} = \{(17, 1), (21, 2)\}$ and
$B^{f_5} = \{(17, 2), (21, 2)\}$.

Now, we can define some binary operations between bags.

Definition 3 ([5]) Let $* \in \{\cup, \cap, \setminus\}$. Then

$$\mathcal{B}^f * \mathcal{B}^g = \mathcal{B}^{f*g} = (f * g, B^{f*g}),$$

where $f * g : P \to \mathcal{P}(O)$ such that $(f * g)(p) = f(p) * g(p)$ for all $p \in P$.

Example 2 ([4]) We can obtain some new bags from operations among bags in Example 1, where their functions are shown in Table 2 and the corresponding summaries are as follows
$B^{f_1 \cup f_2} = \{(17, 2), (21, 3), (35, 1)\}$,
$B^{f_2 \cap f_3} = \{(21, 1), (35, 1)\}$,
$B^{f_1 \setminus f_3} = \{(17, 2), (21, 2)\}$,
$B^{f_3 \setminus f_2} = \{(27, 1)\}$,
$B^{f_1 \cup f_5} = \{(17, 4), (21, 4)\}$,
$B^{f_1 \cap f_5} = \{(17, 0), (21, 0), (27, 0), (35, 0)\}$.

It should be noted that the values of function for different properties need not be disjoint. This means $f(p) \cap f(p')$ may be a non-empty set. As an example consider the bag $\mathcal{B}^{f_1 \cup f_5}$ in Example 2.

Table 2 Operations on functions from Example 1

p	17	21	27	35
$(f_1 \cup f_2)(p)$	{Bill, Sue}	{John, Tom, Stan}	∅	{Harry}
$(f_2 \cap f_3)(p)$	∅	{Stan}	∅	{Harry}
$(f_1 \setminus f_3)(p)$	{Bill, Sue}	{John, Tom}	∅	∅
$(f_3 \setminus f_2)(p)$	∅	∅	{Mary}	∅
$(f_1 \cup f_5)(p)$	{Bill, Sue, John, Tom}	{John, Tom, Ana, Stan}	∅	∅
$(f_1 \cap f_5)(p)$	∅	∅	∅	∅

From the point of view of the functions associated to a bag, we have the following definition.

Definition 4 ([5]) (i) A bag $\mathcal{B}^f$ is a sub bag of $\mathcal{B}^g$, denoted by $\mathcal{B}^f \sqsubseteq \mathcal{B}^g$, if $f(p) \subseteq g(p)$ for all $p \in P$.
(ii) Two bags $\mathcal{B}^f$ and $\mathcal{B}^g$ are equal, denoted by $\mathcal{B}^f = \mathcal{B}^g$ if $\mathcal{B}^f \sqsubseteq \mathcal{B}^g$ and $\mathcal{B}^g \sqsubseteq \mathcal{B}^f$ that means if $f = g$.

Remark 1 ([5]) Operations ∩ and ∪ in $\mathbf{B}(P, O)$ satisfy the laws of idempotency, commutativity, associativity, monotonicity and distributivity. Moreover, $\mathcal{B}^0$ is neutral for operation ∪ and $\mathcal{B}^1$ is neutral for operation ∩.

Definition 5 ([5]) Let $\mathcal{B}^f = (f, B^f)$. Then, complement of $\mathcal{B}^f$ is $\mathcal{B}^{f^c} = (\mathcal{B}^f)^c = (f^c, B^{f^c})$, where $f^c : P \to \mathcal{P}(O)$ is such that $f^c(p) = O \setminus f(p)$ for all $p \in P$.

As an example, observe that $(\mathcal{B}^0)^c = \mathcal{B}^1$.

In what follows, L is a complete lattice and $\mathcal{F}_L(O) = \{A | A : O \to L\}$ is the set of all L-fuzzy subsets of O. In the case of $L = [0, 1]$, we write $\mathcal{F}(O)$.

Definition 6 ([10]) An L-fuzzy bag $\tilde{\mathcal{B}}^{\tilde{f}}$ is a pair $(\tilde{f}, B^{\tilde{f}})$, where $\tilde{f} : P \to \mathcal{F}_L(O)$ is a function and $B^{\tilde{f}}$ is the following subset of $P \times L \times \mathcal{N}_0$

$$B^{\tilde{f}} = \{(p, \delta, card(O^p_\delta)) | p \in P, \delta \in L\}.$$

where, $O^p_\delta = \{o \in O | \tilde{f}(p)(o) = \delta\}$.

Obviously, a bag is a particular case of L-fuzzy bag where, for all $p \in P$, $\tilde{f}(p)$ is a crisp subset of O. Similar to bags, an L-fuzzy bag $\tilde{\mathcal{B}}^{\tilde{f}}$ consists of two parts. The first one is the function $\tilde{f}$ that can be seen as an information source about the relation between objects and properties. The second part $B^{\tilde{f}}$ is a summary of the information in $\tilde{f}$ obtained by means of the count operation $card(.)$.

Note 2 ([10]) In the case that $L = [0, 1]$, the defined bag in Definition 6 is called fuzzy bag.

Table 3 The degrees of memberships for Example 3

p	o								
	Ben	Sue	Tom	John	Stan	Bill	Kim	Ana	Sara
Young	0.7	0.2	0.4	0.0	0.7	0.4	0.2	0.7	0.1
Middle age	0.3	0.8	0.7	0.3	0.3	0.7	0.8	0.3	0.5
Old	0.1	0.2	0.1	0.9	0.1	0.1	0.2	0.1	0.5

Table 4 The degrees of memberships for Example 4

p	o								
	Ben	Sue	Tom	John	Stan	Bill	Kim	Ana	Sara
Tall	0.8	0.6	0.0	0.1	0.8	0.6	0.5	0.7	0.5
Medium	0.3	0.1	0.1	0.6	0.3	0.1	0.8	0.1	0.5
Short	0.1	0.0	0.9	0.4	0.1	0.0	0.2	0.0	0.1

Here, the concept of L-fuzzy bag is illustrated by two examples.

Example 3 ([10]) Let $L = [0, 1]$, $O = \{$Ben, Sue, Tom, John, Stan, Bill, Kim, Ana, Sara$\}$ and $P = \{$young, middle age, old$\}$ is the set of some linguistic descriptions of age. Let the degrees of membership of all $o \in O$ in the set of each property $p \in P$ are given as in Table 3.

So, by Definition 6, we can define fuzzy bag $\tilde{\mathcal{B}}^{\tilde{f}} = (\tilde{f}, B^{\tilde{f}})$ where,

$$\tilde{f}(\text{young}) = \{\frac{0.7}{\text{Ben}}, \frac{0.2}{\text{Sue}}, \frac{0.4}{\text{Tom}}, \frac{0.7}{\text{Stan}}, \frac{0.4}{\text{Bill}}, \frac{0.2}{\text{Kim}}, \frac{0.7}{\text{Ana}}, \frac{0.1}{\text{Sara}}\},$$
$$\tilde{f}(\text{middle age}) = \{\frac{0.3}{\text{Ben}}, \frac{0.8}{\text{Sue}}, \frac{0.7}{\text{Tom}}, \frac{0.3}{\text{John}}, \frac{0.3}{\text{Stan}}, \frac{0.7}{\text{Bill}}, \frac{0.8}{\text{Kim}}, \frac{0.3}{\text{Ana}}, \frac{0.5}{\text{Sara}}\},$$
$$\tilde{f}(\text{old}) = \{\frac{0.1}{\text{Ben}}, \frac{0.2}{\text{Sue}}, \frac{0.1}{\text{Tom}}, \frac{0.9}{\text{John}}, \frac{0.1}{\text{Stan}}, \frac{0.1}{\text{Bill}}, \frac{0.2}{\text{Kim}}, \frac{0.1}{\text{Ana}}, \frac{0.5}{\text{Sara}}\},$$

and

$$B^{\tilde{f}} = \{(\text{young}, 0.7, 3), (\text{young}, 0.4, 2), (\text{young}, 0.2, 2), (\text{young}, 0.1, 1), (\text{middle age}, 0.8, 2), (\text{middle age}, 0.7, 2), (\text{middle age}, 0.5, 1), (\text{middle age}, 0.3, 4), (\text{old}, 0.9, 1), (\text{old}, 0.5, 1), (\text{old}, 0.2, 2), (\text{old}, 0.1, 5)\}.$$

Example 4 ([10]) Let $L = [0, 1]$, O be as in Example 3 and $P = \{$tall, medium, short$\}$ is the set of some linguistic descriptions of height. Let the degrees of membership of all $o \in O$ in the set of each property $p \in P$ be given as in Table 4.

So, by Definition 6, we can define fuzzy bag $\tilde{\mathcal{B}}^{\tilde{g}} = (\tilde{g}, B^{\tilde{g}})$ where,

Table 5 The membership fuzzy sets for Example 5

p	o				
	Ben	Sue	Tom	John	Stan
Young	$\{\frac{0.4}{0.6}, \frac{0.5}{0.7}, \frac{0.6}{0.8}\}$	$\{\frac{0.9}{0.2}, \frac{0.8}{0.3}\}$	$\{\frac{0.7}{0.4}, \frac{0.8}{0.5}\}$	$\{\frac{1}{0.0}\}$	$\{\frac{0.5}{0.6}, \frac{0.5}{0.7}, \frac{0.6}{0.8}\}$
Middle age	$\{\frac{0.7}{0.2}, \frac{0.8}{0.3}, \frac{0.7}{0.4}\}$	$\{\frac{0.7}{0.8}, \frac{0.8}{0.9}\}$	$\{\frac{0.7}{0.7}, \frac{0.8}{0.8}\}$	$\{\frac{0.8}{0.3}, \frac{0.9}{0.4}, \frac{0.8}{0.5}\}$	$\{\frac{0.7}{0.2}, \frac{0.8}{0.3}, \frac{0.7}{0.4}\}$
Old	$\{\frac{1}{0.1}\}$	$\{\frac{0.8}{0.1}, \frac{0.9}{0.2}\}$	$\{\frac{0.8}{0.1}, \frac{0.9}{0.2}, \frac{0.8}{0.3}\}$	$\{\frac{1}{0.9}\}$	$\{\frac{0.9}{0.1}\}$

$$\tilde{g}(\text{tall}) = \{\frac{0.8}{\text{Ben}}, \frac{0.6}{\text{Sue}}, \frac{0.1}{\text{John}}, \frac{0.8}{\text{Stan}}, \frac{0.6}{\text{Bill}}, \frac{0.5}{\text{Kim}}, \frac{0.7}{\text{Ana}}, \frac{0.5}{\text{Sara}}\},$$
$$\tilde{g}(\text{medium}) = \{\frac{0.3}{\text{Ben}}, \frac{0.1}{\text{Sue}}, \frac{0.1}{\text{Tom}}, \frac{0.6}{\text{John}}, \frac{0.3}{\text{Stan}}, \frac{0.1}{\text{Bill}}, \frac{0.8}{\text{Kim}}, \frac{0.1}{\text{Ana}}, \frac{0.5}{\text{Sara}}\},$$
$$\tilde{g}(\text{short}) = \{\frac{0.1}{\text{Ben}}, \frac{0.9}{\text{Tom}}, \frac{0.4}{\text{John}}, \frac{0.1}{\text{Stan}}, \frac{0.2}{\text{Kim}}, \frac{0.1}{\text{Sara}}\},$$

and

$$\begin{aligned} B^{\tilde{g}} = \{&(\text{tall}, 0.8, 2), (\text{tall}, 0.7, 1), (\text{tall}, 0.6, 2), (\text{tall}, 0.5, 2), (\text{tall}, 0.1, 1), \\ &(\text{medium}, 0.8, 1), (\text{medium}, 0.6, 1), (\text{medium}, 0.5, 1), (\text{medium}, 0.3, 2), \\ &(\text{medium}, 0.1, 4), (\text{short}, 0.9, 1), (\text{short}, 0.4, 1), (\text{short}, 0.2, 1), (\text{short}, 0.1, 3)\}. \end{aligned}$$

Remark 2 Let in Definition 6, the lattice is $\mathcal{F}_L(L)$. Then, we have type-2 L-fuzzy bag $\tilde{\mathcal{B}}^{\tilde{f}^2} = (\tilde{f}^2, B^{\tilde{f}^2})$.

Example 5 Let $L = \mathcal{F}([0, 1])$, $O = \{\text{Ben, Sue, Tom, John, Stan}\}$ and P be as in the Example 3. Let the membership of each $o \in O$ in the set of each property $p \in P$ be given as in Table 5.

So, by Definition 6 and Remark 2, we can define type-2 L-fuzzy bag $\tilde{\mathcal{B}}^{\tilde{f}^2} = (\tilde{f}^2, B^{\tilde{f}^2})$ where,

$$\tilde{f}^2(\text{Young}) = \{\frac{\{\frac{0.4}{0.6}, \frac{0.5}{0.7}, \frac{0.6}{0.8}\}}{\text{Ben}}, \frac{\{\frac{0.9}{0.2}, \frac{0.8}{0.3}\}}{\text{Sue}}, \frac{\{\frac{0.7}{0.4}, \frac{0.8}{0.5}\}}{\text{Tom}}, \frac{\{\frac{1}{0.0}\}}{\text{John}}, \frac{\{\frac{0.5}{0.6}, \frac{0.5}{0.7}, \frac{0.6}{0.8}\}}{\text{Stan}}\},$$
$$\tilde{f}^2(\text{Middle age}) = \{\frac{\{\frac{0.7}{0.2}, \frac{0.8}{0.3}, \frac{0.7}{0.4}\}}{\text{Ben}}, \frac{\{\frac{0.7}{0.8}, \frac{0.8}{0.9}\}}{\text{Sue}}, \frac{\{\frac{0.7}{0.7}, \frac{0.8}{0.8}\}}{\text{Tom}}, \frac{\{\frac{0.8}{0.3}, \frac{0.9}{0.4}, \frac{0.8}{0.5}\}}{\text{John}}, \frac{\{\frac{0.7}{0.2}, \frac{0.8}{0.3}, \frac{0.7}{0.4}\}}{\text{Stan}}\},$$
$$\tilde{f}^2(\text{Old}) = \{\frac{\{\frac{1}{0.1}\}}{\text{Ben}}, \frac{\{\frac{0.8}{0.1}, \frac{0.9}{0.2}\}}{\text{Sue}}, \frac{\{\frac{0.8}{0.1}, \frac{0.9}{0.2}, \frac{0.8}{0.3}\}}{\text{Tom}}, \frac{\{\frac{1}{0.9}\}}{\text{John}}, \frac{\{\frac{0.9}{0.1}\}}{\text{Stan}}\},$$

and

$$B^{\tilde{f}^2} = \{(\text{young}, \{\frac{0.6}{0.8}\}, 2), (\text{young}, \{\frac{0.5}{0.7}\}, 2), (\text{young}, \{\frac{0.5}{0.6}\}, 1), (\text{young}, \{\frac{0.4}{0.6}\}, 1),$$
$$(\text{young}, \{\frac{0.8}{0.5}\}, 1), (\text{young}, \{\frac{0.7}{0.4}\}, 1), (\text{young}, \{\frac{0.8}{0.3}\}, 1), (\text{young}, \{\frac{0.9}{0.2}\}, 1),$$
$$(\text{young}, \{\frac{1.0}{0.0}\}, 1), (\text{middle age}, \{\frac{0.8}{0.9}\}, 1), (\text{middle age}, \{\frac{0.8}{0.8}\}, 1),$$
$$(\text{middle age}, \{\frac{0.7}{0.8}\}, 1), (\text{middle age}, \{\frac{0.7}{0.7}\}, 1), (\text{middle age}, \{\frac{0.8}{0.5}\}, 1),$$
$$(\text{middle age}, \{\frac{0.9}{0.4}\}, 1), (\text{middle age}, \{\frac{0.7}{0.4}\}, 2), (\text{middle age}, \{\frac{0.8}{0.3}\}, 3),$$
$$(\text{middle age}, \{\frac{0.7}{0.2}\}, 2), (\text{old}, \{\frac{1.0}{0.9}\}, 1), (\text{old}, \{\frac{0.8}{0.3}\}, 1), (\text{old}, \{\frac{0.9}{0.2}\}, 2),$$
$$(\text{old}, \{\frac{1.0}{0.1}\}, 1), (\text{old}, \{\frac{0.9}{0.1}\}, 1), (\text{old}, \{\frac{0.8}{0.1}\}, 2)\}.$$

Remark 3 ([10]) As it can be seen, the more important part of an L-fuzzy bag is information function $\tilde{f}$. Therefore, it is possible to study the properties of L-fuzzy bags just by considering their information functions.

Notation 2 ([10]) We set $\tilde{\mathbf{B}}_L(P, O)$ as the set of all L-fuzzy bags $\tilde{\mathcal{B}}^{\tilde{f}} = (\tilde{f}, B^{\tilde{f}})$. Where, $\tilde{f} : P \to \mathcal{F}_L(O)$ and $B^{\tilde{f}}$ are as defined in Definition 6. Also, we set $\tilde{\mathbf{B}}(P, O)$ as the set of all fuzzy bags.

The following theorem gives the relation among bags, fuzzy bags and L-fuzzy bags.

Theorem 1 ([10]) *Let a complete lattice L_1 be a sub lattice of a complete lattice L_2. Then, $\tilde{\boldsymbol{B}}_{L_1}(P, O) \subseteq \tilde{\boldsymbol{B}}_{L_2}(P, O)$. In particular, $\boldsymbol{B}(P, O) = \tilde{\boldsymbol{B}}_{\{0,1\}}(P, O) \subseteq \tilde{\boldsymbol{B}}_{[0,1]}(P, O) = \tilde{\boldsymbol{B}}(P, O)$.*

Here, we define the binary operations among L-fuzzy bags.

Definition 7 Let $\tilde{\mathcal{B}}^{\tilde{f}_i} \in \tilde{\mathbf{B}}_L(P_i, O_i)$ for all $i \in I_n$ be given L-fuzzy bags, $\overline{O} = \cap_{i \in I_n} O_i \neq \emptyset$ and $\overline{P} = \cap_{i \in I_n} P_i \neq \emptyset$. Then, their intersection is L-fuzzy bag

$$\cap_{i \in I_n} \tilde{\mathcal{B}}^{\tilde{f}_i} = (\cap_{i \in I_n} \tilde{f}_i, B^{\cap_{i \in I_n} \tilde{f}_i}), \quad (1)$$

where $\cap_{i \in I_n} \tilde{f}_i : \overline{P} \to \mathcal{F}_L(\overline{O})$ such that $(\cap_{i \in I_n} \tilde{f}_i)(p) = \cap_{i \in I_n} \tilde{f}_i(p)$. Also,

$$B^{\cap_{i \in I_n} \tilde{f}_i} = \{(p, \delta, card(O_\delta^p)) | p \in \overline{P}, \delta \in L\},$$

where $O_\delta^p = \{o \in \overline{O} | (\cap_{i \in I_n} \tilde{f}_i)(p)(o) = \delta\}$.

Note that by Definition 6, $\cap_{i \in I_n} \tilde{\mathcal{B}}^{\tilde{f}_i} = \tilde{\mathcal{B}}^{\cap_{i \in I_n} \tilde{f}_i}$.

Table 6 Values of $\tilde{f}_1(p)(o)$

p	o				
	Nancy	Lia	Elena	Suzi	Sam
Tall	0.6	0.8	0.3	0.0	0.6
Medium	0.8	0.4	0.6	0.2	0.4
Short	0.0	0.0	0.8	1.0	0.3

Table 7 Values of $\tilde{f}_2(p)(o)$

p	o			
	Liu	Sam	Bob	Suzi
Extremly tall	0.9	0.2	0.4	0.0
Tall	1.0	0.7	0.7	0.0
Medium	0.1	0.3	0.2	0.3
Short	0.0	0.2	0.1	0.9

Definition 8 Let $\tilde{\mathcal{B}}^{\tilde{f}_i} \in \tilde{\mathbf{B}}_L(P_i, O_i)$ for all $i \in I_n$ be given L-fuzzy bags, $\overline{O} = \cup_{i \in I_n} O_i$ and $\overline{P} = \cup_{i \in I_n} P_i$. Then, their union is L-fuzzy bag

$$\cup_{i \in I_n} \tilde{\mathcal{B}}^{\tilde{f}_i} = (\cup_{i \in I_n} \tilde{f}_i, B^{\cup_{i \in I_n} \tilde{f}_i}), \tag{2}$$

where $\cup_{i \in I_n} \tilde{f}_i : \overline{P} \to \mathcal{F}_L(\overline{O})$ such that $(\cup_{i \in I_n} \tilde{f}_i)(p) = \cup_{i \in I_n} \tilde{f}_i(p)$. Also,

$$B^{\cup_{i \in I_n} \tilde{f}_i} = \{(p, \delta, card(O_\delta^p)) | p \in \overline{P}, \delta \in L\},$$

where $O_\delta^p = \{o \in \overline{O} | (\cup_{i \in I_n} \tilde{f}_i)(p)(o) = \delta\}$.

Note that by Definition 6, $\cup_{i \in I_n} \tilde{\mathcal{B}}^{\tilde{f}_i} = \tilde{\mathcal{B}}^{\cup_{i \in I_n} \tilde{f}_i}$.

Example 6 Let $O_1 = \{\text{Nancy, Lia, Sam, Elena, Suzi}\}$, $O_2 = \{\text{Liu, Sam, Bob, Suzi}\}$, $P_1 = \{\text{tall, medium, short}\}$, $P_2 = \{\text{extremely tall, tall, medium, short}\}$ and $L = [0, 1]$. Consider $\tilde{\mathcal{B}}^{\tilde{f}_1} \in \tilde{\mathbf{B}}(P_1, O_1)$ and $\tilde{\mathcal{B}}^{\tilde{f}_2} \in \tilde{\mathbf{B}}(P_2, O_2)$ in which the values of $\tilde{f}_1$ and $\tilde{f}_2$ are as in Tables 6 and 7.

So, the intersection is $\tilde{\mathcal{B}}^{\tilde{f}_1 \cap \tilde{f}_2} = (\tilde{f}_1 \cap \tilde{f}_2, B^{\tilde{f}_1 \cap \tilde{f}_2})$ where,

$$(\tilde{f}_1 \cap \tilde{f}_2)(\text{tall}) = \{\frac{0.6}{\text{Sam}}\},$$
$$(\tilde{f}_1 \cap \tilde{f}_2)(\text{medium}) = \{\frac{0.2}{\text{Suzi}}, \frac{0.3}{\text{Sam}}\},$$
$$(\tilde{f}_1 \cap \tilde{f}_2)(\text{short}) = \{\frac{0.9}{\text{Suzi}}, \frac{0.2}{\text{Sam}}\}.$$

and

$$B^{\tilde{f}_1\cap\tilde{f}_2} = \{(\text{tall}, 0.6, 1), (\text{medium}, 0.2, 1), (\text{medium}, 0.3, 1), (\text{short}, 0.9, 1), (\text{short}, 0.2, 1)\}.$$

And the union is $\tilde{\mathcal{B}}^{\tilde{f}_1\cup\tilde{f}_2} = (\tilde{f}_1 \cup \tilde{f}_2, B^{\tilde{f}_1\cup\tilde{f}_2})$ where,

$$(\tilde{f}_1 \cup \tilde{f}_2)(\text{extremely tall}) = \{\frac{0.9}{\text{Liu}}, \frac{0.2}{\text{Sam}}, \frac{0.4}{\text{Bob}}\},$$
$$(\tilde{f}_1 \cup \tilde{f}_2)(\text{tall}) = \{\frac{1.0}{\text{Liu}}, \frac{0.7}{\text{Sam}}, \frac{0.7}{\text{Bob}}, \frac{0.6}{\text{Nancy}}, \frac{0.8}{\text{Lia}}, \frac{0.3}{\text{Elena}}\},$$
$$(\tilde{f}_1 \cup \tilde{f}_2)(\text{medium}) = \{\frac{0.8}{\text{Nancy}}, \frac{0.4}{\text{Lia}}, \frac{0.6}{\text{Elena}}, \frac{0.3}{\text{Suzi}}, \frac{0.1}{\text{Liu}}, \frac{0.4}{\text{Sam}}, \frac{0.2}{\text{Bob}}\},$$
$$(\tilde{f}_1 \cup \tilde{f}_2)(\text{short}) = \{\frac{0.8}{\text{Elena}}, \frac{1.0}{\text{Suzi}}, \frac{0.3}{\text{Sam}}, \frac{0.1}{\text{Bob}}\}.$$

and

$$B^{\tilde{f}_1\cup\tilde{f}_2} = \{(\text{extremely tall}, 0.2, 1), (\text{extremely tall}, 0.4, 1), (\text{extremely tall}, 0.9, 1), (\text{tall}, 0.3, 1), (\text{tall}, 0.6, 1), (\text{tall}, 0.7, 2), (\text{tall}, 0.8, 1), (\text{tall}, 1.0, 1), (\text{medium}, 0.1, 1), (\text{medium}, 0.2, 1), (\text{medium}, 0.3, 1), (\text{medium}, 0.4, 2), (\text{medium}, 0.6, 1), (\text{medium}, 0.8, 1), (\text{short}, 0.1, 1), (\text{short}, 0.3, 1), (\text{short}, 0.8, 1), (\text{short}, 1.0, 1)\}.$$

The following definition equips the set of all L-fuzzy bags with an order.

Definition 9 ([10]) (i) An L-fuzzy bag $\tilde{\mathcal{B}}^{\tilde{f}}$ is an L-fuzzy sub bag of $\tilde{\mathcal{B}}^{\tilde{g}}$, denoted by $\tilde{\mathcal{B}}^{\tilde{f}} \tilde{\sqsubseteq} \tilde{\mathcal{B}}^{\tilde{g}}$ if and only if $\tilde{f}(p) \tilde{\subseteq} \tilde{g}(p)$ for all $p \in P$. That means $\tilde{\mathcal{B}}^{\tilde{f}} \tilde{\sqsubseteq} \tilde{\mathcal{B}}^{\tilde{g}}$ if and only if for all $p \in P$, $\tilde{f}(p)$ be an L-fuzzy subset of $\tilde{g}(p)$.
(ii) Two L-fuzzy bags $\tilde{\mathcal{B}}^{\tilde{f}}$ and $\tilde{\mathcal{B}}^{\tilde{g}}$ are equal, denoted by $\tilde{\mathcal{B}}^{\tilde{f}} \cong \tilde{\mathcal{B}}^{\tilde{g}}$ if $\tilde{\mathcal{B}}^{\tilde{f}} \tilde{\sqsubseteq} \tilde{\mathcal{B}}^{\tilde{g}}$ and $\tilde{\mathcal{B}}^{\tilde{g}} \tilde{\sqsubseteq} \tilde{\mathcal{B}}^{\tilde{f}}$ that means if $\tilde{f} = \tilde{g}$.

The next theorem gives some useful results about L-fuzzy bags.

Theorem 2 ([10]) *Operations $\cup$ and $\cap$ in $\tilde{\boldsymbol{B}}_L(P, O)$ satisfy the laws of idempotency, commutativity, associativity, monotonicity and distributivity. Moreover, $\mathcal{B}^0$ is neutral for operation $\cup$ and $\mathcal{B}^1$ is neutral for operation $\cap$.*

In the following definition, we review the concept of the complement of an L-fuzzy bag.

Definition 10 ([10]) Let $\eta : L \to L$ be a fixed strong negation [1], this means an involutive decreasing bijection. Consider $\tilde{\mathcal{B}}^{\tilde{f}} = (\tilde{f}, B^{\tilde{f}})$. Then, the $\eta-$complement of $\tilde{\mathcal{B}}^{\tilde{f}}$ is L-fuzzy bag $(\tilde{\mathcal{B}}^{\tilde{f}})^c = (\tilde{f}^c, B^{\tilde{f}^c})$, where $\tilde{f}^c : P \to \mathcal{F}_L(O)$ such that $\tilde{f}^c(p)(o) = \eta(\tilde{f}(p)(o))$ for all $p \in P$ and $o \in O$.

Note that by Definition 6, $(\tilde{\mathcal{B}}^{\tilde{f}})^c = \tilde{\mathcal{B}}^{\tilde{f}^c}$.

Note 3 ([10]) In Definition 10, if $L = [0, 1]$ and η is the standard negation, $\eta(x) = 1 - x$ for all $x \in [0, 1]$ [1], then $\tilde{\mathcal{B}}^{\tilde{f}^c}$ is called the complement of $\tilde{\mathcal{B}}^{\tilde{f}}$.

Example 7 ([10]) The complement of the fuzzy bag in Example 4 is $\tilde{\mathcal{B}}^{\tilde{g}^c} = (\tilde{g}^c, B^{\tilde{g}^c})$ where,

$$\tilde{g}^c(\text{tall}) = \{\frac{0.2}{\text{Ben}}, \frac{0.4}{\text{Sue}}, \frac{1.0}{\text{Tom}}, \frac{0.9}{\text{John}}, \frac{0.2}{\text{Stan}}, \frac{0.4}{\text{Bill}}, \frac{0.5}{\text{Kim}}, \frac{0.3}{\text{Ana}}, \frac{0.5}{\text{Sara}}\},$$
$$\tilde{g}^c(\text{medium}) = \{\frac{0.7}{\text{Ben}}, \frac{0.9}{\text{Sue}}, \frac{0.9}{\text{Tom}}, \frac{0.4}{\text{John}}, \frac{0.7}{\text{Stan}}, \frac{0.9}{\text{Bill}}, \frac{0.2}{\text{Kim}}, \frac{0.9}{\text{Ana}}, \frac{0.5}{\text{Sara}}\},$$
$$\tilde{g}^c(\text{short}) = \{\frac{0.9}{\text{Ben}}, \frac{1.0}{\text{Sue}}, \frac{0.1}{\text{Tom}}, \frac{0.6}{\text{John}}, \frac{0.9}{\text{Stan}}, \frac{1.0}{\text{Bill}}, \frac{0.8}{\text{Kim}}, \frac{1.0}{\text{Ana}}, \frac{0.9}{\text{Sara}}\},$$

and

$$\begin{aligned} B^{\tilde{g}^c} = \{&(\text{tall}, 1.0, 1), (\text{tall}, 0.9, 1), (\text{tall}, 0.5, 2), (\text{tall}, 0.4, 2), (\text{tall}, 0.3, 1),\\ &(\text{tall}, 0.2, 2), (\text{medium}, 0.9, 4), (\text{medium}, 0.7, 2), (\text{medium}, 0.5, 1),\\ &(\text{medium}, 0.4, 1), (\text{medium}, 0.2, 1), (\text{short}, 1.0, 3), (\text{short}, 0.9, 3),\\ &(\text{short}, 0.8, 1), (\text{ short}, 0.6, 1), (\text{short}, 0.1, 1)\}. \end{aligned}$$

Note 4 In the process of determining the degrees of membership in Definition 6, some degrees are very close to each other and may be they are not different in the decision maker's point of view. This situation appears specially when the cardinality of O is big. In this case, we can cluster the objects based on their degrees of membership. For example consider Example 5 in the case that we have $card(O) = 100$.

Table 8 Clusters

Cluster head	Cluster members
$\frac{0.06}{70}$	$\frac{0.08}{12}, \frac{0.08}{19}, \frac{0.00}{23}, \frac{0.08}{27}, \frac{0.08}{46}, \frac{0.05}{52}, \frac{0.06}{70}, \frac{0.02}{74}, \frac{0.04}{75}, \frac{0.08}{90}$
$\frac{0.11}{21}$	$\frac{0.11}{21}, \frac{0.12}{48}, \frac{0.11}{60}, \frac{0.10}{65}, \frac{0.13}{66}$
$\frac{0.15}{15}$	$\frac{0.16}{1}, \frac{0.15}{15}, \frac{0.15}{35}, \frac{0.14}{36}, \frac{0.14}{40}$
$\frac{0.18}{33}$	$\frac{0.17}{5}, \frac{0.18}{33}, \frac{0.18}{49}, \frac{0.17}{76}, \frac{0.19}{84}, \frac{0.18}{86}$
$\frac{0.24}{47}$	$\frac{0.26}{7}, \frac{0.23}{13}, \frac{0.26}{29}, \frac{0.26}{34}, \frac{0.24}{47}, \frac{0.24}{50}, \frac{0.24}{63}, \frac{0.23}{71}$
$\frac{0.34}{57}$	$\frac{0.31}{3}, \frac{0.35}{43}, \frac{0.34}{57}, \frac{0.37}{59}, \frac{0.35}{72}, \frac{0.30}{82}, \frac{0.37}{87}, \frac{0.31}{96}$
$\frac{0.42}{51}$	$\frac{0.45}{11}, \frac{0.44}{20}, \frac{0.40}{28}, \frac{0.43}{31}, \frac{0.40}{45}, \frac{0.42}{51}, \frac{0.39}{62}, \frac{0.40}{64}, \frac{0.45}{80}, \frac{0.44}{94}, \frac{0.45}{95}$
$\frac{0.51}{44}$	$\frac{0.53}{4}, \frac{0.54}{17}, \frac{0.51}{44}, \frac{0.49}{55}, \frac{0.49}{56}, \frac{0.49}{93}, \frac{0.51}{97}, \frac{0.51}{98}$
$\frac{0.58}{38}$	$\frac{0.60}{6}, \frac{0.58}{38}, \frac{0.55}{39}, \frac{0.62}{42}, \frac{0.58}{69}, \frac{0.55}{81}, \frac{0.63}{88}$
$\frac{0.69}{9}$	$\frac{0.65}{8}, \frac{0.69}{9}, \frac{0.65}{77}, \frac{0.73}{78}, \frac{0.65}{79}, \frac{0.74}{83}, \frac{0.69}{85}$
$\frac{0.79}{2}$	$\frac{0.79}{2}, \frac{0.75}{10}, \frac{0.83}{16}, \frac{0.77}{24}, \frac{0.82}{25}, \frac{0.80}{30}, \frac{0.78}{61} \frac{0.82}{73}, \frac{0.78}{89}, \frac{0.78}{92}, \frac{0.82}{99}, \frac{0.79}{100}$
$\frac{0.90}{53}$	$\frac{0.91}{14}, \frac{0.87}{26}, \frac{0.91}{32}, \frac{0.87}{37}, \frac{0.85}{41}, \frac{0.90}{53}, \frac{0.90}{58}, \frac{0.93}{91}$
$\frac{0.96}{22}$	$\frac{1.00}{18}, \frac{0.96}{22}, \frac{0.94}{54}, \frac{0.94}{67}, \frac{0.96}{68}$

Let us have the following fuzzy set for the property "young".

$$\tilde{f}(\text{young})=\{\frac{0.16}{o_1},\frac{0.79}{o_2},\frac{0.31}{o_3},\frac{0.53}{o_4},\frac{0.17}{o_5},\frac{0.60}{o_6},\frac{0.26}{o_7},\frac{0.65}{o_8},\frac{0.69}{o_9},\frac{0.75}{o_{10}},$$
$$\frac{0.45}{o_{11}},\frac{0.08}{o_{12}},\frac{0.23}{o_{13}},\frac{0.91}{o_{14}},\frac{0.15}{o_{15}},\frac{0.83}{o_{16}},\frac{0.54}{o_{17}},\frac{1.00}{o_{18}},\frac{0.08}{o_{19}},\frac{0.44}{o_{20}},\frac{0.11}{o_{21}},$$
$$\frac{0.96}{o_{22}},\frac{0.00}{o_{23}},\frac{0.77}{o_{24}},\frac{0.82}{o_{25}},\frac{0.87}{o_{26}},\frac{0.08}{o_{27}},\frac{0.40}{o_{28}},\frac{0.26}{o_{29}},\frac{0.80}{o_{30}},\frac{0.43}{o_{31}},\frac{0.91}{o_{32}},$$
$$\frac{0.18}{o_{33}},\frac{0.26}{o_{34}},\frac{0.15}{o_{35}},\frac{0.14}{o_{36}},\frac{0.87}{o_{37}},\frac{0.58}{o_{38}},\frac{0.55}{o_{39}},\frac{0.14}{o_{40}},\frac{0.85}{o_{41}},\frac{0.62}{o_{42}},\frac{0.35}{o_{43}},$$
$$\frac{0.51}{o_{44}},\frac{0.40}{o_{45}},\frac{0.08}{o_{46}},\frac{0.24}{o_{47}},\frac{0.12}{o_{48}},\frac{0.18}{o_{49}},\frac{0.24}{o_{50}},\frac{0.42}{o_{51}},\frac{0.05}{o_{52}},\frac{0.90}{o_{53}},\frac{0.94}{o_{54}},$$
$$\frac{0.49}{o_{55}},\frac{0.49}{o_{56}},\frac{0.34}{o_{57}},\frac{0.90}{o_{58}},\frac{0.37}{o_{59}},\frac{0.11}{o_{60}},\frac{0.78}{o_{61}},\frac{0.39}{o_{62}},\frac{0.24}{o_{63}},\frac{0.40}{o_{64}},\frac{0.10}{o_{65}},$$
$$\frac{0.13}{o_{66}},\frac{0.94}{o_{67}},\frac{0.96}{o_{68}},\frac{0.58}{o_{69}},\frac{0.06}{o_{70}},\frac{0.23}{o_{71}},\frac{0.35}{o_{72}},\frac{0.82}{o_{73}},\frac{0.02}{o_{74}},\frac{0.04}{o_{75}},\frac{0.17}{o_{76}},$$
$$\frac{0.65}{o_{77}},\frac{0.73}{o_{78}},\frac{0.65}{o_{79}},\frac{0.45}{o_{80}},\frac{0.55}{o_{81}},\frac{0.30}{o_{82}},\frac{0.74}{o_{83}},\frac{0.19}{o_{84}},\frac{0.69}{o_{85}},\frac{0.18}{o_{86}},\frac{0.37}{o_{87}},$$
$$\frac{0.63}{o_{88}},\frac{0.78}{o_{89}},\frac{0.08}{o_{90}},\frac{0.93}{o_{91}},\frac{0.78}{o_{92}},\frac{0.49}{o_{93}},\frac{0.44}{o_{94}},\frac{0.45}{o_{95}},\frac{0.31}{o_{96}},\frac{0.51}{o_{97}},\frac{0.51}{o_{98}},$$
$$\frac{0.82}{o_{99}},\frac{0.79}{o_{100}}\}$$

By K-medoids method and choosing 13 clusters, we have the results of Table 8.

3 Relations on Bags and Fuzzy Bags

Let P_i and O_i be the sets of properties and objectives for all $i \in I_n$, respectively. We have the following results.

Definition 11 ([10]) An n-dimensional bag is the pair $\mathcal{B}^l = (l, B^l)$ where,

$$l : \Pi_{i\in I_n} P_i \to \Pi_{i\in I_n}\mathcal{P}(O_i)$$

and

$$B^l = \{((p_1, \ldots, p_n), card(l(p_1, \ldots, p_n)))|p_i \in P_i,\ i \in I_n\}.$$

It should be mentioned that in what follows, for convenience, we use both notations Π and $\times$ for Cartesian product.

Table 9 Values of $(f_1 \times f_2)((p_1, p_2))$

(p_1, p_2)	$(f_1 \times f_2)((p_1, p_2))$
(male, 18)	{A, C, D, E, G, H, I, J, K} × {B, J}
(male, 19)	{A, C, D, E, G, H, I, J, K} × {E, K, L}
(male, 20)	{A, C, D, E, G, H, I, J, K} × {A, D}
(male, 21)	{A, C, D, E, G, H, I, J, K} × {I}
(male, 22)	{A, C, D, E, G, H, I, J, K} × {C, F, G, H, M}
(female, 18)	{B, F, L, M} × {B, J}
(female, 19)	{B, F, L, M} × {E, K, L}
(female, 20)	{B, F, L, M} × {A, D}
(female, 21)	{B, F, L, M} × {I}
(female, 22)	{B, F, L, M} × {C, F, G, H, M}

Definition 12 ([10]) Let $\mathcal{B}^{f_i} \in \mathbf{B}(P_i, O_i)$ for all $i \in I_n$. Define bag $\Pi_{i\in I_n}\mathcal{B}^{f_i} = (\Pi_{i\in I_n} f_i, B^{\Pi_{i\in I_n} f_i})$ as the Cartesian product of $\{\mathcal{B}^{f_i}\}_{i\in I_n}$ which is called C_n-bag. Where, $(\Pi_{i\in I_n} f_i)((p_1, \dots, p_n)) = \Pi_{i\in I_n} f_i(p_i)$ as the Cartesian product of n sets and

$$B^{\Pi_{i\in I_n} f_i} = \{((p_1, \dots, p_n), card(\Pi_{i\in I_n} f_i(p_i))|p_i \in P_i, \text{for all } i \in I_n\}.$$

Note that by Definition 1, $\mathcal{B}^{\Pi_{i\in I_n} f_i} = \Pi_{i\in I_n}\mathcal{B}^{f_i}$.

Theorem 3 ([10]) *C_n-bag $\mathcal{B}^{\Pi_{i\in I_n} f_i}$ is an n-dimensional bag.*

Example 8 ([10]) Let $O = \{A, B, C, D, E, F, G, H, I, J, K, L, M\}$, $P_1 = \{\text{male, female}\}$ and $P_2 = \{18, 19, 20, 21, 22\}$. Let $\mathcal{B}^{f_1} \in \mathbf{B}(P_1, O)$ and $\mathcal{B}^{f_2} \in \mathbf{B}(P_2, O)$, where

$$f_1(\text{male}) = \{A, C, D, E, G, H, I, J, K\}, \quad f_1(\text{female}) = \{B, F, L, M\},$$
$$f_2(18) = \{B, J\}, \quad f_2(19) = \{E, K, L\}, \quad f_2(20) = \{A, D\},$$
$$f_2(21) = \{I\}, \quad f_2(22) = \{C, F, G, H, M\}.$$

Hence, the C_2-bag of $\mathcal{B}^{f_1}$ and $\mathcal{B}^{f_2}$ is $\mathcal{B}^{f_1\times f_2} = (f_1 \times f_2, B^{f_1\times f_2})$, where the values of $(f_1 \times f_2)((p_1, p_2))$ are as in Table 9. So, according to Table 9, $B^{f_1\times f_2}$ is as follows.

$$\begin{aligned} B^{f_1\times f_2} =\{&((\text{male}, 18), 18), ((\text{male}, 19), 27), ((\text{male}, 20), 18), ((\text{male}, 21), 9),\\ &((\text{male}, 22), 45), ((\text{female}, 18), 8), ((\text{female}, 19), 12), ((\text{female}, 20), 8),\\ &((\text{female}, 21), 4), ((\text{female}, 22), 20)\}\end{aligned}$$

Definition 13 Let $\mathcal{B}^{f_i} \in \mathbf{B}(P_i, O_i)$ for all $i \in I_n$ and $\overline{O} = \cup_{i\in I_n} O_i$. Define bag conjunctive Cartesian product of $\{\mathcal{B}^{f_i}\}_{i\in I_n}$, which is called C_n^c-bag, by

Table 10 Values of $(f_1 \times^c f_2)((p_1, p_2))$

(p_1, p_2)	$(f_1 \times^c f_2)((p_1, p_2))$
(male, 18)	{J}
(male, 19)	{E, K}
(male, 20)	{A, D}
(male, 21)	{I}
(male, 22)	{C, G, H}
(female, 18)	{B}
(female, 19)	{L}
(female, 20)	∅
(female, 21)	∅
(female, 22)	{F, M}

$$\Pi^c_{i\in I_n}\mathcal{B}^{f_i} = (\Pi^c_{i\in I_n} f_i, B^{\Pi^c_{i\in I_n} f_i}), \tag{3}$$

where $\Pi^c_{i\in I_n} f_i : \Pi_{i\in I_n} P_i \to P(\overline{O})$ such that $(\Pi^c_{i\in I_n} f_i)((p_1, p_2, \ldots, p_n)) = \cap_{i\in I_n} f_i(p_i)$ for all $p_i \in P_i$. Also,

$$B^{\Pi^c_{i\in I_n} f_i} = \{((p_1, p_2, \ldots, p_n), card(\Pi_{i\in I_n} f_i(p_i)))|p_i \in P_i \text{ for all } i \in I_n\}.$$

Note that by Definition 1, $\Pi^c_{i\in I_n}\mathcal{B}^{f_i} = \mathcal{B}^{\Pi^c_{i\in I_n} f_i}$.

Definition 14 Let $\mathcal{B}^{f_i} \in \mathbf{B}(P_i, O_i)$ for all $i \in I_n$ and $\overline{O} = \cup_{i\in I_n} O_i$. Define bag disjunctive Cartesian product of $\{\mathcal{B}^{f_i}\}_{i\in I_n}$, which is called C^d_n-bag, by

$$\Pi^d_{i\in I_n}\mathcal{B}^{f_i} = (\Pi^d_{i\in I_n} f_i, B^{\Pi^d_{i\in I_n} f_i}), \tag{4}$$

where $\Pi^d_{i\in I_n} f_i : \Pi_{i\in I_n} P_i \to P(\overline{O})$ such that $(\Pi^d_{i\in I_n} f_i)((p_1, p_2, \ldots, p_n)) = \cup_{i\in I_n} f_i(p_i)$ for all $p_i \in P_i$. Also,

$$B^{\Pi^d_{i\in I_n} f_i} = \{((p_1, p_2, \ldots, p_n), card(\Pi_{i\in I_n} f_i(p_i)))|p_i \in P_i \text{ for all } i \in I_n\}.$$

Note that by Definition 1, $\Pi^d_{i\in I_n}\mathcal{B}^{f_i} = \mathcal{B}^{\Pi^d_{i\in I_n} f_i}$. Moreover, $f^i(p_i) \subseteq O_i \subseteq \overline{O}$ for all $i \in I_n$ and thus, $\cap_{i\in I_n}$ is well defined.

Example 9 Let $\mathcal{B}^{f_1} \in \mathbf{B}(P_1, O)$ and $\mathcal{B}^{f_2} \in \mathbf{B}(P_2, O)$ be as in Example 8. The C^c_n-bag of $\mathcal{B}^{f_1}$ and $\mathcal{B}^{f_2}$ is $\mathcal{B}^{f_1\times^c f_2} = (f_1 \times^c f_2, B^{f_1\times^c f_2})$, where the values of $(f_1 \times^c f_2)((p_1, p_2))$ are as in Table 10.

So, according to Table 10, $B^{f_1\times^c f_2}$ is as follows.

$$B^{f_1 \times^c f_2} = \{((\text{male}, 18), 1), ((\text{male}, 19), 2), ((\text{male}, 20), 2), ((\text{male}, 21), 1), ((\text{male}, 22), 3), ((\text{female}, 18), 1), ((\text{female}, 19), 1), ((\text{female}, 20), 0), ((\text{female}, 21), 0), ((\text{female}, 22), 2)\}.$$

Definition 15 ([10]) Fix C_n-bag $\mathcal{B}^{\Pi_{i=1}^n f_i}$. An $n-$ary bag relation is a sub bag of C_n-bag $\mathcal{B}^{\Pi_{i=1}^n f_i}$ which is denoted by $\mathcal{B}^{f_R} = (f_R, B^{f_R})$.

Example 10 ([10]) Consider C_2-bag of Example 8. The bag, $\mathcal{B}^{f_R} = (f_R, B^{f_R})$, is a $2-$ary bag relation which introduces people who are older than twenty, where

$$\begin{aligned}
f_R((\text{male}, 20)) &= \{\text{A, C, D, E, G, H, I, J, K}\} \times \{\text{A, D}\},\\
f_R((\text{male}, 21)) &= \{\text{A, C, D, E, G, H, I, J, K}\} \times \{\text{I}\},\\
f_R((\text{male}, 22)) &= \{\text{A, C, D, E, G, H, I, J, K}\} \times \{\text{C, F, G, H, M}\},\\
f_R((\text{female}, 20)) &= \{\text{B, F, L, M}\} \times \{\text{A, D}\},\\
f_R((\text{female}, 21)) &= \{\text{B, F, L, M}\} \times \{\text{I}\},\\
f_R((\text{female}, 22)) &= \{\text{B, F, L, M}\} \times \{\text{C, F, G, H, M}\}
\end{aligned}$$

and

$$B^{f_R} = \{((\text{male}, 20), 18), ((\text{male}, 21), 9), ((\text{male}, 22), 45), ((\text{female}, 20), 8), ((\text{female}, 21), 4), ((\text{female}, 22), 20)\}.$$

Definition 16 ([10]) Let $R_n \subseteq \Pi_{i\in I_n}\mathcal{P}(O_i)$ and $l_n : \Pi_{i\in I_n} P_i \to \Pi_{i\in I_n}\mathcal{P}(O_i)$. Then, $\mathcal{B}_{R_n}^{l_n} = (l_n, B_{R_n}^{l_n})$, where

$$B_{R_n}^{l_n} = \{((p_1, \ldots, p_n), card(l_n(p_1, \ldots, p_n)))|\ l_n(p_1, \ldots, p_n) \in R_n\}$$

is called the bag induced by R_n.

Example 11 ([10]) Let $O_1 = \{m_i | i \in I_{40}\}$ and $O_2 = \{w_i | i \in I_{40}\}$ where, m_i, w_i for all $i \in I_{40}$ are man and woman, respectively. Let $R_2 = \{(m_i, w_i) | i \in I_{40}\}$ shows the relation of "spouse". Now, Table 11 gives $l_2 : P_1 \times P_2 \to R_2 \subseteq \mathcal{P}(O_1) \times \mathcal{P}(O_2)$, where $P_1 = P_2 = \{A, B, AB, O\}$ is the set of all blood groups.

Thus, by Definition 16, we can present this information by the bag $\mathcal{B}_{R_2}^{l_2} = (l_2, B_{R_2}^{l_2})$, where $B_{R_2}^{l_2}$ is as follows.

$$B_{R_2}^{l_2} = \{((A, A), 3), ((A, B), 3), ((A, AB), 3), ((B, A), 5), ((B, B), 1), ((B, AB), 2), ((B, O), 3), ((AB, A), 4), ((AB, B), 2), ((AB, AB), 2), ((AB, O), 1), ((O, A), 3), ((O, B), 2), ((O, AB), 2), ((O, O), 4)\}.$$

Table 11 function l_2

(p_1, p_2)	$l_2(p_1, p_2)(o_1, o_2)$
(A, A)	$\{(m_5, w_5), (m_{11}, w_{11}), (m_{26}, w_{26})\}$
(A, B)	$\{(m_4, w_4), (m_{18}, w_{18}), (m_{36}, w_{36})\}$
(A, AB)	$\{(m_{12}, w_{12}), (m_{27}, w_{27}), (m_{39}, w_{39})\}$
(A, O)	$\emptyset$
(B, A)	$\{(m_8, w_8), (m_{25}, w_{25}), (m_{28}, w_{28}), (m_{31}, w_{31}), (m_{35}, w_{35})\}$
(B, B)	$\{(m_{14}, w_{14})\}$
(B, AB)	$\{(m_{17}, w_{17}), (m_{23}, w_{23})\}$
(B, O)	$\{(m_9, w_9), (m_{20}, w_{20}), (m_{32}, w_{32})\}$
(AB, A)	$\{(m_3, w_3), (m_{22}, w_{22}), (m_{38}, w_{38}), (m_{40}, w_{40})\}$
(AB, B)	$\{(m_6, w_6), (m_{29}, w_{29})\}$
(AB, AB)	$\{(m_2, w_2), (m_{15}, w_{15})\}$
(AB, O)	$\{(m_{37}, w_{37})\}$
(O, A)	$\{(m_{10}, w_{10}), (m_{24}, w_{24}), (m_{30}, w_{30})\}$
(O, B)	$\{(m_1, w_1), (m_{19}, w_{19})\}$
(O, AB)	$\{(m_{13}, w_{13}), (m_{21}, w_{21})\}$
(O, O)	$\{(m_7, w_7), (m_{16}, w_{16}), (m_{33}, w_{33}), (m_{34}, w_{34})\}$

Remark 4 ([10]) As a matter of fact, if people eat food that is not compatible with their blood type, they will experience many health problems. On the other hand, if a person eats food that is compatible, he/she will be healthier [20]. Since an appropriate diet can affect the unborn child's health, giving a proposal of a special diet to spouses can be helpful. Using the concept of relations on bags, one can screen all spouses with the similar blood groups.

Now, we study relations on L-fuzzy bags and give some results about them. First, we should review the concept of n-dimensional L-fuzzy bag.

Definition 17 ([10]) An n-dimensional L-fuzzy bag is the pair $\tilde{\mathcal{B}}^{\tilde{l}} = (\tilde{l}, B^{\tilde{l}})$ where,

$$\tilde{l} : \Pi_{i\in I_n} P_i \to \Pi_{i\in I_n} \mathcal{F}_L(O_i)$$

and

$$B^{\tilde{l}} = \{((p_1, \ldots, p_n), \delta, card(O_\delta^{p_1,\ldots,p_n}))|p_i \in P_i,\ i \in I_n, \delta \in L,\ O_\delta^{p_1,\ldots,p_n}\}.$$

where, $O_\delta^{p_1,\ldots,p_n} = \{(o_1, \ldots, o_n) \in \Pi_{i\in I_n} O_i | \tilde{l}(p_1, \ldots, p_n)(o_1, \ldots, o_n) = \delta\}$.

Notation 3 ([10]) In the sequel, we use notation $\mathcal{F}_L(O)^n$ for $\underbrace{\mathcal{F}_L(O) \times \mathcal{F}_L(O) \times \cdots \times \mathcal{F}_L(O)}_{n-times}$.

Definition 18 ([10]) Let $\tilde{\mathcal{B}}^{\tilde{f}_i} \in \tilde{\mathbf{B}}_L(P_i, O_i)$ for all $i \in I_n$. Define L-fuzzy bag Cartesian product of $\{\tilde{\mathcal{B}}^{\tilde{f}_i}\}_{i\in I_n}$, which is called C_n-L-fuzzy bag, by

$$\Pi_{i=1}^n \tilde{\mathcal{B}}^{\tilde{f}_i} = (\Pi_{i=1}^n \tilde{f}_i, B^{\Pi_{i=1}^n \tilde{f}_i}).$$

where, $(\Pi_{i=1}^n \tilde{f}_i)((p_1, \ldots, p_n)) = \Pi_{i=1}^n \tilde{f}_i(p_i)$ is the Cartesian product of n L-fuzzy sets and

$$B^{\Pi_{i=1}^n \tilde{f}_i} = \{((p_1, \ldots, p_n), \delta, card(O_\delta^{p_1,\ldots,p_n}))| p_i \in P_i,\ i \in I_n, \delta \in L, O_\delta^{p_1,\ldots,p_n}\},$$

and $O_\delta^{p_1,\ldots,p_n} = \{(o_1, \ldots, o_n) \in \Pi_{i=1}^n O_i | \min\{\tilde{f}_1(p_1)(o_1), \ldots, \tilde{f}_n(p_n)(o_n)\} = \delta\}$.

Note that by Definition 6, $\mathcal{B}^{\Pi_{i=1}^n \tilde{f}_i} = \Pi_{i=1}^n \mathcal{B}^{\tilde{f}_i}$.

Theorem 4 ([10]) *C_n-L-fuzzy bag is an n-dimensional L-fuzzy bag.*

An example of a 2-dimensional L-fuzzy bag is given in the following example.

Example 12 ([10]) Consider the fuzzy bags of Examples 3 and 4. The C_2-fuzzy bag of $\tilde{\mathcal{B}}^{\tilde{f}}$ and $\tilde{\mathcal{B}}^{\tilde{g}}$ is $\tilde{\mathcal{B}}^{\tilde{f}\times\tilde{g}} = (\tilde{f} \times \tilde{g}, B^{\tilde{f}\times\tilde{g}})$, where the values of $(\tilde{f} \times \tilde{g})((p_1, p_2))$ are as in Table 12. According to Table 12, $B^{\tilde{f}\times\tilde{g}}$ can be easily given.

Definition 19 Let $\tilde{\mathcal{B}}^{\tilde{f}_i} \in \tilde{\mathbf{B}}_L(P_i, O_i)$ for all $i \in I_n$ and $\overline{O} = \cup_{i\in I_n} O_i$. Define L-fuzzy bag conjunctive Cartesian product of $\{\tilde{\mathcal{B}}^{\tilde{f}_i}\}_{i\in I_n}$, which is called C_n^c-L-fuzzy bag, by

$$\Pi_{i\in I_n}^c \tilde{\mathcal{B}}^{\tilde{f}_i} = (\Pi_{i\in I_n}^c \tilde{f}_i, B^{\Pi_{i\in I_n}^c \tilde{f}_i}), \tag{5}$$

where $\Pi_{i\in I_n}^c \tilde{f}_i : \Pi_{i\in I_n} P_i \to \mathcal{F}_L(\overline{O})$ is such that $(\Pi_{i\in I_n}^c \tilde{f}_i)((p_1, p_2, \ldots, p_n)) = \cap_{i\in I_n} \tilde{f}_i(p_i)$ for all $p_i \in P_i$. Also,

$$B^{\Pi_{i\in I_n}^c \tilde{f}_i} = \{((p_1, p_2, \ldots, p_n), \delta, card(O_\delta^{p_1,p_2,\ldots,p_n}))| p_i \in P_i, \delta \in L\},$$

where $O_\delta^{p_1,p_2,\ldots,p_n} = \{o \in \overline{O} | (\Pi_{i\in I_n}^d \tilde{f}_i)((p_1, p_2, \ldots, p_n))(o) = \delta\}$.

Note that by Definition 6, $\Pi_{i\in I_n}^d \tilde{\mathcal{B}}^{\tilde{f}_i} = \tilde{\mathcal{B}}^{\Pi_{i\in I_n}^d \tilde{f}_i}$.

Definition 20 Let $\tilde{\mathcal{B}}^{\tilde{f}_i} \in \tilde{\mathbf{B}}_L(P_i, O_i)$ for all $i \in I_n$ and $\overline{O} = \cup_{i\in I_n} O_i$. Define L-fuzzy bag disjunctive Cartesian product of $\{\tilde{\mathcal{B}}^{\tilde{f}_i}\}_{i\in I_n}$, which is called C_n^d-L-fuzzy bag, by

$$\Pi_{i\in I_n}^d \tilde{\mathcal{B}}^{\tilde{f}_i} = (\Pi_{i\in I_n}^d \tilde{f}_i, B^{\Pi_{i\in I_n}^d \tilde{f}_i}), \tag{6}$$

where $\Pi_{i\in I_n}^d \tilde{f}_i : \Pi_{i\in I_n} P_i \to \mathcal{F}_L(\overline{O})$ such that $(\Pi_{i\in I_n}^d \tilde{f}_i)((p_1, p_2, \ldots, p_n)) = \cup_{i\in I_n} \tilde{f}_i(p_i)$ for all $p_i \in P_i$. Also,

Table 12 Values of $(\tilde{f} \times \tilde{g})((p_1, p_2))$

(p_1, p_2)	$(\tilde{f} \times \tilde{g})((p_1, p_2))$
(young, tall)	$\{\frac{0.7}{\text{Ben}}, \frac{0.2}{\text{Sue}}, \frac{0.4}{\text{Tom}}, \frac{0}{\text{John}}, \frac{0.7}{\text{Stan}}, \frac{0.4}{\text{Bill}}, \frac{0.2}{\text{Kim}}, \frac{0.7}{\text{Ana}}, \frac{0.1}{\text{Sara}}\} \times \{\frac{0.8}{\text{Ben}}, \frac{0.6}{\text{Sue}}, \frac{0}{\text{Tom}}, \frac{0.1}{\text{John}}, \frac{0.8}{\text{Stan}}, \frac{0.6}{\text{Bill}}, \frac{0.5}{\text{Kim}}, \frac{0.7}{\text{Ana}}, \frac{0.5}{\text{Sara}}\}$
(young, medium)	$\{\frac{0.7}{\text{Ben}}, \frac{0.2}{\text{Sue}}, \frac{0.4}{\text{Tom}}, \frac{0}{\text{John}}, \frac{0.7}{\text{Stan}}, \frac{0.4}{\text{Bill}}, \frac{0.2}{\text{Kim}}, \frac{0.7}{\text{Ana}}, \frac{0.1}{\text{Sara}}\} \times \{\frac{0.3}{\text{Ben}}, \frac{0.1}{\text{Sue}}, \frac{0.1}{\text{Tom}}, \frac{0.6}{\text{John}}, \frac{0.3}{\text{Stan}}, \frac{0.1}{\text{Bill}}, \frac{0.8}{\text{Kim}}, \frac{0.1}{\text{Ana}}, \frac{0.5}{\text{Sara}}\}$
(young, short)	$\{\frac{0.7}{\text{Ben}}, \frac{0.2}{\text{Sue}}, \frac{0.4}{\text{Tom}}, \frac{0}{\text{John}}, \frac{0.7}{\text{Stan}}, \frac{0.4}{\text{Bill}}, \frac{0.2}{\text{Kim}}, \frac{0.7}{\text{Ana}}, \frac{0.1}{\text{Sara}}\} \times \{\frac{0.1}{\text{Ben}}, \frac{0}{\text{Sue}}, \frac{0.9}{\text{Tom}}, \frac{0.4}{\text{John}}, \frac{0.1}{\text{Stan}}, \frac{0}{\text{Bill}}, \frac{0.2}{\text{Kim}}, \frac{0}{\text{Ana}}, \frac{0.1}{\text{Sara}}\}$
(middle age, tall)	$\{\frac{0.3}{\text{Ben}}, \frac{0.8}{\text{Sue}}, \frac{0.7}{\text{Tom}}, \frac{0.3}{\text{John}}, \frac{0.3}{\text{Stan}}, \frac{0.7}{\text{Bill}}, \frac{0.8}{\text{Kim}}, \frac{0.3}{\text{Ana}}, \frac{0.5}{\text{Sara}}\} \times \{\frac{0.8}{\text{Ben}}, \frac{0.6}{\text{Sue}}, \frac{0}{\text{Tom}}, \frac{0.1}{\text{John}}, \frac{0.8}{\text{Stan}}, \frac{0.6}{\text{Bill}}, \frac{0.5}{\text{Kim}}, \frac{0.7}{\text{Ana}}, \frac{0.5}{\text{Sara}}\}$
(middle age, medium)	$\{\frac{0.3}{\text{Ben}}, \frac{0.8}{\text{Sue}}, \frac{0.7}{\text{Tom}}, \frac{0.3}{\text{John}}, \frac{0.3}{\text{Stan}}, \frac{0.7}{\text{Bill}}, \frac{0.8}{\text{Kim}}, \frac{0.3}{\text{Ana}}, \frac{0.5}{\text{Sara}}\} \times \{\frac{0.3}{\text{Ben}}, \frac{0.1}{\text{Sue}}, \frac{0.1}{\text{Tom}}, \frac{0.6}{\text{John}}, \frac{0.3}{\text{Stan}}, \frac{0.1}{\text{Bill}}, \frac{0.8}{\text{Kim}}, \frac{0.1}{\text{Ana}}, \frac{0.5}{\text{Sara}}\}$
(middle age, short)	$\{\frac{0.3}{\text{Ben}}, \frac{0.8}{\text{Sue}}, \frac{0.7}{\text{Tom}}, \frac{0.3}{\text{John}}, \frac{0.3}{\text{Stan}}, \frac{0.7}{\text{Bill}}, \frac{0.8}{\text{Kim}}, \frac{0.3}{\text{Ana}}, \frac{0.5}{\text{Sara}}\} \times \{\frac{0.1}{\text{Ben}}, \frac{0}{\text{Sue}}, \frac{0.9}{\text{Tom}}, \frac{0.4}{\text{John}}, \frac{0.1}{\text{Stan}}, \frac{0}{\text{Bill}}, \frac{0.2}{\text{Kim}}, \frac{0}{\text{Ana}}, \frac{0.1}{\text{Sara}}\}$
(old, tall)	$\{\frac{0.1}{\text{Ben}}, \frac{0.2}{\text{Sue}}, \frac{0.1}{\text{Tom}}, \frac{0.9}{\text{John}}, \frac{0.1}{\text{Stan}}, \frac{0.1}{\text{Bill}}, \frac{0.2}{\text{Kim}}, \frac{0.1}{\text{Ana}}, \frac{0.5}{\text{Sara}}\} \times \{\frac{0.8}{\text{Ben}}, \frac{0.6}{\text{Sue}}, \frac{0}{\text{Tom}}, \frac{0.1}{\text{John}}, \frac{0.8}{\text{Stan}}, \frac{0.6}{\text{Bill}}, \frac{0.5}{\text{Kim}}, \frac{0.7}{\text{Ana}}, \frac{0.5}{\text{Sara}}\}$
(old, medium)	$\{\frac{0.1}{\text{Ben}}, \frac{0.2}{\text{Sue}}, \frac{0.1}{\text{Tom}}, \frac{0.9}{\text{John}}, \frac{0.1}{\text{Stan}}, \frac{0.1}{\text{Bill}}, \frac{0.2}{\text{Kim}}, \frac{0.1}{\text{Ana}}, \frac{0.5}{\text{Sara}}\} \times \{\frac{0.3}{\text{Ben}}, \frac{0.1}{\text{Sue}}, \frac{0.1}{\text{Tom}}, \frac{0.6}{\text{John}}, \frac{0.3}{\text{Stan}}, \frac{0.1}{\text{Bill}}, \frac{0.8}{\text{Kim}}, \frac{0.1}{\text{Ana}}, \frac{0.5}{\text{Sara}}\}$
(old, short)	$\{\frac{0.1}{\text{Ben}}, \frac{0.2}{\text{Sue}}, \frac{0.1}{\text{Tom}}, \frac{0.9}{\text{John}}, \frac{0.1}{\text{Stan}}, \frac{0.1}{\text{Bill}}, \frac{0.2}{\text{Kim}}, \frac{0.1}{\text{Ana}}, \frac{0.5}{\text{Sara}}\} \times \{\frac{0.1}{\text{Ben}}, \frac{0}{\text{Sue}}, \frac{0.9}{\text{Tom}}, \frac{0.4}{\text{John}}, \frac{0.1}{\text{Stan}}, \frac{0}{\text{Bill}}, \frac{0.2}{\text{Kim}}, \frac{0}{\text{Ana}}, \frac{0.1}{\text{Sara}}\}$

Table 13 The values of $(\tilde{f} \times^c \tilde{g})((p_1, p_2))(o)$ with minimum

(p_1, p_2)	o								
	Ben	Sue	Tom	John	Stan	Bill	Kim	Ana	Sara
(young, tall)	0.8	0.6	0.4	0.1	0.8	0.6	0.5	0.7	0.5
(young, medium)	0.7	0.2	0.4	0.6	0.7	0.4	0.8	0.7	0.5
(young, short)	0.7	0.2	0.9	0.4	0.7	0.4	0.2	0.7	0.1
(middle age, tall)	0.8	0.8	0.7	0.3	0.8	0.7	0.8	0.7	0.5
(middle age, medium)	0.3	0.8	0.7	0.6	0.3	0.7	0.8	0.3	0.5
(middle age, short)	0.3	0.8	0.9	0.4	0.3	0.7	0.8	0.3	0.5
(old, tall)	0.8	0.6	0.1	0.9	0.8	0.6	0.5	0.7	0.5
(old, medium)	0.3	0.2	0.1	0.9	0.3	0.1	0.8	0.1	0.5
(old, short)	0.1	0.2	0.9	0.9	0.1	0.1	0.2	0.1	0.5

Table 14 The values of $(\tilde{f} \times^c \tilde{g})((p_1, p_2))(o)$ with product

(p_1, p_2)	o								
	Ben	Sue	Tom	John	Stan	Bill	Kim	Ana	Sara
(young, tall)	0.56	0.12	0.0	0.0	0.56	0.24	0.1	0.49	0.05
(young, medium)	0.21	0.02	0.04	0.0	0.21	0.04	0.01	0.49	0.05
(young, short)	0.07	0.0	0.36	0.0	0.07	0.0	0.04	0.0	0.01
(middle age, tall)	0.21	0.24	0.0	0.03	0.24	0.42	0.4	0.21	0.25
(middle age, medium)	0.09	0.08	0.07	0.18	0.09	0.07	0.16	0.03	0.25
(middle age, short)	0.03	0.0	0.63	0.12	0.03	0.0	0.16	0.0	0.05
(old, tall)	0.08	0.12	0.0	0.09	0.08	0.06	0.10	0.07	0.25
(old, medium)	0.03	0.02	0.01	0.54	0.03	0.01	0.16	0.01	0.25
(old, short)	0.1	0.0	0.09	0.36	0.01	0.0	0.04	0.0	0.05

$$B^{\Pi^d_{i\in I_n}\tilde{f}_i} = \{((p_1, p_2, \dots, p_n), \delta, card(O_\delta^{p_1,p_2,\dots,p_n}))|p_i \in P_i, \delta \in L\},$$

where $O_\delta^{p_1,p_2,\dots,p_n} = \{o \in \overline{O}|(\Pi^d_{i\in I_n}\tilde{f}_i)((p_1, p_2, \dots, p_n))(o) = \delta\}$.

Note that by Definition 6, $\Pi^d_{i\in I_n}\tilde{\mathcal{B}}^{\tilde{f}_i} = \tilde{\mathcal{B}}^{\Pi^d_{i\in I_n}\tilde{f}_i}$.

Remark 5 Definitions 19 and 20 can be defined with t-norm T or t-conorm S, see [7], instead of minimum or maximum, respectively, i.e. we can consider $T(\tilde{f}_1(p_1)(o), \dots, \tilde{f}_n(p_n)(o))$ and $S(\tilde{f}_1(p_1)(o), \dots, \tilde{f}_n(p_n)(o))$ for all $o \in O$, respectively.

Example 13 Consider the fuzzy bags of Examples 3 and 4. The C_n^c-L-fuzzy bag of $\tilde{\mathcal{B}}^{\tilde{f}}$ and $\tilde{\mathcal{B}}^{\tilde{g}}$ is $\tilde{\mathcal{B}}^{\tilde{f}\times^c\tilde{g}} = (\tilde{f} \times^c \tilde{g}, B^{\tilde{f}\times^c\tilde{g}})$, where the values of $(\tilde{f} \times^c \tilde{g})((p_1, p_2))$ for three different t-norms are given in Tables 13, 14 and 15. It is easy to write $\tilde{B}^{\tilde{f}\times^d\tilde{g}}$ using the tables.

Definition 21 ([10]) Fix C_n- L-fuzzy bag $\mathcal{B}^{\Pi_{i\in I_n}\tilde{f}_i}$. An $n-$ary L-fuzzy bag relation is a L-fuzzy sub bag of C_n- L-fuzzy bag $\mathcal{B}^{\Pi_{i\in I_n}\tilde{f}_i}$ which is denoted by $\tilde{\mathcal{B}}^{\tilde{f}_R} = (\tilde{f}_R, B^{\tilde{f}_R})$.

Table 15 The values of $(\tilde{f} \times^c \tilde{g})((p_1, p_2))(o)$ with Lukasiewicz t-norm

(p_1, p_2)	o								
	Ben	Sue	Tom	John	Stan	Bill	Kim	Ana	Sara
(young, tall)	0.5	0.0	0.0	0.0	0.5	0.0	0.0	0.4	0.0
(young, medium)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
(young, short)	0.0	0.0	0.3	0.0	0.0	0.0	0.0	0.0	0.0
(middle age, tall)	0.1	0.4	0.0	0.0	0.1	0.3	0.3	0.0	0.0
(middle age, medium)	0.0	0.0	0.0	0.0	0.0	0.0	0.6	0.0	0.0
(middle age, short)	0.0	0.0	0.6	0.0	0.0	0.0	0.0	0.0	0.0
(old, tall)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
(old, medium)	0.0	0.0	0.0	0.5	0.0	0.0	0.0	0.0	0.0
(old, short)	0.0	0.0	0.0	0.3	0.0	0.0	0.0	0.0	0.0

Definition 22 ([10]) Let $\tilde{R}_n \subseteq \Pi_{i \in I_n} \mathcal{F}_L(O_i)$ and $\tilde{l}_n : \Pi_{i \in I_n} P_i \to \Pi_{i \in I_n} \mathcal{F}_L(O_i)$. Then, $\mathcal{B}^{\tilde{l}_n}_{\tilde{R}_n} = (\tilde{l}_n, B^{\tilde{l}_n}_{\tilde{R}_n})$ is the fuzzy bag induced by $\tilde{R}_n$ and $\tilde{l}_n$. Where,

$$B^{\tilde{l}_n}_{\tilde{R}_n} = \{((p_1, \ldots, p_n), \delta, card(O^{p_1,\ldots,p_n}_{\delta,\tilde{l}_n}))|\ p_i \in P_i,\ i \in I_n, \delta \in L\}$$

and $O^{p_1,\ldots,p_n}_{\delta,\tilde{l}_n} = \{(o_1, \ldots, o_n) \in \Pi_{i \in I_n} O_i | \tilde{l}((p_1, \ldots, p_n))(o_1, \ldots, o_n) = \delta, \tilde{l}_n ((p_1, \ldots, p_n)) \in \tilde{R}_n\}$.

Example 14 ([10]) Let $L = [0, 1]$, O be as in Example 4 and $R_2 = \{(o_1, o_2)|o_1, o_2 \in O, o_1 = o_2\}$. Now, let Table 16 gives $\tilde{l}_2 : P_1 \times P_2 \to \tilde{R}_2 \subseteq \mathcal{F}(O)^2$, where $P_1 = \{\text{young, middle age, old}\}$ and $P_2 = \{\text{tall, medium, short}\}$.

Thus, by Definition 22, we can present this information by the fuzzy bag $\mathcal{B}^{\tilde{l}_2}_{\tilde{R}_2} = (\tilde{l}_2, B^{\tilde{l}_2}_{\tilde{R}_2})$, where $B^{\tilde{l}_2}_{\tilde{R}_2}$ can be easily given using information of Table 16.

4 Alpha-Cuts of L-Fuzzy Bags

The notion of α-cut plays a fairly big role in the fuzzy theory. So, in this section, we define this notion for the bags. Here are some notations.

Notation 4 ([12]) If $\alpha \in L$, then $\uparrow \alpha = \{c \in L | c \geq \alpha\}$. Thus, $\uparrow$ is a mapping from L into $\mathcal{P}(L)$ and $\uparrow \alpha$ is called the up set of α.

Definition 23 ([10]) The α-cut of an L-fuzzy bag $\tilde{\mathcal{B}}^{\tilde{f}} = (\tilde{f}, B^{\tilde{f}}) \in \tilde{\mathbf{B}}_L(P, O)$ is defined as the crisp bag $(\tilde{\mathcal{B}}^{\tilde{f}})_\alpha = (\tilde{f}_\alpha, B^{\tilde{f}_\alpha})$, where for all $p \in P$

$$\tilde{f}_\alpha(p) = \tilde{f}(p)^{-1}(\uparrow \alpha),$$

for all $\alpha \in L$.

Table 16 Function $\tilde{l}_2$

(p_1, p_2)	$\tilde{l}_2((p_1, p_2))(o_1, o_2)$
(young, tall)	$\{\frac{0.7}{(Ben,Ben)}, \frac{0.2}{(Sue,Sue)}, \frac{0}{(Tom,Tom)}, \frac{0}{(John,John)}, \frac{0.7}{(Stan,Stan)}, \frac{0.4}{(Bill,Bill)}, \frac{0.2}{(Kim,Kim)}, \frac{0.7}{(Ana,Ana)}, \frac{0.1}{(Sara,Sara)}\}$
(young, medium)	$\{\frac{0.3}{(Ben,Ben)}, \frac{0.1}{(Sue,Sue)}, \frac{0.1}{(Tom,Tom)}, \frac{0}{(John,John)}, \frac{0.3}{(Stan,Stan)}, \frac{0.1}{(Bill,Bill)}, \frac{0.2}{(Kim,Kim)}, \frac{0.1}{(Ana,Ana)}, \frac{0.1}{(Sara,Sara)}\}$
(young, short)	$\{\frac{0.1}{(Ben,Ben)}, \frac{0}{(Sue,Sue)}, \frac{0.4}{(Tom,Tom)}, \frac{0}{(John,John)}, \frac{0.1}{(Stan,Stan)}, \frac{0}{(Bill,Bill)}, \frac{0.2}{(Kim,Kim)}, \frac{0}{(Ana,Ana)}, \frac{0.1}{(Sara,Sara)}\}$
(middle age, tall)	$\{\frac{0.3}{(Ben,Ben)}, \frac{0.6}{(Sue,Sue)}, \frac{0}{(Tom,Tom)}, \frac{0.1}{(John,John)}, \frac{0.3}{(Stan,Stan)}, \frac{0.6}{(Bill,Bill)}, \frac{0.5}{(Kim,Kim)}, \frac{0.3}{(Ana,Ana)}, \frac{0.5}{(Sara,Sara)}\}$
(middle age, medium)	$\{\frac{0.3}{(Ben,Ben)}, \frac{0.1}{(Sue,Sue)}, \frac{0.1}{(Tom,Tom)}, \frac{0.3}{(John,John)}, \frac{0.3}{(Stan,Stan)}, \frac{0.1}{(Bill,Bill)}, \frac{0.8}{(Kim,Kim)}, \frac{0.1}{(Ana,Ana)}, \frac{0.5}{(Sara,Sara)}\}$
(middle age, short)	$\{\frac{0.1}{(Ben,Ben)}, \frac{0}{(Sue,Sue)}, \frac{0.7}{(Tom,Tom)}, \frac{0.3}{(John,John)}, \frac{0.1}{(Stan,Stan)}, \frac{0}{(Bill,Bill)}, \frac{0.2}{(Kim,Kim)}, \frac{0}{(Ana,Ana)}, \frac{0.1}{(Sara,Sara)}\}$
(old, tall)	$\{\frac{0.1}{(Ben,Ben)}, \frac{0.2}{(Sue,Sue)}, \frac{0}{(Tom,Tom)}, \frac{0.1}{(John,John)}, \frac{0.1}{(Stan,Stan)}, \frac{0.1}{(Bill,Bill)}, \frac{0.2}{(Kim,Kim)}, \frac{0.1}{(Ana,Ana)}, \frac{0.5}{(Sara,Sara)}\}$
(old, medium)	$\{\frac{0.1}{(Ben,Ben)}, \frac{0.1}{(Sue,Sue)}, \frac{0.1}{(Tom,Tom)}, \frac{0.6}{(John,John)}, \frac{0.1}{(Stan,Stan)}, \frac{0.1}{(Bill,Bill)}, \frac{0.2}{(Kim,Kim)}, \frac{0.1}{(Ana,Ana)}, \frac{0.5}{(Sara,Sara)}\}$
(old, short)	$\{\frac{0.1}{(Ben,Ben)}, \frac{0}{(Sue,Sue)}, \frac{0.1}{(Tom,Tom)}, \frac{0.4}{(John,John)}, \frac{0.1}{(Stan,Stan)}, \frac{0}{(Bill,Bill)}, \frac{0.2}{(Kim,Kim)}, \frac{0}{(Ana,Ana)}, \frac{0.1}{(Sara,Sara)}\}$

Theorem 5 ([10]) *Let $\tilde{\mathcal{B}}^{\tilde{f}}$ and $\tilde{\mathcal{B}}^{\tilde{g}} \in \tilde{\boldsymbol{B}}_L(P, O)$. If $\mathcal{B}^{\tilde{f}_\alpha} = \mathcal{B}^{\tilde{g}_\alpha}$ for all $\alpha \in L$, then $\tilde{\mathcal{B}}^{\tilde{f}} = \tilde{\mathcal{B}}^{\tilde{g}}$.*

Thus, we have the following situation. A function $\tilde{f}(p) : O \to L$ induces a function $\tilde{f}(p)^{-1} \uparrow: L \to \mathcal{P}(O)$. We already know from the Theorem 5 that associating $\tilde{f}(p)$ with the function $\tilde{f}(p)^{-1} \uparrow$ is an injection.

Theorem 6 ([10]) *Let L be a complete lattice, $\mathcal{F}_L(O)$ be the set of all mappings from O to L, and $\mathcal{L}(O)$ be the set of all mappings $g : L \to \mathcal{P}(O)$ such that for all subsets D of L,*

$$g(\vee D) = \cap_{d \in D} g(d).$$

Then, the mapping $\Phi : \mathcal{F}_L(O) \to \mathcal{L}(O)$ given by $\Phi(\tilde{f}(p)) = \tilde{f}(p)^{-1} \uparrow$ is a bijection.

In the case of fuzzy bags, we study them more specifically see the following.

Definition 24 ([10]) Let $\alpha \in [0, 1]$. Then, α-cut of fuzzy bag $\tilde{\mathcal{B}}^{\tilde{f}} \in \tilde{\mathbf{B}}(P, O)$ is a crisp bag $(\tilde{\mathcal{B}}^{\tilde{f}})_\alpha = (\tilde{f}_\alpha, B^{\tilde{f}_\alpha})$ where, $\tilde{f}_\alpha : P \to \mathcal{P}(O)$ is a function in which for all $p \in P$, $\tilde{f}_\alpha(p) = \{o \in O | \tilde{f}(p)(o) \geqslant \alpha\}$ and

$$B^{\tilde{f}_\alpha} = \{(p, card(\tilde{f}_\alpha(p))) | p \in P\}.$$

Definition 25 ([10]) Let $\alpha \in [0, 1]$. Then, strong α-cut of fuzzy bag $\tilde{\mathcal{B}}^{\tilde{f}} \in \tilde{\mathbf{B}}(P, O)$ is the crisp bag $(\tilde{\mathcal{B}}^{\tilde{f}})_{\alpha^\cdot} = (\tilde{f}_{\alpha^\cdot}, B^{\tilde{f}_{\alpha^\cdot}})$ where, $\tilde{f}_{\alpha^\cdot} : P \to \mathcal{P}(O)$ is a function which for all $p \in P$, $\tilde{f}_{\alpha^\cdot}(p) = \{o \in O | \tilde{f}(p)(o) > \alpha\}$ and

$$B^{\tilde{f}_{\alpha^\cdot}} = \{(p, card(\tilde{f}_{\alpha^\cdot}(p))) | p \in P\}.$$

Note that by Definition 1, we have $\mathcal{B}^{\tilde{f}_\alpha} = (\tilde{\mathcal{B}}^{\tilde{f}})_\alpha$ and $\mathcal{B}^{\tilde{f}_{\alpha^\cdot}} = (\tilde{\mathcal{B}}^{\tilde{f}})_{\alpha^\cdot}$.

Notation 5 ([10]) For all $p \in P$, we set
$\tilde{f}_{[\alpha,\beta)}(p) = \{o \in O | \alpha \le \tilde{f}(p)(o) < \beta\}$ and
$\tilde{f}_{(\alpha,\beta]}(p) = \{o \in O | \alpha < \tilde{f}(p)(o) \le \beta\}$.

Some useful results for the fuzzy bags are given in the next theorem.

Theorem 7 ([10]) *Let $\tilde{\mathcal{B}}^{\tilde{f}}, \tilde{\mathcal{B}}^{\tilde{g}} \in \tilde{\boldsymbol{B}}(P, O)$, $\alpha, \beta \in [0, 1]$ and $\alpha \leqslant \beta$.*

(i) $\mathcal{B}^{\tilde{f}_{\beta^\cdot}} \tilde{\sqsubseteq} \mathcal{B}^{\tilde{f}_\beta} \tilde{\sqsubseteq} \mathcal{B}^{\tilde{f}_{\alpha^\cdot}} \tilde{\sqsubseteq} \mathcal{B}^{\tilde{f}_\alpha}$,
(ii) $\mathcal{B}^{\tilde{f}_\alpha} = \mathcal{B}^{\tilde{f}_\beta}$ *if and only if* $\mathcal{B}^{\tilde{f}_{[\alpha,\beta)}} = \mathcal{B}^0$,
(iii) $\mathcal{B}^{\tilde{f}_{\alpha^\cdot}} = \mathcal{B}^{\tilde{f}_{\beta^\cdot}}$ *if and only if* $\mathcal{B}^{\tilde{f}_{(\alpha,\beta]}} = \mathcal{B}^0$,
(iv) $(\tilde{\mathcal{B}}^{\tilde{f}} \cup \tilde{\mathcal{B}}^{\tilde{g}})_\alpha = \mathcal{B}^{\tilde{f}_\alpha} \cup \mathcal{B}^{\tilde{g}_\alpha}$ *and* $(\tilde{\mathcal{B}}^{\tilde{f}} \cup \tilde{\mathcal{B}}^{\tilde{g}})_{\alpha^\cdot} = \mathcal{B}^{\tilde{f}_{\alpha^\cdot}} \cup \mathcal{B}^{\tilde{g}_{\alpha^\cdot}}$,
(v) $(\tilde{\mathcal{B}}^{\tilde{f}} \cap \tilde{\mathcal{B}}^{\tilde{g}})_\alpha = \mathcal{B}^{\tilde{f}_\alpha} \cap \mathcal{B}^{\tilde{g}_\alpha}$ *and* $(\tilde{\mathcal{B}}^{\tilde{f}} \cap \tilde{\mathcal{B}}^{\tilde{g}})_{\alpha^\cdot} = \mathcal{B}^{\tilde{f}_{\alpha^\cdot}} \cap \mathcal{B}^{\tilde{g}_{\alpha^\cdot}}$.

In the following example, we compute α-cuts of a fuzzy bag.

Table 17 The values of $\tilde{f}_\alpha(p)$ for Example 15

α	p		
	Young	Middle age	Old
0.0	O	O	O
0.1	O \ {John}	O	O
0.2	O \ {John, Sara}	O	{Sue, John, Kim, Sara}
0.3	O \ {Sue, John, Kim, Sara}	O	{John, Sara}
0.4	O \ {Sue, John, Kim, Sara}	{Sue, Tom, Bill, Kim, Sara}	{John, Sara}
0.5	{Ben, Stan, Ana}	{Sue, Tom, Bill, Kim, Sara}	{John, Sara}
0.7	{Ben, Stan, Ana}	{Sue, Tom, Bill, Kim}	{John}
0.8	∅	{Sue, Kim}	{John}
0.9	∅	∅	{John}

Example 15 ([10]) Consider the fuzzy bag of Example 3. We compute α-cuts, $\mathcal{B}^{\tilde{f}_\alpha} = (\tilde{f}_\alpha, B^{\tilde{f}_\alpha})$. Where, $\tilde{f}_\alpha(p)$ is presented in Table 17 and $B^{\tilde{f}_\alpha}$ is as follows

$$B^{\tilde{f}_0} = \{(\text{young}, 9), (\text{middle age}, 9), (\text{old}, 9)\}, \quad B^{\tilde{f}_{0.1}} = \{(\text{young}, 8), (\text{middle age}, 9), (\text{old}, 9)\},$$
$$B^{\tilde{f}_{0.2}} = \{(\text{young}, 7), (\text{middle age}, 9), (\text{old}, 4)\}, \quad B^{\tilde{f}_{0.3}} = \{(\text{young}, 5), (\text{middle age}, 9), (\text{old}, 2)\},$$
$$B^{\tilde{f}_{0.4}} = \{(\text{young}, 5), (\text{middle age}, 5), (\text{old}, 2)\}, \quad B^{\tilde{f}_{0.5}} = \{(\text{young}, 3), (\text{middle age}, 5), (\text{old}, 2)\},$$
$$B^{\tilde{f}_{0.7}} = \{(\text{young}, 3), (\text{middle age}, 4), (\text{old}, 1)\}, \quad B^{\tilde{f}_{0.8}} = \{(\text{middle age}, 2), (\text{old}, 1)\},$$
$$B^{\tilde{f}_{0.9}} = \{(\text{old}, 1)\}.$$

Definition 26 ([10]) Let $\mathcal{B}^f \in \mathbf{B}(P, O)$ and $\alpha \in [0, 1]$. We define fuzzy bag $\widetilde{\alpha\mathcal{B}^f} = \tilde{\mathcal{B}}^{\widetilde{\alpha f}} = (\widetilde{\alpha f}, B^{\widetilde{\alpha f}})$. Where,

$$\widetilde{\alpha f}(p)(o) = \min(\alpha, \chi_{f_{(p)}}(o)) = \alpha\chi_{f_{(p)}}(o),$$

for all $o \in O$ and $p \in P$.

Theorem 8 ([10]) *Let $\tilde{\mathcal{B}}^{\tilde{f}}$ be a fuzzy bag and let $\mathcal{B}^{\tilde{f}_\alpha}$ be α-cut of $\tilde{\mathcal{B}}^{\tilde{f}}$. Then,*

$$\tilde{\mathcal{B}}^{\tilde{f}} = \bigcup_{\alpha\in[0,1]} \widetilde{\alpha\mathcal{B}^{\tilde{f}_\alpha}}.$$

Theorem 9 ([10]) *Let $\tilde{\mathcal{B}}^{\tilde{f}}$ be a fuzzy bag and let $\mathcal{B}^{\tilde{f}_{\alpha\cdot}}$ be a strong α-cut of $\tilde{\mathcal{B}}^{\tilde{f}}$. Then,*

$$\tilde{\mathcal{B}}^{\tilde{f}} = \bigcup_{\alpha\in[0,1]} \widetilde{\alpha\mathcal{B}^{\tilde{f}_{\alpha\cdot}}}.$$

Theorem 10 ([10]) *Let* $\tilde{\mathcal{B}}^{\tilde{f}} \in \tilde{\boldsymbol{B}}(P, O)$ *and* $\{\mathcal{B}^{\tilde{g}_\alpha} | \alpha \in [0, 1]\}$ *be a class of elements of* $\boldsymbol{B}(P, O)$ *such that* $\mathcal{B}^{\tilde{f}_{\alpha^\cdot}} \sqsubseteq \mathcal{B}^{\tilde{g}_\alpha} \sqsubseteq \mathcal{B}^{\tilde{f}_\alpha}$. *Then,*

$$\tilde{\mathcal{B}}^{\tilde{f}} = \bigcup_{\alpha \in [0,1]} \widetilde{\alpha \mathcal{B}}^{\tilde{g}_\alpha}.$$

Theorem 11 ([10]) *Let* $\{\mathcal{B}^{g_\alpha} | \alpha \in [0, 1]\}$ *be a class of elements of* $\boldsymbol{B}(P, O)$. *There exists* $\tilde{\mathcal{B}}^{\tilde{f}} \in \tilde{\boldsymbol{B}}(P, O)$ *such that for all* $\alpha \in [0, 1]$, $\mathcal{B}^{\tilde{f}_\alpha} = \mathcal{B}^{g_\alpha}$ *if and only if for all* $\alpha, \beta \in [0, 1]$ *such that* $\alpha \leq \beta$, $\mathcal{B}^{g_\beta} \sqsubseteq \mathcal{B}^{g_\alpha}$ *and* $\mathcal{B}^{g_0} = \mathcal{B}^1$.

So far, L-fuzzy bags and some basic concepts relevant to them are given. In the next section, the algebraic structure of the L-fuzzy bags is studied.

5 Algebraic Structure of Bags and L-Fuzzy Bags

In this section, we study the algebraic structure of bags and L-fuzzy bags. Let $\sqsubseteq$ and $\mathbf{B}(P, O)$ be as in Notation 1 and Definition 4. We have the following results. For terminology of this section, see [12].

Corollary 1 ([9]) $(\boldsymbol{B}(P, O), \cup, \cap, ^c, \mathcal{B}^0, \mathcal{B}^1)$ *is a De Morgan algebra.*

Theorem 12 ([9]) $(\boldsymbol{B}(P, O), \cup, \cap, ^c, \mathcal{B}^0, \mathcal{B}^1)$ *is a complete Boolean algebra.*

Hence, the set of all bags equipped with the proposed order is a complete Boolean algebra. Now, let $\tilde{\mathbf{B}}_L(P, O)$ and $\tilde{\sqsubseteq}$ be as in Notation 2 and Definition 9.

Theorem 13 ([10]) $(\tilde{\boldsymbol{B}}_L(P, O), \tilde{\sqsubseteq})$ *is a bounded distributive lattice.*

Definition 27 ([12]) Let X be a bounded lattice and let $x \in X$. Then, an element x^* is a pseudocomplement of x if $x \wedge x^* = 0$ and $y \leq x^*$ whenever $x \wedge y = 0$. That is, for each $x \in X$, there is a unique largest element whose meet with x is 0.

Theorem 14 ([10]) $(\tilde{\boldsymbol{B}}_L(P, O), \cup, \cap, \mathcal{B}^0, \mathcal{B}^1)$ *is pseudocomplemented.*

An element in a bounded lattice has at most one pseudocomplement since two pseudocomplements must each be less or equal to the other and hence equal. If every element has a pseudocomplement then the bounded lattice is pseudocomplemented and the unary operation * is called a pseudocomplement. The equation $x^* \vee x^{**} = 1$ is called Stone's identity and a Stone algebra is a pseudocomplemented distributive lattice satisfying this identity [12].

Definition 28 ([12]) If $(S, \vee, \wedge, ^*, 0, 1)$ is a Stone algebra, then for $S^* = \{s^* \in S | s \in S\}$, $(S^*, \vee, \wedge, ^*, 0, 1)$ is a Boolean algebra and it is called center of S.

Theorem 15 ([10]) $(\tilde{\boldsymbol{B}}_L(P, O), \cup, \cap, ^*, \mathcal{B}^0, \mathcal{B}^1)$ *is a Stone algebra whose center consists of the bags in* $\boldsymbol{B}(P, O)$.

Here, consider $F_L(O)^n$ as in Notation 3. We have the following order on it.

Definition 29 ([10]) Let $(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n), (\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n) \in F_L(O)^n$. Then, $(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n) \tilde{\preceq} (\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n)$ if and only if $\tilde{A}_i \tilde{\subseteq} \tilde{B}_i$ for all $i \in I_n$.

Note that since $\mathcal{F}_L(O)$ is a Stone algebra, $\mathcal{F}_L(O)^n$ is so as well [12]. The next theorem shows that the lattice of all L-fuzzy bags is isomorphic to the lattice of n-Cartesian product of $F_L(O)$.

Theorem 16 ([10]) *$\tilde{\boldsymbol{B}}_L(P, O)$ is isomorphic to $F_L(O)^n$, where $n = card(P)$.*

Now, let $\tilde{\mathcal{B}}^{\tilde{f}}, \tilde{\mathcal{B}}^{\tilde{g}} \in \tilde{\mathbf{B}}_L(P, O)$ and $n = card(P)$. Then, by Definition 9, $\tilde{\mathcal{B}}^{\tilde{f}} \tilde{\sqsubseteq} \tilde{\mathcal{B}}^{\tilde{g}}$ if and only if $\tilde{f}(p) \tilde{\subseteq} \tilde{g}(p)$ for all $p \in P$. That means $\tilde{\mathcal{B}}^{\tilde{f}} \tilde{\sqsubseteq} \tilde{\mathcal{B}}^{\tilde{g}}$ if and only if $\tilde{f}(p_i) \tilde{\subseteq} \tilde{g}(p_i)$ for all $i \in I_n$ if and only if $\Psi(\tilde{\mathcal{B}}^{\tilde{f}}) \tilde{\preceq} \Psi(\tilde{\mathcal{B}}^{\tilde{g}})$. Since Ψ is one to one and onto, Ψ is an order isomorphism and thus a lattice isomorphism. So, $\tilde{\mathbf{B}}_L(P, O)$ is a Stone algebra as already observed in Theorem 15.

6 Concluding Remarks

The notions of bags, L-fuzzy bags and some of their applications in which L is a complete lattice have been given. Furthermore, the concepts of α-cuts, (L-fuzzy) bag relations and related theorems were given and by some examples, these definitions have been illustrated. Finally, the algebraic structure of bags and L-fuzzy bags have been studied.

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