Curvature of the Heisenberg Group

Bartek Ewertowski and M.W. Wong

Abstract We compute the Riemannian curvature of the Heisenberg group and then contract it to the sectional curvature, Ricci curvature and the scalar curvature of the Heisenberg group. The main result so obtained is that the Heisenberg group is a space of constant positive scalar curvature.

Keywords Heisenberg group • Left-invariant vector fields • Riemannian metric • Levi–Civita connection • Riemannian curvature • Sectional curvature • Ricci curvature • Scalar curvature

Mathematics Subject Classification (2000). 53C21

1 The Heisenberg Group

If we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ via the obvious identification

$$\mathbb{R}^2 \ni (x, y) \leftrightarrow z = x + iy \in \mathbb{C},$$

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B. Ewertowski
Department of Mathematics, The University of Auckland, Private Bag 92019, 1142, Auckland, New Zealand
e-mail: bewe542@aucklanduni.ac.nz

M.W. Wong (✉)
Department of Mathematics and Statistics, York University, 4700 Keele Street, Toronto, ON, M3J 1P3, Canada
e-mail: mwwong@mathstat.yorku.ca

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and we let

$$\mathbb{H}^1 = \mathbb{C} \times \mathbb{R},$$

then $\mathbb{H}^1$ becomes a noncommutative group when equipped with the multiplication $\cdot$ given by

$$(z,t) \cdot (w,s) = \left(z + w, t + s + \frac{1}{4}[z,w]\right), \quad (z,t), (w,s) \in \mathbb{H}^1,$$

where $[z,w]$ is the symplectic form of z and w defined by

$$[z,w] = 2\,\mathrm{Im}\,(z\overline{w}).$$

In fact, $\mathbb{H}^1$ is a unimodular Lie group on which the Haar measure is just the ordinary Lebesgue measure $dz\,dt$.

Let $\mathfrak{h}$ be the Lie algebra of left-invariant vector fields on $\mathbb{H}^1$. Then a basis for $\mathfrak{h}$ is given by X, Y and T, where

$$X = \frac{\partial}{\partial x} + \frac{1}{2} y \frac{\partial}{\partial t},$$
$$Y = \frac{\partial}{\partial y} - \frac{1}{2} x \frac{\partial}{\partial t}$$

and

$$T = \frac{\partial}{\partial t}.$$

It can be checked easily that

$$[X, Y] = -T$$

and all other commutators among X, Y and T are equal to 0. References for the Heisenberg group, its Lie algebra, the sub-Laplacian $-(X^2 + Y^2)$ and the full Laplacian $-(X^2 + Y^2 + T^2)$ can be found in [2–4, 10] among many others. Compact and lucid accounts of Lie groups in [1, 8] are highly recommended.

The aim of this paper is to prove that the scalar curvature of the Heisenberg group is a positive number. This is achieved by contracting from the Riemannian curvature to the scalar curvature through the sectional curvature and the Ricci curvature. The interest in curvature of the Heisenberg group $\mathbb{H}^1$ stems from the fact [9] that $\mathbb{H}^1$ can be thought of as the three-dimensional surface that is the boundary of the four-dimensional Siegel domain, so curvature of the Heisenberg group $\mathbb{H}^1$ may be of some interest in physics.

Results on curvature of the Heisenberg group exist in the literature with a host of different notation and convention. See, for instance, [2, 6, 7]. This paper, which is very similar to the section on Riemannian approximants in [2], is another attempt using notions that can be found in any graduate textbook on Riemannian geometry.

Since indices permeate Riemannian geometry, we find it convenient to label the vector fields X, Y and T by X_1, X_2 and X_3, respectively.

2 The Riemannian Metric

We begin with the fact that there exists a left-invariant Riemannian metric g on $\mathbb{H}^1$ that turns X_1, X_2 and X_3 into an orthonormal basis for $\mathfrak{h}$ with respect to an inner product denoted by $(\ ,\ )$. In fact,

$$g(x, y, t) = \begin{bmatrix} 1 + (y^2/4) & -xy/4 & -y^2/2 \\ -xy/4 & 1 + (x^2/4) & x/2 \\ -y^2/2 & x/2 & 1 \end{bmatrix}$$

for all $(x, y, t) \in \mathbb{H}^1$.

3 The Levi–Civita Connection

A *connection* ∇ on $\mathbb{H}^1$ is a mapping

$$\mathfrak{h} \times \mathfrak{h} \ni (X, Y) \mapsto \nabla_X Y \in \mathfrak{h}$$

such that

- $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$,
- $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$,
- $\nabla_X(fY) = X(f)Y + f\nabla_X Y$,
- $\nabla_{fX} Y = f\nabla_X Y$

for all vector fields X, Y and Z in $\mathfrak{h}$ and all C^∞ real-valued functions f on $\mathbb{H}^1$. The *torsion* T of the connection ∇ is a mapping that assigns to two vector fields X and Y in $\mathfrak{h}$ another vector field $T(X, Y)$ in $\mathfrak{h}$ given by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

A connection ∇ on $\mathbb{H}^1$ is said to be *compatible* with the Riemannian metric g on $\mathbb{H}^1$ if

$$X(Y,Z) = (\nabla_X Y, Z) + (Y, \nabla_X Z) \tag{1}$$

for all vector fields X Y and Z in $\mathfrak{h}$.

The following result is crucial.

Theorem 3.1 *There exists a unique connection ∇ on $\mathbb{H}^1$ such that ∇ is torsion-free, i.e.,*

$$T(X,Y) = 0$$

for all vector fields X and Y in $\mathfrak{h}$ and ∇ is compatible with the Riemannian metric g on $\mathbb{H}^1$.

Proof For $i, j \in \{1,2,3\}$, let γ_{ij} be the real number given by

$$\gamma_{ij} = (X_i, X_j).$$

Then by compatibility,

$$X_i\gamma_{jk} = (\nabla_{X_i}X_j, X_k) + (X_j, \nabla_{X_i}X_k), \quad i,j,k \in \{1,2,3\}. \tag{2}$$

Since ∇ is torsion-free, it follows that

$$\nabla_{X_i}X_j - \nabla_{X_j}X_i = [X_i, X_j], \quad i,j \in \{1,2,3\}.$$

Permuting the indices in (2), we obtain

$$X_j\gamma_{ik} = (\nabla_{X_j}X_i, X_k) + (X_i, \nabla_{X_j}X_k) \tag{3}$$

and

$$X_k\gamma_{ij} = (\nabla_{X_k}X_i, X_j) + (X_i, \nabla_{X_k}X_j). \tag{4}$$

By (2), (3) and (4), we get

$$\begin{aligned} &X_i\gamma_{jk} + X_j\gamma_{ik} - X_k\gamma_{ij} \\ &= 2(\nabla_{X_i}X_j, X_k) - ([X_i, X_j], X_k) + ([X_j, X_k], X_i) - ([X_k, X_i], X_j). \end{aligned} \tag{5}$$

Thus, the uniqueness of ∇ follows. It remains to prove the existence. For $i, j, k \in \{1, 2, 3\}$, let

$$\Gamma_{ij}^{k} = \frac{1}{2}\sum_{l=1}^{3}(X_i g_{jl} + X_j g_{il} - X_l g_{ij})g^{lk},$$

where $[g^{jk}]$ is the inverse of $g = [g_{ij}]$, and we define $\nabla_{X_i}X_j$ by

$$\nabla_{X_i}X_j = \sum_{k=1}^{3}\Gamma_{ij}^{k}X_k.$$

□

The connection alluded to in Theorem 3.1 is known as the *Levi-Civita connection*. The functions Γ_{ij}^{k} are called the *Christoffel symbols*. We shall work with the Levi–Civita connection from now on.

4 The Riemannian Curvature

Let ∇ be the Levi–Civita connection on $\mathbb{H}^1$. Then the *Riemannian curvature* R on $\mathbb{H}^1$ is the mapping that assigns three vector fields X, Y and Z in $\mathfrak{h}$ another vector field in $\mathfrak{h}$ denoted by $R(X, Y)Z$ and given by

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{6}$$

Remark 4.1 An intuitive way to think of the Riemannian curvature R is that it measures the deviation of $\nabla_X\nabla_Y - \nabla_Y\nabla_X$ from $\nabla_{[X,Y]}$. It should be noted that the opposite sign of R is also common in the literature. For example, the sign used in [5, 7] is different from the one used in this paper.

The Riemannian curvature has many symmetries as given by the following theorem, which can be proved easily using (6).

Theorem 4.2 *Let X, Y Z and W be in* $\mathfrak{h}$. *Then we have the following symmetries.*

- $R(X, Y)Z + R(Y, X)Z = 0,$
- $(R(X, Y)Z, W) + (R(X, Y)W, Z) = 0,$
- $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0,$
- $(R(X, Y)Z, W) = (R(Z, W)X, Y).$

In order to perform computations on the Heisenberg group $\mathbb{H}^1$, the following theorem is very useful. It is the *Koszul formula* for the Heisenberg group $\mathbb{H}^1$.

Theorem 4.3 *For all vector fields X, Y and Z in* $\mathfrak{h}$, *we have*

$$(\nabla_X Y, Z) = \frac{1}{2}\{(Z, [X, Y]) - (Y, [X, Z]) - (X, [Y, Z])\}.$$

Proof By compatibility,

$$X(Y, Z) = (\nabla_X Y, Z) + (Y, \nabla_X Z) = 0,$$
$$Y(X, Z) = (\nabla_Y X, Z) + (X, \nabla_Y Z) = 0$$

and

$$Z(X, Y) = (\nabla_Z X, Y) + (X, \nabla_Z Y) = 0.$$

Since ∇ is torsion-free, we use the Jacobi identity, i.e.,

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

to get

$$0 = 2(\nabla_X Y, Z) - (Z, [X, Y]) + (Y, [X, Z]) + (X, [Y, Z]),$$

as asserted. □

The following two theorems can be proved by means of the Koszul formula and direct computations. The first theorem gives a useful formula for the Levi-Civita connection and the second theorem provides an explicit formula for the Riemannian curvature.

Theorem 4.4 *The Levi-Civita connection* ∇ *is given by*

$$\begin{aligned}
&\nabla_{X_1} X_1 = 0, && \nabla_{X_1} X_2 = -\tfrac{1}{2} X_3, && \nabla_{X_1} X_3 = \tfrac{1}{2} X_2,\\
&\nabla_{X_2} X_1 = \tfrac{1}{2} X_3, && \nabla_{X_2} X_2 = 0, && \nabla_{X_2} X_3 = -\tfrac{1}{2} X_1,\\
&\nabla_{X_3} X_1 = \tfrac{1}{2} X_2, && \nabla_{X_3} X_2 = -\tfrac{1}{2} X_1, && \nabla_{X_3} X_3 = 0.
\end{aligned}$$

Theorem 4.5 *For all vector fields X, Y and Z in* $\mathfrak{h}$,

$$\begin{aligned}
R(X, Y)Z = &-\frac{3}{4}((Y, Z)X - (X, Z)Y)\\
&+(Y, X_3)(Z, X_3)X - (X, X_3)(Z, X_3)Y\\
&+(X, X_3)(Y, Z)X_3 - (Y, X_3)(X, Z)X_3.
\end{aligned}$$

5 The Sectional Curvature

Let X and Y be two orthonormal vector fields in $\mathfrak{h}$. Then X and Y determine a *plane* in $\mathfrak{h}$. Using left translations, we get a *plane bundle* on $\mathbb{H}^1$. Let $(z,t) \in \mathbb{H}^1$. Then we can find a neighborhood U of the origin in $T_{(z,t)}\mathbb{H}^1$ and a neighborhood N of (z,t) in $\mathbb{H}^1$ such that the exponential mapping $\exp : U \to N$ is a diffeomorphism. As such, the *plane* (a subspace of $T_{(z,t)}\mathbb{H}^1$) induces a submanifold of $\mathbb{H}^1$ locally and its curvature is given by the so-called *sectional curvature* that we can now define.

Definition 5.1 Let X and Y be orthonormal vector fields in $\mathfrak{h}$. Then the sectional curvature $S(X,Y)$ determined by X and Y is the number given by

$$S(X,Y) = (R(X,Y)X, Y).$$

We can now compute the sectional curvature of the Heisenberg group.

Theorem 5.2 *Let X and Y be orthonormal vector fields in $\mathfrak{h}$. Then*

$$S(X,Y) = \frac{3}{4} - (X, X_3)^2 - (Y, X_3)^2.$$

Proof By Theorem 4.5,

$$\begin{aligned} &R(X,Y)X \\ = &-\frac{3}{4}[(Y,X)X - (X,X)Y] \\ &+(Y,X_3)(X,X_3)X - (X,X_3)(X,X_3)Y \\ &+(X,X_3)(Y,X)X_3 - (Y,X_3)(X,X)X_3. \end{aligned}$$

So,

$$S(X,Y) = (R(X,Y)X, Y) = \frac{3}{4} - (X,X_3)^2 - (Y,X_3)^2.$$

□

6 The Ricci Curvature

Let X and Y be vector fields in $\mathfrak{h}$. Then we consider the linear mapping

$$\mathfrak{h} \ni Z \mapsto R(X,Z)Y \in \mathfrak{h}.$$

We denote this mapping by $M(X, Y) : \mathfrak{h} \to \mathfrak{h}$ and we define the *Ricci curvature* $r(X, Y)$ of the Heisenberg group by

$$r(X, Y) = \operatorname{tr} M(X, Y).$$

Theorem 6.1 *Let X and Y be vector fields in* $\mathfrak{h}$. *Then the Ricci curvature* $r(X, Y)$ *of the Heisenberg group is given by*

$$r(X, Y) = \frac{1}{2}(X, Y) - (X, X_3)(Y, X_3).$$

Proof Using the orthonormal basis X_1, X_2 and X_3 for $\mathbb{H}^1$, we get by means of Theorem 4.5

$$\begin{aligned} &R(X, X_j)Y \\ &= -\frac{3}{4}[(X_j, Y)X - (X, Y)X_j] \\ &\quad +(X_j, X_3)(Y, X_3)X - (X, X_3)(Y, X_3)X_j \\ &\quad +(X, X_3)(X_j, Y)X_3 - (X_j, X_3)(X, Y)X_3. \end{aligned}$$

So, for $j \in \{1, 2, 3\}$,

$$\begin{aligned} &(R(X, X_j)Y, X_j) \\ &= -\frac{3}{4}[(X_j, Y)(X, X_j) - (X, Y)] \\ &\quad +(X_j, X_3)(Y, X_3)(X, X_j) - (X, X_3)(Y, X_3) \\ &\quad +(X, X_3)(X_j, Y)(X_3, X_j) - (X_j, X_3)(X, Y)(X_3, X_j). \end{aligned}$$

Therefore by Parseval's identity,

$$\begin{aligned} &\operatorname{tr} M(X, Y) \\ &= -\frac{3}{4}[(X, Y) - 3(X, Y)] \\ &\quad +(X, X_3)(Y, X_3) - 3(X, X_3)(Y, X_3) \\ &\quad +(X, X_3)(Y, X_3) - (X, Y) \\ &= \frac{1}{2}(X, Y) - \frac{5}{4}(X, X_3)(Y, X_3), \end{aligned}$$

as required. □

By Theorem 6.1, the Ricci curvature is the mapping Ric : $\mathfrak{h} \to \mathfrak{h}$ given by

$$\mathrm{Ric}(X) = \frac{1}{2}X - (X, X_3)X_3$$

for all X in $\mathfrak{h}$.

7 The Scalar Curvature

The scalar curvature κ of the Heisenberg group is defined by

$$\kappa = \mathrm{tr}\,(\mathrm{Ric}).$$

Theorem 7.1 *The scalar curvature κ of the Heisenberg group is given by*

$$\kappa = \frac{1}{2}.$$

Proof Let V_1, V_2 and V_3 be an orthonormal basis for $\mathbb{H}^1$. Then

$$\begin{aligned}\kappa &= \mathrm{tr}\,(\mathrm{Ric}) \\ &= \sum_{j=1}^{3}(\mathrm{Ric}\, V_j, V_j) \\ &= \frac{1}{2}\sum_{j=1}^{3}(V_j, V_j) - \sum_{j=1}^{3}(V_j, V_3)^2 \\ &= \frac{3}{2} - 1 = \frac{1}{2}.\end{aligned}$$

□

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