Pseudo-differential Operators on Non-isotropic Heisenberg Groups with Multi-dimensional Centers

Shahla Molahajloo

Abstract We introduce non-isotropic Heisenberg groups with multi-dimensional centers and the corresponding Schrödinger representations. The Wigner and Weyl transforms are then defined. We prove the Stone-von Neumann theorem for the non-isotropic Heisenebrg group by means of Stone-von Neumann theorem for the ordinary Heisenebrg group. Using this theorem, the Fourier transform is defined in terms of these representations and the Fourier inversion formula is given. Pseudo-differential operators with operator-valued symbols are introduced and can be thought of as non-commutative quantization. We give necessary and sufficient conditions on the symbols for which these operators are in the Hilbert-Schmidt class. We also give a characterization of trace class pseudo-differential operators and a trace formula for these trace class operators.

Keywords Pseudo-differential operators • Heisenberg group • Schrödinger representations • Wigner transforms • Weyl transforms • Fourier transforms • Hilbert-Schmidt operators

Mathematics Subject Classification (2000). Primary 47G30; Secondary 35S05

1 Introduction

The Heisenberg group is the simplest non-commutative nilpotent Lie group. It is actually the first locally compact group whose infinite-dimensional, irreducible representations were classified. Harmonic analysis on the Heisenberg group is a subject of constant interest in various areas of mathematics, from Partial Differential Equations to Geometry and Number Theory.

S. Molahajloo (✉)
Department of Mathematics, Institute for Advanced Studies in Basic Sciences, No. 444, Yousef Sobouti Boulevard, 45137-66731, Zanjan, Iran
e-mail: molahajloo@iasbs.ac.ir; smollaha@gmail.com

© Springer International Publishing AG 2017
M.W. Wong, Hongmei Zhu (eds.), *Pseudo-Differential Operators: Groups, Geometry and Applications*, Trends in Mathematics,
DOI 10.1007/978-3-319-47512-7_2

We fix the vector $(a_1, a_2, \cdots, a_n)$ in $\mathbb{R}^n$. The non-isotropic Heisenberg group on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ is defined by the group law

$$(z,t)\cdot(z',t') = \left(z + z', t + t' + \frac{1}{2}\sum_{j=1}^{n} a_j(x_j y_j' - x_j' y_j)\right),$$

for all $z = (x, y)$, $z' = (x', y')$ in $\mathbb{R}^n \times \mathbb{R}^n$ and t, t' are in $\mathbb{R}$. If we let $a_j = 1$, for all $1 \le j \le n$, then we get the ordinary Heisenberg group $\mathbb{H}^n$ see [4]. The center of the non-isotropic Heisenberg group $\mathbb{H}^n$ is the 1-dimensional subgroup Z given by

$$Z = \{(0, 0, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} : t \in \mathbb{R}\}.$$

In the non-isotropic Heisenberg group the terms $x_k y_l'$ for $l \neq k$, do not appear in the group law. In other words we do not consider these directions in the group law. We want to generalize this group to a group that has changes in other directions as well. Moreover, we want to look at a group with a multi-dimensional center which is of interest in Geometry. To do this, we consider $n \times n$ orthogonal matrices $B_1, B_2, \ldots, B_m$ such that

$$B_j^{-1} B_k = -B_k^{-1} B_j, \quad j \neq k. \tag{1}$$

Example 1.1 Let $m = 2$, then
$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ satisfy the above conditions.

Then we define the non-isotropic Heisenberg group with multi-dimensional center $\mathbb{G}$ on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ by

$$(z,t)\cdot(z',t') = \left(z + z', t + t' + \frac{1}{2}[z, z']\right),$$

for (z, t) and (z', t') in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ where $z = (x, y)$, $z' = (x', y')$ in $\mathbb{R}^n \times \mathbb{R}^n$, $t, t' \in \mathbb{R}^m$ and $[z, z'] \in \mathbb{R}^m$ is defined by

$$[z, z']_j = x' \cdot B_j y - x \cdot B_j y', \quad j = 1, 2, \ldots, m.$$

The center of the non-isotropic Heisenberg group with multi-dimensional center is of dimension m and of the form $(0, 0, t)$, $t \in \mathbb{R}^m$. To see this, we denote the center of $\mathbb{G}$ by $C(\mathbb{G})$. Let (z_0, t_0) be in $C(\mathbb{G})$, then for all $(z, t) \in \mathbb{G}$

$$(z, t)\cdot(z_0, t_0) = (z_0, t_0)\cdot(z, t).$$

Hence, $[z, z_0] = 0$. Therefore, for all $x, y \in \mathbb{R}^n$

$$x_0 B_j y - x B_j y_0 = 0, \quad 1 \le j \le n.$$

In particular for $x = x_0$, and for all $y \in \mathbb{R}^n$

$$\big(x_0, B_j(y - y_0)\big) = 0.$$

So, $B_j^{-1}x_0 = 0$, which implies $x_0 = 0$. Similarly we get $y_0 = 0$.

In fact, $\mathbb{G}$ is a unimodular Lie group on which the Haar measure is just the ordinary Lebesgue measure $dzdt$. Moreover, this is a special case of the Heisenberg type group. The Heisenberg type group first was introduced by A. Kaplan [6]. The geometric properties of the H-type group is studied in e.g. [7].

Note that if we let $m = 1$ and $B_1 = -I_n$ where I_n is the $n \times n$ identity matrix. Then we get the ordinary Heisenberg group $\mathbb{H}^n$.

It is well-known from [9, 10, 13] that Weyl transforms have intimate connections with analysis on the Heisenberg group and with the so-called twisted Laplacian studied in, e.g., [1, 11, 12]. We begin with a recall of the basic definitions and properties of Weyl transforms and Wigner transforms in, for instance, the book [13]. Let $\sigma \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then the Weyl transform $W_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is defined by

$$(W_\sigma f, g)_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(x, \xi) W(f, g)(x, \xi)\, dx\, d\xi, \quad f, g \in L^2(\mathbb{R}^n),$$

where $W(f, g)$ is the Wigner transform of f and g defined by

$$W(f, g)(x, \xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi\cdot p} f\left(x + \frac{p}{2}\right) \overline{g\left(x - \frac{p}{2}\right)} dp, \quad x, \xi \in \mathbb{R}^n.$$

Closely related to the Wigner transform $W(f, g)$ of f and g in $L^2(\mathbb{R}^n)$ is the Fourier–Wigner transform $V(f, g)$ given by

$$V(f, g)(q, p) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{iq\cdot y} f\left(y + \frac{p}{2}\right) \overline{g\left(y - \frac{p}{2}\right)} dy, \quad q, p \in \mathbb{R}^n.$$

It is easy to see that

$$W(f, g) = V(f, g)^\wedge$$

for all f and g in $L^2(\mathbb{R}^n)$, where $\wedge$ denotes the Fourier transform given by

$$\widehat{F}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} F(x)\, dx, \quad \xi \in \mathbb{R}^n,$$

for all F in $L^1(\mathbb{R}^n)$.

Let σ be a measurable function on $\mathbb{R}^n \times \mathbb{R}^n$. Then the classical pseudo-differential operator T_σ associated to the symbol σ is defined by

$$(T_\sigma \varphi)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma(x, \xi) \hat{\varphi}(x) \, d\xi, \quad x \in \mathbb{R}^n,$$

for all φ in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, provided that the integral exists. Once the Fourier inversion formula is in place, a symbol σ defined on the phase space $\mathbb{R}^n \times \mathbb{R}^n$ is inserted into the integral for the purpose of localization and a pseudo-differential operator is obtained. Another basic ingredient of pseudo-differential operators on $\mathbb{R}^n$ in the genesis is the phase space $\mathbb{R}^n \times \mathbb{R}^n$, which we can look at as the Cartesian product of the additive group $\mathbb{R}^n$ and its dual that is also the additive group $\mathbb{R}^n$. These observations allow in principle extensions of pseudo-differential operators to other groups G provided that we have an explicit formula for the dual of G and an explicit Fourier inversion formula for the Fourier transform on the group G. This program has been carried out in, e.g., [2, 3, 8, 14]. The aim of this paper is to look at pseudo-differential operators on the non-isotropic Heisenberg group with multi-dimensional center.

In Sect. 2, We define the Schrödinger representation corresponding to the non-isotropic Heisenberg group. Using the representation, we define the λ-Wigner and λ-Weyl transform related the non-isotropic Heisenberg group. The Moyal identity for the λ-Wigner transform and Hilbert-Schmidt properties of the λ-Weyl transform are proved. In Sect. 3, Using the Schrödinger represenation for the ordinary Heisenberg group we prove the Stone-von Neumann theorem on $\mathbb{G}$. Using the Von-Neumann theorem for the non-isotropic group with multi-dimensional center, we define the operator-valued Fourier transform of $\mathbb{G}$ in Sect. 4. Then, in Sect. 5, we define pseudo-differential operators corresponding to the operator-valued symbols. Then the L^2-boundedness and the Hilbert-Schmidt properties of pseudo-differential operators on the group $\mathbb{G}$ are given. Trace class pseudo-differential operators on the group $\mathbb{G}$ are given and a trace formula is given for them.

2 Schrödiner Representations for Non-isotropic Heisenberg Groups with Multi-dimensional Centers

Let

$$\mathbb{R}^{m*} = \mathbb{R}^m \setminus \{0\}$$

and let $\lambda \in \mathbb{R}^{m*}$. We define the Schrödinger representation of $\mathbb{G}$ on $L^2(\mathbb{R}^n)$ by

$$(\pi_\lambda(q, p, t)\varphi)(x) = e^{i\lambda\cdot t} e^{iq\cdot B_\lambda(x+p/2)} \varphi(x + p), \quad x \in \mathbb{R}^n$$

for all $\varphi \in L^2(\mathbb{R}^n)$ and $(q,p,t) \in \mathbb{G}$, where $z = (q,p) \in \mathbb{R}^n \times \mathbb{R}^n$ and $B_\lambda = \sum_{j=1}^m \lambda_j B_j$. If we let

$$(\pi_\lambda(q,p)\varphi)(x) = e^{iq\cdot B_\lambda(x+p/2)}\varphi(x+p).$$

Then

$$\pi_\lambda(q,p,t) = e^{i\lambda\cdot t}\pi_\lambda(q,p).$$

To prove that π_λ is a group homomorphism, we need the following easy lemma.

Lemma 2.1 *For all $z, z' \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \in \mathbb{R}^{m*}$ we have*

$$\pi_\lambda(z)\pi_\lambda(z') = e^{\frac{i}{2}\lambda\cdot[z,z']}\pi_\lambda(z+z').$$

The following theorem tells us that π_λ is in fact a unitary group representation of $\mathbb{G}$ on $L^2(\mathbb{R}^n)$.

Theorem 2.2 *π_λ is a unitary group representation of $\mathbb{G}$ on $L^2(\mathbb{R}^n)$.*

Proof By Lemma 2.1, it is easy to see that for all (z,t) and (z',t') in $\mathbb{G}$,

$$\pi_\lambda((z,t)\cdot(z',t')) = \pi_\lambda(z,t)\pi_\lambda(z',t').$$

Now let $\varphi, \psi \in L^2(\mathbb{R}^n)$. Then for all $(q,p,t) \in \mathbb{G}$,

$$\begin{aligned}
(\pi_\lambda(q,p,t)\varphi, \psi) &= \int_{\mathbb{R}^n} e^{i\lambda\cdot t}e^{iq\cdot B_\lambda(x+p/2)}\varphi(x+p)\overline{\psi(x)}\,dx \\
&= \int_{\mathbb{R}^n} \varphi(y)\overline{e^{-i\lambda\cdot t}e^{-iq\cdot B_\lambda(y-p/2)}\psi(y-p)}\,dy \\
&= \int_{\mathbb{R}^n} \varphi(y)\overline{(\pi_\lambda(-z,-t)\psi)(y)}\,dy \\
&= (\varphi, \pi_\lambda(-z,-t)\psi).
\end{aligned}$$

Hence $\pi_\lambda(z,t)^* = \pi_\lambda((z,t)^{-1})$. □

In fact π_λ is an irreducible representation of $\mathbb{G}$ on $L^2(\mathbb{R}^n)$. To prove this we need some preparation. Let $f, g \in L^2(\mathbb{R}^n)$. We define the λ-Fourier Wigner transform of f and g on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$V_\lambda(f,g)(q,p) = (2\pi)^{-n/2}(\pi_\lambda(q,p)f, g).$$

In fact,

$$V^\lambda(f,g)(q,p) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{iB_\lambda^t q\cdot x} f(x+\frac{p}{2})\overline{g(x-\frac{p}{2})}\, dx.$$

Therefore, the λ-Fourier Wigner transform is related to the ordinary Fourier Wigner transform by

$$V^\lambda(f,g)(q,p) = V(f,g)(B_\lambda^t q, p). \tag{2}$$

Note that

$$V^\lambda(f,g)(q,-p) = \overline{V^\lambda(g,f)}(q,p), \quad q,p \in \mathbb{R}^n.$$

Now, we define the λ-Wigner transform of $f, g \in L^2(\mathbb{R}^n)$ by

$$W^\lambda(f,g) = \widehat{V_\lambda(f,g)}.$$

In fact, λ-Wigner transform has the form

$$W^\lambda\,(f,g)\,(x,\xi) = |\lambda|^{-n}(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ip\cdot\xi} f(\frac{B_\lambda^t x}{|\lambda|^2}+\frac{p}{2})\overline{g(\frac{B_\lambda^t x}{|\lambda|^2}-\frac{p}{2})}\, dp$$

and it is related to the ordinary Wigner trasform by

$$W^\lambda(f,g)(x,\xi) = |\lambda|^{-n} W(f,g)(\frac{B_\lambda^t x}{|\lambda|^2},\xi)$$

for all x, ξ in $\mathbb{R}^n$. Moreover,

$$W^\lambda(f,g) = \overline{W^\lambda(g,f)}.$$

By using (1) and the fact that B_j, $1 \leq j \leq n$ are orthogonal matrices, we get the following result.

Proposition 2.1 *$B_\lambda B_\lambda^t = |\lambda|^2 I$, where I is the identity $n \times n$ matrix. In particular* $\det B_\lambda = |\lambda|^n$.

The following proposition gives us the relation between the dimesion of the center of the non-isotropic Heisenebrg group and its phase space.

Proposition 2.2 *Let $\mathbb{G}$ be the non-isotropic Heisenberg group on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$. Then $m \leq n^2$.*

Proof For all $1 \leq k \leq m$ and $1 \leq i, j \leq n$, let $(B_k)_{ij}$ be the entry of the matrix B_k in the i-th row and j-th column. Then the $n^2 \times m$ matrix

$$C = \begin{bmatrix} (B_1)_{11} & (B_2)_{11} & \dots & (B_m)_{11} \\ (B_1)_{12} & (B_2)_{12} & \dots & (B_m)_{12} \\ \vdots & \vdots & \ddots & \vdots \\ (B_1)_{1n} & (B_2)_{1n} & \dots & (B_m)_{1n} \\ (B_1)_{21} & (B_2)_{21} & \dots & (B_m)_{21} \\ (B_1)_{22} & (B_2)_{22} & \dots & (B_m)_{22} \\ \vdots & \vdots & \vdots & \vdots \\ (B_1)_{nn} & (B_2)_{nn} & \dots & (B_m)_{nn} \end{bmatrix}$$

has rank m. To prove this, it is enough to show that the columns of C are linearly independent. Let C^i be the i-th column of C and let $\lambda \in \mathbb{R}^m$ be such that

$$\sum_{i=1}^{m} \lambda_i C^i = 0.$$

It follows that $B_\lambda = 0$. Therefore by Proposition 2.1, we get $\lambda = 0$. □

Let $\sigma \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$, then we define the λ-Weyl transform $W_\sigma^\lambda f$ of f corresponding to the symbol σ by

$$\left(W_\sigma^\lambda f, g\right)_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(x,\xi) W^\lambda(f,g)(x,\xi)\, dx\, d\xi,$$

for all $g \in \mathcal{S}(\mathbb{R}^n)$. Therefore, using the Parseval's identity, we have

$$\left(W_\sigma^\lambda f, g\right)_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\sigma}(q,p) V^\lambda(f,g)(q,p)\, dq\, dp.$$

Hence, formally we can write,

$$\left(W_\sigma^\lambda f\right)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\sigma}(q,p) \left(\pi_\lambda(q,p) f\right)(x)\, dq\, dp.$$

Proposition 2.3 *Let $\sigma \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$. Then the λ-Weyl transform W_σ^λ is given by*

$$W_\sigma^\lambda = W_{\sigma_\lambda},$$

where W_{σ_λ} is the ordinary Weyl transform corresponding to the symbol

$$\sigma_\lambda(x,\xi) = \sigma(B_\lambda x, \xi).$$

Proposition 2.4 *Let $\sigma \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$. Then the λ-Weyl transform W_σ^λ is a Hilber-Schmidt operator with kernel*

$$k_\sigma^\lambda(x,p) = (\mathcal{F}_2\sigma)\left(B_\lambda(\frac{x+p}{2}), p-x\right),$$

where $\mathcal{F}_2\sigma$ is the ordinary Fourier transform of σ with respect to the second variable, i.e.,

$$(\mathcal{F}_2\sigma)(x,p) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\xi\cdot p}\sigma(x,\xi)\,d\xi.$$

Moreover,

$$\|W_\sigma^\lambda\|_{HS} = |\lambda|^{-n/2}\|\sigma\|_{L^2(\mathbb{R}^n\times\mathbb{R}^n)}$$

Proof By Proposition 2.4 and the kernel of the ordinary Weyl transform (see [13] for details), we have

$$\begin{aligned} k_\sigma^\lambda(x,p) &= (\mathcal{F}_2\sigma_\lambda)\left(\frac{x+p}{2}, p-x\right) \\ &= (\mathcal{F}_2\sigma)\left(B_\lambda(\frac{x+p}{2}), p-x\right). \end{aligned}$$

Hence,

$$\begin{aligned} \|W_\sigma^\lambda\|_{HS}^2 &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |k_\sigma^\lambda(x,p)|^2\,dx\,dp \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|(\mathcal{F}_2\sigma)\left(B_\lambda(\frac{x+p}{2}), p-x\right)\right|^2\,dx\,dp \\ &= |\lambda|^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |(\mathcal{F}_2\sigma)(x,p)|^2\,dx\,dp \\ &= |\lambda|^{-n}\|\sigma\|_{L^2(\mathbb{R}^n\times\mathbb{R}^n)}^2, \end{aligned}$$

which completes the proof. □

Let F and G be functions in $L^2(\mathbb{R}^{2n})$. The λ-twisted convolution of F and G denoted by $F *_\lambda G$ on $\mathbb{R}^{2n}$ is defined by

$$(F *_\lambda G)(z) = \int_{\mathbb{R}^{2n}} F(z-w)G(w)e^{\frac{i}{2}\lambda.[z,w]}\,dw.$$

By Lemma 2.1 we get the following theorem.

Theorem 2.3 *Let σ and τ be in $L^2(\mathbb{R}^{2n})$. Then*

$$W_\sigma^\lambda W_\tau^\lambda = W_\omega^\lambda,$$

where $\hat{\omega} = (2\pi)^{-n}(\hat{\sigma} *_\lambda \hat{\tau})$.

Using the Moyal identity for the ordinary Wigner transform we have the following Moyal identity for the λ-Wigner transform and λ-Fourier Wigner transform.

Proposition 2.5 *For all f_1, f_2, g_1, g_2 in $L^2(\mathbb{R}^n)$*

$$(W_\lambda(f_1, g_1), W_\lambda(f_2, g_2)) = |\lambda|^{-n}\,(f_1, f_2)\,\overline{(g_1, g_2)},$$

and

$$(V_\lambda(f_1, g_1), V_\lambda(f_2, g_2)) = |\lambda|^{-n}\,(f_1, f_2)\,\overline{(g_1, g_2)}.$$

Now, we are ready to prove the following theorem.

Theorem 2.4 *For all $\lambda \in \mathbb{R}^{m*}$, π_λ is a unitary irreducible representation of $\mathbb{G}$ on $L^2(\mathbb{R}^n)$.*

Proof suppose $M \subset L^2(\mathbb{R}^n)$ is a nonzero closed invariant subspace of π_λ and $f \in M \setminus \{0\}$. Then

$$\pi_\lambda(q, p, t)M \subset M, \quad (q, p, t) \in \mathbb{G}.$$

If $M \neq L^2(\mathbb{R}^n)$, then we can find $g \in L^2(\mathbb{R}^n)$ such that

$$(\pi_\lambda(q, p, t)f, g) = 0, \quad (q, p, t) \in \mathbb{G}.$$

But,

$$\begin{aligned}(\pi_\lambda(q, p, t)f, g) &= e^{i\lambda\cdot t}\,(\pi_\lambda(q, p)f, g)\\ &= e^{i\lambda\cdot t}(2\pi)^{n/2}V_\lambda(f, g)(p, q).\end{aligned}$$

So,

$$V_\lambda(f, g)(q, p) = 0$$

for all $(p, q) \in \mathbb{R}^n \times \mathbb{R}^n$. By the Moyal identity,

$$\|V_\lambda(f, g)\|^2_{L^2(\mathbb{R}^n\times\mathbb{R}^n)} = |\lambda|^{-n}\|f\|^2_{L^2(\mathbb{R}^n)}\|g\|^2_{L^2(\mathbb{R}^n)} = 0.$$

So, $f = 0$ or $g = 0$ which is a contradiction. □

3 Stone-Von Neumann Theorem on $\mathbb{G}$

Let $U(L^2(\mathbb{R}^n))$ be the space of unitary operators on $L^2(\mathbb{R}^n)$. Let $h \in \mathbb{R}^*$, then the Schrödinger representation $\rho_h : \mathbb{H}^n \to U(L^2(\mathbb{R}^n))$ on the ordinary Heisenebrg group is defined by

$$(\rho_h(q,p,t)\varphi)(x) = e^{iht}e^{iq\cdot(x+hp/2)}f(x+hp), \quad x \in \mathbb{R}^n,$$

for all $f \in L^2(\mathbb{R}^n)$. Then ρ_h is an irreducible unitary representation of $\mathbb{H}^n$ on $L^2(\mathbb{R}^n)$. By the Stone-von Neumann theorem, any irreducible unitary representation of $\mathbb{H}^n$ on a Hilbert space that is non-trivial on the center is equivalent to some ρ_h. More precisely we have

Theorem 3.1 *Let π be an irreducible unitary represenatation of $\mathbb{H}^n$ on a Hilbert space $\mathcal{H}$, such that $\pi(0,0,t) = e^{iht}I$ for some $h \in \mathbb{R}^*$. Then π is unitarily equivalent to ρ_h.*

Similarly, we prove the Stone-von Neumann theorem for the non-isotropic Heisenberg group $\mathbb{G}$. To prove we use the following lemma.

Lemma 3.2 *Let $\lambda \in \mathbb{R}^{m*}$. The mapping $\alpha_\lambda : \mathbb{G} \to \mathbb{H}^n$ defined by*

$$\alpha_\lambda(q,p,t) = (B^t_\lambda q, \frac{p}{|\lambda|}, \frac{\lambda \cdot t}{|\lambda|}), \quad (q,p,t) \in \mathbb{G}$$

is a surjective homomorphism of Lie groups. In particular, $G/\ker\alpha_\lambda$ is isomorphic to $\mathbb{H}^n$ where

$$\ker\alpha_\lambda = \{(0,0,t) : (t,\lambda) = 0\}.$$

Proof To prove α_λ is a group homomorphism, let $(q,p,t), (q',p'.t') \in \mathbb{G}$. Then

$$\alpha_\lambda((q,p,t) \cdot_{\mathbb{G}} (q',p',t')) = \alpha_\lambda(q+q', p+p', t+t' + \frac{1}{2}[z,z'])$$
$$= \left(B^t_\lambda(q+q'), \frac{p+p'}{|\lambda|}, \lambda \cdot (t+t' + \frac{1}{2}[z,z'])/|\lambda|\right)$$

Since $\lambda \cdot [z,z'] = (q', B_\lambda p) - (q, B_\lambda p')$, therefore

$$\begin{aligned} &\alpha_\lambda((q,p,t) \cdot_{\mathbb{G}} (q',p',t')) \\ &= (B^t_\lambda q, \frac{p}{|\lambda|}, \frac{\lambda \cdot t}{|\lambda|}) \cdot_{\mathbb{H}^n} (B^t_\lambda q', \frac{p'}{|\lambda|}, \frac{\lambda \cdot t'}{|\lambda|}) \\ &= \alpha_\lambda((q,p,t) \cdot_{\mathbb{H}^n} \alpha_\lambda(q',p',t')). \end{aligned} \tag{3}$$

Surjectivity is easy to see, since B_λ is invertible. □

The following lemma gives the connection between the Schrödinger representation on the ordinary Heisenberg group $\mathbb{H}^n$ and the representations π_λ on the non-isotropic Heisenberg group $\mathbb{G}$.

Lemma 3.3 *For all* $\lambda \in \mathbb{R}^{m*}$,

$$\pi_\lambda = \rho_{|\lambda|} \circ \alpha_\lambda.$$

Now, we are ready to prove the Stone von-Neumann theorem for the non-isotropic Heiseneberg group.

Theorem 3.4 *Let* Π_λ *be an irreducible unitary group representation of* $\mathbb{G}$ *on a Hilbert space* $\mathcal{H}$ *such that* $\Pi_\lambda(0, 0, t) = e^{i\lambda\cdot t}I$*, for some* $\lambda \in \mathbb{R}^m$*. Then* Π_λ *is unitarily equivalent to* π_λ

Proof Let $\Pi_{|\lambda|} : \mathbb{H}^n \to U(\mathcal{H})$ be defined by $\Pi_{|\lambda|} = \Pi_\lambda PT$ where T is the isomorphism of $\mathbb{H}^n$ onto $G/\ker\alpha_\lambda$ (see Lemma 3.2) and P is the projection from $\mathbb{G}/\ker\alpha_\lambda$ onto $\mathbb{G}$. Then $\Pi_{|\lambda|}(0, 0, t_0) = e^{i|\lambda|t_o}I$, for all $t_0 \in \mathbb{R}$. Moreover, $\Pi_{|\lambda|}$ is an irreducible unitary representation of $\mathbb{H}^n$ on the Hilbert space $\mathcal{H}$. This can be easily seen by using the fact that Π_λ is an irreducible unitary representation of $\mathbb{G}$ on $\mathcal{H}$.

□

4 Fourier Transforms and the Fourier Inversion Formula on $\mathbb{G}$

By the Stone-von Neumann theorem every irreducible unitary representation of $\mathbb{G}$ which acts non-trivially on the center is in fact unitarily equivalent to exactly one of π_λ, $\lambda \in \mathbb{R}^{m*}$. Hence, the identification of $\{\pi_\lambda : \lambda \in \mathbb{R}^{m*}\}$ with $\mathbb{R}^{m*}$ will be used. Let $f \in L^1(\mathbb{G})$ and $\lambda \in \mathbb{R}^{m*}$. We define the Fourier transform of f at λ to be the bounded linear operator $\hat{f}(\lambda)$ from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ given by

$$\hat{f}(\lambda)\varphi = \int_{\mathbb{R}^m}\int_{\mathbb{R}^{2n}} f(z, t)\,(\pi_\lambda(z, t)\varphi)\;dz\,dt, \quad \varphi \in L^2(\mathbb{R}^n).$$

To see the boundedness of $\hat{f}(\lambda)$, let $\varphi, \psi \in L^2(\mathbb{R}^n)$. Then By Schwarz inequality

$$\begin{aligned}\left|\left(\hat{f}(\lambda)\varphi, \psi\right)\right| &\le \int_{\mathbb{R}^m}\int_{\mathbb{R}^{2n}} |f(z, t)|\;|\,(\pi_\lambda(z, t)\varphi, \psi)\,|\,dz\,dt\\ &\le \int_{\mathbb{R}^m}\int_{\mathbb{R}^{2n}} |f(z, t)|\;\|\pi_\lambda(z, t)\varphi\|_{L^2(\mathbb{R}^n)}\|\psi\|_{L^2(\mathbb{R}^n)}\,dz\,dt.\\ &\le \|f\|_{L^1(\mathbb{G})}\|\varphi\|_{L^2(\mathbb{R}^n)}\|\psi\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Set

$$f^\lambda(z) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{i\lambda\cdot t} f(z,t)\,dt.$$

Then $\hat{f}(\lambda)\varphi$ has the form

$$\hat{f}(\lambda)\varphi = (2\pi)^{m/2} \int_{\mathbb{R}^{2n}} f^\lambda(z)\,(\pi_\lambda(z)\varphi)\;dz.$$

Therefore we have following proposition relating the Fourier transform $\hat{f}(\lambda)$ to the λ-Weyl transform.

Proposition 4.1 *Let $f \in L^1(\mathbb{G})$. Then for all $\lambda \in \mathbb{R}^{m*}$*

$$\hat{f}(\lambda) = (2\pi)^{(2n+m)/2} W^\lambda_{(f^\lambda)^\vee},$$

where $(f^\lambda)^\vee$ is the inverse Fourier transform of f^λ on $\mathbb{R}^{2n}$.

We have the following Plancheral's formula for the Fourier transform on the non-isotropic Heisenberg group with multi-dimensional center.

Theorem 4.1 *Let $f \in L^2(\mathbb{G})$ and $\lambda \in \mathbb{R}^{m*}$. Then $\hat{f}(\lambda) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a Hilbert-Schmidt operator. In fact we have*

(i) The kernel of $\hat{f}(\lambda)$ is given by

$$k_\lambda(x,p) = (2\pi)^{(n+m)/2} \left(\mathcal{F}_1^{-1} f^\lambda\right) \left(B_\lambda(\frac{x+p}{2}), p-x\right)$$

where $\mathcal{F}_1^{-1} f^\lambda$ is the ordinary inverse Fourier transform of f^λ with respect to the first variable, i.e.,

$$\left(\mathcal{F}_1^{-1} f^\lambda\right)(x,p) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot q} f^\lambda(q,p)\,dq. \quad (x,p) \in \mathbb{R}^n \times \mathbb{R}^n.$$

(ii) The Hilbert-Schmidt norm of $\hat{f}(\lambda)$ is given by

$$\|\hat{f}(\lambda)\|^2_{HS} = (2\pi)^{m+n} |\lambda|^{-n} \|f^\lambda\|^2_{L^2(\mathbb{R}^{2n})}.$$

(iii) Let $d\mu(\lambda) = (2\pi)^{-(n+m)}|\lambda|^n\,d\lambda$. We have the following Plancheral's formula

$$\int_{\mathbb{R}^m} \|\hat{f}(\lambda)\|^2_{HS}\,d\mu(\lambda) = \|f\|^2_{L^2(\mathbb{G})}.$$

Proof Let φ be in $L^2(\mathbb{R}^n)$. Then for all $x \in \mathbb{R}^n$,

$$\begin{aligned}\left(\hat{f}(\lambda)\varphi\right)(x) &= (2\pi)^{m/2}\int_{\mathbb{R}^{2n}} f^\lambda(q,p)\,(\pi_\lambda(q,p)\varphi)\,(x)\,dq\,dp \\ &= (2\pi)^{m/2}\int_{\mathbb{R}^{2n}} f^\lambda(q,p)e^{iq\cdot B_\lambda(x+\frac{p}{2})}\varphi(x+p)\,dq\,dp \\ &= \int_{\mathbb{R}^n}\left((2\pi)^{m/2}\int_{\mathbb{R}^n} e^{iq\cdot B_\lambda(\frac{x+p}{2})}f^\lambda(q,p-x)\,dq\right)\varphi(p)\,dp \\ &= \int_{\mathbb{R}^n} k_\lambda(x,p)\varphi(p)\,dp\end{aligned}$$

where

$$k_\lambda(x,p) = (2\pi)^{(n+m)/2}\left(\mathcal{F}_1^{-1}f^\lambda\right)\left(B_\lambda(\frac{x+p}{2}),p-x\right).$$

Hence the Hilbert-Schmidt norm of $\hat{f}(\lambda)$ is given by

$$\begin{aligned}\|\hat{f}(\lambda)\|\|_{HS}^2 &= \|k_\lambda\|^2_{L^2(\mathbb{R}^n\times\mathbb{R}^n)} \\ &= (2\pi)^{(n+m)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|\left(\mathcal{F}_1^{-1}f^\lambda\right)\left(B_\lambda(\frac{x+p}{2}),p-x\right)\right|^2 dx\,dp \\ &= (2\pi)^{(n+m)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|\left(\mathcal{F}_1^{-1}f^\lambda\right)(x,p)\right|^2|\lambda|^{-n}\,dx\,dp \\ &= |\lambda|^{-n}(2\pi)^{(n+m)}\|f^\lambda\|^2_{L^2(\mathbb{R}^{2n})} \qquad (4)\end{aligned}$$

where in (4) we used the Parseval's identity for the ordinary Fourier transform. □

Now we are ready to prove the inversion formula for the non-isotropic group Fourier transform.

Theorem 4.2 *Let f be a Schwartz function on $\mathbb{G}$. Then we have*

$$f(z,t) = \int_{\mathbb{R}^m} tr\left(\pi_\lambda(z,t)^*\hat{f}(\lambda)\right)\,d\mu(\lambda), \quad (z,t)\in\mathbb{G}.$$

Proof For all $(z,t)\in\mathbb{G}$,

$$\begin{aligned}\pi_\lambda(z,t)^*\hat{f}(\lambda) &= \pi_\lambda(-z,-t)\int_{\mathbb{R}^m}\int_{\mathbb{R}^{2n}} f(\tilde{z},\tilde{t})\,\pi_\lambda(\tilde{z},\tilde{t})\,d\tilde{z}\,d\tilde{t} \\ &= \int_{\mathbb{R}^m}\int_{\mathbb{R}^{2n}} f(\tilde{z},\tilde{t})\,\pi_\lambda\left((-z,-t))\cdot(\tilde{z},\tilde{t})\right)\,d\tilde{z}\,d\tilde{t}\end{aligned}$$

$$= \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} f(\tilde{z}, \tilde{t})\ \pi_\lambda \left(-z + \tilde{z}, -t + \tilde{t} + \frac{1}{2}[-z, \tilde{z}] \right) d\tilde{z}\, d\tilde{t}$$

$$= \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} f(\tilde{z}, \tilde{t}) e^{i\frac{\lambda}{2}\cdot[-z,\tilde{z}]} \pi_\lambda \left(-z + \tilde{z}, -t + \tilde{t} \right) d\tilde{z}\, d\tilde{t}.$$

Now, we let $z' = -z + \tilde{z}$ and $t' = -t + \tilde{t}$. W get

$$\pi_\lambda(z,t)^* \hat{f}(\lambda) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} g(z', t') \pi_\lambda(z', t')\, dz'\, dt',$$

where

$$g(z', t') = e^{-i\frac{\lambda}{2}\cdot[z,z']} f(z' + z, t' + t).$$

Hence,

$$\pi_\lambda(z,t)^* \hat{f}(\lambda) = \hat{g}(\lambda).$$

By Theorem 4.1, the kernel of $\hat{g}(\lambda)$ is given by

$$k_\lambda(x, p) = (2\pi)^{(n+m)/2} \left(\mathcal{F}_1^{-1} g^\lambda \right) \left(B_\lambda(\frac{x+p}{2}), p - x \right).$$

Therefore,

$$tr \left(\pi_\lambda(z,t)^* \hat{f}(\lambda) \right) = \int_{\mathbb{R}^n} k_\lambda(x, x)\, dx.$$

So, for $z = (u, v) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$k_\lambda(x, x) = (2\pi)^{(n+m)/2} \left(\mathcal{F}_1^{-1} g^\lambda \right) (B_\lambda x, 0)$$
$$= (2\pi)^{m/2} \int_{\mathbb{R}^n} e^{iB_\lambda x\cdot\xi} g^\lambda(\xi, 0)\, d\xi.$$

On the other hand, it is easy to see that

$$g^\lambda(z') = e^{-i\frac{\lambda}{2}\cdot[z,z']} e^{-i\lambda\cdot t} f^\lambda(z + z').$$

So, for $z = (u, v) \in \mathbb{R}^n \times \mathbb{R}^n$, and $z' = (\xi, 0)$, we get

$$g^\lambda(\xi, 0) = e^{\frac{-i}{2} B_\lambda v\cdot\xi} e^{-i\lambda\cdot t} f^\lambda(\xi + u, v).$$

Hence,

$$k_\lambda(x,x) = (2\pi)^{m/2} \int_{\mathbb{R}^n} e^{iB_\lambda x\cdot\xi} e^{\frac{-i}{2}B_\lambda v\cdot\xi} e^{-i\lambda\cdot t} f^\lambda(\xi+u,v)\, d\xi$$

$$= (2\pi)^{m/2} e^{-i\lambda\cdot t} e^{i(-B_\lambda x + B_\lambda v/2)\cdot u} \int_{\mathbb{R}^n} e^{i(B_\lambda x - B_\lambda v/2)\cdot\xi} f^\lambda(\xi,v)\, d\xi \qquad (5)$$

Therefore,

$$tr\left(\pi_\lambda(z,t)^*\hat{f}(\lambda)\right)$$

$$= (2\pi)^{m/2} e^{-i\lambda\cdot t} e^{iB_\lambda v/2\cdot u} \int_{\mathbb{R}^n} e^{-ix\cdot B_\lambda^t u} \left\{ \int_{\mathbb{R}^n} e^{i\xi\cdot(-B_\lambda v/2 + B_\lambda x)} f^\lambda(\xi,v)\, d\xi \right\} dx$$

$$= (2\pi)^{(m+n)/2} e^{-i\lambda\cdot t} e^{iB_\lambda v/2\cdot u} \int_{\mathbb{R}^n} e^{-ix\cdot B_\lambda^t u} \left(\mathcal{F}_1^{-1} f^\lambda\right)(-B_\lambda v/2 + B_\lambda x, v)\, dx$$

$$= (2\pi)^{(m+n)/2} e^{-i\lambda\cdot t} |\lambda|^{-n} \int_{\mathbb{R}^n} e^{-ix\cdot u} \left(\mathcal{F}_1^{-1} f^\lambda\right)(x,v)\, dx$$

$$= (2\pi)^{m/2+n} e^{-i\lambda\cdot t} |\lambda|^{-n} f^\lambda(u,v).$$

By integrating both sides of

$$tr\left(\pi_\lambda(z,t)^*\hat{f}(\lambda)\right)(2\pi)^{-(n+m)}|\lambda|^n = (2\pi)^{-m/2} e^{-i\lambda\cdot t} f^\lambda(z)$$

with respect to λ, we get the Fourier inversion formula. □

5 Pseudo-differential Operators on Non-isotropic Heisenberg Groups with Multi-dimensional Centers

Let $B(L^2(\mathbb{R}^n))$ be the C^*-algebra of all bounded linear operators on $L^2(\mathbb{R}^n)$. Then consider the operator valued symbol

$$\sigma : \mathbb{G} \times \mathbb{R}^{m*} \to B(L^2(\mathbb{R}^n)).$$

We define the pseudo-differential operator $T_\sigma : L^2(\mathbb{G}) \to L^2(\mathbb{G})$ corresponding to the symbol σ by

$$(T_\sigma f)(z,t) = \int_{\mathbb{R}^m} tr\left(\pi_\lambda(z,t)^*\sigma(z,t,\lambda)\hat{f}(\lambda)\right) d\mu(\lambda), \quad (z,t) \in \mathbb{G}$$

for all $f \in L^2(\mathbb{G})$. Let $HS(L^2(\mathbb{R}^n))$ be the space of Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$. We have the following theorem on L^2-boundedness of pseudo-differential operators.

Theorem 5.1 *Let* $\sigma : \mathbb{G} \times \mathbb{R}^{m*} \to HS(L^2(\mathbb{R}^n))$ *be such that*

$$C_\sigma^2 = \int_{\mathbb{R}^m} \int_{\mathbb{G}} \|\sigma(z,t,\lambda)\|_{HS}^2 \, dz \, dt \, d\mu(\lambda) < \infty.$$

Then $T_\sigma : L^2(\mathbb{G}) \to L^2(\mathbb{G})$ *is a bounded linear operator and*

$$\|T_\sigma\|_{op} \le C_\sigma,$$

where $\|\cdot\|_{op}$ *is the operator norm on the* C^**-algebra of bounded linear operators on* $L^2(\mathbb{G})$.

Proof Let $f \in L^2(\mathbb{G})$. Then by Minkowski's inequality we have

$$\begin{aligned}
&\|T_\sigma f\|_{L^2(\mathbb{G})} = \\
&\left\{ \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} \left| \int_{\mathbb{R}^m} tr\left(\pi_\lambda(z,t)^* \sigma(z,t,\lambda) \hat{f}(\lambda)\right) \, d\mu(\lambda) \right|^2 dz \, dt \right\}^{1/2} \\
&\le \int_{\mathbb{R}^m} \left\{ \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} \left| tr\left(\pi_\lambda(z,t)^* \sigma(z,t,\lambda) \hat{f}(\lambda)\right) \right|^2 dz \, dt \right\}^{1/2} d\mu(\lambda) \\
&\le \int_{\mathbb{R}^m} \left\{ \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} \|\sigma(z,t,\lambda)\|_{HS}^2 \|\hat{f}(\lambda)\|_{HS}^2 \, dz \, dt \right\}^{1/2} d\mu(\lambda) \\
&= \int_{\mathbb{R}^m} \|\hat{f}(\lambda)\|_{HS} \left\{ \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} \|\sigma(z,t,\lambda)\|_{HS}^2 \, dz \, dt \right\}^{1/2} d\mu(\lambda). \\
&\le C_\sigma \|f\|_{L^2(\mathbb{G})} \qquad (6)
\end{aligned}$$

where in (6), we used Hölder's inequality. □

The following result tells us that under suitable conditions, two symbols of the same pseudo-differential operator are equal.

Proposition 5.1 *Let* $\sigma : \mathbb{G} \times \mathbb{R}^{m*} \to HS(L^2(\mathbb{R}^n))$ *be such that*

$$\int_{\mathbb{R}^m} \int_{\mathbb{G}} \|\sigma(z,t,\lambda)\|_{HS}^2 \, dz \, dt \, d\mu(\lambda) < \infty.$$

Furthermore suppose that

$$\int_{\mathbb{R}^m} \|\sigma(z,t,\lambda)\|_{HS}\, d\mu(\lambda) < \infty, \quad (z,t) \in \mathbb{G}, \tag{7}$$

$$\sup_{(z,t,\lambda)\in\mathbb{G}\times\mathbb{R}^{m*}} \|\sigma(z,t,\lambda)\|_{HS} < \infty, \tag{8}$$

and the mapping

$$\mathbb{G} \times \mathbb{R}^{m*} \ni (z,t,\lambda) \mapsto \pi_\lambda(z,t)^* \sigma(z,t,\lambda) \in HS(L^2(\mathbb{R}^n)) \tag{9}$$

is weakly continuous. Then $T_\sigma f = 0$ *for all* f *only if*

$$\sigma(z,t,\lambda) = 0$$

for almost all $(z,t,\lambda) \in \mathbb{G} \times \mathbb{R}^{m*}$.

Proof For all $(z,t) \in \mathbb{G}$, we define $f_{z,t} \in L^2(\mathbb{G})$ by

$$\widehat{f_{z,t}}(\lambda) = \sigma(z,t,\lambda)^* \pi_\lambda(z,t).$$

Then, for all $(w,s) \in \mathbb{G}$

$$(T_\sigma f_{z,t})\,(w,s) = \int_{\mathbb{R}^m} A^\lambda_{z,t}(w,s)\, d\mu(\lambda),$$

where

$$A^\lambda_{z,t}(w,s) = tr\left(\pi_\lambda(w,s)^* \sigma(w,s,\lambda)\sigma(z,t,\lambda)^* \pi_\lambda(z,t)\right).$$

Let $(z_0, w_0) \in \mathbb{G}$. Then by the weak-continuity of the mapping (9),

$$A^\lambda_{z,t}(w,s) \to A^\lambda_{z,t}(z_0,t_0)$$

as $(w,s) \to (z_0,t_0)$. Moreover, by (8), there exits $C > 0$ such that

$$|A^\lambda_{z,t}(w,s)| \le C\|\sigma(z,t,\lambda)\|_{HS}$$

Therefore, by (7) and Lebesgue's dominated convergence theorem,

$$(T_\sigma f_{z,t})\,(w,s) \to (T_\sigma f_{z,t})\,(z_0,t_0)$$

as $(w, s) \to (z_0, t_0)$. Therefore $T_\sigma f_{z,t}$ is continuous on $\mathbb{G}$ and since by the assumption of the proposition $T_\sigma f_{z,t} = 0$ almost every where, hence

$$(T_\sigma f_{z,t})\,(z,t) = 0.$$

But

$$\begin{aligned}(T_\sigma f_{z,t})\,(z,t) &= \int_{\mathbb{R}^m} tr\left(\pi_\lambda(z,t)^*\sigma(z,t,\lambda)\sigma(z,t,\lambda)^*\pi_\lambda(z,t)\right)\,d\mu(\lambda)\\ &= \int_{\mathbb{R}^m} tr\left(\sigma(z,t,\lambda)^*\sigma(z,t,\lambda)\right)\,d\mu(\lambda)\\ &= \int_{\mathbb{R}^m} \|\sigma(z,t,\lambda)\|_{HS}^2\,d\mu(\lambda) = 0\end{aligned}$$

Hence, $\|\sigma(z,t,\lambda)\|_{HS} = 0$ for almost all $\lambda \in \mathbb{R}^{m*}$ and therefore,

$$\sigma(z,t,\lambda) = 0$$

for almost all $(z,t,\lambda) \in \mathbb{G} \times \mathbb{R}^{m*}$ □

The following theorem gives necessary and sufficient conditions on a symbol σ for $T_\sigma : L^2(\mathbb{G}) \to L^2(\mathbb{G})$ to be a Hilbert-Schmidt operator.

Theorem 5.2 *Let $\sigma : \mathbb{G} \times \mathbb{R}^{m*} \to HS\left(L^2(\mathbb{R}^n)\right)$ be a symbol satisfying the hypothesis of Proposition 5.1. Then $T_\sigma : L^2(\mathbb{G}) \to L^2(\mathbb{G})$ is a Hilbert-Schmidt operator if and only if*

$$\sigma(z,t,\lambda) = \pi_\lambda(z,t)W^\lambda_{(\alpha(z,t)^{-\lambda})^\wedge}, \quad (z,t,\lambda) \in \mathbb{G} \times \mathbb{R}^{m*},$$

where $\alpha : \mathbb{G} \to L^2(\mathbb{G})$ is weakly continuous mapping for which

$$\int_{\mathbb{R}^m}\int_{\mathbb{R}^{2n}} \|\alpha(z,t)\|^2_{L^2(\mathbb{G})}\,dz\,dt < \infty,$$

$$\sup_{(z,t,\lambda)\in\mathbb{G}\times\mathbb{R}^{m*}} |\lambda|^{-n/2}\|(\alpha(z,t))^{-\lambda}\|_{L^2(\mathbb{R}^{2n})} < \infty$$

and

$$\int_{\mathbb{R}^m} |\lambda|^{n/2}\|(\alpha(z,t))^{-\lambda}\|_{L^2(\mathbb{R}^{2n})}\,d\lambda < \infty.$$

Proof We first prove the sufficiently. Let $f \in \mathcal{S}(\mathbb{G})$. Then by Proposition 4.1,

$$(T_\sigma f)\,(z,t) = |\lambda|^n(2\pi)^{-m/2}\int_{\mathbb{R}^m} tr\left(W^\lambda_{(\alpha(z,t))^{-\lambda})^\wedge}W^\lambda_{(f^\lambda)^\vee}\right)\,d\lambda.$$

By Proposition 2.3 and the trace formula in [5], we get

$$
\begin{aligned}
& tr\left(W^{\lambda}_{(\alpha(z,t)^{-\lambda})^{\wedge}} W^{\lambda}_{(f^{\lambda})^{\vee}}\right) \\
&= (2\pi)^{-n} \int_{\mathbb{R}^{2n}} (\alpha(z,t)^{-\lambda})^{\wedge}(B_{\lambda}x,\xi)\ (f^{\lambda})^{\vee}(B_{\lambda}x,\xi)\, dx\, d\xi \\
&= (2\pi)^{-n}|\lambda|^{-n} \int_{\mathbb{R}^{2n}} (\alpha(z,t)^{-\lambda})^{\wedge}(x,\xi)\ (f^{\lambda})^{\vee}(x,\xi)\, dx\, d\xi \\
&= (2\pi)^{-n}|\lambda|^{-n} \int_{\mathbb{R}^{2n}} (\alpha(z,t)^{-\lambda})(z')\ (f^{\lambda})(z')\, dz'.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
(T_{\sigma}f)\,(z,t) &= (2\pi)^{-(m+2n)/2} \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} (\alpha(z,t)^{-\lambda})(z')\ (f^{\lambda})(z')\, dz'\, d\lambda \\
&= (2\pi)^{-(m+2n)/2} \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} \alpha(z,t)(z',\lambda) f(z',\lambda)\, dz'\, d\lambda.
\end{aligned}
$$

So, the kernel of T_{σ} is a function on $\mathbb{R}^{2n+m} \times \mathbb{R}^{2n+m}$ given by

$$
k(z,t,z',t') = (2\pi)^{-(m+2n)/2} \alpha(z,t)(z',\lambda), \quad (z,t),(z',t') \in \mathbb{R}^{2n+m}. \tag{10}
$$

Therefore,

$$
\begin{aligned}
& \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} |k(z,t,z',\lambda)|^2\, dz\, dt\, dz'\, d\lambda \\
&= (2\pi)^{-(2n+m)} \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} |\alpha(z,t)(z',\lambda)|^2\, dz\, dt\, dz'\, d\lambda \\
&= (2\pi)^{-(2n+m)} \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} \|\alpha(z,t)\|^2_{L^2(\mathbb{G})}\, dz\, dt < \infty.
\end{aligned}
$$

Thus, T_{σ} is a Hilbert-Schmidt operator. Conversely, suppose that $T_{\sigma} : L^2(\mathbb{G}) \to L^2(\mathbb{G})$ is a Hilbert Schmidt operator. Then there exists a function k in $L^2(\mathbb{R}^{2n+m} \times \mathbb{R}^{2n+m})$ such that

$$
(T_{\sigma}f)\,(z,t) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} k(z,t,z',\lambda) f(z',\lambda)\, dz'\, d\lambda, \quad (z,t) \in \mathbb{G},
$$

for all $f \in L^2(\mathbb{G})$. We define $\alpha : \mathbb{G} \to L^2(\mathbb{G})$ by

$$
\alpha(z,t)(z',\lambda) = (2\pi)^{(m+2n)/2} k(z,t,z',\lambda).
$$

Then reversing the argument in the proof of the sufficiency and using Proposition 5.1, we have

$$\sigma(z,t,\lambda) = \pi_\lambda(z,t) W^\lambda_{(\alpha(z,t)^{-\lambda})^\wedge}, \quad (z,t,\lambda) \in \mathbb{G} \times \mathbb{R}^{m*}.$$

□

Corollary 5.3 *Let* $\beta \in L^2(\mathbb{G} \times \mathbb{G})$ *be such that*

$$\int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} |\beta(z,t,z,t)|\, dz\, dt < \infty.$$

Let

$$\sigma(z,t,\lambda) = \pi_\lambda(z,t) W^\lambda_{(\alpha(z,t)^{-\lambda})^\wedge}, \quad (z,t,\lambda) \in \mathbb{G} \times \mathbb{R}^{m*},$$

where

$$\alpha(z,t)(z',\lambda) = \beta(z,t,z',\lambda), \quad (z,t),(z',\lambda) \in \mathbb{G} \times \mathbb{R}^{m*}.$$

Then $T_\sigma : L^2(\mathbb{G}) \to L^2(\mathbb{G})$ *is a trace class operator and*

$$tr(T_\sigma) = (2\pi)^{-(2n+m)} \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} \beta(z,t,z,t)\, dz\, dt.$$

Corollary 5.3 follows from the formula (10) on the kernel of the pseudo-differential operator in the proof of the preceding theorem.

Theorem 5.4 *Let* $\sigma : \mathbb{G} \times \mathbb{R}^{m*} \to HS\left(L^2(\mathbb{R}^n)\right)$ *be a symbol satisfying the hypothesis of Proposition 5.1. Then* $T_\sigma : L^2(\mathbb{G}) \to L^2(\mathbb{G})$ *is a trace class operator if and only if*

$$\sigma(z,t,\lambda) = \pi_\lambda(z,t) W^\lambda_{(\alpha(z,t)^{-\lambda})^\wedge}, \quad (z,t,\lambda) \in \mathbb{G} \times \mathbb{R}^{m*},$$

where $\alpha : \mathbb{G} \to L^2(\mathbb{G})$ *is a mapping such that the conditions of Theorem 5.2 are satisfied and*

$$\alpha(z,t)(z',\lambda) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} \alpha_1(z,t)(w,s)\alpha_2(w,s)(z',\lambda)\, dw\, ds$$

for all (z,t) *and* (z',λ) *in* $\mathbb{G} \times \mathbb{R}^{m*}$*, where* $\alpha_1 : \mathbb{G} \to L^2(\mathbb{G})$ *and* $\alpha_2 : \mathbb{G} \to L^2(\mathbb{G})$ *are such that*

$$\int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} \|\alpha_j(z,t)\|^2_{L^2(\mathbb{G})}\, dz\, dt < \infty, \quad j = 1,2.$$

Moreover, the trace of T_σ *is given by*

$$\begin{aligned} tr(T_\sigma) &= \int_{\mathbb{R}^m}\int_{\mathbb{R}^{2n}} \alpha(z,t)(z,t)\,dz\,dt \\ &= \int_{\mathbb{R}^m}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^m}\int_{\mathbb{R}^{2n}} \alpha_1(z,t)(w,s)\alpha_2(w,s)(z,t)\,dw\,ds\,dz\,dt. \end{aligned}$$

Theorem 5.4 follows from Theorem 5.2 and the fact that every trace class operator is a product of two Hilbert-Schmidt operators.

References

1. R. Beals, P.C. Greiner, *Calculus on Heisenberg Manifolds*. Annals of Mathematics Studies, vol. 119 (Princeton University Press, Princeton, 1988)
2. R.T. Carlos Andres, L^p-estimates for pseudo-differential operators on $\mathbb{Z}^n$. J. Pseudo Differ. Oper. Appl. **2**, 367–375 (2011)
3. Z. Chen, M.W. Wong, Traces of pseudo-differential operators on $\mathbb{S}^{n-1}$. J. Pseudo Differ. Oper. Appl. **4**, 13–24 (2013)
4. A. Dasgupta, M.W. Wong, Hilbert-Schmidt and trace class pseudo-differential operators on the Heisenberg group. J. Pseudo-Differ. Oper. Appl. **4**, 345–359 (2013)
5. J. Du, M.W. Wong, A trace formula for Weyl transforms. Approx. Theory Appl. **16**, 41–45 (2000)
6. A. Kaplan, Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms. Trans. Am. Math. Soc. **258**, 147–153 (1980)
7. K.A. Korànyi, Geometric properties of Heisenberg type groups. Adv. Math. **56**, 28–38 (1986)
8. S. Molahajloo, M.W. Wong, Ellipticity, Fredholmness and spectral invariance of pseudo-differential operators on $\mathbb{S}^1$. J. Pseudo Differ. Oper. Appl. **1**, 183–205 (2010)
9. S. Thangavelu, *Lectures on Hermite and Laguerre Functions* (Princeton University Press, Princeton, 1993)
10. S. Thangavelu, *Harmonic Analysis on the Heisenberg Group* (Birkhäuser, Boston, 1998)
11. J. Tie, The non-isotropic twisted Laplacian on $\mathbb{C}^n$ and the sub-Laplacian on $\mathbf{H}_n$. Commun. Partial Differ. Equ. **31**, 1047–1069 (2006)
12. J. Tie, M.W. Wong, Wave kernels of the twisted Laplacian, in *Modern Trends in Pseudo-differential Operators*. Operator Theory: Advances and Applications, vol. 172 (Birkhäuser, Basel/Boston, 2007), pp. 107–115
13. M.W. Wong, *Weyl Transforms* (Springer, New York, 1998)
14. M.W. Wong, *Discrete Fourier Analysis* (Birkhaüser, Basel, 2011)