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M. W. Wong
Hongmei Zhu
Editors

Pseudo-Differential Operators: Groups, Geometry and Applications

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Preface

Two major congresses in mathematics took place back to back in the summer of 2015. The Tenth Congress of the International Society for Analysis, Applications and Computation (ISAAC) was hosted by the University of Macau in China on August 3–8, 2015, and this was followed by the Eighth International Congress on Industrial and Applied Mathematics (ICIAM) held at the China National Convention Center in Beijing on August 10–14, 2015. Presented at these two congresses were, respectively, the special session "Pseudo-Differential Operators" and the mini-symposium "Pseudo-Differential Operators in Industries and Technologies". The former was broader in nature embracing all aspects of pseudo-differential operators in analysis, applications and computations, and the latter was skewed towards "real-life" applications. The two camps of participants in these two events had a significant and viable intersection. The editors of this volume belonged to this intersection and decided to present a volume reflecting the core of the vibrant events with papers most appropriately classified as Groups, Geometry and Applications. The overarching themes are of course on pseudo-differential operators understood by us as always in the broadest sense. These papers are complete papers by either the participants or invited contributors. They are peer-reviewed using the standards well established in pure and applied mathematics.

Categorized under Groups are two papers in ascending order of complexities of the groups being considered. They are the affine group and the non-isotropic Heisenberg group with multidimensional centre. Classified under Geometry are also three papers with one on the curvatures of the Heisenberg group, the second one on ellipticity on compact manifolds with boundary or edge and the final one on localization operators related to the quaternion Fourier transforms. Last but not least, the section that is best named as Applications in this volume contains more than half of the papers. We begin with a time-frequency approach to the study of the relationship of the Langevin equation and the simple harmonic oscillator. This is then followed by two papers on the role of pseudo-differential operators in probability with the first one in quantum probability and the second one in mathematical probability. A paper on nonlinear systems of integro-differential equations with anomalous diffusion modelled by fractional powers of the negative of the Laplacian is presented. The volume ends with one paper on the wavelet transforms and two papers on the Stockwell transforms.

The aim of this volume is to show that the development of pseudo-differential operators, which have been studied since the 1960s, is gaining momentum in its ramifications not only on different settings such as groups and geometry, but also in diverse applications such as quantum physics, probability and statistics and time-frequency analysis. This resonates well with the contemporary societal needs in basic research in mathematics relevant to the proliferation of interdisciplinary sciences. This trend fits in well with previously published volumes on pseudo-differential operators by Birkhäuser in Basel, and it is our vision that this endeavour will result in quantum leaps in the years to come.

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Pseudo-Differential Operators on the Affine Group

Aparajita Dasgupta and M.W. Wong

Abstract Pseudo-differential operators are defined on the affine group using the Fourier inversion formula for the Fourier transform on the affine group. The Weyl transform on the affine group is given and so are the L^2-L^p estimates for pseudo-differential operators on the affine group.

Keywords Affine group • SL(2, $\mathbb{R}$) • Semi-direct product • Fourier transform • Plancherel formula • Fourier inversion formula • Pseudo-differential operator • Weyl transform • L^2-L^p estimates

Mathematics Subject Classification (2000). Primary 47G30

1 Introduction

It is a well-known fact from [16] that pseudo-differential operators on $\mathbb{R}^n$ are based on the Plancherel formula for the Fourier transform on $\mathbb{R}^n$. The Plancherel formula gives rise to the Fourier inversion formula, which says that the identity operator for $L^2(\mathbb{R}^n)$ can be expressed in terms of the Fourier transform on $\mathbb{R}^n$. The Fourier inversion formula, albeit useful in many situations, gives a perfect

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symmetry, namely, the identity operator. By inserting a symbol, which is a suitable function on the phase space $\mathbb{R}^n \times \mathbb{R}^n$, we break the symmetry and obtain a much more interesting and meaningful operator with many applications in sciences and engineering. Such an operator is a pseudo-differential operator on $\mathbb{R}^n$. To extend pseudo-differential operators to other settings, we first observe that $\mathbb{R}^n$ is a group and can also be identified with the set of all irreducible and unitary representations that produce the Fourier inversion formula. So, it is natural to extend pseudo-differential operators from $\mathbb{R}^n$ to other groups with explicit irreducible and unitary representations that give Fourier inversion formulas for the Fourier transforms on the groups. Such a program has been carried out in some detail for $\mathbb{S}^1$, $\mathbb{Z}$, $\mathbb{Z}_N$, finite abelian groups, compact groups and Heisenberg groups [1, 2, 6, 8–12] among others.

The aim of this paper is to move the program forward with the affine group. In Sect. 2, we recall the basics of the affine group. The Schatten von-Neumann classes that we need to study pseudo-differential operators on affine groups are recalled in Sect. 3. The Fourier analysis that we need in this paper to define pseudo-differential operators on the affine group are given in Sect. 4. Good references are [5, 14]. That the Fourier inversion formula follows from the Plancherel formula for the Fourier transform on the affine group is shown in Sect. 6. In Sect. 5, we prove that the Fourier transform on the affine group is a Weyl transform on $L^2(\mathbb{R})$ [13]. L^2-L^p estimates for pseudo-differential operators on the affine group are given in Sects. 7 and 8.

2 The Affine Group

Let U be the upper half plane given by

$$U = \{(b, a) : b \in \mathbb{R},\ a > 0\}.$$

Then we define the binary operation $\cdot$ on U by

$$(b_1, a_1) \cdot (b_2, a_2) = (b_1 + a_1 b_2, a_1 a_2)$$

for all points (b_1, a_1) and (b_2, a_2) in U. With respect to the multiplication $\cdot$, U is a non-abelian group in which $(0, 1)$ is the identity element and the inverse element of (b, a) is $\left(-\frac{b}{a}, \frac{1}{a}\right)$ for all (b, a) in U. We call U the *affine group*. The left and right Haar measures on U are given by

$$d\mu = \frac{db\, da}{a^2}$$

and

$$d\nu = \frac{db\, da}{a}$$

respectively. Let $H^2_+(\mathbb{R})$ be the subspace of $L^2(\mathbb{R})$ defined by

$$H^2_+(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \operatorname{supp}(\hat{f}) \subseteq [0, \infty)\},$$

where $\operatorname{supp}(\hat{f})$ is the set of every x in $\mathbb{R}$ for which there is no neighborhood of x on which $\hat{f}$ is equal to zero almost everywhere. Similarly, we define $H^2_-(\mathbb{R})$ to be the subspace of $L^2(\mathbb{R})$ by

$$H^2_-(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \operatorname{supp}(\hat{f}) \subseteq (-\infty, 0]\}.$$

Obviously, $H^2_+(\mathbb{R})$ and $H^2_-(\mathbb{R})$ are closed subspaces of $L^2(\mathbb{R})$.

Let $\pi_\pm : U \to U(H^2_\pm(\mathbb{R}))$ be mappings defined by

$$(\pi_\pm(b,a)f)(x) = \frac{1}{\sqrt{a}} f\left(\frac{x-a}{b}\right), \quad x \in \mathbb{R},$$

for all points (b, a) in U and all functions f in $H^2_\pm(\mathbb{R})$. It can be shown that $\pi_\pm :$ $U \to U(H^2_\pm)$ are irreducible and unitary representations of U on $H^2_\pm(\mathbb{R})$.

Details of the affine group and its representations can be found in [5, 14].

For a geometric understanding of the affine group, we look at the set G of all affine mappings given by

$$G = \{T_{b,a} : \mathbb{R} \ni x \mapsto T_{b,a}x = ax + b \in \mathbb{R},\ b \in \mathbb{R}, a > 0\}.$$

G is a group with respect to the composition of mappings. Computing explicitly the composition of the mappings T_{b_1,a_1} and T_{b_2,a_2} in G, we get for all $x \in \mathbb{R}$,

$$\begin{aligned}(T_{b_1,a_1} \circ T_{b_2,a_2})(x) &= T_{b_1,a_1}(T_{b_2,a_2}x) = T_{b_1,a_1}(a_2x + b_2) \\ &= a_1a_2x + b_1 + a_1b_2 = T_{b_1+a_2b_1,a_1a_2}x.\end{aligned}$$

Therefore

$$T_{b_1,a_1} \circ T_{b_2,a_2} = T_{b_1+a_1b_2,a_1a_2}.$$

Thus, the group U is isomorphic to G and this is precisely the justification for calling U the affine group.

We can give another way to look at the affine group. The set $\mathbb{R}$ of all positive numbers is clearly an additive group isomorphic to the group $\{T_{b,1} : b \in \mathbb{R}\}$ of translations, which we denote by N. That N is a normal subgroup of G is easy to check. The set $\mathbb{R}^+$ of all positive real numbers is a group with respect to multiplication and is isomorphic to the group $\{T_{0,a} : a > 0\}$ of dilations, which we denote by A. Since $N \cap A = \{T_{0,1}\}$, it follows that the affine group G is given by

$$G = AN,$$

and we call G the *internal semi-direct product* of A and N and we write

$$G = A \ltimes N$$

or

$$G = \mathbb{R}^+ \ltimes \mathbb{R}.$$

More information about semi-direct products can be found in Section 5.5 of the book [4].

It should also be mentioned that the affine group is closely related to the special linear group SL(2, $\mathbb{R}$) given by

$$\mathrm{SL}\,(2,\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R},\ ad - bc = 1 \right\}.$$

By the Iwasawa decomposition, we can write

$$\mathrm{SL}\,(2,\mathbb{R}) = KAN,$$

where

$$K = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} : \theta \in \mathbb{R} \right\},$$

$$A = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{bmatrix} : \alpha > 0 \right\}$$

and

$$N = \left\{ \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix} : \beta \in \mathbb{R} \right\}.$$

The group AN is in fact the affine group. See [7] and page 136 of the book [15].

3 Schatten-von Neumann Classes

Let X be an infinite-dimensional, separable and complex Hilbert space with inner product $(\,,\,)_X$ and norm $\|\ \|_X$. Let $A : X \to X$ be a compact operator. Then $\sqrt{A^*A}$: $X \to X$ is a positive and compact operator. Hence the spectral theorem gives an orthonormal basis $\{\varphi_k : k = 1, 2, \dots\}$ for X consisting of eigenvectors of $\sqrt{A^*A}$. For $k = 1, 2, \dots$, let s_k be the eigenvalue of $\sqrt{A^*A}$ corresponding to the eigenvector φ_k. Then for $1 \le p < \infty$, we say that A is in the *Schatten von-Neumann class S_p* if

$$\sum_{k=1}^{\infty} s_k^p < \infty.$$

If $A \in S_p$, then the Schatten-von Neumann norm $\|A\|_{S_p}$ of A is defined by

$$\|A\|_{S_p} = \left(\sum_{k=1}^{\infty} s_k^p\right)^{1/p}.$$

By convention, S_∞ is taken to be the C^*-algebra of all bounded linear operators on X and the norm in S_∞ is simply the operator norm $\| \|_*$.

4 Fourier Analysis on the Affine Group

We give in this section the Fourier analysis on the affine group emphasizing the Fourier transform, the Plancherel formula and the Fourier inversion formula. To this end, we find it convenient to reformulate the irreducible and unitary representations of the affine group U on $U(H^2_\pm(\mathbb{R}))$. Let

$$\mathbb{R}_+ = [0, \infty)$$

and

$$\mathbb{R}_- = (-\infty, 0].$$

Then we look at the equivalents of $\pi_+ : U \to U(H^2_+(\mathbb{R}))$ and $\pi_- : U \to U(H^2_-(\mathbb{R}))$ denoted by, respectively, $\rho_+ : U \to U(L^2(\mathbb{R}_+))$ and $\rho_- : U \to U(L^2(\mathbb{R}_-))$, and given by

$$(\rho_+(b,a)u)(s) = a^{1/2} e^{-ibs} u(as), \quad s \in \mathbb{R}_+,$$

for all $u \in L^2(\mathbb{R}_+)$, and

$$(\rho_-(b,a)v)(s) = a^{1/2} e^{-ibs} v(as), \quad s \in \mathbb{R}_-,$$

for all $v \in L^2(\mathbb{R}_-)$. For all $\varphi \in L^2(\mathbb{R}_\pm)$, we define the functions $D_\pm\varphi$ on $\mathbb{R}_\pm$ by

$$(D_\pm\varphi)(s) = |s|^{1/2}\varphi(s), \quad s \in \mathbb{R}_\pm.$$

The unbounded linear operators $D_\pm$ on $L^2(\mathbb{R}_\pm)$ are known as the *Duflo–Moore operators* [3].

Let $f \in L^2(U, d\mu)$. Then we define the Fourier transform $\hat{f}$ on $\{\rho_+, \rho_-\}$ by

$$(\hat{f}(\rho_\pm)\varphi)(s) = \int_0^\infty \int_{-\infty}^\infty f(b,a)(\rho_\pm(b,a)D_\pm\varphi)(s)\,\frac{db\,da}{a^2}, \quad s \in \mathbb{R}_\pm,$$

for all $\varphi \in L^2(\mathbb{R}_\pm)$. We have the Plancherel formula to the effect that

$$\|\hat{f}(\rho_+)\|_{S_2}^2 + \|\hat{f}(\rho_-)\|_{S_2}^2 = \|f\|_{L^2(U,d\mu)}^2,$$

where $\| \, \|_{S_2}$ is the norm in the Hilbert space S_2 of all Hilbert-Schmidt operators on $L^2(\mathbb{R})$.

The Fourier inversion formula states that for all $f \in L^2(U, d\mu)$, we get

$$f(b,a) = \frac{\sqrt{a}}{2\pi}\mathrm{tr}(D_+\hat{f}(\rho_+)\rho_+(b,a)^*) + \frac{\sqrt{a}}{2\pi}\mathrm{tr}(D_-\hat{f}(\rho_-)\rho_-(b,a)^*)$$

for all $(b,a) \in U$.

We find it convenient to denote $\{\rho_+, \rho_-\}$ by $\{\pm\}$. Let $\sigma : U \times \{\pm\} \to B(L^2(\mathbb{R}))$ be a mapping, where $B(L^2(\mathbb{R}))$ is the C^*-algebra of all bounded linear operators on $L^2(\mathbb{R})$. Then for all $f \in L^2(U, d\mu)$, we define $T_\sigma f$ *formally* to be the function on U by

$$(T_\sigma f)(b,a) = \frac{\sqrt{a}}{2\pi}\sum_{j=\pm}\mathrm{tr}\,(\sigma(b,a,j)D_j\hat{f}(\rho_j)\rho_j(b,a)^*), \quad (b,a) \in U.$$

We call T_σ the pseudo-differential operator on the affine group U corresponding to the operator-valued symbol σ.

5 The Fourier Transform on the Affine Group

Let $f \in L^2(U, d\mu)$. Then for all $\varphi \in L^2(\mathbb{R}_+)$, we get for all $s \in (0, \infty)$,

$$(\hat{f}(\rho_+)\varphi)(s) = \int_{-\infty}^\infty \int_0^\infty f(b,a)a^{1/2}e^{-ibs}(as)^{1/2}\varphi(as)\frac{da\,db}{a^2}.$$

Let $as = t$. Then $da = \frac{dt}{s}$ and we have

$$\begin{aligned}
&(\hat{f}(\rho_+)\varphi)(s) \\
&= \int_{-\infty}^{\infty}\int_0^{\infty} f\left(b, \frac{t}{s}\right)\left(\frac{t}{s}\right)^{1/2} e^{-ibs} t^{1/2}\varphi(t) s \frac{dt\,db}{t^2} \\
&= \int_{-\infty}^{\infty}\int_0^{\infty} f\left(b, \frac{t}{s}\right) s^{1/2} e^{-ibs}\varphi(t)\frac{dt\,db}{t}
\end{aligned}$$

for all $s \in (0, \infty)$. Thus, for all $s \in (0, \infty)$,

$$(\hat{f}(\rho_+)\varphi)(s) = \int_0^{\infty} K_+^f(s,t)\varphi(t)\,dt, \tag{1}$$

where

$$K_+^f(s,t) = \frac{\sqrt{s}}{t}\int_{-\infty}^{\infty} f\left(b, \frac{t}{s}\right) e^{-ibs} db = \frac{\sqrt{s}}{t}(2\pi)^{1/2}(\mathcal{F}_1 f)\left(s, \frac{t}{s}\right) \tag{2}$$

for $0 < s, t < \infty$, where $\mathcal{F}_1 f$ denotes the Fourier transform of f with respect to the first variable.

Similarly, for all $\varphi \in L^2(\mathbb{R}_-)$, we obtain for all $s \in (-\infty, 0)$,

$$(\hat{f}(\rho_-)\varphi)(s) = \int_{-\infty}^{0} K_-^f(s,t)\varphi(t)\,dt,$$

where

$$K_-^f(s,t) = \frac{\sqrt{|s|}}{|t|}(2\pi)^{1/2}(\mathcal{F}_1 f)\left(s, \frac{t}{s}\right)$$

for $-\infty < s, t < 0$.

Let $f \in L^2(U, d\mu)$. Then we define the bounded linear operator $\hat{f}(\rho) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$\hat{f}(\rho)\varphi = \hat{f}(\rho_+)\varphi_+ + \hat{f}(\rho_-)\varphi_-,$$

where

$$\varphi_\pm = \varphi\chi_{\mathbb{R}_\pm}.$$

Here,

$$\chi_{\mathbb{R}_\pm}(s) = \begin{cases} 1, & s \in \mathbb{R}_\pm, \\ 0, & s \notin \mathbb{R}_\pm. \end{cases}$$

Thus, we have the following result.

Theorem 5.1 *Let $f \in L^2(U, d\mu)$. Then for all $\varphi \in L^2(\mathbb{R})$,*

$$(\hat{f}(\rho)\varphi)(s) = \int_{-\infty}^{\infty} K^f(s,t)\,\varphi(t)\,dt, \quad s \in \mathbb{R},$$

where

$$K^f(s,t) = \begin{cases} K_+^f(s,t), & s > 0,\ t > 0, \\ K_-^f(s,t), & s < 0,\ t < 0, \\ 0, & s > 0,\ t < 0, \\ 0, & s < 0,\ t > 0. \end{cases} \tag{3}$$

That the Fourier transform on the affine group is a Weyl transform on $L^2(\mathbb{R})$ is the content of the following theorem. First we recall the twisting operator T in [13] given by

$$(Tf)(x,y) = f\left(x + \frac{y}{2}, x - \frac{y}{2}\right), \quad x, y \in \mathbb{R},$$

for all measurable functions f on $\mathbb{R} \times \mathbb{R}$.

Theorem 5.2 *Let $f \in L^2(U, d\mu)$. Then for all $\varphi \in L^2(\mathbb{R})$,*

$$\hat{f}(\rho)\varphi = W_{\sigma_f}\varphi, \quad \varphi \in L^2(\mathbb{R}),$$

where

$$\sigma_f(x,\xi) = (2\pi)^{-1/2}(\mathcal{F}_2 T K^f)(x,\xi), \quad x, \xi \in \mathbb{R}.$$

6 The Fourier Inversion Formula

We prove in this section the Fourier inversion formula for the Fourier transform on the affine group.

Theorem 6.1 *For all $f \in L^2(U, d\mu)$, we have*

$$f(b,a) = \frac{\sqrt{a}}{2\pi} \sum_{j=\pm} \operatorname{tr}(D_j \hat{f}(\rho_j)\rho_j(b,a)^*).$$

Proof Let $\varphi \in L^2(\mathbb{R}_+)$. Then by (1) and (2), we get for all $s > 0$,

$$\begin{aligned}
&(D_+\hat{f}(\rho_+)\rho_+(b,a)^*\varphi)(s) \\
&= s^{1/2}(\hat{f}(\rho_+)\rho_+(b,a)^*\varphi)(s) \\
&= s^{1/2}\int_0^\infty K_+^f(s,t)(\rho_+(b,a)^*\varphi)(t)\,dt \\
&= s^{1/2}\int_0^\infty K_+^f(s,t)\left(\rho_+\left(-\frac{b}{a},\frac{1}{a}\right)\varphi\right)(t)\,dt \\
&= \int_0^\infty \frac{s}{t}(2\pi)^{1/2}(\mathcal{F}_1 f)\left(s,\frac{t}{s}\right)a^{-1/2}e^{i(b/a)t}\varphi\left(\frac{t}{a}\right)\,dt \\
&= \int_0^\infty \frac{s}{at}(2\pi)^{1/2}(\mathcal{F}_1 f)\left(s,\frac{at}{s}\right)a^{-1/2}e^{ibt}\varphi(t)a\,dt \\
&= \int_0^\infty \frac{s}{t}a^{-1/2}(2\pi)^{1/2}(\mathcal{F}_1 f)\left(s,\frac{at}{s}\right)e^{ibt}\varphi(t)\,dt.
\end{aligned}$$

So, the kernel $K_+^{f,b,a}$ of $D_+\hat{f}(\rho_+)\rho_+(b,a)^*$ is given by

$$K_+^{f,b,a}(s,t) = \frac{s}{t}a^{-1/2}(2\pi)^{1/2}(\mathcal{F}_1 f)\left(s,\frac{at}{s}\right)e^{ibt}$$

for all s and t in $(0,\infty)$. Since

$$\int_0^\infty K_+^{f,b,a}(s,s)\,ds = \int_0^\infty a^{-1/2}(2\pi)^{1/2}(\mathcal{F}_1 f)(s,a)e^{ibs}ds,$$

it follows that for all $(b,a) \in U$, $D_+\hat{f}(\rho_+)\rho_+(b,a)^*$ is a trace class operator and

$$\operatorname{tr}(D_+\hat{f}(\rho_+)\rho_+(b,a)^*) = (2\pi)^{1/2}\int_0^\infty a^{-1/2}(\mathcal{F}_1 f)(s,a)e^{ibs}ds.$$

Similarly, for all $(b,a) \in U$, $D_-\hat{f}(\rho_-)(b,a)\rho_-(b,a)^*$ is a trace class operator and

$$\operatorname{tr}(D_-\hat{f}(\rho_-)\rho_-(b,a)^*) = (2\pi)^{1/2}\int_{-\infty}^0 a^{-1/2}(\mathcal{F}_1 f)(s,a)e^{ibs}ds.$$

Therefore for all $(b,a) \in U$,

$$\sum_{j=\pm} \mathrm{tr}(D_j \hat{f}(\rho_j)\rho_j(b,a)^*) = a^{-1/2}(2\pi)^{1/2} \int_{-\infty}^{\infty} e^{ibs}(\mathcal{F}_1 f)(s,a)\, ds = 2\pi a^{-1/2} f(b,a)$$

and this completes the proof. □

Remark 6.2 It should be pointed out that in the preceding proof, we assume that the function f is sufficiently well-behaved and a limiting argument should be used. The details are omitted in order to manifest the essential fabric of the proof.

7 L^2 Boundedness

Theorem 7.1 *Let σ be a measurable function on $U \times \{\pm\}$ such that*

$$\sum_{j=\pm} \int_0^{\infty} \int_{-\infty}^{\infty} \|\sigma(b,a,j)D_j\|_{S_p}^2 \frac{db\, da}{a} < \infty,$$

where $1 \le p \le 2$. Then $T_\sigma : L^2(U, d\mu) \to L^2(U, d\mu)$ is a bounded linear operator. Moreover,

$$\|T_\sigma\|_* \le \left\{ \sum_{j=\pm} \int_0^{\infty} \int_{-\infty}^{\infty} \|\sigma(b,a,j)D_j\|_{S_p}^2 \frac{db\, da}{a} \right\}^{1/2},$$

where $\| \ \|_*$ is the norm in the C^*-algebra of all bounded linear operators on $L^2(\mathbb{R})$.

Proof Let $f \in L^2(U, d\mu)$. Then using Minkowski's inequality in integral form, we get

$$\begin{aligned}
&\|T_\sigma f\|_{L^2(U,d\mu)} \\
&= \left\{ \int_0^{\infty} \int_{-\infty}^{\infty} |(T_\sigma f)(b,a)|^2 \frac{db\, da}{a^2} \right\}^{1/2} \\
&= \frac{1}{2\pi} \left\{ \int_0^{\infty} \int_{-\infty}^{\infty} \left| \sum_{j=\pm} \mathrm{tr}\,(\sigma(b,a,j)D_j \hat{f}(\rho_j)\rho_j(b,a)^*) \right|^2 \frac{db\, da}{a} \right\}^{1/2} \\
&\le \frac{1}{2\pi} \sum_{j=\pm} \left\{ \int_0^{\infty} \int_{-\infty}^{\infty} |\mathrm{tr}\,(\sigma(b,a,j)D_j \hat{f}(\rho_j)\rho_j(b,a)^*)|^2 \frac{db\, da}{a} \right\}^{1/2}. \qquad (4)
\end{aligned}$$

For $1 \le p \le q \le \infty$, it follows from the definition of the Schatten-von Neumann classes that

$$S_p \subseteq S_q$$

and

$$\|A\|_{S_q} \le \|A\|_{S_p}, \quad A \in S_p.$$

Thus, it follows from (4) that

$$\begin{aligned}
&\|T_\sigma f\|_{L^2(U,d\mu)} \\
&\le \frac{1}{2\pi} \sum_{j=\pm} \left\{ \int_0^\infty \int_{-\infty}^\infty \|\hat{f}(\rho_j)\|_{S_2}^2 \|\sigma(b,a,j)D_j\|_{S_2}^2 \frac{db\,da}{a} \right\}^{1/2} \\
&\le \frac{1}{2\pi} \sum_{j=\pm} \left\{ \int_0^\infty \int_{-\infty}^\infty \|\hat{f}(\rho_j)\|_{S_2}^2 \|\sigma(b,a,j)D_j\|_{S_p}^2 \frac{db\,da}{a} \right\}^{1/2} \\
&= \frac{1}{2\pi} \sum_{j=\pm} \|\hat{f}(\rho_j)\|_{S_2} \left\{ \int_0^\infty \int_{-\infty}^\infty \|\sigma(b,a,j)D_j\|_{S_p}^2 \frac{db\,da}{a} \right\}^{1/2} \\
&\le \frac{1}{2\pi} \left\{ \sum_{j=\pm} \|\hat{f}(\rho_j)\|_{S_2}^2 \right\}^{1/2} \left\{ \sum_{j=\pm} \int_0^\infty \int_{-\infty}^\infty \|\sigma(b,a,j)D_j\|_{S_p}^2 \frac{db\,da}{a} \right\}^{1/2} \\
&= \frac{1}{2\pi} \left\{ \sum_{j=\pm} \int_0^\infty \int_{-\infty}^\infty \|\sigma(b,a,j)D_j\|_{S_p}^2 \frac{db\,da}{a} \right\}^{1/2} \|f\|_{L^2(U,d\mu)}.
\end{aligned}$$

□

8 L^2-L^p Estimates, $2 \le p \le \infty$

Theorem 8.1 *Let σ be a measurable function on $U \times \{\pm\}$ such that*

$$\sum_{j=\pm} \int_0^\infty \int_{-\infty}^\infty \|\sigma(b,a,j)D_j\|_{S_{p'}}^p \frac{db\,da}{a^{2-(p/2)}} < \infty,$$

where $2 \le p < \infty$ *and* p' *is the conjugate index of* p, *i.e.,* $\frac{1}{p} + \frac{1}{p'} = 1$. *Then* $T_\sigma : L^2(U, d\mu) \to L^p(U, d\mu)$ *is a bounded linear operator and*

$$\|T_\sigma\|_{B(L^2(U,d\mu),L^p(U,d\mu))} \le \frac{1}{2\pi} \left\{ \sum_{j=\pm} \left[\int_0^\infty \int_{-\infty}^\infty \|\sigma(b,a,j)D_j\|_{S_{p'}}^p \frac{db\,da}{a^{2-(p/2)}} \right]^{2/p} \right\}^{1/2},$$

where $\| \, \|_{B(L^2(U,d\mu),L^p(U,d\mu))}$ *is the norm in the Banach space of all bounded linear operators from* $L^2(U, d\mu)$ *into* $L^p(U, d\mu)$.

Proof Let $f \in L^p(U, d\mu)$. Then using Minkowski's inequality in integral form, we get

$$\begin{aligned}
&\|T_\sigma f\|_{L^p(U,d\mu)} \\
&= \left\{ \int_0^\infty \int_{-\infty}^\infty |(T_\sigma f)(b,a)|^p \frac{db\,da}{a^2} \right\}^{1/p} \\
&= \frac{1}{2\pi} \left\{ \int_0^\infty \int_{-\infty}^\infty \left| \sum_{j=\pm} \mathrm{tr}\,(\sigma(b,a,j)D_j \hat{f}(\rho_j)\rho_j(b,a)^*)) \right|^p \frac{db\,da}{a^{2-(p/2)}} \right\}^{1/p} \\
&\le \frac{1}{2\pi} \sum_{j=\pm} \left\{ \int_0^\infty \int_{-\infty}^\infty |\mathrm{tr}\,(\sigma(b,a,j)D_j \hat{f}(\rho_j)\rho_j(b,a)^*)|^p \frac{db\,da}{a^{2-(p/2)}} \right\}^{1/p}. \qquad (5)
\end{aligned}$$

Now, using Hölder's inequality and the Plancherel theorem, it follows from (5) that

$$\begin{aligned}
&\|T_\sigma f\|_{L^p(U,d\mu)} \\
&\le \frac{1}{2\pi} \sum_{j=\pm} \left\{ \int_0^\infty \int_{-\infty}^\infty \|\hat{f}(\rho_j)\|_{S_p}^p \|\sigma(b,a,jD_j)\|_{S_{p'}}^p \frac{db\,da}{a^{2-(p/2)}} \right\}^{1/p} \\
&= \frac{1}{2\pi} \sum_{j=\pm} \|\hat{f}(\rho_j)\|_{S_p} \left\{ \int_0^\infty \int_{-\infty}^\infty \|\sigma(b,a,j)D_j\|_{S_{p'}}^p \frac{db\,da}{a^{2-(p/2)}} \right\}^{1/p} \\
&\le \frac{1}{2\pi} \sum_{j=\pm} \|\hat{f}(\rho_j)\|_{S_2} \left\{ \int_0^\infty \int_{-\infty}^\infty \|\sigma(b,a,j)D_j\|_{S_{p'}}^p \frac{db\,da}{a^{2-(p/2)}} \right\}^{1/p} \\
&= \frac{1}{2\pi} \left\{ \sum_{j=\pm} \|\hat{f}(\rho_j)\|_{S_2}^2 \right\}^{1/2} \left\{ \sum_{j=\pm} \left[\int_0^\infty \int_{-\infty}^\infty \|\sigma(b,a,j)D_j\|_{S_{p'}}^p \frac{db\,da}{a^{2-(p/2)}} \right]^{2/p} \right\}^{1/2}
\end{aligned}$$

and this completes the proof. □

Remark 8.2 The estimate in Theorem 8.1 can be reformulated as

$$\|T_\sigma\|_{B(L^2(U,d\mu),L^p(U,d\mu))} \leq \frac{1}{2\pi}\left\{\sum_{j=\pm} \|\,\|\sigma(\cdot,\cdot,j)D_j\|_{S_{p'}}\|_{L^p(U,d\nu_p)}\right\}^{1/2},$$

where $d\nu_p = \frac{db\,da}{a^{2-(p/2)}}$ for $2 \leq p < \infty$. We fill in the endpoint $p = \infty$ in the following theorem.

Theorem 8.3 *Let σ be a measurable function on $U \times \{\pm\}$ be such that*

$$\sum_{j=\pm} \|\,\|\sigma(\cdot,\cdot,j)D_j\|_{S_1}\|^2_{L^\infty(U,d\nu_1)} < \infty.$$

Then $T_\sigma : L^2(U, d\mu) \to L^\infty(U, d\mu)$ is a bounded linear operator and

$$\|T_\sigma\|_{B(L^2(U,d\mu),L^\infty(U,d\mu))} \leq \frac{1}{2\pi}\left\{\sum_{j=\pm} \|\,\|\sigma(\cdot,\cdot,j)D_j\|_{S_1}\|^2_{L^\infty(U,d\nu_1)}\right\}^{1/2}.$$

Proof Let $f \in L^\infty(U, d\mu)$. Then by Minkowski's inequality,

$$\begin{aligned}
&\|T_\sigma f\|_{L^\infty(U,d\mu)}\\
&= \frac{1}{2\pi}\left\|\sum_{j=\pm} \mathrm{tr}\,(\sigma(\cdot,\cdot,j)D_j\hat{f}(\rho_j)\rho_j(\cdot,\cdot)^*)\right\|_{L^\infty(U,d\nu_1)}\\
&\leq \frac{1}{2\pi}\sum_{j=\pm} \|\mathrm{tr}\,(\sigma(\cdot,\cdot,j)D_j\hat{f}(\rho_j)\rho_j(\cdot,\cdot)^*)\|_{L^\infty(U,d\nu_1)}.
\end{aligned} \tag{6}$$

Using Hölder's inequality and Plancherel's theorem, it follows from (6) that

$$\begin{aligned}
&\|T_\sigma f\|_{L^\infty(U,d\mu)}\\
&\leq \frac{1}{2\pi}\sum_{j=\pm} \left\|\|\hat{f}(\rho_j)\|_{S_\infty}\|\sigma(\cdot,\cdot,j)D_j\|_{S_1}\right\|_{L^\infty(U,d\nu_1)}\\
&= \frac{1}{2\pi}\sum_{j=\pm} \|\hat{f}(\rho_j)\|_{S_\infty}\,\|\,\|\sigma(\cdot,\cdot,j)\|_{S_1}\|_{L^\infty(U,d\nu_1)}\\
&\leq \sum_{j=\pm} \|\hat{f}(\rho_j)\|_{S_2}\left\|\|\sigma(\cdot,\cdot,j)D_j\|_{S_1}\right\|_{L^\infty(U,d\nu_1)}
\end{aligned}$$

$$\leq \frac{1}{2\pi} \left\{ \sum_{j=\pm} \|\hat{f}(\rho_j)\|_{S_2}^2 \right\}^{1/2} \left\{ \sum_{j=\pm} \| \|\sigma(\cdot,\cdot,j)\|_{S_1} \|_{L^\infty(U,d\nu_1)}^2 \right\}^{1/2}$$

$$= \frac{1}{2\pi} \left\{ \sum_{j=\pm} \| \|\sigma(\cdot,\cdot,j)\|_{S_1} \|_{L^\infty(U,d\nu_1)}^2 \right\}^{1/2} \|f\|_{L^2(U,d\mu)}.$$

□

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Pseudo-differential Operators on Non-isotropic Heisenberg Groups with Multi-dimensional Centers

Shahla Molahajloo

Abstract We introduce non-isotropic Heisenberg groups with multi-dimensional centers and the corresponding Schrödinger representations. The Wigner and Weyl transforms are then defined. We prove the Stone-von Neumann theorem for the non-isotropic Heisenebrg group by means of Stone-von Neumann theorem for the ordinary Heisenebrg group. Using this theorem, the Fourier transform is defined in terms of these representations and the Fourier inversion formula is given. Pseudo-differential operators with operator-valued symbols are introduced and can be thought of as non-commutative quantization. We give necessary and sufficient conditions on the symbols for which these operators are in the Hilbert-Schmidt class. We also give a characterization of trace class pseudo-differential operators and a trace formula for these trace class operators.

Keywords Pseudo-differential operators • Heisenberg group • Schrödinger representations • Wigner transforms • Weyl transforms • Fourier transforms • Hilbert-Schmidt operators

Mathematics Subject Classification (2000). Primary 47G30; Secondary 35S05

1 Introduction

The Heisenberg group is the simplest non-commutative nilpotent Lie group. It is actually the first locally compact group whose infinite-dimensional, irreducible representations were classified. Harmonic analysis on the Heisenberg group is a subject of constant interest in various areas of mathematics, from Partial Differential Equations to Geometry and Number Theory.

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We fix the vector $(a_1, a_2, \cdots, a_n)$ in $\mathbb{R}^n$. The non-isotropic Heisenberg group on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ is defined by the group law

$$(z,t)\cdot(z',t') = \left(z+z', t+t' + \frac{1}{2}\sum_{j=1}^{n} a_j(x_j y_j' - x_j' y_j)\right),$$

for all $z = (x,y)$, $z' = (x',y')$ in $\mathbb{R}^n \times \mathbb{R}^n$ and t, t' are in $\mathbb{R}$. If we let $a_j = 1$, for all $1 \le j \le n$, then we get the ordinary Heisenberg group $\mathbb{H}^n$ see [4]. The center of the non-isotropic Heisenberg group $\mathbb{H}^n$ is the 1-dimensional subgroup Z given by

$$Z = \{(0,0,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} : t \in \mathbb{R}\}.$$

In the non-isotropic Heisenberg group the terms $x_k y_l'$ for $l \neq k$, do not appear in the group law. In other words we do not consider these directions in the group law. We want to generalize this group to a group that has changes in other directions as well. Moreover, we want to look at a group with a multi-dimensional center which is of interest in Geometry. To do this, we consider $n \times n$ orthogonal matrices $B_1, B_2, \ldots, B_m$ such that

$$B_j^{-1} B_k = -B_k^{-1} B_j, \quad j \neq k. \tag{1}$$

Example 1.1 Let $m = 2$, then
$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ satisfy the above conditions.

Then we define the non-isotropic Heisenberg group with multi-dimensional center $\mathbb{G}$ on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ by

$$(z,t)\cdot(z',t') = \left(z+z', t+t' + \frac{1}{2}[z,z']\right),$$

for (z,t) and (z',t') in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ where $z = (x,y)$, $z' = (x',y')$ in $\mathbb{R}^n \times \mathbb{R}^n$, $t, t' \in \mathbb{R}^m$ and $[z,z'] \in \mathbb{R}^m$ is defined by

$$[z,z']_j = x' \cdot B_j y - x \cdot B_j y', \quad j = 1, 2, \ldots, m.$$

The center of the non-isotropic Heisenberg group with multi-dimensional center is of dimension m and of the form $(0,0,t)$, $t \in \mathbb{R}^m$. To see this, we denote the center of $\mathbb{G}$ by $C(\mathbb{G})$. Let (z_0, t_0) be in $C(\mathbb{G})$, then for all $(z,t) \in \mathbb{G}$

$$(z,t)\cdot(z_0,t_0) = (z_0,t_0)\cdot(z,t).$$

Hence, $[z, z_0] = 0$. Therefore, for all $x, y \in \mathbb{R}^n$

$$x_0 B_j y - x B_j y_0 = 0, \quad 1 \le j \le n.$$

In particular for $x = x_0$, and for all $y \in \mathbb{R}^n$

$$\big(x_0, B_j(y - y_0)\big) = 0.$$

So, $B_j^{-1}x_0 = 0$, which implies $x_0 = 0$. Similarly we get $y_0 = 0$.

In fact, $\mathbb{G}$ is a unimodular Lie group on which the Haar measure is just the ordinary Lebesgue measure $dzdt$. Moreover, this is a special case of the Heisenberg type group. The Heisenberg type group first was introduced by A. Kaplan [6]. The geometric properties of the H-type group is studied in e.g. [7].

Note that if we let $m = 1$ and $B_1 = -I_n$ where I_n is the $n \times n$ identity matrix. Then we get the ordinary Heisenberg group $\mathbb{H}^n$.

It is well-known from [9, 10, 13] that Weyl transforms have intimate connections with analysis on the Heisenberg group and with the so-called twisted Laplacian studied in, e.g., [1, 11, 12]. We begin with a recall of the basic definitions and properties of Weyl transforms and Wigner transforms in, for instance, the book [13]. Let $\sigma \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then the Weyl transform $W_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is defined by

$$(W_\sigma f, g)_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(x, \xi) W(f, g)(x, \xi)\, dx\, d\xi, \quad f, g \in L^2(\mathbb{R}^n),$$

where $W(f, g)$ is the Wigner transform of f and g defined by

$$W(f, g)(x, \xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi\cdot p} f\left(x + \frac{p}{2}\right) \overline{g\left(x - \frac{p}{2}\right)} dp, \quad x, \xi \in \mathbb{R}^n.$$

Closely related to the Wigner transform $W(f, g)$ of f and g in $L^2(\mathbb{R}^n)$ is the Fourier–Wigner transform $V(f, g)$ given by

$$V(f, g)(q, p) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{iq\cdot y} f\left(y + \frac{p}{2}\right) \overline{g\left(y - \frac{p}{2}\right)} dy, \quad q, p \in \mathbb{R}^n.$$

It is easy to see that

$$W(f, g) = V(f, g)^\wedge$$

for all f and g in $L^2(\mathbb{R}^n)$, where $\wedge$ denotes the Fourier transform given by

$$\widehat{F}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} F(x)\, dx, \quad \xi \in \mathbb{R}^n,$$

for all F in $L^1(\mathbb{R}^n)$.

Let σ be a measurable function on $\mathbb{R}^n \times \mathbb{R}^n$. Then the classical pseudo-differential operator T_σ associated to the symbol σ is defined by

$$(T_\sigma \varphi)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma(x,\xi)\hat{\varphi}(x)\, d\xi, \quad x \in \mathbb{R}^n,$$

for all φ in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, provided that the integral exists. Once the Fourier inversion formula is in place, a symbol σ defined on the phase space $\mathbb{R}^n \times \mathbb{R}^n$ is inserted into the integral for the purpose of localization and a pseudo-differential operator is obtained. Another basic ingredient of pseudo-differential operators on $\mathbb{R}^n$ in the genesis is the phase space $\mathbb{R}^n \times \mathbb{R}^n$, which we can look at as the Cartesian product of the additive group $\mathbb{R}^n$ and its dual that is also the additive group $\mathbb{R}^n$. These observations allow in principle extensions of pseudo-differential operators to other groups G provided that we have an explicit formula for the dual of G and an explicit Fourier inversion formula for the Fourier transform on the group G. This program has been carried out in, e.g., [2, 3, 8, 14]. The aim of this paper is to look at pseudo-differential operators on the non-isotropic Heisenberg group with multi-dimensional center.

In Sect. 2, We define the Schrödinger representation corresponding to the non-isotropic Heisenberg group. Using the representation, we define the λ-Wigner and λ-Weyl transform related the non-isotropic Heisenberg group. The Moyal identity for the λ-Wigner transform and Hilbert-Schmidt properties of the λ-Weyl transform are proved. In Sect. 3, Using the Schrödinger represenation for the ordinary Heisenberg group we prove the Stone-von Neumann theorem on $\mathbb{G}$. Using the Von-Neumann theorem for the non-isotropic group with multi-dimensional center, we define the operator-valued Fourier transform of $\mathbb{G}$ in Sect. 4. Then, in Sect. 5, we define pseudo-differential operators corresponding to the operator-valued symbols. Then the L^2-boundedness and the Hilbert-Schmidt properties of pseudo-differential operators on the group $\mathbb{G}$ are given. Trace class pseudo-differential operators on the group $\mathbb{G}$ are given and a trace formula is given for them.

2 Schrödiner Representations for Non-isotropic Heisenberg Groups with Multi-dimensional Centers

Let

$$\mathbb{R}^{m*} = \mathbb{R}^m \setminus \{0\}$$

and let $\lambda \in \mathbb{R}^{m*}$. We define the Schrödinger representation of $\mathbb{G}$ on $L^2(\mathbb{R}^n)$ by

$$(\pi_\lambda(q,p,t)\varphi)(x) = e^{i\lambda\cdot t} e^{iq\cdot B_\lambda(x+p/2)} \varphi(x+p), \quad x \in \mathbb{R}^n$$

for all $\varphi \in L^2(\mathbb{R}^n)$ and $(q, p, t) \in \mathbb{G}$, where $z = (q, p) \in \mathbb{R}^n \times \mathbb{R}^n$ and $B_\lambda = \sum_{j=1}^m \lambda_j B_j$. If we let

$$(\pi_\lambda(q, p)\varphi)(x) = e^{iq\cdot B_\lambda(x+p/2)}\varphi(x+p).$$

Then

$$\pi_\lambda(q, p, t) = e^{i\lambda\cdot t}\pi_\lambda(q, p).$$

To prove that π_λ is a group homomorphism, we need the following easy lemma.

Lemma 2.1 *For all $z, z' \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \in \mathbb{R}^{m*}$ we have*

$$\pi_\lambda(z)\pi_\lambda(z') = e^{\frac{i}{2}\lambda\cdot[z,z']}\pi_\lambda(z+z').$$

The following theorem tells us that π_λ is in fact a unitary group representation of $\mathbb{G}$ on $L^2(\mathbb{R}^n)$.

Theorem 2.2 *π_λ is a unitary group representation of $\mathbb{G}$ on $L^2(\mathbb{R}^n)$.*

Proof By Lemma 2.1, it is easy to see that for all (z, t) and (z', t') in $\mathbb{G}$,

$$\pi_\lambda((z, t)\cdot(z', t')) = \pi_\lambda(z, t)\pi_\lambda(z', t').$$

Now let $\varphi, \psi \in L^2(\mathbb{R}^n)$. Then for all $(q, p, t) \in \mathbb{G}$,

$$\begin{aligned}
(\pi_\lambda(q, p, t)\varphi, \psi) &= \int_{\mathbb{R}^n} e^{i\lambda\cdot t}e^{iq\cdot B_\lambda(x+p/2)}\varphi(x+p)\overline{\psi(x)}\,dx \\
&= \int_{\mathbb{R}^n} \varphi(y)\overline{e^{-i\lambda\cdot t}e^{-iq\cdot B_\lambda(y-p/2)}\psi(y-p)}\,dy \\
&= \int_{\mathbb{R}^n} \varphi(y)\overline{(\pi_\lambda(-z, -t)\psi)(y)}\,dy \\
&= (\varphi, \pi_\lambda(-z, -t)\psi).
\end{aligned}$$

Hence $\pi_\lambda(z, t)^* = \pi_\lambda((z, t)^{-1})$. □

In fact π_λ is an irreducible representation of $\mathbb{G}$ on $L^2(\mathbb{R}^n)$. To prove this we need some preparation. Let $f, g \in L^2(\mathbb{R}^n)$. We define the λ-Fourier Wigner transform of f and g on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$V_\lambda(f, g)(q, p) = (2\pi)^{-n/2}(\pi_\lambda(q, p)f, g).$$

In fact,

$$V^\lambda(f,g)(q,p) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{iB_\lambda^t q\cdot x} f(x+\frac{p}{2})\overline{g(x-\frac{p}{2})}\,dx.$$

Therefore, the λ-Fourier Wigner transform is related to the ordinary Fourier Wigner transform by

$$V^\lambda(f,g)(q,p) = V(f,g)(B_\lambda^t q,p). \tag{2}$$

Note that

$$V^\lambda(f,g)(q,-p) = \overline{V^\lambda(g,f)}(q,p), \quad q,p\in\mathbb{R}^n.$$

Now, we define the λ-Wigner transform of $f,g\in L^2(\mathbb{R}^n)$ by

$$W^\lambda(f,g) = \widehat{V_\lambda(f,g)}.$$

In fact, λ-Wigner transform has the form

$$W^\lambda\,(f,g)\,(x,\xi) = |\lambda|^{-n}(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ip\cdot\xi} f(\frac{B_\lambda^t x}{|\lambda|^2}+\frac{p}{2})\overline{g(\frac{B_\lambda^t x}{|\lambda|^2}-\frac{p}{2})}\,dp$$

and it is related to the ordinary Wigner trasform by

$$W^\lambda(f,g)(x,\xi) = |\lambda|^{-n}W(f,g)(\frac{B_\lambda^t x}{|\lambda|^2},\xi)$$

for all x,ξ in $\mathbb{R}^n$. Moreover,

$$W^\lambda(f,g) = \overline{W^\lambda(g,f)}.$$

By using (1) and the fact that B_j, $1\le j\le n$ are orthogonal matrices, we get the following result.

Proposition 2.1 *$B_\lambda B_\lambda^t = |\lambda|^2 I$, where I is the identity $n\times n$ matrix. In particular* $\det B_\lambda = |\lambda|^n$.

The following proposition gives us the relation between the dimesion of the center of the non-isotropic Heisenebrg group and its phase space.

Proposition 2.2 *Let $\mathbb{G}$ be the non-isotropic Heisenberg group on $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m$. Then $m\le n^2$.*

Proof For all $1 \leq k \leq m$ and $1 \leq i, j \leq n$, let $(B_k)_{ij}$ be the entry of the matrix B_k in the i-th row and j-th column. Then the $n^2 \times m$ matrix

$$C = \begin{bmatrix} (B_1)_{11} & (B_2)_{11} & \dots & (B_m)_{11} \\ (B_1)_{12} & (B_2)_{12} & \dots & (B_m)_{12} \\ \vdots & \vdots & \ddots & \vdots \\ (B_1)_{1n} & (B_2)_{1n} & \dots & (B_m)_{1n} \\ (B_1)_{21} & (B_2)_{21} & \dots & (B_m)_{21} \\ (B_1)_{22} & (B_2)_{22} & \dots & (B_m)_{22} \\ \vdots & \vdots & \vdots & \vdots \\ (B_1)_{nn} & (B_2)_{nn} & \dots & (B_m)_{nn} \end{bmatrix}$$

has rank m. To prove this, it is enough to show that the columns of C are linearly independent. Let C^i be the i-th column of C and let $\lambda \in \mathbb{R}^m$ be such that

$$\sum_{i=1}^{m} \lambda_i C^i = 0.$$

It follows that $B_\lambda = 0$. Therefore by Proposition 2.1, we get $\lambda = 0$. □

Let $\sigma \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$, then we define the λ-Weyl transform $W_\sigma^\lambda f$ of f corresponding to the symbol σ by

$$\left(W_\sigma^\lambda f, g\right)_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(x, \xi) W^\lambda(f, g)(x, \xi)\, dx\, d\xi,$$

for all $g \in \mathcal{S}(\mathbb{R}^n)$. Therefore, using the Parseval's identity, we have

$$\left(W_\sigma^\lambda f, g\right)_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\sigma}(q, p) V^\lambda(f, g)(q, p)\, dq\, dp.$$

Hence, formally we can write,

$$\left(W_\sigma^\lambda f\right)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\sigma}(q, p)\, (\pi_\lambda(q, p) f)\,(x)\, dq\, dp.$$

Proposition 2.3 *Let $\sigma \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$. Then the λ-Weyl transform W_σ^λ is given by*

$$W_\sigma^\lambda = W_{\sigma_\lambda},$$

where W_{σ_λ} is the ordinary Weyl transform corresponding to the symbol

$$\sigma_\lambda(x, \xi) = \sigma(B_\lambda x, \xi).$$

Proposition 2.4 *Let $\sigma \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$. Then the λ-Weyl transform W_σ^λ is a Hilber-Schmidt operator with kernel*

$$k_\sigma^\lambda(x,p) = (\mathcal{F}_2\sigma)\left(B_\lambda(\frac{x+p}{2}), p-x\right),$$

where $\mathcal{F}_2\sigma$ is the ordinary Fourier transform of σ with respect to the second variable, i.e.,

$$(\mathcal{F}_2\sigma)\,(x,p) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi\cdot p}\sigma(x,\xi)\, d\xi.$$

Moreover,

$$\|W_\sigma^\lambda\|_{HS} = |\lambda|^{-n/2}\|\sigma\|_{L^2(\mathbb{R}^n\times\mathbb{R}^n)}$$

Proof By Proposition 2.4 and the kernel of the ordinary Weyl transform (see [13] for details), we have

$$\begin{aligned} k_\sigma^\lambda(x,p) &= (\mathcal{F}_2\sigma_\lambda)\left(\frac{x+p}{2}, p-x\right) \\ &= (\mathcal{F}_2\sigma)\left(B_\lambda(\frac{x+p}{2}), p-x\right). \end{aligned}$$

Hence,

$$\begin{aligned} \|W_\sigma^\lambda\|_{HS}^2 &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |k_\sigma^\lambda(x,p)|^2\, dx\, dp \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|(\mathcal{F}_2\sigma)\left(B_\lambda(\frac{x+p}{2}), p-x\right)\right|^2 dx\, dp \\ &= |\lambda|^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |(\mathcal{F}_2\sigma)\,(x,p)|^2\; dx\, dp \\ &= |\lambda|^{-n}\|\sigma\|_{L^2(\mathbb{R}^n\times\mathbb{R}^n)}^2, \end{aligned}$$

which completes the proof. □

Let F and G be functions in $L^2(\mathbb{R}^{2n})$. The λ-twisted convolution of F and G denoted by $F *_\lambda G$ on $\mathbb{R}^{2n}$ is defined by

$$(F *_\lambda G)(z) = \int_{\mathbb{R}^{2n}} F(z-w)G(w)e^{\frac{i}{2}\lambda.[z,w]}\, dw.$$

By Lemma 2.1 we get the following theorem.

Theorem 2.3 *Let σ and τ be in $L^2(\mathbb{R}^{2n})$. Then*

$$W_\sigma^\lambda W_\tau^\lambda = W_\omega^\lambda,$$

where $\hat{\omega} = (2\pi)^{-n}(\hat{\sigma} *_\lambda \hat{\tau})$.

Using the Moyal identity for the ordinary Wigner transform we have the following Moyal identity for the λ-Wigner transform and λ-Fourier Wigner transform.

Proposition 2.5 *For all f_1, f_2, g_1, g_2 in $L^2(\mathbb{R}^n)$*

$$(W_\lambda(f_1, g_1), W_\lambda(f_2, g_2)) = |\lambda|^{-n} (f_1, f_2) \overline{(g_1, g_2)},$$

and

$$(V_\lambda(f_1, g_1), V_\lambda(f_2, g_2)) = |\lambda|^{-n} (f_1, f_2) \overline{(g_1, g_2)}.$$

Now, we are ready to prove the following theorem.

Theorem 2.4 *For all $\lambda \in \mathbb{R}^{m*}$, π_λ is a unitary irreducible representation of $\mathbb{G}$ on $L^2(\mathbb{R}^n)$.*

Proof suppose $M \subset L^2(\mathbb{R}^n)$ is a nonzero closed invariant subspace of π_λ and $f \in M \setminus \{0\}$. Then

$$\pi_\lambda(q, p, t)M \subset M, \quad (q, p, t) \in \mathbb{G}.$$

If $M \neq L^2(\mathbb{R}^n)$, then we can find $g \in L^2(\mathbb{R}^n)$ such that

$$(\pi_\lambda(q, p, t)f, g) = 0, \quad (q, p, t) \in \mathbb{G}.$$

But,

$$\begin{aligned}(\pi_\lambda(q, p, t)f, g) &= e^{i\lambda \cdot t} (\pi_\lambda(q, p)f, g) \\ &= e^{i\lambda \cdot t}(2\pi)^{n/2} V_\lambda(f, g)(p, q).\end{aligned}$$

So,

$$V_\lambda(f, g)(q, p) = 0$$

for all $(p, q) \in \mathbb{R}^n \times \mathbb{R}^n$. By the Moyal identity,

$$\|V_\lambda(f, g)\|^2_{L^2(\mathbb{R}^n \times \mathbb{R}^n)} = |\lambda|^{-n} \|f\|^2_{L^2(\mathbb{R}^n)} \|g\|^2_{L^2(\mathbb{R}^n)} = 0.$$

So, $f = 0$ or $g = 0$ which is a contradiction. $\square$

3 Stone-Von Neumann Theorem on $\mathbb{G}$

Let $U(L^2(\mathbb{R}^n))$ be the space of unitary operators on $L^2(\mathbb{R}^n)$. Let $h \in \mathbb{R}^*$, then the Schrödinger representation $\rho_h : \mathbb{H}^n \to U(L^2(\mathbb{R}^n))$ on the ordinary Heisenebrg group is defined by

$$(\rho_h(q,p,t)\varphi)(x) = e^{iht} e^{iq\cdot(x+hp/2)} f(x+hp), \quad x \in \mathbb{R}^n,$$

for all $f \in L^2(\mathbb{R}^n)$. Then ρ_h is an irreducible unitary representation of $\mathbb{H}^n$ on $L^2(\mathbb{R}^n)$. By the Stone-von Neumann theorem, any irreducible unitary representation of $\mathbb{H}^n$ on a Hilbert space that is non-trivial on the center is equivalent to some ρ_h. More precisely we have

Theorem 3.1 *Let π be an irreducible unitary represenatation of $\mathbb{H}^n$ on a Hilbert space $\mathcal{H}$, such that $\pi(0,0,t) = e^{iht} I$ for some $h \in \mathbb{R}^*$. Then π is unitarily equivalent to ρ_h.*

Similarly, we prove the Stone-von Neumann theorem for the non-isotropic Heisenberg group $\mathbb{G}$. To prove we use the following lemma.

Lemma 3.2 *Let $\lambda \in \mathbb{R}^{m*}$. The mapping $\alpha_\lambda : \mathbb{G} \to \mathbb{H}^n$ defined by*

$$\alpha_\lambda(q,p,t) = (B_\lambda^t q, \frac{p}{|\lambda|}, \frac{\lambda\cdot t}{|\lambda|}), \quad (q,p,t) \in \mathbb{G}$$

is a surjective homomorphism of Lie groups. In particular, $G/\ker\alpha_\lambda$ is isomorphic to $\mathbb{H}^n$ where

$$\ker\alpha_\lambda = \{(0,0,t) : (t,\lambda) = 0\}.$$

Proof To prove α_λ is a group homomorphism, let $(q,p,t), (q',p'.t') \in \mathbb{G}$. Then

$$\alpha_\lambda((q,p,t)\cdot_{\mathbb{G}}(q',p',t')) = \alpha_\lambda(q+q', p+p', t+t' + \frac{1}{2}[z,z'])$$
$$= \left(B_\lambda^t(q+q'), \frac{p+p'}{|\lambda|}, \lambda\cdot(t+t'+\frac{1}{2}[z,z'])/|\lambda|\right)$$

Since $\lambda\cdot[z,z'] = (q', B_\lambda p) - (q, B_\lambda p')$, therefore

$$\begin{aligned}&\alpha_\lambda((q,p,t)\cdot_{\mathbb{G}}(q',p',t'))\\ &= (B_\lambda^t q, \frac{p}{|\lambda|}, \frac{\lambda\cdot t}{|\lambda|})\cdot_{\mathbb{H}^n}(B_\lambda^t q', \frac{p'}{|\lambda|}, \frac{\lambda\cdot t'}{|\lambda|})\\ &= \alpha_\lambda((q,p,t)\cdot_{\mathbb{H}^n}\alpha_\lambda(q',p',t')).\end{aligned} \tag{3}$$

Surjectivity is easy to see, since B_λ is invertible. □

The following lemma gives the connection between the Schrödinger representation on the ordinary Heisenberg group $\mathbb{H}^n$ and the representations π_λ on the non-isotropic Heisenberg group $\mathbb{G}$.

Lemma 3.3 *For all* $\lambda \in \mathbb{R}^{m*}$,

$$\pi_\lambda = \rho_{|\lambda|} \circ \alpha_\lambda.$$

Now, we are ready to prove the Stone von-Neumann theorem for the non-isotropic Heiseneberg group.

Theorem 3.4 *Let* Π_λ *be an irreducible unitary group representation of* $\mathbb{G}$ *on a Hilbert space* $\mathcal{H}$ *such that* $\Pi_\lambda(0,0,t) = e^{i\lambda\cdot t}I$, *for some* $\lambda \in \mathbb{R}^m$. *Then* Π_λ *is unitarily equivalent to* π_λ

Proof Let $\Pi_{|\lambda|} : \mathbb{H}^n \to U(\mathcal{H})$ be defined by $\Pi_{|\lambda|} = \Pi_\lambda PT$ where T is the isomorphism of $\mathbb{H}^n$ onto $G/\ker\alpha_\lambda$ (see Lemma 3.2) and P is the projection from $\mathbb{G}/\ker\alpha_\lambda$ onto $\mathbb{G}$. Then $\Pi_{|\lambda|}(0,0,t_0) = e^{i|\lambda|t_o}I$, for all $t_0 \in \mathbb{R}$. Moreover, $\Pi_{|\lambda|}$ is an irreducible unitary representation of $\mathbb{H}^n$ on the Hilbert space $\mathcal{H}$. This can be easily seen by using the fact that Π_λ is an irreducible unitary representation of $\mathbb{G}$ on $\mathcal{H}$. □

4 Fourier Transforms and the Fourier Inversion Formula on $\mathbb{G}$

By the Stone-von Neumann theorem every irreducible unitary representation of $\mathbb{G}$ which acts non-trivially on the center is in fact unitarily equivalent to exactly one of π_λ, $\lambda \in \mathbb{R}^{m*}$. Hence, the identification of $\{\pi_\lambda : \lambda \in \mathbb{R}^{m*}\}$ with $\mathbb{R}^{m*}$ will be used. Let $f \in L^1(\mathbb{G})$ and $\lambda \in \mathbb{R}^{m*}$. We define the Fourier transform of f at λ to be the bounded linear operator $\hat{f}(\lambda)$ from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ given by

$$\hat{f}(\lambda)\varphi = \int_{\mathbb{R}^m}\int_{\mathbb{R}^{2n}} f(z,t)\,(\pi_\lambda(z,t)\varphi)\;dz\,dt, \quad \varphi \in L^2(\mathbb{R}^n).$$

To see the boundedness of $\hat{f}(\lambda)$, let $\varphi, \psi \in L^2(\mathbb{R}^n)$. Then By Schwarz inequality

$$\begin{aligned}\left|\left(\hat{f}(\lambda)\varphi,\psi\right)\right| &\le \int_{\mathbb{R}^m}\int_{\mathbb{R}^{2n}} |f(z,t)|\;|\,(\pi_\lambda(z,t)\varphi,\psi)\,|\,dz\,dt \\ &\le \int_{\mathbb{R}^m}\int_{\mathbb{R}^{2n}} |f(z,t)|\;\|\pi_\lambda(z,t)\varphi\|_{L^2(\mathbb{R}^n)}\|\psi\|_{L^2(\mathbb{R}^n)}\,dz\,dt. \\ &\le \|f\|_{L^1(\mathbb{G})}\|\varphi\|_{L^2(\mathbb{R}^n)}\|\psi\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Set

$$f^{\lambda}(z) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{i\lambda \cdot t} f(z,t)\, dt.$$

Then $\hat{f}(\lambda)\varphi$ has the form

$$\hat{f}(\lambda)\varphi = (2\pi)^{m/2} \int_{\mathbb{R}^{2n}} f^{\lambda}(z)\, (\pi_{\lambda}(z)\varphi)\, dz.$$

Therefore we have following proposition relating the Fourier transform $\hat{f}(\lambda)$ to the λ-Weyl transform.

Proposition 4.1 *Let $f \in L^1(\mathbb{G})$. Then for all $\lambda \in \mathbb{R}^{m*}$*

$$\hat{f}(\lambda) = (2\pi)^{(2n+m)/2} W^{\lambda}_{(f^{\lambda})^{\vee}},$$

where $(f^{\lambda})^{\vee}$ is the inverse Fourier transform of f^{λ} on $\mathbb{R}^{2n}$.

We have the following Plancheral's formula for the Fourier transform on the non-isotropic Heisenberg group with multi-dimensional center.

Theorem 4.1 *Let $f \in L^2(\mathbb{G})$ and $\lambda \in \mathbb{R}^{m*}$. Then $\hat{f}(\lambda) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a Hilbert-Schmidt operator. In fact we have*

(i) The kernel of $\hat{f}(\lambda)$ is given by

$$k_{\lambda}(x,p) = (2\pi)^{(n+m)/2} \left(\mathcal{F}_1^{-1} f^{\lambda}\right) \left(B_{\lambda}(\frac{x+p}{2}), p-x\right)$$

where $\mathcal{F}_1^{-1} f^{\lambda}$ is the ordinary inverse Fourier transform of f^{λ} with respect to the first variable, i.e.,

$$\left(\mathcal{F}_1^{-1} f^{\lambda}\right)(x,p) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot q} f^{\lambda}(q,p)\, dq. \quad (x,p) \in \mathbb{R}^n \times \mathbb{R}^n.$$

(ii) The Hilbert-Schmidt norm of $\hat{f}(\lambda)$ is given by

$$\|\hat{f}(\lambda)\|^2_{HS} = (2\pi)^{m+n} |\lambda|^{-n} \|f^{\lambda}\|^2_{L^2(\mathbb{R}^{2n})}.$$

(iii) Let $d\mu(\lambda) = (2\pi)^{-(n+m)} |\lambda|^n\, d\lambda$. We have the following Plancheral's formula

$$\int_{\mathbb{R}^m} \|\hat{f}(\lambda)\|^2_{HS}\, d\mu(\lambda) = \|f\|^2_{L^2(\mathbb{G})}.$$

Proof Let φ be in $L^2(\mathbb{R}^n)$. Then for all $x \in \mathbb{R}^n$,

$$
\begin{aligned}
\left(\hat{f}(\lambda)\varphi\right)(x) &= (2\pi)^{m/2} \int_{\mathbb{R}^{2n}} f^\lambda(q,p)\,(\pi_\lambda(q,p)\varphi)\,(x)\,dq\,dp \\
&= (2\pi)^{m/2} \int_{\mathbb{R}^{2n}} f^\lambda(q,p) e^{iq\cdot B_\lambda(x+\frac{p}{2})}\varphi(x+p)\,dq\,dp \\
&= \int_{\mathbb{R}^n} \left((2\pi)^{m/2} \int_{\mathbb{R}^n} e^{iq\cdot B_\lambda(\frac{x+p}{2})} f^\lambda(q,p-x)\,dq \right) \varphi(p)\,dp \\
&= \int_{\mathbb{R}^n} k_\lambda(x,p)\varphi(p)\,dp
\end{aligned}
$$

where

$$
k_\lambda(x,p) = (2\pi)^{(n+m)/2} \left(\mathcal{F}_1^{-1} f^\lambda\right) \left(B_\lambda(\frac{x+p}{2}), p-x\right).
$$

Hence the Hilbert-Schmidt norm of $\hat{f}(\lambda)$ is given by

$$
\begin{aligned}
\|\hat{f}(\lambda)|\|_{HS}^2 &= \|k_\lambda\|^2_{L^2(\mathbb{R}^n\times\mathbb{R}^n)} \\
&= (2\pi)^{(n+m)} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left| \left(\mathcal{F}_1^{-1} f^\lambda\right)\left(B_\lambda(\frac{x+p}{2}), p-x\right)\right|^2 dx\,dp \\
&= (2\pi)^{(n+m)} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|\left(\mathcal{F}_1^{-1} f^\lambda\right)(x,p)\right|^2 |\lambda|^{-n}\,dx\,dp \\
&= |\lambda|^{-n}(2\pi)^{(n+m)}\|f^\lambda\|^2_{L^2(\mathbb{R}^{2n})} \qquad (4)
\end{aligned}
$$

where in (4) we used the Parseval's identity for the ordinary Fourier transform. □

Now we are ready to prove the inversion formula for the non-isotropic group Fourier transform.

Theorem 4.2 *Let f be a Schwartz function on $\mathbb{G}$. Then we have*

$$
f(z,t) = \int_{\mathbb{R}^m} tr\left(\pi_\lambda(z,t)^*\hat{f}(\lambda)\right)\,d\mu(\lambda), \quad (z,t) \in \mathbb{G}.
$$

Proof For all $(z,t) \in \mathbb{G}$,

$$
\begin{aligned}
\pi_\lambda(z,t)^*\hat{f}(\lambda) &= \pi_\lambda(-z,-t) \int_{\mathbb{R}^m}\int_{\mathbb{R}^{2n}} f(\tilde{z},\tilde{t})\,\pi_\lambda(\tilde{z},\tilde{t})\,d\tilde{z}\,d\tilde{t} \\
&= \int_{\mathbb{R}^m}\int_{\mathbb{R}^{2n}} f(\tilde{z},\tilde{t})\,\pi_\lambda\left((-z,-t))\cdot(\tilde{z},\tilde{t})\right)\,d\tilde{z}\,d\tilde{t}
\end{aligned}
$$

$$= \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} f(\tilde{z}, \tilde{t})\ \pi_\lambda \left(-z + \tilde{z}, -t + \tilde{t} + \frac{1}{2}[-z, \tilde{z}] \right) d\tilde{z}\, d\tilde{t}$$

$$= \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} f(\tilde{z}, \tilde{t}) e^{i\frac{\lambda}{2}\cdot[-z,\tilde{z}]} \pi_\lambda \left(-z + \tilde{z}, -t + \tilde{t} \right) d\tilde{z}\, d\tilde{t}.$$

Now, we let $z' = -z + \tilde{z}$ and $t' = -t + \tilde{t}$. W get

$$\pi_\lambda(z,t)^* \hat{f}(\lambda) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} g(z', t') \pi_\lambda(z', t')\, dz'\, dt',$$

where

$$g(z', t') = e^{-i\frac{\lambda}{2}\cdot[z,z']} f(z' + z, t' + t).$$

Hence,

$$\pi_\lambda(z,t)^* \hat{f}(\lambda) = \hat{g}(\lambda).$$

By Theorem 4.1, the kernel of $\hat{g}(\lambda)$ is given by

$$k_\lambda(x,p) = (2\pi)^{(n+m)/2} \left(\mathcal{F}_1^{-1} g^\lambda \right) \left(B_\lambda(\frac{x+p}{2}), p - x \right).$$

Therefore,

$$tr\left(\pi_\lambda(z,t)^* \hat{f}(\lambda) \right) = \int_{\mathbb{R}^n} k_\lambda(x,x)\, dx.$$

So, for $z = (u, v) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$k_\lambda(x,x) = (2\pi)^{(n+m)/2} \left(\mathcal{F}_1^{-1} g^\lambda \right) (B_\lambda x, 0)$$
$$= (2\pi)^{m/2} \int_{\mathbb{R}^n} e^{iB_\lambda x\cdot\xi} g^\lambda(\xi, 0)\, d\xi.$$

On the other hand, it is easy to see that

$$g^\lambda(z') = e^{-i\frac{\lambda}{2}\cdot[z,z']} e^{-i\lambda\cdot t} f^\lambda(z + z').$$

So, for $z = (u, v) \in \mathbb{R}^n \times \mathbb{R}^n$, and $z' = (\xi, 0)$, we get

$$g^\lambda(\xi, 0) = e^{\frac{-i}{2} B_\lambda v\cdot\xi} e^{-i\lambda\cdot t} f^\lambda(\xi + u, v).$$

Hence,

$$
\begin{aligned}
k_\lambda(x,x) &= (2\pi)^{m/2} \int_{\mathbb{R}^n} e^{iB_\lambda x\cdot\xi} e^{\frac{-i}{2}B_\lambda v\cdot\xi} e^{-i\lambda\cdot t} f^\lambda(\xi+u,v)\,d\xi \\
&= (2\pi)^{m/2} e^{-i\lambda\cdot t} e^{i(-B_\lambda x + B_\lambda v/2)\cdot u} \int_{\mathbb{R}^n} e^{i(B_\lambda x - B_\lambda v/2)\cdot\xi} f^\lambda(\xi,v)\,d\xi \qquad (5)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& tr\left(\pi_\lambda(z,t)^*\hat{f}(\lambda)\right) \\
&= (2\pi)^{m/2} e^{-i\lambda\cdot t} e^{iB_\lambda v/2\cdot u} \int_{\mathbb{R}^n} e^{-ix\cdot B_\lambda^t u} \left\{ \int_{\mathbb{R}^n} e^{i\xi\cdot(-B_\lambda v/2 + B_\lambda x)} f^\lambda(\xi,v)\,d\xi \right\} dx \\
&= (2\pi)^{(m+n)/2} e^{-i\lambda\cdot t} e^{iB_\lambda v/2\cdot u} \int_{\mathbb{R}^n} e^{-ix\cdot B_\lambda^t u} \left(\mathcal{F}_1^{-1} f^\lambda\right)(-B_\lambda v/2 + B_\lambda x, v)\,dx \\
&= (2\pi)^{(m+n)/2} e^{-i\lambda\cdot t} |\lambda|^{-n} \int_{\mathbb{R}^n} e^{-ix\cdot u} \left(\mathcal{F}_1^{-1} f^\lambda\right)(x,v)\,dx \\
&= (2\pi)^{m/2+n} e^{-i\lambda\cdot t} |\lambda|^{-n} f^\lambda(u,v).
\end{aligned}
$$

By integrating both sides of

$$
tr\left(\pi_\lambda(z,t)^*\hat{f}(\lambda)\right)(2\pi)^{-(n+m)}|\lambda|^n = (2\pi)^{-m/2} e^{-i\lambda\cdot t} f^\lambda(z)
$$

with respect to λ, we get the Fourier inversion formula. □

5 Pseudo-differential Operators on Non-isotropic Heisenberg Groups with Multi-dimensional Centers

Let $B(L^2(\mathbb{R}^n))$ be the C^*-algebra of all bounded linear operators on $L^2(\mathbb{R}^n)$. Then consider the operator valued symbol

$$
\sigma : \mathbb{G} \times \mathbb{R}^{m*} \to B(L^2(\mathbb{R}^n)).
$$

We define the pseudo-differential operator $T_\sigma : L^2(\mathbb{G}) \to L^2(\mathbb{G})$ corresponding to the symbol σ by

$$
(T_\sigma f)(z,t) = \int_{\mathbb{R}^m} tr\left(\pi_\lambda(z,t)^*\sigma(z,t,\lambda)\hat{f}(\lambda)\right) d\mu(\lambda), \quad (z,t) \in \mathbb{G}
$$

for all $f \in L^2(\mathbb{G})$. Let $HS(L^2(\mathbb{R}^n))$ be the space of Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$. We have the following theorem on L^2-boundedness of pseudo-differential operators.

Theorem 5.1 *Let $\sigma : \mathbb{G} \times \mathbb{R}^{m*} \to HS(L^2(\mathbb{R}^n))$ be such that*

$$C_\sigma^2 = \int_{\mathbb{R}^m} \int_{\mathbb{G}} \|\sigma(z,t,\lambda)\|_{HS}^2 \, dz\, dt\, d\mu(\lambda) < \infty.$$

Then $T_\sigma : L^2(\mathbb{G}) \to L^2(\mathbb{G})$ is a bounded linear operator and

$$\|T_\sigma\|_{op} \leq C_\sigma,$$

where $\|\cdot\|_{op}$ is the operator norm on the C^*-algebra of bounded linear operators on $L^2(\mathbb{G})$.

Proof Let $f \in L^2(\mathbb{G})$. Then by Minkowski's inequality we have

$$\begin{aligned}
&\|T_\sigma f\|_{L^2(\mathbb{G})} = \\
&\left\{ \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} \left| \int_{\mathbb{R}^m} tr\left(\pi_\lambda(z,t)^* \sigma(z,t,\lambda) \hat{f}(\lambda)\right) \, d\mu(\lambda) \right|^2 dz\, dt \right\}^{1/2} \\
&\leq \int_{\mathbb{R}^m} \left\{ \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} \left| tr\left(\pi_\lambda(z,t)^* \sigma(z,t,\lambda) \hat{f}(\lambda)\right) \right|^2 dz\, dt \right\}^{1/2} d\mu(\lambda) \\
&\leq \int_{\mathbb{R}^m} \left\{ \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} \|\sigma(z,t,\lambda)\|_{HS}^2 \|\hat{f}(\lambda)\|_{HS}^2 \, dz\, dt \right\}^{1/2} d\mu(\lambda) \\
&= \int_{\mathbb{R}^m} \|\hat{f}(\lambda)\|_{HS} \left\{ \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} \|\sigma(z,t,\lambda)\|_{HS}^2 \, dz\, dt \right\}^{1/2} d\mu(\lambda). \\
&\leq C_\sigma \|f\|_{L^2(\mathbb{G})} \qquad (6)
\end{aligned}$$

where in (6), we used Hölder's inequality. □

The following result tells us that under suitable conditions, two symbols of the same pseudo-differential operator are equal.

Proposition 5.1 *Let $\sigma : \mathbb{G} \times \mathbb{R}^{m*} \to HS(L^2(\mathbb{R}^n))$ be such that*

$$\int_{\mathbb{R}^m} \int_{\mathbb{G}} \|\sigma(z,t,\lambda)\|_{HS}^2 \, dz\, dt\, d\mu(\lambda) < \infty.$$

Furthermore suppose that

$$\int_{\mathbb{R}^m} \|\sigma(z,t,\lambda)\|_{HS}\, d\mu(\lambda) < \infty, \quad (z,t) \in \mathbb{G}, \tag{7}$$

$$\sup_{(z,t,\lambda)\in\mathbb{G}\times\mathbb{R}^{m*}} \|\sigma(z,t,\lambda)\|_{HS} < \infty, \tag{8}$$

and the mapping

$$\mathbb{G}\times\mathbb{R}^{m*} \ni (z,t,\lambda) \mapsto \pi_\lambda(z,t)^*\sigma(z,t,\lambda) \in HS(L^2(\mathbb{R}^n)) \tag{9}$$

is weakly continuous. Then $T_\sigma f = 0$ for all f only if

$$\sigma(z,t,\lambda) = 0$$

for almost all $(z,t,\lambda) \in \mathbb{G}\times\mathbb{R}^{m*}$.

Proof For all $(z,t) \in \mathbb{G}$, we define $f_{z,t} \in L^2(\mathbb{G})$ by

$$\widehat{f_{z,t}}(\lambda) = \sigma(z,t,\lambda)^*\pi_\lambda(z,t).$$

Then, for all $(w,s) \in \mathbb{G}$

$$(T_\sigma f_{z,t})\,(w,s) = \int_{\mathbb{R}^m} A^\lambda_{z,t}(w,s)\, d\mu(\lambda),$$

where

$$A^\lambda_{z,t}(w,s) = tr\left(\pi_\lambda(w,s)^*\sigma(w,s,\lambda)\sigma(z,t,\lambda)^*\pi_\lambda(z,t)\right).$$

Let $(z_0,w_0) \in \mathbb{G}$. Then by the weak-continuity of the mapping (9),

$$A^\lambda_{z,t}(w,s) \to A^\lambda_{z,t}(z_0,t_0)$$

as $(w,s) \to (z_0,t_0)$. Moreover, by (8), there exits $C > 0$ such that

$$|A^\lambda_{z,t}(w,s)| \le C\|\sigma(z,t,\lambda)\|_{HS}$$

Therefore, by (7) and Lebesgue's dominated convergence theorem,

$$(T_\sigma f_{z,t})\,(w,s) \to (T_\sigma f_{z,t})\,(z_0,t_0)$$

as $(w, s) \to (z_0, t_0)$. Therefore $T_\sigma f_{z,t}$ is continuous on $\mathbb{G}$ and since by the assumption of the proposition $T_\sigma f_{z,t} = 0$ almost every where, hence

$$(T_\sigma f_{z,t})\,(z, t) = 0.$$

But

$$\begin{aligned}(T_\sigma f_{z,t})\,(z, t) &= \int_{\mathbb{R}^m} tr\left(\pi_\lambda(z, t)^* \sigma(z, t, \lambda)\sigma(z, t, \lambda)^* \pi_\lambda(z, t)\right)\, d\mu(\lambda)\\ &= \int_{\mathbb{R}^m} tr\left(\sigma(z, t, \lambda)^* \sigma(z, t, \lambda)\right)\, d\mu(\lambda)\\ &= \int_{\mathbb{R}^m} \|\sigma(z, t, \lambda)\|_{HS}^2\, d\mu(\lambda) = 0\end{aligned}$$

Hence, $\|\sigma(z, t, \lambda)\|_{HS} = 0$ for almost all $\lambda \in \mathbb{R}^{m*}$ and therefore,

$$\sigma(z, t, \lambda) = 0$$

for almost all $(z, t, \lambda) \in \mathbb{G} \times \mathbb{R}^{m*}$ □

The following theorem gives necessary and sufficient conditions on a symbol σ for $T_\sigma : L^2(\mathbb{G}) \to L^2(\mathbb{G})$ to be a Hilbert-Schmidt operator.

Theorem 5.2 *Let $\sigma : \mathbb{G} \times \mathbb{R}^{m*} \to HS\left(L^2(\mathbb{R}^n)\right)$ be a symbol satisfying the hypothesis of Proposition 5.1. Then $T_\sigma : L^2(\mathbb{G}) \to L^2(\mathbb{G})$ is a Hilbert-Schmidt operator if and only if*

$$\sigma(z, t, \lambda) = \pi_\lambda(z, t) W^\lambda_{(\alpha(z,t)^{-\lambda})^\wedge}, \quad (z, t, \lambda) \in \mathbb{G} \times \mathbb{R}^{m*},$$

where $\alpha : \mathbb{G} \to L^2(\mathbb{G})$ is weakly continuous mapping for which

$$\int_{\mathbb{R}^m}\int_{\mathbb{R}^{2n}} \|\alpha(z, t)\|^2_{L^2(\mathbb{G})}\, dz\, dt < \infty,$$

$$\sup_{(z,t,\lambda)\in\mathbb{G}\times\mathbb{R}^{m*}} |\lambda|^{-n/2} \|(\alpha(z, t))^{-\lambda}\|_{L^2(\mathbb{R}^{2n})} < \infty$$

and

$$\int_{\mathbb{R}^m} |\lambda|^{n/2} \|(\alpha(z, t))^{-\lambda}\|_{L^2(\mathbb{R}^{2n})}\, d\lambda < \infty.$$

Proof We first prove the sufficiently. Let $f \in \mathcal{S}(\mathbb{G})$. Then by Proposition 4.1,

$$(T_\sigma f)\,(z, t) = |\lambda|^n (2\pi)^{-m/2} \int_{\mathbb{R}^m} tr\left(W^\lambda_{(\alpha(z,t))^{-\lambda})^\wedge} W^\lambda_{(f^\lambda)^\vee}\right)\, d\lambda.$$

By Proposition 2.3 and the trace formula in [5], we get

$$
\begin{aligned}
& tr\left(W^{\lambda}_{(\alpha(z,t)^{-\lambda})^{\wedge}} W^{\lambda}_{(f^{\lambda})^{\vee}}\right) \\
&= (2\pi)^{-n} \int_{\mathbb{R}^{2n}} (\alpha(z,t)^{-\lambda})^{\wedge}(B_{\lambda}x,\xi)\, (f^{\lambda})^{\vee}(B_{\lambda}x,\xi)\, dx\, d\xi \\
&= (2\pi)^{-n}|\lambda|^{-n} \int_{\mathbb{R}^{2n}} (\alpha(z,t)^{-\lambda})^{\wedge}(x,\xi)\, (f^{\lambda})^{\vee}(x,\xi)\, dx\, d\xi \\
&= (2\pi)^{-n}|\lambda|^{-n} \int_{\mathbb{R}^{2n}} (\alpha(z,t)^{-\lambda})(z')\, (f^{\lambda})(z')\, dz'.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
(T_{\sigma}f)\,(z,t) &= (2\pi)^{-(m+2n)/2} \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} (\alpha(z,t)^{-\lambda})(z')\, (f^{\lambda})(z')\, dz'\, d\lambda \\
&= (2\pi)^{-(m+2n)/2} \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} \alpha(z,t)(z',\lambda) f(z',\lambda)\, dz'\, d\lambda.
\end{aligned}
$$

So, the kernel of T_{σ} is a function on $\mathbb{R}^{2n+m} \times \mathbb{R}^{2n+m}$ given by

$$
k(z,t,z',t') = (2\pi)^{-(m+2n)/2} \alpha(z,t)(z',\lambda), \quad (z,t),(z',t') \in \mathbb{R}^{2n+m}. \tag{10}
$$

Therefore,

$$
\begin{aligned}
& \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} |k(z,t,z',\lambda)|^2\, dz\, dt\, dz'\, d\lambda \\
&= (2\pi)^{-(2n+m)} \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} |\alpha(z,t)(z',\lambda)|^2\, dz\, dt\, dz'\, d\lambda \\
&= (2\pi)^{-(2n+m)} \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} \|\alpha(z,t)\|^2_{L^2(\mathbb{G})}\, dz\, dt < \infty.
\end{aligned}
$$

Thus, T_{σ} is a Hilbert-Schmidt operator. Conversely, suppose that $T_{\sigma} : L^2(\mathbb{G}) \to L^2(\mathbb{G})$ is a Hilbert Schmidt operator. Then there exists a function k in $L^2(\mathbb{R}^{2n+m} \times \mathbb{R}^{2n+m})$ such that

$$
(T_{\sigma}f)\,(z,t) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} k(z,t,z',\lambda) f(z',\lambda)\, dz'\, d\lambda, \quad (z,t) \in \mathbb{G},
$$

for all $f \in L^2(\mathbb{G})$. We define $\alpha : \mathbb{G} \to L^2(\mathbb{G})$ by

$$
\alpha(z,t)(z',\lambda) = (2\pi)^{(m+2n)/2} k(z,t,z',\lambda).
$$

Then reversing the argument in the proof of the sufficiency and using Proposition 5.1, we have

$$\sigma(z,t,\lambda) = \pi_\lambda(z,t)W^\lambda_{(\alpha(z,t)^{-\lambda})^\wedge}, \quad (z,t,\lambda) \in \mathbb{G} \times \mathbb{R}^{m*}.$$

□

Corollary 5.3 *Let $\beta \in L^2(\mathbb{G} \times \mathbb{G})$ be such that*

$$\int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} |\beta(z,t,z,t)|\, dz\, dt < \infty.$$

Let

$$\sigma(z,t,\lambda) = \pi_\lambda(z,t)W^\lambda_{(\alpha(z,t)^{-\lambda})^\wedge}, \quad (z,t,\lambda) \in \mathbb{G} \times \mathbb{R}^{m*},$$

where

$$\alpha(z,t)(z',\lambda) = \beta(z,t,z',\lambda), \quad (z,t),(z',\lambda) \in \mathbb{G} \times \mathbb{R}^{m*}.$$

Then $T_\sigma : L^2(\mathbb{G}) \to L^2(\mathbb{G})$ is a trace class operator and

$$tr(T_\sigma) = (2\pi)^{-(2n+m)} \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} \beta(z,t,z,t)\, dz\, dt.$$

Corollary 5.3 follows from the formula (10) on the kernel of the pseudo-differential operator in the proof of the preceding theorem.

Theorem 5.4 *Let $\sigma : \mathbb{G} \times \mathbb{R}^{m*} \to HS\left(L^2(\mathbb{R}^n)\right)$ be a symbol satisfying the hypothesis of Proposition 5.1. Then $T_\sigma : L^2(\mathbb{G}) \to L^2(\mathbb{G})$ is a trace class operator if and only if*

$$\sigma(z,t,\lambda) = \pi_\lambda(z,t)W^\lambda_{(\alpha(z,t)^{-\lambda})^\wedge}, \quad (z,t,\lambda) \in \mathbb{G} \times \mathbb{R}^{m*},$$

where $\alpha : \mathbb{G} \to L^2(\mathbb{G})$ is a mapping such that the conditions of Theorem 5.2 are satisfied and

$$\alpha(z,t)(z',\lambda) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} \alpha_1(z,t)(w,s)\alpha_2(w,s)(z',\lambda)\, dw\, ds$$

for all (z,t) and (z',λ) in $\mathbb{G} \times \mathbb{R}^{m*}$, where $\alpha_1 : \mathbb{G} \to L^2(\mathbb{G})$ and $\alpha_2 : \mathbb{G} \to L^2(\mathbb{G})$ are such that

$$\int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} \|\alpha_j(z,t)\|^2_{L^2(\mathbb{G})}\, dz\, dt < \infty, \quad j = 1,2.$$

Moreover, the trace of T_σ is given by

$$\begin{aligned} tr(T_\sigma) &= \int_{\mathbb{R}^m}\int_{\mathbb{R}^{2n}} \alpha(z,t)(z,t)\,dz\,dt \\ &= \int_{\mathbb{R}^m}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^m}\int_{\mathbb{R}^{2n}} \alpha_1(z,t)(w,s)\alpha_2(w,s)(z,t)\,dw\,ds\,dz\,dt. \end{aligned}$$

Theorem 5.4 follows from Theorem 5.2 and the fact that every trace class operator is a product of two Hilbert-Schmidt operators.

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Curvature of the Heisenberg Group

Bartek Ewertowski and M.W. Wong

Abstract We compute the Riemannian curvature of the Heisenberg group and then contract it to the sectional curvature, Ricci curvature and the scalar curvature of the Heisenberg group. The main result so obtained is that the Heisenberg group is a space of constant positive scalar curvature.

Keywords Heisenberg group • Left-invariant vector fields • Riemannian metric • Levi–Civita connection • Riemannian curvature • Sectional curvature • Ricci curvature • Scalar curvature

Mathematics Subject Classification (2000). 53C21

1 The Heisenberg Group

If we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ via the obvious identification

$$\mathbb{R}^2 \ni (x, y) \leftrightarrow z = x + iy \in \mathbb{C},$$

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and we let

$$\mathbb{H}^1 = \mathbb{C} \times \mathbb{R},$$

then $\mathbb{H}^1$ becomes a noncommutative group when equipped with the multiplication $\cdot$ given by

$$(z,t) \cdot (w,s) = \left(z + w, t + s + \frac{1}{4}[z,w]\right), \quad (z,t),(w,s) \in \mathbb{H}^1,$$

where $[z,w]$ is the symplectic form of z and w defined by

$$[z,w] = 2\,\mathrm{Im}\,(z\overline{w}).$$

In fact, $\mathbb{H}^1$ is a unimodular Lie group on which the Haar measure is just the ordinary Lebesgue measure $dz\,dt$.

Let $\mathfrak{h}$ be the Lie algebra of left-invariant vector fields on $\mathbb{H}^1$. Then a basis for $\mathfrak{h}$ is given by X, Y and T, where

$$X = \frac{\partial}{\partial x} + \frac{1}{2} y \frac{\partial}{\partial t},$$
$$Y = \frac{\partial}{\partial y} - \frac{1}{2} x \frac{\partial}{\partial t}$$

and

$$T = \frac{\partial}{\partial t}.$$

It can be checked easily that

$$[X,Y] = -T$$

and all other commutators among X, Y and T are equal to 0. References for the Heisenberg group, its Lie algebra, the sub-Laplacian $-(X^2 + Y^2)$ and the full Laplacian $-(X^2 + Y^2 + T^2)$ can be found in [2–4, 10] among many others. Compact and lucid accounts of Lie groups in [1, 8] are highly recommended.

The aim of this paper is to prove that the scalar curvature of the Heisenberg group is a positive number. This is achieved by contracting from the Riemannian curvature to the scalar curvature through the sectional curvature and the Ricci curvature. The interest in curvature of the Heisenberg group $\mathbb{H}^1$ stems from the fact [9] that $\mathbb{H}^1$ can be thought of as the three-dimensional surface that is the boundary of the four-dimensional Siegel domain, so curvature of the Heisenberg group $\mathbb{H}^1$ may be of some interest in physics.

Results on curvature of the Heisenberg group exist in the literature with a host of different notation and convention. See, for instance, [2, 6, 7]. This paper, which is very similar to the section on Riemannian approximants in [2], is another attempt using notions that can be found in any graduate textbook on Riemannian geometry.

Since indices permeate Riemannian geometry, we find it convenient to label the vector fields X, Y and T by X_1, X_2 and X_3, respectively.

2 The Riemannian Metric

We begin with the fact that there exists a left-invariant Riemannian metric g on $\mathbb{H}^1$ that turns X_1, X_2 and X_3 into an orthonormal basis for $\mathfrak{h}$ with respect to an inner product denoted by $(\ ,\ )$. In fact,

$$g(x,y,t) = \begin{bmatrix} 1+(y^2/4) & -xy/4 & -y^2/2 \\ -xy/4 & 1+(x^2/4) & x/2 \\ -y^2/2 & x/2 & 1 \end{bmatrix}$$

for all $(x,y,t) \in \mathbb{H}^1$.

3 The Levi–Civita Connection

A *connection* ∇ on $\mathbb{H}^1$ is a mapping

$$\mathfrak{h} \times \mathfrak{h} \ni (X,Y) \mapsto \nabla_X Y \in \mathfrak{h}$$

such that

- $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$,
- $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$,
- $\nabla_X(fY) = X(f)Y + f\nabla_X Y$,
- $\nabla_{fX} Y = f\nabla_X Y$

for all vector fields X, Y and Z in $\mathfrak{h}$ and all C^∞ real-valued functions f on $\mathbb{H}^1$. The *torsion* T of the connection ∇ is a mapping that assigns to two vector fields X and Y in $\mathfrak{h}$ another vector field $T(X,Y)$ in $\mathfrak{h}$ given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

A connection ∇ on $\mathbb{H}^1$ is said to be *compatible* with the Riemannian metric g on $\mathbb{H}^1$ if

$$X(Y, Z) = (\nabla_X Y, Z) + (Y, \nabla_X Z) \tag{1}$$

for all vector fields X Y and Z in $\mathfrak{h}$.

The following result is crucial.

Theorem 3.1 *There exists a unique connection ∇ on $\mathbb{H}^1$ such that ∇ is torsion-free, i.e.,*

$$T(X, Y) = 0$$

for all vector fields X and Y in $\mathfrak{h}$ and ∇ is compatible with the Riemannian metric g on $\mathbb{H}^1$.

Proof For $i, j \in \{1, 2, 3\}$, let γ_{ij} be the real number given by

$$\gamma_{ij} = (X_i, X_j).$$

Then by compatibility,

$$X_i \gamma_{jk} = (\nabla_{X_i} X_j, X_k) + (X_j, \nabla_{X_i} X_k), \quad i, j, k \in \{1, 2, 3\}. \tag{2}$$

Since ∇ is torsion-free, it follows that

$$\nabla_{X_i} X_j - \nabla_{X_j} X_i = [X_i, X_j], \quad i, j \in \{1, 2, 3\}.$$

Permuting the indices in (2), we obtain

$$X_j \gamma_{ik} = (\nabla_{X_j} X_i, X_k) + (X_i, \nabla_{X_j} X_k) \tag{3}$$

and

$$X_k \gamma_{ij} = (\nabla_{X_k} X_i, X_j) + (X_i, \nabla_{X_k} X_j). \tag{4}$$

By (2), (3) and (4), we get

$$\begin{aligned} &X_i \gamma_{jk} + X_j \gamma_{ik} - X_k \gamma_{ij} \\ &= 2(\nabla_{X_i} X_j, X_k) - ([X_i, X_j], X_k) + ([X_j, X_k], X_i) - ([X_k, X_i], X_j). \end{aligned} \tag{5}$$

Thus, the uniquness of ∇ follows. It remains to prove the existence. For $i, j, k \in \{1, 2, 3\}$, let

$$\Gamma_{ij}^{k} = \frac{1}{2}\sum_{l=1}^{3}(X_i g_{jl} + X_j g_{il} - X_l g_{ij})g^{lk},$$

where $[g^{jk}]$ is the inverse of $g = [g_{ij}]$, and we define $\nabla_{X_i}X_j$ by

$$\nabla_{X_i}X_j = \sum_{k=1}^{3}\Gamma_{ij}^{k}X_k.$$

□

The connection alluded to in Theorem 3.1 is known as the *Levi-Civita connection*. The functions Γ_{ij}^{k} are called the *Christoffel symbols*. We shall work with the Levi–Civita connection from now on.

4 The Riemannian Curvature

Let ∇ be the Levi–Civita connection on $\mathbb{H}^1$. Then the *Riemannian curvature R* on $\mathbb{H}^1$ is the mapping that assigns three vector fields X, Y and Z in $\mathfrak{h}$ another vector field in $\mathfrak{h}$ denoted by $R(X, Y)Z$ and given by

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{6}$$

Remark 4.1 An intuitive way to think of the Riemannian curvature R is that it measures the deviation of $\nabla_X\nabla_Y - \nabla_Y\nabla_X$ from $\nabla_{[X,Y]}$. It should be noted that the opposite sign of R is also common in the literature. For example, the sign used in [5, 7] is different from the one used in this paper.

The Riemannian curvature has many symmetries as given by the following theorem, which can be proved easily using (6).

Theorem 4.2 *Let X, Y Z and W be in* $\mathfrak{h}$*. Then we have the following symmetries.*

- $R(X, Y)Z + R(Y, X)Z = 0,$
- $(R(X, Y)Z, W) + (R(X, Y)W, Z) = 0,$
- $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0,$
- $(R(X, Y)Z, W) = (R(Z, W)X, Y).$

In order to perform computations on the Heisenberg group $\mathbb{H}^1$, the following theorem is very useful. It is the *Koszul formula* for the Heisenberg group $\mathbb{H}^1$.

Theorem 4.3 *For all vector fields X, Y and Z in* $\mathfrak{h}$, *we have*

$$(\nabla_X Y, Z) = \frac{1}{2}\{(Z, [X, Y]) - (Y, [X, Z]) - (X, [Y, Z])\}.$$

Proof By compatibility,

$$X(Y, Z) = (\nabla_X Y, Z) + (Y, \nabla_X Z) = 0,$$
$$Y(X, Z) = (\nabla_Y X, Z) + (X, \nabla_Y Z) = 0$$

and

$$Z(X, Y) = (\nabla_Z X, Y) + (X, \nabla_Z Y) = 0.$$

Since ∇ is torsion-free, we use the Jacobi identity, i.e.,

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

to get

$$0 = 2(\nabla_X Y, Z) - (Z, [X, Y]) + (Y, [X, Z]) + (X, [Y, Z]),$$

as asserted. □

The following two theorems can be proved by means of the Koszul formula and direct computations. The first theorem gives a useful formula for the Levi-Civita connection and the second theorem provides an explicit formula for the Riemannian curvature.

Theorem 4.4 *The Levi-Civita connection* ∇ *is given by*

$$\begin{aligned}
\nabla_{X_1} X_1 &= 0, \quad \nabla_{X_1} X_2 = -\tfrac{1}{2} X_3, \; \nabla_{X_1} X_3 = \tfrac{1}{2} X_2, \\
\nabla_{X_2} X_1 &= \tfrac{1}{2} X_3, \; \nabla_{X_2} X_2 = 0, \quad \nabla_{X_2} X_3 = -\tfrac{1}{2} X_1, \\
\nabla_{X_3} X_1 &= \tfrac{1}{2} X_2, \; \nabla_{X_3} X_2 = -\tfrac{1}{2} X_1, \; \nabla_{X_3} X_3 = 0.
\end{aligned}$$

Theorem 4.5 *For all vector fields X, Y and Z in* $\mathfrak{h}$,

$$\begin{aligned}
R(X, Y)Z = -\frac{3}{4}&((Y, Z)X - (X, Z)Y) \\
&+ (Y, X_3)(Z, X_3)X - (X, X_3)(Z, X_3)Y \\
&+ (X, X_3)(Y, Z)X_3 - (Y, X_3)(X, Z)X_3.
\end{aligned}$$

5 The Sectional Curvature

Let X and Y be two orthonormal vector fields in $\mathfrak{h}$. Then X and Y determine a *plane* in $\mathfrak{h}$. Using left translations, we get a *plane bundle* on $\mathbb{H}^1$. Let $(z, t) \in \mathbb{H}^1$. Then we can find a neighborhood U of the origin in $T_{(z,t)}\mathbb{H}^1$ and a neighborhood N of (z, t) in $\mathbb{H}^1$ such that the exponential mapping $\exp : U \to N$ is a diffeomorphism. As such, the *plane* (a subspace of $T_{(z,t)}\mathbb{H}^1$) induces a submanifold of $\mathbb{H}^1$ locally and its curvature is given by the so-called *sectional curvature* that we can now define.

Definition 5.1 Let X and Y be orthonormal vector fields in $\mathfrak{h}$. Then the sectional curvature $S(X, Y)$ determined by X and Y is the number given by

$$S(X, Y) = (R(X, Y)X, Y).$$

We can now compute the sectional curvature of the Heisenberg group.

Theorem 5.2 *Let X and Y be orthonormal vector fields in $\mathfrak{h}$. Then*

$$S(X, Y) = \frac{3}{4} - (X, X_3)^2 - (Y, X_3)^2.$$

Proof By Theorem 4.5,

$$\begin{aligned} &R(X, Y)X \\ = &-\frac{3}{4}[(Y, X)X - (X, X)Y] \\ &+(Y, X_3)(X, X_3)X - (X, X_3)(X, X_3)Y \\ &+(X, X_3)(Y, X)X_3 - (Y, X_3)(X, X)X_3. \end{aligned}$$

So,

$$S(X, Y) = (R(X, Y)X, Y) = \frac{3}{4} - (X, X_3)^2 - (Y, X_3)^2.$$

□

6 The Ricci Curvature

Let X and Y be vector fields in $\mathfrak{h}$. Then we consider the linear mapping

$$\mathfrak{h} \ni Z \mapsto R(X, Z)Y \in \mathfrak{h}.$$

We denote this mapping by $M(X,Y) : \mathfrak{h} \to \mathfrak{h}$ and we define the *Ricci curvature* $r(X,Y)$ of the Heisenberg group by

$$r(X,Y) = \operatorname{tr} M(X,Y).$$

Theorem 6.1 *Let X and Y be vector fields in* $\mathfrak{h}$. *Then the Ricci curvature* $r(X,Y)$ *of the Heisenberg group is given by*

$$r(X,Y) = \frac{1}{2}(X,Y) - (X,X_3)(Y,X_3).$$

Proof Using the orthonormal basis X_1, X_2 and X_3 for $\mathbb{H}^1$, we get by means of Theorem 4.5

$$\begin{aligned} &R(X,X_j)Y \\ &= -\frac{3}{4}[(X_j,Y)X - (X,Y)X_j] \\ &\quad +(X_j,X_3)(Y,X_3)X - (X,X_3)(Y,X_3)X_j \\ &\quad +(X,X_3)(X_j,Y)X_3 - (X_j,X_3)(X,Y)X_3. \end{aligned}$$

So, for $j \in \{1,2,3\}$,

$$\begin{aligned} &(R(X,X_j)Y,X_j) \\ &= -\frac{3}{4}[(X_j,Y)(X,X_j) - (X,Y)] \\ &\quad +(X_j,X_3)(Y,X_3)(X,X_j) - (X,X_3)(Y,X_3) \\ &\quad +(X,X_3)(X_j,Y)(X_3,X_j) - (X_j,X_3)(X,Y)(X_3,X_j). \end{aligned}$$

Therefore by Parseval's identity,

$$\begin{aligned} &\operatorname{tr} M(X,Y) \\ &= -\frac{3}{4}[(X,Y) - 3(X,Y)] \\ &\quad +(X,X_3)(Y,X_3) - 3(X,X_3)(Y,X_3) \\ &\quad +(X,X_3)(Y,X_3) - (X,Y) \\ &= \frac{1}{2}(X,Y) - \frac{5}{4}(X,X_3)(Y,X_3), \end{aligned}$$

as required. □

By Theorem 6.1, the Ricci curvature is the mapping $\mathrm{Ric} : \mathfrak{h} \to \mathfrak{h}$ given by

$$\mathrm{Ric}(X) = \frac{1}{2}X - (X, X_3)X_3$$

for all X in $\mathfrak{h}$.

7 The Scalar Curvature

The scalar curvature κ of the Heisenberg group is defined by

$$\kappa = \mathrm{tr}\,(\mathrm{Ric}).$$

Theorem 7.1 *The scalar curvature κ of the Heisenberg group is given by*

$$\kappa = \frac{1}{2}.$$

Proof Let V_1, V_2 and V_3 be an orthonormal basis for $\mathbb{H}^1$. Then

$$\begin{aligned}\kappa &= \mathrm{tr}\,(\mathrm{Ric}) \\ &= \sum_{j=1}^{3} (\mathrm{Ric}\, V_j, V_j) \\ &= \frac{1}{2}\sum_{j=1}^{3}(V_j, V_j) - \sum_{j=1}^{3}(V_j, V_3)^2 \\ &= \frac{3}{2} - 1 = \frac{1}{2}.\end{aligned}$$

□

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Ellipticity with Global Projection Conditions

B.-W. Schulze

Abstract We give an overview of elliptic operators on a compact smooth manifold with boundary or edge, with elliptic boundary or edge conditions in global projection approach, introduced in Schulze (J Funct Anal 179:374–408, 2001) and then continued in Schulze and Seiler (J Funct Anal 206(2):449–498, 2004; J Inst Math Jussieu 5(1):101–123, 2006). Such conditions are motivated by the fact that important elliptic operators do not admit Shapiro-Lopatinskii elliptic conditions, though they always admit global projection conditions. Basically Shapiro-Lopatinskii conditions are a special case, such that the global projection idea unifies different concepts. There is a similarity to Toeplitz operators whence it also makes sense to talk about boundary or edge problems of Toeplitz type. Another stream of investigations goes back to Atiyah-Patodi-Singer (Math Proc Camb Philos Soc 77/78/79:43–69/405–432/315–330, 1975/1976/1976), though the analytical ideas and intentions from there are quite different. In our approach we focus on the aspect of operator algebras in scales of Sobolev spaces or subspaces induced by pseudo-differential projections, on parametrices within those structures, and on the role of principal symbolic hierarchies coming from the singular analysis. Ellipticity with global projection conditions in the singular analysis is also a source of new challenges. A few of them are indicated in the final part of this exposition.

Keywords Pseudo-differential operators • Toeplitz operators • Ellipticity • Fredholm operators • Manifolds with boundary or edge

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1 Introduction

Standard ellipticity of a differential (or classical pseudo-differential) operator A on a smooth manifold X is defined in terms of non-vanishing (or bijectivity) of its homogeneous principal symbol $\sigma_\psi(A)$. We then obtain a parametrix P within the algebra of pseudo-differential operators, where $\sigma_\psi(P) = \sigma_\psi^{-1}(A)$. If X is closed ellipticity is equivalent to the Fredholm property in Sobolev spaces, and invertibility of such an A implies that A^{-1} belongs to the algebra again. However, if X is not closed or non-smooth, e.g., a compact smooth manifold with boundary, or a manifold with singularities, such as conical points or edges, we cannot expect results of this kind, unless we do not refer to adequate new inventions, in particular, to enlarging the concept of ellipticity in terms of new symbolic structures. An example of such a generalization is Boutet de Monvel's algebra of boundary value problems (BVPs) with the transmission property, cf. [5], which developes a concept of ellipticity of 2×2 block matrix operators $\mathcal{A}$, based on a principal symbolic hierarchy $(\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A}))$, consisting of the interior symbol $\sigma_\psi(\mathcal{A})$, only defined in terms of the upper left corner A, and the operator-valued boundary symbol $\sigma_\partial(\mathcal{A})$ which reflects the role of boundary conditions of trace and potential type. Already Boutet de Monvel's calculus is far from being elementary, and there are some monographs on this topic, e.g., of Rempel and Schulze [16] or Grubb [9]. Other generalizations, e.g., operators without the transmission property at the boundary or operators of the edge calculus induce a new philosophy on how to understand ellipticity and the nature of Sobolev spaces. This is, in fact, topic of an entire stream of new developments. A common feature of what we intend to discuss in this article is the nature of ellipticity of the additional symbolic level analogously as $\sigma_\partial(\cdot)$, and the key-word is global projection ellipticity, originally introduced in [24] in the framework of Boutet de Monvel's calculus. The σ_ψ-elliptic operators in corresponding upper left corners are quite arbitrary, of any order, and they do not necessarily admit elliptic boundary conditions of Shapiro-Lopatinskii type. Specific σ_ψ-elliptic operators without Shapiro-Lopatinskii ellipticity at the boundary have been studied before by Atiyah, Patodi, Singer [2], in connection with index theory, using a completely different approach, and later on by Grubb, Seeley [10], Nazaikinskij, Schulze, Sternin, Shatalov [14]. More references can be found in the articles [18] of Savin, Schulze, Sternin, and [17] of Savin, Sternin. The idea of algebras with involved projection operators has been extended by Seiler in a number of recent papers [30, 31].

An elementary example of a σ_ψ-elliptic operator is the Cauchy-Riemann operator in a disk in the complex plane. It does not admit Shapiro-Lopatinskii elliptic boundary conditions, but there are elliptic global projection conditions, as is shown in [24]. The novelty in [24] is a unification of Shapiro-Lopatinskii and global projection ellipticity in an operator algebra, extending the one of Boutet de Monvel, including the computation of parametrices within the structure.

In the present article we give a survey on this development and subsequent results for operators without the transmission property and on manifolds with edge, obtained in [27, 28], jointly with Seiler, and we comment new achievements on global projection conditions for elliptic complexes [29]. Similarly as in the theory for "scalar" operators, every elliptic complex of differential operators on a smooth compact manifold with boundary – e.g., the Dolbeault complex – admits elliptic boundary conditions of global projection type and as such induces Fredholm complexes with additional spaces on the boundary that appear as images under projections of Sobolev spaces on the boundary. Their nature will be explained below in connection with projection data in Sect. 2.

2 Projections and Toeplitz Operators

Let $\mathrm{Vect}(M)$ denote the set of smooth complex vector bundles over a smooth manifold M. Let T^*M be the cotangent bundle over M with its canonical projection $\pi : T^*M \to M$.

Theorem 2.1 *Let $J \in \mathrm{Vect}(M)$ for a smooth closed manifold M, and let $p : \pi_M^* J \to \pi_M^* J$ be a smooth bundle morphism of homogeneity 0 in the covariable $\xi \neq 0$, i.e., $p(x, \lambda\xi) = p(x, \xi)$ for all $(x, \xi) \in T^*M \backslash 0$, $\lambda \in \mathbb{R}_+$, and $p^2 = p$. Then there exists a $P \in L^0_{\mathrm{cl}}(M; J, J)$ such that $\sigma_\psi(P) = p$ and $P^2 = P$. Moreover, if p satisfies the condition $p^* = p$, the operator P can be chosen in such a way that $P^* = P$.*

This result is well-known, cf. [3, 33]. An explicit proof can be found in [24]. Concerning other known relations in connection with projection operators see the book [4] of Booss-Bavnbek and K. Wojciechowski.

Proposition 2.2 *Let P, Q be projections in a Hilbert space H such that $P - Q$ is a compact operator. Then the restrictions of P to $\mathrm{im}\, Q$ and of Q to $\mathrm{im}\, P$ are Fredholm operators*

$$P_Q : \mathrm{im}\, Q \to \mathrm{im}\, P, \quad Q_P : \mathrm{im}\, P \to \mathrm{im}\, Q$$

between the respective closed subspaces of H, and Q_P is a parametrix of P_Q, and vice versa.

In fact, the operator Q obviously acts as the identity on $\mathrm{im}\, Q$. This gives us

$$Q_P P_Q - 1_{\mathrm{im}\, Q} = Q_P P_Q - Q^2 = Q_P (P_Q - Q_P) : \mathrm{im}\, Q \to \mathrm{im}\, Q,$$

i.e., $Q_P P_Q - 1_{\mathrm{im}\, Q}$ is a compact operator in $\mathrm{im}\, Q$. It follows that Q_P is a left parametrix of P_Q. In an analogous manner we see that $P_Q Q_P - 1_{\mathrm{im}\, P} = P_Q(Q_P - P_Q) : \mathrm{im}\, P \to \mathrm{im}\, P$ is compact, i.e., Q_P is also a right parametrix of P_Q which means that P_Q is a Fredholm operator. At the same time we see that Q_P is also Fredholm.

Remark 2.3 Let $\mathrm{ind}\,(P,Q)$ denote the index of $P_Q : \mathrm{im}\,Q \to \mathrm{im}\,P$. Then we have

$$\mathrm{ind}\,(P,Q) = -\mathrm{ind}\,(Q,P).$$

Let $J \in \mathrm{Vect}(M)$ and

$$p : \pi_M^* J \to \pi_M^* J \tag{1}$$

be a projection as in Theorem 2.1. Then we have

$$L := \mathrm{im}\,p \in \mathrm{Vect}(T^*M \backslash 0) \tag{2}$$

which is a subbundle of $\pi_M^* J$. Conversely for every $L \in \mathrm{Vect}(T^*M\backslash 0)$ there exists a $J \in \mathrm{Vect}(M)$ such that L is a subbundle of $\pi_M^* J$. In fact, there exists an N and an $L^\perp \in \mathrm{Vect}(T^*M\backslash 0)$ such that $L \oplus L^\perp = (T^*M\backslash 0) \times \mathbb{C}^N$.

Definition 2.4 A triple $\mathbb{L} := (P, J, L)$ will be called projection data on M when $P \in L^0_{\mathrm{cl}}(M; J, J)$ is a projection as in Theorem 2.1, and L defined by (2). Let $\mathbb{P}(M)$ denote the set of all such projection data.

Proposition 2.5

(i) *For every* $J \in \mathrm{Vect}(M)$ *we have* $(\mathrm{id}, J, \pi_M^* J) \in \mathbb{P}(M)$.
(ii) *For every* $\mathbb{L} = (P, J, L)$, $\widetilde{\mathbb{L}} = (\tilde{P}, \tilde{J}, \tilde{L}) \in \mathbb{P}(M)$ *we have*

$$\mathbb{L} \oplus \widetilde{\mathbb{L}} := (P \oplus \tilde{P}, J \oplus \tilde{J}, L \oplus \tilde{L}) \in \mathbb{P}(M).$$

(iii) *Every* $\mathbb{L} = (P, J, L) \in \mathbb{P}(M)$ *admits complementary projection data* $\mathbb{L}^\perp \in \mathbb{P}(M)$ *in the sense that* $\mathbb{L} \oplus \mathbb{L}^\perp = (\mathrm{id}, F, \pi_M^* F)$ *for some* $F \in \mathrm{Vect}(M)$.
(iv) *Every* $\mathbb{L} = (P, J, L) \in \mathbb{P}(M)$ *has an adjoint* $\mathbb{L}^* = (P^*, J, L^*) \in \mathbb{P}(M)$ *where* P^* *is the adjoint of* P *in* $L^2(M, J)$ *and* $L^* \in \mathrm{Vect}(T^*M\backslash 0)$ *given by* $\mathrm{im}\,p^*$ *for* $p^* = \sigma_\psi(P^*)$. *Note that* $(\mathbb{L}^*)^* = \mathbb{L}$.
(v) *For every subbundle* L *of* $\pi_M^* J$, $J \in \mathrm{Vect}(M)$, *there exist projection data* $\mathbb{L} = (P, J, L) \in \mathbb{P}(M)$.

Let $J \in \mathrm{Vect}(M)$ and (1) be a projection as in Theorem 2.1. Then

$$L := \mathrm{im}\,p \in \mathrm{Vect}(T^*M \backslash 0) \tag{3}$$

is a subbundle of $\pi_M^* J$. Conversely for every $L \in \mathrm{Vect}(T^*M\backslash 0)$ there exists a $J \in \mathrm{Vect}(M)$ such that L is a subbundle of $\pi_M^* J$. In fact, there exists an N and an $L^\perp \in \mathrm{Vect}(T^*M\backslash 0)$ such that $L \oplus L^\perp = (T^*M\backslash 0) \times \mathbb{C}^N$.

Remark 2.6 For every $\mathbb{L} = (P, J, L) \in \mathbb{P}(M)$ we find complementary projection data $\mathbb{L}^\perp = (1 - P, J, L^\perp)$ by setting $L^\perp = \mathrm{im}\,\sigma_\psi(1 - P)$ where we have $L \oplus L^\perp = \pi_M^* J$.

Remark 2.7 For every $\mathbb{L} = (P, J, L)$ we can form continuous projections also in Sobolev spaces $H^s(M, J)$ of distributional sections in the bundle J,

$$P : H^s(M, J) \to H^s(M, J),$$

$s \in \mathbb{R}$. Let us set

$$H^s(M, \mathbb{L}) := PH^s(M, J). \tag{4}$$

This is a closed subspace of $H^s(M, J)$, in fact, a Hilbert space with the scalar product induced by $H^s(M, J)$. Occasionally if P is regarded as an operator on $H^s(M, J)$ we also write P^s.

Remark 2.8 Let $\mathbb{L} = (P, J, L)$ and $\tilde{\mathbb{L}} = (\tilde{P}, \tilde{J}, L) \in \mathbb{P}(M)$ where J is a subbundle of $\tilde{J}$ (such that L is a subbundle also of $\pi_M^* \tilde{J}$) and $P : H^s(M, J) \to H^s(M, \mathbb{L})$ the restriction of $\tilde{P} : H^s(M, \tilde{J}) \to H^s(M, \tilde{\mathbb{L}})$ to $H^s(M, J)$. Then $H^s(M, \mathbb{L}) = H^s(M, \tilde{\mathbb{L}})$.

Let us now formulate other properties of the spaces $H^s(M, \mathbb{L})$.

Proposition 2.9 *We have continuous embeddings*

$$H^{s'}(M, \mathbb{L}) \hookrightarrow H^s(M, \mathbb{L}) \tag{5}$$

for every $s' \geq s$ that are compact for $s' > s$.

Proposition 2.10

(i) *The space $H^\infty(M, \mathbb{L}) = \bigcap_{s \in \mathbb{R}} H^s(M, \mathbb{L})$ is dense in $H^s(M, \mathbb{L})$ for every $s \in \mathbb{R}$.*
(ii) *Let $H^0(M, \mathbb{L})$ be endowed with the scalar product from $H^0(M, J)$, and let $V \subset H^\infty(M, \mathbb{L})$ be a subspace of finite dimension. Then the orthogonal projection $C_V : H^0(M, \mathbb{L}) \to V$ induces continuous operators $C_V : H^s(M, \mathbb{L}) \to V$ for all $s \in \mathbb{R}$, and C_V is a compact operator $H^s(M, \mathbb{L}) \to H^s(M, \mathbb{L})$ for every $s \in \mathbb{R}$.*

Remark 2.11 Let $\mathbb{L} = (P, J, L) \in \mathbb{P}(M)$ and $\mathbb{L}^* = (P^*, J, L^*)$ its adjoint in the sense of Proposition 2.5 (iv). Then we have the subspaces

$$H^s(M, \mathbb{L}) \subseteq H^s(M, J),\ H^{-s}(M, \mathbb{L}^*) \subseteq H^{-s}(M, J)$$

for every $s \in \mathbb{R}$. The sesquilinear pairing $(\cdot, \cdot) : H^\infty(M, \mathbb{L}) \times H^\infty(M, \mathbb{L}^*) \to \mathbb{C}$ induced by

$$H^s(M, J) \times H^{-s}(M, J) \to \mathbb{C} \tag{6}$$

extends to a non-degenerate sesquilinear pairing

$$(\cdot, \cdot) : H^s(M, \mathbb{L}) \times H^{-s}(M, \mathbb{L}^*) \to \mathbb{C} \tag{7}$$

for every $s \in \mathbb{R}$ (which is nothing else than the restriction of (6) to the respective subspaces). If necessary the pairing will also be denoted by $(\cdot,\cdot)_{H^s(M,\mathbb{L})\times H^{-s}(M,\mathbb{L}^*)}$.

Let $\mathbb{L}_i = (P_i, J_i, L_i) \in \mathbb{P}(M)$, $i = 1,2$, and let $A : H^\infty(M,\mathbb{L}_1) \to H^\infty(M,\mathbb{L}_2)$ be an operator that extends to a continuous operator $A : H^s(M,\mathbb{L}_1) \to H^{s-\mu}(M,\mathbb{L}_2)$ for all $s \in \mathbb{R}$ and some $\mu \in \mathbb{R}$. Then there is a (unique) $A^* : H^\infty(M,\mathbb{L}_2^*) \to H^\infty(M,\mathbb{L}_1^*)$ defined by

$$(Au, v)_{H^0(M,\mathbb{L}_2)\times H^0(M,\mathbb{L}_2^*)} = (u, A^*v)_{H^0(M,\mathbb{L}_1)\times H^0(M,\mathbb{L}_1^*)} \tag{8}$$

for all $u \in H^\infty(M,\mathbb{L}_1)$, $v \in H^\infty(M,\mathbb{L}_2^*)$, that extends to a continuous operator

$$A^* : H^s(M,\mathbb{L}_2^*) \to H^{s-\mu}(M,\mathbb{L}_1^*) \tag{9}$$

for every $s \in \mathbb{R}$, called the formal adjoint of A.

Given an $\mathbb{L} = (P, J, L) \in \mathbb{P}(M)$ with the subspaces

$$H^s(M,\mathbb{L}) \hookrightarrow H^s(M,J), \quad s \in \mathbb{R}, \tag{10}$$

we consider the embedding operator E given by (10). Analogously we observe the embedding

$$e : L \to \pi_M^* J \tag{11}$$

as a subbundle, where (11) is assumed to be homogeneous of order 0 in ξ. More precisely, if S^*M denotes the unit cosphere bundle induced by $T^*M\backslash 0$ (with respect to a fixed Riemiannian metric on M) and if $\pi_1 : T^*M\backslash 0 \to S^*M$ denotes the canonical projection, defined by $(x,\xi) \to (x, \xi/|\xi|)$, then we have $L = \pi_1^* L_1$ for $L_1 := L|_{S^*M} \in \mathrm{Vect}(S^*M)$. Similarly we have $\pi_M^* J = \pi_1^*((\pi_M^* J)_1)$; then we obtain an embedding

$$e_1(x,\xi) : (L_1)_{(x,\xi)} \to ((\pi_M^* J)_1)_{(x,\xi)}, \quad (x,\xi) \in S^*M, \tag{12}$$

which induces embeddings

$$e(x,\xi) : L_{(x,\xi)} \to (\pi_M^* J)_{(x,\xi)}, \quad (x,\xi) \in T^*M\backslash 0, \tag{13}$$

defined by the composition of linear mappings

$$e(x,\xi) : L_{(x,\xi)} \to L_{(x,\xi/|\xi|)} \to ((\pi_M^* J)_1)_{(x,\xi/|\xi|)} \to (\pi_M^* J)_{(x,\xi)}$$

where the first mapping is defined by the bundle pull back under the embedding $S^*M \hookrightarrow T^*M\backslash 0$, the second one by (12) and the third one by the identification $(\pi_M^* J)_{(x,\xi)} = J_x$, $x \in M$. Then we have $e(x,\lambda\xi) = e(x,\xi)$ for all $\lambda \in \mathbb{R}$, $(x,\xi) \in T^*M\backslash 0$.

In the following we formulate operators which appear as lower right corners of 2×2 matrices of the Toeplitz analogue of Boutet de Monvel's calculus below.

Definition 2.12 Let $\mathbb{L}_i := (P_i, J_i, L_i)$, $i = 1, 2$, with the corresponding operators $E_1 : H^\infty(M, \mathbb{L}_1) \to H^\infty(M, J_1)$ and $P_2 : H^\infty(M, J_2) \to H^\infty(M, \mathbb{L}_2)$, respectively. An operator of the form

$$A = P_2 \tilde{A} E_1$$

for some $\tilde{A} \in L^\mu_{\mathrm{cl}}(M; J_1, J_2)$, $\mu \in \mathbb{R}$, is called a Toeplitz operator of order $\mu \in \mathbb{R}$ associated with the projection data $\mathbb{L}_1$, $\mathbb{L}_2$. Let $\mathcal{T}^\mu(M; \mathbb{L}_1, \mathbb{L}_2)$ denote the set of all Toeplitz operators on M of order μ. Moreover, we set

$$\mathcal{T}^{-\infty}(M; \mathbb{L}_1, \mathbb{L}_2) := \{P_2 \tilde{C} E_1 : \tilde{C} \in L^{-\infty}(M; J_1, J_2)\}. \tag{14}$$

Observe that $\mathcal{T}^{\mu-j}(M; \mathbb{L}_1, \mathbb{L}_2) \subseteq \mathcal{T}^\mu(M; \mathbb{L}_1, \mathbb{L}_2)$ for every positive integer j and $\mathcal{T}^{-\infty}(M; \mathbb{L}_1, \mathbb{L}_2)$ for every μ. Thus

$$\mathcal{T}^{-\infty}(M; \mathbb{L}_1, \mathbb{L}_2) \subseteq \bigcap_{\mu \in \mathbb{R}} \mathcal{T}^\mu(M; \mathbb{L}_1, \mathbb{L}_2).$$

Note that $\mathcal{T}^{-\infty}(M; \mathbb{L}_1, \mathbb{L}_2)$ can be equivalently defined as the set of all $A \in \mathcal{T}^\infty(M; \mathbb{L}_1, \mathbb{L}_2) := \bigcup_{\mu \in \mathbb{R}} \mathcal{T}^\mu(M; \mathbb{L}_1, \mathbb{L}_2)$ such that there is an $\tilde{A} \in L^\mu_{\mathrm{cl}}(M; J_1, J_2)$ for some $\mu \in \mathbb{R}$ with $A = P_2 \tilde{A} E_1$ where

$$\tilde{C} := P_2 \tilde{A} P_1 \in L^{-\infty}(M; J_1, J_2);$$

then $A = P_2 \tilde{C} E_1$. Moreover,

$$P_2 \tilde{C} E_1 \in \mathcal{T}^{-\infty}(M; \mathbb{L}_1, \mathbb{L}_2) \Leftrightarrow P_2 \tilde{C} P_1 \in L^{-\infty}(M; J_1, J_2). \tag{15}$$

Proposition 2.13 *Given $\mathbb{L}_i \in \mathbb{P}(M)$, $i = 1, 2$, we have a canonical isomorphism*

$$\mathcal{T}^\mu(M; \mathbb{L}_1, \mathbb{L}_2) \to \{P_2 \tilde{A} P_1 : \tilde{A} \in L^\mu_{\mathrm{cl}}(M; J_1, J_2)\}. \tag{16}$$

Remark 2.14 The space $\mathcal{T}^\mu(M; \mathbb{L}_1, \mathbb{L}_2)$ can be identified with the corresponding quotient space $L^\mu_{\mathrm{cl}}(M; J_1, J_2)/\sim$.

Observe that for $\mathbb{L}_i \in \mathbb{P}(M)$, $i = 1, 2$, and $\tilde{A} \in L^\mu_{\mathrm{cl}}(M; J_1, J_2)$ we have $\tilde{A} \sim P_2 \tilde{A} P_1$.

Theorem 2.15 *An $A \in \mathcal{T}^\mu(M; \mathbb{L}_1, \mathbb{L}_2)$ extends to a continuous operator*

$$A : H^s(M, \mathbb{L}_1) \to H^{s-\mu}(M, \mathbb{L}_2) \tag{17}$$

for every $s \in \mathbb{R}$.

Remark 2.16 For every pair of projection data $\mathbb{L}_i = (P_i, J_i, L_i) \in \mathbb{P}(M)$, $i = 1, 2$, there exist $\mathbb{M}_i = (Q_i, \mathbb{C}^m, L_i) \in \mathbb{P}(M)$, such that

$$\mathcal{T}^\mu(M; \mathbb{L}_1, \mathbb{L}_2) = \mathcal{T}^\mu(M; \mathbb{M}_1, \mathbb{M}_2). \tag{18}$$

In fact, every two bundles J_1, J_2 over M can be regarded as subbundles of a trivial bundle $\mathbb{C}^m$; it suffices to use the fact that $J_1 \oplus J_2$ has a complementary bundle $(J_1 \oplus J_2)^\perp$ where $J_1 \oplus J_2 \oplus (J_1 \oplus J_2)^\perp = \mathbb{C}^m$ for a resulting m. Let $J_i^\perp$ be the complementary bundle of J_i in $\mathbb{C}^m$, $i = 1, 2$. According to Theorem 2.1 with the projection $\pi_M^* \mathbb{C}^m \to \pi_M^* J_i$ along $\pi_M^* J_i^\perp$ we can associate pseudo-differential projections $\tilde{P}_i \in L^0_{\mathrm{cl}}(M; \mathbb{C}^m, \mathbb{C}^m)$, $\tilde{P}_i : H^s(M, \mathbb{C}^m) \to H^s(M, \mathbb{C}^m)$, $i = 1, 2$. Moreover, we have our original projections $P_i : \pi_M^* J_i \to L_i$ which gives us projections $\pi_M^* \mathbb{C}^m \to L_i$ and associated pseudo-differential projections $Q_i \in L^0_{\mathrm{cl}}(M; \mathbb{C}^m, \mathbb{C}^m)$, where $Q_i = P_i \tilde{P}_i$, and $Q_i : H^s(M, \mathbb{C}^m) \to H^s(M, L_i)$, $i = 1, 2$. Relation (18) then follows from the fact that every $\tilde{A} \in L^\mu_{\mathrm{cl}}(M; J_1, J_2)$ can be identified with some $\tilde{\tilde{A}} \in L^\mu_{\mathrm{cl}}(M; \mathbb{C}^m, \mathbb{C}^m)$ by setting $\tilde{\tilde{A}} = \tilde{A}\tilde{P}_1$.

Recall that when $A \in L^\mu_{\mathrm{cl}}(M; J_1, J_2)$, $A' \in L^\mu_{\mathrm{cl}}(M; J_1', J_2')$ are pseudo-differential operators, J_i, $J_i' \in \mathrm{Vect}(M)$, $i = 1, 2$, we have the direct sum

$$A \oplus A' := \mathrm{diag}(A, A') \in L^\mu_{\mathrm{cl}}(M; J_1 \oplus J_1', J_2 \oplus J_2').$$

A similar operation is possible on the level of Toeplitz operators. In fact, let $\mathbb{L}_i = (P_i, J_i, L_i)$, $\mathbb{L}_i' = (P_i', J_i', L_i')$, $i = 1, 2$, be projection data on M; then for $A := P_2 \tilde{A} E_1 \in \mathcal{T}^\mu(M; \mathbb{L}_1, \mathbb{L}_2)$ and $A' := P_2' \tilde{A}' R_1' \in \mathcal{T}^\mu(M; \mathbb{L}_1', \mathbb{L}_2')$ we have the direct sum

$$A \oplus A' := \mathrm{diag}(A, A') \in \mathcal{T}^\mu(M; \mathbb{L}_1' \oplus \mathbb{L}_1', \mathbb{L}_2 \oplus \mathbb{L}_2').$$

Proposition 2.17 *Let $A_j \in \mathcal{T}^{\mu - j}(M; \mathbb{L}_1, \mathbb{L}_2)$, $j \in \mathbb{N}$, be an arbitrary sequence. Then there exists an $A \in \mathcal{T}^\mu(M; \mathbb{L}_1, \mathbb{L}_2)$ such that*

$$A - \sum_{j=1}^N A_j \in \mathcal{T}^{\mu - (N+1)}(M; \mathbb{L}_1, \mathbb{L}_2) \tag{19}$$

for every $N \in \mathbb{N}$, and A is unique mod $\mathcal{T}^{-\infty}(M; \mathbb{L}_1, \mathbb{L}_2)$.

Definition 2.18 Let $A \in \mathcal{T}^\mu(M; \mathbb{L}_1, \mathbb{L}_2)$, $A = P_2 \tilde{A} E_1$ for some $\tilde{A} \in L^\mu_{\mathrm{cl}}(M; J_1, J_2)$. We define the homogeneous principal symbol of A as the bundle morphism

$$\sigma_\psi(A) : L_1 \to L_2,$$

fibrewise over $(x, \xi) \in T^*M \setminus 0$ given as the composition

$$\sigma_\psi(A)(x, \xi) = \sigma_\psi(P_2)(x, \xi) \sigma_\psi(\tilde{A})(x, \xi) \sigma_\psi(E_1)(x, \xi). \tag{20}$$

Here $\sigma_\psi(E_1)(x,\xi)$ is interpreted as (13) for L_1 and J_1 instead of L and J, respectively, while $\sigma_\psi(\tilde{A})(x,\xi)$ and $\sigma_\psi(P_2)(x,\xi)$ are the standard homogeneous principal symbols of the corresponding classical pseudo-differential operators.

For simplicity, we occasionally identify $\sigma_\psi(A)$ with $\sigma_\psi(P_2)\sigma_\psi(\tilde{A})\sigma_\psi(P_1)$, where $\sigma_\psi(P_1)|_{L_1} : L_1 \to \pi_M^* J_1$ is identified with the embedding $\sigma_\psi(E_1)$.

Let $S^{(\mu)}(T^*M \setminus 0; L_1, L_2)$ for $L_1,\ L_2 \in \mathrm{Vect}(T^*M \setminus 0)$ denote the space of all bundle morphisms

$$\sigma : L_1 \to L_2$$

such that $\sigma(x,\lambda\xi) = \lambda^\mu \sigma(x,\xi)$, $\lambda \in \mathbb{R}_+$, as a linear mapping $L_{1,(x,\xi)} \to L_{2,(x,\xi)}$ for every $(x,\xi) \in T^*M \setminus 0$.

Then σ_ψ gives us a linear map

$$\sigma_\psi : \mathcal{T}^\mu(M; \mathbb{L}_1, \mathbb{L}_2) \to S^{(\mu)}(T^*M \setminus 0; L_1, L_2). \tag{21}$$

Theorem 2.19

(i) *The principal symbolic map* (21) *is surjective, and there is a right inverse*

$$\mathrm{op} : S^{(\mu)}(T^*M \setminus 0; L_1, L_2) \to \mathcal{T}^\mu(M; \mathbb{L}_1, \mathbb{L}_2).$$

(ii) *The kernel of* (21) *coincides with* $\mathcal{T}^{\mu-1}(M; \mathbb{L}_1, \mathbb{L}_2)$.

Remark 2.20 Let $A \in \mathcal{T}^\mu(M; \mathbb{L}_1, \mathbb{L}_2)$ be an operator such that $\sigma_\psi(A) = 0$. Then (17) is a compact operator for every $s \in \mathbb{R}$.

In fact, Theorems 2.19 and 2.15 show that $A \in \mathcal{T}^{\mu-1}(M; \mathbb{L}_1, \mathbb{L}_2)$ and $A : H^s(M, \mathbb{L}_1) \to H^{s-\mu+1}(M, \mathbb{L}_2)$ is continuous; then the compactness of (17) is a consequence of Proposition 2.9.

Theorem 2.21 *Let* $A \in \mathcal{T}^\mu(M; \mathbb{L}_0, \mathbb{L}_2)$ *and* $B \in \mathcal{T}^\nu(M; \mathbb{L}_1, \mathbb{L}_0)$ *for* $\mu,\ \nu \in \mathbb{R}$, *and* $\mathbb{L}_1,\ \mathbb{L}_0,\ \mathbb{L}_2 \in \mathbb{P}(M)$. *Then we have* $AB \in \mathcal{T}^{\mu+\nu}(M; \mathbb{L}_1, \mathbb{L}_2)$ *and*

$$\sigma_\psi(AB) = \sigma_\psi(A)\sigma_\psi(B). \tag{22}$$

Theorem 2.22 *Given* $A \in \mathcal{T}^\mu(M; \mathbb{L}_1, \mathbb{L}_2)$, $\mathbb{L}_i = (P_i, J_i, L_i) \in \mathbb{P}(M)$, $i = 1, 2$, *for the formal adjoint in the sense of* Remark 2.11 *we have* $A^* \in \mathcal{T}^\mu(M; \mathbb{L}_2^*, \mathbb{L}_1^*)$ *where* $\mathbb{L}_i^* \in \mathbb{P}(M)$ (*see* Proposition 2.5 (iv)) *for* $i = 1, 2$, *and*

$$\sigma_\psi(A^*) = \sigma_\psi(A)^*.$$

Definition 2.23 An operator $A \in \mathcal{T}^\mu(M; \mathbb{L}_1, \mathbb{L}_2)$, $\mu \in \mathbb{R}$, for $\mathbb{L}_i \in \mathbb{P}(M)$, $i = 1, 2$, is called elliptic (of order μ) if $\sigma_\psi(A) : L_1 \to L_2$ is an isomorphism.

Example 2.1 Let $\mathbb{L} = (P, J, L) \in \mathbb{P}(M)$, and let

$$a_{(\mu)} : \pi_M^* J \to \pi_M^* J$$

for any fixed $\mu \in \mathbb{R}$ denote the unique smooth bundle morphism such that $a_{(\mu)} : \pi_1^* J \to \pi_1^* J$ for $\pi_1 : S^*M \to M$ is the identity map and $a_{(\mu)}(x, \lambda\xi) = \lambda^\mu a_{(\mu)}(x, \xi)$ for all $(x, \xi) \in T^*M \setminus 0$, $\lambda \in \mathbb{R}_+$. Let $\tilde{A} \in L^\mu_{\mathrm{cl}}(M; J, J)$ be any element with $\sigma_\psi(\tilde{A}) = a_{(\mu)}$ and consider the composition $P\tilde{A}P \in L^\mu_{\mathrm{cl}}(M; J, J)$. Then $P\tilde{A}R$ identified with $P\tilde{A}P|_{H^\infty(M,\mathbb{L})}$ represents an elliptic operator in $\mathcal{T}^\mu(M; \mathbb{L}, \mathbb{L})$.

Definition 2.24 Let $A \in \mathcal{T}^\mu(M; \mathbb{L}_1, \mathbb{L}_2)$, $\mu \in \mathbb{R}$, $\mathbb{L}_i \in \mathbb{P}(M)$, $i = 1, 2$. Then an operator $B \in \mathcal{T}^{-\mu}(M; \mathbb{L}_1, \mathbb{L}_2)$ is called a parametrix of A, if B satisfies the relations

$$C_{\mathrm{L}} := I - BA \in \mathcal{T}^{-\infty}(M; \mathbb{L}_1, \mathbb{L}_1), \;\; C_{\mathrm{R}} := I - AB \in \mathcal{T}^{-\infty}(M; \mathbb{L}_2, \mathbb{L}_2); \tag{23}$$

here I denotes corresponding identity operators.

Remark 2.25 For every $\mathbb{L} := (P, J, L) \in \mathbb{P}(M)$ and every $\mu \in \mathbb{R}$ there exists an elliptic operator $R^\mu_{\mathbb{L}} \in \mathcal{T}^\mu(M; \mathbb{L}, \mathbb{L})$.

In fact, let $a_{(\mu)} \in S^\mu_{(\mathrm{cl})}(T^*M \setminus 0; J, J)$ be the unique element that restricts to the identity map on $\pi_1^* J$ where $\pi_1 : S^*M \to M$ is the canonical projection of the unit cosphere bundle S^*M induced by T^*M to M. Set $\tilde{A} := \mathrm{op}(a_{(\mu)})$. Then $P\tilde{A}E$ for the embedding $E : H^s(M, \mathbb{L}) \to H^s(M, J)$ is elliptic because $\sigma_\psi(P\tilde{A}E) : L \to L$ is an isomorphism.

Proposition 2.26 *Let $A \in \mathcal{T}^\mu(M; \mathbb{L}_1, \mathbb{L}_2)$ be an elliptic operator, and represent A as an element $A \in \mathcal{T}^\mu(M; \mathbb{M}_1, \mathbb{M}_2)$ for $\mathbb{M}_i := (Q_i, \mathbb{C}^m, L_i)$, $i = 1, 2$, for a sufficiently large m (cf.* Remark 2.16*). Then there exists an elliptic operator $A^\perp \in \mathcal{T}^\mu(M; \mathbb{M}_1^\perp, \mathbb{M}_2^\perp)$ for suitable $\mathbb{M}_i^\perp \in \mathbb{P}(M)$ such that $A \oplus A^\perp \in L^\mu_{\mathrm{cl}}(M; \mathbb{C}^m, \mathbb{C}^m)$ is elliptic in the standard sense.*

Theorem 2.27 *Let $A \in \mathcal{T}^\mu(M; \mathbb{L}_1, \mathbb{L}_2)$, $\mu \in \mathbb{R}$, $\mathbb{L}_i = (P_i, J_i, L_i) \in \mathbb{P}(M)$.*

(i) *The operator A is elliptic (of order μ) if and only if*

$$A : H^s(M, \mathbb{L}_1) \to H^{s-\mu}(M, \mathbb{L}_2) \tag{24}$$

is a Fredholm operator for some $s = s_0 \in \mathbb{R}$.

(ii) *If A is elliptic, then (24) is Fredholm for all $s \in \mathbb{R}$, and* $\dim\ker A$ *(as well as the kernel itself) and* $\dim\operatorname{coker} A$ *are independent of s.*

(iii) *An elliptic operator $A \in \mathcal{T}^\mu(M; \mathbb{L}_1, \mathbb{L}_2)$ has a parametrix $B \in \mathcal{T}^{-\mu}(M; \mathbb{L}_2, \mathbb{L}_1)$, and B can be chosen in such a way that the remainders in relation (23) are projections*

$$C_{\mathrm{L}} : H^s(M, \mathbb{L}_1) \to V, \; C_{\mathrm{R}} : H^{s-\mu}(M, \mathbb{L}_2) \to W$$

for all $s \in \mathbb{R}$, for $V := \ker A \subset H^{\infty}(M, \mathbb{L}_1)$, and a finite-dimensional subspace $W \subset H^{\infty}(M, \mathbb{L}_2)$ such that $W + \operatorname{im} A = H^{s-\mu}(M, \mathbb{L}_2)$ and $W \cap \operatorname{im} A = \{0\}$ for all $s \in \mathbb{R}$.

Remark 2.28 The ellipticity of an operator $A \in \mathcal{T}^{\mu}(M; \mathbb{L}_1, \mathbb{L}_2)$ for $\mathbb{L}_i = (P_i, J_i, L_i)$, $i = 1, 2$ only depends on the bundles L_1, L_2, not on the projections P_1, P_2 or the chosen bundles J_1, J_2 over M. Of course, the spaces $H^s(M, \mathbb{L}_1)$ and $H^{s-\mu}(M, \mathbb{L}_2)$ depend on the choice of projections P_1, P_2. So the Fredholm index of (24) may change under varying projections.

The corresponding effect can be illustrated in functional analytic terms.

Theorem 2.29 *Let H_i, $i = 1, 2$ be Hilbert spaces, and let P_i, $Q_i \in \mathcal{L}(H_i)$ be continuous projections such that $P_i - Q_i \in \mathcal{K}(H_i)$, $i = 1, 2$. Moreover, let $A \in \mathcal{L}(H_1, H_2)$ be an operator such that*

$$A := P_2 \tilde{A} : \operatorname{im} P_1 \to \operatorname{im} P_2$$

is a Fredholm operator. Then also

$$B := Q_2 \tilde{A} : \operatorname{im} Q_1 \to \operatorname{im} Q_2$$

is a Fredholm operator, and we have

$$\operatorname{ind} A - \operatorname{ind} B = \operatorname{ind}(P_1, Q_1) - \operatorname{ind}(P_2, Q_2). \tag{25}$$

Corollary 2.30 *Let $A \in \mathcal{T}^{\mu}(M; \mathbb{L}_1, \mathbb{L}_2)$ for $\mathbb{L}_j := (P_j, J_j, L_j)$; $B \in \mathcal{T}^{\mu}(M; \mathbb{M}_1, \mathbb{M}_2)$ for $\mathbb{M}_j := (Q_j, J_j, L_j)$, and assume that $\sigma_{\psi}(A) = \sigma_{\psi}(B)$. Then the Fredholm indices of A and B as operators*

$$A : H^s(M, \mathbb{L}_1) \to H^{s-\mu}(M, \mathbb{L}_2), \quad B : H^s(M, \mathbb{M}_1) \to H^{s-\mu}(M, \mathbb{M}_2)$$

are related by formula (25), which is independent of s.

In fact, the Fredholm indices of A and B are independent of s, cf. Theorem 2.27, and hence, we may apply (25) for any fixed s.

Remark 2.31 Let $\mathbb{L} := (P, J, L)$ and $\mathbb{M} := (Q, J, L)$ be projection data; we interpret the operators

$$P : H^s(M, \mathbb{M}) \to H^s(M, \mathbb{L}) \quad \text{and} \quad Q : H^s(M, \mathbb{L}) \to H^s(M, \mathbb{M})$$

as elements of $\mathcal{T}^0(M; \mathbb{M}, \mathbb{L})$ and $\mathcal{T}^0(M; \mathbb{L}, \mathbb{M})$, respectively. Recall that we have an identification

$$\mathcal{T}^0(M; \mathbb{M}, \mathbb{L}) \cong \{P\tilde{A}Q : \tilde{A} \in L^{\mu}_{\mathrm{cl}}(M; J, J)\},$$

cf. Proposition 2.13. Inserting $\tilde{A} = P$ we obtain $P\tilde{A}Q = P : \operatorname{im} Q \to \operatorname{im} P$. For a similar reason we interpret Q as a Toeplitz operator $Q\tilde{A}P = Q : \operatorname{im} P \to \operatorname{im} Q$ for $\tilde{A} = Q$. Since $\sigma_\psi(P) = \sigma_\psi(Q) = \mathrm{id}_L$ the operators P and Q are elliptic in the respective classes. Moreover, we have

$$\operatorname{ind} P = \operatorname{ind}(P, Q), \quad \operatorname{ind} Q = \operatorname{ind}(Q, P).$$

Remark 2.32 Let $A, B \in \mathcal{T}^\mu(M; \mathbb{L}_1, \mathbb{L}_2)$ be elliptic, and assume that the principal symbols

$$\sigma_\psi(A),\ \sigma_\psi(B) : L_1 \to L_2$$

coincide. Then we have

$$\operatorname{ind} A = \operatorname{ind} B.$$

In fact, Proposition 2.20 gives us $\sigma_\psi(A - B) = 0$, i.e., $A - B \in \mathcal{T}^{\mu-1}(M; \mathbb{L}_1, \mathbb{L}_2)$, and hence A is equal to B modulo a compact operator.

3 Operators with the Transmission Property

Let X be a compact C^∞ manifold with boundary Y, and let $2X$ denote the double of X, obtained by gluing together two copies X_+ and X_- of X along their common boundary Y by the identity map. We then identify X with X_+. Moreover, let e^+ denote the operator of extension of functions on $\operatorname{int} X_+$ by zero to the opposite side X_-, and let r^+ denote the operator of restriction of distributions on $2X$ to $\operatorname{int} X_+$. In an analogous manner we define the operators e^- and r^- with respect to the minus-side of $2X$. On $2X$ we choose a Riemannian metric that is equal to the product metric of $Y \times (-1, 1)$ in a neighbourhood of Y for some Riemannian metric on Y.

For a given $E \in \mathrm{Vect}(X)$ we fix any $\tilde{E} \in \mathrm{Vect}(X)$ and $E = \tilde{E}|_X$. Now let E and F in $\mathrm{Vect}(X)$ with fibre dimensions k and m, respectively. Consider the space $L^\mu_{\mathrm{cl}}(2X; \tilde{E}, \tilde{F})$. For every chart $\chi : V \to \Omega$ on $2X$, $\Omega \subset \mathbb{R}^n$ open, and trivialisations $\tilde{E}|_V \cong \Omega \times \mathbb{C}^k$, $\tilde{F}|_V \cong \Omega \times \mathbb{C}^m$, the push forward $\chi_* A$ of an operator $A \in L^\mu_{\mathrm{cl}}(2X; \tilde{E}, \tilde{F})$ belongs to $L^\mu_{\mathrm{cl}}(\Omega; \mathbb{C}^k, \mathbb{C}^m)$. By notation the push forward χ_* also refers to the chosen trivialisations of $\tilde{E}|_V$ and $\tilde{F}|_V$; for simplicity those are not explicitly indicated here. This should not cause any confusion.

Let $V \cap Y \neq \emptyset$, $V := V' \times (-1, 1)$, where V' is a coordinate neighbourhood on the boundary Y, and assume that χ restricts to charts $\chi_\pm : V_\pm \to \Omega \times \overline{\mathbb{R}}_\pm$ on $X_\pm$ for $V_\pm := X_\pm \cap V$, and to a chart $\chi' := \chi|_{V'} : V' \to \Omega$ on Y, $\Omega \subset \mathbb{R}^{n-1}$.

Definition 3.1 For $\mu \in \mathbb{Z}$ we define $L^{\mu}_{\mathrm{tr}}(2X; \tilde{E}, \tilde{F})$ to be the space of all elements $\tilde{A} \in L^{\mu}_{\mathrm{cl}}(2X; \tilde{E}, \tilde{F})$ such that for every chart $\chi : V \to \Omega \times \mathbb{R}$ of the described kind and $\varphi,\ \psi \in C_0^{\infty}(\Omega \times \mathbb{R})$ the symbol

$$\tilde{a}(x, \xi) := e_{-\xi}\{\varphi(\chi_* \tilde{A}\psi\} e_{\xi} \ \text{ for } e_{\xi} := e^{ix\xi}$$

is an $m \times k$ matrix of elements in $S^{\mu}_{\mathrm{tr}}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$. Moreover, we set

$$L^{\mu}_{\mathrm{tr}}(X; E, F) := \{\mathrm{r}^+ \tilde{A} \mathrm{e}^+ : \tilde{A} \in L^{\mu}_{\mathrm{tr}}(2X; \tilde{E}, \tilde{F})\}. \tag{26}$$

Remark 3.2 The space $L^{\mu}_{\mathrm{tr}}(2X; \tilde{E}, \tilde{F})$ is closed in $L^{\mu}_{\mathrm{cl}}(2X; \tilde{E}, \tilde{F})$. Moreover, the space

$$\{\tilde{A} \in L^{\mu}_{\mathrm{tr}}(2X; \tilde{E}, \tilde{F}) : \mathrm{r}^+ \tilde{A} \mathrm{e}^+ = 0\} \tag{27}$$

is closed in $L^{\mu}_{\mathrm{tr}}(2X; \tilde{E}, \tilde{F})$, and we have

$$L^{\mu}_{\mathrm{tr}}(X; E, F) = L^{\mu}_{\mathrm{tr}}(2X; \tilde{E}, \tilde{F})/\sim, \tag{28}$$

where $/\sim$ indicates the quotient space with respect to the equivalence relation $\tilde{A} \sim \tilde{B} \Leftrightarrow \mathrm{r}^+(\tilde{A} - \tilde{B})\mathrm{e}^+ = 0$. This gives us a natural Fréchet topology also in the space (28).

Proposition 3.3 *Let $\Omega \subset \mathbb{R}^q$ be an open set, and assume that $a(y, t, \eta, \tau) \in S^{\mu}_{\mathrm{tr}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n)$ is independent of t for $t > c$ for some constant $c > 0$. Then* $\mathrm{Op}^+(a) := \mathrm{r}^+ \mathrm{Op}(\tilde{a}) \mathrm{e}^+$ *(for any $\tilde{a}(y, t, \eta, \tau) \in S^{\mu}_{\mathrm{tr}}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ such that $a = \tilde{a}|_{\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n}$) induces a continuous operator*

$$\mathrm{Op}^+(a) : H^s_{\mathrm{comp}(y)}(\Omega \times \mathbb{R}_+) \to: H^{s-\mu}_{\mathrm{loc}(y)}(\Omega \times \mathbb{R}_+)$$

for every $s \in \mathbb{R}$, $s > -1/2$.

Theorem 3.4 *An $A \in L^{\mu}_{\mathrm{tr}}(X; E, F)$ induces a continuous operator*

$$A : H^s(X, E) \to H^{s-\mu}(X, F)$$

for every $s \in \mathbb{R}$, $s > -1/2$.

Given an $A \in L^{\mu}_{\mathrm{tr}}(X; E, F)$, $A = \mathrm{r}^+ \tilde{A} \mathrm{e}^+$ for an $\tilde{A} \in L^{\mu}_{\mathrm{tr}}(2X; \tilde{E}, \tilde{F})$ we first have the homogeneous principal symbol $\sigma_{\psi}(\tilde{A}) : \pi^*_{2X} \tilde{E} \to \pi^*_{2X} \tilde{F}$ for $\pi_{2X} : T^*(2X) \setminus 0 \to 2X$, and we set $\sigma_{\psi}(A) := \sigma_{\psi}(\tilde{A})|_{T^*X \setminus 0}$,

$$\sigma_{\psi}(A) : \pi^*_X E \to \pi^*_X F, \tag{29}$$

$\pi_X : T^*X \setminus 0 \to X$ $(T^*X = T^*(2X)|_X)$. With (29) we associate a family of operators

$$\sigma_{\partial}(A)(y, \eta) := \mathrm{r}^+ \sigma_{\psi}(A)(y, 0, \eta, D_t) \mathrm{e}^+ = \mathrm{r}^+ \mathrm{op}(\sigma_{\psi}(A)|_{t=0})(y, \eta) \mathrm{e}^+ \tag{30}$$

for $(y,\eta) \in T^*Y \setminus 0$. This refers to the variables $(y,t) \in Y \times [0,1)$ of a collar neighbourhood of the boundary. The family of operators (30) represents a bundle morphism

$$\sigma_\partial(A) : \pi_Y^* E' \otimes H^s(\mathbb{R}_+) \to \pi_Y^* F' \otimes H^{s-\mu}(\mathbb{R}_+)$$

for every fixed $s \in \mathbb{R}$, $s > -1/2$, $\pi_Y : T^*Y \setminus 0 \to Y$, $E' := E|_Y$, $F' := F|_Y$. Alternatively, we interpret $\sigma_\partial(A)$ as a morphism

$$\sigma_\partial(A) : \pi_Y^* E' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \to \pi_Y^* F' \otimes \mathcal{S}(\overline{\mathbb{R}}_+),$$

$\mathcal{S}(\overline{\mathbb{R}}_+) := \mathcal{S}(\mathbb{R})|_{\overline{\mathbb{R}}_+}$. We call $\sigma_\psi(A)$ the principal interior symbol and $\sigma_\partial(A)$ the principal boundary symbol of the operator A.

Example 3.1 Let $A = \sum_{|\alpha|\le\mu} a_\alpha(x) D_x^\alpha$ be a differential operator in $\mathbb{R}^n_+ = \{x = (y,t) \in \mathbb{R}^n : t > 0\}$,
$a_\alpha \in C^\infty(\overline{\mathbb{R}}^n_+)$. Then

$$\sigma_\psi(A)(x,\xi) = \sum_{|\alpha|=\mu} a_\alpha(x)\xi^\alpha,$$

$$\sigma_\partial(A)(y,\eta) = \sum_{|\alpha|=\mu} a_\alpha(y,0)(\eta, D_t)^\alpha.$$

Remark 3.5 For $(\kappa_\lambda u)(t) := \lambda^{1/2} u(\lambda t)$, $\lambda \in \mathbb{R}_+$, we have

$$\sigma_\partial(A)(y,\lambda\eta) = \lambda^\mu \kappa_\lambda \sigma_\partial(A)(y,\eta)\kappa_\lambda^{-1}$$

for all $\lambda \in \mathbb{R}_+$.

We now formulate the 2×2 block matrix algebra of boundary value problems on X with trace and potential conditions. The motivation is similar to that of classical pseudo-differential operators on an open manifold where we complete the algebra of differential operators to an algebra that contains the parametrices of elliptic elements. In the present case it is the set of differential boundary value problems with differential boundary conditions (up to an order reduction on the boundary) that we complete to an algebra of pseudo-differential boundary value problems that contains the parametrices of elliptic elements.

The spaces $L^\mu_{\mathrm{tr}}(X; E, F)$ belong to the upper left corners. However, if we compose two elements of that kind there appear remainder terms, here called Green operators. In addition boundary operators as in classical BVPs (like Dirichlet or Neumann for Laplacians) have to be designed. This is just the topic of the following discussion. Also the respective 2×2 block matrices will be called Green operators since the 12- and 21-entries have some similarity with the above-mentioned Green operators in the upper left corners.

First we need smoothing Green operators. They will also have a so-called type $d \in \mathbb{N}$; and we begin with the case $d = 0$.

Let $\mathcal{B}^{-\infty,0}(X; \boldsymbol{v})$ for $\boldsymbol{v} := (E, F, J_1, J_2)$ for E, $F \in \mathrm{Vect}(X)$, $J_1 J_2 \in \mathrm{Vect}(Y)$, denote the space of all operators

$$\mathcal{G} := \begin{pmatrix} G & K \\ T & Q \end{pmatrix} : \begin{matrix} C^\infty(X, E) \\ \oplus \\ C^\infty(Y, J_1) \end{matrix} \longrightarrow \begin{matrix} C^\infty(X, F) \\ \oplus \\ C^\infty(Y, J_2) \end{matrix}$$

such that $\mathcal{G}$ and $\mathcal{G}^*$ extend to continuous operators

$$\mathcal{G} : \begin{matrix} H^s(X, E) \\ \oplus \\ H^s(Y, J_1) \end{matrix} \longrightarrow \begin{matrix} C^\infty(X, F) \\ \oplus \\ C^\infty(Y, J_2) \end{matrix}, \quad \mathcal{G}^* : \begin{matrix} H^s(X, F) \\ \oplus \\ H^s(Y, J_2) \end{matrix} \longrightarrow \begin{matrix} C^\infty(X, E) \\ \oplus \\ C^\infty(Y, J_1) \end{matrix}$$

for all $s \in \mathbb{R}$, $s > -1/2$. Here $\mathcal{G}^*$ is the formal adjoint of $\mathcal{G}$ in the sense

$$(u, \mathcal{G}^* v)_{L^2(X,E) \oplus L^2(Y,J_1)} = (\mathcal{G} u, v)_{L^2(X,F) \oplus L^2(Y,J_2)} \tag{31}$$

for all $u \in C^\infty(X, E) \oplus C^\infty(Y, J_1)$, $v \in C^\infty(X, F) \oplus C^\infty(Y, J_2)$; the L^2-scalar products refers to the chosen Riemannian metrics on X and Y and to the Hermitean metrics in the respective vector bundles.

In order to pass to operators of type $d \in \mathbb{N} \setminus \{0\}$ for every $E \in \mathrm{Vect}(X)$ we fix an operator $T : C^\infty(X, E) \to C^\infty(X, E)$ that is equal to $\mathrm{id}_{E'} \otimes \partial_t$ in a collar neighbourhood of Y, in the splitting of variables $(y, t) \in Y \times [0, 1)$.

The space $\mathcal{B}^{-\infty,d}(X; \boldsymbol{v})$ of smoothing "BVPs" of type $d \in \mathbb{N} \setminus \{0\}$ is defined to be the set of all operators

$$\mathcal{G} = \mathcal{G}_0 + \sum_{j=1}^{d} \mathcal{G}_j \mathrm{diag}(T^j, 0) \tag{32}$$

for arbitrary $\mathcal{G}_j \in \mathcal{B}^{-\infty,0}(X; \boldsymbol{v}), j = 0, \dots, d$.

In order to formulate the case of order $\mu \in \mathbb{R}$, $d \in \mathbb{N}$, we first introduce corresponding operator-valued symbols. The notion refers to the spaces

$$H := L^2(\mathbb{R}_+, \mathbb{C}^k) \oplus \mathbb{C}^{j_1}, \quad \tilde{H} := L^2(\mathbb{R}_+, \mathbb{C}^m) \oplus \mathbb{C}^{j_2}$$

or Fréchet subspaces

$$\mathcal{S}(\overline{\mathbb{R}}_+, \mathbb{C}^k) \oplus \mathbb{C}^{j_1}, \quad \mathcal{S}(\overline{\mathbb{R}}_+, \mathbb{C}^m) \oplus \mathbb{C}^{j_2}$$

where the group actions are defined by

$$u(t) \oplus c \to \lambda^{1/2} u(\lambda t) \oplus c, \quad \lambda \in \mathbb{R}_+.$$

Definition 3.6 Let $k, m, j_1, j_2 \in \mathbb{N}$, $\mu \in \mathbb{R}$, $\Omega \subset \mathbb{R}^q$ open, $q = \dim Y$. The space $\mathcal{R}_G^{\mu,0}(\Omega \times \mathbb{R}^q; \boldsymbol{w})$, $\boldsymbol{w} = (k, m; j_1, j_2)$, of Green symbols of order μ and type 0 is defined to be the set of all

$$g(y, \eta) \in S_{\mathrm{cl}}^{\mu}(\Omega \times \mathbb{R}^q; L^2(\mathbb{R}_+, \mathbb{C}^k) \oplus \mathbb{C}^{j_1}, \mathcal{S}(\overline{\mathbb{R}}_+, \mathbb{C}^m) \oplus \mathbb{C}^{j_2})$$

such that

$$g^*(y, \eta) \in S_{\mathrm{cl}}^{\mu}(\Omega \times \mathbb{R}^q; L^2(\mathbb{R}_+, \mathbb{C}^m) \oplus \mathbb{C}^{j_2}, L^2(\mathbb{R}_+, \mathbb{C}^k) \oplus \mathbb{C}^{j_1}).$$

Here $g^*(y, \eta)$ is the pointwise adjoint in the sense

$$(u, g^*(y, \eta)v)_{L^2(\mathbb{R}_+, \mathbb{C}^k) \oplus \mathbb{C}^{j_1}} = (g(y, \eta)u, v)_{L^2(\mathbb{R}_+, \mathbb{C}^m) \oplus \mathbb{C}^{j_2}} \tag{33}$$

for all $u \in L^2(\mathbb{R}_+, \mathbb{C}^k) \oplus \mathbb{C}^{j_1}$, $v \in L^2(\mathbb{R}_+, \mathbb{C}^m) \oplus \mathbb{C}^{j_2}$.

Remark 3.7 It can be proved that every $g(y, \eta) \in \mathcal{R}_G^{\mu,0}(\Omega \times \mathbb{R}^q; \boldsymbol{w})$ induced elements

$$g(y, \eta) \in S_{\mathrm{cl}}^{\mu}(\Omega \times \mathbb{R}^q; H^s(\mathbb{R}_+, \mathbb{C}^k) \oplus \mathbb{C}^{j_1}, \mathcal{S}(\overline{\mathbb{R}}_+, \mathbb{C}^m) \oplus \mathbb{C}^{j_2})$$

for all $s \in \mathbb{R}$, $s > -1/2$.

Remark 3.8 The operator $\mathrm{id}_{\mathbb{C}^k} \otimes \partial_t^j$ represents an operator-valued symbol

$$\mathrm{id}_{\mathbb{C}^k} \otimes \partial_t^j \in S_{\mathrm{cl}}^{j}(\Omega \times \mathbb{R}^q; H^s(\mathbb{R}_+, \mathbb{C}^k), H^{s-j}(\overline{\mathbb{R}}_+, \mathbb{C}^k))$$

for every $s \in \mathbb{R}$ (there is in this case no dependence on $(y, \eta) \in \Omega \times \mathbb{R}^q$).

In fact, the operator $T^j := \partial_t^j : H^s(\mathbb{R}_+) \to H^{s-j}(\mathbb{R}_+)$ belongs to $C^\infty(\Omega \times \mathbb{R}^q, \mathcal{L}(H^s(\mathbb{R}_+), H^{s-j}(\mathbb{R}_+))$ and satisfies the relation

$$T^j = \lambda^j \kappa_\lambda T^j \kappa_\lambda^{-1} \text{ for all } \lambda \in \mathbb{R}_+.$$

Definition 3.9

(i) By $\mathcal{R}_G^{\mu,d}(\Omega \times \mathbb{R}^q; \boldsymbol{w})$ for $\mu \in \mathbb{R}$, $d \in \mathbb{N}$, we denote the space of all operator functions

$$g(y, \eta) := g_0(y, \eta) + \sum_{j=1}^{d} g_j(y, \eta) \mathrm{diag}(\partial_t^j, 0)$$

for arbitrary $g_j(y,\eta) \in \mathcal{R}^{\mu-j,0}(\Omega \times \mathbb{R}^q; \boldsymbol{w})$. The elements of $\mathcal{R}_G^{\mu,d}(\Omega \times \mathbb{R}^q; \boldsymbol{w})$ are called Green symbols of order μ and type d.

(ii) By $\mathcal{R}^{\mu,d}(\Omega \times \mathbb{R}^q; \boldsymbol{w})$ for $\mu \in \mathbb{Z}$, $d \in \mathbb{N}$, we denote the space of all operator functions

$$a(y,\eta) := \mathrm{Op}^+(p)(y,\eta) + g(y,\eta)$$

for arbitrary $p(y,t,\eta,\tau) \in S^{\mu}_{\mathrm{tr}}(\Omega \times \mathbb{R} \times \mathbb{R}^n; \mathbb{C}^k, \mathbb{C}^m)$ and $g(y,\eta) \in \mathcal{R}_G^{\mu,d}(\Omega \times \mathbb{R}^q; \boldsymbol{w})$.

Remark 3.10

(i) Observe that elements of $\mathcal{R}_G^{\mu,d}(\Omega \times \mathbb{R}^q; \boldsymbol{w})$ or $\mathcal{R}^{\mu,d}(\Omega \times \mathbb{R}^q; \boldsymbol{w})$ can be composed by functions in $C^\infty(\Omega)$.

(ii) For $a(y,\eta) \in \mathcal{R}_G^{\mu,d}(\Omega \times \mathbb{R}^q; \boldsymbol{w})$ we have $D_y^\alpha D_\eta^\beta a(y,\eta) \in \mathcal{R}^{\mu-|\beta|,d}(\Omega \times \mathbb{R}^q; \boldsymbol{w})$ for every $\alpha, \beta \in \mathbb{N}^q$. Moreover for

$$a(y,\eta) \in \mathcal{R}^{\mu,d}(\Omega \times \mathbb{R}^q; \boldsymbol{v}_0),\ b(y,\eta) \in \mathcal{R}^{\nu,e}(\Omega \times \mathbb{R}^q; \boldsymbol{w}_0),$$

$$\boldsymbol{v}_0 := (k_0, m; j_0, j_2),\ \boldsymbol{w}_0 := (k, k_0; j_1, j_0), \tag{34}$$

we have

$$D_\eta^\alpha a(y,\eta) D_y^\alpha b(y,\eta) \in \mathcal{R}_G^{\mu+\nu-|\alpha|,h}(\Omega \times \mathbb{R}^q; \boldsymbol{v}_0 \circ \boldsymbol{w}_0) \tag{35}$$

for $\boldsymbol{v}_0 \circ \boldsymbol{w}_0 = (k, m; j_1, j_2)$,

Observe that $g(y,\eta) \in \mathcal{R}_G^{\mu,d}(\Omega \times \mathbb{R}^q; \boldsymbol{w})$ implies

$$g(y,\eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; H^s(\mathbb{R}_+, \mathbb{C}^k) \oplus \mathbb{C}^{j_1}, \mathcal{S}(\overline{\mathbb{R}}_+, \mathbb{C}^m) \oplus \mathbb{C}^{j_2}) \tag{36}$$

for every $s \in \mathbb{R}$, $s > d - 1/2$.

Proposition 3.11

(i) *Let $g_l(y,\eta) \in \mathcal{R}_G^{\mu-l,d}(\Omega \times \mathbb{R}^q; \boldsymbol{w})$, $l \in \mathbb{N}$, be an arbitrary sequence. Then there is a $g(y,\eta) \in \mathcal{R}_G^{\mu,d}(\Omega \times \mathbb{R}^q; \boldsymbol{w})$ such that*

$$g(y,\eta) - \sum_{l=0}^{N} g_l(y,\eta) \in \mathcal{R}_G^{\mu-(N+1),d}(\Omega \times \mathbb{R}^q; \boldsymbol{w})$$

for every $N \in \mathbb{N}$, and $g(y,\eta)$ is unique modulo $\mathcal{R}_G^{-\infty,d}(\Omega \times \mathbb{R}^q; \boldsymbol{w})$.

(i) *For arbitrary* $a_l(y,\eta) \in \mathcal{R}^{\mu-l,d}(\Omega \times \mathbb{R}^q; \boldsymbol{w})$, $l \in \mathbb{N}$, *there exists an* $a(y,\eta) \in \mathcal{R}^{\mu,d}(\Omega \times \mathbb{R}^q; \boldsymbol{w})$ *such that*

$$a(y,\eta) - \sum_{l=0}^{N} a_l(y,\eta) \in \mathcal{R}^{\mu-(N+1),d}(\Omega \times \mathbb{R}^q; \boldsymbol{w})$$

for every $N \in \mathbb{N}$, *and* $a(y,\eta)$ *is unique modulo* $\mathcal{R}_G^{-\infty,d}(\Omega \times \mathbb{R}^q; \boldsymbol{w})$.

With Green symbols we now associate so-called Green operators of order μ and type d, namely,

$$\mathcal{G} = \mathrm{Op}(g), \quad \mathrm{Op}(g)u(y) = \iint e^{i(y-y')\eta} g(y,\eta)u(y')dy'đ\eta,$$

first for $u \in C_0^\infty(\Omega, H^s(\mathbb{R}_+, \mathbb{C}^k)) \oplus C_0^\infty(\Omega, \mathbb{C}^{j_1})$; then

$$\mathrm{Op}(g)u \in C^\infty(\Omega, H^{s-\mu}(\mathbb{R}_+, \mathbb{C}^m)) \oplus C^\infty(\Omega, \mathbb{C}^{j_2}).$$

Theorem 3.12 *For every* $g(y,\eta) \in \mathcal{R}_G^{\mu,d}(\Omega \times \mathbb{R}^q; \boldsymbol{w})$, $\boldsymbol{w} = (k, m; j_1, j_2)$, *the operator* $\mathcal{G} = \mathrm{Op}(g)$ *extends to a continuous operator*

$$\mathcal{G} : \begin{array}{c} H^s_{\mathrm{comp}(y)}(\Omega \times \mathbb{R}_+, \mathbb{C}^k) \\ \oplus \\ H^s_{\mathrm{comp}}(\Omega, \mathbb{C}^{j_1}) \end{array} \longrightarrow \begin{array}{c} H^{s-\mu}_{\mathrm{loc}(y)}(\Omega \times \mathbb{R}_+, \mathbb{C}^m) \\ \oplus \\ H^{s-\mu}_{\mathrm{loc}}(\Omega, \mathbb{C}^{j_2}) \end{array}$$

for every $s \in \mathbb{R}$, $s > d - 1/2$.

The symbolic structure of a Green operator $\mathcal{G} = \mathrm{Op}(g)$ allows us to define its boundary symbol

$$\sigma_\partial(\mathcal{G})(y,\eta) : H^s(\mathbb{R}_+, \mathbb{C}^k) \oplus \mathbb{C}^{j_1} \to \mathcal{S}(\overline{\mathbb{R}}_+, \mathbb{C}^m) \oplus \mathbb{C}^{j_2} \tag{37}$$

for $(y,\eta) \in T^*\Omega \setminus 0$, namely, as $\sigma_\partial(\mathcal{G})(y,\eta) = g_{(\mu)}(y,\eta)$, the homogeneous principal component of $g(y,\eta)$. Alternatively we also write

$$\sigma_\partial(\mathcal{G})(y,\eta) : H^s(\mathbb{R}_+, \mathbb{C}^k) \oplus \mathbb{C}^{j_1} \to H^{s-\mu}(\overline{\mathbb{R}}_+, \mathbb{C}^m) \oplus \mathbb{C}^{j_2}$$

or

$$\sigma_\partial(\mathcal{G})(y,\eta) : \mathcal{S}(\overline{\mathbb{R}}_+, \mathbb{C}^k) \oplus \mathbb{C}^{j_1} \to \mathcal{S}(\overline{\mathbb{R}}_+, \mathbb{C}^m) \oplus \mathbb{C}^{j_2}.$$

Now we pass to global Green operators on a smooth compact manifold X with boundary Y. Let $Y \times [0,1)$ be a collar neighbourhood of Y and $V' \subset Y$ a coordinate neighbourhood, $V := V' \times [0,1)$, and $\chi : V \to \Omega \times \overline{\mathbb{R}}_+$ a chart that restricts to a chart $\chi' : V \to \Omega$. For our vector bundles $E, F \in \mathrm{Vect}(X)$ and $J_{1,2} \in \mathrm{Vect}(Y)$ we have trivializations

$$E|_V \cong \Omega \times \overline{\mathbb{R}}_+ \times \mathbb{C}^k,\ F|_V \cong \Omega \times \overline{\mathbb{R}}_+ \times \mathbb{C}^m,\ \text{and } J_{1,2}|_{V'} \cong \Omega \times \mathbb{C}^{j_{1,2}}.$$

Green operators $\mathcal{G}$ can be interpreted as operators between sections in the corresponding bundles over V and V', respectively, namely,

$$\mathcal{G}_V : C_0^\infty(V, E|_V) \oplus C_0^\infty(V', J_1|_{V'}) \to C^\infty(V, F|_V) \oplus C^\infty(V', J_2|_{V'}).$$

Let us write $\mathcal{G}_V = (\chi^{-1})_*\mathrm{Op}(g)$, where the push forward $(\chi^{-1})_*$ is an abbreviation of $\mathrm{diag}(((\chi^{-1})_*, (\chi')^{-1})_*)$ that also includes the cocycles of transition maps of the involved bundles.

Let us fix a finite system of coordinate neighbourhoods $\{V_j\}_{j=1,\dots,L}$, $V_j = V_j' \times [0,1)$, for an open covering $\{V_j'\}_{j=1,\dots,L}$ of Y by coordinate neighbourhoods. Moreover, choose functions $\varphi_j \prec \psi_j$ in $C_0^\infty(V_j)$ and set $\varphi_j' := \varphi_j|_{V_j'}$, $\psi_j' := \psi_j|_{V_j'}$, and assume that $\{\varphi_j'\}_{j=1,\dots,L}$ is a partition of unity subordinate to $\{V_j'\}_{j=1,\dots,L}$.

Definition 3.13 The space of $\mathcal{B}_G^{\mu,d}(X; \boldsymbol{v})$, $\boldsymbol{v} = (E, F; J_1, J_2)$, of Green operators of order μ and type d is defined to be the set of all operators

$$\mathcal{G} := \sum_{j=0}^{L} \mathrm{diag}(\varphi_j, \varphi_j')(\chi_j^{-1})_*\mathrm{Op}(g_j)\mathrm{diag}(\psi_j, \psi_j') + \mathcal{C}$$

for arbitrary $g_j(y,\eta) \in \mathcal{R}_G^{\mu,d}(\Omega \times \overline{\mathbb{R}}_+; \boldsymbol{w})$, $\boldsymbol{w} = (k, m; j_1, j_2)$, $1 \le j \le L$, and $\mathcal{C} \in \mathcal{B}_G^{-\infty,d}(X; \boldsymbol{v})$. The space of upper left corners of elements in $\mathcal{B}_G^{\mu,d}(X; \boldsymbol{v})$ will also be denoted by $\mathcal{B}_G^{\mu,d}(X; E, F)$.

The families of maps (37) have an invariant meaning as bundle morphisms

$$\sigma_\partial(\mathcal{G}) : \pi_Y^* \begin{pmatrix} E' \otimes H^s(\mathbb{R}_+) \\ \oplus \\ J_1 \end{pmatrix} \longrightarrow \pi_Y^* \begin{pmatrix} F' \otimes H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ J_2 \end{pmatrix}, \tag{38}$$

$\pi_Y : T^*Y \setminus 0 \to Y$, $s > d - 1/2$ (alternatively, we may write $\mathcal{S}(\overline{\mathbb{R}}_+)$ instead of $H^{s-\mu}(\mathbb{R}_+)$ on the right of (38), or $\mathcal{S}(\overline{\mathbb{R}}_+)$ on both sides).

Let us now define the spaces of pseudo-differential BVPs for operators with the transmission property at the boundary, also referred to as Boutet de Monvel's calculus.

Definition 3.14 The space $\mathcal{B}^{\mu,d}(X;\boldsymbol{v})$, $\mu \in \mathbb{Z}$, $d \in \mathbb{N}$, $\boldsymbol{v} = (E, F; J_1, J_2)$, is defined to be the set of operators of the form

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{G} \tag{39}$$

for arbitrary $A \in L^{\mu}_{\mathrm{tr}}(X; E, F)$, cf. notation (26), and $\mathcal{G} \in \mathcal{B}^{\mu,d}_{G}(X;\boldsymbol{v})$, cf. Definition 3.13. The elements of $\mathcal{B}^{\mu,d}(X;\boldsymbol{v})$ are called pseudo-differential BVPs of order μ and type d. The space of upper left corners of elements in $\mathcal{B}^{\mu,d}(X;\boldsymbol{v})$ will also be denoted by $\mathcal{B}^{\mu,d}(X; E, F)$.

For $\mathcal{A} \in \mathcal{B}^{\mu,d}(X;\boldsymbol{v})$ we set

$$\sigma(\mathcal{A}) := (\sigma_{\psi}(\mathcal{A}), \sigma_{\partial}(\mathcal{A})),$$

where $\sigma_{\psi}(\mathcal{A}) := \sigma_{\psi}(A)$, cf. formula (29), called the (homogeneous principal) interior symbol of $\mathcal{A}$ of order μ, and

$$\sigma_{\partial}(\mathcal{A}) := \begin{pmatrix} \sigma_{\partial}(A) & 0 \\ 0 & 0 \end{pmatrix} + \sigma_{\partial}(\mathcal{G}),$$

called the (homogeneous principal) boundary symbol of $\mathcal{A}$ of order μ.

The homogeneity of $\sigma_{\psi}(\mathcal{A})$ is as usual, i.e., $\sigma_{\psi}(\mathcal{A})(x, \lambda\xi) = \lambda^{\mu}\sigma_{\psi}(\mathcal{A})(x, \xi)$ for all $\lambda \in \mathbb{R}_{+}$, $(x, \xi) \in T^{*}X \setminus 0$. For $\sigma_{\partial}(\mathcal{A})$ we have

$$\sigma_{\partial}(\mathcal{A})(y, \lambda\eta) = \lambda^{\mu}\mathrm{diag}(\kappa_{\lambda}, \mathrm{id})\sigma_{\partial}(\mathcal{A})(y, \eta)\mathrm{diag}(\kappa_{\lambda}^{-1}, \mathrm{id}) \tag{40}$$

for all $\lambda \in \mathbb{R}_{+}$, $(y, \eta) \in T^{*}Y \setminus 0$.

Remark 3.15

(i) We have

$$\mathcal{B}^{\mu-1,d}(X;\boldsymbol{v}) = \{\mathcal{A} \in \mathcal{B}^{\mu,d}(X;\boldsymbol{v}) : \sigma(\mathcal{A}) = 0\}.$$

Setting $\sigma(\mathcal{B}^{\mu,d}(X;\boldsymbol{v})) := \{\sigma(\mathcal{A}) : \mathcal{A} \in \mathcal{B}^{\mu,d}(X;\boldsymbol{v})\}$ there is an operator convention

$$\mathrm{op} : \sigma(\mathcal{B}^{\mu,d}(X;\boldsymbol{v})) \to \mathcal{B}^{\mu,d}(X;\boldsymbol{v})$$

in form of a linear operator (non-canonical) such that $\sigma \circ \mathrm{op} = \mathrm{id}_{\sigma(\mathcal{B}^{\mu,d}(X;\boldsymbol{v}))}$. The principal symbolic map

$$\sigma : \mathcal{B}^{\mu,d}(X;\boldsymbol{v}) \to \sigma(\mathcal{B}^{\mu,d}(X;\boldsymbol{v}))$$

gives rise to an exact sequence

$$0 \to \mathcal{B}^{\mu-1,d}(X;\boldsymbol{v}) \to \mathcal{B}^{\mu,d}(X;\boldsymbol{v}) \xrightarrow{\sigma} \sigma(\mathcal{B}^{\mu,d}(X;\boldsymbol{v})) \to 0.$$

(ii) If we replace in (i) $\mathcal{B}^{\mu,d}(X;\boldsymbol{v})$ by $\mathcal{B}_G^{\mu,d}(X;\boldsymbol{v})$ and σ by σ_∂ then we obtain an analogue of (i) for Green operators.

Let $\mathcal{B}^{\mu,d}(X;E,F)$ and $\mathcal{B}_G^{\mu,d}(X;E,F)$ denote the spaces of upper left corners of $\mathcal{B}^{\mu,d}(X;\boldsymbol{v})$ and $\mathcal{B}_G^{\mu,d}(X;\boldsymbol{v})$, respectively. By definition we have

$$\mathcal{B}^{\mu,d}(X;E,F) = L^{\mu}_{\mathrm{tr}}(X;E,F) + \mathcal{B}_G^{\mu,d}(X;E,F).$$

Observe that

$$\mathcal{B}_G^{\mu,d}(X;E,F) \subset L^{-\infty}(\mathrm{int}\,X;E,F)$$

and hence $\mathcal{B}^{\mu,d}(X;E,F) \subset L^{\mu}_{\mathrm{cl}}(\mathrm{int}\,X;E,F)$.

Remark 3.16 $A \in \mathcal{B}^{\mu,d}(X;E,F) \cap L^{-\infty}(\mathrm{int}\,X;E,F)$ is equivalent to

$$A \in \mathcal{B}_G^{\mu,d}(X;E,F).$$

Writing an $\mathcal{A} \in \mathcal{B}^{\mu,d}(X;\boldsymbol{v})$ in the form $\mathcal{A} = (\mathcal{A}_{ij})_{i,j=1,2}$, we also call $\mathcal{A}_{21}$ a trace and $\mathcal{A}_{12}$ a potential operator. For the lower right corner $\mathcal{A}_{22}$ we simply have $\mathcal{A}_{22} \in L^{\mu}_{\mathrm{cl}}(Y;J_1,J_2)$.

Proposition 3.17 *Every $G \in \mathcal{B}_G^{\mu,d}(X;E,F)$ has a unique representation*

$$G = G_0 + \sum_{j=0}^{d-1} K_j \circ T^j$$

for a $G_0 \in \mathcal{B}_G^{\mu,0}(X;E,F)$, potential operators $K_j \in \mathcal{B}^{\mu-j-1/2,0}(X;(0,F;E',0))$ and trace operators T^j of the same form as in (32).

Theorem 3.18 *An $\mathcal{A} \in \mathcal{B}^{\mu,d}(X;\boldsymbol{v})$ for $\boldsymbol{v} = (E,F;J_1,J_2)$ induces a continuous operator*

$$\mathcal{A}: \begin{array}{c} H^s(X,E) \\ \oplus \\ H^s(Y,J_1) \end{array} \longrightarrow \begin{array}{c} H^{s-\mu}(X,F) \\ \oplus \\ H^{s-\mu}(Y,J_2) \end{array} \tag{41}$$

for every $s > d - 1/2$.

Remark 3.19 Under the conditions of Theorem 3.18 the operator (41) is compact when $\mathcal{A} \in \mathcal{B}^{\mu-1,d}(X; \boldsymbol{v})$.

In fact, the compactness is a consequence of Theorem 3.18, applied for $\mu - 1$, and the compactness of embeddings

$$H^{s-(\mu+1)}(X, F) \oplus H^{s-(\mu+1)}(Y, J_2) \hookrightarrow H^{s-\mu}(X, F) \oplus H^{s-\mu}(Y, J_2)$$

Theorem 3.20 *Let $\mathcal{A}_j \in \mathcal{B}^{\mu-j,d}(X; \boldsymbol{v})$, $j \in \mathbb{N}$, be an arbitrary sequence. Then there exists an asymptotic sum $\mathcal{A} \sim \sum_{j=0}^{\infty} \mathcal{A}_j$ in $\mathcal{B}^{\mu,d}(X; \boldsymbol{v})$, i.e., an $\mathcal{A} \in \mathcal{B}^{\mu,d}(X; \boldsymbol{v})$ such that*

$$\mathcal{A} - \sum_{j=0}^{\infty} \mathcal{A}_j \in \mathcal{B}^{\mu-(N+1),d}(X; \boldsymbol{v})$$

for every $N \in \mathbb{N}$, and $\mathcal{A}$ is unique mod $\mathcal{B}^{-\infty,d}(X; \boldsymbol{v})$.

Theorem 3.21

(i) *Let $\mathcal{A} \in \mathcal{B}^{\mu,d}(X; \boldsymbol{v})$ for $\boldsymbol{v} = (E_0, F; J_0, J_2)$, $\mathcal{B} \in \mathcal{B}^{\nu,e}(X; \boldsymbol{w})$ for $\boldsymbol{w} = (E, E_0; J_1, J_0)$. Then $\mathcal{A}\mathcal{B} \in \mathcal{B}^{\mu+\nu,h}(X; \boldsymbol{v} \circ \boldsymbol{w})$ for $\boldsymbol{v} \circ \boldsymbol{w} := (E, F; J_1, J_2)$ and $h := \max(\nu + d, e)$, and we have*

$$\sigma(\mathcal{A}\mathcal{B}) = \sigma(\mathcal{A})\sigma(\mathcal{B})$$

(*with componentwise multiplication*). *If $\mathcal{A}$ or $\mathcal{B}$ is Green then so is $\mathcal{A}\mathcal{B}$.*

(ii) *Let $\mathcal{A} \in \mathcal{B}^{0,0}(X; \boldsymbol{v})$ for $\boldsymbol{v} = (E, F; J_1, J_2)$. Then for the adjoint* (*analogously defined as* (31)) *we have $\mathcal{A}^* \in \mathcal{B}^{0,0}(X; \boldsymbol{v}^*)$ for $\boldsymbol{v}^* = (F, E; J_2, J_1)$, and*

$$\sigma(\mathcal{A}^*) = \sigma(\mathcal{A})^*$$

(*with componentwise adjoint, cf. also formula* (33)).

We now define Shapiro-Lopatinskii-ellipticity (also referred to as SL-ellipticity) of boundary conditions for an operator in Boutet de Monvel's calculus on a smooth manifold X with boundary Y.

Definition 3.22 Let $\mathcal{A} \in \mathcal{B}^{\mu,d}(X; \boldsymbol{v})$, $\mu \in \mathbb{Z}$, $d \in \mathbb{N}$, $\boldsymbol{v} = (E, F; J_1, J_2)$ for E, $F \in \text{Vect}(X)$, $J_1, J_2 \in \text{Vect}(Y)$. The operator $\mathcal{A}$ is called elliptic if

(i) $\mathcal{A}$ is σ_ψ-elliptic, i.e.,

$$\sigma_\psi(\mathcal{A}) : \pi_X^* E \to \pi_X^* F \tag{42}$$

for $\pi_X : T^*X \setminus 0 \to X$ is an isomorphism.

(ii) $\mathcal{A}$ is σ_∂-elliptic, i.e.,

$$\sigma_\partial(\mathcal{A}) : \pi_Y^* \begin{pmatrix} E' \otimes H^s(\mathbb{R}_+) \\ \oplus \\ J_1 \end{pmatrix} \longrightarrow \pi_Y^* \begin{pmatrix} F' \otimes H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ J_2 \end{pmatrix} \tag{43}$$

for $\pi_Y : T^*Y \setminus 0 \to Y$ is an isomorphism for some $s > \max\{\mu, d\} - 1/2$.

Condition (ii) for some $s = s_0 > \max\{\mu, d\} - 1/2$ is equivalent to this property for all $s > \max\{\mu, d\} - 1/2$. This in turn is equivalent to the bijectivity of

$$\sigma_\partial(\mathcal{A}) : \pi_Y^* \begin{pmatrix} E' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ J_1 \end{pmatrix} \longrightarrow \pi_Y^* \begin{pmatrix} F' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ J_2 \end{pmatrix}. \tag{44}$$

Definition 3.23 Let $\mathcal{A} \in \mathcal{B}^{\mu,d}(X; \boldsymbol{v})$, $\boldsymbol{v} = (E, F; J_1, J_2)$; then a $\mathcal{P} \in \mathcal{B}^{-\mu,e}(X; \boldsymbol{v}^{-1})$ for $\boldsymbol{v}^{-1} = (F, E; J_2, J_1)$ and some $e \in \mathbb{N}$ is called a parametrix of $\mathcal{A}$, if

$$\mathcal{C}_{\mathrm{L}} := \mathcal{I} - \mathcal{P}\mathcal{A} \in \mathcal{B}^{-\infty,d_{\mathrm{L}}}(X; \boldsymbol{v}_{\mathrm{L}}), \;\; \mathcal{C}_{\mathrm{R}} := \mathcal{I} - \mathcal{A}\mathcal{P} \in \mathcal{B}^{-\infty,d_{\mathrm{R}}}(X; \boldsymbol{v}_{\mathrm{R}})$$

for certain $d_{\mathrm{L}}, d_{\mathrm{R}} \in \mathbb{N}$ and $\boldsymbol{v}_{\mathrm{L}} := (E, E; J_1, J_1)$, $\boldsymbol{v}_{\mathrm{R}} := (F, F; J_2, J_2)$, where $\mathcal{I}$ denotes the respective identity operators.

Theorem 3.24 *Let $\mathcal{A} \in \mathcal{B}^{\mu,d}(X; \boldsymbol{v})$, $\boldsymbol{v} = (E, F; J_1, J_2)$, be elliptic. Then there is a parametrix $\mathcal{B} \in \mathcal{B}^{-\mu,(d-\mu)^+}(X; \boldsymbol{v}^{-1})$ for $\boldsymbol{v}^{-1} := (F, E; J_2, J_1)$, where*

$$\sigma(\mathcal{B}) = \sigma(\mathcal{A})^{-1}$$

with componentwise inverses, and $\mathcal{C}_{\mathrm{L}} := 1 - \mathcal{B}\mathcal{A} \in \mathcal{B}^{-\infty,d_{\mathrm{L}}}(X, \boldsymbol{v}_{\mathrm{L}})$, $\mathcal{C}_{\mathrm{R}} := 1 - \mathcal{A}\mathcal{B} \in \mathcal{B}^{-\infty,d_{\mathrm{R}}}(X, \boldsymbol{v}_{\mathrm{R}})$ for $\boldsymbol{v}_{\mathrm{L}} := (E, E; J_1, J_1)$, $\boldsymbol{v}_{\mathrm{R}} := (F, F; J_2, J_2)$, $d_{\mathrm{L}} = \max\{\mu, d\}$, $d_{\mathrm{R}} = (d - \mu)^+$ (where $\nu^+ := \max\{\nu, 0\}$). More precisely, $\mathcal{B}$ may be found in such a way that

$$\mathcal{C}_{\mathrm{L}} : H^s(X, E) \oplus H^s(Y, J_1) \to \mathcal{V}, \;\; \mathcal{C}_{\mathrm{R}} : H^{s-\mu}(X, F) \oplus H^{s-\mu}(Y, J_2) \to \mathcal{W}$$

are projections for $s > \max\{\mu, d\} - 1/2$.

Theorem 3.25 *Let X be compact. For an operator $\mathcal{A} \in \mathcal{B}^{\mu,d}(X; \boldsymbol{v})$, where*

$$\boldsymbol{v} = (E, F; J_1, J_2)$$

the following conditions are equivalent:

(i) $\mathcal{A}$ *is elliptic.*
(ii) *The operator*

$$\mathcal{A} : H^s(X, E) \oplus H^s(Y, J_1) \to H^{s-\mu}(X, F) \oplus H^{s-\mu}(Y, J_2) \tag{45}$$

is Fredholm for some $s = s_0 > \max\{\mu, d\} - 1/2$.

Remark 3.26 Let X be compact and $\mathcal{A} \in \mathcal{B}^{\mu,d}(X; \boldsymbol{v})$ elliptic.

(i) The operator $\mathcal{A}$ is Fredholm for all $s > \max\{\mu, d\} - 1/2$.
(ii) $\mathcal{V} := \ker_s \mathcal{A} = \{u \in H^s(X, E) \oplus H^s(Y, J_1) : \mathcal{A}u = 0\}$ is a finite-dimensional subspace $H^s(X, E) \oplus H^s(Y, J_1) \subset C^\infty(X, E) \oplus C^\infty(Y, J_1)$ independent of s, and there is a finite-dimensional $\mathcal{W} \subset C^\infty(X, F) \oplus C^\infty(Y, J_2)$ independent of s such that $\mathrm{im}_s\, \mathcal{A} + \mathcal{W} = H^{s-\mu}(X, F) \oplus H^{s-\mu}(Y, J_2)$ for every s; here $\mathrm{im}_s\, \mathcal{A} = \{\mathcal{A}u : u \in H^s(X, E) \oplus H^s(Y, J_1)\}$.

Theorem 3.27 *Let* X *be compact, and assume that* $\mathcal{A} \in \mathcal{B}^{\mu,d}(X; \boldsymbol{v})$ *induces an isomorphism* (45) *for some* $s = s_0 > \max\{\mu, d\} - 1/2$. *Then* (45) *is an isomorphism for all* $s > \max\{\mu, d\} - 1/2$, *and we have* $\mathcal{A}^{-1} \in \mathcal{B}^{-\mu,(\mu-d)^+}(X; \boldsymbol{v}^{-1})$.

4 The Atiyah-Bott Obstruction and Boundary Symbols

Let us now discuss the question to what extent a σ_ψ-elliptic operator on a smooth (compact) manifold with boundary admits Shapiro-Lopatinskij elliptic boundary conditions cf. also [15].

Theorem 4.1 *Let* $A \in \mathcal{B}^{\mu,d}(X; E, F)$ *be* σ_ψ*- elliptic, cf. Definition 3.22. Then the boundary symbol*

$$\sigma_\partial(A) : \pi_Y^*(E' \otimes H^s(\mathbb{R}_+)) \to \pi_Y^*(F' \otimes H^{s-\mu}(\mathbb{R}_+)) \tag{46}$$

represents a family of Fredholm operators for every

$$s > \max\{\mu, d\} - 1/2,$$

parametrized by $(y, \eta) \in T^*Y \setminus 0$, *and* $\ker \sigma_\partial(A)(y, \eta)$, $\mathrm{coker}\, \sigma_\partial(A)(y, \eta)$ *are independent of s.*

By virtue of (40) we have

$$\sigma_\partial(A)(y, \lambda\eta) = \lambda^\mu \kappa_\lambda \sigma_\partial(A)(y, \eta) \kappa_\lambda^{-1} \tag{47}$$

for all $\lambda \in \mathbb{R}_+$. It follows that

$$\sigma_\partial(A)\left(y, \frac{\eta}{|\eta|}\right) : E'_y \otimes H^s(\mathbb{R}_+) \to F'_y \otimes H^{s-\mu}(\mathbb{R}_+)$$

is a family of Fredholm operators parametrised by $(y, \eta) \in S^*Y$, the unit cosphere bundle induced by $T^*Y \setminus 0$ (referring to the fixed Riemannian metric). This corresponds to a standard situation of K-theory, discussed in abstract terms in [11, Subsection 3.3.4]. The space S^*Y is compact, and hence there is a K-theoretic index element

$$\mathrm{ind}_{S^*Y}\sigma_\partial(A) \in K(S^*Y). \tag{48}$$

Let

$$\pi : S^*Y \to Y \tag{49}$$

denote the canonical projection. Then the bundle pull back induces a homomorphism

$$\pi^* : K(Y) \to K(S^*Y).$$

The condition

$$\mathrm{ind}_{S^*Y}\sigma_\partial(A) \in \pi^*K(Y), \tag{50}$$

is referred to as the Atiyah-Bott obstruction.

The following result was first established by Atiyah and Bott in [1] in the case of differential operators, later on formulated for pseudo-differential operators with the transmission property at the boundary by Boutet de Monvel [5], and an edge analogue may be found in [21].

Let X be a smooth manifold with compact boundary Y, and $E, F \in \mathrm{Vect}(Y)$.

Theorem 4.2 *A σ_ψ-elliptic operator A in $\mathcal{B}^{\mu,d}(X; E, F)$ has a Shapiro-Lopatinskij elliptic BVP $\mathcal{A} \in \mathcal{B}^{\mu,d}(X; \boldsymbol{v})$, $\boldsymbol{v} := (E, F; J_1, J_2)$, for certain $J_1, J_2 \in$* Vect(Y) (*i.e., the upper left corner of $\mathcal{A}$ is of the form $A + G$, for some $G \in \mathcal{B}^{\mu,d}_G(X; E, F)$ and $\sigma_\partial(\mathcal{A})$ is an isomorphism*) *if and only if* (50) *holds.*

Notation refers to [13]. The proof is based on the following auxiliary considerations.

Remark 4.3 Let $A \in \mathcal{B}^{\mu,d}(X; E, F)$ be σ_ψ-elliptic, and consider the family of Fredholm operators $\sigma_\partial(A)(y, \eta) : E'_y \otimes H^s(\mathbb{R}_+) \to F'_y \otimes H^{s-\mu}(\mathbb{R}_+)$, $s > \max\{\mu, d\} - 1/2$. Then there exists a subbundle $\tilde{W} \subset \pi^*F' \otimes \mathcal{S}(\overline{\mathbb{R}}_+)$ of finite fibre dimension such that

$$\tilde{W}_{y,\eta} + \mathrm{im}\, \sigma_\partial(A)(y, \eta) = F'_y \otimes H^{s-\mu}(\mathbb{R}_+)$$

for all $(y, \eta) \in S^*Y$.

Proposition 4.4 *Let $A \in \mathcal{B}^{\mu,d}(X; E, F)$ be σ_ψ-elliptic and let $L_1, L_2 \in \mathrm{Vect}(S^*Y)$ be such that*

$$\mathrm{ind}_{S^*Y}\sigma_\partial(A) = [L_2] - [L_1]. \tag{51}$$

Then there exists an element $G \in \mathcal{B}_G^{\mu,0}(X; E, F)$ such that

$$\ker_{S^*Y}\sigma_\partial(A+G) \cong L_2, \quad \mathrm{coker}_{S^*Y}\sigma_\partial(A+G) \cong L_1. \tag{52}$$

Proposition 4.5 *Let $A \in \mathcal{B}^{\mu,d}(X; E, F)$ be a σ_ψ-elliptic operator. Then there exist vector bundles $J_1, J_2 \in \mathrm{Vect}(Y)$, $L_1, L_2 \in \mathrm{Vect}(T^*Y \setminus 0)$, where L_i is a subbundle of $\pi_Y^* J_i$, $i = 1, 2$, and an operator*

$$\mathcal{A} = \begin{pmatrix} A+G & K \\ T & 0 \end{pmatrix} \tag{53}$$

in $\mathcal{B}^{\mu,d}(X; \boldsymbol{v})$ for $\boldsymbol{v} := (E, F; J_1, J_2)$ such that $\sigma_\partial(\mathcal{A})$ restricts to an isomorphism

$$\begin{matrix} \pi_Y^* E' \otimes H^s(\mathbb{R}_+) & & \pi_Y^* F' \otimes H^{s-\mu}(\mathbb{R}_+) \\ \oplus & \longrightarrow & \oplus \\ L_1 & & L_2 \end{matrix} \tag{54}$$

for every $s > \max\{\mu, d\} - 1/2$.

5 Boundary Problems with Global Projection Conditions

Definition 5.1 Let $\mathbb{L}_i = (P_i, J_i, L_i) \in \mathbb{P}(Y)$ be projection data (cf. Definition 2.4), $V_i \in \mathrm{Vect}(X)$, $J_i \in \mathrm{Vect}(Y)$, $i = 1, 2$, and set

$$\begin{aligned} \boldsymbol{v} &:= (V_1, V_2; J_1, J_2), \ \boldsymbol{l} := (V_1, V_2; \mathbb{L}_1, \mathbb{L}_2), \\ \mathcal{P}_2 &:= \mathrm{diag}(1, P_2), \ \mathcal{E}_1 := \mathrm{diag}(1, E_1). \end{aligned} \tag{55}$$

Then $\mathcal{T}^{\mu,d}(X; \boldsymbol{l})$ for $\mu \in \mathbb{Z}$, $d \in \mathbb{N}$, is defined to be the set of all operators

$$\mathcal{A} := \mathcal{P}_2 \tilde{\mathcal{A}} \mathcal{E}_1 \tag{56}$$

for arbitrary $\tilde{\mathcal{A}} \in \mathcal{B}^{\mu,d}(X; \boldsymbol{v})$. The elements of $\mathcal{T}^{\mu,d}(X; \boldsymbol{l})$ will be called boundary value problems of order μ and type d with global projection (boundary) conditions.

Moreover, set

$$\mathcal{T}^{-\infty,d}(X;\boldsymbol{l}) := \{\mathcal{P}_2\tilde{\mathcal{C}}\mathcal{E}_1 : \tilde{\mathcal{C}} \in \mathcal{B}^{-\infty,d}(X;\boldsymbol{v})\}, \tag{57}$$

and $\mathcal{T}^{\infty,d}(X;\boldsymbol{l}) = \bigcup_{\mu\in\mathbb{Z}} \mathcal{T}^{\mu,d}(X;\boldsymbol{l})$.

Remark 5.2 The system of spaces $\mathcal{T}^{\mu,d}(X;\boldsymbol{l})$ represents an extension of $\mathcal{B}^{\mu,d}(X;\boldsymbol{v})$.

In fact, the special case of operators in Boutet de Monvel's calculus is attained by the case $J_i = L_i$ and $P_i = \mathrm{id}$ the identity operators in $L^0_{\mathrm{cl}}(Y;J_i)$, $i = 1,2$. The spaces $\mathcal{T}^{\mu,d}(X;\boldsymbol{l})$ may be regarded a Toeplitz analogue of Boutet de Monvel's calculus, but they are unifying both concepts. This point of view has been first introduced in [27]. Similarly as in connection with Definition 2.12 an operator (56) first represents a continuous operator

$$\mathcal{A} : \begin{array}{c} H^\infty(X,V_1) \\ \oplus \\ H^\infty(Y,\mathbb{L}_1) \end{array} \to \begin{array}{c} H^\infty(X,V_2) \\ \oplus \\ H^\infty(Y,\mathbb{L}_2) \end{array},$$

using the respective continuity of operators in $\mathcal{B}^{\mu,d}(X;\boldsymbol{v})$ and of the involved embedding and projection operators.

Observe that the space (57) can be equivalently characterised as the set of all $\mathcal{A} \in \mathcal{T}^{\infty,d}(X;\boldsymbol{l})$, $\mathcal{A} = \mathcal{P}_2\tilde{\mathcal{A}}\mathcal{E}_1$ for some $\tilde{\mathcal{A}} \in \mathcal{B}^{\mu,d}(X;\boldsymbol{v})$ such that $\mathcal{P}_2\tilde{\mathcal{A}}\mathcal{P}_1 \in \mathcal{B}^{-\infty,d}(X;\boldsymbol{v})$; then $\mathcal{A} = \mathcal{P}_2(\mathcal{P}_2\tilde{\mathcal{A}}\mathcal{P}_1)\mathcal{E}_1$. Moreover,

$$\mathcal{P}_2(\mathcal{P}_2\tilde{\mathcal{A}}\mathcal{P}_1)\mathcal{E}_1 \in \mathcal{T}^{-\infty,d}(X;\boldsymbol{l}) \Rightarrow \mathcal{P}_2\tilde{\mathcal{A}}\mathcal{P}_1 \in \mathcal{B}^{-\infty,d}(X;\boldsymbol{v}).$$

Proposition 5.3 *Given $V_i \in \mathrm{Vect}(X)$, $\mathbb{L}_i \in \mathbb{P}(Y)$, $i = 1,2$, we have a canonical isomorphism*

$$\mathcal{T}^{\mu,d}(X;\boldsymbol{l}) \to \{\mathcal{P}_2\tilde{A}\mathcal{P}_1 : \tilde{\mathcal{A}} \in \mathcal{B}^{\mu,d}(X;\boldsymbol{v})\}.$$

Remark 5.4 We have an identification

$$\mathcal{T}^{\mu,d}(X;\boldsymbol{l}) = \mathcal{B}^{\mu,d}(X;\boldsymbol{v})/\sim .$$

Proposition 5.5 *Every $\mathcal{A} \in \mathcal{T}^{\mu,d}(X;\boldsymbol{l})$ induces continuous operators*

$$\mathcal{A} : \begin{array}{c} H^s(X,V_1) \\ \oplus \\ H^s(Y,\mathbb{L}_1) \end{array} \to \begin{array}{c} H^{s-\mu}(X,V_2) \\ \oplus \\ H^{s-\mu}(Y,\mathbb{L}_2) \end{array}, \tag{58}$$

for every $s \in \mathbb{R}$, $s > d - \frac{1}{2}$.

Let us now introduce the principal symbolic structure of $\mathcal{T}^{\mu,d}(X;\boldsymbol{l})$. In the 2×2 block matrix structure $\mathcal{A} = (A_{ij})_{i,j=1,2}$ of our operators we have $A_{11} \in \mathcal{B}^{\mu,d}(X;V_1,V_2)$ and the (homogeneous principal) symbol

$$\sigma_\psi(\mathcal{A}) := \sigma_\psi(A_{11}) : \pi_X^* V_1 \to \pi_X^* V_2$$

as in (29). Occasionally we also call $\sigma_\psi(\mathcal{A})$ the interior symbol of $\mathcal{A}$. Moreover, the family of operators

$$\sigma_\partial(\mathcal{A}) := \begin{pmatrix} 1 & 0 \\ 0 & p_2 \end{pmatrix} \sigma_\partial(\tilde{\mathcal{A}}) \begin{pmatrix} 1 & 0 \\ 0 & e_1 \end{pmatrix} : \begin{matrix} \pi_Y^* V_1' \otimes H^s(\mathbb{R}_+) \\ \oplus \\ L_1 \end{matrix} \to \begin{matrix} \pi_Y^* V_2' \otimes H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ L_2 \end{matrix}$$

is called the (homogeneous principal) boundary symbol of $\mathcal{A}$ (recall that $p_2(y,\eta)$ is the homogeneous principal symbol of order zero of the projection $P_2 \in L^0_{\mathrm{cl}}(Y;J_2,J_2)$ while $e_1 : L_1 \to \pi_Y^* J_1$ is the canonical embedding). Similarly as in $\mathcal{B}^{\mu,d}$ we set

$$\sigma(\mathcal{A}) := (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A})).$$

Remark 5.6 Identifying an operator $\mathcal{A} = \mathcal{P}_2\tilde{\mathcal{A}}\mathcal{E}_1 \in \mathcal{T}^{\mu,d}(X;\boldsymbol{l})$ with $\tilde{\tilde{\mathcal{A}}} := \mathcal{P}_2\tilde{\mathcal{A}}\mathcal{P}_1 \in \mathcal{B}^{\mu,d}(X;\boldsymbol{v})$ (cf. Proposition 5.3), relation $\sigma(\tilde{\tilde{\mathcal{A}}}) = 0$ in the sense of $\mathcal{B}^{\mu,d}$ is equivalent to $\sigma(\mathcal{A}) = 0$ in the sense of $\mathcal{T}^{\mu,d}$.

Theorem 5.7

(i) *$\mathcal{A} \in \mathcal{T}^{\mu,d}(X;\boldsymbol{l})$ and $\sigma(\mathcal{A}) = 0$ imply $\mathcal{A} \in \mathcal{T}^{\mu-1,d}(X;\boldsymbol{l})$ and the operator* (58) *is compact for every $s > d - 1/2$.*

(ii) *$\mathcal{A} \in \mathcal{T}^{\mu,d}(X;\boldsymbol{l}_0)$, $\mathcal{B} \in \mathcal{T}^{\nu,e}(X;\boldsymbol{l}_1)$ implies $\mathcal{A}\mathcal{B} \in \mathcal{T}^{\mu+\nu,h}(X;\boldsymbol{l}_0 \circ \boldsymbol{l}_1)$ (when the bundle and projection data in the middle fit together such that $\boldsymbol{l}_0 \circ \boldsymbol{l}_1$ makes sense), $h = \max\{\nu + d, e\}$, and we have $\sigma(\mathcal{A}\mathcal{B}) = \sigma(\mathcal{A})\sigma(\mathcal{B})$ with componentwise multiplication.*

(iii) *$\mathcal{A} \in \mathcal{T}^{0,0}(X;\boldsymbol{l})$ for $\boldsymbol{l} = (V_1, V_2; \mathbb{L}_1, \mathbb{L}_2)$ implies $\mathcal{A}^* \in \mathcal{T}^{0,0}(X;\boldsymbol{l}^*)$ for $\boldsymbol{l}^* = (V_2, V_1; \mathbb{L}_2^*, \mathbb{L}_1^*)$, with $\mathcal{A}^*$ being defined by*

$$(u, \mathcal{A}^* v)_{L^2(X,V_1)\oplus H^0(Y,\mathbb{L}_1)} = (\mathcal{A}u, v)_{L^2(X,V_2)\oplus H^0(Y,\mathbb{L}_2)}$$

for all $u \in L^2(X,V_1) \oplus H^0(Y,\mathbb{L}_1)$, $v \in L^2(X,V_2) \oplus H^0(Y,\mathbb{L}_2)$, and we have $\sigma(\mathcal{A}^*) = \sigma(\mathcal{A})^*$ with componentwise adjoint (cf. Theorem 3.21 and Theorem 2.22).

Theorem 5.8 *Let $\mathcal{A}_j \in \mathcal{T}^{\mu-j,d}(X;\boldsymbol{l})$, $j \in \mathbb{N}$, be an arbitrary sequence. Then there exists an $\mathcal{A} \in \mathcal{T}^{\mu,d}(X;\boldsymbol{l})$ such that*

$$\mathcal{A} - \sum_{j=0}^{N} \mathcal{A}_j \in \mathcal{T}^{\mu-(N+1),d}(X;\boldsymbol{l})$$

for every $N \in \mathbb{N}$, and $\mathcal{A}$ is unique mod $\mathcal{T}^{-\infty,d}(X;\boldsymbol{l})$.

Similarly as in the Toeplitz operator calculus on a closed manifold it will be useful to form direct sums. Given $\mathcal{A} \in \mathcal{T}^{\mu,d}(X;\boldsymbol{l})$, $\mathcal{B} \in \mathcal{T}^{\mu,d}(X;\boldsymbol{m})$ for bundle and projection data

$$\boldsymbol{l} := (V_1, V_2; \mathbb{L}_1, \mathbb{L}_2),\ \mathbb{L}_i = (P_i, J_i, L_i) \in \mathbb{P}(Y), i = 1, 2,$$
$$\boldsymbol{m} := (W_1, W_2; \mathbb{M}_1, \mathbb{M}_2),\ \mathbb{M}_i = (Q_i, G_i, M_i) \in \mathbb{P}(Y), i = 1, 2,$$

we set

$$\boldsymbol{l} \oplus \boldsymbol{m} := (V_1 \oplus W_1, V_2 \oplus W_2; \mathbb{L}_1 \oplus \mathbb{M}_1, \mathbb{L}_2 \oplus \mathbb{M}_2).$$

Then for the direct sum of operators we obtain

$$\mathcal{A} \oplus \mathcal{B} \in \mathcal{T}^{\mu,d}(X;\boldsymbol{l} \oplus \boldsymbol{m})$$

and

$$\sigma_\psi(\mathcal{A} \oplus \mathcal{B}) = \sigma_\psi(\mathcal{A}) \oplus \sigma_\psi(\mathcal{B}), \sigma_\partial(\mathcal{A} \oplus \mathcal{B}) = \sigma_\partial(\mathcal{A}) \oplus \sigma_\partial(\mathcal{B}),$$

with obvious meaning of notation.

Let us study ellipticity in our Toeplitz calculus of boundary value problems.

Definition 5.9 An $\mathcal{A} \in \mathcal{T}^{\mu,d}(X;\boldsymbol{l})$ for $\boldsymbol{l} := (V_1, V_2; \mathbb{L}_1, \mathbb{L}_2)$, $\mathbb{L}_i = (P_i, J_i, L_i) \in \mathbb{P}(Y), i = 1, 2$, is said to be elliptic if

(i) the interior symbol

$$\sigma_\psi(\mathcal{A}) : \pi_X^* V_1 \to \pi_X^* V_2 \tag{59}$$

is an isomorphism;

(ii) the boundary symbol

$$\sigma_\partial(\mathcal{A}) : \begin{array}{c} \pi_Y^* V_1' \otimes H^s(\mathbb{R}_+) \\ \oplus \\ L_1 \end{array} \to \begin{array}{c} \pi_Y^* V_2' \otimes H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ L_2 \end{array} \tag{60}$$

is an isomorphism for every $s > \max\{\mu, d\} - 1/2$.

Remark 5.10 Similarly as in the $\mathcal{B}^{\mu,d}$-case the condition (ii) in Definition 5.9 is equivalent to the bijectivity of

$$\sigma_\partial(\mathcal{A}) : \begin{array}{c} \pi_Y^* V_1' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ L_1 \end{array} \to \begin{array}{c} \pi_Y^* V_2' \otimes \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ L_2 \end{array} . \tag{61}$$

Theorem 5.11 *For every $A \in L^\mu_{\mathrm{tr}}(X; V_1, V_2)$ elliptic with respect to σ_ψ (i.e., (59) is an isomorphism) there exist projection data $\mathbb{L}_1, \mathbb{L}_2 \in \mathbb{P}(Y)$ and an element $\mathcal{A} \in \mathcal{T}^{\mu,0}(X; \boldsymbol{l})$ with A as the upper left corner, $\boldsymbol{l} := (V_1, V_2; \mathbb{L}_1, \mathbb{L}_2)$, which is elliptic in the sense of* Definition 5.9.

In fact, we can choose elements $L_1, L_2 \in \mathrm{Vect}(S^*Y)$ such that

$$\mathrm{ind}_{S^*Y}\sigma_\partial(A) = [L_2] - [L_1]$$

holds in $K(S^*Y)$. Then for the operator $G \in \mathcal{B}^{\mu,d}_G(X; V_1, V_2)$ of Proposition 4.4 applied in the proof of Proposition 4.5 we obtain an operator (53) that we now denote by $\tilde{\mathcal{A}}$, and it suffices to set $\mathcal{A} := \mathcal{P}_2 \tilde{\mathcal{A}} \mathcal{E}_1$.

Theorem 5.12 *For every $\mu \in \mathbb{Z}, V \in \mathrm{Vect}(X)$ there exists an elliptic $R^\mu_V \in \mathcal{B}^{\mu,0}(X; V, V)$ which induces isomorphisms*

$$R^\mu_V(\lambda) : H^s(X, V) \to H^{s-\mu}(X, V)$$

for all $\lambda \in \mathbb{R}^l$, $s \in \mathbb{R}$.

Proposition 5.13 *For every $\mu \in \mathbb{Z}$, $V \in \mathrm{Vect}(X)$ and $\mathbb{L} \in \mathbb{P}(Y)$ there exists an elliptic element $\mathcal{R}^\mu_{V,\mathbb{L}} \in \mathcal{T}^{\mu,0}(X; \boldsymbol{l})$ for $\boldsymbol{l} := (V, V; \mathbb{L}, \mathbb{L})$ which induces a Fredholm operator*

$$\mathcal{R}^\mu_{V,\mathbb{L}} : \begin{array}{c} H^s(X, V) \\ \oplus \\ H^s(Y, \mathbb{L}) \end{array} \to \begin{array}{c} H^{s-\mu}(X, V) \\ \oplus \\ H^{s-\mu}(Y, \mathbb{L}) \end{array}$$

for every $s > \max\{\mu, 0\} - 1/2$.

Remark 5.14 By virtue of ellipticity of $R^\mu_V \in \mathcal{B}^{\mu,0}(X; V, V)$ the boundary symbol

$$\sigma_\partial(R^\mu_V) : \pi_Y^* V' \otimes H^s(\mathbb{R}_+) \to \pi_Y^* V' \otimes H^{s-\mu}(\mathbb{R}_+)$$

is an isomorphism for $s > \max\{\mu, 0\} - 1/2$. Thus $\mathrm{ind}_{S^*Y}\sigma_\partial(R^\mu_V) = 0$.

Similarly as R^μ_V we can form an operator $S^\mu_V \in \mathcal{B}^{\mu,0}(X; V, V)$ the local symbol of which close to Y is equal to $\mathrm{r}^\mu_+(\eta, \tau) := \overline{\mathrm{r}^\mu_-(\eta, \tau)}$ (the complex conjugate). This operator can chosen in such a way that

$$\sigma_\partial(S_V^\mu) : \pi_Y^* V' \otimes H^s(\mathbb{R}_+) \to \pi_Y^* V' \otimes H^{s-\mu}(\mathbb{R}_+)$$

is surjective and $\ker_{S^*Y}\sigma_\partial(S_V^\mu) = \mu[\pi_1^* V']$; thus

$$\mathrm{ind}_{S^*Y}\sigma_\partial(S_V^\mu) = \mu[\pi^* V'],\ \pi : S^*Y \to Y. \tag{62}$$

Theorem 5.15 *For every elliptic operator* $\mathcal{A} \in \mathcal{T}^{\mu,d}(X;\boldsymbol{l})$*,* $\boldsymbol{l} := (V_1, V_2; \mathbb{L}_1, \mathbb{L}_2)$*, there exists an elliptic operator* $\mathcal{B} \in \mathcal{T}^{\mu,d}(X;\boldsymbol{m})$*,* $\boldsymbol{m} := (V_2, V_1; \mathbb{M}_1, \mathbb{M}_2)$*, for certain projection data* $\mathbb{M}_1, \mathbb{M}_2 \in \mathbb{P}(Y)$ *of the form* $\mathbb{M}_i := (Q_i, \mathbb{C}^N, M_i)$*,* $i = 1, 2$*, for some* $N \in \mathbb{N}$*, such that* $\mathcal{A} \oplus \mathcal{B} \in \mathcal{B}^{\mu,d}(X; \boldsymbol{v})$ *for* $\boldsymbol{v} = (V_1 \oplus V_2, V_2 \oplus V_1; \mathbb{C}^N, \mathbb{C}^N)$*, is Shapiro-Lopatinskii elliptic.*

In fact, let A denote the upper left corner of $\mathcal{A}$ which belongs to $\mathcal{B}^{\mu,d}(X; V_1, V_2)$ and is σ_ψ-elliptic, cf. Definition 3.22 (i). By assumption the boundary symbol

$$\sigma_\partial(A) : \pi^* V_1' \otimes H^s(\mathbb{R}_+) \to \pi^* V_2' \otimes H^{s-\mu}(\mathbb{R}_+),$$

$\pi : S^*Y \to Y$, represents a family of Fredholm operators on S^*Y for every $s > \max\{\mu, d\} - 1/2$. The specific choice of s is not essential at this moment, but in connection with reductions of orders below we assume $s \in \mathbb{N}$ sufficiently large. Choose any $B \in \mathcal{B}^{\mu,d}(X; V_2, V_1)$ with the property

$$\mathrm{ind}_{S^*Y}\sigma_\partial(B) = -\mathrm{ind}_{S^*Y}\sigma_\partial(A).$$

A way to find such a B is as follows. First consider the case $\mu = d = 0$. Then we can set $B := A^*$, cf. Theorem 3.21 (ii). In fact, we have $A^*A \in \mathcal{B}^{0,0}(X; V_1, V_1)$, and $\sigma_\partial(A^*A) = \sigma_\partial(A^*)\sigma_\partial(A)$, cf. Theorem 3.21 (i). From self-adjointness it follws that

$$\mathrm{ind}_{S^*Y}\sigma_\partial(A^*A) = 0 = \mathrm{ind}_{S^*Y}\sigma_\partial(A^*) + \mathrm{ind}_{S^*Y}\sigma_\partial(A).$$

Now for arbitrary μ, d we write $A = A_\mu + G$ for $A_\mu \in \mathcal{B}^{\mu,0}(X; V_1, V_2)$, $G \in \mathcal{B}_G^{\mu,g}(X; V_1, V_2)$. We realise A_μ as a continuous operator

$$A_\mu : H^s(X, V_1) \to H^{s-\mu}(X, V_1)$$

for some fixed $s \in \mathbb{N}$ sufficiently large. Since $\sigma_\partial(G)$ is a family of compact operators, we have $\mathrm{ind}_{S^*Y}\sigma_\partial(A) = \mathrm{ind}_{S^*Y}\sigma_\partial(A_\mu)$. Thus we may ignore d, i.e., assume $d = 0$. Form the operator

$$A_0 := R_{V_2}^{s-\mu} A_\mu R_{V_1}^{-s} : L^2(X, V_1) \to L^2(X, V_2)$$

for order reducing operators $R_{V_1}^{-s}, R_{V_2}^{s-\mu}$, obtained from Theorem 5.12 for the corresponding choice of V. We have $R_{V_1}^{-s} \in \mathcal{B}^{-s,0}(X; V_1, V_1), R_{V_2}^{s-\mu} \in \mathcal{B}^{s-\mu,0}(X; V_2, V_2)$, and isomorphisms

$$R_{V_1}^{-s} : L^2(X, V_1) \to H^s(X, V_1), \;\; R_{V_2}^{s-\mu} : H^{s-\mu}(X, V_2) \to L^2(X, V_2).$$

Then $A_0 \in \mathcal{B}^{0,0}(X; V_1, V_2)$ and

$$\mathrm{ind}_{S^*Y}\sigma_\partial(A_0) = \mathrm{ind}_{S^*Y}\sigma_\partial(A) = [L_2] - [L_1].$$

For the L^2-adjoint $A_0^* \in \mathcal{B}^{0,0}(X; V_2, V_1)$ it follows that

$$\mathrm{ind}_{S^*Y}\sigma_\partial(A_0^*) = [L_1] - [L_2].$$

and $\mathrm{ind}_{S^*Y}\sigma_\partial(B_1) = [L_1] - [L_2]$ for $B_1 := R_{V_1}^{-s+\mu} A_0^* R_{V_2}^s \in \mathcal{B}^{\mu,s}(X; V_2, V_1)$, cf. Theorem 3.21 (i). The operator B_1 can be written as $B_1 = B + G$ for a $B \in \mathcal{B}^{\mu,0}(X; V_2, V_1)$ and a $G \in \mathcal{B}_G^{\mu,s}(X; V_2, V_1)$. Then

$$\mathrm{ind}_{S^*Y}\sigma_\partial(B) = [L_1] - [L_2],$$

since $\sigma_\partial(G_1)$ takes values in compact operators. There are bundles $M_2, M_1 \in \mathrm{Vect}(S^*Y)$ such that $M_1 \oplus L_1 \cong M_2 \oplus L_2 \cong \mathbb{C}^N$. Since $[M_1] + [L_1] = [\mathbb{C}^N], [M_2] + [L_2] = [\mathbb{C}^N]$ we obtain $[L_1] - [L_2] = \big([\mathbb{C}^N] - [M_1]\big) - \big([\mathbb{C}^N] - [M_2]\big)$, i.e.,

$$\mathrm{ind}_{S^*Y}\sigma_\partial(B) = [M_2] - [M_1].$$

Applying Theorem 5.11 we find an elliptic operator $\mathcal{B} \in \mathcal{T}^{\mu,0}(X; \boldsymbol{m})$ for $\boldsymbol{m} := (V_2, V_1; \mathbb{M}_1, \mathbb{M}_2)$, $\mathbb{M}_i := (Q_i, \mathbb{C}^N, M_i)$, $i = 1, 2$, such that $\ker_{S^*Y}\sigma_\partial(\mathcal{B}) \cong M_2$, $\mathrm{coker}_{S^*Y}\sigma_\partial(\mathcal{B}) \cong M_1$. Taking for Q_1 (Q_2) the complementary projection to P_2 (P_1) it follows that $\mathcal{A} \oplus \mathcal{B}$ is elliptic in $\mathcal{B}^{\mu,d}(X; \boldsymbol{v})$ for $\boldsymbol{v} = (V_1 \oplus V_2, V_2 \oplus V_1; \mathbb{C}^N, \mathbb{C}^N)$. The operator $\mathcal{B}$ is then as asserted.

Note that Grubb and Seeley [10] used a similar idea to embed an elliptic boundary value problem with projection conditions into a standard one by using the adjoint operator and the complementary projection.

Definition 5.16 Let $\mathcal{A} \in \mathcal{T}^{\mu,d}(X; \boldsymbol{l})$, $\boldsymbol{l} := (V_1, V_2; \mathbb{L}_1, \mathbb{L}_2)$, $\mathbb{L}_1, \mathbb{L}_2 \in \mathbb{P}(Y)$. An operator $\mathcal{P} \in \mathcal{T}^{-\mu,e}(X; \boldsymbol{l}^{-1})$ for $\boldsymbol{l}^{-1} := (V_2, V_1; \mathbb{L}_2, \mathbb{L}_1)$ and some $e \in \mathbb{N}$ is called a parametrix of $\mathcal{A}$, if the operators

$$\mathcal{C}_{\mathrm{L}} := \mathcal{I} - \mathcal{P}\mathcal{A} \;\text{ and }\; \mathcal{C}_{\mathrm{R}} := \mathcal{I} - \mathcal{A}\mathcal{P} \tag{63}$$

belong to $\mathcal{T}^{-\infty,d_{\mathrm{L}}}(X; \boldsymbol{m}_{\mathrm{L}})$ and $\mathcal{T}^{-\infty,d_{\mathrm{R}}}(X; \boldsymbol{m}_{\mathrm{R}})$, respectively, for

$$\boldsymbol{m}_{\mathrm{L}} := (V_1, V_1; \mathbb{L}_1, \mathbb{L}_1), \;\; \boldsymbol{m}_{\mathrm{R}} := (V_2, V_2; \mathbb{L}_2, \mathbb{L}_2),$$

and certain $d_{\mathrm{L}}, d_{\mathrm{R}} \in \mathbb{N}$.

Theorem 5.17 *Let* $\mathcal{A} \in \mathcal{T}^{\mu,d}(X;\boldsymbol{l})$, $\mu \in \mathbb{Z}$, $d \in \mathbb{N}$, $\boldsymbol{l} := (V_1, V_2; \mathbb{L}_1, \mathbb{L}_2)$ *for* $V_1,\ V_2 \in \mathrm{Vect}(X)$, $\mathbb{L}_1,\ \mathbb{L}_2 \in \mathbb{P}(Y)$.

(i) *The operator* $\mathcal{A}$ *is elliptic if and only if*

$$\mathcal{A} : \begin{array}{c} H^s(X, V_1) \\ \oplus \\ H^s(Y, \mathbb{L}_1) \end{array} \longrightarrow \begin{array}{c} H^{s-\mu}(X, V_2) \\ \oplus \\ H^{s-\mu}(Y, \mathbb{L}_2) \end{array} \tag{64}$$

is a Fredholm operator for an $s = s_0,\ s_0 > \max\{\mu, d\} - 1/2$.

(ii) *If* $\mathcal{A}$ *is elliptic,* (64) *is Fredholm for all* $s > \max\{\mu, d\} - 1/2$, *and* $\dim\ker\mathcal{A}$ *and* $\dim\operatorname{coker}\mathcal{A}$ *are independent of s.*

(iii) *An elliptic operator* $\mathcal{A}$ *has a parametrix* $\mathcal{P} \in \mathcal{T}^{-\mu,(d-\mu)^+}(X;\boldsymbol{l}^{-1})$ *in the sense of* Definition 5.16 *for* $d_{\mathrm{L}} = \max\{\mu, d\}$, $d_{\mathrm{R}} = (d-\mu)^+$, *and* $\mathcal{P}$ *can be chosen in such a way that the remainders in* (63) *are projections*

$$\mathcal{C}_{\mathrm{L}} : H^s(X, V_1) \oplus H^s(Y, \mathbb{L}_1) \to \mathcal{V}_1,\ \ \mathcal{C}_{\mathrm{R}} : H^{s-\mu}(X, V_2) \oplus H^{s-\mu}(Y, \mathbb{L}_2) \to \mathcal{V}_2$$

for all $s > \max\{\mu, d\} - 1/2$, *for* $\mathcal{V}_1 = \ker\mathcal{A} \subset C^\infty(X, V_1) \oplus H^\infty(Y, \mathbb{L}_1)$ *and a finite-dimensional subspace* $\mathcal{V}_2 \subset C^\infty(X, V_2) \oplus H^\infty(Y, \mathbb{L}_2)$ *with the property* $\mathcal{V}_2 + \operatorname{im}\mathcal{A} = H^{s-\mu}(X, V_2) \oplus H^{s-\mu}(Y, \mathbb{L}_2)$, $\mathcal{V}_2 \cap \operatorname{im}\mathcal{A} = \{0\}$ *for every* $s > \max\{\mu, d\} - 1/2$.

In fact, let us first show that an elliptic operator $\mathcal{A} \in \mathcal{T}^{\mu,d}(X;\boldsymbol{l})$ has a parametrix

$$\mathcal{P} \in \mathcal{T}^{-\mu,(d-\mu)^+}(X;\boldsymbol{l}^{-1}).$$

We apply Theorem 5.15 to $\mathcal{A}$ and choose a complementary operator

$$\mathcal{B} \in \mathcal{T}^{\mu,d}(X;\boldsymbol{m}),\ \boldsymbol{m} = (V_2, V_1; \mathbb{M}_1, \mathbb{M}_2)$$

such that $\tilde{\mathcal{A}} := \mathcal{A} \oplus \mathcal{B} \in \mathcal{B}^{\mu,d}(X;\boldsymbol{v})$ for $\boldsymbol{v} = (V_1 \oplus V_2, V_2 \oplus V_1; \mathbb{C}^N, \mathbb{C}^N)$ is SL-elliptic. Then

$$\mathcal{A} = \operatorname{diag}(1, P_2)\,\tilde{\mathcal{A}}\,\operatorname{diag}(1, E_1). \tag{65}$$

From Theorem 3.24 we obtain a parametrix $\tilde{\mathcal{P}} \in \mathcal{B}^{-\mu,(d-\mu)^+}(X;\boldsymbol{v}^{-1})$ for $\boldsymbol{v}^{-1} := (V_2 \oplus V_1, V_1 \oplus V_2; \mathbb{C}^N, \mathbb{C}^N)$, where $\sigma(\tilde{\mathcal{P}}) = \sigma(\tilde{\mathcal{A}})^{-1}$. Let us set

$$\mathcal{P}_0 := \operatorname{diag}(1, P_1)\,\tilde{\mathcal{P}}\,\operatorname{diag}(1, E_2) \in \mathcal{T}^{-\mu,(d-\mu)^+}(X;\boldsymbol{l}^{-1}),$$

where $E_2 : H^{s-\mu}(Y, \mathbb{L}_2) \to H^{s-\mu}(Y, J_2)$ is the canonical embedding and $P_1 : H^s(Y, J_1) \to H^s(Y, \mathbb{L}_1)$ the projection involved in $\mathbb{L}_1$, cf. notation in Definition 5.9. This yields

$$\mathcal{P}_0\mathcal{A} = \operatorname{diag}(1, P_1)\,\tilde{\mathcal{P}}\,\operatorname{diag}(1, P_2)\,\tilde{\mathcal{A}}\,\operatorname{diag}(1, E_1).$$

Thus for $\mathcal{C}_{\mathrm{L}} := \mathcal{I} - \mathcal{P}_0\mathcal{A} \in \mathcal{T}^{0,h}(X; \boldsymbol{v}_{\mathrm{L}})\,\mathrm{diag}\,(1, E_1)$ for $\boldsymbol{v}_{\mathrm{L}} = (V_1, V_1; \mathbb{L}_1, \mathbb{L}_1)$, $h = \max\{\mu, d\}$ we have $\sigma(\mathcal{C}_{\mathrm{L}}) = 0$, i.e., $\mathcal{C}_{\mathrm{L}} \in \mathcal{T}^{-1,h}(X; \boldsymbol{v}_{\mathrm{L}})$, cf. Theorem 5.7 (i). Applying Theorem 5.8 we find an operator $\mathcal{D}_{\mathrm{L}} \in \mathcal{T}^{-1,h}(X; \boldsymbol{v}_{\mathrm{L}})$ such that $(\mathcal{I} + \mathcal{D}_{\mathrm{L}})(\mathcal{I} - \mathcal{C}_{\mathrm{L}}) = \mathcal{I} \mod \mathcal{T}^{-\infty,h}(X; \boldsymbol{v}_{\mathrm{L}})$. We can define $\mathcal{D}_{\mathrm{L}}$ as an asymptotic sum $\sum_{j=1}^{\infty} \mathcal{C}_{\mathrm{L}}^j$. Thus $(\mathcal{I} + \mathcal{D}_{\mathrm{L}})\mathcal{P}_0\mathcal{A} = \mathcal{I} \mod \mathcal{T}^{-\infty,h}(X; \boldsymbol{v}_{\mathrm{L}})$, and hence $\mathcal{P}_{\mathrm{L}} := \mathcal{I} + \mathcal{D}_{\mathrm{L}}\mathcal{P}_0 \in \mathcal{T}^{-\mu,(d-\mu)^+}(X; \boldsymbol{l}^{-1})$ is a left parametrix of $\mathcal{A}$. In a similar manner we find a right parametrix. Thus we may take $\mathcal{P} := \mathcal{P}_{\mathrm{L}}$.

The Fredholm property of (64) is a direct consequence of the compactness of the remainders $\mathcal{C}_{\mathrm{L}}$, $\mathcal{C}_{\mathrm{R}}$ in relation (63), cf. also Theorem 5.7. The second part of (iii) is a consequence of general facts on elliptic operators that are always satisfied when elliptic regularity holds in the respective scales of spaces, see, for instance, [12, Subsection 1.2.7]. This confirms, in particular, assertion (ii).

It remains to verify that the Fredholm property of (64) for $s = s_0$, $s_0 > \max\{\mu, d\} - 1/2$ entails ellipticity. We reduce order and type to 0 by means of elliptic operators from Proposition 5.13, namely,

$$\mathcal{R}^{-s_0}_{V_1,\mathbb{L}_1} : \begin{matrix} L^2(X, V_1) \\ \oplus \\ H^0(Y, \mathbb{L}_1) \end{matrix} \to \begin{matrix} H^{s_0}(X, V_1) \\ \oplus \\ H^{s_0}(Y, \mathbb{L}_1) \end{matrix}, \quad \mathcal{R}^{s_0-\mu}_{V_2,\mathbb{L}_2} : \begin{matrix} H^{s_0-\mu}(X, V_2) \\ \oplus \\ H^{s_0-\mu}(Y, \mathbb{L}_2) \end{matrix} \to \begin{matrix} L^2(X, V_2) \\ \oplus \\ H^0(Y, \mathbb{L}_2) \end{matrix} \tag{66}$$

which are both Fredholm. The composition

$$\mathcal{A}_0 := \mathcal{R}^{s_0-\mu}_{V_2,\mathbb{L}_2}\, \mathcal{A}\, \mathcal{R}^{-s_0}_{V_1,\mathbb{L}_1} : \begin{matrix} L^2(X, V_1) \\ \oplus \\ H^0(Y, \mathbb{L}_1) \end{matrix} \to \begin{matrix} L^2(X, V_2) \\ \oplus \\ H^0(Y, \mathbb{L}_2) \end{matrix} \tag{67}$$

is again a Fredholm operator. In addition it belongs to $\mathcal{T}^{0,0}(X; (V_1, V_2; \mathbb{L}_1, \mathbb{L}_2))$ (the type in the upper left corner is necessarily 0 since it is acting in L^2). It suffices to show the ellipticity of $\mathcal{A}_0$. We now employ the fact that every $\mathbb{L} \in \mathbb{P}(Y)$ admits complementary projection data $\mathbb{L}^{\perp} \in \mathbb{P}(Y)$, cf. Proposition 2.5 (iii). In particular, for $\mathbb{L}_1 = (P_1, J_1, \sigma_\psi(P_1)J_1)$ we form $\mathbb{L}_1^{\perp} = (1 - P_1, J_1, \sigma_\psi(1 - P_1)J_1)$. Then $L^2(Y, J_1) = H^0(Y, \mathbb{L}_1) \oplus H^0(Y, \mathbb{L}_1^{\perp})$. We define an operator $\mathcal{B} := \mathcal{I}_2\mathcal{E}\mathcal{C}\mathcal{I}_1 : \begin{matrix} L^2(X, V_1) \\ \oplus \\ L^2(Y, J_1)) \end{matrix} \to \begin{matrix} L^2(X, V_2) \\ \oplus \\ L^2(Y, J_2 \oplus J_1) \end{matrix}$ where

$$\mathcal{I}_1 : \begin{matrix} L^2(X, V_1) \\ \oplus \\ L^2(Y, J_1) \end{matrix} \to \begin{matrix} L^2(X, V_1) \\ \oplus \\ H^0(Y, \mathbb{L}_1) \\ \oplus \\ H^0(Y, \mathbb{L}_1^{\perp}) \end{matrix}, \quad \mathcal{I}_2 : \begin{matrix} L^2(X, V_2) \\ \oplus \\ L^2(Y, J_2) \\ \oplus \\ L^2(Y, J_1) \end{matrix} \to \begin{matrix} L^2(X, V_2) \\ \oplus \\ L^2(Y, J_2 \oplus J_1) \end{matrix},$$

are canonical identifications, and

$$\begin{array}{ccc} L^2(X,V_1) & & L^2(X,V_2) \\ \oplus & & \oplus \\ \mathcal{C}: H^0(Y,\mathbb{L}_1) & \to & H^0(Y,\mathbb{L}_2) \\ \oplus & & \oplus \\ H^0(Y,\mathbb{L}_1^\perp) & & H^0(Y,\mathbb{L}_1^\perp) \end{array}, \quad \begin{array}{ccc} L^2(X,V_2) & & L^2(X,V_2) \\ \oplus & & \oplus \\ \mathcal{E}: H^0(Y,\mathbb{L}_2) & \hookrightarrow & L^2(Y,J_2) \\ \oplus & & \oplus \\ H^0(Y,\mathbb{L}_1^\perp) & & L^2(Y,J_1) \end{array}$$

with $\mathcal{E}$ being a canonical embedding, and $\mathcal{C} := \operatorname{diag}(\mathcal{A}_0, \mathrm{id}_{H^0(Y,\mathbb{L}_1^\perp)})$. We obviously have $\dim\ker\mathcal{B} = \dim\ker\mathcal{A}_0 < \infty$. Moreover, $\ker\mathcal{B}^*\mathcal{B} = \ker\mathcal{B} = \operatorname{im}(\mathcal{B}^*\mathcal{B})^\perp$, and $\mathcal{B}^*\mathcal{B}$ has closed range since $\mathcal{C}^*\mathcal{C}$ has. Therefore, $\mathcal{B}^*\mathcal{B} \in \mathcal{B}^{0,0}(X;(V_1,V_1;J_1,J_1))$ is a Fredholm operator and hence elliptic by Theorem 3.25. Therefore, both $\sigma_\psi(\mathcal{A}_0)$ and $\sigma_\partial(\mathcal{A}_0)$ are injective. Analogous arguments for adjoint operators show that $\sigma_\psi(\mathcal{A}_0)$ and $\sigma_\partial(\mathcal{A}_0)$ are also surjective.

The operator algebra furnished by the spaces $\mathcal{T}^{\mu,d}(X;\boldsymbol{l})$ for

$$\boldsymbol{l} := (V_1,V_2;\mathbb{L}_1,\mathbb{L}_2),\ \mathbb{L}_i = (P_i,J_i,L_i) \in \mathbb{P}(Y),\ i = 1,2,$$

contains the subalgebra of right lower corners, consisting of the spaces $\mathcal{T}^\mu(Y;\mathbb{L}_1,\mathbb{L}_2)$, studied in Sect. 2, cf. Definition 2.12. For the spaces $\mathcal{T}^{\mu,d}(X;\boldsymbol{l})$ it is not essential that $\mu \in \mathbb{Z}$. The principle of reducing a boundary value problem (BVP) to the boundary by means of another BVP has been well-known since a very long time, see, for instance, the monograph [11] and the bibliography there. Let us illustrate this in the case of the Neumann problem on a smooth manifold X with boundary Y using the Dirichlet problem. In this case the result is, that the potential operator K_0 contained in the inverse ${}^{\mathrm{t}}(\Delta\ \ T_0)^{-1} =: (P_0\ \ K_0)$ of the Dirichlet problem (with T_0 being the restriction of a function to the boundary) is composed from the left with T_1 the boundary operator of the Neumann problem. The result is T_1K_0 which is a first order classical elliptic pseudo-differential operator on the boundary. In other words, we talk about the composition

$$\begin{pmatrix} \Delta \\ T_1 \end{pmatrix} (P_0 \quad K_0) = \begin{pmatrix} 1 & 0 \\ T_1P_0 & T_1K_0 \end{pmatrix}.$$

Clearly if we replace Δ by another elliptic operator A with the transmission property at the boundary and if $\mathcal{A}_i := \begin{pmatrix} A \\ T_i \end{pmatrix}$, $i = 0, 1$, are SL-elliptic BVPs for A, then for a parametrix $\mathcal{P}_0 =: (P_0\ \ K_0)$ of $\mathcal{A}_0$ we can form the composition

$$\mathcal{A}_1\mathcal{P}_0 = \begin{pmatrix} A \\ T_1 \end{pmatrix} (P_0 \quad K_0) = \begin{pmatrix} 1 & 0 \\ T_1P_0 & T_1K_0 \end{pmatrix} + \mathcal{C}_{\mathrm{R}}, \tag{68}$$

where $\mathcal{C}_{\mathrm{R}}$ is a compact remainder in Boutet de Monvel's calculus. The operator $R := T_1K_0$ lives on the boundary Y and is elliptic. Equation (68) entails the

Agranovich-Dynin formula for the Fredholm indices

$$\operatorname{ind}\mathcal{A}_1 - \operatorname{ind}\mathcal{A}_0 = \operatorname{ind} R. \tag{69}$$

Note that from a parametrix $\mathcal{P}_0$ of $\mathcal{A}_0$ we get a parametrix $\mathcal{P}_1 =: (P_1 \quad K_1)$ of $\mathcal{A}_0$ by a simple algebraic consideration, namely,

$$(P_1 \quad K_1) = (P_0 - K_0 R^{(-1)} T_1 P_0 \quad K_0 R^{(-1)}), \tag{70}$$

where $R^{(-1)}$ means a parametrix of R which is obtained in a more elementary way compared with the parametrix construction in Boutet de Monvel's calculus.

Let us now consider elliptic operators

$$\mathcal{A}_i = \begin{pmatrix} A \\ T_i \end{pmatrix} \in \mathcal{T}^{\mu,d_i}(X; \boldsymbol{l}_i),\ i = 0, 1, \quad \mathcal{A}_i : H^s(X, V_1) \to \begin{matrix} H^{s-\mu}(X, V_2) \\ \oplus \\ H^{s-\mu}(Y, \mathbb{L}_i) \end{matrix}, \tag{71}$$

for $\mu > \max\{\mu, d\} - 1/2$, $\boldsymbol{l}_i = (V_1, V_2; \mathbb{O}, \mathbb{L}_i)$, $i = 0, 1$, $\mathbb{L}_i = (Q_i, J, L_i) \in \mathbb{P}(Y)$, where $\mathbb{O}$ indicates the case where the fibre dimension of the bundle in the middle is zero. For convenience we assume the trace operators to be of the same orders as A. However, a simple reduction of order allows us to pass to arbitrary orders, cf. Remark 2.25. By virtue of Theorem 5.17 (iii) the operators $\mathcal{A}_i$ have parametrices $\mathcal{P}_i \in \mathcal{T}^{-\mu,(d_i-\mu)^+}(X; \boldsymbol{l}_i^{-1})$ for $\boldsymbol{l}_i^{-1} = (V_2, V_1; \mathbb{L}_i, \mathbb{O})$, $i = 0, 1$,

$$\mathcal{P}_i =: (P_i \quad K_i),\ i = 0, 1.$$

Because of

$$\mathcal{A}_0\mathcal{P}_0 = \operatorname{diag}(\mathrm{id}_{H^{s-\mu}(X,V_2)}, \mathrm{id}_{H^{s-\mu}(Y,\mathbb{L}_0)}) \operatorname{mod} \mathcal{T}^{-\infty,(d-\mu)^+}(X; V_2, V_2; \mathbb{L}_0, \mathbb{L}_0)$$

it follows that

$$\mathcal{A}_1\mathcal{P}_0 = \begin{pmatrix} \mathrm{id}_{H^{s-\mu}(X;V_2)} & 0 \\ T_1 P_0 & T_1 K_0 \end{pmatrix} \bmod \mathcal{T}^{-\infty,(d-\mu)^+}(X; V_2, V_2; \mathbb{L}_0, \mathbb{L}_1).$$

Since the latter operator is elliptic, also $R := T_1 K_0 \in \mathcal{T}^0(Y; \mathbb{L}_0, \mathbb{L}_1)$ is elliptic, now in the Toeplitz calculus on the boundary, developed in Sect. 2. In particular,

$$R : H^{s-\mu}(Y, \mathbb{L}_0) \to H^{s-\mu}(Y, \mathbb{L}_1) \tag{72}$$

is a Fredholm operator, and we have an analogue of the Agranovich-Dynin formula (69). Moreover, knowing a parametrix $\mathcal{P}_0$ of $\mathcal{A}_0$ we can easily express a parametrix $\mathcal{P}_1$ of $\mathcal{A}_1$ by applying the corresponding analogue of relation (70), here using a parametrix $R^{(-1)} \in \mathcal{T}^0(Y; \mathbb{L}_1, \mathbb{L}_0)$ of the operator R.

Remark 5.18 Reductions of boundary conditions to the boundary in the Toeplitz analogue of Boutet de Monvel's calculus are possible also for 2×2 block matrix operators, containing trace and potential operators at the same time. The corresponding algebraic arguments are similar to those in [16, pages 252–254], and there is then also an analogue of the Agranovich-Dynin formula, cf. also the final part of Sect. 6 below.

6 The Edge Algebra with Global Projection Conditions

We now consider the edge algebra on a compact manifold M with edge Y. Recall that a manifold M with smooth edge Y can be regarded as a quotient space

$$M = \mathbb{M}/\sim, \tag{73}$$

where $\mathbb{M}$ is a smooth manifold with boundary $\partial\mathbb{M}$, and $\partial\mathbb{M}$ in turn is an X-bundle over Y, where N is a smooth closed manifold. Denoting by $p : \partial\mathbb{M} \to Y$ the bundle projection, the equivalence relation $\sim$ in (73) means that points $m, m' \in \partial\mathbb{M}$ are equivalent when $p(m) = p(m')$. A special case is a smooth manifold M with boundary Y; then N is of dimension 0, and we have a canonical identification $\mathbb{M} = M$.

The operator calculus on a manifold M with edge Y, called the edge algebra, consists of 2×2 block matrices $\mathcal{A}$ with edge-degenerate pseudo-differential operators, together with Mellin and Green operators in the upper left corners, and entries representing trace and potential operators referring to the edge Y. This calculus has been introduced in [20] and later on deepened in [21, 23], and other monographs and articles, cf. [11, 12], and the respective biliographies.

To be more precise, the edge algebra is furnished by spaces

$$L^{\mu}(M, \boldsymbol{g}; \boldsymbol{v}) \tag{74}$$

of operators

$$\mathcal{A} : \begin{array}{c} H^{s,\gamma}(M, E) \\ \oplus \\ H^{s}(Y, J_1) \end{array} \longrightarrow \begin{array}{c} H^{s-\mu,\gamma-\mu}(M, F) \\ \oplus \\ H^{s-\mu}(Y, J_2) \end{array}, \tag{75}$$

where $\boldsymbol{g} := (\gamma, \gamma - \mu, \Theta)$ are weight data, $\boldsymbol{v} := (E, F; J_1, J_2)$ is the involved tuple of vector bundles over M and Y, respectively, and $H^{s,\gamma}(M, E)$, etc., are weighted Sobolev spaces (of distributional sections) of smoothness s on $M \setminus Y$. By $\Theta := (\vartheta, 0]$, $-\infty < \vartheta < 0$, we understand a weight interval which determines a strips on the left of weight lines

$$\Gamma_{(n+1)/2-\gamma} \quad \text{and} \quad \Gamma_{(n+1)/2-(\gamma-\mu)}$$

for $n := \dim N$, $\Gamma_\beta := \{w \in \mathbb{C} : \operatorname{Re} w = \beta\}$ in the complex w-plane of the Mellin covariable, where asymptotic data of the involved Mellin and Green operators are controlled. More details can be found in the above-mentioned monographs.

Writing M locally near a point $y \in Y$ as a wedge

$$N^\Delta \times \Omega,$$

where $\Omega \subseteq \mathbb{R}^q$ open represents a chart on Y and $N^\Delta := (\overline{\mathbb{R}}_+ \times N)/(\{0\} \times N)$ the local model cone with base N, then in stretched variables $(r, x, y) \in N^\wedge \times \Omega$ for $N^\wedge := \mathbb{R}_+ \times N$ the edge-degenerate pseudo-differential operators have local amplitude functions

$$r^{-\mu} p(r, x, y, \rho, \xi, \eta) \quad \text{for} \quad p(r, x, y, \rho, \xi, \eta) := \tilde{p}(r, x, y, r\rho, \xi, r\eta)$$

for standard symbols

$$\tilde{p}(r, x, y, \tilde{\rho}, \xi, \tilde{\eta}) \in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{R}^{1+n+q}),$$

where $\Sigma \subseteq \mathbb{R}^n$ corresponds to a chart on N.

Modulo lower order terms the operators in $\mathcal{A} \in L^\mu(M, \boldsymbol{g}; \boldsymbol{v})$ are determined by a principal symbolic hierarchy

$$\sigma(\mathcal{A}) := (\sigma_\psi(\mathcal{A}), \sigma_\wedge(\mathcal{A})). \tag{76}$$

Here

$$\sigma_\psi(\mathcal{A}) : \pi^* E \to \pi^* F, \quad \text{for} \quad \pi : T^*(M \setminus Y) \to M \setminus Y \tag{77}$$

is the standard homogeneous principal symbol of order μ of the upper left corner of $\mathcal{A}$ and

$$\sigma_\wedge(\mathcal{A})(y, \eta) : \begin{array}{c} \mathcal{K}^{s,\gamma}(N^\wedge) \otimes E^\wedge_y \\ \oplus \\ J_{1,y} \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{s-\mu,\gamma-\mu}(N^\wedge) \otimes F^\wedge_y \\ \oplus \\ J_{2,y} \end{array}, \tag{78}$$

$(y, \eta) \in T^*Y \setminus 0$, where $E^\wedge_y$, etc., means the pull back of E_y to $N^\wedge \times \{y\}$ for any $y \in Y$, and $\mathcal{K}^{s,\gamma}(N^\wedge)$ are weighted Kegel Sobolev spaces over $N^\wedge$.

The symbolic structure (76) gives rise to a filtration of the spaces $L^\mu(M, \boldsymbol{g}; \boldsymbol{v})$. Let $\sigma^\mu(\mathcal{A}) := \sigma(\mathcal{A})$, and set

$$L^{\mu-1}(M, \boldsymbol{g}; \boldsymbol{v}) := \{\mathcal{A} \in L^\mu(M, \boldsymbol{g}; \boldsymbol{v}) : \sigma^\mu(\mathcal{A}) = 0\}. \tag{79}$$

In (79) we have again a pair of principal symbols $\sigma^{\mu-1}(\mathcal{A}) := (\sigma_\psi^{\mu-1}(\mathcal{A}), \sigma_\wedge^{\mu-1}(\mathcal{A}))$. Successively we then obtain

$$L^{\mu-(j+1)}(M, \boldsymbol{g}; \boldsymbol{v}) := \{\mathcal{A} \in L^{\mu-j}(M, \boldsymbol{g}; \boldsymbol{v}) : \sigma^{\mu-j}(\mathcal{A}) = 0\}$$

for every $j \in \mathbb{N}$. In other words we have spaces

$$L^\nu(M, \boldsymbol{g}; \boldsymbol{v}) \quad \text{for} \quad \mu - \nu \in \mathbb{N},\ \boldsymbol{g} = (\gamma, \gamma - \mu, \Theta). \tag{80}$$

The spaces (80) have subspaces

$$L^\nu_{\mathrm{M+G}}(M, \boldsymbol{g}; \boldsymbol{v}) \quad \text{and} \quad L^\nu_{\mathrm{G}}(M, \boldsymbol{g}; \boldsymbol{v}) \tag{81}$$

of smoothing Mellin plus Green and Green operators, respectively, cf. [23]. For convenience in the definition of (81) we refer to the case of continuous asymptotics.

Similarly as in Boutet de Monvel's calculus we have ellipticity with respect to both components of (76). An $\mathcal{A}$ is called σ_ψ-elliptic, if (77) is an isomorphism and if in addition locally close to Y the homogeneous principal part $\tilde{p}_{(\mu)}(r, x, y, \tilde{\rho}, \xi, \tilde{\eta})$ of $\tilde{p}(r, x, y, \tilde{\rho}, \xi, \tilde{\eta})$ of order μ bijectively maps $E_{(r,x,y)}$ to $F_{(r,x,y)}$ up to $r = 0$.

Concerning the second component of (76) the operator $\mathcal{A}$ is called Shapiro-Lopatinskii elliptic if it is σ_ψ-elliptic and if (78) is a family of isomorphisms for some $s \in \mathbb{R}$ (which is then the case for all s). Shapiro-Lopatinskii ellipticity will also be denoted by $(\sigma_\psi, \sigma_\wedge)$-ellipticity.

Among the known results in this context we have the following theorem.

Theorem 6.1 *A $(\sigma_\psi, \sigma_\wedge)$-elliptic operator $\mathcal{A} \in L^\mu(M, \boldsymbol{g}; \boldsymbol{v})$, $\boldsymbol{v} := (E, F; J_1, J_2)$, has a properly supported parametrix $\mathcal{P} \in L^{-\mu}(M, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1})$. If M is compact the following conditions are equivalent:*

(i) *$\mathcal{A}$ is $(\sigma_\psi, \sigma_\wedge)$-elliptic,*
(ii) *The operator (75) is Fredholm for some $s = s_0 \in \mathbb{R}$.*

Remark 6.2 Let $A \in L^\mu(M, \boldsymbol{g}; E, F)$ be σ_ψ-elliptic in the above-mentioned sense. Then $\sigma_\wedge(A)(y, \eta)$ is elliptic in the cone calculus for every fixed $(y, \eta) \in T^*Y \setminus 0$ which includes exit ellipticity at the conical exit $r \to \infty$. Moreover, for every $y \in Y$ there is a discrete set $D_A(y) \subset \mathbb{C}$ such that

$$\sigma_M \sigma_\wedge(A)(y, z) : H^s(N, E') \to H^{s-\mu}(N, F')$$

is an isomorphism, $s \in \mathbb{R}$, if and only if $z \notin D_A(y)$. Here $E', F' \in \mathrm{Vect}(N)$ are bundles induced by E, F on stretched cones $N^\wedge \times \{y\}$ for $y \in Y$.

This observation gives us the following result.

Proposition 6.3 *Let $A \in L^\mu(M, \boldsymbol{g}; E, F)$ be σ_ψ-elliptic. Then*

$$\sigma_\wedge(A)(y, \eta) : \mathcal{K}^{s,\gamma}(N^\wedge) \otimes E_y^\wedge \to \mathcal{K}^{s-\mu,\gamma-\mu}(N^\wedge) \otimes F_y^\wedge, \tag{82}$$

is a Fredholm operator for $y \in Y$, $\eta \neq 0$, $s \in \mathbb{R}$, *if and only if* $\gamma \in \mathbb{R}$ *satisfies the condition*

$$\Gamma_{\frac{n+1}{2}-\gamma} \cap D_A(y) = \emptyset. \tag{83}$$

In the following we assume that our operator $A \in L^\mu(M, \boldsymbol{g}; E, F)$ is σ_ψ-elliptic and satisfies the condition (83) for every $y \in Y$. Because of the homogeneity

$$\sigma_\wedge(A)(y, \lambda\eta) = \lambda^\mu \kappa_\lambda \sigma_\wedge(A)(y, \eta)\kappa_\lambda^{-1}$$

we have

$$\operatorname{ind} \sigma_\wedge(A)(y, \eta) = \sigma_\wedge(A)(y, \eta/|\eta|).$$

More precisely, the dimensions of $\ker \sigma_\wedge(A)(y, \eta)$ and $\operatorname{coker} \sigma_\wedge(A)(y, \eta)$ only depend on $\eta/|\eta|$. Therefore, (82) may be regarded as a family of Fredholm operators depending on the parameters $(y, \eta) \in S^*Y$, the unit cosphere bundle of Y which is a compact topological space. This gives rise to an index element

$$\operatorname{ind}_{S^*Y} \sigma_\wedge(A) \in K(S^*Y).$$

The property

$$\operatorname{ind}_{S^*Y} \sigma_\wedge(A) \in \pi^* K(Y), \tag{84}$$

$\pi : S^*Y \to Y$, is of analogous meaning for the edge calculus as the corresponding condition in Theorem 4.2. If A satisfies relation (84) we say that the Atiyah-Bott obstruction vanishes.

Theorem 6.4 *Let* $A \in L^\mu(M, \boldsymbol{g}; E, F)$ *be* σ_ψ*-elliptic, and* (83) *be satisfied for some* $\gamma \in \mathbb{R}$ *and all* $y \in Y$*, and denote the family of Fredholm operators* (82) *for the moment by*

$$\sigma_\wedge(A)^\gamma(y, \eta) : \mathcal{K}^{s,\gamma}(N^\wedge) \otimes E_y^\wedge \to \mathcal{K}^{s-\mu,\gamma-\mu}(N^\wedge) \otimes F_y^\wedge,$$

$(y, \eta) \in S^*Y$*. Then if* $\tilde{\gamma} \in \mathbb{R}$ *is another weight satisfying*

$$\Gamma_{\frac{n+1}{2}-\tilde{\gamma}} \cap D_A(y) = \emptyset$$

for all $y \in Y$*, we have*

$$\operatorname{ind}_{S^*Y} \sigma_\wedge(A)^\gamma \in \pi^* K(Y) \Leftrightarrow \operatorname{ind}_{S^*Y} \sigma_\wedge(A)^{\tilde{\gamma}} \in \pi^* K(Y).$$

In addition if $\tilde{A} \in L^{\mu}(M, \boldsymbol{g}; E, F)$ *satisfies* $\tilde{A} = A \bmod L^{\mu}_{M+G}(M, \boldsymbol{g}; E, F)$ *then*

$$\Gamma_{\frac{n+1}{2}-\gamma} \cap D_{\tilde{A}}(y) = \emptyset, \quad \Gamma_{\frac{n+1}{2}-\gamma} \cap D_{A}(y) = \emptyset$$

for some $\gamma \in \mathbb{R}$ *and all* $y \in Y$ *has the consequence*

$$\mathrm{ind}_{S^*Y}\sigma_{\wedge}(\tilde{A})^{\gamma} \in \pi^* K(Y) \Leftrightarrow \mathrm{ind}_{S^*Y}\sigma_{\wedge}(A)^{\gamma} \in \pi^* K(Y).$$

Theorem 6.5 *Let* $A \in L^{\mu}(M, \boldsymbol{g}; E, F)$ *be* σ_{ψ}*-elliptic, and let*

$$\sigma_{\wedge}(A)(y, \eta) : \mathcal{K}^{s,\gamma}(N^{\wedge}) \otimes E'^{\wedge}_{y} \to \mathcal{K}^{s-\mu,\gamma-\mu}(N^{\wedge}) \otimes F'^{\wedge}_{y}, \tag{85}$$

$(y, \eta) \in S^*Y$, *be a family of Fredholm operators. Choose* $L_1, L_2 \in \mathrm{Vect}(T^*Y \setminus 0)$ *such that*

$$\mathrm{ind}_{S^*Y}\sigma_{\wedge}(A) = [L_2|_{S^*Y}] - [L_1|_{S^*Y}]. \tag{86}$$

Then there exists an element $G \in L^{\mu}_{\mathrm{G}}(M, \boldsymbol{g}; E, F)$ *such that*

$$\ker_{S^*Y}\sigma_{\wedge}(A+G) \cong L_2|_{S^*Y}, \quad \mathrm{coker}_{S^*Y}\sigma_{\wedge}(A+G) \cong L_1|_{S^*Y}. \tag{87}$$

If $J_1, J_2 \in \mathrm{Vect}(Y)$ *are bundles such that* L_i *are subbundles of* $\pi_Y^* J_i$ *for* $\pi_Y : T^*Y \setminus 0 \to Y$, $i = 1, 2$, *there exists an* $\mathcal{A} \in L^{\mu}(M, \boldsymbol{g}; \boldsymbol{v})$ *for* $\boldsymbol{v} = (E, F; J_1, J_2)$ *with* $A + G$ *as the upper left corner of* $\mathcal{A}$, *such that* $\sigma_{\wedge}(\mathcal{A})$ *induces an isomorphism*

$$\sigma_{\wedge}(\mathcal{A}) : \begin{array}{c} \mathcal{K}^{s,\gamma}(N^{\wedge}) \otimes \pi_Y^* E^{\wedge} \\ \oplus \\ L_1 \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{s-\mu,\gamma-\mu}(N^{\wedge}) \otimes \pi_Y^* F^{\wedge} \\ \oplus \\ L_2 \end{array}. \tag{88}$$

Proof Since (85) is Fredholm we find a potential symbol

$$k_{(\mu)}(y, \eta) : \mathbb{C}^{N_-} \to \mathcal{K}^{s-\mu,\gamma-\mu}(N^{\wedge}) \otimes F^{\wedge}_{y}$$

in the sense of an upper right corner of a Green symbol of order μ with weight data $\boldsymbol{g}$, and for a suitable N_-, such that

$$\begin{pmatrix} \sigma_{\wedge}(A)(y, \eta) & k_{(\mu)}(y, \eta) \end{pmatrix} : \begin{array}{c} \mathcal{K}^{s,\gamma}(N^{\wedge}) \otimes E^{\wedge}_{y} \\ \oplus \\ \mathbb{C}^{N_-} \end{array} \to \mathcal{K}^{s-\mu,\gamma-\mu}(N^{\wedge}) \otimes F^{\wedge}_{y}$$

is surjective for all $(y, \eta) \in S^*Y$. This holds for all s. Let

$$p_2 : \mathcal{K}^{0,\gamma-\mu}(N^{\wedge}) \otimes \pi^* F^{\wedge} \to \mathrm{im}_{S^*Y} k_{(\mu)}$$

be the orthogonal projection with respect to the scalar product in the fibres

$$\mathcal{K}^{0,\gamma-\mu}(N^\wedge) \otimes \pi^* F_y^\wedge.$$

Since its kernel is smooth p_2 extends to $\mathcal{K}^{s,\gamma-\mu}(N^\wedge) \otimes \pi^* F^\wedge$ for every s. There are subbundles

$$\tilde{L}_1, \tilde{L}_1^\perp \subset \mathcal{K}^{\infty,\gamma}(N^\wedge) \otimes \pi^* E^\wedge, \quad \tilde{L}_2, \tilde{L}_2^\perp \subset \mathcal{K}^{\infty,\gamma-\mu}(N^\wedge) \otimes \pi^* F^\wedge$$

such that $\tilde{L}_{1,2} \cong L_{1,2}$, $\tilde{L}_2^\perp \cong \tilde{L}_1^\perp$, and

$$\mathrm{im}_{S^*Y}\big((1-p_2)\sigma_\wedge(A)\big) \cong \tilde{L}_2 \oplus \tilde{L}_2^\perp, \quad \mathrm{im}_{S^*Y} k_{(\mu)} \cong \tilde{L}_1 \oplus \tilde{L}_1^\perp.$$

Let $p_1 : \mathcal{K}^{s,\gamma}(N^\wedge) \otimes \pi^* E^\wedge \to \tilde{L}_2^\perp$ be induced by the corresponding orthogonal projection for $s = 0$, moreover, $\lambda : \tilde{L}_2^\perp \to \tilde{L}_1^\perp$ any smooth isomorphism, and $\iota : \tilde{L}_1^\perp \to \mathcal{K}^{s-\mu,\gamma-\mu}(N^\wedge) \otimes \pi^* F'^\wedge$ the canonical embedding. Set $q := \iota \circ \lambda \circ p_1$, and form

$$g_1 := -p_2 \sigma_\wedge(A) + q : \mathcal{K}^{s,\gamma}(N^\wedge) \otimes \pi^* E^\wedge \to \mathcal{K}^{s,\gamma-\mu}(N^\wedge) \otimes \pi^* F^\wedge.$$

Then g_1 can be regarded as the restriction of S^* of the homogeneous principal symbol $g_{(\mu)}$ of a Green operator $G \in L_{\mathrm{G}}^\mu(M, \boldsymbol{g}; E, F)$, and by construction we have relation (87). In order to construct the operator $\mathcal{A}$ it suffices to define its principal edge symbol $\sigma_\wedge(\mathcal{A}) := (\sigma_\wedge(\mathcal{A})_{ij})_{i,j=1,2}$ for $\sigma_\wedge(\mathcal{A})_{11} := \sigma_\wedge(A+G)$. For the remaining entries we choose arbitrary $J_{1,2} \in \mathrm{Vect}(Y)$ such that $L_{1,2}$ are subbundles of $\pi_Y^* J_{1,2}$. Similarly as in the standard calculus of pseudo-differential BVPs, outlined before, there is a potential symbol

$$\sigma_\wedge(\mathcal{A})_{12} : \pi_Y^* J_1 \to \mathcal{K}^{s-\mu,\gamma-\mu}(N^\wedge) \otimes \pi_Y^* F^\wedge,$$

and a trace symbol

$$\sigma_\wedge(\mathcal{A})_{21} : \mathcal{K}^{s,\gamma}(N^\wedge) \otimes \pi_Y^* E^\wedge \to \pi_Y^* J_2,$$

such that, if we set $\sigma_\wedge(\mathcal{A})_{22} := 0$ the matrix $\sigma_\wedge(\mathcal{A})$ induces an isomorphism (88). □

Let M be a compact manifold with edge Y.

Definition 6.6 Let $\mathbb{L}_i := (P_i, J_i, L_i) \in \mathbb{P}(Y)$ be projection data, cf. Definition 2.4, $V_1, V_2 \in \mathrm{Vect}(\mathbb{M})$, $i = 1, 2$, and set

$$\boldsymbol{v} := (V_1, V_2; J_1, J_2), \quad \boldsymbol{l} := (V_1, V_2; \mathbb{L}_1, \mathbb{L}_2).$$

Then $\mathcal{T}^{\mu}(M,\boldsymbol{g};\boldsymbol{l})$, $\mu \in \mathbb{R}$, for $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$ or $\boldsymbol{g} = (\gamma, \gamma - \mu)$ is defined to be the set of all operators

$$\mathcal{A} := \mathcal{P}_2 \tilde{\mathcal{A}} \mathcal{E}_1 \tag{89}$$

for $\mathcal{P}_2 := \operatorname{diag}(1, P_2)$, $\mathcal{E}_1 := \operatorname{diag}(1, E_1)$, cf. formula (55), for arbitrary $\tilde{\mathcal{A}} \in L^{\mu}(M,\boldsymbol{g};\boldsymbol{v})$. The elements of $\mathcal{T}^{\mu}(M,\boldsymbol{g};\boldsymbol{l})$ will be called edge problems of order μ with global projection conditions. Moreover, set

$$\mathcal{T}^{-\infty}(M,\boldsymbol{g};\boldsymbol{l}) := \{\mathcal{P}_2 \tilde{\mathcal{C}} \mathcal{E}_1 : \tilde{\mathcal{C}} \in L^{-\infty}(M,\boldsymbol{g};\boldsymbol{l})\}. \tag{90}$$

Observe that the space (90) can be equivalently characterised as the set of all $\mathcal{A} \in \mathcal{T}^{\mu}(M,\boldsymbol{g};\boldsymbol{l})$, $\mathcal{A} := \mathcal{P}_2 \tilde{\mathcal{A}} \mathcal{E}_1$ for some $\tilde{\mathcal{A}} \in L^{\mu}(M,\boldsymbol{g};\boldsymbol{v})$, such that $\mathcal{P}_2 \tilde{\mathcal{A}} \mathcal{P}_1 \in L^{-\infty}(M,\boldsymbol{g};\boldsymbol{v})$; then $\mathcal{A} = \mathcal{P}_2(\mathcal{P}_2 \tilde{\mathcal{A}} \mathcal{P}_1)\mathcal{E}_1$. Moreover,

$$\mathcal{P}_2(\mathcal{P}_2 \tilde{\mathcal{A}} \mathcal{P}_1)\mathcal{E}_1 \in \mathcal{T}^{-\infty}(M,\boldsymbol{g};\boldsymbol{l}) \Rightarrow \mathcal{P}_2 \tilde{\mathcal{A}} \mathcal{P}_1 \in L^{-\infty}(M,\boldsymbol{g};\boldsymbol{v}).$$

Theorem 6.7 *Every $\mathcal{A} \in \mathcal{T}^{\mu}(M,\boldsymbol{g};\boldsymbol{l})$ induces continuous operators*

$$\mathcal{A} : \begin{array}{c} H^{s,\gamma}(M, V_1) \\ \oplus \\ H^{s}(Y, \mathbb{L}_1) \end{array} \longrightarrow \begin{array}{c} H^{s-\mu,\gamma-\mu}(M, V_2) \\ \oplus \\ H^{s-\mu}(Y, \mathbb{L}_2) \end{array} \tag{91}$$

for every $s \in \mathbb{R}$.

Proof The proof is evident after the continuity of (104). □

Remark 6.8

(i) Let $\mathbb{L}_{1,2} := (P_{1,2}, J_{1,2}, L_{1,2})$, $\tilde{\mathbb{L}}_{1,2}(\tilde{P}_{1,2}, \tilde{J}_{1,2}, \tilde{L}_{1,2}) \in \mathbb{P}(Y)$, such that $J_{1,2}$ are subbundles of $\tilde{J}_{1,2}$, and

$$\tilde{P}_{1,2}|_{H^s(Y,J_{1,2})} = P_{1,2}. \tag{92}$$

Then we have a canonical isomorphism

$$\mathcal{T}^{\mu}(M,\boldsymbol{g};\boldsymbol{l}) \cong \mathcal{T}^{\mu}(M,\boldsymbol{g};\tilde{\boldsymbol{l}})$$

for $\tilde{\boldsymbol{l}} := (V_1, V_2; \tilde{\mathbb{L}}_1, \tilde{\mathbb{L}}_2)$, $\boldsymbol{l} := (V_1, V_2; \mathbb{L}_1, \mathbb{L}_2)$

(ii) If $\mathbb{L}_{1,2} \in \mathbb{P}(Y)$ and $\tilde{J}_{1,2} \in \operatorname{Vect}(Y)$ contain $J_{1,2}$ as subbundles, we find projections $\tilde{P}_{1,2} \in L^0_{\mathrm{cl}}(Y; \tilde{J}_{1,2}, \tilde{J}_{1,2})$ with the property (92).

Proposition 6.9 *Given $V_i \in \operatorname{Vect}(\mathbb{M})$, $i = 1, 2$, we have a canonical isomorphism*

$$\mathcal{T}^{\mu}(M,\boldsymbol{g};\boldsymbol{l}) \to \{\mathcal{P}_2 \tilde{\mathcal{A}} \mathcal{P}_1 : \tilde{\mathcal{A}} \in L^{\mu}(M,\boldsymbol{g};\boldsymbol{v})\}.$$

Proof The proof is analogous to that of Proposition 2.13, cf. also Proposition 5.3. □

In other words we have an identification

$$\mathcal{T}^{\mu}(M, \boldsymbol{g}; \boldsymbol{l}) = L^{\mu}(M, \boldsymbol{g}; \boldsymbol{v})/\sim \tag{93}$$

with the equivalence relation

$$\tilde{\mathcal{A}} \sim \tilde{\mathcal{B}} \Leftrightarrow \mathcal{P}_2 \tilde{\mathcal{A}} \mathcal{P}_1 = P_2 \tilde{\mathcal{B}} \mathcal{P}_1. \tag{94}$$

The space $\mathcal{T}^{\mu}(M, \boldsymbol{g}; \boldsymbol{l})$ is equipped with the principal symbolic structure

$$\sigma(\mathcal{A}) = (\sigma_{\psi}(\mathcal{A}), \sigma_{\wedge}(\mathcal{A}))$$

with the interior and the edge symbolic component. Writing $\mathcal{A} = (A_{ij})_{i,j=1,2}$ we first set $\sigma_{\psi}(\mathcal{A}) := \sigma_{\psi}(A_{11})$, i.e.,

$$\sigma_{\psi}(\mathcal{A}) : \pi^*_{M \setminus Y} V_1 \to \pi^*_{M \setminus Y} V_2$$

for $\pi_{M \setminus Y} : T^*(M \setminus Y) \to M \setminus Y$.

The edge symbol of $\mathcal{A}$, represented as $\mathcal{A} = \mathcal{P}_2 \tilde{\mathcal{A}} \mathcal{E}_1$, is defined as

$$\sigma_{\wedge}(\mathcal{A}) = \mathrm{diag}(1, p_2) \sigma_{\wedge}(\tilde{\mathcal{A}}) \mathrm{diag}(1, e_1),$$

$$\sigma_{\wedge}(\mathcal{A})(y, \eta) : \begin{array}{c} \mathcal{K}^{s,\gamma}(N^{\wedge}) \otimes V^{\wedge}_{1,y} \\ \oplus \\ L_{1,(y,\eta)} \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{s-\mu,\gamma-\mu}(N^{\wedge}) \otimes V^{\wedge}_{2,y} \\ \oplus \\ L_{2,(y,\eta)} \end{array}, \tag{95}$$

where $p_2(y, \eta)$ is the homogeneous principal symbol of order zero of the projection $P_2 \in L^0_{\mathrm{cl}}(Y; J_2, J_2)$, and $e_1 : L_{1,(y,\eta)} \to (\pi^*_Y J_1)_{(y,\eta)}$ is the canonical embedding.

Remark 6.10 Identifying an operator $\mathcal{A} = \mathcal{P}_2 \tilde{\mathcal{A}} \mathcal{E}_1 \in \mathcal{T}^{\mu}(M, \boldsymbol{g}; \boldsymbol{l})$ with $\tilde{\tilde{\mathcal{A}}} := \mathcal{P}_2 \tilde{\mathcal{A}} \mathcal{P}_1 \in L^{\mu}(M, \boldsymbol{g}; \boldsymbol{v})$, cf. Proposition 6.9, then $\sigma(\tilde{\tilde{\mathcal{A}}}) = 0$ in the sense of $L^{\mu}(M, \boldsymbol{g}; \boldsymbol{v})$ is equivalent to $\sigma(\mathcal{A}) = 0$ in the sense of $T^{\mu}(M, \boldsymbol{g}; \boldsymbol{l})$.

Analogously as (89) using (80) we define $T^{\nu}(M, \boldsymbol{g}; \boldsymbol{l})$ for $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$ or $\boldsymbol{g} = (\gamma, \gamma - \mu)$, $\mu - \nu \in \mathbb{N}$,

$$T^{\nu}(M, \boldsymbol{g}; \boldsymbol{l}) := \{\mathcal{P}_2 \tilde{\mathcal{A}} \mathcal{E}_1 : \tilde{\mathcal{A}} \in L^{\nu}(M, \boldsymbol{g}; \boldsymbol{v})\}.$$

Remark 6.11 $\mathcal{A} \in \mathcal{T}^{\mu}(M, \boldsymbol{g}; \boldsymbol{l})$ and $\sigma(\mathcal{A}) = 0$ imply $\mathcal{A} \in \mathcal{T}^{\mu-1}(M, \boldsymbol{g}; \boldsymbol{l})$, and the operator (91) is compact for every $s \in \mathbb{R}$.

Theorem 6.12 *Let $\mathcal{A}_j \in \mathcal{T}^{\mu-j}(M, \boldsymbol{g}; \boldsymbol{l})$, $j \in \mathbb{N}$, be an arbitrary sequence, $\boldsymbol{g} := (\gamma, \gamma - \mu, (-(k+1), 0])$ for a finite k, or $\boldsymbol{g} := (\gamma, \gamma - \mu)$, and assume that the asymptotic types involved in the Green operators are independent of j. Then there* *exists an* $\mathcal{A} \in \mathcal{T}^{\mu}(M, \boldsymbol{g}; \boldsymbol{l})$ *such that*

$$\mathcal{A} - \sum_{j=0}^{N} \mathcal{A}_j \in \mathcal{T}^{\mu-(N+1)}(M, \boldsymbol{g}; \boldsymbol{l})$$

for every $N \in \mathbb{N}$, *and* $\mathcal{A}$ *is unique* mod $\mathcal{T}^{-\infty}(M, \boldsymbol{g}; \boldsymbol{l})$.

Proof The proof is analogous to the proof of Theorem 5.8. ☐

Theorem 6.13

(i) $\mathcal{A} \in \mathcal{T}^{\mu}(M, \boldsymbol{g}_0; \boldsymbol{l}_0)$, $\mathcal{B} \in \mathcal{T}^{\rho}(M, \boldsymbol{g}_1; \boldsymbol{l}_1)$ *for* $\boldsymbol{g}_0 = (\gamma - \mu, \gamma - (\mu + \rho), \Theta)$, $\boldsymbol{g}_1 = (\gamma, \gamma - \rho, \Theta)$, *or* $\boldsymbol{g}_0 = (\gamma - \mu, \gamma - (\mu + \rho))$, $\boldsymbol{g}_1 = (\gamma, \gamma - \rho)$, *implies* $\mathcal{A}\mathcal{B} \in \mathcal{T}^{\mu+\nu}(M, \boldsymbol{g}_0 \circ \boldsymbol{g}_1; \boldsymbol{l}_0 \circ \boldsymbol{l}_1)$ (*when the projection data in the middle fit together such that* $\boldsymbol{l}_0 \circ \boldsymbol{l}_1$ *makes sense*), *and we have* $\sigma(\mathcal{A}\mathcal{B}) = \sigma(\mathcal{A})\sigma(\mathcal{B})$ *with componentwise multiplication.*

(ii) $\mathcal{A} \in \mathcal{T}^{\mu}(M, \boldsymbol{g}; \boldsymbol{l})$ *for* $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$ *or* $\boldsymbol{g} = (\gamma, \gamma - \mu)$, *and* $\boldsymbol{l} = (V_1, V_2; \mathbb{L}_1, \mathbb{L}_2)$ *implies* $\mathcal{A}^* \in \mathcal{T}^{\mu}(M, \boldsymbol{g}^*; \boldsymbol{l}^*)$ *for* $\boldsymbol{l}^* = (V_2, V_1; \mathbb{L}_2^*, \mathbb{L}_1^*)$, *where* $\mathcal{A}^*$ *is the formal adjoint in the sense*

$$(u, \mathcal{A}^* v)_{H^{0,0}(M,V_1) \oplus H^0(Y,\mathbb{L}_1)} = (\mathcal{A}u, v)_{H^{0,0}(M,V_2) \oplus H^0(Y,\mathbb{L}_2)}$$

for all $u \in H^{\infty,\infty}(M, V_1) \oplus H^{\infty}(Y, \mathbb{L}_1)$, $v \in H^{\infty,\infty}(M, V_2) \oplus H^{\infty}(Y, \mathbb{L}_2)$, *and we have* $\sigma(\mathcal{A}^*) = \sigma(\mathcal{A})^*$ *with componentwise formal adjoint.*

Proof The proof is formally analogous to the proof of Theorem 5.7 when we take into account the corresponding results from the edge calculus. ☐

Remark 6.14 Similarly as in the Toeplitz algebras before we have a natural notion of direct sum

$$\mathcal{T}^{\mu}(M, \boldsymbol{g}; \boldsymbol{l}) \oplus \mathcal{T}^{\mu}(M, \boldsymbol{g}; \boldsymbol{m}) = \mathcal{T}^{\mu}(M, \boldsymbol{g}; \boldsymbol{l} \oplus \boldsymbol{m})$$

where $\sigma(\mathcal{A} \oplus \mathcal{B}) = \sigma(\mathcal{A}) \oplus \sigma(\mathcal{B})$ with the componentwise direct sum of symbols.

We now turn to ellipticity in the Toeplitz calculus of edge problems.

Definition 6.15 Let $\mathcal{A} \in \mathcal{T}^{\mu}(M, \boldsymbol{g}; \boldsymbol{l})$ for $\boldsymbol{g} := (\gamma, \gamma - \mu, \Theta)$ or $\boldsymbol{g} := (\gamma, \gamma - \mu)$, $\mu \in \mathbb{R}$, and $\boldsymbol{l} := (V_1, V_2; \mathbb{L}_1, \mathbb{L}_2)$, $V_i \in \mathrm{Vect}(\mathbb{M})$, $\mathbb{L}_i = (P_i, J_i, L_i) \in \mathbb{P}(Y)$, $i = 1, 2$. The operator $\mathcal{A}$ is called elliptic if the upper left corner $A \in L^{\mu}(M, \boldsymbol{g}; V_1, V_2)$ is σ_{ψ}-elliptic and if the edge symbol (95) is an isomorphism for every $(y, \eta) \in T^*Y \setminus 0$ and some $s = s_0 \in \mathbb{R}$.

Remark 6.16 The bijectivity of (95) for some $s = s_0 \in \mathbb{R}$ is equivalent to the bijectivity of (95) for every $s \in \mathbb{R}$. The latter property is equivalent to the bijectivity

$$\sigma_\wedge(\mathcal{A})(y,\eta) : \begin{matrix} \mathcal{K}^{\infty,\gamma;\infty}(N^\wedge) \otimes V_{1,y}^\wedge \\ \oplus \\ L_{1,(y,\eta)} \end{matrix} \longrightarrow \begin{matrix} \mathcal{K}^{\infty,\gamma-\mu;\infty}(N^\wedge) \otimes V_{2,y}^\wedge \\ \oplus \\ L_{2,(y,\eta)} \end{matrix}. \tag{96}$$

Theorem 6.17

(i) *For every σ_ψ-elliptic $A \in L^\mu(M, \boldsymbol{g}; V_1, V_2)$ such that*

$$\sigma_\wedge(A)(y,\eta) : \mathcal{K}^{s,\gamma}(N^\wedge) \otimes V_{1,y}^\wedge \to \mathcal{K}^{s-\mu,\gamma-\mu}(N^\wedge) \otimes V_{2,y}^\wedge, \tag{97}$$

$(y,\eta) \in T^*Y \setminus 0$, *is a family of Fredholm operators with*

$$\mathrm{ind}_{S^*Y}\sigma_\wedge(A) = [L_2|_{S^*Y}] - [L_1|_{S^*Y}]$$

for some $L_1, L_2 \in \mathrm{Vect}(T^*Y \setminus 0)$ *there exist a Green operator*

$$G \in L^\mu_{\mathrm{G}}(M, \boldsymbol{g}; V_1, V_2),$$

projection data $\mathbb{L}_1, \mathbb{L}_2 \in \mathbb{P}(Y)$, *and an elliptic*

$$\mathcal{A} \in \mathcal{T}^\mu(M, \boldsymbol{g}; \boldsymbol{l}),\ \boldsymbol{l} := (V_1, V_2; \mathbb{L}_1, \mathbb{L}_2)$$

with $A + G$ as the upper left corner, such that $\mathcal{A}$ is $(\sigma_\psi, \sigma_\wedge)$-elliptic.

(ii) *For A as in* (i) *and suitable projection data $\mathbb{L}_i = (P_i, J_i, L_i) \in \mathbb{P}(Y)$, $i = 1, 2$, there exists an elliptic operator $\mathcal{A} \in \mathcal{T}^\mu(M, \boldsymbol{g}; \boldsymbol{l})$ containing $\mathcal{A}$ as the upper left corner.*

Proof (i) is a consequence of Theorem 6.5. For (ii) we choose $J_{1,2} \in \mathrm{Vect}(Y)$ of sufficiently large fibre dimension and a potential edge symbol

$$\sigma_\wedge(K) : \pi_Y^* J_1 \to \pi_Y^* \mathcal{K}^{s-\mu,\gamma-\mu}(N^\wedge) \otimes V_2^\wedge$$

of κ_δ-homogeneity μ such that

$$\begin{pmatrix} \sigma_\wedge(A) & \sigma_\wedge(K) \end{pmatrix} : \pi_Y^* \begin{pmatrix} \mathcal{K}^{s,\gamma}(N^\wedge) \otimes V_1^\wedge \\ \oplus \\ J_1 \end{pmatrix} \to \mathcal{K}^{s-\mu,\gamma-\mu}(N^\wedge) \otimes V_2^\wedge$$

is surjective; this is always possible, also when J_1 is trivial and of sufficiently large fibre dimension. Then

$$\ker_{S^*Y}\begin{pmatrix} \sigma_\wedge(A) & \sigma_\wedge(K) \end{pmatrix} =: L_2$$

has the property $\ker_{S^*Y} = [L_2|_{S^*Y}] - [\pi^* J_1]$. This allows us to apply (i) for $\mathbb{L}_1 := (\mathrm{id}, J_1, J_1)$. ◻

Proposition 6.18 *For every $\mu, \gamma \in \mathbb{R}$, $\boldsymbol{g} := (\gamma, \gamma - \mu, \Theta)$, $V \in \mathrm{Vect}(\mathbb{M})$ and $\mathbb{L} \in \mathbb{P}(Y)$ there exists an elliptic element $\mathcal{R}^{\mu}_{V,\mathbb{L}} \in \mathcal{T}^{\mu}(M, \boldsymbol{g}; \boldsymbol{l})$ for $\boldsymbol{l} := (V, V; \mathbb{L}, \mathbb{L})$ which induces a Fredholm operator*

$$\mathcal{R}^{\mu}_{V,\mathbb{L}} : \begin{array}{c} H^{s,\gamma}(M, V) \\ \oplus \\ H^{s}(Y, \mathbb{L}) \end{array} \to \begin{array}{c} H^{s-\mu,\gamma-\mu}(M, V) \\ \oplus \\ H^{s-\mu}(Y, \mathbb{L}) \end{array}$$

for every $s \in \mathbb{R}$.

Proof It suffices to set

$$\mathcal{R}^{\mu}_{V,\mathbb{L}} := \mathrm{diag}(R^{\mu}_{V}, R^{\mu}_{\mathbb{L}})$$

where R^{μ}_{V} is an order reducing operator from the edge calculus and $R^{\mu}_{\mathbb{L}}$ from Remark 2.25. ◻

Theorem 6.19 *For every elliptic operator $\mathcal{A} \in \mathcal{T}^{\mu}(M, \boldsymbol{g}; \boldsymbol{l})$, $\boldsymbol{l} = (V_1, V_2; \mathbb{L}_1, \mathbb{L}_2)$, there exists an elliptic operator $\mathcal{B} \in \mathcal{T}^{\mu}(M; \boldsymbol{g}; \boldsymbol{m})$, $\boldsymbol{m} := (V_2, V_1; \mathbb{M}_1, \mathbb{M}_2)$, for certain projection data $\mathbb{M}_1, \mathbb{M}_2 \in \mathbb{P}(Y)$ of the form $\mathbb{M}_i := (Q_i, \mathbb{C}^N, M_i)$, $i = 1, 2$, for some $N \in \mathbb{N}$, such that $\mathcal{A} \oplus \mathcal{B} \in L^{\mu}(M, \boldsymbol{g}; \boldsymbol{v})$ for $\boldsymbol{v} = (V_1 \oplus V_2, V_2 \oplus V_1; \mathbb{C}^N, \mathbb{C}^N)$, is $(\sigma_\psi, \sigma_\wedge)$-elliptic.*

Proof The operator $A \in L^{\mu}(M, \boldsymbol{g}; V_1, V_2)$ in the upper left corner of $\mathcal{A}$ induces continuous maps

$$A : H^{s,\gamma}(M, V_1) \to H^{s-\mu,\gamma-\mu}(M, V_2) \tag{98}$$

for all $s \in \mathbb{R}$. This will be applied for $s = \gamma$. We have order reducing isomorphisms

$$R^{\gamma}_{V_1} : H^{\gamma,\gamma}(M, V_1) \to H^{0,0}(M, V_1), \quad R^{\gamma-\mu}_{V_2} : H^{\gamma-\mu,\gamma-\mu}(M, V_2) \to H^{0,0}(M, V_2),$$

belonging to $L^{\gamma}(M, (\gamma, 0, \Theta); V_1, V_1)$ and $L^{\gamma-\mu}(M, (\gamma - \mu, 0, \Theta); V_2, V_2)$, respectively. These operators are elliptic in the edge calculus. According to Theorem 6.13 (i) we can form

$$A_0 := R^{\gamma-\mu}_{V_2} A (R^{\gamma}_{V_1})^{-1} \in L^{0}(M, (0, 0, \Theta); V_1, V_2),$$

$A_0 : H^{0,0}(M) \to H^{0,0}(M)$. It follows that $\sigma(A_0) = \sigma(R^{\gamma-\mu}_{V_2})\sigma(A)\sigma((R^{\gamma}_{V_1})^{-1})$. Applying this for the $\sigma_\wedge$-components we see that

$$\mathrm{ind}_{S^*Y}\sigma_\wedge(A_0) = \mathrm{ind}_{S^*Y}\sigma_\wedge(A) = [L_2|_{S^*Y}] - [L_1|_{S^*Y}]. \tag{99}$$

There is an N such that the bundles J_i contained in $\mathbb{L}_i$, $i = 1, 2$, are subbundles of $\mathbb{C}^N$. Because of Remark 6.8 without loss of generality we assume $\mathbb{L}_{1,2} = (P_{1,2}, \mathbb{C}^N, L_{1,2})$. For complementary bundles $L_{1,2}^{\perp}$ of $L_{1,2}$ in $\mathbb{C}^N$ we have

$$[L_2|_{S^*Y}] - [L_1|_{S^*Y}] = [L_1^{\perp}|_{S^*Y}] - [L_2^{\perp}|_{S^*Y}]. \tag{100}$$

For the adjoint of A_0 we have $A_0^* \in L^0(M, (0, 0, \Theta); V_2, V_1)$, and relations (99), (100) imply

$$\operatorname{ind}_{S^*Y}\sigma_{\wedge}(A_0^*) = [L_2^{\perp}|_{S^*Y}] - [L_1^{\perp}|_{S^*Y}].$$

By virtue of Theorem 6.5 we find a Green operator $G_0 \in L^0(M, (0, 0, \Theta); V_2, V_1)$ such that

$$\ker_{S^*Y}\sigma_{\wedge}(A_0^* + G_0) \cong L_2^{\perp}|_{S^*Y}, \quad \operatorname{coker}_{S^*Y}\sigma_{\wedge}(A_0^* + G_0) \cong L_1^{\perp}|_{S^*Y}.$$

For $B := (R_{V_1}^{\gamma-\mu})^{-1}(A_0^* + G_0)R_{V_2}^{\gamma} \in L^{\mu}(M, \boldsymbol{g}; V_2, V_1)$ we also have

$$\ker_{S^*Y}\sigma_{\wedge}(B) \cong L_2^{\perp}|_{S^*Y}, \quad \operatorname{coker}_{S^*Y}\sigma_{\wedge}(B) \cong L_1^{\perp}|_{S^*Y}.$$

Let us set $\mathbb{M}_{1,2} := (P_{1,2}^{\perp}, \mathbb{C}^N, L_{1,2}^{\perp})$, where $P_{1,2}^{\perp}$ are complementary projections to $P_{1,2}$. Because of Theorem 6.17 there is now an elliptic operator $\mathcal{B} \in \mathcal{T}^{\mu}(M, \boldsymbol{g}; \boldsymbol{m})$ for $\boldsymbol{m} := (V_2, V_1; \mathbb{M}_1, \mathbb{M}_2)$ containing B as upper left corner. From the construction it is then evident that $\mathcal{A} \oplus \mathcal{B}$ has the desired properties. $\square$

Definition 6.20 Let $\mathcal{A} \in \mathcal{T}^{\mu}(M, \boldsymbol{g}; \boldsymbol{l})$ be as in Definition 6.15. Then a $\mathcal{P} \in \mathcal{T}^{-\mu}(M, \boldsymbol{g}^{-1}; \boldsymbol{l}^{-1})$ for $\boldsymbol{g}^{-1} = (\gamma - \mu, \gamma, \Theta)$ or $(\gamma - \mu, \gamma)$ and $\boldsymbol{l}^{-1} = (V_2, V_1; \mathbb{L}_2, \mathbb{L}_1)$, is called a parametrix of $\mathcal{A}$, if

$$\mathcal{C}_{\mathrm{L}} := \mathcal{I} - \mathcal{P}\mathcal{A} \in \mathcal{T}^{-\infty}(M, \boldsymbol{g}_{\mathrm{L}}; \boldsymbol{l}_{\mathrm{L}}), \quad \mathcal{C}_{\mathrm{R}} := \mathcal{I} - \mathcal{A}\mathcal{P} \in \mathcal{T}^{-\infty}(M, \boldsymbol{g}_{\mathrm{R}}; \boldsymbol{l}_{\mathrm{R}}) \tag{101}$$

for $\boldsymbol{g}_{\mathrm{L}} = (\gamma, \gamma, \Theta)$ or $\boldsymbol{g}_{\mathrm{R}} = (\gamma - \mu, \gamma - \mu, \Theta)$ and similarly without Θ, with $\mathcal{I}$ being the respective identity operators, and $\boldsymbol{l}_{\mathrm{L}} := (V_1, V_1; \mathbb{L}_1, \mathbb{L}_1)$, $\boldsymbol{l}_{\mathrm{R}} := (V_2, V_2; \mathbb{L}_2, \mathbb{L}_2)$.

Theorem 6.21 *Let* $\mathcal{A} \in \mathcal{T}^{\mu}(M, \boldsymbol{g}; \boldsymbol{l})$, $\mu \in \mathbb{R}$, $\boldsymbol{l} := (V_1, V_2; \mathbb{L}_1, \mathbb{L}_2)$ *for* V_1, $V_2 \in \operatorname{Vect}(\mathbb{M})$, $\mathbb{L}_1$, $\mathbb{L}_2 \in \mathbb{P}(Y)$.

(i) *Let* $\mathcal{A}$ *be elliptic; then*

$$\mathcal{A} : \begin{array}{c} H^{s,\gamma}(M, V_1) \\ \oplus \\ H^s(Y, \mathbb{L}_1) \end{array} \longrightarrow \begin{array}{c} H^{s-\mu,\gamma-\mu}(X, V_2) \\ \oplus \\ H^{s-\mu}(Y, \mathbb{L}_2) \end{array} \tag{102}$$

is a Fredholm operator for every $s \in \mathbb{R}$. *Moreover, if* (102) *is Fredholm for* $s = \gamma$ *then the operator* $\mathcal{A}$ *is elliptic.*

(ii) *If $\mathcal{A}$ is elliptic,* (64) *is Fredholm for all $s \in \mathbb{R}$, and* $\dim\ker\mathcal{A}$ *and* $\dim\operatorname{coker}\mathcal{A}$ *are independent of s.*

(iii) *An elliptic operator $\mathcal{A} \in \mathcal{T}^{\mu}(M, \boldsymbol{g}; \boldsymbol{l})$ has a parametrix $\mathcal{P} \in \mathcal{T}^{-\mu}(M, \boldsymbol{g}^{-1}; \boldsymbol{l}^{-1})$ in the sense of* Definition 5.16*, and $\mathcal{P}$ can be chosen in such a way that the remainders in* (63) *are projections*

$$\mathcal{C}_{\mathrm{L}} : H^{s,\gamma}(M, V_1) \oplus H^{s}(Y, \mathbb{L}_1) \to \mathcal{V}_1, \ \mathcal{C}_{\mathrm{R}} : H^{s-\mu,s-\mu}(M, V_2) \oplus H^{s-\mu}(Y, \mathbb{L}_2) \to \mathcal{V}_2$$

for all $s \in \mathbb{R}$, for $\mathcal{V}_1 = \ker\mathcal{A} \subset H^{\infty,\gamma}(M, V_1) \oplus H^{\infty}(Y, \mathbb{L}_1)$ and a finite-dimensional subspace $\mathcal{V}_2 \subset H^{\infty,\gamma-\mu}(M, V_2) \oplus H^{\infty}(Y, \mathbb{L}_2)$ with the property $\mathcal{V}_2 + \operatorname{im}\mathcal{A} = H^{s-\mu,\gamma-\mu}(M, V_2) \oplus H^{s-\mu}(Y, \mathbb{L}_2)$, $\mathcal{V}_2 \cap \operatorname{im}\mathcal{A} = \{0\}$ for every $s \in \mathbb{R}$.

Proof The proof is formally analogous to that of Theorem 5.17. Let us nevertheless carry out a few steps. We first show that an elliptic operator $\mathcal{A} \in \mathcal{T}^{\mu}(M, \boldsymbol{g}; \boldsymbol{l})$ has a parametrix

$$\mathcal{P} \in \mathcal{T}^{-\mu}(M, \boldsymbol{g}^{-1}; \boldsymbol{l}^{-1}).$$

We apply Theorem 6.19 and choose a complementary operator

$$\mathcal{B} \in \mathcal{T}^{\mu,d}(M, \boldsymbol{g}; \boldsymbol{m}), \ \boldsymbol{m} = (V_2, V_1; \mathbb{M}_1, \mathbb{M}_2)$$

such that $\tilde{\mathcal{A}} := \mathcal{A} \oplus \mathcal{B} \in L^{\mu}(M, \boldsymbol{g}; \boldsymbol{v})$ for $\boldsymbol{v} = (V_1 \oplus V_2, V_2 \oplus V_1; \mathbb{C}^N, \mathbb{C}^N)$ is $(\sigma_\psi, \sigma_\wedge)$-elliptic. Then

$$\mathcal{A} = \operatorname{diag}(1, P_2)\, \tilde{\mathcal{A}} \operatorname{diag}(1, E_1). \tag{103}$$

From Theorem 6.1 we obtain a parametrix $\tilde{\mathcal{P}} \in L^{-\mu}(M, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1})$ for $\boldsymbol{v}^{-1} := (V_2 \oplus V_1, V_1 \oplus V_2; \mathbb{C}^N, \mathbb{C}^N)$, where $\sigma(\tilde{\mathcal{P}}) = \sigma(\tilde{\mathcal{A}})^{-1}$. Let us set

$$\mathcal{P}_0 := \operatorname{diag}(1, P_1)\, \tilde{\mathcal{P}} \operatorname{diag}(1, E_2) \in \mathcal{T}^{-\mu}(M, \boldsymbol{g}^{-1}; \boldsymbol{l}^{-1}),$$

where $E_2 : H^{s-\mu}(Y, \mathbb{L}_2) \to H^{s-\mu}(Y, J_2)$ is the canonical embedding and $P_1 : H^{s}(Y, J_1) \to H^{s}(Y, \mathbb{L}_1)$ the projection involved in $\mathbb{L}_1$. This yields

$$\mathcal{P}_0\mathcal{A} = \operatorname{diag}(1, P_1)\, \tilde{\mathcal{P}} \operatorname{diag}(1, P_2)\, \tilde{\mathcal{A}} \operatorname{diag}(1, E_1).$$

Thus for $\mathcal{C}_{\mathrm{L}} := \mathcal{I} - \mathcal{P}_0\mathcal{A} \in \mathcal{T}^{0}(M, \boldsymbol{g}_{\mathrm{L}}; \boldsymbol{v}_{\mathrm{L}}) \operatorname{diag}(1, E_1)$ for $\boldsymbol{v}_{\mathrm{L}} = (V_1, V_1; \mathbb{L}_1, \mathbb{L}_1)$, we have $\sigma(\mathcal{C}_{\mathrm{L}}) = 0$, i.e., $\mathcal{C}_{\mathrm{L}} \in \mathcal{T}^{-1}(M, \boldsymbol{g}_{\mathrm{L}}; \boldsymbol{v}_{\mathrm{L}})$, cf. Remark 6.11. Applying Theorem 6.12 we find an operator $\mathcal{D}_{\mathrm{L}} \in \mathcal{T}^{-1}(M, \boldsymbol{g}_{\mathrm{L}}; \boldsymbol{v}_{\mathrm{L}})$ such that $(\mathcal{I} + \mathcal{D}_{\mathrm{L}})(\mathcal{I} - \mathcal{C}_{\mathrm{L}}) = \mathcal{I} \mod \mathcal{T}^{-\infty}(M, \boldsymbol{g}_{\mathrm{L}}; \boldsymbol{v}_{\mathrm{L}})$. We can define $\mathcal{D}_{\mathrm{L}}$ as an asymptotic sum $\sum_{j=1}^{\infty} \mathcal{C}_{\mathrm{L}}^{j}$. Thus $(\mathcal{I} + \mathcal{D}_{\mathrm{L}})\mathcal{P}_0\mathcal{A} = \mathcal{I} \mod \mathcal{T}^{-\infty}(M, \boldsymbol{g}_{\mathrm{L}}; \boldsymbol{v}_{\mathrm{L}})$, and hence $\mathcal{P}_{\mathrm{L}} := \mathcal{I} + \mathcal{D}_{\mathrm{L}}\mathcal{P}_0 \in \mathcal{T}^{-\mu}(M, \boldsymbol{g}_{\mathrm{L}}; \boldsymbol{l}^{-1})$ is a left parametrix of $\mathcal{A}$. In a similar manner we find a right parametrix. Thus we may take $\mathcal{P} := \mathcal{P}_{\mathrm{L}}$.

The Fredholm property of (102) is a direct consequence of the compactness of the remainders $\mathcal{C}_{\mathrm{L}}$, $\mathcal{C}_{\mathrm{R}}$ in relation (101), cf. also Remark 6.11. The second part of (iii) is a consequence of general facts on elliptic operators that are always satisfied when elliptic regularity holds in the respective scales of spaces, see, for instance, [12, Subsection 1.2.7]. This confirms, in particular, assertion (ii).

The proof that the Fredholm property of (102) for $s = \gamma$ entails ellipticity is of the same structure as the last part of the proof of Theorem 5.17, left to the reader.

□

Let us now consider elliptic operators

$$\mathcal{A}_i = \begin{pmatrix} A \\ T_i \end{pmatrix} \in \mathcal{T}^{\mu}(M, \boldsymbol{g}; \boldsymbol{l}_i),\ i = 0, 1, \quad \mathcal{A}_i : H^{s,\gamma}(M, V_1) \to \begin{array}{c} H^{s-\mu,\gamma-\mu}(M, V_2) \\ \oplus \\ H^{s-\mu}(Y, \mathbb{L}_i) \end{array}, \tag{104}$$

for $\boldsymbol{l}_i = (V_1, V_2; \mathbb{O}, \mathbb{L}_i)$, $i = 0, 1$, $\mathbb{L}_i = (Q_i, J, L_i) \in \mathbb{P}(Y)$, where $\mathbb{O}$ indicates the case where the fibre dimension of the bundle in the middle is zero. For convenience we assume the trace operators to be of the same orders as A. However, a simple reduction of order allows us to pass to arbitrary orders, cf. Remark 2.25. By virtue of Theorem 6.21 (iii) the operators $\mathcal{A}_i$ have parametrices $\mathcal{P}_i \in \mathcal{T}^{-\mu}(M, \boldsymbol{g}^{-1}; \boldsymbol{l}_i^{-1})$ for $\boldsymbol{l}_i^{-1} = (V_2, V_1; \mathbb{L}_i, \mathbb{O})$, $i = 0, 1$,

$$\mathcal{P}_i =: (P_i \quad C_i),\ i = 0, 1.$$

Because of

$$\mathcal{A}_0 \mathcal{P}_0 = \operatorname{diag}(\mathrm{id}_{H^{s-\mu}(X, V_2)}, \mathrm{id}_{H^{s-\mu}(Y, \mathbb{L}_0)}) \operatorname{mod} \mathcal{T}^{-\infty}(M, \boldsymbol{g}_{\mathrm{L}}; (V_2, V_2; \mathbb{L}_0, \mathbb{L}_0))$$

for $\boldsymbol{g}_{\mathrm{L}} = (\gamma - \mu, \gamma - \mu, \Theta)$ it follows that

$$\mathcal{A}_1 \mathcal{P}_0 = \begin{pmatrix} \mathrm{id}_{H^{s-\mu}(M; V_2)} & 0 \\ T_1 P_0 & T_1 C_0 \end{pmatrix} \bmod \mathcal{T}^{-\infty}(M, \boldsymbol{g}_{\mathrm{L}}; (V_2, V_2; \mathbb{L}_0, \mathbb{L}_1)).$$

Since the latter operator is elliptic, also $R := T_1 C_0 \in \mathcal{T}^0(Y; \mathbb{L}_0, \mathbb{L}_1)$ is elliptic, now in the Toeplitz calculus on Y, developed in Sect. 2. In particular,

$$R : H^{s-\mu}(Y, \mathbb{L}_0) \to H^{s-\mu}(Y, \mathbb{L}_1) \tag{105}$$

is a Fredholm operator, and we have an analogue of the Agranovich-Dynin formula (69). Moreover, knowing a parametrix $\mathcal{P}_0$ of $\mathcal{A}_0$ we can easily express a parametrix $\mathcal{P}_1$ of $\mathcal{A}_1$ by applying the corresponding analogue of relation (70), here using a parametrix $R^{(-1)} \in \mathcal{T}^0(Y; \mathbb{L}_1, \mathbb{L}_0)$ of the operator R.

Let us extend reducing of operators to the edge to elliptic operators in block matrix form. For simplicity we assume orders to be zero; the general case can be achieved by reduction of orders to 0.

$$\mathcal{A}_i = \begin{pmatrix} A & K_i \\ T_i & Q_i \end{pmatrix} \in \mathcal{T}^0(M, \boldsymbol{g}; \boldsymbol{l}_i), \quad \mathcal{A}_i : \begin{matrix} H^{0,\gamma}(M, V_1) \\ \oplus \\ H^0(Y, \mathbb{K}_i) \end{matrix} \to \begin{matrix} H^{0,\gamma}(M, V_2) \\ \oplus \\ H^0(Y, \mathbb{L}_i) \end{matrix}, \quad i = 0, 1, \tag{106}$$

for $\boldsymbol{l}_i = (V_1, V_2; \mathbb{K}_i, \mathbb{L}_i)$, $i = 0, 1$, $\mathbb{K}_i = (P_i, J, K_i)$, $\mathbb{L}_i = (Q_i, J, L_i) \in \mathbb{P}(Y)$, where the upper left corner is the same for $i = 1, 2$. In order to achieve an analogue of the Agranovich-Dynin formula for the Fredholm indices we pass to the operators

$$\tilde{\mathcal{A}}_0 = \begin{pmatrix} A & K_1 & K_0 \\ T_0 & 0 & Q_0 \\ 0 & 1 & 0 \end{pmatrix} : \begin{matrix} H^{0,\gamma}(M, V_1) \\ \oplus \\ H^0(Y, \mathbb{K}_1) \\ \oplus \\ H^0(Y, \mathbb{K}_0) \end{matrix} \to \begin{matrix} H^{0,\gamma}(M, V_2) \\ \oplus \\ H^0(Y, \mathbb{L}_0) \\ \oplus \\ H^0(Y, \mathbb{K}_1) \end{matrix}, \tag{107}$$

$\tilde{\mathcal{A}}_0 \in \mathcal{T}^\mu(M, \boldsymbol{g}; \boldsymbol{l}_i)$, $\boldsymbol{g} = (\gamma, \gamma, \Theta)$, and

$$\tilde{\mathcal{A}}_1 = \begin{pmatrix} A & K_1 & K_0 \\ T_1 & Q_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \begin{matrix} H^{0,\gamma}(M, V_1) \\ \oplus \\ H^0(Y, \mathbb{K}_1) \\ \oplus \\ H^0(Y, \mathbb{K}_0) \end{matrix} \to \begin{matrix} H^{0,\gamma}(M, V_2) \\ \oplus \\ H^0(Y, \mathbb{L}_1) \\ \oplus \\ H^0(Y, \mathbb{K}_0) \end{matrix}, \tag{108}$$

$\tilde{\mathcal{A}}_1 \in \mathcal{T}^0(M, \boldsymbol{g}; \boldsymbol{l}_i)$. If $\mathcal{P}_0 = \begin{pmatrix} P_0 & C_0 \\ B_0 & Q_0 \end{pmatrix} \in \mathcal{T}^0(M, \boldsymbol{g}; \boldsymbol{l}_0^{-1})$ is a parametrix of $\mathcal{A}_0$ which exists by Theorem 6.21, we obtain a parametrix $\tilde{\mathcal{P}}_0$ of $\tilde{\mathcal{A}}_0$ in the form

$$\tilde{\mathcal{P}}_0 = \begin{pmatrix} P_0 & C_0 & -P_0K_1 \\ 0 & 0 & 1 \\ B_0 & Q_0 & -B_0K_1 \end{pmatrix}. \tag{109}$$

It follows that

$$\tilde{\mathcal{A}}_1\tilde{\mathcal{P}}_0 = \begin{pmatrix} 1 & 0 & 0 \\ T_1P_0 & T_1C_0 & -T_1P_0K_1 + Q_1 \\ B_0 & Q_0 & -B_0K_1 \end{pmatrix} \tag{110}$$

mod $\mathcal{T}^{-\infty}(M, \boldsymbol{g}; \boldsymbol{n})$ for $\boldsymbol{n} = (V_2, V_2; \mathbb{L}_0 \oplus \mathbb{K}_1, \mathbb{L}_1 \oplus \mathbb{K}_0)$, where the lower right corner

$$\mathcal{R} = \begin{pmatrix} T_1 C_0 & -T_1 P_0 K_1 + Q_1 \\ Q_0 & -B_0 K_1 \end{pmatrix} : \begin{matrix} H^0(Y, \mathbb{L}_0) \\ \oplus \\ H^0(Y, \mathbb{K}_1) \end{matrix} \rightarrow \begin{matrix} H^0(Y, \mathbb{L}_1) \\ \oplus \\ H^0(Y, \mathbb{K}_0) \end{matrix} \tag{111}$$

is elliptic and belongs to $\mathcal{T}^0(Y; \boldsymbol{r})$ for $\boldsymbol{r} = (\mathbb{L}_0 \oplus \mathbb{K}_1, \mathbb{L}_1 \oplus \mathbb{K}_0)$. The analogue of the Agranovich-Dynin formula in this case is as follows.

Theorem 6.22 *For every two elliptic operators* (106) *the reduction to the edge* (110) *is elliptic, and we have*

$$\operatorname{ind} \mathcal{A}_1 - \operatorname{ind} \mathcal{A}_0 = \operatorname{ind} \mathcal{R}.$$

Proof The result is a consequence of $\operatorname{ind} \mathcal{A}_1 - \operatorname{ind} \mathcal{A}_0 = \operatorname{ind} \mathcal{A}_1 \mathcal{P}_0 = \operatorname{ind} \tilde{\mathcal{A}}_1 \tilde{\mathcal{P}}_0 = \operatorname{ind} \mathcal{R}$. □

7 Operators Without the Transmission Property

In Sect. 3 we studied operators with the transmission property on a manifold X with boundary. Those are the background of Boutet de Monvel's calculus on X, including ellipticity in the Shapiro-Lopatinskii or global projection sense. However, if we replace (26) by

$$\{\tilde{A}|_{\operatorname{int} X} : \tilde{A} \in L^\mu_{\mathrm{cl}}(2X; \tilde{E}, \tilde{F})\} \tag{112}$$

then the transmission property is violated in general, see [32], and a priori it is by no means clear how to organize a calculus of boundary value problems with ellipticity, parametrices, etc. It will be more natural to start from

$$\begin{aligned} L^\mu_{\mathrm{cl}}(X; E, F)_{\mathrm{smooth}} := \{&A \in L^\mu_{\mathrm{cl}}(\operatorname{int} X; E, F) : \\ &A = \tilde{A}|_{\operatorname{int} X} + C, \tilde{A} \in L^\mu_{\mathrm{cl}}(2X; \tilde{E}, \tilde{F}), C \in L^{-\infty}(\operatorname{int} X; E, F)\} \end{aligned} \tag{113}$$

rather than (112) and to single out suitable operator conventions which specify the admitted spaces of smoothing operators. Such a program is voluminous, and we only sketch a few ideas here. It turns out that different approaches lead to more or less the same answers, namely, that essentially the arising operator theories are special cases of the edge algebra. This already tells us that Shapiro-Lopatinskii or global projection ellipticity in boundary value problems without the transmission property are of a similar structure as corresponding ellipticities in the edge calculus, outlined in Sect. 6. The compelling aspect lies in the fact that in the case of a manifold with smooth boundary, interpreted as a manifold with edge (which is just the boundary), there is a rich variety of interesting subalgebras of the general edge algebra, and analytic tools in such substructures reveal more explicit information than in the general case, cf. [7]. One of the unexpected phenomena, first observed by Eskin [8] on the half axis, is that Fourier based zero order pseudo-differential operators which are realized in truncated form

$$\mathrm{r}^+\mathrm{Op}(a)\mathrm{r}^+ \quad \text{for} \quad a(t,\tau) \in S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \mathbb{R}) \tag{114}$$

can be expressed in terms of Mellin operators with meromorphic symbols, modulo some remainders which are under control. This structure has been deepened and applied in the analysis on manifolds with conical singularities, see, for instance, the monograph [22]. Moreover, truncations as in (112) give rise to operators in the edge calculus. This is the main topic of a joint article [26] with Seiler. In that sense we have Shapiro-Lopatinskii elliptic boundary problems inherited from the edge algebra. Boundary problems in the global projection set-up have been studied in another paper [27]. In this connection it turned out that it is also convenient to produce substructures of the edge calculus on a manifold with boundary by directly organizing edge quantizations via edge-degenerate symbols with some extra properties close to the boundary.

In order to illustrate such properties we consider the case $X := \overline{\mathbb{R}}_+ \times \Omega,\ \Omega \subseteq \mathbb{R}^q$ open, $q = \dim \partial X$. Then

$$\begin{aligned} &L^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Omega)_{\mathrm{smooth}} \\ &= \{\mathrm{Op}_{r,y}(a) + C : a(r,y,\rho,\eta) \in S^{\mu}_{\mathrm{cl}}((\overline{\mathbb{R}}_+ \times \Omega) \times \mathbb{R}^{1+q}_{\rho,\eta}), C \in L^{-\infty}(\mathbb{R}_+ \times \Omega)\}. \end{aligned}$$

Theorem 7.1 *Let us fix* $\mu, \gamma \in \mathbb{R}$, *and set* $\boldsymbol{g} := (\gamma, \gamma-\mu, (-\infty, 0])$. *Then for every* $A \in L^{\mu}_{\mathrm{cl}}(X; E, F)_{\mathrm{smooth}}$ *there exists a* $C_\gamma \in L^{-\infty}(\mathrm{int}\, X; E, F)$ *such that*

$$A - C_\gamma \in L^{\mu}(X, \boldsymbol{g}; E, F), \tag{115}$$

cf. notation (74) *for* $\boldsymbol{v} = (E, F)$.

Proof For simplicity we assume $E = F = \mathbb{C}$; the arguments in the general case are completely analogous. It suffices to consider the case $A \in L^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Omega)_{\mathrm{smooth}}$, and we may assume $A := \mathrm{Op}_{r,y}(a)$ for some $a(r,y,\rho,\eta) \in S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{R}^{1+q}_{\rho,\eta})$. The dependence of the symbol a on y does not affect the arguments; so we assume $a(r,\rho,\eta) \in S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \mathbb{R}^{1+q}_{\rho,\eta})$. Let $\chi(\rho,\eta)$ be an excision function, and write a as an asymptotic expansion

$$a(r,\rho,\eta) \sim \sum_{j=0}^{\infty} \chi(\rho,\eta) a_{(\mu-j)}(r,\rho,\eta)$$

in $S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \mathbb{R}^{1+q}_{\rho,\eta})$ where $a_{(\mu-j)}$ is the homogeneous component of a of order $\mu - j$. This allows us to write

$$a_{(\mu-j)}(r,\rho,\eta) = r^{-\mu}(r^j a_{(\mu-j)}(r, r\rho, r\eta)) \quad \text{for} \ \ r > 0. \tag{116}$$

The functions $\tilde{p}_{(\mu-j)}(r,\tilde{\rho},\tilde{\eta}) := r^j a_{(\mu-j)}(r,\tilde{\rho},\tilde{\eta})$ are homogeneous in $(\tilde{\rho},\tilde{\eta}) \neq 0$ and smooth up to $r = 0$, and we can form an asymptotic expansion

$$\tilde{p}(r,\tilde{\rho},\tilde{\eta}) \sim \sum_{j=0}^{\infty} \chi(\tilde{\rho},\tilde{\eta})\tilde{p}_{(\mu-j)}(r,\tilde{\rho},\tilde{\eta})$$

in $S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})$. We have

$$p(r,\rho,\eta) := \tilde{p}(r, r\rho, r\eta) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}_+ \times \mathbb{R}^{1+q}_{\rho,\eta})$$

and

$$a(r,\rho,\eta) = r^{-\mu} p(r,\rho,\eta) \mod S^{-\infty}(\mathbb{R}_+ \times \mathbb{R}^{1+q}_{\rho,\eta}).$$

In fact, first we have

$$a(r,\rho,\eta) - \sum_{j=0}^{N} \chi(\rho,\eta)a_{(\mu-j)}(r,\rho,\eta) \in S^{-(N+1)}_{\mathrm{cl}}(\mathbb{R}_+ \times \mathbb{R}^{1+q}_{\rho,\eta}), \tag{117}$$

and, similarly,

$$\tilde{p}(r,\tilde{\rho},\tilde{\eta}) - \sum_{j=0}^{N} \chi(\tilde{\rho},\tilde{\eta})\tilde{p}_{(\mu-j)}(r,\tilde{\rho},\tilde{\eta}) \in S^{-(N+1)}(\mathbb{R}_+ \times \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}}). \tag{118}$$

This entails

$$r^{-\mu}\tilde{p}(r, r\rho, r\eta) - r^{-\mu} \sum_{j=0}^{N} \chi(r\rho, r\eta)\tilde{p}_{(\mu-j)}(r, r\rho, r\eta) \in S^{-(N+1)}(\mathbb{R}_+ \times \mathbb{R}^{1+q}_{\rho,\eta}). \tag{119}$$

From (117) and (119) it follows that $a(r,\rho,\eta) - r^{-\mu}\tilde{p}(r, r\rho, r\eta) \in S^{-(N+1)}(\mathbb{R}_+ \times \mathbb{R}^{1+q}_{\rho,\eta})$ since

$$\chi(\rho,\eta)a_{(\mu-j)}(r,\rho,\eta) - r^{-\mu}\chi(r\rho, r\eta)\tilde{p}_{(\mu-j)}(r, r\rho, r\eta) \in S^{-\infty}(\mathbb{R}_+ \times \mathbb{R}^{1+q}_{\rho,\eta}),$$

cf. relation (116). Thus we obtain

$$\mathrm{Op}_{r,y}(a) = r^{-\mu}\mathrm{Op}_{r,y}(p) \mod L^{-\infty}(\mathbb{R}_+ \times \Omega). \tag{120}$$

From the Mellin quantization of the edge calculus, cf. [23, Theorem 3.2.7], it follows that there is an $\tilde{h}(r,z,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+, M^\mu_{\mathcal{O}}(\mathbb{R}^q_{\tilde{\eta}}))$ such that for $h(r,z,\eta) := \tilde{h}(r,z,r\eta)$ we have

$$\mathrm{Op}_{r,y}(p)(\eta) = \mathrm{Op}_y \mathrm{op}^\beta_M(h)(\eta) \mod L^{-\infty}(\mathbb{R}_+ \times \Omega) \tag{121}$$

for any real β. We now obtain that $f(\eta) := r^{-\mu}\omega \mathrm{op}^\gamma_M(h)(\eta)\omega'$ is an edge amplitude function for any choice of cut-off functions ω, ω' and that

$$r^{-\mu}\omega \mathrm{Op}_{r,y}(p)\omega' = \mathrm{Op}_y(f) \mod L^{-\infty}(\mathbb{R}_+ \times \Omega).$$

This gives us altogether a $C_\gamma \in L^{-\infty}(\mathbb{R}_+ \times \Omega)$ such that

$$\omega \mathrm{Op}_{r,y}(a)\omega' - C_\gamma \in L^\mu(\overline{\mathbb{R}}_+ \times \Omega, \boldsymbol{g}).$$

Since $\mathrm{Op}_{r,y}(a) = \omega \mathrm{Op}_{r,y}(a)\omega' + (1-\omega)\mathrm{Op}_{r,y}(a)(1-\omega'') \mod L^{-\infty}(\mathbb{R}_+ \times \Omega)$ for cut-off functions $\omega'' \prec \omega \prec \omega'$ we finally obtain $\mathrm{Op}_{r,y}(a) - C_\gamma \in L^\mu(\overline{\mathbb{R}}_+ \times \Omega, \boldsymbol{g})$ for a suitable $C_\gamma \in L^{-\infty}(\mathbb{R}_+ \times \Omega)$. □

Remark 7.2 By definition we have $L^\mu_{\mathrm{cl}}(X; E, F)_{\mathrm{smooth}} \subset L^\mu_{\mathrm{cl}}(\mathrm{int}\, X; E, F)$ and we first interpret the operators A in that space as continuous operators

$$A : C^\infty_0(\mathrm{int}\, X, E) \to C^\infty(\mathrm{int}\, X, F).$$

Because of Theorem 7.1 any choice of C_γ represents an edge quantisation of A depending on the weight γ. According to the results of the edge calculus we therefore have extensions as continuous operators

$$A - C_\gamma : H^{s,\gamma}(X, E) \to H^{s-\mu,\gamma-\mu}(X, F)$$

between weighted edge spaces for all $s \in \mathbb{R}$ (when X is compact, cf. the formula (75), otherwise between corresponding comp/loc-spaces).

Relation (116) suggests to introduce the following space of edge-degenerate symbols:

Definition 7.3 Let $S^\mu_{\mathrm{cl}}(\mathbb{R}_+ \times \Omega \times \mathbb{R}^{1+q}_{\rho,\eta})_{\mathrm{smooth}}$ denote the set of all $p(r,y,\rho,\eta) \in S^\mu_{\mathrm{cl}}(\mathbb{R}_+ \times \Omega \times \mathbb{R}^{1+q}_{\rho,\eta})$ which are of the form

$$p(r,y,\rho,\eta) = \tilde{p}(r,y,r\rho,r\eta) \quad \text{for some } \tilde{p}(r,y,\tilde{\rho},\tilde{\eta}) \in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})$$

such that the homogeneous components $\tilde{p}_{(\mu-j)}(r,y,\tilde{\rho},\tilde{\eta}), j \in \mathbb{N}$, have the properties

$$\tilde{p}_{(\mu-j)}(r,y,\tilde{\rho},\tilde{\eta}) = r^j \tilde{\tilde{p}}_{(\mu-j)}(r,y,\tilde{\rho},\tilde{\eta})$$

and

$$\tilde{\tilde{p}}_{(\mu-j)}(r, y, \tilde{\rho}, \tilde{\eta}) \in S^{(\mu-j)}(\overline{\mathbb{R}}_+ \times \Omega \times (\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}} \setminus \{0\})).$$

Proposition 7.4 *For every $a(r, y, \rho, \eta) \in S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{R}^{1+q}_{\rho,\eta})$ there exists a*

$$p(r, y, \rho, \eta) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}_+ \times \Omega \times \mathbb{R}^{1+q}_{\rho,\eta})_{\mathrm{smooth}}$$

satisfying the relation

$$a(r, y, \rho, \eta) = r^{-\mu} p(r, y, \rho, \eta) \in S^{-\infty}(\mathbb{R}_+ \times \Omega \times \mathbb{R}^{1+q}_{\rho,\eta}). \tag{122}$$

Conversely for every $p(r, y, \rho, \eta) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}_+ \times \Omega \times \mathbb{R}^{1+q}_{\rho,\eta})_{\mathrm{smooth}}$ there exists an $a(r, y, \rho, \eta) \in S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{R}^{1+q}_{\rho,\eta})$ such that (122) holds.

Proof The first part of Proposition 7.4 is contained in the proof of Theorem 7.1. However, the relation between $r^{-\mu}p(r, y, \rho, \eta)$ and $a(r, y, \rho, \eta)$ can be established the other way around, using asymptotic summations in $S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{R}^{1+q}_{\rho,\eta})$. □

Let

$$L^{\mu}_{\mathrm{cl}}(\mathbb{R}_+ \times \Omega)_{\mathrm{smooth}} := \{\mathrm{Op}_{r,y}(b) + C : \\ b(r, y, \rho, \eta) \in r^{-\mu} S^{\mu}_{\mathrm{cl}}(\mathbb{R}_+ \times \Omega \times \mathbb{R}^{1+q}_{\rho,\eta})_{\mathrm{smooth}}, C \in L^{-\infty}(\mathbb{R}_+ \times \Omega)\}.$$

By virtue of Propsition 7.4 every $A \in L^{\mu}_{\mathrm{cl}}(\mathbb{R}_+ \times \Omega)_{\mathrm{smooth}}$ has the form

$$A = \mathrm{Op}_{r,y}(a) + C \tag{123}$$

for an $a(r, y, \rho, \eta) \in S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{R}^{1+q}_{\rho,\eta})$ and $C \in L^{-\infty}(\mathbb{R}_+ \times \Omega)$. Operators of this kind are interpreted as maps

$$A : C^{\infty}_0(\mathbb{R}_+ \times \Omega) \to C^{\infty}(\mathbb{R}_+ \times \Omega).$$

Every $A \in L^{\mu}_{\mathrm{cl}}(\mathbb{R}_+ \times \Omega)_{\mathrm{smooth}}$ can be written as $A = A_0 + C$ for a properly supported $A_0 \in L^{\mu}_{\mathrm{cl}}(\mathbb{R}_+ \times \Omega)_{\mathrm{smooth}}$ and a $C \in L^{-\infty}(\mathbb{R}_+ \times \Omega)$.

An $A \in L^{\mu}_{\mathrm{cl}}(\mathbb{R}_+ \times \Omega)_{\mathrm{smooth}}$ is called elliptic if the symbol $a(r, y, \rho, \eta) \in S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Omega \times \mathbb{R}^{1+q}_{\rho,\eta})$ in the representation (123) is elliptic in the standard sense, more precisely, $a_{(\mu)}(r, y, \rho, \xi) \neq 0$ for all $(r, y, \rho, \xi) \in \overline{\mathbb{R}}_+ \times \Omega \times (\mathbb{R}^{1+q}_{\rho,\eta} \setminus \{0\})$.

Corollary 7.5 *Let $A \in L^{\mu}_{\mathrm{cl}}(\mathbb{R}_+ \times \Omega)_{\mathrm{smooth}}$, $B \in L^{\nu}_{\mathrm{cl}}(\mathbb{R}_+ \times \Omega)_{\mathrm{smooth}}$, and let A or B be properly supported. Then we have $AB \in A \in L^{\mu+\nu}_{\mathrm{cl}}(\mathbb{R}_+ \times \Omega)_{\mathrm{smooth}}$. Moreover, an elliptic $A \in L^{\mu}_{\mathrm{cl}}(\mathbb{R}_+ \times \Omega)_{\mathrm{smooth}}$ has a properly supported parametrix $P \in L^{-\mu}_{\mathrm{cl}}(\mathbb{R}_+ \times \Omega)_{\mathrm{smooth}}$, where $AP - 1,\ PA - 1 \in L^{-\infty}(\mathbb{R}_+ \times \Omega)$.*

It is now clear that in the case of a manifold X with boundary the edge calculus contains substructures with upper left corners belonging to

$$L^{\mu}_{\mathrm{cl}}(X; E, F)_{\mathrm{smooth}} \tag{124}$$

modulo $L^{\mu}_{\mathrm{M+G}}(X; \boldsymbol{g}; E, F)$, where $L^{\mu}_{\mathrm{cl}}(X; E, F)_{\mathrm{smooth}}$ means a global analogue of the above-mentioned $L^{\mu}_{\mathrm{cl}}(\mathbb{R}_+ \times \Omega)_{\mathrm{smooth}}$, taking into account also bundles $E, F \in \mathrm{Vect}(X)$. The remaining entries, especially those of trace and potential type, are as in the general edge calculus. Summing up, on a manifold with boundary we have a large variety of specific 2×2-block matrix algebras, representing BVPs, here for operators without transmission property at the boundary.

8 Elliptic Complexes and Other New Challenges

In classical ellipticity it is natural not only to consider single operators but also complexes

$$0 \to H_0 \to \cdots \to H_i \to H_{i+1} \to \cdots \to H_{N+1} \to 0 \tag{125}$$

of operators $A_i : H_i \to H_{i+1}$. In simple cases those may be pseudo-differential operators on a smooth closed manifold acting in standard Sobolev spaces of distributional sections of vector bundles. It is also interesting to consider complexes of BVPs on a smooth manifold with boundary, see [16, Section 3.2.3], concerning complexes in Boutet de Monvel's algebra, with (an analogue of) Shapiro-Lopatinskii elliptic boundary conditions. In a recent paper [29] jointly with Seiler we studied elliptic complexes of operators with the transmission property on a compact smooth manifold X with boundary Y with global projection conditions. Similarly as in [24] a new information is that every elliptic complex of differential operators of order μ, i.e., with an exact complex of principal symbols, admits global projection conditions. Those turn it to a Fredholm complex

$$0 \to \begin{matrix} H_0 \\ \oplus \\ H'_0 \end{matrix} \to \cdots \to \begin{matrix} H_i \\ \oplus \\ H'_i \end{matrix} \to \begin{matrix} H_{i+1} \\ \oplus \\ H'_{i+1} \end{matrix} \to \cdots \to \begin{matrix} H_{N+1} \\ \oplus \\ H'_{N+1} \end{matrix} \to 0 \tag{126}$$

of 2×2 matrices

$$\mathcal{A}_i : \begin{pmatrix} A_i & K_i \\ T_i & Q_i \end{pmatrix} \in \mathcal{T}^{\mu, d_i}(X; \boldsymbol{l_i}), \tag{127}$$

cf. Definition 5.16, for $\boldsymbol{l_i} := (E_i, E_{i+1}; \mathbb{L}_i, \mathbb{L}_{i+1})$, $\mathbb{L}_i$, $\mathbb{L}_{i+1} \in \mathbb{P}(Y)$, cf. Definition 2.4, and spaces $H_i = H^{s-i\mu}(X, E_i)$, $E_i \in \mathrm{Vect}(X)$ and $H'_i = H^{s-i\mu}(Y, \mathbb{L}_i)$, $\mathbb{L}_i \in \mathbb{P}(Y)$, cf. formula (4). If an analogue of the Atiyah-Bott obstruction [1] vanishes then there exist Shapiro-Lopatinskii elliptic boundary conditions. The latter is the case for the de-Rham complex; this has been known for a long time, see Dynin [6], while there are interesting complexes such as the Dolbeault complex where the Atiyah-Bott obstruction does not vanish.

The program of extending different kind of ellipticity both for complexes and for the special case of single operators also makes sense on manifolds with singularities, e.g., of conical or edge type, cf. [19] for the case of cones with a smooth closed base. For instance, it seems to be difficult to extend the approach of [24] to the case of manifolds with non-smooth boundary, e.g., with conical or edge singularities. Another difficulty is to establish a parameter-dependent variant of elliptic operators with global projection conditions.

Ellipticity of operators $\mathcal{A}$ on a corner manifold M of singularity order $k \in \mathbb{N}$ (with $k = 0$ indicating smoothness, $k = 1$ conical or edge singularities) is connected with a principal symbolic hierarchy

$$\sigma(\mathcal{A}) = (\sigma_0(\mathcal{A}), \ldots, \sigma_k(\mathcal{A})),$$

cf. [25]. In this framework M has a stratification

$$s(M) = (s_0(M), \ldots, s_k(M)),$$

where the strata $s_j(M)$, $j = 1, \ldots, k-1$, (and also $s_k(M)$ when $\dim s_k(M) > 0$) play the role of higher edges. It is natural for the Fredholm property of operators to pose along $s_j(M)$ Shapiro-Lopatinkii or global projection conditions. The corresponding operators furnish the entries of the block matrix operator $\mathcal{A}$, together with the interior operator A contained in the upper left corner. Their nature is expected to depend on the behaviour of higher edge symbols, more precisely, on whether or not higher analogues of the Atiyah-Bott obstruction vanish. Thus on a manifold M of singularity order k we can expect a corresponding hierarchy of topological obstructions for the existence of Shapiro-Lopatinskii elliptic edge conditions, while in case of non-vanishing of those obstructions we have to expect global projection conditions, in order that the corresponding block matrix operators $\mathcal{A}$ are Fredholm and have parametrices in the respective operator algebras.

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Multilinear Localization Operators Associated to Quaternion Fourier Transforms

Guangsheng Ma and Jiman Zhao

Abstract In this article, we study the multilinear localization operator $L^F_{\varphi,\psi}f$ associated to quaternion Fourier transform(QFT). If F satisfies some conditions, we prove this kind of multilinear operator is bounded on $L^2(\mathbb{R}^2;\mathbb{H}) \times L^2(\mathbb{R}^2;\mathbb{H}) \times L^2(\mathbb{R}^2;\mathbb{H})$. Further more, if we fix F and φ, then $L^F_{\varphi,\psi}f$ is a bilinear compact operator.

Keywords Multilinear • Localization operator • Quaternion Fourier transform

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1 Introduction

Localization operators have been studied in many different settings by Daubechies [7], Cordero and Nicola [5], Cordero and Gröchenig [6], Boggiatto, Cordero and Gröchenig [2], Janson and Peetre [14], Peng [18, 19], Peng and Wong [20], Wong [23–25], and Zhao [26], etc. As a paracommutator, the localization operator L^F_{φ} has been studied by Janson, Peetre, Peng, Wong [14, 18–20]. And in [25], Wong has proved if F is a suitable function, L^F_{φ} is a Fourier multiplier. In the respect of time-frequency analysis (see [11] for more details), each localization operator $L^F_{\varphi,\psi}f$, as the windowed Fourier transform, is defined by a symbol function and two window functions. If we fix F and ψ, $L^F_{\varphi,\psi}f$ can be viewed as a bilinear operator. In [9], Fernández, Galbis and Martínez considered multilinear multipliers

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associated to localization operators, and obtained the boundedness and compactness of the multilinear localization operators. Inspired by [9], we define the localization operator associated to quaternion Fourier transform, and study the boundedness and compactness of it. Due to the lack of traditional commutative, proofs become more complicated.

Firstly, let's recall some related basic definitions and properties.

Quaternions As general notation (more details and some proofs of propositions in this section can be found in [4, 10, 12]), if a is a quaternion, then a can be represented as $a = a_0 + a_1 i + a_2 j + a_3 k$, where i, j, k are basis elements, satisfied $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$, $ki = j$ and anti-commutative, a_0, a_1, a_2, a_3 are real numbers. We call a a pure quaternion when $a_0 = 0$. We define $|a| = \sqrt{a_0{}^2 + a_1{}^2 + a_2{}^2 + a_3{}^2}$ as the norm of a, and denote the conjugate number of a by $\bar{a}$. It's easy to verify: $\overline{ab} = \bar{b}\bar{a}$, for any a, b in quaternion.

Quaternion-Valued Function Spaces The following spaces will be used in this article:

$$L^p(\mathbb{R}^2; \mathbb{H}) = \{f \mid \|f(x)\|_p = (\int_{\mathbb{R}^2} (\sum_{j=0}^{3} |f_j(x)|^2)^{\frac{p}{2}} dx)^{1/p} < \infty\},$$

where $f = f_0 + f_1 i + f_2 j + f_3 k$, f_0, f_1, f_2, f_3 are real-valued functions, $1 \leq p < \infty$. Obviously, $f \in L^2(\mathbb{R}^2; \mathbb{H})$ if and only if $f_0 \in L^2(\mathbb{R}^2)$, $f_1 \in L^2(\mathbb{R}^2)$, $f_2 \in L^2(\mathbb{R}^2)$, $f_3 \in L^2(\mathbb{R}^2)$. Further more, if f_0,f_1,f_2,f_3 are real-valued Schwartz functions, we call f as quaternion-valued Schwartz function, denote by $f \in \mathbb{S}(\mathbb{R}^{\mathbf{2}}; \mathbb{H})$.

$$L^\infty(\mathbb{R}^2; \mathbb{H}) = \{f \mid \|f(x)\|_\infty = \inf_{\mu(E)=0} \sup_{x \in \mathbb{R}^2 \setminus E} (\sum_{j=0}^{3} |f_j(x)|^2)^{\frac{1}{2}} < \infty\},$$

where E is a subset of $\mathbb{R}^2$, and $\mu(E)$ is the Lebesgue measure of E.

$$L^{\infty,1}(\mathbb{R}^2 \times \mathbb{R}^2; \mathbb{H}) = \{f \mid \int_{\mathbb{R}^2} ||f(\cdot, x)||_\infty dx < \infty\}.$$

Remark In the following, the domain of quaternion-valued function is defined on $\mathbb{R}^2$. In fact, almost all the following propositions can be established with the domain in $\mathbb{R}^1$.

Although commutative law is not always right for general quaternions, such as $e^i \cdot e^j \neq e^{i+j}$, the following identity related to commutative law is still right:

Proposition 1.1 *$e^{\mu a} \cdot e^{\mu b} = e^{\mu(a+b)} = e^{\mu b} \cdot e^{\mu a}$, where a, b are reals, μ is an unit pure quaternion, $e^{a\mu} = cosa + \mu sina$.*

Now, we will introduce some quaternionic version of inequalities (more details can be found in [1, 4, 22]):

Proposition 1.2 *Cauchy inequality: If* $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$*, then*
$\int_{\mathbb{R}^2} |fg|dx \leq ||f||_2||g||_2$.
Young inequality: If $f \in L^1(\mathbb{R}^2; \mathbb{H})$*,* $g \in L^2(\mathbb{R}^2; \mathbb{H})$*, then*
$||f * g||_2 = || \int_{\mathbb{R}^2} f(t)g(x-t)dt||_2 \leq ||f||_1||g||_2$.
We also have the following inequality: If μ *is an unit pure quaternion, then*

$$|e^{a\mu} - 1| = |cosa + \ \mu sina - 1| \leq |2sin\frac{a}{2}(sin\frac{a}{2} + \mu cos\frac{a}{2})| \leq |a|. \tag{1}$$

In order to study Wigner transforms and localization operators, we first recall some properties about quaternion Fourier transform. As well known, there are more than one kind of quaternion Fourier transform, such as left-side QFT, two-side QFT, their definitions can be found in [3, 13, 17, 21]. In this paper, we use right-side QFT. The difference with traditional Fourier transform is that the element number i in the exponent is replaced by an unit pure quaternion μ. In this paper, we choose $\mu = \frac{i+j+k}{\sqrt{3}}$ for its effect in RGB image processing, which corresponds to the luminance, orgrayline, axis of the unit RGB color cube (which was introduced in [15]). By calculation, 1, $\mu = \frac{i+j+k}{\sqrt{3}}$, $\mu' = \frac{i-j}{\sqrt{2}}$, $\mu'' = \frac{i+j-2k}{\sqrt{6}}$ form a basis of quaternions, they satisfy anti-commutative, such as $\mu\mu' = -\mu'\mu$, $\mu'\mu'' = -\mu''\mu'$, and $\mu^2 = \mu'^2 = \mu''^2 = -1$.

Depending on the fixed μ, for each quaternion-valued function f, it can be written as: $f(x) = f_0(x) + f_1(x)\mu + f_2(x)\mu' + f_3(x)\mu''$, where $f_i(x)$ are real-valued functions, $i = 0, 1, 2, 3$. Moreover, we define that $f_\parallel = f_0 + f_1\mu$, $f_\perp = f_2\mu' + f_3\mu''$, f can be resolved into $f_\parallel$ and $f_\perp$ (more details can be found in [15, p. 1943]). It's clear that for any quaternion f, f can be uniquely decomposed into these two parts.

Remark As mentioned above the parallel and perpendicular decomposition holds for any unit pure quaternion μ. So the results of this paper also hold for any unit pure quaternion μ.

Definition 1.3 Right-side quternion Fourier transform of $f \in \mathbb{S}(\mathbb{R}^2)$ is defined by
$\widehat{f}(x) = \int_{\mathbb{R}^2} f(t)e^{-2\pi\mu(x\cdot t)}dt$

Using Proposition 1.1, we can get the Fourier inversion:

$$f(y) = \int_{\mathbb{R}^2} \widehat{f}(x)e^{2\pi\mu(y\cdot x)}dx \tag{2}$$

By direct calculation, we can get Propositions 1.4 and 1.5.

Proposition 1.4 $\widehat{f_\parallel} = \hat{f}_\parallel, \widehat{f_\perp} = \hat{f}_\perp$

Proposition 1.5 $f_\perp e^{2\pi\mu a} = e^{-2\pi\mu a}f_\perp, \widehat{f_\perp}e^{2\pi\mu a} = e^{-2\pi\mu a}\widehat{f_\perp}$*, where* $a \in \mathbb{R}$.

Proposition 1.6 ([3]) $\widehat{f * \varphi}(y) = \widehat{f}(y)\widehat{\varphi_\parallel}(y) + \widehat{f}(-y)\widehat{\varphi_\perp}(y)$

2 Quaternionic Multilinear Localization Operators

Like [9], in this section, we define quaternionic multilinear localization operators and study the properties of their related Wigner transform. Each localization operator $L^F_{\varphi,\psi}f$ is depend on a symbol function F, an analysis window functions φ, and a synthesis window ψ. Fixed F, it can be viewed as a multilinear operator $L^F_{\cdot,\cdot}\cdot$.

Definition 2.1 Let f, $\varphi \in \mathbb{S}(\mathbb{R}^{\mathbf{2}}; \mathbb{H})$, then the Wigner transform is defined by

$$V_\varphi f(x, \omega) = \int_{\mathbb{R}^2} f(y)\overline{\varphi}(y-x)e^{-2\pi\mu(\omega\cdot y)}dy.$$

Definition 2.2 Let φ, ψ, $f \in \mathbb{S}(\mathbb{R}^{\mathbf{2}}; \mathbb{H})$, $F \in \mathbb{S}(\mathbb{R}^{\mathbf{2}} \times \mathbb{R}^{2}; \mathbb{H})$, then the localization operator is defined by

$$L^F_{\varphi,\psi}f(t) = \iint_{\mathbb{R}^2\times\mathbb{R}^2} F(x, \omega)V_\varphi f(x, \omega)e^{-2\pi\mu(\omega\cdot t)}\psi(t-x)dxd\omega.$$

Proposition 2.3 *If* $\varphi, f \in \mathbb{S}(\mathbb{R}^{\mathbf{2}}; \mathbb{H})$*, the Wigner transform* $V_\varphi f(x, \omega)$ *can be written as:* $V_\varphi f(x, \omega) = \int_{\mathbb{R}^2} \widehat{f}(y)\overline{\widehat{\varphi_\perp}(y+\omega)e^{-2\pi\mu[x\cdot(y+\omega)]}}dy + \int_{\mathbb{R}^2} \widehat{f}(y)\overline{\widehat{\varphi_\|}(y-\omega)}$ $e^{2\pi\mu[x\cdot(y-\omega)]}dy$.

Proof

$$\begin{aligned} V_\varphi f(x, \omega) &= \int_{\mathbb{R}^2} f(t)(\overline{\varphi_\|}(t-x) + \overline{\varphi_\perp}(t-x))e^{-2\pi\mu(\omega\cdot t)}dt \\ &= V_{\varphi_\|}f(x, \omega) + V_{\varphi_\perp}f(x, \omega) \end{aligned}$$

By Proposition 1.1, (2) and Fubini theorem, we can write $V_{\varphi_\|}f(x, \omega)$ as following:

$$\begin{aligned} V_{\varphi_\|}f(x, \omega) &= \int_{\mathbb{R}^2} f(t)\overline{\varphi_\|}(t-x)e^{-2\pi\mu(\omega\cdot t)}dt \\ &= \int_{\mathbb{R}^2} \widehat{f}(y)\overline{\int_{\mathbb{R}^2} e^{2\pi\mu(\omega\cdot t)}\varphi_\|(t-x)e^{-2\pi\mu(y\cdot t)}dtdy} \\ &= \int_{\mathbb{R}^2} \widehat{f}(y)\overline{\int_{\mathbb{R}^2} \varphi_\|(t')e^{-2\pi\mu[(y-\omega)\cdot t']}dt'}e^{2\pi\mu[x\cdot(y-\omega)]}dy \\ &= \int_{\mathbb{R}^2} \widehat{f}(y)\overline{\widehat{\varphi_\|}(y-\omega)}e^{2\pi\mu[x\cdot(y-\omega)]}dy \end{aligned} \tag{3}$$

By Proposition 1.5, we can find the alternative expression of the $V_{\varphi_\perp}f(x,\omega)$:

$$
\begin{aligned}
V_{\varphi_\perp}f(x,\omega) &= \int_{\mathbb{R}^2} f(t)\overline{\varphi_\perp}(t-x)e^{-2\pi\mu(\omega\cdot t)}dt \\
&= \int_{\mathbb{R}^2} \widehat{f}(y)\overline{\int_{\mathbb{R}^2} e^{2\pi\mu(\omega\cdot t)}\varphi_\perp(t-x)e^{-2\pi\mu(y\cdot t)}dtdy} \\
&= \int_{\mathbb{R}^2} \widehat{f}(y)\overline{\int_{\mathbb{R}^2} \varphi_\perp(t')^{-2\pi\mu[(\omega+y)\cdot t']}e^{-2\pi\mu[(\omega+y)\cdot x]}dt'dy} \\
&= \int_{\mathbb{R}^2} \widehat{f}(y)\overline{\widehat{\varphi_\perp}(y+\omega)}e^{-2\pi\mu[x\cdot(y+\omega)]}dy
\end{aligned}
\tag{4}
$$

Combine (3) and (4), we finish the proof. □

In addition, f can be recovered from Wigner transform by the following:

Proposition 2.4 *If $(\overline{\widehat{\varphi_\parallel}(y)}+\overline{\widehat{\varphi_\perp}(-y)})e^{2\pi\mu(y\cdot x)} \neq 0$ for alomst every x, then*

$$
f(x) = \widehat{V_\varphi}(y,-x)[(\overline{\widehat{\varphi_\parallel}(y)}+\overline{\widehat{\varphi_\perp}(-y)})e^{-2\pi\mu(y\cdot x)}]^{-1},
$$

where $[(\overline{\widehat{\varphi_\parallel}(y)}+\overline{\widehat{\varphi_\perp}(-y)})e^{-2\pi\mu(y\cdot x)}]^{-1}$ represents the inverse number of $[(\overline{\widehat{\varphi_\parallel}(y)}+\overline{\widehat{\varphi_\perp}(-y)})e^{-2\pi\mu(y\cdot x)}]$.

Proof

$$
\begin{aligned}
\widehat{V_\varphi f}(a,b) &= \int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} f(t)\overline{\varphi}(t-x)e^{-2\pi\mu(\omega\cdot t)}e^{-2\pi\mu(a\cdot x+b\cdot\omega)}dxdtdw \\
&= \int_{\mathbb{R}^2} \widehat{f}(a+w)\overline{\widehat{\varphi_\parallel}(a)}e^{-2\pi\mu(b\cdot\omega)}d\omega + \int_{\mathbb{R}^2} \widehat{f}(-a-w)\overline{\widehat{\varphi_\perp}(-a)}e^{-2\pi\mu(b\cdot\omega)}d\omega \\
&= \int_{\mathbb{R}^2} [\widehat{f}(a+w)\overline{\widehat{\varphi_\parallel}(a)}+\widehat{f}(-a-w)\overline{\widehat{\varphi_\perp}(-a)}]e^{-2\pi\mu(b\cdot\omega)}dw \\
&= f(-b)(\overline{\widehat{\varphi_\parallel}(a)}+\overline{\widehat{\varphi_\perp}(-a)})e^{2\pi\mu(a\cdot b)}
\end{aligned}
$$

□

Now, we prove the Fourier transform of localization operator can be written as following:

Theorem 2.5 *For any $\varphi,\psi,F,f \in \mathbb{S}(\mathbb{R}^{\mathbf{2}};\mathbb{H})$, we have*

$$
\begin{aligned}
\widehat{L^F_{\varphi,\psi}f(s)} &= \iint_{\mathbb{R}^2\times\mathbb{R}^2} (\int_{\mathbb{R}^2} F(x,\omega)\widehat{f}(y)e^{-2\pi\mu[(s-y)\cdot x]}dx)\widehat{\psi_\parallel}(s-\omega)\overline{\widehat{\varphi_\parallel}}(y-\omega)dyd\omega \\
&+ \iint_{\mathbb{R}^2\times\mathbb{R}^2} (\int_{\mathbb{R}^2} F(x,\omega)\widehat{f}(y)e^{2\pi\mu[(s+y)\cdot x]}dx)\widehat{\psi_\perp}(s+\omega)\widehat{\varphi_\parallel}(y-\omega)dyd\omega
\end{aligned}
$$

$$+ \iint_{\mathbb{R}^2\times\mathbb{R}^2} (\int_{\mathbb{R}^2} F(x,\omega)\widehat{f}(y)e^{2\pi\mu[(s+y)\cdot x]}dx)\overline{\widehat{\psi_{\parallel}}}(s-\omega)\overline{\widehat{\varphi_{\perp}}}(y+\omega)dyd\omega$$
$$+ \iint_{\mathbb{R}^2\times\mathbb{R}^2} (\int_{\mathbb{R}^2} F(x,\omega)\widehat{f}(y)e^{-2\pi\mu[(s-y)\cdot x]}dx)\widehat{\psi_{\perp}}(-s-\omega)\overline{\widehat{\varphi_{\perp}}}(y+\omega)dyd\omega \quad (5)$$

Proof We decompose φ, ψ as $\varphi = \varphi_{\parallel} + \varphi_{\perp}$, $\psi = \psi_{\parallel} + \psi_{\perp}$, so that

$$L^F_{\varphi,\psi}f(t) = L^F_{\varphi_{\parallel},\psi_{\parallel}}f(t) + L^F_{\varphi_{\parallel},\psi_{\perp}}f(t) + L^F_{\varphi_{\perp},\psi_{\parallel}}f(t) + L^F_{\varphi_{\perp},\psi_{\perp}}f(t).$$

Let us see how to deal with the four parts:

(i) $L^F_{\varphi_{\parallel},\psi_{\parallel}}f(t)$. By (3), Proposition 1.1 and Fubini theorem, we get

$$\begin{aligned} &L^F_{\varphi,\psi}f(t)\\ &= \iint_{\mathbb{R}^2\times\mathbb{R}^2} F(x,\omega)(\int_{\mathbb{R}^2} \widehat{f}(y)\overline{\widehat{\varphi_{\parallel}}(y-\omega)}e^{2\pi\mu[x\cdot(y-\omega)]}dy)e^{2\pi\mu(\omega\cdot t)}\psi_{\parallel}(t-x)dxd\omega\\ &= \iint_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} F(x,\omega)\widehat{f}(y)\psi_{\parallel}(t-x)e^{-2\pi\mu[(t-x)\cdot(y-\omega)]}\overline{\widehat{\varphi_{\parallel}}(y-\omega)}e^{2\pi\mu(y\cdot t)}dydxd\omega, \end{aligned}$$

by Fourier transform, we have

$$\begin{aligned} \widehat{L^F_{\varphi,\psi}f(s)} &= \iint_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} F(x,\omega)\widehat{f}(y)\psi_{\parallel}(t-x)e^{-2\pi\mu[(t-x)\cdot(y-\omega)]}\overline{\widehat{\varphi_{\parallel}}(y-\omega)}\\ &\times e^{2\pi\mu(y\cdot t)}e^{-2\pi\mu(s\cdot t)}dydxd\omega dt. \end{aligned}$$

It can be viewed as the Fourier transform of convolution of $F(x,\omega)\widehat{f}(y)$ and $\psi_{\parallel}(x)e^{-2\pi\mu(y-\omega)\cdot x}$ about x. Finally, we obtain that

$$\widehat{L^F_{\varphi,\psi}f(s)} = \iint_{\mathbb{R}^2\times\mathbb{R}^2} (\int_{\mathbb{R}^2} F(x,\omega)\widehat{f}(y)e^{-2\pi\mu[(s-y)\cdot x]}dx)\widehat{\psi_{\parallel}}(s-\omega)\overline{\widehat{\varphi_{\parallel}}(y-\omega)}d\omega dy.$$

(ii) $L^F_{\varphi_{\parallel},\psi_{\perp}}f(t)$. Similarly to (i) and combine with Proposition 1.6 we can get

$$\widehat{L^F_{\varphi,\psi}f(s)} = \iint_{\mathbb{R}^2\times\mathbb{R}^2} (\int_{\mathbb{R}^2} F(x,\omega)\widehat{f}(y)e^{2\pi\mu[(s+y)\cdot x]}dx)\widehat{\psi_{\perp}}(s+\omega)\widehat{\varphi_{\parallel}}(y-\omega)d\omega dy.$$

(iii) $L^F_{\varphi_{\perp},\psi_{\parallel}}f(t)$. Using Proposition 1.5 we have

$$\widehat{L^F_{\varphi,\psi}f(s)} = \iint_{\mathbb{R}^2\times\mathbb{R}^2} (\int_{\mathbb{R}^2} F(x,\omega)\widehat{f}(y)e^{2\pi\mu[(s+y)\cdot x]}dx)\overline{\widehat{\psi_{\parallel}}}(s-\omega)\overline{\widehat{\varphi_{\perp}}}(y+\omega)d\omega dy.$$

(iv) $L^F_{\varphi_\perp,\psi_\perp}f(t)$. Combine with Propositions 1.5 and 1.6, we obtain that

$$\widehat{L^F_{\varphi,\psi}f(s)} = \iint_{\mathbb{R}^2\times\mathbb{R}^2} (\int_{\mathbb{R}^2} F(x,\omega)\widehat{f}(y)e^{-2\pi\mu[(s-y)\cdot x]}dx)\widehat{\psi_\perp}(-s-\omega)\overline{\widehat{\varphi_\perp}(y+\omega)}d\omega dy.$$

In summary, we get (5). □

By Theorem 2.5, like in the reference [9], we can prove the boundedness of $L^F_{\varphi,\psi}f(t)$ if F satisfies some conditions. We denote

$$a(x,\omega) = \int_{\mathbb{R}^2} F(y,\omega)e^{-2\pi\mu(x\cdot y)}dy,$$

and $a(x,\omega)$ can also be viewed as partial Fourier transform of F about x.

Theorem 2.6 *If $F(x,\omega)$ is an even function about x, then we have*

$$\int_{\mathbb{R}^2} F(x,\omega)\widehat{f}(y)e^{-2\pi\mu[(s-y)\cdot x]}dx = a(s-y,\omega)\widehat{f}(y)$$

for any $f \in \mathbb{S}(\mathbb{R}^{\mathbf{2}};\mathbb{H})$.

Proof We have

$$\begin{aligned}
&\int_{\mathbb{R}^2} F(x,\omega)\widehat{f}(y)e^{-2\pi\mu[(s-y)\cdot x]}dx = \int_{\mathbb{R}^2} F(x,\omega)\widehat{f}(y)e^{-2\pi\mu[(s-y)\cdot x]}dx \\
&= \int_{\mathbb{R}^2} F(x,\omega)(\widehat{f_\parallel}(y)+\widehat{f_\perp}(y))e^{-2\pi\mu[(s-y)\cdot x]}dx \\
&= \int_{\mathbb{R}^2} F(x,\omega)e^{-2\pi\mu[(s-y)\cdot x]}\widehat{f_\parallel}(y)dx + \int_{\mathbb{R}^2} F(x,\omega)e^{2\pi\mu[(s-y)\cdot x]}\widehat{f_\perp}(y)dx \\
&= a(s-y,\omega)\widehat{f}(y)
\end{aligned}$$

□

Remark In the case that $F(x,\omega)$ is an even function about x, we can obtain the boundedness of $L^F_{\varphi,\psi}f(t)$. Indeed, combine with Theorem 2.6, (5) can be written as:

$$\begin{aligned}
\widehat{L^F_{\varphi,\psi}f(s)} = &\iint_{\mathbb{R}^2\times\mathbb{R}^2} a(s-y,\omega)\widehat{f}(y)\widehat{\psi_\parallel}(s-\omega)\overline{\widehat{\varphi_\parallel}}(y-\omega)dyd\omega \\
&+ \iint_{\mathbb{R}^2\times\mathbb{R}^2} a(-s-y,\omega)\widehat{f}(y)\widehat{\psi_\perp}(s+\omega)\widehat{\varphi_\parallel}(y-\omega)dyd\omega
\end{aligned}$$

$$+\iint_{\mathbb{R}^2\times\mathbb{R}^2} a(-s-y,\omega)\widehat{f}(y)\overline{\widehat{\psi_\parallel}}(s-\omega)\overline{\widehat{\varphi_\perp}}(y+\omega)dyd\omega$$
$$+\iint_{\mathbb{R}^2\times\mathbb{R}^2} a(s-y,\omega)\widehat{f}(y)\widehat{\psi_\perp}(-s-\omega)\overline{\widehat{\varphi_\perp}}(y+\omega)dyd\omega$$

Noting that right-side QFT, conjugate, opposition of independent variable all are L^2-norm preserving operations, for the first part of $\widehat{L^F_{\varphi,\psi}f(s)}$:

$$\begin{aligned}
&||\iint_{\mathbb{R}^2\times\mathbb{R}^2} a(s-y,\omega)\widehat{f}(y)\widehat{\psi_\parallel}(s-\omega)\overline{\widehat{\varphi_\parallel}}(y-\omega)dyd\omega||_2\\
&\le \|\widehat{f}\|_2\|\widehat{\varphi_\parallel}\|_2||\int_{\mathbb{R}^2} a(s-y,\omega)\widehat{\psi_\parallel}(s-\omega)d\omega||_2\\
&\le \|a\|_{L^{\infty,1}}\,\|\widehat{f}\|_2\|\widehat{\varphi_\parallel}\|_2\|\widehat{\psi_\parallel}\|_2\\
&\le \|a\|_{L^{\infty,1}}\,\|f\|_2\,\|\varphi\|_2\,\|\psi\|_2
\end{aligned}\tag{6}$$

Similarily, each of other parts' L^2-norm is less than or equal to

$$\|a\|_{L^{\infty,1}}\,\|f\|_2\,\|\varphi\|_2\,\|\psi\|_2\,.$$

Now, we obtain the boundedness of $L^F_{\varphi,\psi}f(t)$. Due to the norm preserving of QFT and $\mathbb{S}(\mathbb{R}^2;\mathbb{H})$ is dense in $L^2(\mathbb{R}^2;\mathbb{H})$, $L^F_{\varphi,\psi}f$ which is defined in $\mathbb{S}(\mathbb{R}^2;\mathbb{H})\times\mathbb{S}(\mathbb{R}^{\mathbf{2}};\mathbb{H})\times\mathbb{S}(\mathbb{R}^{\mathbf{2}};\mathbb{H})$ can be extend to a multilinear boundedness operator on $L^2(\mathbb{R}^2;\mathbb{H})\times L^2(\mathbb{R}^2;\mathbb{H})\times L^2(\mathbb{R}^2;\mathbb{H})$.

If F isn't an even function, we have a more general theorem about the boundedness of $L^F_{\varphi,\psi}f(t)$:

Theorem 2.7 *For fixed $a\in L^{\infty,1}(\mathbb{R}^2\times\mathbb{R}^2;\mathbb{H})$, $L^F_{\varphi,\psi}f$ can be extended as a bounded multilinear operator on quaternion-valued function: $L^F : L^2(\mathbb{R}^2;\mathbb{H})\times L^2(\mathbb{R}^2;\mathbb{H})\times L^2(\mathbb{R}^2;\mathbb{H})\to L^2(\mathbb{R}^2;\mathbb{H})$ $L^F:(\varphi,\psi,f)\to L^F_{\varphi,\psi}f$ where φ, ψ, $f\in L^2(\mathbb{R}^2;\mathbb{H})$.*

Proof In order to avoid technicalities, we first assume that φ, ψ, $f\in\mathbb{S}(\mathbb{R}^2;\mathbb{H})$. We decompose F into $f_\parallel+f_\perp$. For example, in Theorem 2.5 (i):

$$\begin{aligned}
L^F_{\varphi_\parallel,\psi_\parallel}f(t) &= L^F_{\varphi_\parallel,\psi_\parallel}f_\parallel(t)+L^F_{\varphi_\parallel,\psi_\parallel}f_\perp(t)\\
&=\iint_{\mathbb{R}^2\times\mathbb{R}^2} a(s-y,\omega)\widehat{f_\parallel}(y)\widehat{\psi_\parallel}(s-\omega)\overline{\widehat{\varphi_\parallel}}(y-\omega)dyd\omega\\
&+\iint_{\mathbb{R}^2\times\mathbb{R}^2} a(y-s,\omega)\widehat{f_\perp}\widehat{\psi_\parallel}(s-\omega)\overline{\widehat{\varphi_\parallel}}(y-\omega)dyd\omega
\end{aligned}$$

is resolved into two parts. Imitating above decomposition, formula (5) can be resolved into eight similar parts. Like the norm estimate about (6), we can obtain the boundedness of every parts. In summary, we have

$$||L^F_{\varphi,\psi}f(t)||_2 \leq 8\, \|a\|_{L^{\infty,1}}\, \|f\|_2\, \|\varphi\|_2\, \|\psi\|_2$$

This theorem has been proved. □

Like [16], we introduce the following results, And we will use them to prove the compactness of multilinear localization.

Definition 2.8 Let K is a bounded subset of $L^2(\mathbb{R}^2;\mathbb{H})$,

$$\int_{\mathbb{R}^2} |f(x+y)-f(x)|^2 dx \to 0 \text{ } as \text{ } y \ \to 0 \text{ } uniformly \text{ } for \text{ } F \text{ } in \text{ } K. \tag{7}$$

$$\int_{|x|>R} |f(x)|^2 dx \to \ 0 \text{ } as \text{ } R \to \infty \text{ } uniformly \text{ } for \text{ } F \text{ } in \text{ } K. \tag{8}$$

Property (7) is called L^2-equicontinuous and property (8) is an uniform decay property.

Combine the definition of quaternionic L^2-norm and the Riesz-Tamarkin theorem (see [8]), we obtain a criterion of quaternionic compact subset in $L^2(\mathbb{R}^2;\mathbb{H})$:

Theorem 2.9 *A bounded subset K of $L^2(\mathbb{R}^2;\mathbb{H})$ is relatively compact if and only if K has property* (7) *and property* (8)*.*

Proof For any $f \in K$, we can uniquely decompose $f = f_0 + f_1 i + f_2 j + f_3 k$, where f_0, f_1, f_2, f_3 are real-valued functions. We denote $\{f_m | f = f_0 + f_1 i + f_2 j + f_3 k, f \in K\}$ by K_m, m = 0,1,2,3. In the complete metric space $L^2(\mathbb{R}^2;\mathbb{H})$, a subset K is relatively compact if and only if K_m is relatively compact in $L^2(\mathbb{R}^2)$, $m = 0, 1, 2, 3$. By Riesz-Tamarkin theorem, K_m are relatively compact if and only if K_m have property (7) and property (8). Noting that K_0, K_1, K_2, K_3 satisfy property (7) and property (8) if and only if K satisfies property (7) and property (8). Hence, K is relatively compact if and only if K has property (7) and property (8). □

Like [16], the following theorem related to quaternion Fourier transform is still right.

Theorem 2.10 *Let K is a bounded subset of $L^2(\mathbb{R}^2;\mathbb{H})$, and $\widehat{f}$ is the right-side quaternion Fourier transform of f, $\widehat{K}$=$\{\widehat{f} | f \in K\}$. Then, K is compact if and only if K and $\widehat{K}$ satisfy the property* (8)*.*

Proof Due to Theorem 2.9, we just need to prove K satisfies property (7) if and only if $\widehat{K}$ satisfies the property (8). Like [16, p. 253], combine with (1) and (2), we have

$$\begin{aligned}\|f(x+y)-f(x)\|_2 &= ||(e^{\mu(y\cdot t)}-1)\widehat{f}(t)||_2\\ &\leq (\int_{|t|\leq R}|y||t||\widehat{f}(t)|^2dt+2\int_{|t|>R}|\widehat{f}(t)|^2dt)^{\frac{1}{2}}.\end{aligned} \tag{9}$$

For arbitrary $\varepsilon >\ 0$, we can choose R so large enough that the second part in (9) is less than $\frac{\varepsilon}{2}$, because $\widehat{K}$ satisfies the property (8). And if y is small enough, the first part in (9) will less than $\frac{\varepsilon}{2}$. Hence, K satisfies property (7).

On the other hand, if K satisfies property (7), the proof is similar with in [16]. □

If we fixed F, then ψ, $L^F_{\varphi,\psi}f(t)$ can be viewed as an operator act on F and φ. In order to prove it is compact bilinear operator, we can verify $K = \{g(t)|g(t) = L^F_{\varphi,\psi}f(t), \forall||\varphi||_2 \leq 1, ||f||_2 \leq 1\}$ and $\widehat{K}$ satisfy the property (8).

Lemma 2.11 *We denote the function as QFT of $F(x,\omega)$ about ω by $\tilde{F}$. $G(x)$ represents $||\tilde{F}(x,\cdot)||_\infty$. If $G(x) \in L^2(\mathbb{R}^2,\mathbb{H})$ and $\psi \in L^1(\mathbb{R}^2;\mathbb{H})$, then the bilinear operator can be extended to a boundedness operator: $L^F_{\psi,\cdot}\cdot : L^2(\mathbb{R}^2;\mathbb{H}) \times L^2(\mathbb{R}^2;\mathbb{H}) \to L^2(\mathbb{R}^2;\mathbb{H})$*

Proof We assume F and ψ are in $\mathbb{S}(\mathbb{R}^2;\mathbb{H})$, if we write $f = f_{\parallel} + f_{\perp}$, $\varphi = \varphi_{\parallel} + \varphi_{\perp}$ and $\psi = \psi_{\parallel} + \psi_{\perp}$:

$$\begin{aligned}&L^F_{\varphi,\psi}f(t)\\ &= \iiint_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} F(x,\omega)(f_{\parallel}(y)+f_{\perp}(y))(\overline{\varphi_{\parallel}}(y-x)+\overline{\varphi_{\perp}}(y-x))e^{-2\pi\mu(\omega\cdot y)}\\ &\times e^{2\pi\mu(\omega\cdot t)}(\psi_{\parallel}(t-x)+\psi_{\perp}(t-x))dydxd\omega,\end{aligned}$$

$L^F_{\varphi,\psi}f(t)$ is decomposed into eight similar parts, we only prove one of the eight part and omit the similar proofs for the rest parts. Combine with (3), Cauchy inequality and Fubini theorem, the first part can be written as:

$$\begin{aligned}&|\iiint_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} F(x,\omega)\widehat{f_{\parallel}}(y)\psi_{\parallel}(t-x)e^{-2\pi\mu[(x-t)\cdot\omega-x\cdot y]}\overline{\widehat{\varphi_{\parallel}}(y-\omega)}dydxd\omega|\\ =&|\iint_{\mathbb{R}^2\times\mathbb{R}^2} F(x,\omega)\psi_{\parallel}(t-x)e^{-2\pi\mu[(x-t)\cdot\omega]}(e^{2\pi\mu(x\cdot y)}\int_{\mathbb{R}^2}\widehat{f_{\parallel}}(y)\overline{\widehat{\varphi_{\parallel}}(y-\omega)}dy)dxd\omega|\\ \leq&||f||_2||\varphi||_2|\int_{\mathbb{R}^2}\tilde{F}(x,x-t)\psi_{\parallel}(t-x)dx|\\ \leq&||f||_2||\varphi||_2|G*|\psi_{\parallel}|(t)|\end{aligned} \tag{10}$$

due to Young inequality, $||L^F_{\varphi,\psi}f(t)||_2$ is controlled by $||f||_2||\varphi||_2||G||_2||\psi_{||}||_1$. □

Theorem 2.12 *If* $\psi \in L^1(\mathbb{R}^2;\mathbb{H})$, $\widehat{\psi} \in L^1(\mathbb{R}^2;\mathbb{H})$ *and* $G(x) \in L^2(\mathbb{R}^2)$, *then the operator* $L^F_{\psi,\cdot}\cdot : L^2(\mathbb{R}^2;\mathbb{H}) \times L^2(\mathbb{R}^2;\mathbb{H}) \to L^2(\mathbb{R}^2;\mathbb{H})$ *is a bilinear compact operator.*

Proof We define K=$\{g(t)|g(t) = L^F_{\varphi,\psi}f(t), \forall ||\varphi||_2 \leq 1, ||f||_2 \leq 1\}$. By Lemma 2.11, we have:

$$(\int_{|t|>R} |g(t)|^2)^{\frac{1}{2}} dt \leq 8||f||_2||\varphi||_2 (\int_{|t|>R} |(G * \psi)(t)|^2 dt)^{\frac{1}{2}} \to 0 \; as \; R \to \infty$$

uniformly for g in K

Indeed, this convergence only depend on G, ψ and share no relation with g. So K satisfies property (8). Similarly, we can prove that $\widehat{K}$ shares the same property (8) with K by using Theorem 2.5. According to Theorem 2.9, K is relatively compact subset of $L^2(\mathbb{R}^2;\mathbb{H})$. Therefore $L^F_{\psi,\cdot}\cdot$ is a compact bilinear operator. □

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A Time-Frequency Relationship Between the Langevin Equation and the Harmonic Oscillator

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Abstract We derive a simple relationship between the Wigner distribution of the Green's function of the Langevin equation and of the harmonic oscillator. This relationship shows that the Wigner distribution of the Green's function of the harmonic oscillator consists of the sum of two terms obtained by translating the Wigner distribution of the Green's function of the Langevin equation at the resonant frequencies of the harmonic oscillator, plus an interference term. This result paves the way for a simplification of the time-frequency representation of differential equations, as well as for a better understanding and filtering of interference terms.

Keywords Time-frequency snalysis • Langevin equation • Harmonic oscillator

Mathematics Subject Classification (2000). Primary 60H10

1 Introduction

Consider the Langevin equation

$$\frac{dx(t)}{dt} + \beta x(t) = f(t), \tag{1}$$

where $\beta > 0$, and the harmonic oscillator defined by the equation

$$\frac{d^2x(t)}{dt^2} + 2\mu \frac{dx(t)}{dt} + \omega_0^2 x(t) = f(t), \tag{2}$$

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where we consider the underdamped case with $0 < \mu < \omega_0$. In the classic Langevin equation, the forcing term $f(t)$ is a white Gaussian noise, and the equation describes Brownian motion [1]. In this article, though, we consider $f(t)$ to be any deterministic or stochastic forcing term, and we refer to the Langevin equation as the first-order differential equation defined in Eq. (1). Equivalently, the term harmonic oscillator represents typically any second-order differential equation of the form given in Eq. (2), with a deterministic or a stochastic forcing term $f(t)$.

Both the Langevin equation and the harmonic oscillator are fundamental models for the analysis and design of physical systems and devices. Aside from describing Brownian motion, the Langevin equation is a first approximation for decaying phenomena, as well as for signals obtained by *low-pass filtering* the forcing term $f(t)$. The harmonic oscillator is instead the most fundamental resonant system. When $\mu < \omega_0$, it resonates in fact at the frequency value

$$\omega_c = \sqrt{\omega_0^2 - \mu^2}, \tag{3}$$

and at $-\omega_c$. In this work, we obtain a simple connection between the time-frequency spectrum of the harmonic oscillator and of the Langevin equation. Specifically, we show that the Wigner distribution [2–4] of the Green's function of the harmonic oscillator is made by the sum of three terms. The first two terms are the Wigner distributions of the Green's function of the Langevin equation translated at the resonant frequencies ω_c and $-\omega_c$. The third is an interference term due to the quadratic nature of the Wigner distribution, centered about the frequency $\omega = 0$. We note that the Green's function is also referred to as the impulse response.

This connection is useful for several reasons. First, it clarifies the time-frequency structure of the Green's function of the harmonic oscillator. Second, it paves the way for a simplification of the time-frequency representation of differential equations. Currently, one of the major drawbacks of applying time-frequency analysis to differential equations is the large size of the mathematical expressions involved, caused by the nonlinear nature of the time-frequency distributions commonly used. The extension of the result presented in this article to n-th order differential equations can simplify dramatically their time-frequency representation, because any solution can be written as the convolution of the Green's function with the forcing term, both in time [5] and in the Wigner distribution domain [2]. We are currently working at this extension. Third, our result provides an explicit expression for the interference terms of the Green's function, useful for their characterization and filtering. Much work has been done in time-frequency analysis to characterize and mitigate the interference terms [6, 7], because their oscillatory nature mixes up with the key time-frequency features of a signal, making them often very hard to understand.

The article is organized as follows. First, in Sect. 2 we summarize the main results. Second, in Sect. 3 we define the Green's function and the Wigner distribution, and we establish our notation. Third, in Sect. 4 we derive the connection between the Green's function of the Langevin equation and of the harmonic oscillator, we obtain the connection between their Wigner distributions, and we check the obtained result by comparing it with that obtained by transforming the harmonic oscillator to the time-frequency domain. Finally, in Sect. 5 we give more details on the signal processing aspects of the Langevin equation and the harmonic oscillator.

2 Summary of Results

For the Langevin equation (with β replaced by μ)

$$\frac{dx(t)}{dt} + \mu x(t) = f(t), \tag{4}$$

and the harmonic oscillator equation

$$\frac{d^2x(t)}{dt^2} + 2\mu \frac{dx(t)}{dt} + \omega_0^2 x(t) = f(t), \tag{5}$$

the respective Green's functions are defined by

$$x(t) = \int_{-\infty}^{+\infty} h_1(t-t')f(t')dt' \qquad \text{Langevin equation} \tag{6}$$

$$x(t) = \int_{-\infty}^{+\infty} h_2(t-t')f(t')dt' \qquad \text{Harmonic oscillator} \tag{7}$$

We show that

$$h_2(t) = \frac{1}{\omega_c} h_1(t) \sin \omega_c t. \tag{8}$$

The Wigner distribution is defined by [2–4]

$$W_x(t,\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} x^*(t-\tau/2)x(t+\tau/2)e^{-i\tau\omega}d\tau, \tag{9}$$

where the star sign indicates complex conjugation. For Eqs. (6) and (7), it is [2]

$$W_x(t,\omega) = \int_{-\infty}^{+\infty} W_{h_1}(t-t',\omega)W_f(t',\omega)dt' \qquad \text{Langevin equation} \tag{10}$$

$$W_x(t,\omega) = \int_{-\infty}^{+\infty} W_{h_2}(t-t',\omega)W_f(t',\omega)dt' \qquad \text{Harmonic oscillator} \tag{11}$$

We show that, for the Langevin equation,

$$W_{h_1}(t,\omega) = u(t)e^{-2\mu t}\frac{\sin 2\omega t}{\pi\omega}, \tag{12}$$

where $u(t)$ is the Heaviside step function defined as $u(t) = 1$ for $t \geq 0$, and $u(t) = 0$ for $t < 0$, whereas for the harmonic oscillator the Wigner distribution $W_{h_2}(t,\omega)$ can be expressed in terms of $W_{h_1}(t,\omega)$, namely

$$W_{h_2}(t,\omega) = \frac{1}{4\omega_c^2}W_{h_1}(t,\omega-\omega_c) + \frac{1}{4\omega_c^2}W_{h_1}(t,\omega+\omega_c) - \frac{1}{2\omega_c^2}\cos 2\omega_c t W_{h_1}(t,\omega). \tag{13}$$

3 Definitions and Basic Properties

We give the main definitions for the Green's function and the Wigner distribution.

3.1 Green's Function

Consider the class of differential equations defined by

$$\frac{d^n x(t)}{dt^n} + a_{n-1}\frac{d^{n-1}x(t)}{dt^{n-1}} \ldots + a_0 x(t) = f(t), \tag{14}$$

where $a_0,\ldots,a_{n-1}$ are real constant coefficients. The Green's function $h_n(t)$ is obtained as the solution of the equation when the forcing term is a Dirac delta function [5, 8],

$$\frac{d^n h_n(t)}{dt^n} + a_{n-1}\frac{d^{n-1}h_n(t)}{dt^{n-1}} \ldots + a_0 h_n(t) = \delta(t). \tag{15}$$

The notation $h_n(t)$ denotes that the Green's function refers to the equation of order n. Equivalently, the Green's function can be obtained by solving the homogeneous equation

$$\frac{d^n h_n(t)}{dt^n} + a_{n-1}\frac{d^{n-1}h_n(t)}{dt^{n-1}} \ldots + a_0 h_n(t) = 0, \tag{16}$$

with the initial conditions

$$\left.\frac{d^k h_n(t)}{dt^k}\right|_{t=0} = 0, \quad \text{for } k = 0, \ldots, n-2 \tag{17}$$

$$\left.\frac{d^{n-1} h_n(t)}{dt^{n-1}}\right|_{t=0} = 1 \tag{18}$$

where

$$\frac{d^k h_n(t)}{dt^k} = h_n(t), \quad \text{for } k = 0. \tag{19}$$

The Green's function is a fundamental tool in the study of differential equations and in the analysis and design of physical systems, because any solution to Eq. (14) can be written through the convolution integral [5]

$$x(t) = \int_{-\infty}^{+\infty} h_n(t-t')f(t')dt'. \tag{20}$$

3.2 Wigner Distribution

The Wigner distribution for deterministic signals is defined in Eq. (9), whereas, when $x(t)$ is a random process, we use the Wigner spectrum [9–11], defined as

$$\overline{W}_x(t,\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} E\left[x^*(t-\tau/2)x(t+\tau/2)\right]e^{-i\tau\omega}d\tau, \tag{21}$$

where E is the expected value. When

$$x(t) = x_1(t) + x_2(t), \tag{22}$$

it follows that

$$W_x(t,\omega) = W_{x_1}(t,\omega) + W_{x_2}(t,\omega) + 2\Re\{W_{x_1,x_2}(t,\omega)\}, \tag{23}$$

where the cross-Wigner distribution is defined as

$$W_{x_1,x_2}(t,\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} x_1^*(t-\tau/2)x_2(t+\tau/2)e^{-i\tau\omega}d\tau. \tag{24}$$

The term of Eq. (23) depending on the cross-Wigner distribution is a consequence of the quadratic nature of the Wigner distribution, and it is often referred to as interference term.

Moreover, when

$$y(t) = ax(t), \tag{25}$$

it is

$$W_y(t,\omega) = |a|^2 \, W_x(t,\omega). \tag{26}$$

Finally, when

$$y(t) = x(t)e^{i\omega_1 t}, \tag{27}$$

we have [2]

$$W_y(t,\omega) = W_x(t,\omega-\omega_1). \tag{28}$$

4 Time-Frequency Connection Between the Langevin Equation and the Harmonic Oscillator

We first derive a simple connection between the Green's function of the Langevin equation and of the harmonic oscillator, then we obtain its time-frequency representation.

4.1 Relationship Between the Green's Functions in Time

From the initial condition problem of Eqs. (16), (17) and (18), the Green's function of the Langevin equation is obtained by solving

$$\frac{dh_1(t)}{dt} + \mu h_1(t) = 0, \tag{29}$$

with the initial condition

$$h_1(0) = 1. \tag{30}$$

The solution is straightforwardly obtained as

$$h_1(t) = u(t)e^{-\mu t}, \tag{31}$$

Similarly, the Green's function for the harmonic oscillator is obtained by solving

$$\frac{d^2h_2(t)}{dt^2} + 2\mu\frac{dh_2(t)}{dt} + \omega_0^2 h_2(t) = 0, \tag{32}$$

with the initial conditions

$$h_2(0) = 0, \tag{33}$$

$$\left.\frac{dh_2(t)}{dt}\right|_{t=0} = 1. \tag{34}$$

The solution is [5]

$$h_2(t) = u(t)\left(Ae^{i\lambda_1 t} + Be^{i\lambda_2 t}\right), \tag{35}$$

where the poles λ_1, λ_2 are given by

$$\lambda_{1,2} = -\mu \pm i\omega_c. \tag{36}$$

The complex constants A and B are determined by using the initial conditions given in Eqs. (33) and (34), obtaining

$$h_2(t) = u(t)e^{-\mu t}\left(\frac{1}{2i\omega_c}e^{i\omega_c t} - \frac{1}{2i\omega_c}e^{-i\omega_c t}\right). \tag{37}$$

Trigonometric identities and Eq. (31) give

$$h_2(t) = \frac{1}{\omega_c}h_1(t)\sin\omega_c t. \tag{38}$$

Therefore, the Green's function of the underdamped harmonic oscillator is a frequency modulated version of the Green's function of the Langevin equation. As known from the properties of the Fourier transform, this frequency modulation generates two replicas of the frequency spectrum of $h_1(t)$ centered about the frequencies ω_c and $-\omega_c$.

4.2 Relationship Between the Green's Functions in the Time-Frequency Domain

By replacing Eq. (31), we rewrite Eq. (37) as

$$h_2(t) = \frac{1}{2i\omega_c}h_1(t)e^{i\omega_c t} - \frac{1}{2i\omega_c}h_1(t)e^{-i\omega_c t}. \tag{39}$$

We now set

$$h_{21}(t) = \frac{1}{2i\omega_c} h_1(t) e^{i\omega_c t}, \tag{40}$$

$$h_{22}(t) = -\frac{1}{2i\omega_c} h_1(t) e^{-i\omega_c t}. \tag{41}$$

Therefore

$$h_2(t) = h_{21}(t) + h_{22}(t). \tag{42}$$

From Eq. (23), it is

$$W_{h_2}(t,\omega) = W_{h_{21}}(t,\omega) + W_{h_{22}}(t,\omega) + 2\Re\{W_{h_{21},h_{22}}(t,\omega)\}, \tag{43}$$

where, by using Eqs. (25), (26), (27) and (28), we have

$$W_{h_{21}}(t,\omega) = \frac{1}{4\omega_c^2} W_{h_1}(t,\omega-\omega_c), \tag{44}$$

$$W_{h_{22}}(t,\omega) = \frac{1}{4\omega_c^2} W_{h_1}(t,\omega+\omega_c), \tag{45}$$

and, from Eq. (24), noting that $h_1(t)$, $h_{21}(t)$, and $h_{22}(t)$ are real, we obtain

$$W_{h_{21},h_{22}}(t,\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} h_{21}(t-\tau/2)h_{22}(t+\tau/2)e^{-i\tau\omega}d\tau, \tag{46}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{-2i\omega_c} h_1(t-\tau/2)e^{-i\omega_c(t-\tau/2)} \times\left(-\frac{1}{2i\omega_c}\right) h_1(t+\tau/2)e^{-i\omega_c(t+\tau/2)}e^{-i\tau\omega}d\tau \tag{47}$$

$$= -\frac{1}{4\omega_c^2}e^{-2i\omega_c t}\frac{1}{2\pi}\int_{-\infty}^{+\infty} h_1(t-\tau/2)h_1(t+\tau/2)e^{-i\tau\omega}d\tau, \tag{48}$$

$$= -\frac{1}{4\omega_c^2}e^{-2i\omega_c t}W_{h_1}(t,\omega). \tag{49}$$

Replacing these results in Eq. (43) gives

$$W_{h_2}(t,\omega) = \frac{1}{4\omega_c^2}W_{h_1}(t,\omega-\omega_c) + \frac{1}{4\omega_c^2}W_{h_1}(t,\omega+\omega_c) - \frac{1}{2\omega_c^2}\cos 2\omega_c t W_{h_1}(t,\omega), \tag{50}$$

which represents the desired time-frequency relationship between the Wigner representation of the Green's functions of the Langevin equation and the harmonic oscillator. The first two terms are obtained by translating the Wigner distribution of the Green's function of the Langevin equation at the resonant frequencies $\pm\omega_c$, whereas the third term is an interference term.

4.3 Explicit Expressions for the Green's Functions in the Time-Frequency Domain

We obtain the explicit Wigner distributions of the Green's functions for the Langevin equation and the harmonic oscillator, and, to validate our results, we compare the latter one with the solution obtained by transforming the differential equation to the Wigner distribution domain.

By replacing $h_1(t)$ in Eq. (9), we have

$$W_{h_1}(t,\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} u(t-\tau/2)u(t+\tau/2)e^{-\mu(t-\tau/2)-\mu(t+\tau/2)-i\tau\omega}d\tau, \quad (51)$$

$$= e^{-2\mu t}\frac{1}{2\pi}\int_{-\infty}^{+\infty} u(t-\tau/2)u(t+\tau/2)e^{-i\tau\omega}d\tau, \quad (52)$$

$$= u(t)e^{-2\mu t}\frac{1}{2\pi}\int_{-2t}^{2t} e^{-i\tau\omega}d\tau. \quad (53)$$

Evaluating the integral gives the Wigner distribution of the Green's function of the Langevin equation

$$W_{h_1}(t,\omega) = u(t)e^{-2\mu t}\frac{\sin 2\omega t}{\pi\omega}. \quad (54)$$

We note that for every given time $t > 0$, $W_{h_1}(t,\omega)$ is a sinc function centered about $\omega = 0$. Substituting in Eq. (50), we obtain the Wigner distribution of the Green's function for the harmonic oscillator

$$W_{h_2}(t,\omega) = \frac{1}{2\pi\omega_c^2}u(t)e^{-2\mu t}\left(\frac{\sin 2(\omega-\omega_c)t}{2(\omega-\omega_c)} + \frac{\sin 2(\omega+\omega_c)t}{2(\omega+\omega_c)} - \cos 2\omega_c t\frac{\sin 2\omega t}{\omega}\right). \quad (55)$$

We verify this result by transforming the equation defining the Green's function of the harmonic oscillator to the Wigner distribution domain [12–15], and by showing that its solution corresponds to $W_{h_2}(t,\omega)$. For convenience, we repeat the Green's function problem for the general class of differential equations defined in Eq. (14)

$$\frac{d^n h_n(t)}{dt^n} + a_{n-1}\frac{d^{n-1}h_n(t)}{dt^{n-1}} \ldots + a_0 h_n(t) = \delta(t). \tag{56}$$

We factor this equation as

$$(D-\lambda_1)\cdots(D-\lambda_n)h_n(t) = \delta(t), \tag{57}$$

where $D = \frac{d}{dt}$ and the poles $\lambda_1,\ldots,\lambda_n$ are the solutions of the characteristic equation

$$\lambda^n + a_{n-1}\lambda^{n-1}\ldots + a_0 = 0. \tag{58}$$

The equation for the Wigner distribution of $h_n(t)$ is [13, 15]

$$\frac{1}{4^n}\prod_{m=1}^{n}\left(\partial_t - p_m(\omega)\right)\left(\partial_t - p_m^*(\omega)\right)W_{h_n}(t,\omega) = \frac{1}{2\pi}\delta(t), \tag{59}$$

where $\partial_t = \frac{\partial}{\partial t}$ and the generic time-frequency pole $p_m(\omega)$ is defined as [16]

$$p_m(\omega) = 2\alpha_m + 2i(\beta_m - \omega), \tag{60}$$

with α_m and β_m being the real and imaginary part of λ_m, respectively. The solution of the equation for the Wigner distribution is [13]

$$\begin{aligned} W_{h_n}(t,\omega) = {} & \frac{1}{2\pi}4^n u(t)\sum_{m=1}^{n} e^{\Re\{p_m\}t} \\ & \times\left(C_m\cos\left(\Im\{p_m\}t\right) + \frac{D_m + C_m\Re\{p_m\}}{\Im\{p_m\}}\sin\left(\Im\{p_m\}t\right)\right), \end{aligned} \tag{61}$$

where

$$C_m = 2\Re\{r_m\}, \tag{62}$$

$$D_m = -2\Re\{r_m p_m^*\}, \tag{63}$$

$$r_m = \frac{1}{(p_m - p_m^*)\prod_{k=1,\, k\neq m}^{n}(p_m - p_k)(p_m - p_k^*)}. \tag{64}$$

For the harmonic oscillator it is $n = 2$, the time-frequency poles are

$$p_1(\omega) = -2\mu + 2i(\omega_c - \omega), \tag{65}$$

$$p_2(\omega) = -2\mu - 2i(\omega_c + \omega), \tag{66}$$

and

$$C_1 = C_2 = 0, \tag{67}$$

$$D_1 = \frac{1}{16\omega\omega_c}, \tag{68}$$

$$D_2 = -\frac{1}{16\omega\omega_c}. \tag{69}$$

By replacing these quantities in Eq. (61) we obtain

$$W_{h_2}(t,\omega) = \frac{1}{2\pi}\frac{1}{\omega\omega_c}u(t)e^{-2\mu t}\left(\frac{\sin 2(\omega-\omega_c)t}{2(\omega-\omega_c)} - \frac{\sin 2(\omega+\omega_c)t}{2(\omega+\omega_c)}\right). \tag{70}$$

This equation is identical to Eq. (55) if we use the identity

$$\cos 2\omega_c t \sin 2\omega t = \frac{1}{2}\sin 2(\omega_c+\omega)t + \frac{1}{2}\sin 2(\omega-\omega_c)t. \tag{71}$$

5 Signal Processing Interpretation of the Langevin Equation and the Harmonic Oscillator

The frequency representation of the Langevin equation is given by [5]

$$X(\omega) = \frac{1}{\beta + i\omega}F(\omega), \tag{72}$$

where we define the Fourier transform as

$$X(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} x(t)e^{-i\omega t}dt. \tag{73}$$

The high frequencies of the spectrum $F(\omega)$ are *filtered out* (attenuated) by the transfer function

$$H(\omega) = \frac{1}{\beta + i\omega}, \tag{74}$$

whereas the low frequencies are allowed to *pass*. For such reason, in signal processing the Langevin equation is referred to as a low-pass filter, and is used in a variety of fields, such as, among the others, circuit theory [17], optimal estimation [18], vibrations of structures [19], mechanical systems [20], and control systems [21].

The harmonic oscillator is instead the most fundamental resonant system, because, when $\mu < \omega_0$, it resonates at the frequency ω_c [5], and at the symmetric negative frequency $-\omega_c$. Resonances are fundamental tools in signal and system theory, because they correspond to the peaks observed in the frequency spectrum, and can hence be used for system design and identification. Similarly to the Langevin equation, also the harmonic oscillator is widely applied in all of the fields of science and engineering.

6 Conclusions

We have shown that a simple relationship exists between the time-frequency representations of the Green's functions of the Langevin equation and the harmonic oscillator. Specifically, we have proved that the Wigner distribution of the Green's function of the harmonic oscillator is made by two frequency translated versions of the Wigner distribution of the Green's function of the Langevin equation, plus an interference term due to the quadratic nature of the Wigner distribution. We have also verified our result by comparing it with the result obtained by transforming the differential equation governing the harmonic oscillator directly to the Wigner distribution domain.

This result, if extended to n-th order differential equations, could simplify the complexity of time-frequency representations of differential equations, often made by a large number of terms due to the nonlinear nature of the quadratic time-frequency distributions used. We are currently working at the extension of this method to n-th order differential equations of the form given in Eq. (14). Our result also provides a simple analytic expression for the interference terms arising in the Green's function, which can be useful for their characterization and filtering.

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Are There Quantum Operators and Wave Functions in Standard Probability Theory

Leon Cohen

Abstract The methods of quantum probability theory are radically different from standard probability as developed over the last 300 years. While the results of quantum probability, such as expectation values, are the same as standard probability theory, the methods used are strange, as they deal with operators and wave functions and use strange rules of manipulation. We ask whether there are operators and wave functions in standard probability theory. By generalizing a theorem of Khinchine on characteristic functions, we show that indeed the strange probabilistic methods of quantum mechanics follow from standard probability theory.

Keywords Probability theory • Operators • Quantum mechanics • Khinchine theorem

Mathematics Subject Classification (2000). Primary 47G30; Secondary 60A05

1 Introduction

Quantum mechanics is the most successful theory ever devised, by far. It explains everything that we know about matter, atoms, stars, the universe, chemistry, and indeed all physical phenomena that it has been applied to. Moreover, quantum mechanics predicts bizarre phenomena, such as vacuum fluctuations, that have been experimentally observed.

Quantum mechanics is a probability theory. While the probabilistic "results" of quantum mechanics are of the same nature as standard probability theory, for example expectation values and probability densities, the method of calculation is radically different from standard probability theory. Quantum mechanics uses wave

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functions, operators, and methods which are seemingly totally foreign to standard probability theory.

There have been numerous attempts to formulate quantum mechanics as a standard probability theory. It is fair to say that these attempts have not succeeded. We reverse the question and ask: Since quantum mechanics is certainly the most successful probability theory ever devised, we ask whether standard probability theory has the concepts of wave functions and operators. We emphasize that we are not trying to formulate quantum mechanics as a standard probability theory; quite the contrary, we are trying to see if standard probability theory contains the ideas and methods of quantum probability theory and if it could be formulated in quantum mechanical language [2, 3, 8].

Notation Operators will be denoted by bold-face letters and the corresponding random variables by lower case letters. When it is not obvious what random variable the characteristic function and corresponding probability density are referring to, we use the notation $M_a(\theta)$ and $P_a(a)$ where the subscript denotes the random variable. All integrals go from $-\infty$ to ∞ or the appropriate range of the variables. Also, it is assumed that eigenfunctions are normalized to one for the discrete case and to a delta function for the continuous case.

2 Characteristic Functions

For a probability density, $P(x)$, the characteristic function, $M(\theta)$, is the expectation value of $e^{i\theta x}$

$$M(\theta) = \langle e^{i\theta x} \rangle = \int e^{i\theta x} P(x)\, dx \tag{1}$$

and from the characteristic function, one may obtain the probability density by Fourier inversion,

$$P(x) = \frac{1}{2\pi} \int M(\theta) e^{-i\theta x} d\theta \tag{2}$$

The characteristic function is standard in probability theory for many reasons [4, 7]. It is often easier to manipulate probabilistic results by using the characteristic function compared to the probability density function itself. For example, the moments, defined by

$$\langle x^n \rangle = \int x^n P(x)\, dx \tag{3}$$

may be obtained from

$$\langle x^n \rangle = \frac{1}{i^n} \frac{d^n}{d\theta^n} M(\theta) \bigg|_{\theta=0} \tag{4}$$

Since differentiation is easier then integration, Eq. (4) is often easier to use than Eq. (3) if indeed we know $M(\theta)$. Furthermore the characteristic function is very useful for obtaining probability densities for new variables [5].

The characteristic function is generally complex, but not every complex function is a characteristic function since it has to be derivable from a positive density function. What are necessary and sufficient conditions for a function $M(\theta)$ to be a characteristic function? Khinchine solved this problem [6]. A function, $M(\theta)$, is a characteristic function if and only if there exists a function, $g(x)$, so that the characteristic function is expressed in the following form [6, 7]

$$M(\theta) = \int g^*(x) g(x+\theta) dx \tag{5}$$

If there is such a function, it should be normalized to one, which insures that the corresponding density will integrate to one. While this theorem is fundamental in probability theory, it appears that the significance and properties of the $g(x)$ functions have not been extensively studied. We will argue that they are the "wave functions" of quantum mechanics, and that the generalization of Khinchine's theorem that we present in Sect. 3 leads to the concept of operators in standard probability theory. We first present our idea for the Khinchine theorem as originally given, Eq. (5), before we give the general result in the next section.

Rewrite Khinchine's theorem in the following way

$$M(\theta) = \int g^*(x) e^{\theta \frac{d}{dx}} g(x) dx \tag{6}$$

where in going from Eqs. (5)–(6) we have used the fact that $e^{\theta \frac{d}{dx}}$ is the translation operator in that for any function $f(x)$ [10]

$$e^{\theta \frac{d}{dx}} f(x) = f(x+\theta) \tag{7}$$

We now insert i as indicated

$$M(\theta) = \int g^*(x) e^{i\theta \left(\frac{1}{i} \frac{d}{dx}\right)} g(x) dx \tag{8}$$

and write Eq. (6) as

$$M(\theta) = \int g^*(x) e^{i\theta \mathbf{p}} g(x) dx \tag{9}$$

where we have defined the operator, $\mathbf{p}$, by

$$\mathbf{p} = \frac{1}{i}\frac{d}{dx} \tag{10}$$

We now calculate the expectation value by way of Eq. (4). In anticipation of the result we shall use the letter p for the random variables since it will turn out to be momentum and the letter p is standard for momentum. In particular,

$$\langle p \rangle = \frac{1}{i}\frac{d}{d\theta}M(\theta)\bigg|_{\theta=0} \tag{11}$$

$$= \frac{1}{i}\frac{d}{d\theta}\int g^*(x)e^{i\theta\mathbf{p}}g(x)dx\bigg|_{\theta=0} \tag{12}$$

$$= \int g^*(x)\left(\frac{1}{i}\frac{\partial}{\partial\theta}\right)e^{i\theta\mathbf{p}}g(x)dx\bigg|_{\theta=0} \tag{13}$$

$$= \int g^*(x)\mathbf{p}e^{i\theta\mathbf{p}}g(x)dx\bigg|_{\theta=0} \tag{14}$$

or

$$\langle p \rangle = \int g^*(x)\mathbf{p}\, g(x)dx = \int g^*(x)\left(\frac{1}{i}\frac{d}{dx}\right)g(x)dx \tag{15}$$

This is precisely how one calculates the average momentum in quantum mechanics when the system has the "wave function" $g(x)$ [1, 9]. Therefore we argue that for this case (momentum) the g's of the Khinchine theorem are the wave functions of quantum mechanics. Note that the g's are generally complex functions and that the operator $\mathbf{p}$ is self-adjoint, as indeed they should be. The reason for self-adjointness will be discussed in Sect. 3

The Probability Density What is the probability density that corresponds to the characteristic function given by Eq. (5)? Again, we use p for the random variable, which is an ordinary variable, and should not be confused with the operator $\mathbf{p}$. Using Eq. (2) we have

$$P(p) = \frac{1}{2\pi}\int M(\theta)e^{-i\theta p}d\theta \tag{16}$$

$$= \frac{1}{2\pi}\iint g^*(x)\, g(x+\theta)e^{-i\theta p}d\theta dx \tag{17}$$

Making a change of variables

$$x' = x + \theta \qquad\qquad dx' = d\theta \tag{18}$$

we have

$$P(p) = \frac{1}{2\pi}\iint g^*(x)\, g(x')e^{-i(x'-x)p}dx'dx \tag{19}$$

$$= \frac{1}{2\pi}\left(\int g^*(x)e^{ixp}dx\right)\left(\int g(x')e^{-ix'p}dx'\right) \tag{20}$$

that is

$$P(p) = \frac{1}{2\pi}\left|\int g(x)e^{-ixp}dx\right|^2 \tag{21}$$

But this is precisely the probability density of "momentum" in quantum mechanics [1, 9].

Comment Notice that the random variable p is continuous, ranging from $-\infty$ to ∞. That is indeed the case in quantum mechanics, and we say that momentum is not quantized. How quantization for other physical quantities comes in will be clear when we discuss general operators and general random variables in the next section.

3 Generalization of Khinchine's Theorem

We now generalize Khinchine's theorem to apply to arbitrary self adjoint operators. $M_a(\theta)$ is a characteristic function if and only if for a self adjoint operator $\mathbf{A}$ there exists the representation

$$M_a(\theta) = \int g^*(x)e^{i\theta\mathbf{A}}g(x)dx \tag{22}$$

We prove this in Appendix A, "Khinchine Theorem for Operators". For the expectation value we have, using Eq. (2), that

$$\langle a\rangle = \frac{1}{i}\frac{d}{d\theta}M_a(\theta)\bigg|_{\theta=0} \tag{23}$$

$$= \frac{1}{i}\frac{d}{d\theta}\int g^*(x)e^{i\theta\mathbf{A}}\,g(x)dx\bigg|_{\theta=0} \tag{24}$$

$$= \int g^*(x)\left(\frac{1}{i}\frac{\partial}{\partial\theta}\right)e^{i\theta\mathbf{A}}\,g(x)dx\bigg|_{\theta=0} \tag{25}$$

$$= \int g^*(x)\mathbf{A}e^{i\theta\mathbf{A}}\,g(x)dx\bigg|_{\theta=0} \tag{26}$$

giving

$$\langle a \rangle = \int g^*(x)\mathbf{A}\, g(x)\, dx \tag{27}$$

This is precisely the standard manner of calculating expectation values in quantum mechanics for a physical quantity associated with the self-adjoint operator **A** [1, 9].

3.1 Probability Density

We now discuss the probability density that corresponds to the characteristic function given by Eq. (22). Substitute Eq. (22) into Eq. (2) to obtain

$$P(a) = \frac{1}{2\pi} \int M_a(\theta) e^{-i\theta a} d = \frac{1}{2\pi} \iint g^*(x) e^{i\theta \mathbf{A}}\, g(x) e^{-i\theta a} dx\, d\theta \tag{28}$$

We evaluate Eq. (28) in Appendix B, "Probability Density". Here we state the result. There are two cases: namely, if we have discrete or continuous random variables. This follows naturally, as we show in the Appendix "Probability Density". In short, it is the spectrum of the operator **A** which determines whether the random variables are discrete or continuous. Moreover the random variables are the eigenvalues of the operator.

Continuous case If the spectrum of the operator has continuous eigenvalues we write

$$\mathbf{A} u_a(x) = a u_a(x) \tag{29}$$

where a and $u_a(x)$ are the eigenvalues and corresponding eigenfunctions of the operator **A**. The probability density as evaluated by way of Eq. (28) is given by

$$P(a) = |c(a)|^2 \tag{30}$$

where

$$c(a) = \int g(x) u_a^*(x) dx \tag{31}$$

Hence, the random variables are the $a's$ (the eigenvalues) and their range is the range of the eigenvalues.

Discrete case If the spectrum of the operator is discrete, we write

$$\mathbf{A} u_n(x) = a_n u_n(x) \tag{32}$$

then the probability distribution is given by

$$P(a) = \sum_n |c_n|^2 \delta(a - a_n) \tag{33}$$

where

$$c_n = \int g(x) u_n^*(x) dx \tag{34}$$

Notice that the probability density is non-zero only when the random variable, a, is one of the discrete eigenvalues. In this case we have quantization. We may write Eq. (33) as

$$P(a_n) = |c_n|^2 \tag{35}$$

Discussion The probability densities derived above are called the Born rule. We have derived them from the generalization of the Khinchine theorem, Eq. (22). Also, we proved that the random variables are the eigenvalues of the operator **A**, which is usually just assumed in quantum mechanics.

3.2 Two Ways of Calculating Expectation Values

We have shown, using Eq. (22) that one may calculate expectation values by

$$\langle a \rangle = \int g^*(x) \mathbf{A}\, g(x)\, dx \tag{36}$$

which is the standard quantum mechanical way. However in standard probability theory we calculate expectation values by

$$\langle a \rangle = \sum (\text{random variable}) \times (\text{probability}) \qquad \text{(discrete case)} \tag{37}$$

for the discrete case, and by

$$\langle a \rangle = \int (\text{random variable}) \times (\text{probability}) \qquad \text{(continuous case)} \tag{38}$$

for the continuous case. Substituting Eqs. (30) and (35) we have

$$\langle a \rangle = \sum a_n \left| \int g(x) u_n^*(x) dx \right|^2 \qquad \text{(discrete case)} \tag{39}$$

$$\langle a \rangle = \int a \left| \int g(x) u_a^*(x) dx \right|^2 da \qquad \text{(continuous case)} \tag{40}$$

It is well known in quantum mechanics that the two methods are the same, that is that

$$\int g^*(x)\mathbf{A}\, g(x)\, dx = \sum a_n \left| \int g(x)u_n^*(x)dx \right|^2 \qquad \text{(discrete case)} \tag{41}$$

$$\int g^*(x)\mathbf{A}\, g(x)\, dx = \int a \left| \int g(x)u_a^*(x)dx \right|^2 da \qquad \text{(continuous case)} \tag{42}$$

In Appendix C, "Standard vs. Quantum Manner of Calculating Expectation Values" we show the equivalence for the sake of readers that may not be familiar with the result.

4 Conclusion

We summarize the main results. We have generalized Khinchine's theorem for a self-adjoint operator **A** by showing that a function, $M_a(\theta)$, defined by

$$M_a(\theta) = \int g^*(x)e^{i\theta\mathbf{A}}\, g(x)dx \tag{43}$$

is a *proper* characteristic function. From Eq. (43) we have shown that the usual rules of quantum probabilities follow. In particular we have shown that:

1. The expected value is

$$\langle a \rangle = \int g^*(x)\mathbf{A}\, g(x)\, dx \tag{44}$$

2. The random variables are the eigenvalues of the operator **A**.
3. If the eigenvalues, a, are continuous, then the probability density associated with the characteristic function is

$$P(a) = \left| \int g(x)u_a^*(x)dx \right|^2 \tag{45}$$

where $u_a(x)$ are the eigenfunctions.
4. If the eigenvalues, a_n, are discrete with corresponding eigenfunctions $u_n(x)$, the probability density is given by

$$P(a_n) = \left| \int g(x)u_n^*(x)dx \right|^2 \tag{46}$$

The above items are exactly how one obtains the random variables and probabilities in quantum mechanics. We have *derived* them from the characteristic function defined by Eq. (43).

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Appendix A: Khinchine Theorem for Operators

We prove that $M_a(\theta)$ is a characteristic function corresponding to the self-adjoint operator, $\mathbf{A}$, if and only if, there exists the representation

$$M_a(\theta) = \int g^*(x) e^{i\theta \mathbf{A}} g(x) dx \tag{47}$$

for some function $g(x)$. First, we show that $M_a(\theta)$ produces a proper probability density. Substituting Eq. (47) into Eq. (2) the probability density is then

$$P(a) = \frac{1}{2\pi} \int M(\theta) e^{-i\theta a} d\theta = \frac{1}{2\pi} \iint g^*(x) e^{i\theta \mathbf{A}} g(x) e^{-i\theta a} dx \, d\theta \tag{48}$$

We first consider the continuous case. Since the operator is self-adjoint, the solution to the eigenvalue problem

$$\mathbf{A} u_\alpha(x) = \alpha u_\alpha(x) \tag{49}$$

produces real eigenvalues, α, and complete and orthogonal eigenfunctions, $u_\alpha(x)$

$$\int u_\alpha^*(x) u_\beta(x) dx = \delta(\alpha - \beta) \tag{50}$$

$$\int u_\alpha^*(x) u_\alpha(x') d\alpha = \delta(x - x') \tag{51}$$

Since the eigenfunctions are complete and orthogonal, we can expand any function as

$$g(x) = \int u_\alpha(x) c(\alpha) d\alpha \tag{52}$$

and inversely

$$c(\alpha) = \int u_\alpha^*(x) \, g(x) dx \tag{53}$$

Substituting Eq. (52) into Eq. (48) we have

$$P(a) = \frac{1}{2\pi} \iiiint u_\beta^*(x) c^*(\beta) e^{i\theta \mathbf{A}} u_\alpha(x) c(\alpha) e^{-i\theta a} dx d\beta d\alpha d\theta \tag{54}$$

Using the fact that

$$e^{i\theta \mathbf{A}} u_\alpha(x) = e^{i\theta\alpha} u_\alpha(x) \tag{55}$$

we have

$$P(a) = \frac{1}{2\pi} \iiiint u_\beta^*(x) c^*(\beta) e^{i\theta\alpha} u_\alpha(x) c(\alpha) e^{-i\theta a} dx d\beta d\alpha d\theta \tag{56}$$

$$= \iint F^*(\beta) \delta(a-\alpha) \delta(\alpha-\beta) F(\alpha) d\beta d\alpha \tag{57}$$

The θ integration gives

$$\int e^{i\theta\alpha} e^{-i\theta a} d\theta = 2\pi \delta(\alpha - \beta) \tag{58}$$

and using Eq. (50) we have

$$P(a) = \iint c^*(\beta) \delta(a-\alpha) \delta(\alpha-\beta) c(\alpha) d\beta d\alpha \tag{59}$$

Therefore

$$P(a) = |c(a)|^2 \tag{60}$$

Equation (60) shows that we have a manifestly positive density, and that it will be normalized to one if the wave function is normalized to one because

$$\int |c(a)|^2 \, da = \int |\, g(x)|^2 \, dx \tag{61}$$

This proves the sufficiency of the form given by Eq. (47).

To prove the necessity, suppose we have the probability distribution $P(\alpha)$, and hence the characteristic function is given by

$$M(\theta) = \int e^{i\theta\alpha} P(\alpha) \, d\alpha \tag{62}$$

We expand, not the probability distribution but the square root of $P(\alpha)$

$$\sqrt{P(\alpha)} = \int u_\alpha(x) f(x) dx \tag{63}$$

Since $\sqrt{P(\alpha)}$ is real we also have

$$\sqrt{P(\alpha)} = \int u_\alpha^*(x) f^*(x) dx \tag{64}$$

Therefore

$$M(\theta) = \int e^{i\theta\alpha} \sqrt{P(\alpha)} \sqrt{P(\alpha)}\, d\alpha \tag{65}$$

$$= \iiint u_\alpha^*(x') f^*(x')\, u_\alpha(x) f(x) e^{i\theta\alpha} dx d\alpha dx' \tag{66}$$

$$= \iiint u_\alpha^*(x') f^*(x') \left\{ e^{i\theta \mathbf{A}} u_\alpha(x) \right\} f(x) dx dx' d\alpha \tag{67}$$

$$= \iint f^*(x') \left\{ e^{i\theta \mathbf{A}} \delta(x - x') \right\} f(x) dx dx' \tag{68}$$

or

$$M_a(\theta) = \int f^*(x) e^{i\theta \mathbf{A}} f(x) dx \tag{69}$$

which is of the form given by Eq. (47).

A similar proof follows for the discrete case.

Appendix B: Probability Density

We now derive the probability density corresponding to $M_a(\theta)$, where

$$M_a(\theta) = \int g^*(x) e^{i\theta \mathbf{A}} g(x)\, d\theta \tag{70}$$

Using Eq. (2) we have

$$P_a(a) = \frac{1}{2\pi} \iint g^*(x) e^{i\theta \mathbf{A}} g(x) e^{-i\theta a} dx\, d\theta \tag{71}$$

To evaluate Eq. (71) we consider two separate cases depending on whether the spectrum of the operator **A** is continuous or discrete. For a discrete spectrum we write

$$\mathbf{A}u_n(x) = a_n u_n(x) \tag{72}$$

where the eigenfunctions satisfy completeness and orthogonality properties

$$\int u_n^*(x)u_k(x)dx = \delta_{nk} \tag{73}$$

$$\sum_n u_n^*(x)u_n(x') = \delta(x - x') \tag{74}$$

We expand the wave function as

$$g(x) = \sum_n c_n u_n(x) \tag{75}$$

with

$$c_n = \int g(x)u_n^*(x)dx \tag{76}$$

Substituting Eq. (75) into Eq. (71) we have

$$P(a) = \frac{1}{2\pi}\iint \sum_{n,m} c_m^* u_m^*(x)e^{i\theta\mathbf{A}} c_n u_n(x)e^{-i\theta a}dx\,d\theta \tag{77}$$

Using

$$e^{i\theta\mathbf{A}}u_n(x) = e^{i\theta a_n}u_n(x) \tag{78}$$

gives

$$P(a) = \frac{1}{2\pi}\iint \sum_{n,m} c_m^* u_m^*(x)e^{i\theta a_n} c_n u_n(x)e^{-i\theta a}dx\,d\theta \tag{79}$$

$$= \frac{1}{2\pi}\int \sum_{n,m} c_m^* \delta_{nm}(x)e^{i\theta a_n} c_n e^{-i\theta a}\,d\theta \tag{80}$$

$$= \frac{1}{2\pi}\sum_n |c_n|^2 \int e^{i\theta a_n - i\theta a}\,d\theta \tag{81}$$

Therefore

$$P(a) = \sum_n |c_n|^2 \delta(a - a_n) \tag{82}$$

Equation (82) shows that the a_n are the random variables with corresponding probability $|c_n|^2$. This is exactly the quantum mechanical result. One can write Eq. (82) as

$$P(a_n) = |c_n|^2 \tag{83}$$

Note: Since **A** is self-adjoint the eigenvalues are real, as they should be, since they represent measurable quantities.

For the continuous case we write

$$\mathbf{A}u_\alpha(x) = au_\alpha(x) \tag{84}$$

and the eigenfunctions satisfy

$$\int u_\alpha^*(x)u_\beta(x)dx = \delta(\alpha - \beta) \tag{85}$$

$$\int u_\alpha^*(x)u_\alpha(x')d\alpha = \delta(x - x') \tag{86}$$

Expand $g(x)$ as

$$g(x) = \int c(\alpha)u_\alpha(x)d\alpha \tag{87}$$

with

$$c(\alpha) = \int g(x)u_\alpha^*(x)dx \tag{88}$$

and substitute Eq. (87) into (71) to obtain

$$P_a(a) = \frac{1}{2\pi}\iiiint c^*(\alpha)u_\alpha^*(x)e^{i\theta\mathbf{A}}c(\beta)u_\beta(x)e^{-i\theta a}dx\,d\theta d\alpha d\beta \tag{89}$$

$$= \frac{1}{2\pi}\iiiint c^*(\alpha)u_\alpha^*(x)e^{i\theta\beta}c(\beta)u_\beta(x)e^{-i\theta a}dx\,d\theta d\alpha d\beta \tag{90}$$

$$= \iint c^*(\alpha)c(\beta)\delta(\alpha - \beta)\delta(a - \beta)d\alpha d\beta \tag{91}$$

which evaluates to

$$P_a(a) = |c(a)|^2 \tag{92}$$

Appendix C: Standard vs. Quantum Manner of Calculating Expectation Values

We show Eq. (41) of the text, which we repeat here

$$\int g^*(x)\mathbf{A}\, g(x)\, dx = \sum a_n \left| \int g(x) u_n^*(x) dx \right|^2 \qquad \text{discrete case} \tag{93}$$

We expand $g(x)$

$$g(x) = \sum c_n u_n(x) \tag{94}$$

where

$$c_n = \int g(x) u_n^*(x) dx \tag{95}$$

Starting with the left hand side of Eq. (93) we have

$$\int g^*(x)\mathbf{A}\, g(x)\, dx \tag{96}$$

$$= \int \sum_{n,m} c_m^* u_m^*(x) \mathbf{A} c_n u_n(x)\, dx \tag{97}$$

$$= \int \sum_{n,m} c_m^* u_m^*(x) a_n c_n u_n(x)\, dx \tag{98}$$

$$= \sum_{n,m} c_m^* \delta_{nm} a_n c_n \tag{99}$$

$$= \sum_n a_n \left| \int g(x) u_n^*(x) dx \right|^2 \tag{100}$$

which is Eq. (93). The proof for the continuous case, Eq. (42), follows an analogous derivation.

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A Class of Non-Markovian Pseudo-differential Operators of Lévy Type

Rémi Léandre

Abstract We give large deviation estimates for a convolution semigroup, which is not Markovian and of Lévy type, of big order.

Keywords Pseudo-differential operator • Wentzel-Freidlin estimates

Mathematics Subject Classification (2000). Primary 60F10; Secondary 35S05

1 Introduction

There are much more semigroups than semigroups which are represented by stochastic processes. On the other hand, there are a lot of formulas in stochastic analysis which are natural. The theory of pseudo-differential operators [1], [3–7] allow to understand a lot of partial differential equations, including parabolic equations. On the other hand we have imported in the theory of non-markovian semigroups a lot of tools of stochastic analysis [12–26]. Stochastic analysis formulas are valid for the whole process. Their interpretation for non-markovian semigroups work only for the semigroup.

In [23] and [25], we have done with the classical normalization of semi-classical analysis [4] Wentzel-Freidlin estimates [5] for fourth order differential operators. Here we extend the method of [23] to the case of an integro-differential operator of big order which generates a non-markovian convolution semi-group. Normalisation are of Maslov type [4]. This paper presents a generalization of [26] to the multidimensional case.

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2 A Class of Elliptic Pseudo-differential Operators

Let us recall some basis of the pseudo-differential calculus. $\hat{f}$ is the Fourier transform of f. Let L_1 be an operator acting on $C_b^\infty(\mathbb{R}^d)$ by

$$L_1 f(x) = \int_{\mathbb{R}^d} a(x,\xi)\hat{f}(\xi)\exp[2\sqrt{-1}\pi < \xi, x >]d\xi \tag{1}$$

We say that $a(.,.)$ is its symbol. If

$$|\frac{\partial^n}{\partial x^n}\frac{\partial^m}{\partial \xi^m}a(x,\xi)| \leq C|\xi|^{r-m} \tag{2}$$

and if for $|\xi| > C_0$

$$|a(x,\xi)| \geq C|\xi|^{r'} \tag{3}$$

where $r' > 0$, we say that L_1 is an elliptic operator. Let us recall that our thesis underline the relationship between pseudo-differential operators and Poisson processes [11]. See the books [8–10]. We consider the operator:

$$Lf(x) = (-1)^{l+1}\int_{\mathbb{R}^d}(f(x+y) - f(x) - \sum_{i=1}^{2l}\frac{1}{i!} < y^{\otimes i}, f^{(i)}(x) >)\frac{h(x,y)}{|y|^{2l+d+\alpha}}dy \tag{4}$$

where h is a smooth function with bounded derivatives at each order and which is equal to zero if $|y| > C$. We suppose that $\alpha \in]-1,0[$.

Theorem 2.1 *If $h(x,0) = 1$, L is an elliptic pseudo-differential operator.*

Proof Let us compute the symbol of L.

$$f(x) = \int_{\mathbb{R}^d}\hat{f}(\xi)\exp[\sqrt{-1} < x, \xi >]d\xi \tag{5}$$

Therefore

$$Lf(x) = \int_{\mathbb{R}^d}\frac{h(x,y)}{|y|^{2l+d+\alpha}}dy$$

$$\int_{\mathbb{R}^d}(\exp[\sqrt{-1} < (x+y), \xi >] - \sum_{i=0}^{2l}\frac{(\sqrt{-1} < \xi, y >)^i}{i!}$$

$$exp[\sqrt{-1} < x, \xi >]\hat{f}(\xi)d\xi = \int_{\mathbb{R}^d}\hat{f}(\xi)\exp[\sqrt{-1} < x, \xi >]d\xi$$

$$\int_{\mathbb{R}^d} (\exp[\sqrt{-1} < y, \xi >] - \sum_{i=0}^{2l} \frac{(\sqrt{-1} < \xi, y >)^i}{i!}) \frac{h(x,y)}{|y|^{2l+d+\alpha}} dy$$
$$= \int_{\mathbb{R}^d} \hat{f}(\xi) H(x, \sqrt{-1}\xi) \exp[\sqrt{-1} < x, \xi >] d\xi \quad (6)$$

$H(x, \sqrt{-1}\xi)$ is obviously smooth and satisfy obviously to (2) due to the hypothesis on h. Let us show (3).

We put $y|\xi| = z$. We get that

$$H(x, \sqrt{-1}\xi) = |\xi|^{2l+\alpha} H_1(x, \sqrt{-1}\xi) \quad (7)$$

We introduce a smooth function from $\mathbb{R}^+$ into $[0, 1]$ which is equal to 0 in a neighborhood from 0 and to 1 in a neighborhood of ∞. We put $\xi_1 = \frac{\xi}{|\xi|}$. We get that

$$H_1(x, \sqrt{-1}\xi) = H_1^1(x, \sqrt{-1}\xi) + H_1^2(x, \sqrt{-1}\xi) \quad (8)$$

where

$$H_1^1(x, \sqrt{-1}\xi) =$$
$$\int_{\mathbb{R}^d} (\exp[\sqrt{-1} < (z, \xi_1 >] - \sum_{i=0}^{2l} \frac{(\sqrt{-1} < \xi_1, z >)^i}{i!}) \frac{h(x, \frac{z}{|\xi|}) h_1(\frac{|z|}{|\xi|})}{|z|^{2l+d+\alpha}} dy \quad (9)$$

We remark that

$$|H_1^1(x, \sqrt{-1}\xi)| \leq C|\xi|^{-k} \quad (10)$$

for all k since $h(x, y)$ is bounded with all bounded derivatives and with compact support in y. Let us give the details.

$$\exp[\sqrt{-1}r] - \sum_{i=0}^{2l} \frac{(\sqrt{-1}r)^i}{i!} = \int_{0<s_1<..<_{2l+1}<r} \exp[\sqrt{-1}s_1] ds_1..ds_{2l+1} \quad (11)$$

Therefore

$$H_1^1(x, \sqrt{-1}\xi) =$$
$$\int_{\mathbb{R}+} h_1(\frac{r}{|\xi|}) \frac{dr}{r^{2l+1+\alpha}} \int_{S^d} \int_{0<s_1<..<_{2l+1}<r} \exp[\sqrt{-1}s_1 < \theta, \xi_1 >] h(x, \frac{r\theta}{|\xi|}) ds_1..ds_{2l+1} \quad (12)$$

where $d\theta$ is the unit volume element in the unit sphere S^d of $\mathbb{R}^d$. We put $r_1 = r|\xi|$. We get

$$H_1^1(x, \sqrt{-1}\xi) =$$
$$\int_{S^d} d\theta \int_{0<s_1<..<s_{2l+1}<\frac{r_1}{|\xi|}} \int_{\mathbb{R}+} \exp[\sqrt{-1}s_1 < \theta, \xi_1 >]$$
$$h(x, r_1\theta)h_1(r_1)ds_1..ds_{2l+1}dr_1 \frac{|\xi|^{2l+\alpha}}{r_1^{2l+1+\alpha}} =$$
$$\int_{S^d} d\theta \int_{0<s_1<..<s_{2l+1}<r_1} \int_{\mathbb{R}+} \exp[\sqrt{-1}s_1|\xi| < \theta, \xi_1 >]$$
$$h(x, r_1\theta)h_1(r_1))ds_1..ds_{2l+1}dr_1 \frac{|\xi|^{\alpha-1}}{r_1^{2l+1+\alpha}} \tag{13}$$

We consider the primitive of the function

$$r_1 \to \int_{0<s_1<..<s_{2l+1}<r_1} \int_{\mathbb{R}+} \exp[\sqrt{-1}s_1|\xi| < \theta, \xi_1 >]h(x, r_1\theta)h_1(r_1))ds_1..ds_{2l+1} \tag{14}$$

They are given by iterated integrals of the same type but with longer lenght. We integrate by parts and take derivatives in the function

$$r_1 \to \frac{h_1(r_1)h(x, r_1\theta)}{r_1^{2l+1+\alpha}} \tag{15}$$

Therefore the result. Let us study $H_1^2(x, \sqrt{-1}\xi)$. We have if the support of $1 - h_1$ is small enough

$$ReH_1^2(x, \sqrt{-1}\xi) \geq \int_{1\leq|z|\leq\frac{C_1}{|\xi|}} dz \frac{|z|^{2l+1}}{|z|^{2l+d+\alpha}} \geq C|\xi|^{1-\alpha} \tag{16}$$

□

3 A Class of Non-Markovian Lévy Operators

Theorem 3.1 *Let us suppose that $h(x, y) = h(y)$ and that $h(y) = h(-y)$ and that $h \geq 0$. Let us consider the operator*

$$L(x) = (-1)^{l+1} \int_{\mathbb{R}^d} (f(x+y) - f(x) - \sum_{i=1}^{l} \frac{1}{2i!} < y^{\otimes 2i}, f^{(2i)}(x) >) \frac{h(y)}{|y|^{2l+d+\alpha}} dy \tag{17}$$

Then L is positive symmetric on $L^2(\mathbb{R}^d)$.

Proof Let us suppose that f and g are with compact support. We have

$$< Lf, g >=$$

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} g(x)(f(x+y)-f(x)-\sum_{i=1}^{l}\frac{1}{2i!} < y^{\otimes 2i}, f^{(2i)}(x) >)\frac{h(y)}{|y|^{2l+d+\alpha}}dxdy \tag{18}$$

The symmetry arises by doing the change of variable $y \to -y$ since $h(y) = h(-y)$.

Let us show that L is positive. We have

$$f(x+y)-f(x) = \sum_{i=1}^{2l-1}\frac{1}{i!} < y^{\otimes i}, f^{(i)}(x) > + \int_{0<s_1<..<s_{2l}<1} < y^{\otimes 2l}, f^{(2l)}(x+s_1y) > ds_1..ds_{2l} \tag{19}$$

Due to the fact that $h(y) = h(-y)$, we have only to look at the expression

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} f(x)\int_{0<s_1<..<s_{2l}<1} < y^{\otimes 2l}, f^{(2l)}(x+s_1y) > ds_1..ds_{2l}dxdy \tag{20}$$

By integating by part in x, this expression is equal to

$$(-1)^l\int_{\mathbb{R}^d\times\mathbb{R}^d} < y^{\otimes l}, f^{(l)}(x) > \int_{0<s_1<..<s_{2l}<1} < y^{\otimes l}, f^{(l)}(x+s_1y) > ds_1..ds_{2l}dxdy \tag{21}$$

By Cauchy-Schwartz inequality

$$\int_{\mathbb{R}^d} < y^{\otimes l}, f^{(l)}(x) >< y^{\otimes l}, f^{(l)}(x+s_1y) > dx \le \int_{\mathbb{R}^d} < y^{\otimes l}, f^{(l)}(x) >^2 dx \tag{22}$$

This shows that

$$(-1)^{l+1}\int_{\mathbb{R}^d\times\mathbb{R}^d} f(x) \int_{0<s_1<..<s_{2l}<1} < y^{\otimes 2l}, f^{(2l)}(x+s_1y) - f^{(2l)}(x) > ds_1..ds_{2l}dxdy \ge 0 \tag{23}$$

We deduce the result. □

By ellipticity, we get:

Theorem 3.2 *If $l+1$ is even L generates a contraction semigroup* $\exp[-tL]$ *on* $L^2(\mathbb{R}^d)$ *and which acts continuously on* $C_b(\mathbb{R}^d)$

Let us consider an example (with the only difference that h has no compact support!). We consider

$$Lf(x) = (-1)^{l+1}\int_{\mathbb{R}^d}(f(x+y)-f(x)-\sum_{i=1}^{l}\frac{1}{2i!}<y^{\otimes 2i},f^{(2i)}(x)>)\frac{1}{|y|^{2l+d+\alpha}}dy \tag{24}$$

His symbol is

$$H(\sqrt{-1}\xi) = \int_{\mathbb{R}^d}(\cos(<y,\xi>)-\sum_{i=0}^{l}\frac{(-1)^i<\xi,y>)^{2i}}{2i!})\frac{1}{|y|^{2l+d+\alpha}}dy \tag{25}$$

We put $y_1 = y|\xi|$ and $\xi_1 = \frac{\xi}{|\xi|}$ such that

$$H(\sqrt{-1}\xi) = A(\xi_1)|\xi|^{2l+\alpha} \tag{26}$$

where

$$A(\xi_1) = \int_{\mathbb{R}^d}(\cos(<y_1,\xi_1>)-\sum_{i=0}^{l}\frac{(-1)^i<\xi_1,y_1>)^{2i}}{2i!})\frac{1}{|y_1|^{2l+d+\alpha}}dy_1 \tag{27}$$

By rotational invariance $A(\xi_1)$ does not depend of ξ_1. But the symbol of the standard laplacian on $\mathbb{R}^d$ is $|\xi|^2$. So L is a fractional power of L (See [18] for another presentation).

4 The Action Functional

We consider the Hamiltonian

$$H(\xi) = \int_{\mathbb{R}^d}(\exp[<\xi,y>]-1-\sum_{i=1}^{l}\frac{<\xi,y>^{2i}}{2i!})\frac{h(y)}{|y|^{d+2l+\alpha}}dy \tag{28}$$

Theorem 4.1 *$H(\xi)$ is a smooth convex function equals to 1 in 0.*

Proof Since h is with compact support, H is clearly smooth with bounded derivatives at each order. Moreover,

$$F_y(\xi) = \exp[<y,\xi>]+\exp[-<y,\xi>]-2\sum_{i=0}^{l}\frac{<\xi,y>^{2i}}{2i!} = 2\sum_{i=l+1}^{\infty}\frac{<\xi,y>^{2i}}{2i!} \tag{29}$$

Then the function $\xi \to F_y(\xi)$ is a sum of positive convex functions and his therefore a positive convex function. But $h(y) = h(-y)$. It follows that

$$H(\xi) = \int_E F_y(\xi) \frac{h(y)}{|y|^{d+2l+\alpha}} dy \tag{30}$$

(for a convenient subset E of $\mathbb{R}^d$) is a positive convex function since h is positive.

□

Associate to it, we consider its Legendre transform:

$$L(p) = \sup_{\xi \in \mathbb{R}^d} (< \xi, p > -H(\xi)) \tag{31}$$

If ϕ is a finite energy function in $\mathbb{R}^d$, we consider the action functional

$$S(\phi) = \int_0^1 L(\frac{d\phi}{dt})dt \tag{32}$$

According the theory of semi-classical analysis [5], we consider the symbol L_1^ϵ associated to the symbol $\epsilon^{-1}a(x, \epsilon\xi)$. This leads to the operator

$$L^\epsilon f(x) = (-1)^{l+1} \frac{1}{\epsilon} \int_{\mathbb{R}^d} (f(x + \epsilon y) - f(x) - \sum_{i=1}^{l} \frac{\epsilon^{2i}}{2i!} < y^{\otimes 2i}, f^{(2i)}(x) >) \frac{h(y)}{|y|^{2l+d+\alpha}} dy \tag{33}$$

By elliptic theory L^ϵ generates a semigroup on L^2 and even on $C_b(\mathbb{R}^d)$ P_t^ϵ. We consider its absolute value $|P_t^\epsilon|$. We have

Theorem 4.2 (Wentzel-Freidlin estimates) *Let O be the complement in $\mathbb{R}^d$ of the interval of the cube of center x and radius δ. We have when $\epsilon \to 0$*

$$\overline{Lim} \epsilon Log |P_1^\epsilon|[1_O](x) \leq - \inf_{\phi(0)=x; \phi(1) \in O} S(\phi) \tag{34}$$

if $l + 1$ is even.

5 Proof of the Wentzel-Freidlin Estimates

Let us begin by some elementary remarks. We remark that

$$\hat{Lf} = H(\sqrt{-1})\hat{f} \tag{35}$$

such that

$$\hat{P_t f} = \exp[-tH(\sqrt{-1}).]\hat{f} \tag{36}$$

These elementary remarks (which are true a lot of convolution semigroups) will allow us to adapt the proof of [5] and [26].

Theorem 5.1 *For all $\delta > 0$, all C there exist t_δ such that if $t < t_\delta$*

$$|P_t^\epsilon|(1_O)(x) \leq \exp[-\frac{C}{\epsilon}] \tag{37}$$

Proof We consider the semigroup

$$\exp[- < \frac{x, \xi}{\epsilon} >]P_t^\epsilon[\exp[< \frac{x', \xi}{\epsilon} >]f(x')](x) \tag{38}$$

The symbol of its generator is

$$F_\xi^\epsilon(\xi') = \frac{1}{\epsilon}H(\epsilon\sqrt{-1}\xi' + \xi) \tag{39}$$

This is the symbol of an elliptic operator which is positive if $|\xi'|$ is big. It generates therefore a semi-group on $C_b(\mathbb{R}^d)$ $Q_t^{\epsilon,\xi}$. We get the expansion

$$\begin{aligned} F_\xi^\epsilon(\xi') = \frac{H(\xi)}{\epsilon} + \\ H^{(1)}(\xi)\sqrt{-1}\xi' + \epsilon\int_{0<s_1<s_2<1} < \xi'^{\otimes 2}, H^{(2)}(\epsilon s_1\sqrt{-1}\xi') > +\xi)ds_1 ds_2 + \\ \frac{H(\xi)}{\epsilon} + H^{(1)}(\sqrt{-1}\xi') + R_\xi^\epsilon(\xi') \end{aligned} \tag{40}$$

Therefore we get

$$\hat{Q_t^{\epsilon,\xi} f} = \exp[-\frac{tH(\xi)}{\epsilon}]\exp[-t(+H^{(1)}(\sqrt{-1}\xi') + R_\xi^\epsilon(\xi'))]\hat{f} \tag{41}$$

The uniform norm of $\exp[-t(+H^{(1)}\sqrt{-1}\xi' + R_\xi^\epsilon(\xi'))$ is bounded and the uniform norm of its derivative is bounded by $\exp[C|\xi|]/\epsilon$. Therefore the norm on $C_b(\mathbb{R}^d)$ of $Q_t^{\epsilon,\xi}$ is bounded by $\exp[-\frac{CtH(\xi)}{\epsilon}]\exp[C|\xi|]$. Therefore

$$|P_t^\epsilon|(1_O)(x) \leq \exp[-\frac{CtH(\xi)}{\epsilon}]\exp[\frac{|\delta\xi|}{\epsilon}]\exp[C|\xi|] \tag{42}$$

But $H(\xi) \geq C|\xi|^2$ if $\xi > 0$ for some C. □

Remark In this inequality, the classical Davies gauge transform plays a fundamental role [2] and replaces the role of exponential martingales of [5].

Proof When we have proved this lemma, the estimates follow closely the lines of [5] and [26].

We cut the time interval $[0, 1]$ is small intervals of length $[t_i, t_{i+1}]$. By the semigroup property we use that

$$|P_1^\epsilon|[1_O](x) \leq |P_{t_1}^\epsilon|..|P_{1-t_n}^\epsilon|[1_O](x) \tag{43}$$

In $P_{t_{i+1}-t_i}^\epsilon$, we distinguish if $x_{t_{i-1}}$ and x_{t_i} are far or not. If they are far, we use the previous lemma. If they are close, we deduce a positive measures $|W_\epsilon|$ on polygonal paths ϕ_t which joins x_{t_i} to $x_{t_{i+1}}$. By the previous lemma, it remains to estimate $|W_\epsilon|[1_O(\phi_1)]$. But $|W_\epsilon|$ is a positive measure, we have

$$|W_\epsilon|[1_O](\phi_1)] \leq |W_\epsilon|[\exp[\frac{S(\phi)}{\epsilon}][1_O(\phi(1)]\exp[-\inf_{\phi(0)=x;\phi(1)\in O}\frac{S(\phi)}{\epsilon}] \tag{44}$$

Therefore we have only to estimate $|W_\epsilon|[\exp[\frac{S(\phi)}{\epsilon}]1_O(\phi_1)]$. The sequel follows [5, p. 152] and [26]. We can choose some p_i in finite numbers such that if we put

$$L'(p) = \sup_i(L(p_i) + \frac{\partial}{\partial p}L(p_i)(p - p_i)) \tag{45}$$

we have for all polygonal paths considereded for a small χ

$$L(\frac{d\phi_t}{dt}) - L'(\frac{d\phi_t}{dt}) \leq \chi \tag{46}$$

Let us put

$$S'(\phi) = \int_0^1 L'(\frac{d\phi_t}{dt})dt \tag{47}$$

Since $|W_\epsilon|$ is a positive measure, we have only to estimate the quantity

$$|W_\epsilon|[\exp[\frac{S'(\phi)}{\epsilon}]1_O(\phi_1)] \tag{48}$$

We remark that

$$\exp[\sup a_i] \leq \sum \exp[a_i] \tag{49}$$

Moreover

$$L'(p) = \sup(< \xi_i, p > -H(\xi_i)) \tag{50}$$

where $\xi_i = \frac{\partial}{\partial p} L(p_i)$. Therefore it is enough to show that

$$\sup_{x,|\xi|<C} |P^{\epsilon}_{t_\delta}|[\exp[< \frac{\xi}{\epsilon}, (x' - x) > -t_\delta H(\xi)]](x) \tag{51}$$

has a small blowing up when $\epsilon \to 0$. We do as in the previous lemma. We consider the generator of the semigroup

$$f \to P^{\epsilon}_t[\exp[< \frac{\xi}{\epsilon}, (x' - x) > -tH(\xi)]f](x) \tag{52}$$

Its symbol is

$$\frac{1}{\epsilon} H(\epsilon\sqrt{-1}\xi' + \xi) - \frac{1}{\epsilon} H(\xi) \tag{53}$$

Its asymptotic expansion in ϵ is

$$((H^{(1)}(\sqrt{-1}\xi') + R^{\epsilon}_{\xi}(\xi') \tag{54}$$

The result follows as in the lemma. □

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On the Solvability of Some Systems of Integro-Differential Equations with Anomalous Diffusion

Vitali Vougalter and Vitaly Volpert

Abstract The article deals with the existence of solutions of a system of integro-differential equations in the case of anomalous diffusion with the Laplacian in a fractional power. The proof of existence of solutions is based on a fixed point technique. Solvability conditions for non Fredholm elliptic operators in unbounded domains are used.

Keywords Integro-differential equations • Non Fredholm operators • Sobolev spaces

Mathematics Subject Classification (2000). 35J05, 35P30, 47F05

1 Introduction

The present work is devoted to the existence of stationary solutions of the following system of integro-differential equations

$$\frac{\partial u_m}{\partial t} = -D_m \left(-\frac{\partial^2}{\partial x^2}\right)^s u_m + \int_{-\infty}^{\infty} K_m(x-y) g_m(u(y,t))dy + f_m(x), \tag{1}$$

$1 \leq m \leq N$, appearing in cell population dynamics. The space variable x here corresponds to the cell genotype, functions $u_m(x,t)$ describe the cell density

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distributions for various groups of cells as functions of their genotype and time,

$$u(x,t) = (u_1(x,t), u_2(x,t), \dots, u_N(x,t))^T.$$

The right side of this system of equations describes the evolution of cell densities due to cell proliferation, mutations and cell influx or efflux. The anomalous diffusion terms with positive coefficients D_m correspond to the change of genotype due to small random mutations, and the nonlocal production terms describe large mutations. Functions $g_m(u)$ denote the rates of cell birth which depend on u (density dependent proliferation), and the kernels $K_m(x-y)$ express the proportions of newly born cells changing their genotype from y to x. We assume that they depend on the distance between the genotypes. The functions $f_m(x)$ describe the influx or efflux of cells for different genotypes.

The operator $\left(-\frac{\partial^2}{\partial x^2}\right)^s$ in system (1) describes a particular case of anomalous diffusion actively studied in the context of different applications in plasma physics and turbulence [7, 16], surface diffusion [12, 14], semiconductors [15] and so on. Anomalous diffusion can be understood as a random process of particle motion characterized by the probability density distribution of jump length. The moments of this density distribution are finite in the case of normal diffusion, but this is not the case for superdiffusion. Asymptotic behavior at infinity of the probability density function determines the value s of the power of the Laplacian [13]. The operator $\left(-\frac{\partial^2}{\partial x^2}\right)^s$ is defined by means of the spectral calculus. In the present work we will consider the case of $0 < s < 1/4$. A similar problem in the case of the standard Laplace operator in the diffusion term was studied recently in [28]. Note that the restriction on the power s here comes from the solvability conditions of our problem.

Let us set all $D_m = 1$ and establish the existence of solutions of the system of equations

$$-\left(-\frac{d^2}{dx^2}\right)^s u_m + \int_{-\infty}^{\infty} K_m(x-y)g_m(u(y))dy + f_m(x) = 0, \quad 0 < s < \frac{1}{4}, \tag{2}$$

with $1 \le m \le N$. Let us consider the case where the linear part of this operator fails to satisfy the Fredholm property. As a consequence, conventional methods of nonlinear analysis may not be applicable. We use solvability conditions for non Fredholm operators along with the method of contraction mappings.

Consider the equation

$$-\Delta u + V(x)u - au = f, \tag{3}$$

where $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, a is a constant and the scalar potential function $V(x)$ is either zero identically or converges to 0 at infinity. For $a \ge 0$, the essential spectrum of the operator $A : E \to F$ corresponding to the left side of problem (3) contains the origin. As a consequence, such operator fails to satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimension of its kernel and the codimension of its image are not finite. The present work deals with the studies of some properties of the operators of this kind. Note that elliptic problems with non Fredholm operators were treated actively in recent years.

Approaches in weighted Sobolev and Hölder spaces were developed in [2–6]. The non Fredholm Schrödinger type operators were studied with the methods of the spectral and the scattering theory in [17, 22, 24]. The Laplace operator with drift from the point of view of non Fredholm operators was treated in [25] and linearized Cahn-Hilliard problems in [20] and [26]. Nonlinear non Fredholm elliptic problems were studied in [23] and [27]. Important applications to the theory of reaction-diffusion equations were developed in [9, 10]. Non Fredholm operators arise also when studying wave systems with an infinite number of localized traveling waves (see [1]). In particular, when $a = 0$ the operator A is Fredholm in some properly chosen weighted spaces (see [2–6]). However, the case of $a \neq 0$ is significantly different and the approach developed in these articles cannot be used. Front propagation equations with anomalous diffusion were studied largely in recent years (see e.g. [18, 19]).

We set $K_m(x) = \varepsilon_m \mathcal{K}_m(x)$ with $\varepsilon_m \geq 0$, such that

$$\varepsilon := max_{1 \leq m \leq N} \varepsilon_m$$

and suppose that the following assumption is satisfied.

Assumption 1.1 *Let* $1 \leq m \leq N$ *and consider* $0 < s < \frac{1}{4}$. *Let* $f_m(x) : \mathbb{R} \to \mathbb{R}$ *be nontrivial for some m. Let* $f_m(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ *and*

$$\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}-s} f_m(x) \in L^2(\mathbb{R}).$$

Assume also that $\mathcal{K}_m(x) : \mathbb{R} \to \mathbb{R}$, *such that* $\mathcal{K}_m(x) \in L^1(\mathbb{R})$ *and*

$$\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}-s} \mathcal{K}_m(x) \in L^2(\mathbb{R}).$$

Moreover,

$$\mathcal{K}^2 := \sum_{m=1}^{N} \|\mathcal{K}_m(x)\|^2_{L^1(\mathbb{R})} > 0$$

and

$$Q^2 := \sum_{m=1}^{N} \left\| \left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}-s} \mathcal{K}_m(x) \right\|_{L^2(\mathbb{R})}^2 > 0.$$

Let us choose the space dimension $d = 1$, which is related to the solvability conditions for the linear Poisson type problem (34) stated in Lemma 4.1 below. We use the Sobolev spaces for $0 < s \le 1$, namely

$$H^{2s}(\mathbb{R}) := \left\{ \phi(x) : \mathbb{R} \to \mathbb{R} \mid \phi(x) \in L^2(\mathbb{R}),\ \left(-\frac{d^2}{dx^2}\right)^s \phi \in L^2(\mathbb{R}) \right\}$$

equipped with the norm

$$\|\phi\|_{H^{2s}(\mathbb{R})}^2 := \|\phi\|_{L^2(\mathbb{R})}^2 + \left\| \left(-\frac{d^2}{dx^2}\right)^s \phi \right\|_{L^2(\mathbb{R})}^2. \tag{4}$$

For a vector function

$$u(x) = (u_1(x), u_2(x), \dots, u_N(x))^T$$

we will use the norm

$$\|u\|_{H^1(\mathbb{R},\mathbb{R}^N)}^2 := \|u\|_{L^2(\mathbb{R},\mathbb{R}^N)}^2 + \sum_{m=1}^{N} \left\| \frac{du_m}{dx} \right\|_{L^2(\mathbb{R})}^2, \tag{5}$$

where

$$\|u\|_{L^2(\mathbb{R},\mathbb{R}^N)}^2 := \sum_{m=1}^{N} \|u_m\|_{L^2(\mathbb{R})}^2.$$

By means of the standard Sobolev inequality in one dimension (see e.g. Section 8.5 of [11]) we have

$$\|\phi\|_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2}} \|\phi\|_{H^1(\mathbb{R})}. \tag{6}$$

When all the nonnegative parameters ε_m vanish, we obtain the linear Poisson type equations

$$\left(-\frac{d^2}{dx^2}\right)^s u_m = f_m(x), \quad 1 \le m \le N. \tag{7}$$

By virtue of Lemma 4.1 below along with Assumption 1.1 each equation (7) has a unique solution

$$u_{0,m}(x) \in H^{2s}(\mathbb{R}), \quad 0 < s < \frac{1}{4},$$

such that no orthogonality conditions are required. By means of Lemma 4.1, when $\frac{1}{4} \le s < 1$, certain orthogonality relations (36) and (37) are necessary to be able to solve problem (7) in $H^{2s}(\mathbb{R})$. By means of Assumption 1.1, since

$$\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}} u_{0,m}(x) = \left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}-s} f_m(x) \in L^2(\mathbb{R}),$$

we get for the unique solution of linear problem (7) that $u_{0,m}(x) \in H^1(\mathbb{R})$, such that

$$u_0(x) := (u_{0,1}(x), u_{0,2}(x), \dots, u_{0,N}(x))^T \in H^1(\mathbb{R}, \mathbb{R}^N).$$

We seek the resulting solution of nonlinear system of equations (2) as

$$u(x) = u_0(x) + u_p(x), \tag{8}$$

where

$$u_p(x) := (u_{p,1}(x), u_{p,2}(x), \dots, u_{p,N}(x))^T.$$

Clearly, we arrive at the perturbative system of equations

$$\left(-\frac{d^2}{dx^2}\right)^s u_{p,m} = \varepsilon_m \int_{-\infty}^{\infty} \mathcal{K}_m(x-y) g_m(u_0(y) + u_p(y)) dy, \quad 0 < s < \frac{1}{4}, \tag{9}$$

where $1 \le m \le N$. Let us introduce a closed ball in the Sobolev space

$$B_\rho := \{u(x) \in H^1(\mathbb{R}, \mathbb{R}^N) \mid \|u\|_{H^1(\mathbb{R}, \mathbb{R}^N)} \le \rho\}, \quad 0 < \rho \le 1. \tag{10}$$

We seek the solution of problem (9) as the fixed point of the auxiliary nonlinear system of equations

$$\left(-\frac{d^2}{dx^2}\right)^s u_m = \varepsilon_m \int_{-\infty}^{\infty} \mathcal{K}_m(x-y) g_m(u_0(y) + v(y)) dy, \quad 0 < s < \frac{1}{4}, \tag{11}$$

with $1 \le m \le N$ in ball (10). For a given vector function $v(y)$ this is a system of equations with respect to $u(x)$. The left side of (11) involves the non Fredholm operator

$$\left(-\frac{d^2}{dx^2}\right)^s : H^{2s}(\mathbb{R}) \to L^2(\mathbb{R}).$$

Its essential spectrum fills the nonnegative semi-axis $[0, +\infty)$. Therefore, such operator has no bounded inverse. The similar situation appeared in articles [23] and [27] but as distinct from the present situation, the equations studied there required orthogonality conditions. The fixed point technique was used in [21] to estimate the perturbation to the standing solitary wave of the Nonlinear Schrödinger (NLS) equation when either the external potential or the nonlinear term in the NLS were perturbed but the Schrödinger operator involved in the nonlinear equation there had the Fredholm property (see Assumption 1 of [21], also [8]). We define the closed ball in the space of N dimensions as

$$I := \left\{ z \in \mathbb{R}^N \mid |z| \le \frac{1}{\sqrt{2}}(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1) \right\} \tag{12}$$

along with the closed ball in the space of $C^2(I, \mathbb{R}^N)$ functions, namely

$$D_M := \{g(z) := (g_1(z), g_2(z), \dots, g_N(z)) \in C^2(I, \mathbb{R}^N) \mid \|g\|_{C^2(I,\mathbb{R}^N)} \le M\}, \tag{13}$$

where $M > 0$. Here the norms

$$\|g\|_{C^2(I,\mathbb{R}^N)} := \sum_{m=1}^{N} \|g_m\|_{C^2(I)}, \tag{14}$$

$$\|g_m\|_{C^2(I)} := \|g_m\|_{C(I)} + \sum_{n=1}^{N} \left\| \frac{\partial g_m}{\partial z_n} \right\|_{C(I)} + \sum_{n,l=1}^{N} \left\| \frac{\partial^2 g_m}{\partial z_n \partial z_l} \right\|_{C(I)}, \tag{15}$$

where $\|g_m\|_{C(I)} := max_{z \in I}|g_m(z)|$. Let us make the following assumption on the nonlinear part of system (2).

Assumption 1.2 *Let $1 \le m \le N$. Assume that $g_m(z) : \mathbb{R}^N \to \mathbb{R}$, such that $g_m(0) = 0$ and $\nabla g_m(0) = 0$. It is also assumed that $g(z) \in D_M$ and it does not vanish identically in the ball I.*

Let us explain why we assume here that $\nabla g_m(0) = 0,\ 1 \le m \le N$. If $\frac{\partial g_m}{\partial z_k} < 0$, for $1 \le k \le N$, then the essential spectrum of the corresponding linearized operator is in the left-half plane. Such operator satisfies the Fredholm property, and the standard methods of nonlinear analysis are applicable here. When $\frac{\partial g_m}{\partial z_k} \ge 0$, our operator fails to satisfy the Fredholm property and the goal is to establish the existence of solutions in the situation when usual techniques are not applicable. The approach developed in this work can be used when $\nabla g_m(0) = 0,\ 1 \le m \le N$ but not for $\dfrac{\partial g_m}{\partial z_k} > 0$. Therefore we impose the appropriate condition on the nonlinear terms.

We introduce the operator T_g, such that $u = T_g v$, where u is a solution of system (11). Our first main proposition is as follows.

Theorem 1.3 *Let Assumptions 1.1 and 1.2 hold. Then for every $\rho \in (0, 1]$ there exists $\varepsilon^* > 0$, such that system (11) defines the map $T_g : B_\rho \to B_\rho$, which is a strict contraction for all $0 < \varepsilon < \varepsilon^*$. The unique fixed point $u_p(x)$ of this map T_g is the only solution of system (9) in B_ρ.*

Evidently, the resulting solution $u(x)$ of system (2) will be nontrivial because the source terms $f_m(x)$ are nontrivial for some $1 \le m \le N$ and all $g_m(0) = 0$ as assumed. We make use of the following trivial lemma.

Lemma 1.4 *For $R \in (0, +\infty)$ consider the function*

$$\varphi(R) := \alpha R^{1-4s} + \frac{\beta}{R^{4s}}, \quad 0 < s < \frac{1}{4}, \quad \alpha, \beta > 0.$$

It achieves the minimal value at $R^* := \dfrac{4\beta s}{\alpha(1-4s)}$, which is given by

$$\varphi(R^*) = \frac{(1-4s)^{4s-1}}{(4s)^{4s}} \alpha^{4s} \beta^{1-4s}.$$

Our second main result is about the continuity of the fixed point of the map T_g which existence was proved in Theorem 1.3 above with respect to the nonlinear vector function g.

Theorem 1.5 *Let $j = 1, 2$, the assumptions of Theorem 1.3 hold, such that $u_{p,j}(x)$ is the unique fixed point of the map $T_{g_j} : B_\rho \to B_\rho$, which is a strict contraction for all $0 < \varepsilon < \varepsilon_j^*$ and $\delta := min(\varepsilon_1^*, \varepsilon_2^*)$. Then for all $0 < \varepsilon < \delta$ the inequality*

$$\|u_{p,1} - u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)} \le C \|g_1 - g_2\|_{C^2(I,\mathbb{R}^N)} \tag{16}$$

holds, where $C > 0$ is a constant.

We proceed to the proof of our first main proposition.

2 The Existence of the Perturbed Solution (Proof of Theorem 1.3)

We choose arbitrarily $v(x) \in B_\rho$ and designate the term involved in the integral expression in the right side of system (11) as

$$G_m(x) := g_m(u_0(x) + v(x)), \quad 1 \le m \le N.$$

Let us use the standard Fourier transform

$$\widehat{\phi}(p) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x) e^{-ipx} dx. \tag{17}$$

Obviously, we have the inequality

$$\|\widehat{\phi}(p)\|_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2\pi}} \|\phi(x)\|_{L^1(\mathbb{R})}. \tag{18}$$

Let us apply (17) to both sides of system (11) and obtain

$$\widehat{u_m}(p) = \varepsilon_m \sqrt{2\pi} \frac{\widehat{\mathcal{K}_m}(p)\widehat{G_m}(p)}{|p|^{2s}}, \quad 1 \le m \le N.$$

Thus we express the norm as

$$\|u_m\|^2_{L^2(\mathbb{R})} = 2\pi\varepsilon_m^2 \int_{-\infty}^{\infty} \frac{|\widehat{\mathcal{K}_m}(p)|^2 |\widehat{G_m}(p)|^2}{|p|^{4s}} dp, \quad 1 \le m \le N. \tag{19}$$

As distinct from articles [23] and [27] involving the standard Laplace operator in the diffusion term, here we do not try to control the norm

$$\left\| \frac{\widehat{\mathcal{K}_m}(p)}{|p|^{2s}} \right\|_{L^\infty(\mathbb{R})}.$$

Instead, we estimate the right side of (19) using the analog of inequality (18) applied to functions $\mathcal{K}_m$ and G_m with $R > 0$ as

$$2\pi\varepsilon_m^2 \left[\int_{|p| \le R} \frac{|\widehat{\mathcal{K}_m}(p)|^2 |\widehat{G_m}(p)|^2}{|p|^{4s}} dp + \int_{|p| > R} \frac{|\widehat{\mathcal{K}_m}(p)|^2 |\widehat{G_m}(p)|^2}{|p|^{4s}} dp \right] \le$$

$$\le \varepsilon_m^2 \|\mathcal{K}_m\|^2_{L^1(\mathbb{R})} \left\{ \frac{1}{\pi} \|G_m(x)\|^2_{L^1(\mathbb{R})} \frac{R^{1-4s}}{1-4s} + \frac{1}{R^{4s}} \|G_m(x)\|^2_{L^2(\mathbb{R})} \right\}. \tag{20}$$

Using norm definition (5) along with the triangle inequality and due to the fact that $v(x) \in B_\rho$, we easily obtain

$$\|u_0 + v\|_{L^2(\mathbb{R},\mathbb{R}^N)} \le \|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1.$$

Sobolev inequality (6) implies that

$$|u_0 + v| \le \frac{1}{\sqrt{2}}(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1).$$

Let the dot denote the scalar product of two vectors in $\mathbb{R}^N$. Formula

$$G_m(x) = \int_0^1 \nabla g_m(t(u_0(x) + v(x))).(u_0(x) + v(x))dt, \quad 1 \le m \le N$$

with the ball I defined in (12) yields

$$|G_m(x)| \le sup_{z\in I}|\nabla g_m(z)||u_0(x) + v(x)| \le M|u_0(x) + v(x)|.$$

Thus

$$\|G_m(x)\|_{L^2(\mathbb{R})} \le M\|u_0 + v\|_{L^2(\mathbb{R},\mathbb{R}^N)} \le M(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1).$$

Apparently, for $t \in [0, 1]$ and $1 \le m, j \le N$, we have

$$\frac{\partial g_m}{\partial z_j}(t(u_0(x) + v(x))) = \int_0^t \nabla \frac{\partial g_m}{\partial z_j}(\tau(u_0(x) + v(x))).(u_0(x) + v(x))d\tau.$$

This implies

$$\left|\frac{\partial g_m}{\partial z_j}(t(u_0(x) + v(x)))\right| \le sup_{z\in I}\left|\nabla\frac{\partial g_m}{\partial z_j}\right| |u_0(x) + v(x)| \le$$

$$\le \sum_{n=1}^{N} \left\|\frac{\partial^2 g_m}{\partial z_n \partial z_j}\right\|_{C(I)} |u_0(x) + v(x)|.$$

Therefore,

$$|G_m(x)| \le |u_0(x) + v(x)| \sum_{n,j=1}^{N} \left\|\frac{\partial^2 g_m}{\partial z_n \partial z_j}\right\|_{C(I)} |u_{0,j}(x) + v_j(x)| \le M|u_0(x) + v(x)|^2.$$

Hence,

$$\|G_m(x)\|_{L^1(\mathbb{R})} \le M\|u_0 + v\|^2_{L^2(\mathbb{R},\mathbb{R}^N)} \le M(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^2. \tag{21}$$

This enables us to obtain the upper bound for the right side of (20) as

$$\varepsilon_m^2 M^2 \|\mathcal{K}_m\|_{L^1(\mathbb{R})}^2 (\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^2 \left\{ \frac{(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^2 R^{1-4s}}{\pi(1-4s)} + \frac{1}{R^{4s}} \right\},$$

with $R \in (0, +\infty)$. Lemma 1.4 gives us the minimal value of the expression above. Thus,

$$\|u_m\|_{L^2(\mathbb{R})}^2 \leq \varepsilon^2 \|\mathcal{K}_m\|_{L^1(\mathbb{R})}^2 (\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^{2+8s} \frac{M^2}{(1-4s)(4\pi s)^{4s}},$$

such that

$$\|u\|_{L^2(\mathbb{R},\mathbb{R}^N)}^2 \leq \varepsilon^2 \mathcal{K}^2 (\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^{2+8s} \frac{M^2}{(1-4s)(4\pi s)^{4s}}. \tag{22}$$

Clearly, (11) yields

$$\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}} u_m(x) = \varepsilon_m \left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}-s} \int_{-\infty}^{\infty} \mathcal{K}_m(x-y) G_m(y) dy, \quad 1 \leq m \leq N.$$

By means of the analog of inequality (18) applied to function G_m along with (21) we obtain

$$\left\| \frac{du_m}{dx} \right\|_{L^2(\mathbb{R})}^2 \leq \varepsilon_m^2 \|G_m\|_{L^1(\mathbb{R})}^2 \left\| \left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}-s} \mathcal{K}_m \right\|_{L^2(\mathbb{R})}^2 \leq$$

$$\leq \varepsilon^2 M^2 (\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^4 \left\| \left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}-s} \mathcal{K}_m \right\|_{L^2(\mathbb{R})}^2,$$

such that

$$\sum_{m=1}^{N} \left\| \frac{du_m}{dx} \right\|_{L^2(\mathbb{R})}^2 \leq \varepsilon^2 M^2 (\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^4 Q^2. \tag{23}$$

Therefore, by virtue of the definition of the norm (5) along with inequalities (22) and (23) we derive the estimate from above for $\|u\|_{H^1(\mathbb{R},\mathbb{R}^N)}$ as

$$\varepsilon M (\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^2 \left[\frac{\mathcal{K}^2 (\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^{8s-2}}{(1-4s)(4\pi s)^{4s}} + Q^2 \right]^{\frac{1}{2}} \leq \rho \tag{24}$$

for all $\varepsilon > 0$ sufficiently small. Hence, $u(x) \in B_\rho$ as well. If for a certain $v(x) \in B_\rho$ there exist two solutions $u_{1,2}(x) \in B_\rho$ of system (11), their difference $w(x) := u_1(x) - u_2(x) \in L^2(\mathbb{R}, \mathbb{R}^N)$ solves

$$\left(-\frac{d^2}{dx^2}\right)^s w_m = 0, \quad 1 \le m \le N.$$

Because the operator $\left(-\frac{d^2}{dx^2}\right)^s$ considered on the whole real line does not possess nontrivial square integrable zero modes, $w(x)$ vanishes a.e. on $\mathbb{R}$. Thus, system (11) defines a map $T_g : B_\rho \to B_\rho$ for all $\varepsilon > 0$ small enough. Our goal is to establish that this map is a strict contraction. Let us choose arbitrarily $v_{1,2}(x) \in B_\rho$. The argument above implies $u_{1,2} := T_g v_{1,2} \in B_\rho$ as well. By means of (11) we have for $1 \le m \le N$

$$\left(-\frac{d^2}{dx^2}\right)^s u_{1,m} = \varepsilon_m \int_{-\infty}^{\infty} \mathcal{K}_m(x-y) g_m(u_0(y) + v_1(y)) dy, \tag{25}$$

$$\left(-\frac{d^2}{dx^2}\right)^s u_{2,m} = \varepsilon_m \int_{-\infty}^{\infty} \mathcal{K}_m(x-y) g_m(u_0(y) + v_2(y)) dy, \tag{26}$$

$0 < s < \frac{1}{4}$. We introduce

$$G_{1,m}(x) := g_m(u_0(x) + v_1(x)), \quad G_{2,m}(x) := g_m(u_0(x) + v_2(x)), \quad 1 \le m \le N$$

and apply the standard Fourier transform (17) to both sides of systems (25) and (26). This yields

$$\widehat{u}_{1,m}(p) = \varepsilon_m \sqrt{2\pi} \frac{\widehat{\mathcal{K}}_m(p) \widehat{G}_{1,m}(p)}{|p|^{2s}}, \quad \widehat{u}_{2,m}(p) = \varepsilon_m \sqrt{2\pi} \frac{\widehat{\mathcal{K}}_m(p) \widehat{G}_{2,m}(p)}{|p|^{2s}}.$$

Obviously,

$$\|u_{1,m} - u_{2,m}\|_{L^2(\mathbb{R})}^2 = \varepsilon_m^2 2\pi \int_{-\infty}^{\infty} \frac{|\widehat{\mathcal{K}}_m(p)|^2 |\widehat{G}_{1,m}(p) - \widehat{G}_{2,m}(p)|^2}{|p|^{4s}} dp.$$

Evidently, it can be estimated from above by virtue of inequality (18) by

$$\varepsilon^2 \|\mathcal{K}_m\|_{L^1(\mathbb{R})}^2 \left\{ \frac{1}{\pi} \|G_{1,m}(x) - G_{2,m}(x)\|_{L^1(\mathbb{R})}^2 \frac{R^{1-4s}}{1-4s} + \frac{\|G_{1,m}(x) - G_{2,m}(x)\|_{L^2(\mathbb{R})}^2}{R^{4s}} \right\},$$

with $R \in (0, +\infty)$. We will make use of the identity for $1 \le m \le N$

$$G_{1,m}(x) - G_{2,m}(x) = \int_0^1 \nabla g_m(u_0(x) + tv_1(x) + (1-t)v_2(x)).(v_1(x) - v_2(x))dt.$$

Clearly, for $t \in [0, 1]$

$$\|v_2(x) + t(v_1(x) - v_2(x))\|_{H^1(\mathbb{R},\mathbb{R}^N)} \le t\|v_1(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)} + (1-t)\|v_2(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)} \le$$
$$\le \rho,$$

such that $v_2(x) + t(v_1(x) - v_2(x)) \in B_\rho$. Hence,

$$|G_{1,m}(x) - G_{2,m}(x)| \le sup_{z\in I}|\nabla g_m(z)||v_1(x) - v_2(x)| \le M|v_1(x) - v_2(x)|.$$

This yields

$$\|G_{1,m}(x) - G_{2,m}(x)\|_{L^2(\mathbb{R})} \le M\|v_1 - v_2\|_{L^2(\mathbb{R},\mathbb{R}^N)} \le M\|v_1 - v_2\|_{H^1(\mathbb{R},\mathbb{R}^N)}.$$

Evidently, for $1 \le m, j \le N$, we can express $\frac{\partial g_m}{\partial z_j}(u_0(x) + tv_1(x) + (1-t)v_2(x))$ as

$$\int_0^1 \nabla \frac{\partial g_m}{\partial z_j}(\tau[u_0(x) + tv_1(x) + (1-t)v_2(x)]).[u_0(x) + tv_1(x) + (1-t)v_2(x)]d\tau,$$

such that for $t \in [0, 1]$

$$\left|\frac{\partial g_m}{\partial z_j}(u_0(x) + tv_1(x) + (1-t)v_2(x))\right| \le$$
$$\le \sum_{n=1}^N \left\|\frac{\partial^2 g_m}{\partial z_n \partial z_j}\right\|_{C(I)} (|u_0(x)| + t|v_1(x)| + (1-t)|v_2(x)|).$$

We obtain the upper bound for $G_{1,m}(x) - G_{2,m}(x)$ in the absolute value as

$$M|v_1(x) - v_2(x)|\left(|u_0(x)| + \frac{1}{2}|v_1(x)| + \frac{1}{2}|v_2(x)|\right).$$

By means of the Schwarz inequality we arrive at the estimate from above for the norm $\|G_{1,m}(x) - G_{2,m}(x)\|_{L^1(\mathbb{R})}$ as

$$M\|v_1 - v_2\|_{L^2(\mathbb{R},\mathbb{R}^N)}\left(\|u_0\|_{L^2(\mathbb{R},\mathbb{R}^N)} + \frac{1}{2}\|v_1\|_{L^2(\mathbb{R},\mathbb{R}^N)} + \frac{1}{2}\|v_2\|_{L^2(\mathbb{R},\mathbb{R}^N)}\right) \le$$
$$\le M\|v_1 - v_2\|_{H^1(\mathbb{R},\mathbb{R}^N)}(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1).$$

Thus we arrive at the upper bound for the norm $\|u_1(x)-u_2(x)\|^2_{L^2(\mathbb{R},\mathbb{R}^N)}$ given by

$$\varepsilon^2\mathcal{K}^2M^2\|v_1-v_2\|^2_{H^1(\mathbb{R},\mathbb{R}^N)}\left\{\frac{1}{\pi}(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)}+1)^2\frac{R^{1-4s}}{1-4s}+\frac{1}{R^{4s}}\right\}.$$

By means of Lemma 1.4 we minimize the expression above over $R\in(0,+\infty)$ to obtain the estimate from above for $\|u_1(x)-u_2(x)\|^2_{L^2(\mathbb{R},\mathbb{R}^N)}$ as

$$\varepsilon^2\mathcal{K}^2M^2\|v_1-v_2\|^2_{H^1(\mathbb{R},\mathbb{R}^N)}\frac{(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)}+1)^{8s}}{(1-4s)(4\pi s)^{4s}}. \tag{27}$$

By virtue of formulas (25) and (26), for $1\le m\le N$ we have

$$\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}}(u_{1,m}-u_{2,m})=\varepsilon_m\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}-s}\int_{-\infty}^{\infty}\mathcal{K}_m(x-y)[G_{1,m}(y)-G_{2,m}(y)]dy.$$

Inequalities (18) and (2) yield

$$\left\|\frac{d}{dx}(u_{1,m}-u_{2,m})\right\|^2_{L^2(\mathbb{R})}\le\varepsilon^2\|G_{1,m}-G_{2,m}\|^2_{L^1(\mathbb{R})}\left\|\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}-s}\mathcal{K}_m\right\|^2_{L^2(\mathbb{R})}\le$$

$$\le\varepsilon^2M^2\|v_1-v_2\|^2_{H^1(\mathbb{R},\mathbb{R}^N)}(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)}+1)^2\left\|\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}-s}\mathcal{K}_m\right\|^2_{L^2(\mathbb{R})},$$

such that

$$\sum_{m=1}^{N}\left\|\frac{d}{dx}(u_{1,m}-u_{2,m})\right\|^2_{L^2(\mathbb{R})}\le\varepsilon^2M^2\|v_1-v_2\|^2_{H^1(\mathbb{R},\mathbb{R}^N)}(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)}+1)^2Q^2. \tag{28}$$

By virtue of (27) and (28) the norm $\|u_1-u_2\|_{H^1(\mathbb{R},\mathbb{R}^N)}$ can be estimated from above by the expression

$$\varepsilon M(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)}+1)\left\{\frac{\mathcal{K}^2(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)}+1)^{8s-2}}{(1-4s)(4\pi s)^{4s}}+Q^2\right\}^{\frac{1}{2}}\|v_1-v_2\|_{H^1(\mathbb{R},\mathbb{R}^N)}. \tag{29}$$

This yields that the map $T_g : B_\rho\to B_\rho$ defined by system (11) is a strict contraction for all values of $\varepsilon>0$ small enough. Its unique fixed point $u_p(x)$ is the only solution of system (9) in the ball B_ρ. The resulting $u(x) \in H^1(\mathbb{R}, \mathbb{R}^N)$ given by (8) is a solution of system (2). Note that by means of (24) $u_p(x)$ tends to zero in the $H^1(\mathbb{R}, \mathbb{R}^N)$ norm as $\varepsilon \to 0$.

□

Then we turn our attention to the proof of the second main statement of our article.

3 The Continuity of the Fixed Point of the Map T_g (Proof of Theorem 1.5)

Obviously, for all $0 < \varepsilon < \delta$ we have

$$u_{p,1} = T_{g_1} u_{p,1}, \quad u_{p,2} = T_{g_2} u_{p,2}.$$

Hence

$$u_{p,1} - u_{p,2} = T_{g_1} u_{p,1} - T_{g_1} u_{p,2} + T_{g_1} u_{p,2} - T_{g_2} u_{p,2}.$$

Therefore,

$$\|u_{p,1} - u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)} \leq \|T_{g_1} u_{p,1} - T_{g_1} u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)} + \|T_{g_1} u_{p,2} - T_{g_2} u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)}.$$

Inequality (29) yields

$$\|T_{g_1} u_{p,1} - T_{g_1} u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)} \leq \varepsilon\sigma \|u_{p,1} - u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)},$$

with $\varepsilon\sigma < 1$ since the map $T_{g_1} : B_\rho \to B_\rho$ under our assumptions is a strict contraction. Here the positive constant

$$\sigma := M(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1) \left\{ \frac{\mathcal{K}^2 (\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^{8s-2}}{(1-4s)(4\pi s)^{4s}} + Q^2 \right\}^{\frac{1}{2}}.$$

Hence, we obtain

$$(1 - \varepsilon\sigma)\|u_{p,1} - u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)} \leq \|T_{g_1} u_{p,2} - T_{g_2} u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)}. \tag{30}$$

Clearly, for our fixed point $T_{g_2} u_{p,2} = u_{p,2}$. Let us denote $\xi(x) := T_{g_1} u_{p,2}$. For $1 \leq m \leq N$, we arrive at

$$\left(-\frac{d^2}{dx^2}\right)^s \xi_m(x) = \varepsilon_m \int_{-\infty}^{\infty} \mathcal{K}_m(x-y) g_{1,m}(u_0(y) + u_{p,2}(y)) dy, \tag{31}$$

$$\left(-\frac{d^2}{dx^2}\right)^s u_{p,2,m}(x) = \varepsilon_m \int_{-\infty}^{\infty} \mathcal{K}_m(x-y) g_{2,m}(u_0(y) + u_{p,2}(y)) dy, \tag{32}$$

where $0 < s < \frac{1}{4}$. Let us designate here

$$G_{1,2,m}(x) := g_{1,m}(u_0(x) + u_{p,2}(x)), \quad G_{2,2,m}(x) := g_{2,m}(u_0(x) + u_{p,2}(x)).$$

We apply the standard Fourier transform (17) to both sides of (31) and (32). This yields

$$\widehat{\xi}_m(p) = \varepsilon_m \sqrt{2\pi} \frac{\widehat{\mathcal{K}}_m(p)\widehat{G}_{1,2,m}(p)}{|p|^{2s}}, \quad \widehat{u}_{p,2,m}(p) = \varepsilon_m \sqrt{2\pi} \frac{\widehat{\mathcal{K}}_m(p)\widehat{G}_{2,2,m}(p)}{|p|^{2s}}.$$

Evidently,

$$\|\xi_m(x) - u_{p,2,m}(x)\|^2_{L^2(\mathbb{R})} = \varepsilon_m^2 2\pi \int_{-\infty}^{\infty} \frac{|\widehat{\mathcal{K}}_m(p)|^2 |\widehat{G}_{1,2,m}(p) - \widehat{G}_{2,2,m}(p)|^2}{|p|^{4s}} dp.$$

Apparently, it can be bounded from above by means of (18) by

$$\varepsilon^2 \|\mathcal{K}_m\|^2_{L^1(\mathbb{R})} \left\{ \frac{1}{\pi} \|G_{1,2,m} - G_{2,2,m}\|^2_{L^1(\mathbb{R})} \frac{R^{1-4s}}{1-4s} + \|G_{1,2,m} - G_{2,2,m}\|^2_{L^2(\mathbb{R})} \frac{1}{R^{4s}} \right\},$$

with $R \in (0, +\infty)$. We use the formula

$$G_{1,2,m}(x) - G_{2,2,m}(x) = \int_0^1 \nabla[g_{1,m} - g_{2,m}](t(u_0(x) + u_{p,2}(x))).(u_0(x) + u_{p,2}(x))dt,$$

such that

$$|G_{1,2,m}(x) - G_{2,2,m}(x)| \le \|g_{1,m} - g_{2,m}\|_{C^2(I)} |u_0(x) + u_{p,2}(x)|.$$

Therefore,

$$\begin{aligned} \|G_{1,2,m} - G_{2,2,m}\|_{L^2(\mathbb{R})} \le \|g_{1,m} - g_{2,m}\|_{C^2(I)} \|u_0 + u_{p,2}\|_{L^2(\mathbb{R},\mathbb{R}^N)} \le \\ \le \|g_{1,m} - g_{2,m}\|_{C^2(I)} (\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1). \end{aligned}$$

Let us apply another useful representation formula with $1 \le j \le N$ and $t \in [0, 1]$, namely

$$\begin{aligned} \frac{\partial}{\partial z_j}(g_{1,m} - g_{2,m})(t(u_0(x) + u_{p,2}(x))) = \\ = \int_0^t \nabla \left[\frac{\partial}{\partial z_j}(g_{1,m} - g_{2,m}) \right] (\tau(u_0(x) + u_{p,2}(x))).(u_0(x) + u_{p,2}(x))d\tau. \end{aligned}$$

Hence

$$\left|\frac{\partial}{\partial z_j}(g_{1,m} - g_{2,m})(t(u_0(x) + u_{p,2}(x)))\right| \leq$$
$$\leq \sum_{n=1}^{N} \left\| \frac{\partial^2 (g_{1,m} - g_{2,m})}{\partial z_n \partial z_j} \right\|_{C(I)} |u_0(x) + u_{p,2}(x)|,$$

such that

$$|G_{1,2,m}(x) - G_{2,2,m}(x)| \leq \|g_{1,m} - g_{2,m}\|_{C^2(I)} |u_0(x) + u_{p,2}(x)|^2.$$

Thus,

$$\|G_{1,2,m} - G_{2,2,m}\|_{L^1(\mathbb{R})} \leq \|g_{1,m} - g_{2,m}\|_{C^2(I)} \|u_0 + u_{p,2}\|^2_{L^2(\mathbb{R},\mathbb{R}^N)} \leq$$
$$\leq \|g_{1,m} - g_{2,m}\|_{C^2(I)} (\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^2. \tag{33}$$

This enables us to derive the upper bound for the norm $\|\xi(x) - u_{p,2}(x)\|^2_{L^2(\mathbb{R},\mathbb{R}^N)}$ as

$$\varepsilon^2 \mathcal{K}^2 (\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^2 \|g_1 - g_2\|^2_{C^2(I,\mathbb{R}^N)} \left[\frac{(\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^2 R^{1-4s}}{\pi(1-4s)} + \frac{1}{R^{4s}} \right].$$

This expression can be trivially minimized over $R \in (0, +\infty)$ by virtue of Lemma 1.4. We obtain the inequality

$$\|\xi(x) - u_{p,2}(x)\|^2_{L^2(\mathbb{R},\mathbb{R}^N)} \leq \varepsilon^2 \mathcal{K}^2 (\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^{2+8s} \frac{\|g_1 - g_2\|^2_{C^2(I,\mathbb{R}^N)}}{(1-4s)(4\pi s)^{4s}}.$$

Formulas (31) and (32) with $1 \leq m \leq N$ yield

$$\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}} \xi_m(x) = \varepsilon_m \left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}-s} \int_{-\infty}^{\infty} \mathcal{K}_m(x-y) G_{1,2,m}(y) dy,$$
$$\left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}} u_{p,2,m}(x) = \varepsilon_m \left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}-s} \int_{-\infty}^{\infty} \mathcal{K}_m(x-y) G_{2,2,m}(y) dy,$$

such that by means of (18) and (33) the norm $\left\| \frac{d}{dx}\left(\xi_m(x) - u_{p,2,m}(x)\right)\right\|^2_{L^2(\mathbb{R})}$ can be estimated from above by

$$\varepsilon^2 \|G_{1,2,m} - G_{2,2,m}\|^2_{L^1(\mathbb{R})} \left\| \left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}-s} \mathcal{K}_m \right\|^2_{L^2(\mathbb{R})} \leq$$

$$\leq \varepsilon^2 \|g_1 - g_2\|^2_{C^2(I,\mathbb{R}^N)} (\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^4 \left\| \left(-\frac{d^2}{dx^2}\right)^{\frac{1}{2}-s} \mathcal{K}_m \right\|^2_{L^2(\mathbb{R})}.$$

Then

$$\sum_{m=1}^{N} \left\| \frac{d}{dx}\left(\xi_m(x) - u_{p,2,m}(x)\right)\right\|^2_{L^2(\mathbb{R})} \leq \varepsilon^2 \|g_1 - g_2\|^2_{C^2(I,\mathbb{R}^N)} (\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^4 Q^2.$$

Therefore, we arrive at $\|\xi(x) - u_{p,2}(x)\|_{H^1(\mathbb{R},\mathbb{R}^N)} \leq$

$$\leq \varepsilon \|g_1 - g_2\|_{C^2(I,\mathbb{R}^N)} (\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^2 \left[\frac{\mathcal{K}^2 (\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^{8s-2}}{(1-4s)(4\pi s)^{4s}} + Q^2 \right]^{\frac{1}{2}}.$$

By virtue of inequality (30), the norm $\|u_{p,1} - u_{p,2}\|_{H^1(\mathbb{R},\mathbb{R}^N)}$ can be bounded from above by

$$\frac{\varepsilon}{1-\varepsilon\sigma} (\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^2 \left[\frac{\mathcal{K}^2 (\|u_0\|_{H^1(\mathbb{R},\mathbb{R}^N)} + 1)^{8s-2}}{(1-4s)(4\pi s)^{4s}} + Q^2 \right]^{\frac{1}{2}} \|g_1 - g_2\|_{C^2(I,\mathbb{R}^N)},$$

which completes the proof of the theorem. □

4 Auxiliary Results

Below we state the solvability conditions proven easily in [29] by applying the standard Fourier transform (17) to the linear Poisson type equation with a square integrable right side

$$\left(-\frac{d^2}{dx^2}\right)^s \phi = f(x), \quad x \in \mathbb{R}, \quad 0 < s < 1. \tag{34}$$

We denote the inner product as

$$(f(x), g(x))_{L^2(\mathbb{R})} := \int_{-\infty}^{\infty} f(x)\bar{g}(x)dx, \tag{35}$$

with a slight abuse of notations when the functions involved in (35) are not square integrable, like for instance the one involved in orthogonality condition (36) of Lemma 1.4 below. Indeed, if $f(x) \in L^1(\mathbb{R})$ and $g(x)$ is bounded, then the integral in the right side of (35) makes sense. The left side of relation (37) is well defined as well under the stated conditions. We have the following technical proposition.

Lemma 4.1 *Let $f(x) : \mathbb{R} \to \mathbb{R}$ and $f(x) \in L^2(\mathbb{R})$.*

1. *When $0 < s < \frac{1}{4}$ and in addition $f(x) \in L^1(\mathbb{R})$, equation (34) admits a unique solution $\phi(x) \in H^{2s}(\mathbb{R})$.*
2. *When $\frac{1}{4} \leq s < \frac{3}{4}$ and additionally $|x|f(x) \in L^1(\mathbb{R})$, problem (34) possesses a unique solution $\phi(x) \in H^{2s}(\mathbb{R})$ if and only if the orthogonality relation*

$$(f(x), 1)_{L^2(\mathbb{R})} = 0 \tag{36}$$

   holds.
3. *When $\frac{3}{4} \leq s < 1$ and in addition $x^2 f(x) \in L^1(\mathbb{R})$, equation (34) has a unique solution $\phi(x) \in H^{2s}(\mathbb{R})$ if and only if orthogonality conditions (36) and*

$$(f(x), x)_{L^2(\mathbb{R})} = 0 \tag{37}$$

   hold.

Note that for the lower values of the power of the negative second derivative operator $0 < s < \dfrac{1}{4}$ under the conditions stated above no orthogonality relations are required to solve the linear Poisson type equation (34) in $H^{2s}(\mathbb{R})$.

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Visualizing Discrete Complex-Valued Time-Frequency Representations

Yusong Yan and Hongmei Zhu

Abstract Time-frequency analysis techniques are effective in detecting local signal structure and have been applied successfully in a wide range of fields. Different time-frequency analysis transforms yield different time-frequency spectra. However, visualizing a complex-valued time-frequency spectrum is not a trivial task as it requires graphing in a four-dimensional space: two coordinate variables time and frequency and the real and imaginary parts of the spectrum. The most common way to graph such a complex time-frequency spectrum is to plot the amplitude or magnitude spectrum and the phase spectrum separately. Such visualization may cause difficulty in understanding combined information of amplitude and phase in time-frequency domain. In this paper, we propose a new way to visualize a complex-valued time-frequency spectrum in one graph. In particular, we will describe this technique in the context of the discrete generalized Stockwell transforms for simplicity and practical usage. We show that the proposed visualization tool may facilitate better understanding of local signal behavior and become a useful tool for non-stationary signal analysis and processing applications.

Keywords Complex time frequency analysis • Domain coloring • Generalized discrete Stockwell transform • Morlet wavelet transform • Short time Fourier transform • Stockwell transform

Mathematics Subject Classification (2000). Primary 94A12; Secondary 94A15

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1 Introduction

Time-frequency analysis offers a variety of techniques that map a one-dimensional temporal signal into a function of both time and frequency variables. Such a function is called a time-frequency representation or spectrum describing the temporal variation of frequency content within the signal. Time-frequency analysis techniques are effective in detecting local signal structure and have been applied successfully in a wide range of fields including geophysics, speech recognition, music analysis, oceanology and bio-medicine. Comprehensive reviews on the related theory and applications can be found in [1–3].

Different time-frequency analysis transforms yield different time-frequency spectra. Here we are interested in those whose spectra are complex-valued functions of two real-valued variables time and frequency. Visualizing a complex-valued time-frequency spectrum is not a trivial task as it requires graphing in a four-dimensional space: two coordinate variables time and frequency and the real and imaginary parts of the spectrum.

The most common way to graph such a complex time-frequency spectrum is to plot the amplitude or magnitude spectrum and the phase spectrum separately. Interpretation of amplitude spectrum or square of amplitude spectrum as an energy distribution is visually straightforward and can reveal certain important local frequency characteristics. In many published work in time-frequency analysis, only the amplitude spectrum is investigated while the phase spectrum is discarded. Meanwhile phase spectrum also provides a different and stable description on local signal behaviors such as signal offset and image edge localization. Phase information contributes significantly in many applications including speech recognition [4], music analysis [5], and neuroscience [6, 7].

Though phase spectrum of a time-frequency representation is at least as important as its amplitude spectrum, it has not been studied as extensively as the latter. It is partially because it is more complicated to interpret the phase information due to periodicity and non-linearity of phase spectrum.

Motivated by an interesting body of work on visualizing complex functions [8–16], we propose a new way to picture a complex-valued time-frequency spectrum in one graph by mapping amplitude spectrum to gray scale intensity and mapping phase spectrum to hue on saturated color wheel. There are early uses of color to visualize complex functions including complex Fourier spectrum of a given signal. For example, Cowtan, whose work is related to X-ray crystallography, utilizes jointly phase and amplitude representations to visualize "Fourier Duck" [9] and "Fourier Cat" in his picture book of Fourier transforms [10]. However, it is not yet well explored in complex-valued time-frequency spectra. Hence, in this paper, we explore the use of Hue/Luminance scheme to visualize complex-valued time-frequency spectrum. Note that we choose a linear brightness/luminance scheme with black for zero and white for the maximum amplitude to represent amplitude, similar to that for complex numbers described by Farris [11]. It can provide us a straight and simple interpretation of the amplitude while the luminance system for domain coloring is repeated for every integer power of 2. Our phase coloring scheme assembles phase plots [8]. It requires a one-dimensional color space instead of a two-dimensional color space as needed by the domain coloring [14]. We will show that the proposed visualization tool may facilitate better insight about local signal behavior and become a useful tool for non-stationary signal analysis and processing applications. In Sect. 3, we also theoretically investigate phase patterns often appeared in the discrete complex time-frequency representation given by the generalized discrete Stockwell transform (ST, [17–19]) in numerical calculation. Understanding these patterns can help us properly interpreting the phase spectrum. More examples are given in Sect. 4. Further discussions and conclusion are provided in Sect. 5.

To carry out the discussion efficiently, we focus only on the discrete generalized Stockwell transform [19]. Note that the technique and analysi are straightforwardly applicable to the other commonly used complex value time-frequency techniques that generalize the Fourier transforms, such as the short time Fourier transform (STFT, [20]), Morlet wavelet transform [21] and the conventional ST. They are all based on localized Fourier analysis; the STFT has a fixed time-frequency resolution while the ST a frequency-dependent resolution in time-frequency domain. There is only a phase difference between the ST and the Morlet wavelet transform [27]. They can be expressed in a common generalized Stockwell framework [7, 22–31].

2 Visual Encoding of Amplitude and Phase Spectra

Mathematically the continuous ST of a given signal $g(t)$ is defined as:

$$S(\tau, \nu) = \int_{-\infty}^{+\infty} g(t) w_{\sigma(\nu)}(\tau - t, \nu) e^{-2\pi i t \nu} dt \tag{1}$$

Here, ν is frequency variable. The window function $w_{\sigma(\nu)}(t, \sigma)$ is localized at time τ, where $\sigma(\nu)$ controls the window width and is a function of the frequency variable. When $\sigma(\nu)$ is a constant, Eq. 1 gives the STFT; when $\sigma(\nu)$ is inversely proportional to the frequency variable, it is the original form of the ST [18]. Without loss of generality, we replace the notation $w_{\sigma(\nu)}(t, \sigma)$ with $w(t, \sigma)$ to simplify our notation. Note that choices of window functions can be infinite; some popular ones are Gaussian, rectangular, Hanning and Hamming windows. Our theoretical discussion is not limited to a particular window function; but we use Gaussian window functions in all our numerical illustrations.

We rewrite the complex valued time-frequency spectrum $S(\tau, \nu)$ as the following

$$S(\tau, \nu) = |S(\tau, \nu)|e^{-i\phi(\tau,\nu)} \tag{2}$$

where $|S(\tau, \nu)|$ is the amplitude spectrum and $e^{-i\phi(\tau,\nu)}$ is the phase spectrum. The Fourier spectrum $G(\nu)$ of the signal $g(t)$ can be easily recovered by integrating $S(\tau, \nu)$ along time index τ, i.e.,

$$G(\nu) = \int_{-\infty}^{\infty} S(\tau, \nu)d\tau. \tag{3}$$

which shows the absolutely referenced phase property.

Next we will visualize such a complex-valued time-frequency spectrum to capture both amplitude and phase spectra in a single picture using a hue/luminance scheme. It is a slightly different version of this visualization technique used to visual complex Fourier spectrum [11] and complex functions [8]. Given a complex valued time-frequency spectrum $S(\tau, \nu) = |S(\tau, \nu)|e^{i\phi(\tau,\nu)}$, the essential idea of our visual encoding is to map the amplitude spectrum $|S(\tau, \nu)|$ to gray scale intensity (i.e., brightness) and the phase $e^{i\phi(\tau,\nu)}$ to hue (i.e., color).

More specifically, we linearly map $|S(\tau, \nu)|$ to a gray scale of [0, 1] with black 0 corresponding to $|S(\tau, \nu)| = 0$ and white 1 to $max|S(\tau, \nu)|$ as illustrated in Fig. 1a. Note that if the amplitude varies over a wide range, $c_1 log(c_2|S(\tau, \nu)| + 1)$ can be considered in plotting. Here, values of the constants $c_1, c_2 > 1$ or $c_1, c_2 < 1$ can be selected to adjust the intensity gradient in the display in order to magnify or suppress subtle details, respectively.

The mapping between phase and hue is bijective. In fact, a phase value $e^{i\phi(\tau,\nu)}$ can be calculated by

$$e^{i\phi(\tau,\nu)} = S(\tau, \nu)/|S(\tau, \nu)| = Re(e^{i\phi(\tau,\nu)}) + iIm(e^{i\phi(\tau,\nu)}) \tag{4}$$

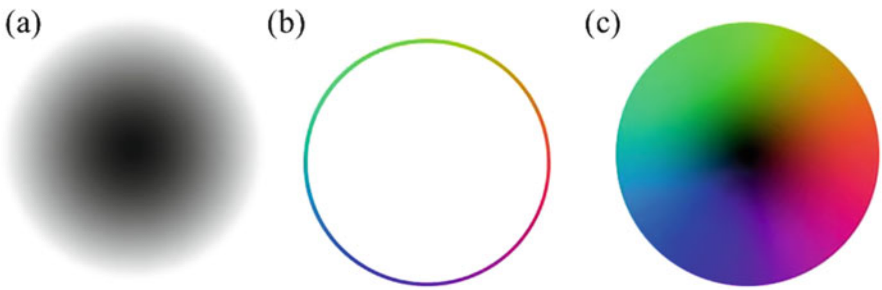

Fig. 1 Indicators of our reference coloring schemes. (**a**) Gray value indicator of magnitude time-frequency spectrum $|S(\tau, \nu)|$ with *black* being 0 and *white* being $max|S(\tau, \nu)|$; (**b**) Continuous color gradient indicator of the phase time-frequency spectrum $e^{i\phi(\tau,\nu)}$, the color of which is determined by the color positioned at$(Re(e^{i\phi(\tau,\nu)}), Im(e^{i\phi(\tau,\nu)}))$ on the color wheel; (**c**) Combined indicator of *gray value* and hue encoded time-frequency spectrum $S(\tau, \nu) = |S(\tau, \nu)|e^{i\phi(\tau,\nu)}$

and is a point $(Re(e^{i\phi(\tau,\nu)}), Im(e^{i\phi(\tau,\nu)}))$ on a unit circle. Figure 1b shows how $e^{i\phi(\tau,\nu)}$ can be naturally represented by a color on the continuous color wheel circle. Note that we plot the phase $e^{i\phi(\tau,\nu)}$ instead of its argument $\phi(\tau,\nu)$. This is because argument $\phi(\tau,\nu)$ is unique up to an additive number of 2π and restricted to an interval, say$[0, 2\pi)$. It causes a big value jump for an argument approaching 0^-. This phenomenon does not exist when plotting the phase $e^{i\phi(\tau,\nu)}$ that is a point rotating along the color unit circle. This is because $e^{i\phi(\tau,\nu)}$ does not distinguish between a multiplication of 2π in its value and is continuous.

Figure 1c provides a reference for visualizing the combined effect of plotting the intensity-encoded amplitude and color-encoded phase. For every point (τ,ν) in the time-frequency domain, we assign

1. a gray intensity value to $|S(\tau,\nu)|$;
2. a color positioned at$(Re(e^{i\phi(\tau,\nu)}), Im(e^{i\phi(\tau,\nu)}))$ on the color wheel to $e^{i\phi(\tau,\nu)}$.

This results in the visual encoded time-frequency spectrum in one graph.

3 Interpreting Color-Encoded Phase Spectrum for a Single Complex Sinusoid

To properly understand the technique, we first examine the visual encoded time-frequency spectrum for a complex sinusoid $g(t) = e^{2\pi i f_0 t}$ of a constant frequency f_0. To simplify the notation and discussions, we assume that the given discrete signal $g[n] = g(n\triangle t)$ is infinitely sampled at a sampling interval $\triangle t$, where $-\infty < n < +\infty$, with loss of generality. If a given signal $g[n]$, where $n = 0, 1, \ldots, L-1$, is finitely sampled, we can periodically extend the discrete signal to infinity.

Given a discrete signal $g[n](-\infty < n < +\infty)$, its time-frequency spectrum is calculated using the discrete ST [19].

$$S[l,m] = \sum_{n=-\infty}^{+\infty} g[n] \cdot e^{-2\pi i \cdot \frac{nm}{N}} \cdot w[l-n,m] \tag{5}$$

where the time index $-\infty < l < +\infty$ and the frequency index $m = 0, 1, \ldots, N-1$. Note that N, the total number of uniformly sampled points in the frequency domain, is usually determined by the number of points involved in the fast Fourier transform (FFT) that is used to compute the time-frequency spectrum. The equivalent formula of Eq. (5) in the Fourier domain is given by

$$S[l,m] = FT^{-1}(G(f+m)W_m(f)) \tag{6}$$

where $G(f)$ and $W_m(f)$ are the Fourier transform of the signal and the window function

$$G(f) = \sum_{n=-\infty}^{+\infty} e^{-2\pi i f \cdot n} \cdot g[n]$$

and

$$W_m(f) = \sum_{n=-\infty}^{+\infty} e^{-2\pi i f \cdot n} \cdot w[n, m]$$

respectively. When the signal is finite-length and the localization window is Gaussian, Eq. (6) coincides with the conventional ST [17, 18].

$$S[l, m] = \sum_{n=0}^{N-1} G[n, m] \cdot e^{-2\pi n^2/m^2} \cdot e^{2\pi inl/N} \tag{7}$$

Figure 2a shows an example of a complex sinusoid with $f_0 = 0.15$ Hz and $t \in [0\,\mathrm{s}, 1023\,\mathrm{s}]$. The sampling rate is 1 Hz. Figure 2b is the amplitude $|S(\tau, \nu)|$ of the Stockwell spectrum with a Gaussian window, showing that one constant

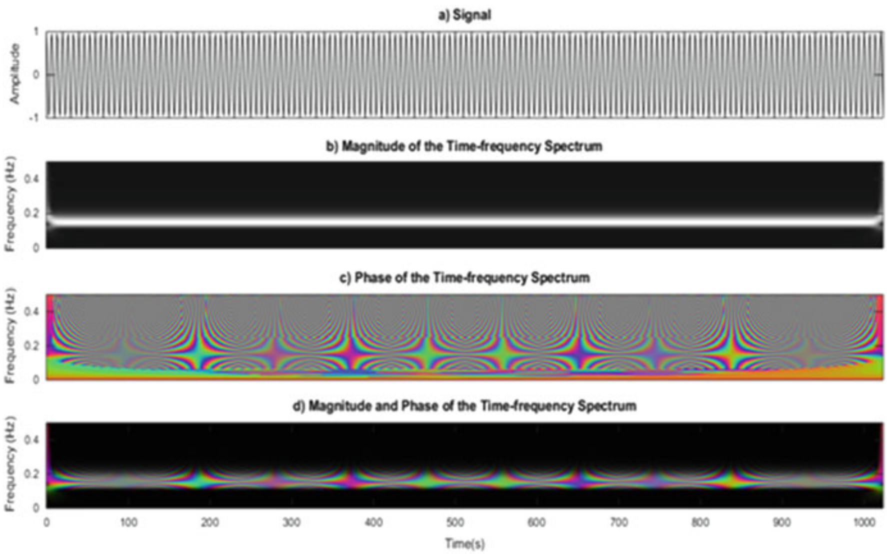

Fig. 2 (**a**) An example of a complex sinusoid $g(t)$ with $f_0 = 0.15$ Hz and $t[0\,\mathrm{s}, 1023\,\mathrm{s}]$. The sampling rate is 1 Hz; (**b**) the amplitude $|S(\tau, \nu)|$ of its time-frequency spectrum; (**c**) the color-encoded phase $e^{i\phi(\tau,\nu)}$ of its time-frequency spectrum; (**d**) the combined color and intensity encoded time-frequency spectrum

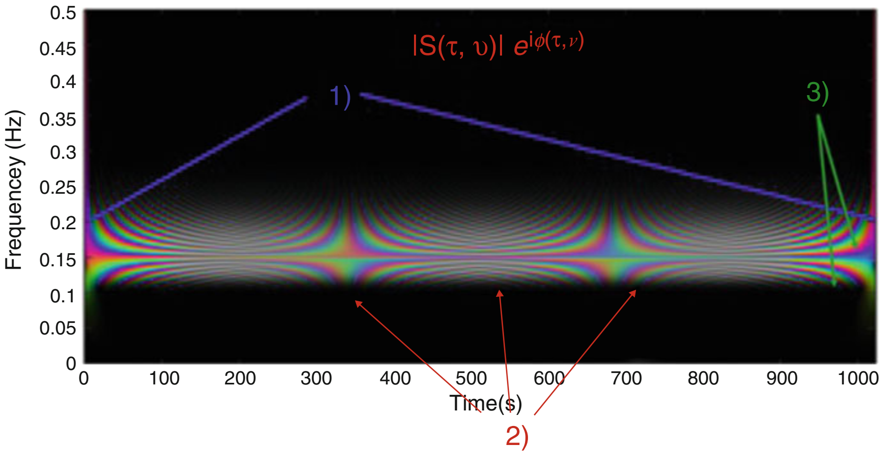

Fig. 3 The zoomed version of the visual encoded time-frequency spectrum for a complex sinusoid $g(t) = e^{2\pi i f_0 t}$ of a constant frequency $f_0 = 0.15$ Hz, the phase of which exhibits three strong patterns as pointed by the arrows in *blue*, *red* and *green*

frequency around 0.15 Hz throughout the time. Figure 2c shows the color-encoded phase $e^{i\phi(\tau,\nu)}$ of the Stockwell spectrum. Figure 2d, the combination of Fig. 2b, c, is the color and intensity encoded Stockwell spectrum, indicating how signal energy distributes and how the argument evolves in the time-frequency domain simultaneously.

The luminance mapping of the amplitude spectrum is easy to understand, but the phase spectrum displayed using the proposed visualizing technique is hard to interpret directly. In particular, as shown in Fig. 3, a zoomed-in version of Fig. 2d, we can observe that the level curves (i.e. contours) of the phase spectrum exhibit strong patterns: (1) diamond shapes in continuous color gradients (i.e. a smooth color transition) occur at the two ends of the signal, i.e., at t = 0 and 1023 s, (2) diamond shapes in discontinuous color gradients (i.e., a non-smooth color transition) appear a number of times over time, and (3) color alternate patterns occur at frequencies closed to the signal frequency $f_0 = 0.15$ Hz. We will explain these phase phenomena theoretically in the rest of the section.

3.1 Hyperbolic Level Curves of the Phase Spectrum

For the discussion in the rest of the section, let $S[l,m] = |S[l,m]|e^{i\phi[l,m]}$ denote the Stockwell spectrum of a discrete complex sinusoid $g[n] = e^{2\pi i f_0 n}$ calculated by (5), where $l \in \mathbb{Z}$ and $m = 0, 1, , N-1$.

Lemma 3.1 *The Stockwell spectrum satisfies*

$$S[l,m] = e^{2\pi i l(f_0 - \frac{m}{N})} \cdot W_m(f_0 - \frac{m}{N}) \tag{8}$$

$$arg(S[l,m]) = \phi[l,m] = 2\pi l(f_0 - \frac{m}{N}) + phase(W_m(f_0 - \frac{m}{N})) \tag{9}$$

Here, $arg(z)$ and z^* are the argument and the complex conjugate of a complex number z, respectively.

Proof From Eq. (5), we compute

$$\begin{aligned}
S[l,m] &= \sum_{n=-\infty}^{+\infty} g[n] \cdot e^{-2\pi \frac{nm}{N}} \cdot w[l-n,m] \\
&= \sum_{n=-\infty}^{+\infty} e^{2\pi i f_0 n} \cdot e^{-2\pi \frac{nm}{N}} \cdot w[l-n,m] \\
&= \sum_{n=-\infty}^{+\infty} e^{2\pi i n(f_0 - \frac{m}{N})} \cdot w[l-n,m] \\
&= \sum_{n=-\infty}^{+\infty} e^{2\pi i (l-n)(f_0 - \frac{m}{N})} \cdot w[n,m] \\
&= e^{2\pi i l(f_0 - \frac{m}{N})} \sum_{n=-\infty}^{+\infty} e^{-2\pi i n(f_0 - \frac{m}{N})} \cdot w[n,m] \\
&= e^{2\pi i l(f_0 - \frac{m}{N})} \cdot W_m(f_0 - \frac{m}{N})
\end{aligned}$$

Therefore, we can obtain the corresponding phase spectrum

$$arg(S[l,m]) = \phi[l,m] = 2\pi l(f_0 - \frac{m}{N}) + phase(W_m(f_0 - \frac{m}{N})). \qquad \square$$

We now investigate the different patterns exhibited in phase of the Stockwell spectrum of a single complex sinusoid. First, we define the level curves of the phase spectrum as

$$L_c = \left\{ (l, \frac{m}{N}) | e^{l\phi[l,m]} = c, where\ c\ is\ a\ complex\ constant \right\}.$$

The following lemma leads to prove that if the window function satisfies certain conditions, then the hyperbolic shaped level curves of the phase occur at certain time indexes that are integer multiples of the FFT length N used in the numerical computations.

Lemma 3.2 (Level Curves of the Phase Spectrum) *Index pairs* $(l, \frac{m}{N}) \in L_c$ *in a small neighborhood of* (kN, f_0) *for some integer* k *satisfy*

$$(l - kN)(f_0 - \frac{m}{N}) + \frac{1}{2\pi} Arg(W_m(f_0 - \frac{m}{N})) = constant$$

where $Arg(.)$ *is the principal argument defined in* $[0, 2\pi)$.

Proof Given any time index l, there exist two integers l^* where $0 \le l^* < N$ and $k \in \mathbb{Z}$, such that $l = l^* + kN$. Let $(l, \frac{m}{N}) \in L_c$ be a point in a small neighborhood of (kN, f_0). From Lemma 3.1, we have

$$\begin{aligned}
&Arg(S[l, m]) \\
&= \left[2\pi l(f_0 - \frac{m}{N}) + Arg(W_m(f_0 - \frac{m}{N}))\right] (mod[0, 2\pi]) \\
&= \left[2\pi(l - kN + kN)(f_0 - \frac{m}{N}) + Arg(W_m(f_0 - \frac{m}{N}))\right] (mod[0, 2\pi]) \\
&= \left[2\pi(l - kN)(f_0 - \frac{m}{N}) + 2\pi kNf_0 - 2\pi km + Arg(W_m(f_0 - \frac{m}{N}))\right] (mod[0, 2\pi]) \\
&= constant
\end{aligned}$$

Since kNf_0 is a constant and $2\pi mk$ is an integer multiple of 2π, we then have

$$\begin{aligned}
&Arg(S[l, m]) \\
&= \left[2\pi(l - kN)(f_0 - \frac{m}{N}) + Arg(W_m(f_0 - \frac{m}{N}))\right] (mod[0, 2\pi]) \\
&= constant
\end{aligned}$$

Hence we proved that

$$(l - kN)(f_0 - \frac{m}{N}) + \frac{1}{2\pi} Arg(W_m(f_0 - \frac{m}{N})) = constant. \qquad \square$$

Theorem 3.3 (Hyperbolic Level Curves of the Phase Spectrum 1) *If the windows function* $w[n, m]$ *is conjugate symmetric, then the* $(l, \frac{m}{N}) \in L_c$ *in a small neighborhood of* (kN, f_0) *for some integer* k *satisfies*

$$(l - kN)(f_0 - \frac{m}{N}) = constant.$$

Proof If the windows function $w[n,m]$ is a conjugated symmetry, then its Fourier spectrum $W_m(f)$ is a real-valued function. Hence the phase of $W_m(f)$ becomes a constant, i.e.

$$Arg(W_m(f_0 - \frac{m}{N})) = 0 \; or \; \pi$$

Therefore, from Lemma 3.2, the $(l, \frac{m}{N}) \in L_c$ in a small neighborhood of (kN, f_0) for some integer k satisfies

$$(l - kN)(f_0 - \frac{m}{N}) = constant.$$

Namely, the level curves of the phase spectrum form rectangular hyperbola with horizontal/vertical asymptotes centered at (kN, f_0). □

Similarly, we can prove the following theorem.

Theorem 3.4 (Hyperbolic Level Curves of the Phase Spectrum 2) *If the Fourier Spectrum $W_m(\cdot)$ of the window function $w[n,m]$ is positive, then the $(l, \frac{m}{N}) \in L_c$ in a small neighborhood of (kN, f_0) for some integer k satisfies*

$$(l - kN)(f_0 - \frac{m}{N}) = constant$$

Theorems 3.3 and 3.4 explain the hyperbolic curves occurred at the two ends of the time-frequency phase spectrum, pointed by the blue arrows in Fig. 3. Note that in this example, $f_0 = 0.15$ Hz and we used a 1024-point FFT which is the same as the total signal length N = 1024. And the window function used in the computation is Gaussian. By Theorem 3.4, we can observe that the level curves of the phase spectrum form rectangular hyperbola with horizontal/vertical asymptotes centered at (0 s,0.15 Hz) and (1023 s,0.15 Hz).

Interestingly, one may also observe a number of hyperbolic-like curves with discontinuous transition of colors occurring in the middle of the time axis, as indicated by the red arrows in Fig. 3. We categorize those hyperbolic-like curves as false or true hyperbolic-like curves with discontinuous color gradients.

The false hyperbolic-like curves are due to spatial aliasing caused by insufficient display resolution. The spatial aliasing occurs especially at locations with a fast color transition. A zoomed-in version of the phase spectrum as shown in Fig. 4 can dissolve those false hyperbolic-like curves that appear at locations.

The true hyperbolic-like curves with discontinuous color gradients as indicated in Fig. 4 exist in both original and zoomed-in versions of the phase spectrum. This can be explained by the relation of the phase spectra with the high frequency-sampling resolution and low frequency-sampling resolution. For instance, the true hyperbolic-like curve at location (512 s, 0.15 Hz) can be explained by the phase relation of the time-frequency spectra $S[l,m]$ of size 1024-by-1024 and 1024-by-512.

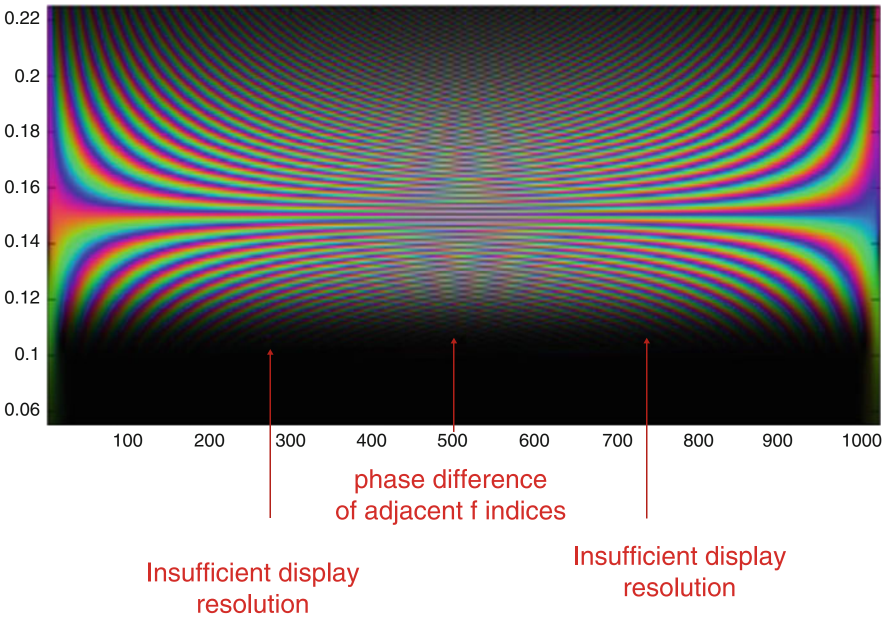

Fig. 4 The false hyperbolic-like curves with discontinuous color transition can be dissolved by sufficient display resolution, while the true hyperbolic-like curve with discontinuous color transition at location (512 s, 0.15 Hz) can be explained by the phase relation of the time-frequency spectra $S[l, m]$ of size 1024-by-1024 and 1024-by-512

We can also prove that the following theorem by similar calculations:

Theorem 3.5 *Let* $S[l, m] = |S[l, m]|e^{i\phi[l,m]}$ *be the time-frequency spectrum of a complex sinusoid* $g[n] = e^{2\pi i f_0 n}$ *calculated by* (5)*, where* $l \in \mathbb{Z}$ *and* $m = 0, 1, , N-1$. *If the Fourier spectrum of the window function* $w[n, m]$ *is positive, then the phase of ST function at odd and even frequency sampling index is given as the following, respectively:*

$$arg(S[l, 2m]) = \pi f_0 Nk + 2\pi(f_0 - \frac{2m}{N}) \cdot (l - k\frac{N}{2}) \qquad (even)$$

$$arg(S[l, 2m + 1]) = \pi f_0 Nk - k\pi + 2\pi(f_0 - \frac{2m + 1}{N}) \cdot (l - k\frac{N}{2}) \quad (odd)$$

From Theorem 3.5, we can draw the following conclusions:

Remark 3.6 If we extract the ST coefficients only at even frequency-sampling indexes to form a new time-frequency spectrum, we will observe a set of rectangular hyperbola centered at position $(k\frac{N}{2}, f_0)$ with continuous color transition along the color wheel. As illustrated in Fig. 5a, the time-frequency spectrum formed by

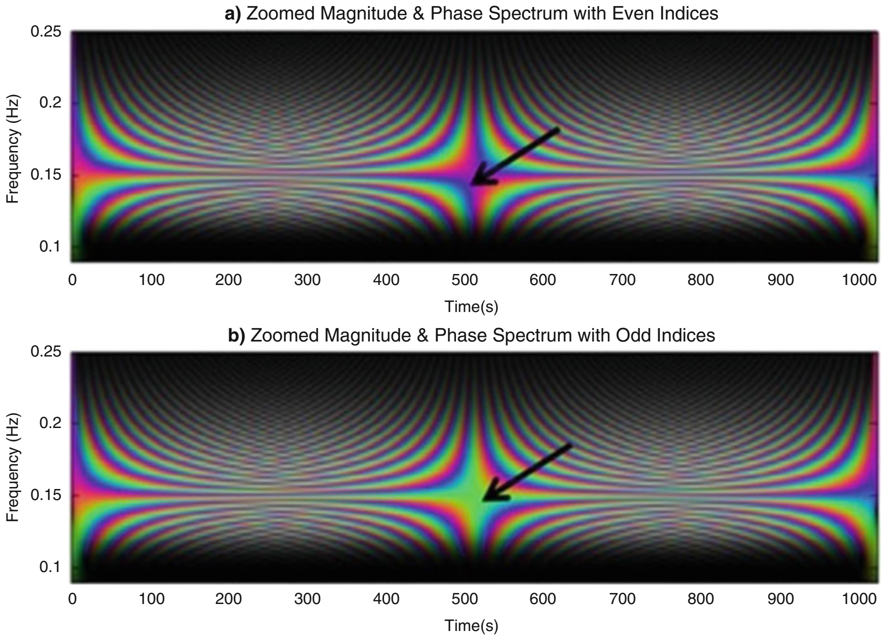

Fig. 5 The time-frequency spectrum formed by extracting the coefficients at (**a**) even and (**b**) odd frequency sampling indexes form a rectangular hyperbola centered at position (512, 0.15 Hz), as pointed by the *arrows*

extracting the coefficients at even frequency sampling indexes has a rectangular hyperbola centered at position (512, 0.15 Hz)

Remark 3.7 If we extract the ST coefficients only at odd frequency-sampling indexes to form a time-frequency spectrum, we will observe a set of rectangular hyperbola centered at position $(k\frac{N}{2}, f_0)$ with continuous color transition along the color wheel; As illustrated in Fig. 5b, the time-frequency spectrum formed by extracting the coefficients at odd frequency sampling indexes has a rectangular hyperbola centered at position (512, 0.15 Hz)

Remark 3.8 More importantly, at the vertical lines $l = \frac{N}{2}k$, the phase difference of the ST coefficients at even and odd frequency-sampling indexes is k causing that the colors assigned at 2 m frequency-sampling indexes and 2 m + 1 frequency-sampling indexes are opposite on the color wheel. Therefore, hyperbolic-like curves with discontinuous color transition occur at time indexes $l = \frac{N}{2}k$, as pointed by the middle arrow in Fig. 4.

3.2 Rotations and Finite Difference of the Phase Spectrum

The following theorem provides an explicit formula for the finite difference (i.e., an approximation to the derivative) with respect to time index of the time-frequency phase spectrum of an analytic signal. This then helps us to explain the third phenomena related to the color rotations of the phase spectrum along the unit color circle.

Theorem 3.9 (Rotations and Finite Difference of the Phase Spectrum) *Let* $0 \le m_0 < N$ *be an integer such that* $\frac{m_0}{N} \le f_0 < \frac{m_0+1}{N}$. *Then the phase of* $S[l,m]$ *with a conjugated symmetric window function has the following properties:*

(i) $\Delta_l \phi[l,m] = \phi[l+1,m] - \phi[l,m] = 2\pi(f_0 - \frac{m}{N})$;
(ii) For any integer $m \ge m_0 + 1$, *the phase of* $S[l,m]$ *rotates clockwise as the time index l increases ;*
(iii) For any integer $m \le m_0 + 1$, *the phase of* $S[l,m]$ *rotates counter-clockwise as the time index l increases ;*
(iv) If $\frac{m_0}{N} = f_0$, *then the phase of* $S[l,m]$ *remains unchanged.*

Proof

(i) From Lemma 3.1, we can obtain the phase

$$arg(S[l,m]) = \phi[l,m] = 2\pi l(f_0 - \frac{m}{N}) + phase(W_m(f_0 - \frac{m}{N}))$$

Since the windows function $w[n,m]$ is a conjugated symmetry, then its Fourier spectrum $W_m(\cdot)$ is a real-valued function. Hence, we have

$$\phi[l,m] = 2\pi l(f_0 - \frac{m}{N}) + constant$$

Then the difference $\Delta_l \phi[l,m] = \phi[l+1,m] - \phi[l,m] = 2\pi(f_0 - \frac{m}{N})$.

(ii) For any integer $m \ge m_0 + 1$, we have $f_0 - \frac{m}{N} \le f_0 - \frac{m_0+1}{N} < 0$; therefore $\Delta_l \phi[l,m] < 0$, i.e., the phase of $S[l,m]$ will rotate clockwise with the increasing of the time index l

(iii) For any integer $m < m_0$, we have $f_0 - \frac{m}{N} > f_0 - \frac{m_0}{N} > 0$; therefore $\Delta_l \phi[l,m] > 0$, i.e., the phase of $S[l,m]$ will rotate counter-clockwise with the increasing of the time index l

(iv) For any integer $f_0 = \frac{m_0}{N}$, we have $f_0 - m/N = 0$; therefore $\Delta_l \phi[l,m] = 0$, i.e., the phase does not vary over time and hence it is stationary with respect to time. □

As illustrated in the zoomed-in phase spectrum in Fig. 6, we can easily observe the phase rotation from the colored spectrum. For instance, as the time index increases, the color of the frequencies above $f_0 = 0.15$ Hz transits smoothly from red, blue to green as the phase rotates clockwise along the unit color wheel, while the color of the frequency below $f_0 = 0.15$ Hz transits smoothly from green, blue

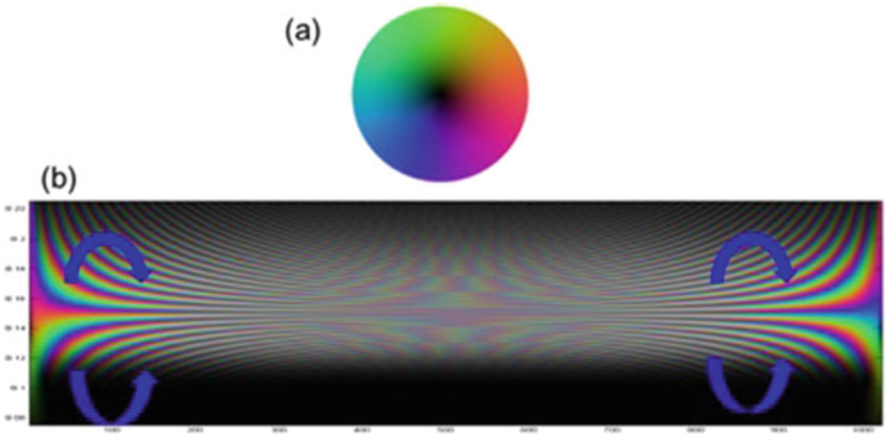

Fig. 6 Near the ends of the time-frequency representation of a discrete signal, the color rotates clockwise in the regions where frequency is above 0.15 Hz, counterclockwise in the regions where frequency is below 0.15 Hz, and stays the same when the frequency is equal to 0.15 Hz

to red as the phase rotates counter clockwise along the unit color wheel. The color of the spectrum at frequency $f_0 = 0.15$ Hz stays orange during the whole duration, indicating the location of the signal frequency in the time-frequency domain.

From the proof of Theorem 3.9, we see that

$$\Delta_l\phi[l,m] = \phi[l+1,m] - \phi[l,m] = 2\pi(f_0 - \frac{m}{N})$$

implying that $\Delta_l\phi[l,m] = 0$ iff $f_0 - \frac{m}{N} = 0$. In other words, as $\Delta_l\phi[l,m]$ approaches to 0, the sample frequency $\frac{m}{N}$ approaches to the signal frequency f_0, i.e., the stationary point of the time-frequency phase or the instantaneous frequency of a signal [7, 27].

4 Examples on Non-stationary Signals

When analyzing signals for real world applications, a general signal can be decomposed or approximated by a sum of simple complex exponential signals by the Fourier analysis. Hence, the features of the phase spectrum that we discussed in Sect. 3 can provide useful insights for interpolating the phase behavior of a general signal.

In this section, we provide the color-encoded time-frequency spectra of more complicated synthetic signals such as mono-component non-stationary and/or multi-component signals and real signals. Note that the sampling rate of synthetic signals is 1 Hz. In all the examples the time-frequency spectra are computed by Eq. (5) using a 512-point FFTs.

4.1 Synthetic Signals

First, we consider a synthetic signal consisting of four components as shown in Fig. 7a: one cosine wave of frequency 0.05 Hz from [0 s, 511 s], two bursts of cosine wave of frequency 0.4 Hz for [161 s, 240 s] and [288 s, 367 s], one cosine wave of frequency 0.2 Hz from [512 s, 1023 s]. Figure 7b is its magnitude spectrum clearly showing the temporal arrivals of different frequency components. Figure 7c is its phase spectrum in which the stationary phase pinpoints the locations of different frequencies contained in the signal. Figure 7d combines the gray-valued magnitude

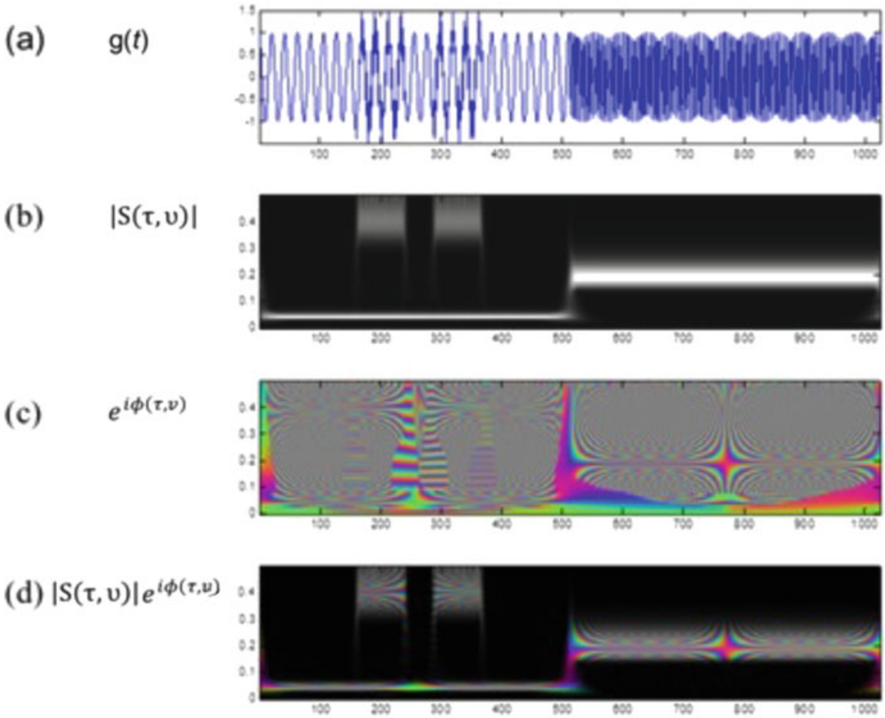

Fig. 7 (**a**) A synthetic signal consisting of four components; (**b**) its magnitude spectrum, (**c**) its phase spectrum, and (**d**) its combined gray-value and hue-encoded time-frequency spectrum

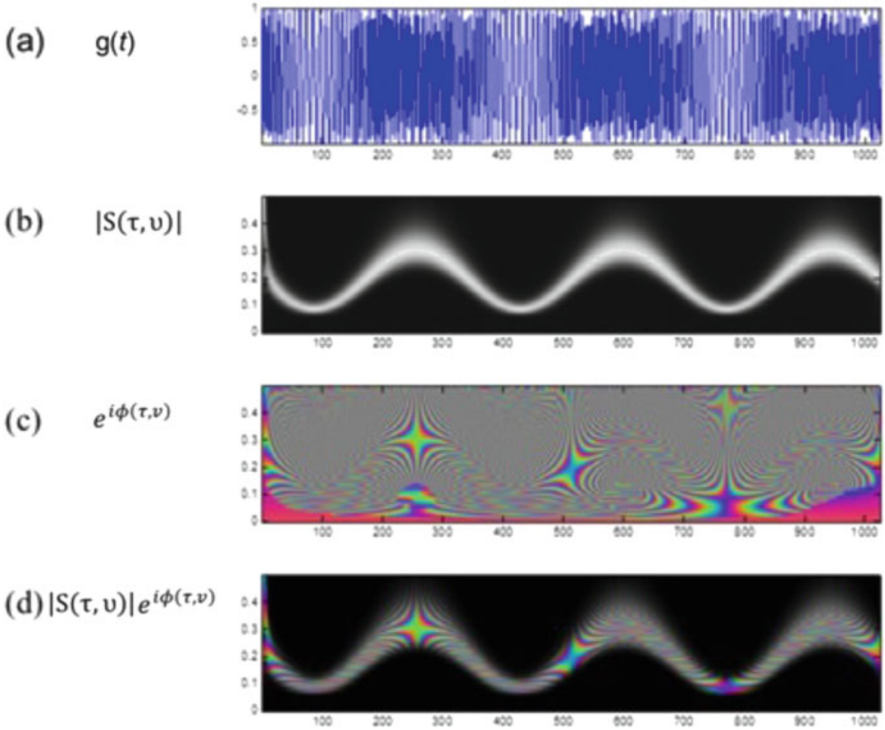

Fig. 8 (**a**) A synthetic signal with a sinusoidal fluctuated instantaneous frequency; (**b**) its magnitude spectrum, (**c**) its phase spectrum, and (**d**) its combined gray-value and hue-encoded time-frequency spectrum

and the hue-encoded phase of the spectrum, which clearly shows what and when different frequencies arrive in the signals.

The second synthetic signal is a mono-component signal whose instantaneous frequency fluctuates like a cosine function:

$$h(t) = cos(2\pi(0.2t + 6cos(2\pi t3/1024)))$$

as shown in Fig. 8a. Figure 8b–d are the magnitude spectrum, the phase spectrum, and the combined gray-value and hue-encoded time-frequency spectrum, respectively. Again, the time-frequency spectrum is computed using a 512-point FFTs.

We construct a multi-component signal with two chirp signals:

$$h(t) = cos(2\pi(40 + t/7))t/1024) + cos(2\pi(1024/2.8 - t/6)t/1024)$$

where $t \in [0, 1023]$ as shown in Fig. 9a. Figure 9b–d are the magnitude spectrum, the phase spectrum, and the combined gray-value and hue-encoded time-frequency spectrum, respectively. Note that the magnitude spectrum as shown in Fig. 9b

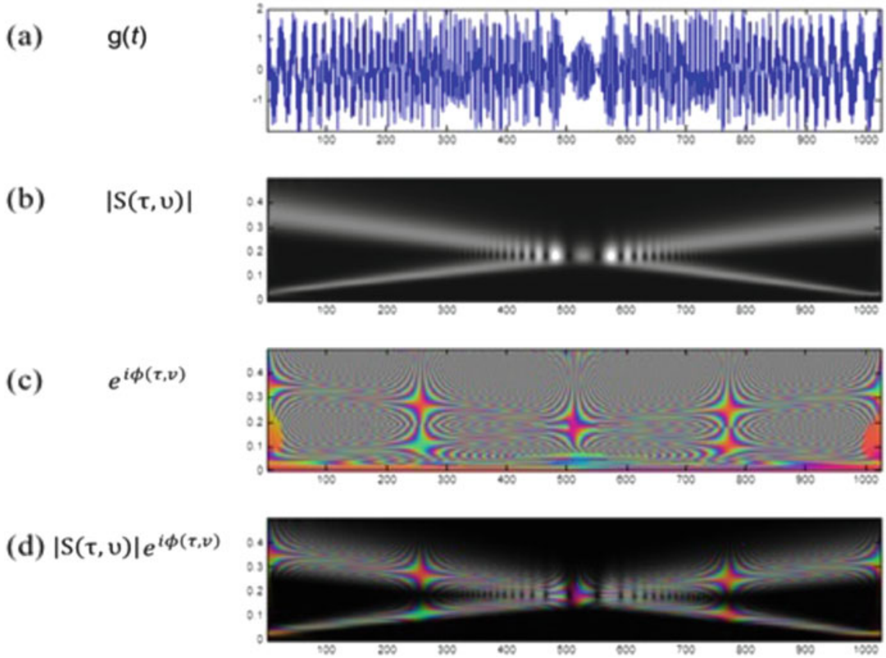

Fig. 9 (**a**) A synthetic signal with two chirps; (**b**) its magnitude spectrum, (**c**) its phase spectrum, and (**d**) its combined gray-value and hue-encoded time-frequency spectrum

exhibits obvious interference near the intersection of two chirps while the phase spectrum as shown in Fig. 9c is less affected by interference. Hence, combining both magnitude and phase information can assist us to interpret the actual temporal variations of the different frequencies contained in a signal.

4.2 A Real Signal

A recording of bat sonar chirping was downloaded from SoundBible.com. The sampling rate is 44.1 kHz and a short segment of the recording from 0.05 s to 0.13 s was selected for illustration. Its time-frequency spectrum is calculated using the infinite length time-frequency analysis as shown in (5) [19]. Due to its large data size and the limitation of hardware, it is not practical to compute the entire spectrum all at once. Hence the computation is done by segment-by-segment with a 512-point FFT (Fig. 10).

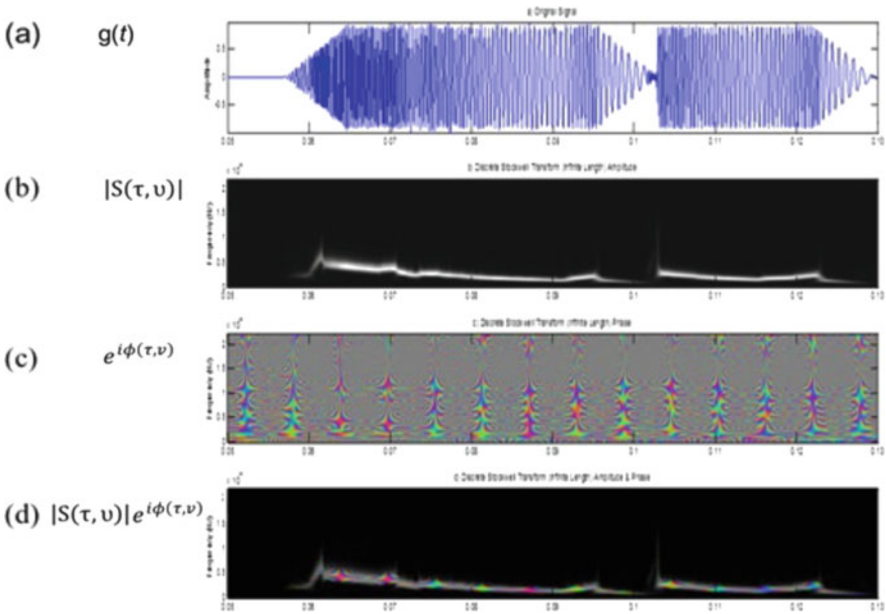

Fig. 10 (**a**) A bat sonar chirping signal; (**b**) its magnitude spectrum, (**c**) its phase spectrum, and (**d**) its combined gray-value and hue-encoded time-frequency spectrum

5 Discussions and Conclusions

In this paper, we proposed to visualize the complex-valued discrete time-frequency spectrum in one two-dimensional graph by encoding amplitude spectrum with gray scale intensity and phase spectrum with the hue of color on a unit circle. We theoretically investigated the patterns appeared in the ST-phase spectrum of an analytic signal with a constant frequency f_0. We proved that the level curves of the phase spectrum form rectangular hyperbola with horizontal/vertical asymptotes centered (kN, f_0) where N is the number of points used in the FFT. We also showed that some hyperbolic alike level curves may be fake due to insufficient display resolution, while some is real due to the $k\pi$ phase difference of the ST coefficients at even and odd frequency-sampling indexes.

Through studying the finite difference of the phase, we explained the color rotations of the phase spectrum. In addition, Theorem 3.9 suggests that the frequency indexes at which the phase spectrum attains its local minimums can be used to estimate the instantaneous frequency (*IF*) laws of a signal. Similarly, we estimate the local group delay by

$$\Delta_m \phi[l, m] = \phi[l, m+1] - \phi[l, m] = -\frac{2\pi l}{N}$$

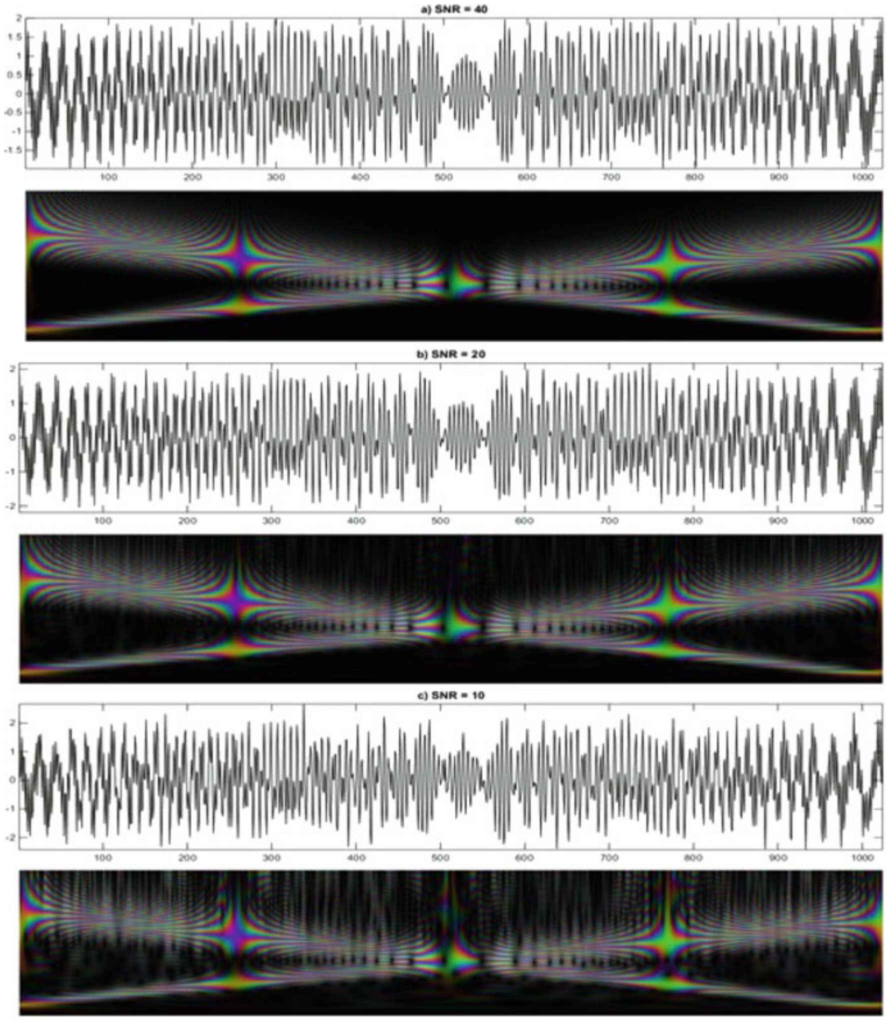

Fig. 11 A synaptic signal of two crossing chirps with additive random noise and its combined gray-value and hue-encoded time-frequency spectrum. The signal-to-noise ratio is (**a**) 40, (**b**) 20 and (**c**) 10, respectively

which is the dual of the IF. Normally the amplitude spectrum plays a dominant role in the estimation of the IF laws. As we can see from the numerical examples, phase spectrum can accurately estimate the time arrivals and locations of different frequency component. Its performance is robust to spectrum interference and noise compared to the performance of the amplitude spectrum.

In Fig. 11, we add (a) mild, (b) moderate and (c) high level of noise to the crossing chirp signal in Fig. 9a. When the signal-to-noise ratio (SNR) is 40, no obvious noise effect is observed in amplitude and phase spectra, as shown in Fig. 11a. When

$SNR = 20$, the amplitude spectrum was obviously distorted by noise while the phase spectrum has no obvious distortion, as shown in Fig. 11b. When $SNR = 10$, effect of noise to the amplitude spectrum is more significant than to the phase spectrum, as shown in Fig. 11c. Therefore utilizing both spectrum and phase information can be beneficial to accurate IF estimation.

Certainly we need to explore the advantages of visualizing color and gradient encoded complex valued time-frequency spectrum in real world applications. In this paper, we examined the full ST spectrum for accurate interpretation of the new visualization method and easy understanding of the phase properties. However due to the intensive computation, the full ST is not practical to use when dealing large data size. Hence various discrete generalized ST transforms have been developed for efficient computation [19–31]. Therefore, we will extend our study to the generalized discrete ST transforms with any amount of information redundancy.

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The Reassigned Spectrogram of the Stockwell Transform

Cheng Liu and Hongmei Zhu

Abstract In this paper, we introduce the reassigned spectrogram of Stockwell transform by re-mapping the surface of the spectrogram of Stockwell transform with the aim to improve its readability. We first define the channelized instantaneous frequency and the local group delay for the Stockwell transform. At any given point in the time-frequency domain, the associated local group delay and channelized instantaneous frequency provide a re-estimation of the time arrival and instantaneous frequency of the signal component observed at that point. The reassigned spectrogram of Stockwell transform therefore has signal energy highly concentrated at the instantaneous frequency/group delay curves and greatly increases the resolution and readability of the time-frequency structure of the underlying signal. The instantaneous frequencies of signal components can then be extracted by detecting the local energy peaks in the reassigned spectrogram of Stockwell transform. The improvement of the reassigned spectrogram of Stockwell transform over the conventional spectrogram of Stockwell transform is illustrated using both synthetic and real signals.

Keywords Instantaneous frequency • Stockwell transform • The reassignment method

Mathematics Subject Classification (2000). Primary 94A12

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1 Introduction

The Stockwell transform or the S transform (ST) was proposed by Stockwell in [1] as a time-frequency analysis method for describing non-stationary signals. Let $\psi \in L^1(\mathcal{R}) \cap L^2(\mathcal{R})$ be such that $\int_{-\infty}^{\infty} \psi(t)dt = 1$, the ST of a signal $x(t)$ in $L^2(\mathcal{R})$ with respect to the window function $\psi(t0$ is defined by

$$ST_x(t,f) = |f| \int_{-\infty}^{\infty} x(\tau)\overline{\psi(|f|(\tau - t))} e^{-j2\pi\tau f} d\tau, \quad t \in \mathcal{R}, \ f \in \mathcal{R}/\{0\}, \tag{1}$$

and at zero frequency $f = 0$, the ST is equal to the average of the signal, i.e.,

$$ST_x(t,0) = \int_{-\infty}^{\infty} x(\tau)d\tau. \tag{2}$$

The ST can also be defined in the frequency domain, i.e.,

$$ST_x(t,f) = \int_{-\infty}^{\infty} X(\alpha + f)\overline{\Psi\left(\frac{\alpha}{|f|}\right)} e^{j2\pi\alpha t} d\alpha, \quad t \in \mathcal{R}, \ f \in \mathcal{R}/\{0\}, \tag{3}$$

where $X(f)$ and $\Psi(f)$ are the Fourier spectrum of the signal x and the window function ψ, respectively. The discrete analog of Equation (3) is often used to compute the ST by taking advantage of the efficiency of the fast Fourier transform (FFT) algorithm. The original Stockwell transform was proposed with the Gaussian window, $\psi(t) = \frac{1}{\sqrt{2\pi}} e^{t^2/2}$. The Gaussian function is a commonly used window function in analog signal processing, as it reaches the lower bound of the uncertainty principle [2].

The ST was first derived as the "phase correction" of the continuous wavelet transform [1] and thus it inherits the multi-scale resolution feature from the wavelet transform. But unlike the wavelet transform, the ST has the absolutely referenced phase information [3], i.e., the phase information at any time given by the ST is always referenced to the Fourier phase of the signal at zero time. The ST can also be interpreted as a modification of the short-time Fourier transform with a frequency-dependent window width. Such interpretation makes the ST a well-received tool for people in the time-frequency analysis community. As a hybrid of short-time Fourier transform and wavelet transform, the ST therefore has quickly gained popularity in the community of non-stationary signal processing: see papers [4–6] for its underlying mathematics and papers [7–9] for its diverse applications.

The instantaneous frequency (IF) is another useful parameter to describe the time-frequency structure of non-stationary signals [10]. The IF is usually defined based on the Hilbert transform, and it provides a unique description of the time-varying frequency characteristic of mono-component signals. However, the definition of IF often fails when the non-stationary signals have multiple components

[11]. By representing the signal in the joint time-frequency domain, time-frequency representations (TFRs) have been widely used to reveal the structure of multi-component signals. For most of the TFRs, a tradeoff between the time and frequency resolutions exists as a consequence of the Heisenberg uncertainty principle. The spread energy in the time-frequency domain may mask the true time-frequency structure of non-stationary signals, which will lead to an erroneous interpretation of the IFs of the signals.

The reassignment method was proposed to enhance the energy concentration of a TFR, which can be traced back to the modified moving window method proposed by Kodera, Gendrin and Villeday [12] 30 years ago. By relocating the spread energy of the spectrogram to the center of gravity in the energy distribution, the developed reassigned spectrogram has the signal energy concentrated at the IF curves in the time-frequency domain. Therefore, the IF of each component of a multi-component signal can be revealed by detecting the local energy peaks in the reassigned spectrogram. In 1995, Auger and Flandrin [13] generalized the method of Kodera et al. to any bilinear time-frequency distributions of the Cohen's class and time-scaling representations [11]. Algorithms of reassigned representations have been investigated in [14, 15], and their applications in processing the speech and music signals can be found in [16, 17].

The reassignment method is a postprocessing technique applied on TFRs, and thus its performance depends on the accomplishment of the underlying TFR. In this paper, we extend the reassignment technique to the ST. In Sect. 2, definitions of instantaneous frequency and group delay (GD) are first reviewed. With mathematical validations, the channelized instantaneous frequency (CIF) and the local group delay (LGD) of ST are then proposed, which are used as the locations in the time-frequency domain to reassign the spread signal energy in the spectrogram of ST. The reassigned spectrogram of Stockwell transform is defined in Sect. 3, and a practical discrete implementation is also provided. Numerical simulations have been performed to demonstrate the performance of the reassigned spectrogram of ST, which are discussed in Sect. 4. The conclusions are provided in Sect. 5.

2 The Channelized Instantaneous Frequency and Local Group Delay of Stockwell Transform

2.1 Instantaneous Frequency and Group Delay

To accurately quantify the time-varying frequency characteristic of non-stationary signals, the instantaneous frequency was proposed by extending the concept of Fourier frequency [10]. For a mono-component signal, the IF is defined as

$$f_i(t) = \frac{1}{2\pi}\frac{d\phi(t)}{dt}, \tag{4}$$

where $\phi(t)$ is the phase of the analytic signal $s(t)$, given by

$$\begin{aligned} s(t) &= x(t) + j\mathcal{H}\{x(t)\} \\ &= a(t)e^{j\phi(t)}, \end{aligned} \tag{5}$$

where $\mathcal{H}$ denotes the Hilbert transform [10].

The IF as a function of time provides a frequency measure at any time instant. Alternatively, the time-frequency structure of a signal can be depicted as a function of frequency, which gives rise to the definition of the group delay. The GD is defined as the derivative of the phase of the frequency domain signal representation, given by

$$t_g(f) = -\frac{1}{2\pi}\frac{d\Phi(f)}{df}, \tag{6}$$

where $\Phi(f)$ is the phase of the Fourier transform $S(f)$ of the analytic signal $s(t)$. For signals with a large bandwidth-duration production value, the IF and the GD agree to each other, which present the same curve in the time-frequency domain [12].

However, the IF and the GD defined by Equations (4) and (6) are only valid for mono-component signals. To apply the definitions of IF and GD to multi-component signals, decomposition of multi-component signals needs to obtained first. Accurate estimation of IF and GD for multi-component signals is still a challenge. One approach is to extend the definitions of the IF and GD from one-dimension to two-dimensions through a joint time-frequency representation of a signal. This yields the definitions of the channelized instantaneous frequency and the local group delay. We have a number of successes using the Stockwell transform in biomedical and industrial applications [7, 9, 18]. It is important to obtain well separated multiple components of the signal in order to accurate estimate their frequencies related to physiological activities or natural vibrations of objects. Hence, in this paper, we are interested in deviations of the CIF and the LGD from the ST of a signal.

2.2 Channelized Instantaneous Frequency of the Stockwell Transform

Similar to other time-frequency analysis techniques, the ST provides a two dimensional description of signal characteristics. At any fixed frequency f_0, the one dimensional time domain function $ST_x(t,f_0)$ is called a voice of the ST. To extend the concept of the IF, we define the channelized instantaneous frequency to quantify the local characteristic of each ST voice, which is given by

$$f_i^{(ST)}(t,f) = f + \frac{1}{2\pi}\frac{\partial\phi^{(ST)}(t,f)}{\partial t}, \tag{7}$$

where $\phi^{(ST)}(t,f)$ is the phase of the ST.

To illustrate the relation between the conventional IF of a mono-component signal and the proposed CIF of the ST, an explicit approximation formula to the CIF of the ST with a Gaussian window function was derived.

Theorem 2.1 *Let the analytic signal $s(t) = a(t)e^{j\phi(t)}$ be such that the amplitude $a(t)$ is slowly-varying, then the following approximation formula can be derived for the ST with the Gaussian window function $\psi(t) = \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}$,*

$$ST_s(t,f) \approx a^{(ST)}(t,f)e^{j\phi^{(ST)}(t,f)}, \tag{8}$$

where

$$a^{(ST)}(t,f) = \left(1 + \frac{\phi''(t)^2}{f^4}\right)^{-\frac{1}{4}} e^{-\frac{\frac{f^2}{2}(\phi'(t)-2\pi f)^2}{f^4+\phi''(t)^2}} a(t), \tag{9}$$

$$\phi^{(ST)}(t,f) = \phi(t) - 2\pi tf + \frac{1}{2}\arctan\left(\frac{\phi''(t)}{f^2}\right) - \frac{\frac{1}{2}(\phi'(t) - 2\pi f)^2\,\phi''(t)}{f^4 + \phi''(t)^2}. \tag{10}$$

Proof In Equation (1), expressing the signal s with its polar form given by (5) and changing the variable $u = \tau - t$, we have

$$ST_s(t,f) = \frac{|f|}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(u+t)e^{-\frac{f^2u^2}{2}} \cdot e^{j[\phi(u+t)-2\pi(u+t)f]}du.$$

Since the amplitude $a(t)$ is slowly-varying, we assume that $a(u+t)$ inside the Gaussian window to be constant, which leads to the approximation

$$a(u+t)e^{-\frac{f^2u^2}{2}} \approx a(t)e^{-\frac{f^2u^2}{2}}.$$

Expanding the phase function $\phi(u+t)$ into a Taylor series at the time t, that is, $\phi(u+t) = \phi(t) + \phi'(t)u + \frac{1}{2}\phi''(t)u^2 + O(u^3)$ and neglecting the high order term $O(u^3)$, we have

$$\begin{aligned} ST_s(t,f) &\approx \frac{|f|}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(t)e^{-\frac{f^2u^2}{2}} \cdot e^{j\left[\phi(t)+\phi'(t)u+\frac{1}{2}\phi''(t)u^2-2\pi(u+t)f\right]}du \\ &= \frac{|f|}{\sqrt{2\pi}}a(t)e^{j(\phi(t)-2\pi tf)} \int_{-\infty}^{\infty} e^{-\frac{f^2u^2}{2}} \cdot e^{j\left(\phi'(t)u+\frac{1}{2}\phi''(t)u^2-2\pi uf\right)}du \\ &= \frac{|f|}{\sqrt{2\pi}}a(t)e^{j(\phi(t)-2\pi tf)} \int_{-\infty}^{\infty} e^{-\left(\frac{f^2}{2}-\frac{1}{2}j\phi''(t)\right)u^2+(j\phi'(t)-j2\pi f)u}du \end{aligned}$$

$$
\begin{aligned}
&= \frac{|f|}{\sqrt{2\pi}} a(t) e^{j(\phi(t)-2\pi tf)} \cdot \sqrt{\frac{\pi}{\frac{f^2}{2} - \frac{1}{2} j\phi''(t)}} \cdot e^{\frac{\frac{1}{2}(j\phi'(t)-j2\pi f)^2}{f^2 - j\phi''(t)}} \\
&= a(t) e^{j(\phi(t)-2\pi tf)} \sqrt{\frac{1 + j\frac{\phi''(t)}{f^2}}{1 + \frac{\phi''(t)^2}{f^4}}} \cdot e^{-\frac{\frac{1}{2}(\phi'(t)-2\pi f)^2 \left(f^2 + j\phi''(t)\right)}{f^4 + \phi''(t)^2}} \\
&= a(t) e^{j(\phi(t)-2\pi tf)} \left(1 + \frac{\phi''(t)^2}{f^4}\right)^{-\frac{1}{4}} \cdot e^{j\frac{1}{2}\arctan\left(\frac{\phi''(t)}{f^2}\right)} \\
&\quad \cdot e^{-\frac{\frac{f^2}{2}(\phi'(t)-2\pi f)^2}{f^4 + \phi''(t)^2}} e^{-j\frac{\frac{1}{2}(\phi'(t)-2\pi f)^2 \phi''(t)}{f^4 + \phi''(t)^2}} \\
&= \left(1 + \frac{\phi''(t)^2}{f^4}\right)^{-\frac{1}{4}} e^{-\frac{\frac{f^2}{2}(\phi'(t)-2\pi f)^2}{f^4 + \phi''(t)^2}} a(t) \\
&\quad \cdot e^{j\left[\phi(t) - 2\pi tf + \frac{1}{2}\arctan\left(\frac{\phi''(t)}{f^2}\right) - \frac{\frac{1}{2}(\phi'(t)-2\pi f)^2 \phi''(t)}{f^4 + \phi''(t)^2}\right]} \\
&= a^{(ST)}(t,f) e^{j\phi^{(ST)}(t,f)}.
\end{aligned}
$$

□

Note that a general approximation formula was derived by Guo, Molahajloo and Wong [19] for the ST with an arbitrary window function where the phase function was expanded up to the first order. With the Gaussian localization window, we are able to approximate the phase functions up to the second order. Theorem 2.1 enables us to have a better understanding to the limitation of applying the ST-based CIF of signals.

In general, the IF of a real signal does not change rapidly with respect to time, which leads to an valid approximation $\phi''(t) \approx 0$ in any short time period. When $\phi''(t) = 0$, the last two terms in Equation (10) vanish, which gives

$$\phi^{(ST)}(t,f) \approx \phi(t) - 2\pi tf. \tag{11}$$

This approximation indicates that the ST holds the absolutely referenced phase information, i.e., the phase information given by the ST at any frequency refers to the argument of the cosinusoid at zero time. Substitution of Equation (11) into Equation (7) gives

$$f_i^{(ST)}(t,f) \approx \frac{1}{2\pi} \frac{d\phi(t)}{dt} = f_i(t). \tag{12}$$

Equation (12) indicates that the CIF of any given ST voice approximates the IF for a mono-component signal.

The approximated amplitude of the ST in Equation (9) also implies that the ST has the spectral energy distributed around its IF laws. Assuming that the effect of the second order derivative of the signal phase is negligible, $a^{(ST)}(t,f)$ reaches its maximum at $f = \frac{1}{2\pi}\phi'(t)$, the IF of the signal. Since the ST is a linear transform, multiple components of a signal are separated in the time-frequency domain by the fact that the energy of each component is distributed around its own IF law.

2.3 Local Group Delay of the Stockwell Transform

On the other hand, the GD can also be extended as a local time measure with the ST. By fixing the value of the time $t = t_0$, a function of frequency is obtained from the ST. The local group delay is then defined by the taking the partial derivative of the phase with respect to the frequency,

$$t_g^{(ST)}(t,f) = -\frac{1}{2\pi}\frac{\partial \Phi^{(ST)}(t,f)}{\partial f}, \tag{13}$$

where $\Phi^{(ST)}(t,f)$ is the phase of the ST along the frequency direction. An approximation of this phase function can be computed for a mono-component signal by using Equation (3) with the Fourier transform of the analytic signal $S(f) = A(f)e^{j\Phi(f)}$. The approximation formula is given by the following theorem.

Theorem 2.2 *Let the spectral representation of the signal $S(f) = A(f)e^{j\Phi(f)}$ be such that the amplitude $A(t)$ is slowly-varying. Then the following approximation formula can be derived for the ST with the Gaussian window function $\psi(t) = \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}$,*

$$ST_s(t,f) \approx A^{(ST)}(t,f)e^{j\Phi^{(ST)}(t,f)}, \tag{14}$$

where

$$A^{(ST)}(t,f) = \sqrt{2\pi}\left(\frac{16\pi^4}{f^4} + \Phi''(f)^2\right)^{-\frac{1}{4}} \cdot e^{-\frac{2\pi^2 f^2(\Phi'(f)+2\pi t)^2}{16\pi^4+f^4\Phi''(f)^2}} A(f), \tag{15}$$

$$\Phi^{(ST)}(t,f) = \Phi(f) + \frac{1}{2}\arctan\left(\frac{f^2\Phi''(f)}{4\pi^2}\right) - \frac{\frac{1}{2}f^4\left(\Phi'(f)+2\pi t\right)^2\Phi''(f)}{16\pi^4 + f^4\Phi''(f)^2}. \tag{16}$$

Proof By replacing the signal X with its polar form in (3) and changing the variable $u = \alpha - f$, we have

$$ST_s(t,f) = \int_{-\infty}^{\infty} A(u+f)e^{j\Phi(u+f)}e^{-\frac{2\pi^2u^2}{f^2}}e^{j2\pi ut}du,$$

Since the amplitude $A(f)$ is slowly-varying, we assume that $A(u+f)$ inside the Gaussian window to be constant, which leads to the approximation $A(u+f)e^{-\frac{2\pi^2u^2}{f^2}} \approx A(f)e^{-\frac{2\pi^2u^2}{f^2}}$. Expanding the phase function $\Phi(u+f)$ into a Taylor series at the frequency f, that is $\Phi(u+f) = \Phi(f) + \Phi'(f)u + \frac{1}{2}\Phi''(f)u^2 + O(u^3)$ and neglecting the high order term $O(u^3)$, we then have

$$\begin{aligned}
ST_x(t,f) &\approx \int_{-\infty}^{\infty} A(f)e^{-\frac{2\pi^2u^2}{f^2}}e^{j\left(\Phi(f)+\Phi'(f)u+\frac{1}{2}\Phi''(f)u^2\right)}e^{j2\pi ut}du \\
&= A(f)e^{j\Phi(f)}\int_{-\infty}^{\infty} e^{-\frac{2\pi^2u^2}{f^2}}e^{j\left(\Phi'(f)u+\frac{1}{2}\Phi''(f)u^2+2\pi ut\right)}du \\
&= A(f)e^{j\Phi(f)}\int_{-\infty}^{\infty} e^{-\left(\frac{2\pi^2}{f^2}-\frac{1}{2}j\Phi''(f)\right)u^2+(j\Phi'(f)+j2\pi t)u}du \\
&= A(f)e^{j\Phi(f)}\sqrt{\frac{\pi}{\frac{2\pi^2}{f^2}-\frac{1}{2}j\Phi''(f)}} \cdot e^{\frac{(j\Phi'(f)+j2\pi t)^2}{\frac{8\pi^2}{f^2}-2j\Phi''(f)}} \\
&= \sqrt{\pi}A(f)e^{j\Phi(f)}\sqrt{\frac{\frac{2\pi^2}{f^2}+j\frac{\Phi''(f)}{2}}{\frac{4\pi^4}{f^4}+\frac{\Phi''(f)^2}{4}}} \cdot e^{-\frac{\frac{1}{2}(\Phi'(f)+2\pi t)^2\left(\frac{4\pi^2}{f^2}+j\Phi''(f)\right)}{\frac{16\pi^4}{f^4}+\Phi''(f)^2}} \\
&= \sqrt{2\pi}\left(\frac{16\pi^4}{f^4}+\Phi''(f)^2\right)^{-\frac{1}{4}} e^{-\frac{2\pi^2f^2(\Phi'(f)+2\pi t)^2}{16\pi^4+f^4\Phi''(f)^2}}A(f) \cdot \\
&\quad e^{j\left(\Phi(f)+\frac{1}{2}\arctan\left(\frac{f^2\Phi''(f)}{4\pi^2}\right)-\frac{\frac{1}{2}f^4(\Phi'(f)+2\pi t)^2\Phi''(f)}{16\pi^4+f^4\Phi''(f)^2}\right)} \\
&= A^{(ST)}(t,f)e^{j\Phi^{(ST)}(t,f)}.
\end{aligned}$$

□

Similarly, by assuming that $\Phi''(f) \approx 0$, we then have another approximation of the local phase given by the ST

$$\Phi^{(ST)}(t,f) \approx \Phi(f). \tag{17}$$

It follows that the LGD of the ST at any time instant gives the GD for a mono-component signal, i.e.,

$$t_g^{(ST)}(t,f) \approx -\frac{1}{2\pi}\frac{d\Phi(f)}{df} = t_g(f). \tag{18}$$

The approximated amplitude $A^{(ST)}(t,f)$ in Equation (15) also shows that the energy of each signal component is distributed around its own GD law in the spectrogram of the ST.

Equations (12) and (18) state that the CIF and the LGD defined by the ST can be used to estimate the IF and the GD for a mono-component signal. As multiple components of a signal can be separated in the time-frequency domain with the ST [9], it is theoretically feasible to estimate the IF and the GD of each component by taking the measures at different voices of the ST. However, in applications, such an approach is impractical due to the following reasons. First, noise is usually presented in real signals. In the time-frequency domain where the signal-to-noise ratio is low, the signal information can be greatly distorted by noise and thus direct IF and GD estimates from the CIF and the LGD fail. Second, for a multi-component signal, the spread energy from different signal components may overlap and introduce the misleading cross-terms. Equations (12) and (18) become invalid in the areas where cross-terms appear. Third, a single voice of the ST may contain energy spread from different components occurred at different time. In the situations mentioned above, the CIF measure may provide mixed IF information from different signal components, which will lead to an erroneous interpretation of the time-frequency structure of the signals.

3 The Reassigned Spectrogram of Stockwell Transform

3.1 The Reassigned Spectrogram of Stockwell Transform

Instead of calculating the IF and the GD of a non-stationary signal directly using their definitions, the reassignment method [13] was proposed to detect the IF from another perspective. As most TFRs have the signal energy peaked around the IFs of their components, it follows that the IF can be estimated by investigating the locations of energy distributions in the time-frequency domain. However, the spread signal energy in TFRs may mask the true locations of the IF. The reassignment method intends to enhance energy localization of a TFR by reassigning the spread energy to the centers of gravities.

At each point in the time-frequency domain, we can define a point

$$\left(t_g^{(ST)}(t,f), f_i^{(ST)}(t,f)\right)$$

from LGD and CIF of the ST, which is located on the IF/GD curves as indicated by Equations (12) and (18). Thus, we can interpret them as the re-estimation of the time arrival and the instantaneous frequency of the signal energy located at the point (t,f) in the ST. In other words, $\left(t_g^{(ST)}(t,f), f_i^{(ST)}(t,f)\right)$ gives the information of the time and the frequency of the signal component contributing to the spectral energy $|ST_x(t,f)|^2$. By applying the reassignment technique, the value of the energy $|ST_x(t,f)|^2$ at each point (t,f) is reassigned to the location $\left(t_g^{(ST)}(t,f), f_i^{(ST)}(t,f)\right)$, which is the essential idea of the reassigned spectrogram of Stockwell transform.

Definition 3.1 Let $ST_s(t,f)$ be the Stockwell transform of the analytic signal $s(t)$, and $t_g^{(ST)}(t,f)$ and $f_i^{(ST)}(t,f)$ be the local group delay and the channelized instantaneous frequency of the ST, then the reassigned Stockwell spectrogram is defined as

$$\begin{aligned} &RSS_s(t',f') \\ &= \int\int_{-\infty}^{\infty} f'|ST_s(t,f)|^2 \delta\left(t' - t_g^{(ST)}(t,f)\right) \delta\left(f' - f_i^{(ST)}(t,f)\right) \frac{dtdf}{f}, \end{aligned} \tag{19}$$

where $\delta(t)$ denotes the Dirac impulse.

The re-mapping of the energy $|ST_s(t,f)|^2$ from the location (t,f) to

$$\left(t_g^{(ST)}(t,f), f_i^{(ST)}(t,f)\right)$$

is many-to-one in general. The measure $\frac{dtdf}{f}$ is used in the reassignment equation because the spectrogram of the ST belongs to the function space $L^2\left(\mathcal{R}^2, \frac{dtdf}{f}\right)$ [6]. As the spread energy in the time-frequency domain has been reassigned to the IF/GD curves, the IFs can be easily obtained by identifying the locations of the local energy peaks in the reassigned spectrogram of ST.

Since the reassignment method is a postprocessing technique applied to the TFR, its performance depends on t the accomplishment of the underlying TFR. As an improved multi-scale time-frequency representation, the ST provides a more accurate description of time-varying frequency characteristics of multi-component signals by diminishing the mismatch between the window width and the local signal characteristics. Therefore, the ST is a good candidate for the reassignment method. The the reassigned spectrogram of ST provides high-resolution ridges in the time-frequency domain that can be further used to extract the IFs.

3.2 Implementation of Reassigned Spectrogram of Stockwell Transform

Implementing the LGD and the CIF of the ST involves calculating the phase derivative. This is often done by finite difference methods that require phase-unwrapping the representation to resolve the discontinuities of the principle-value representation of the argument function. To avoid phase ambiguities, we adopt the cross-spectral method suggested by Nelson [15] in our implementation.

Given the computed ST, two intermediate cross-spectral surfaces are defined as

$$\begin{aligned} C_f(t,f,\epsilon) &= ST_s\left(t+\frac{\epsilon}{2},f\right) ST_s^*\left(t-\frac{\epsilon}{2},f\right), \\ L_f(t,f,\epsilon) &= ST_s\left(t,f+\frac{\epsilon}{2}\right) ST_s^*\left(t,f-\frac{\epsilon}{2}\right) \end{aligned} \tag{20}$$

for small values of ϵ. The spectrogram, the LGD and the CIF of ST are then estimated by

$$|ST_s(t,f)|^2 \approx |C_f(t,f,\epsilon)| \approx |L_f(t,f,\epsilon)|, \tag{21}$$

$$\hat{t}_g^{(ST)}(t,f) \approx -\frac{1}{2\pi\epsilon} \arg[L_f(t,f,\epsilon)], \tag{22}$$

$$\hat{f}_i^{(ST)}(t,f) \approx f + \frac{1}{2\pi\epsilon} \arg[C_f(t,f,\epsilon)]. \tag{23}$$

There is no need to phase unwrap the representation surface to resolve the discontinuity problem.

Let $s[kT], k = 0, 1, \cdots, N-1$ denote the discrete signal of $s(t)$, with a time sampling interval T, the practical discrete implementation of the reassigned spectrogram of ST is summarized as follows:

S1: Compute the discrete Fourier spectrum based on the FFT algorithm, $S[\frac{n}{NT}] = \frac{1}{N}\sum_{k=0}^{N-1} s[kT]e^{-\frac{j2\pi nk}{N}}$, where $n = 0, 1, \cdots, N-1$.
S2: Calculate the Stockwell transform,$ST_s(mT, 0) = \frac{1}{N}\sum_{k=0}^{N-1} s[\frac{k}{NT}]$ for $n = 0$ and $ST_s(mT, \frac{n}{NT}) = \sum_{k=0}^{N-1} S[\frac{k+n}{NT}]e^{-\frac{2\pi^2 k^2}{n^2}} e^{\frac{j2\pi km}{N}}, n \neq 0$.
S3: Calculate the LGD and the CIF,
$t_g^{(ST)}(mT, \frac{n}{NT}) = -\frac{NT}{4\pi}\arg[ST_s(mT, \frac{n+1}{NT}) \cdot \overline{ST_s(mT, \frac{n-1}{NT})}]$ and
$f_i^{(ST)}(mT, \frac{n}{NT}) = \frac{n}{NT} + \frac{1}{4\pi T}\arg[ST_s((m+1)T, \frac{n}{NT}) \cdot \overline{ST_s((m-1)T, \frac{n}{NT})}]$.
S4: For each point $(mT, \frac{n}{NT})$, reassign the spectral energy $|ST_s(mT, \frac{n}{NT})|^2$ to the point $[t_g^{(ST)}(mT, \frac{n}{NT}), f_i^{(ST)}(mT, \frac{n}{NT})]$ such that the discrete reassigned spectrogram of Stockwell transform is obtained.

This implementation requires computing the ST of the signal only once, which is computationally efficient.

4 Numerical Results

In this section, we demonstrate the effectiveness of the reassigned spectrogram of ST compared to the conventional ST using numerical simulations. First, we consider a synthetical signal consisting of two components with different IF laws: one component is a sine wave with frequency 50 Hz, the other component has a complex IF law, which is $f_i(t) = 200 - 16\pi \cdot \cos(16\pi t)$. The sample rate of this synthetic signal is 1024. Figure 1b shows the spectrogram of ST for this signal. This representation provides a rough picture of the time-frequency structure of the signal. However, due to the spread signal energy in the time-frequency domain, the exact IF of each signal component cannot be identified. The reassigned spectrogram of ST of this same signal is given in Fig. 1c. As we can see, the reassigned spectrogram of ST has the energy of each signal component highly concentrated at its IF. Thus, the IF of each component can be revealed by detecting the locations of energy peaks.

The next examples are simulated dual-tone multi-frequency (DTMF) signals. DTMF signaling is used in push-button telephones for tone dialing. A DTMF signal

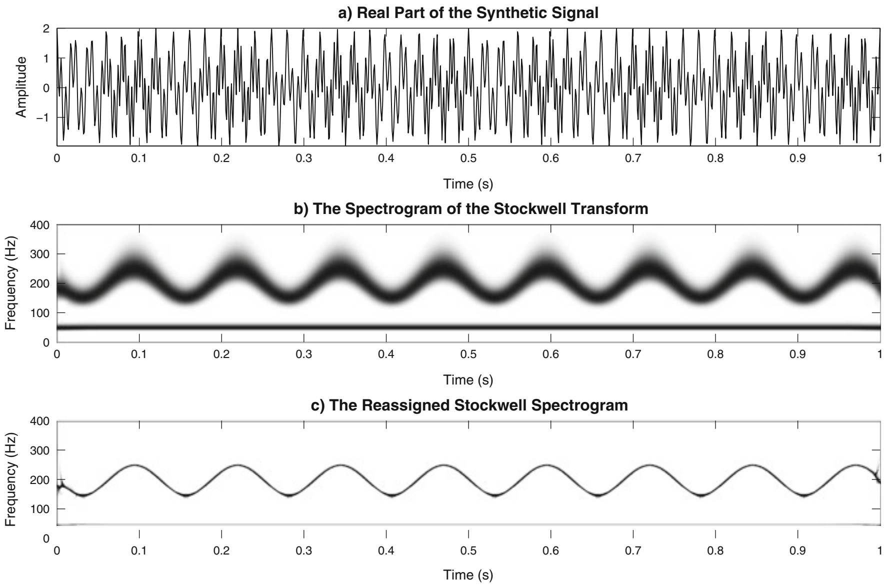

Fig. 1 (**a**) The real part, (**b**) the spectrogram of Stockwell transform and (**c**) the reassigned spectrogram of Stockwell transform of a synthetic signal made up of two components with different instantaneous frequency laws

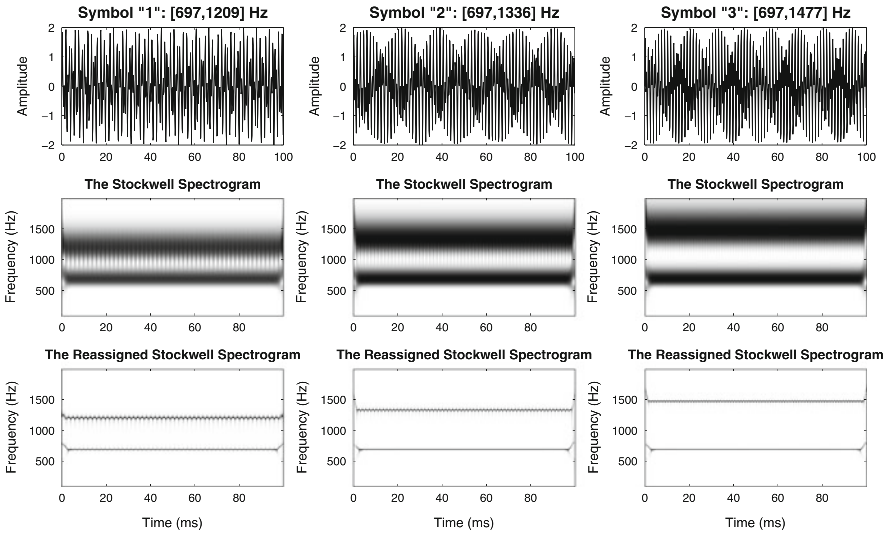

Fig. 2 *Top*: the real parts, *Middle*: the spectrograms of Stockwell transform, *Bottom*: the reassigned spectrograms of Stockwell transform of three simulated dual-tone multi-frequency signals corresponding to the push-buttons 1, 2 and 3 of the telephone pad

consists of the sum of two sinusoids with frequencies taken from two mutually exclusive groups. These frequencies were chosen to prevent any harmonics from being incorrectly detected by the receiver as some other DTMF frequency. The dialing tones of the push-buttons 1, 2 and 3 of the telephone pad have been simulated with the sample rate 8000. These three tones share the same low frequency component at 697 Hz, but the high frequency components are distinct, which are 1209, 1366, and 1477 Hz respectively. Results of the spectrogram of ST and the reassigned spectrogram of ST are presented for each signal in Fig. 2. Due to the energy spread along the frequency domain, the exact frequency of each signal component cannot be directly identified from the spectrogram of ST. But, the locations of the energy peaks in the reassigned spectrogram of ST clearly show the frequency of each signal component for the DTMF signals. As a result, the reassigned spectrogram of ST provides an approach to detect and recognize digital tones.

We further apply the reassigned spectrogram of ST to a real audio signal to illustrate its performance in applications. The signal example presented here is a bat sonar signal [20] recorded by the research program RCP 445 supported by CNRS. Sounds produced by bat in fact are ultrasound which are inaudible for human beings. The real part of this signal is displayed in Fig. 3a. The spectrogram of the ST in Fig. 3b shows the basic time-frequency structure of this signal, which is non-stationary and multi-component. However, the IF of each component embedded in this bat signal cannot be easily identified because of the energy spread in the ST. The reassigned spectrogram of ST of this bat sonar signal is presented in Fig. 3c.

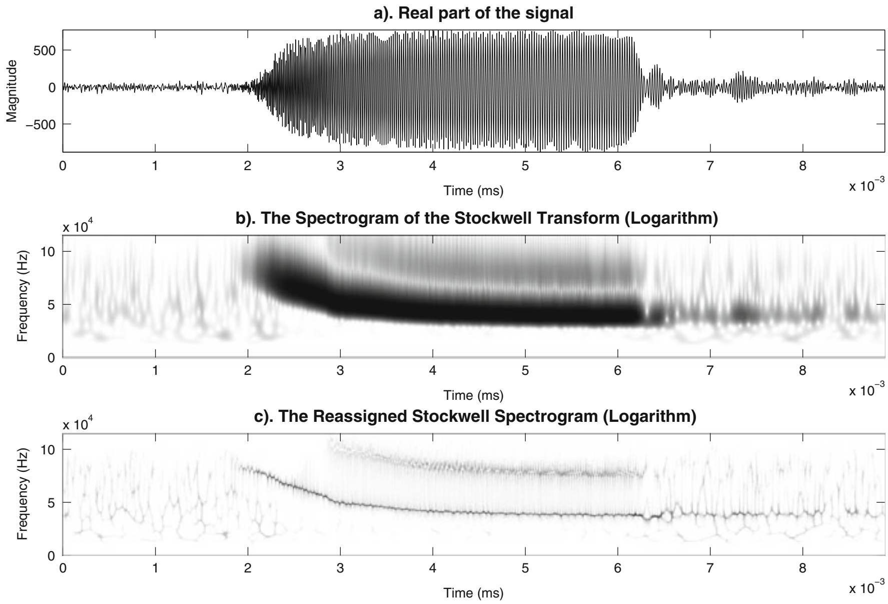

Fig. 3 (**a**) The real part, (**b**) the spectrogram of Stockwell transform, and (**c**) the reassigned spectrogram of Stockwell transform of a bat sonar signal with the sampling rate 230.4 kHZ

As showed by the spectral energy peaks in the reassigned spectrogram of ST, we can clearly identify two significant non-stationary chirp-like components contained in the bat signal. The most significant one starts around time 2 ms with frequency lower than 10×10^4 Hz and decreases to frequency around 4×10^4 Hz and ends after 6 ms. Another component has a higher frequency and shorter duration, which starts around 3 ms and ends after 6 ms, and its frequency decreases from above 10×10^4 Hz to around 8×10^4 Hz. Note that the identified higher frequency component by the reassigned spectrogram of ST has been missed by many commonly used TFRs due to the relatively low energy of this component.

5 Conclusions

In this paper, we propose the local group delay and channelized instantaneous frequency of Stockwell transform, and prove that these two measurements closely assemble the conventional definitions of the group delay and the instantaneous frequency for a mono-component signal. We further applied the reassignment method to the Stockwell transform to obtain a high-resolution time-frequency representation of a signal. Due to its closeness to the traditional group delay and instantaneous frequency and multi-resolution feature of the Stockwell transform, the reassigned spectrogram of Stockwell transform provides a good time-frequency representation as a base to further estimate the instantaneous frequencies for non-stationary and multi-component signals. Numerical simulations have shown the effectiveness of the reassigned spectrogram of Stockwell transform and indicated its great potential in applications.

Although the Stockwell transform provides an improved representation by its frequency-dependent resolution characteristic, it does not assure that multiple signal components can be always well separated in the time-frequency domain. As shown in Fig. 2, the two components of dialing tones of push-buttons 1 and 2 are too close such that some ringing effects are introduced in the spectrogram of the ST. It follows that the performance of the reassignment method is affected. A new variant of the Stockwell transform was recently proposed in [18], which provides an adaptive multi-resolution time-frequency representation that can better separate signal components in the time-frequency domain. We will further extend the reassignment method to this adaptive Stockwell transform to more accurately extract the instantaneous frequencies of multi-component signals and explore other efficient means in estimating instantaneous frequency.

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Continuous Multiwavelet Transform for Blind Signal Separation

Ryuichi Ashino, Takeshi Mandai, and Akira Morimoto

Abstract Observed signals are usually recorded as linear mixtures of original sources. Our purpose is to separate observed signals into original sources. To analyse observed signals, it is important to use several wavelet functions having different characteristics and compare their continuous wavelet transforms. The notion of the continuous multiwavelet transform and its essentials are introduced. An application to blind image separation is presented.

Keywords Blind signal source separation • Continuous multiwavelet transform • Time-frequency analysis

Mathematics Subject Classification (2000). Primary: 42C40; Secondary: 94A12, 65T60

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1 Blind Signal Separation

Assume several persons are talking in a cocktail party. One can focus one's auditory attention on a particular person and understand his or her talk. How one's auditory perception works? This is what we call *cocktail party problem* [8]. One of basic questions for the cocktail party problem is how to build a machine to solve the cocktail party problem in a satisfactory manner. This question corresponds to *blind signal separation* or *blind source separation* (BSS) [9, 10, 17].

As the blind signal separation is an inverse problem, certain a priori knowledge on the original sources is needed for this separation, and the original sources cannot be uniquely estimated.

Besides methods based on independent component analysis [20] which is one of the most powerful tools for blind signal separation, several methods based on time-frequency [11, 12, 15, 28] analysis have been proposed. One of them is the quotient signal estimation method which can estimate the unknown number of sources [7, 21, 27, 29].

1.1 Blind Signal Separation of One-Dimensional Signals

Let us introduce our blind signal separation. Our purpose is to separate and to estimate the original sources (talks by N persons, N is unknown) from the observed signals (recorded talks with M microphones, M is known) as in Fig. 1. To estimate

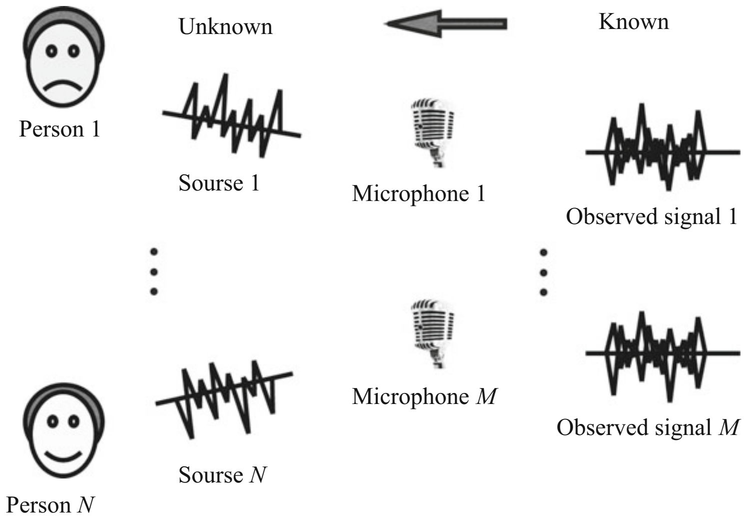

Fig. 1 Blind signal separation

the number N of sources is one of the most difficult procedures in the blind signal separation.

We will explain the blind signal separation of one-dimensional signals based on [2–4]. Let $\{s_k(t)\}_{k=1}^N$ be the original source signals and $\{x_j(t)\}_{j=1}^M$ be the observed signals. Assume that $M \geq N$ and all of these signals are real-valued. Put

$$s(t) = (s_1(t), \dots, s_N(t))^T, \qquad x(t) = (x_1(t), \dots, x_M(t))^T.$$

The *spatial mixture problem* of BSS assumes that the observed signals $\{x_j(t)\}_{j=1}^M$ can be represented as

$$x_j(t) = \sum_{k=1}^{N} a_{j,k} s_k(t), \qquad a_{j,k} \in \mathbb{R}, \tag{1}$$

for unknown matrix $A = (a_{j,k}) \in \mathbb{R}^{M \times N}$, which is called *mixing matrix*. Here, $\mathbb{R}^{M \times N}$ denotes the set of $M \times N$ matrices with real elements. We assume $a_{j,k} \in \mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\}$, for the sake of simplicity.

1.2 Solving Blind Signal Separation

To explain the essential idea to solve the blind signal separation, let us assume that we know the number N and the set of points $\{t_\ell\}_{\ell=1}^N$ such that $s_k(t_\ell) = \delta_{k,\ell}$, for $\ell, k = 1, \dots, N$. Here $\delta_{k,\ell}$ denotes the Kronecker delta. Then, (1) implies

$$x_j(t_\ell) = \sum_{k=1}^{N} a_{j,k} s_k(t_\ell) = \sum_{k=1}^{N} a_{j,k} \delta_{k,\ell} = a_{j,\ell}, \tag{2}$$

which gives us an estimation of $A = \big(x_j(t_\ell)\big)$. Applying the Fourier transform to (1), we have

$$\widehat{x_j}(\omega) = \sum_{k=1}^{N} a_{j,k} \widehat{s_k}(\omega). \tag{3}$$

Again, assume we know the number N and the set of points $\{\omega_\ell\}_{\ell=1}^N$ such that $\widehat{s_k}(\omega_\ell) = \delta_{k,\ell}$, for $\ell, k = 1, \dots, N$. Then, we have

$$\widehat{x_j}(\omega_\ell) = \sum_{k=1}^{N} a_{j,k} \widehat{s_k}(\omega_\ell) = \sum_{k=1}^{N} a_{j,k} \delta_{k,\ell} = a_{j,\ell}, \tag{4}$$

which gives us another estimation of $A = \big(\widehat{x_j}(\omega_\ell)\big)$.

- Which is easy to find such a set of points, $\{t_\ell\}_{\ell=1}^N$ or $\{\omega_\ell\}_{\ell=1}^N$?

If one person is talking all the time, it is impossible to find $\{t_\ell\}_{\ell=1}^N$. Even such a situation, if their voices are different in frequency, it may be possible to find $\{\omega_\ell\}_{\ell=1}^N$.

- Moreover, if we use time-frequency information, finding a set of points $\{(t_\ell, \omega_\ell)\}_{\ell=1}^N$ should be better.

1.3 A Simple Example of Time-Frequency Information

Let us demonstrate the above statement by a simple example. In Fig. 2, the original sources $\{s_j(t)\}_{j=1,2,3,4}$ are shown in the top and their continuous wavelet transforms [18] $\{S_j(t,\omega)\}_{j=1,2,3,4}$ are bottom, where the scale is transformed to frequency using the fact that $1/a$ is in proportion to frequency ω. The overlapping of transformed source signals (bottom) in the time-frequency region are rather small comparing to the overlapping of original source signals (top) in the time region.

2 Time-Frequency Analysis

2.1 Fundamental Unitary Operators and Their Properties

Let us define three fundamental unitary operators in time-frequency analysis. Denote by $\mathcal{T}_b$, the *translation* operator by $b \in \mathbb{R}^n$:

$$(\mathcal{T}_b f)(x) = f(x-b),$$

by $\mathcal{M}_\omega$, the *modulation* operator by $\omega \in \mathbb{R}^n$:

$$(\mathcal{M}_\omega f)(x) = e^{i\omega\cdot x} f(x),$$

by $\mathcal{D}_a$, the *dilation* operator by $a \in \mathbb{R}_+$:

$$(\mathcal{D}_a f)(x) = a^{-n/2} f(x/a).$$

These three operators, $\mathcal{T}_b$, $\mathcal{M}_\omega$, $\mathcal{D}_a$, are unitary on $L^2(\mathbb{R}^n)$, hence their adjoints are given by their inverses:

$$\mathcal{T}_b^* = \mathcal{T}_{-b}, \qquad \mathcal{M}_\omega^* = \mathcal{M}_{-\omega}, \qquad \mathcal{D}_a^* = \mathcal{D}_{1/a}.$$

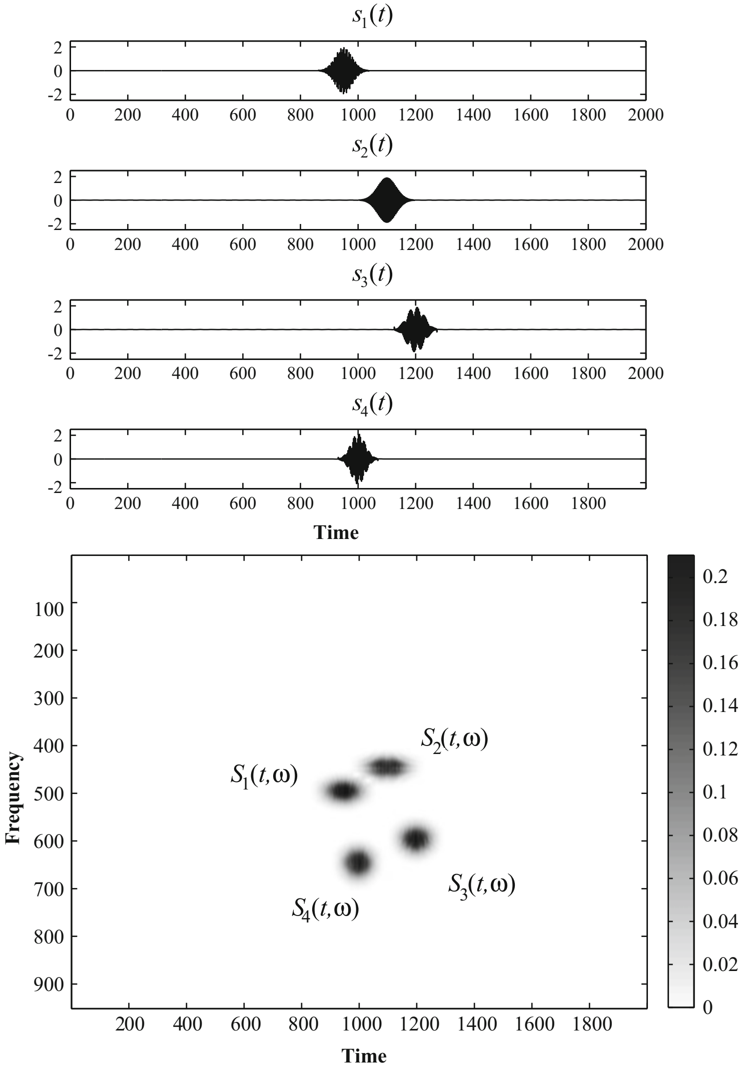

Fig. 2 Original sources (*top*) and their time-frequency information (*bottom*)

Lemma 2.1 (Commutation Relation)

$$\mathcal{T}_y\mathcal{M}_\xi = e^{-i\xi y}\mathcal{M}_\xi\mathcal{T}_y, \qquad \mathcal{M}_\xi\mathcal{T}_y = e^{i\xi y}\mathcal{T}_y\mathcal{M}_\xi,$$
$$\mathcal{T}_y\mathcal{D}_\rho = \mathcal{D}_\rho\mathcal{T}_{y/\rho}, \qquad \mathcal{D}_\rho\mathcal{T}_y = \mathcal{T}_{\rho y}\mathcal{D}_\rho,$$
$$\mathcal{M}_\xi\mathcal{D}_\rho = \mathcal{D}_\rho\mathcal{M}_{\rho\xi}, \qquad \mathcal{D}_\rho\mathcal{M}_\xi = \mathcal{M}_{\xi/\rho}\mathcal{D}_\rho.$$

Define the Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ and the inverse Fourier transform of a function $g \in L^1(\mathbb{R}^n)$ by

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx,$$
$$\mathcal{F}^{-1}(g)(x) = \check{g}(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} g(\xi)\,d\xi,$$

where $i = \sqrt{-1}$.

Lemma 2.2 (Commutation Relation with Fourier Transform)

$$\mathcal{F}[\mathcal{T}_y f] = \mathcal{M}_{-y}\mathcal{F}[f], \qquad \mathcal{F}[\mathcal{M}_\omega f] = \mathcal{T}_\omega \mathcal{F}[f], \qquad \mathcal{F}[\mathcal{D}_\rho f] = \mathcal{D}_{1/\rho}\mathcal{F}[f].$$

As the dilation $\mathcal{D}_a$ and the translation $\mathcal{T}_b$ are unitary, their composition $\mathcal{T}_b\mathcal{D}_a$ is also unitary and called *time-scale operator*.

Lemma 2.3 (Composition of Time-Scale Operators)

$$(\mathcal{T}_b\mathcal{D}_a)(\mathcal{T}_t\mathcal{D}_s) = \mathcal{T}_{at+b}\mathcal{D}_{as}, \qquad b, t \in \mathbb{R}^n, \quad a, s \in \mathbb{R}_+.$$

2.2 *Center and Width of a Window Function*

To investigate properties of a given function $f(x)$, we often represent $f(x)$ as a superposition of well-known functions $g_j(x)$ or $g(s, x)$, $s \in \Omega$, such that $f(x) = \sum_j a_j g_j(x)$ or $f(x) = \int_\Omega a(s)g(s, x)\,ds$. The Fourier inversion formula:

$$f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\widehat{f}(\xi)\,d\xi$$

can be regarded as one of such representations. But the functions $e^{ix\cdot\xi}$ are not localized in space. Therefore, the *windowed Fourier transform*:

$$\begin{aligned} V_w f(b, \omega) &:= \left\langle f(x), e^{ix\cdot\omega} w(x - b)\right\rangle \\ &= \langle f, \mathcal{M}_\omega \mathcal{T}_b w\rangle \end{aligned}$$

was proposed to access the local information both in space and in frequency. Here, we denote the canonical inner product of the Hilbert space $L^2(\mathbb{R}^n)$ by

$$\langle f, g\rangle = \int_{\mathbb{R}^n} f(x)\overline{g(x)}\,dx.$$

In fact, Parseval's formula and the commutation relations $\mathcal{F}\mathcal{M}_\omega = \mathcal{T}_\omega \mathcal{F}$, $\mathcal{F}\mathcal{T}_b = \mathcal{M}_{-b}\mathcal{F}$ imply

$$\begin{aligned} \langle f, \mathcal{M}_\omega \mathcal{T}_b w\rangle &= (2\pi)^{-n} \langle \mathcal{F}f, \mathcal{F}\mathcal{M}_\omega \mathcal{T}_b w\rangle \\ &= (2\pi)^{-n} \left\langle \widehat{f}, \mathcal{T}_\omega \mathcal{M}_{-b}\widehat{w}\right\rangle \\ &= (2\pi)^{-n} e^{-i\omega\cdot b} \left\langle \widehat{f}, \mathcal{M}_{-b} \mathcal{T}_\omega \widehat{w}\right\rangle. \end{aligned}$$

Hence, $V_w f(b,\omega)$ can access the information on both areas localized by $w(x-b)$ in space and by $\widehat{w}(\xi-\omega)$ in frequency.

For a function $w \in L^2(\mathbb{R}^n)$, the *center* c_w of w is defined by

$$\begin{aligned} c_w &= (c_{w,1}, \ldots, c_{w,n}), \\ c_{w,j} &:= \frac{1}{||w||^2} \int_{\mathbb{R}^n} x_j |w(x)|^2\, dx, \quad j = 1, \ldots, n, \end{aligned}$$

and the *width* Δ_w of w is defined by

$$\begin{aligned} \Delta_w &:= (\Delta_{w,1}, \ldots, \Delta_{w,n}), \\ \Delta_{w,j} &:= \frac{1}{||w||} \left(\int_{\mathbb{R}^n} (x_j - c_{w,j})^2 |w(x)|^2\, dx \right)^{1/2}. \end{aligned}$$

A function $w \in L^2(\mathbb{R}^n) \setminus \{0\}$ is called a *window function* if $|x|w(x) \in L^2(\mathbb{R}^n)$, for which we can define c_w and Δ_w. Hereafter, we assume that both w and its Fourier transform $\widehat{w}$ are window functions. The center of w (resp. $\widehat{w}$) is denoted by x^* (resp. ξ^*), and (x^*, ξ^*) is also called the center of w.

2.3 Multi-dimensional Uncertainty Principle in Fourier Analysis

As an analogue of the one dimensional uncertainty principle in Fourier analysis [14], we have

$$\Delta_{w,j} \Delta_{\widehat{w},j} \geq \frac{1}{2}, \qquad j = 1, \ldots, n,$$

which shows that the localization is a trade-off between w and $\widehat{w}$. As a corollary, we also have

$$|\Delta_w|\, |\Delta_{\widehat{w}}| \geq \frac{n}{2},$$

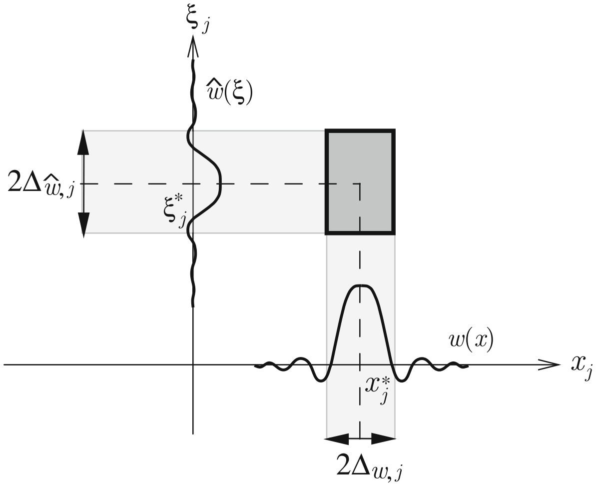

Fig. 3 The projection of the time-frequency window of $w(x)$ to the (x_j, ξ_j)-plane

where $|\cdot|$ denotes the length of a vector. The rectangular parallelepiped defined by

$$\prod_{j=1}^{n} [x_j^* - \Delta_{w,j}, x_j^* + \Delta_{w,j}] \times [\xi_j^* - \Delta_{\widehat{w},j}, \xi_j^* + \Delta_{\widehat{w},j}]$$

is called the *time-frequency window* of $w(x)$, whose projection to the (x_j, ξ_j)-plane is illustrated in Fig. 3.

3 Continuous Wavelet Transform

Wavelet analysis [19, 24] can be used as a tool for time-frequency analysis. The continuous wavelet transform correlates a given function $f \in L^2(\mathbb{R}^n)$ with a family of waveforms $\mathcal{T}_b \mathcal{D}_a \psi$. The function $\psi \in L^2(\mathbb{R}^n)$ is called a *wavelet function*. If the wavelet function ψ is properly chosen so as to be concentrated in "time" x and in "frequency" ξ, the continuous wavelet transform of f can be regarded as a time-frequency information of f. One advantage of the continuous wavelet transform is the availability of variety of wavelet functions. Each wavelet function has its own unique characteristics.

Definition 3.1 The *continuous wavelet transform* of $f \in L^2(\mathbb{R}^n)$ with respect to a wavelet function $\psi \in L^2(\mathbb{R}^n)$ is defined by

$$(W_\psi f)(b,a) := \langle f, \mathcal{T}_b \mathcal{D}_a \psi \rangle$$
$$= (2\pi)^{-n} \left\langle \widehat{f}, \mathcal{M}_{-b} \mathcal{D}_{1/a} \widehat{\psi} \right\rangle, \quad a \in \mathbb{R}_+, \ b \in \mathbb{R}^n.$$

Theorem 3.2 (Inversion Formula) *Assume $\psi \in L^2(\mathbb{R}^n)$ satisfies the following admissibility condition: there exists a positive constant K such that*

$$\int_{\mathbb{R}_+} |\widehat{\psi}(a\omega)|^2 \frac{da}{a} = K, \qquad \text{for almost all } \omega \in S^{n-1},$$

where S^{n-1} is the unit sphere in $\mathbb{R}^n$. Then,

$$f = \frac{1}{K} \int_{\mathbb{R}^n} \int_{\mathbb{R}_+} (W_\psi f)(b,a) \mathcal{T}_b \mathcal{D}_a \psi \frac{db\, da}{a^{n+1}}, \qquad f \in L^2(\mathbb{R}^n).$$

3.1 Time-Frequency Information of Continuous Wavelet Transform

To clarify our problems, let us start with one dimensional continuous wavelet transform. We use the wavelet toolbox in MATLAB. The wavelet toolbox provides a data, named cuspamax, f for demonstration which is illustrated in Fig. 4, the top-left. Since $(W_\psi f)(b,a)$ is two-dimensional data, we need a visual method of displaying the wavelet transform. Usually we use the intensity image, called *scaleogram or scalogram*, of the absolute values $|(W_\psi f)(b,a)|$ or the real part of $(W_\psi f)(b,a)$. Since the frequency ξ is inversely proportional to the scale a, the time-scale plane (scaleogram) can be regarded as the time-frequency plane by flipping upside down.

The bottom-left is the scaleogram $|(W_\psi f)(b,a)|$ using Meyer wavelet ψ_M. The top-right and bottom-right are the scaleograms $|(W_\psi f)(b,a)|$ using Daubechies 3 wavelet ψ_{D3} and Mexican hat wavelet ψ_{MH}, respectively.

We can detect the position of the cusp from each scaleogram, but these three scaleograms look different. Therefore, we should choose a proper wavelet function for our purpose.

When we compare these three scaleograms, we have a problem. The time-frequency windows of these three wavelets are different both in position and in shape. By the following lemma, we can align the centers of time-frequency windows by translation and dilation (see Fig. 5).

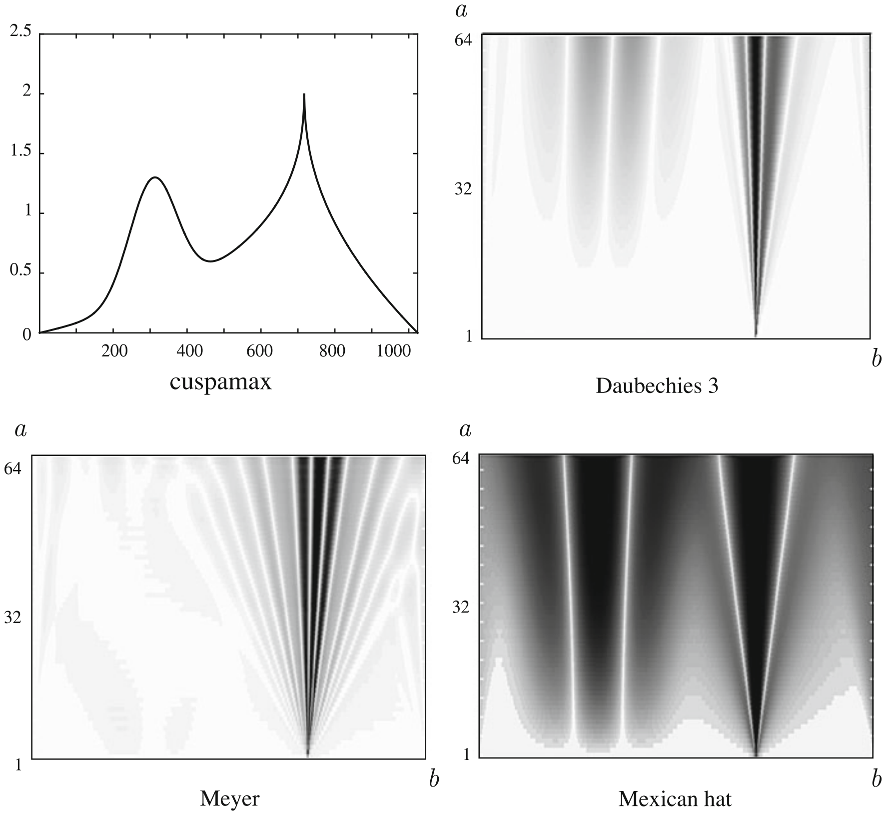

Fig. 4 The data cuspamax and its scaleograms

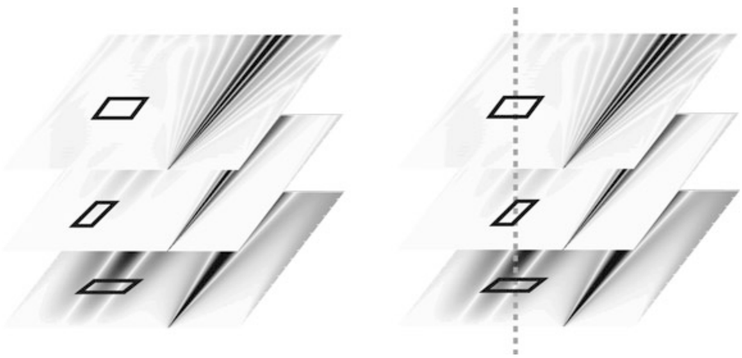

Fig. 5 Conceptual scheme of the time-frequency window (*left*) and center aligned time-frequency windows (*right*) of the three wavelets

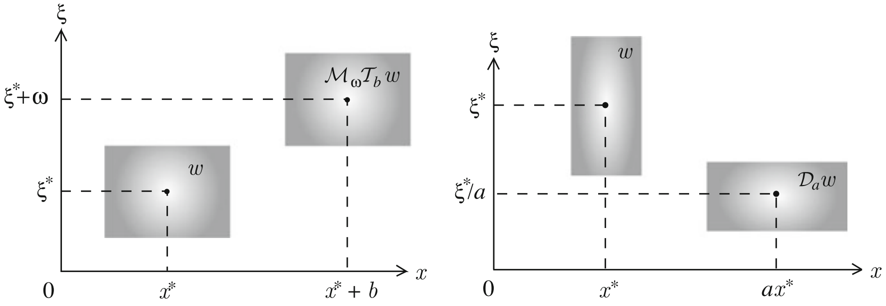

Fig. 6 The time-frequency windows of w, $\mathcal{M}_\omega \mathcal{T}_b w$, $\mathcal{D}_a w$

Lemma 3.3 *Assume that both w and its Fourier transform $\widehat{w}$ are window functions. Denote by x^* and Δ_w, the center and the width of w, and by ξ^* and $\Delta_{\widehat{w}}$, the center and the width of $\widehat{w}$, respectively. Then,*

(i) *$\mathcal{M}_\omega \mathcal{T}_b w$ and $\mathcal{T}_b \mathcal{M}_\omega w$, and their Fourier transforms are window functions, and the time-frequency windows of $\mathcal{M}_\omega \mathcal{T}_b w$ and $\mathcal{T}_b \mathcal{M}_\omega w$ are the same*

$$[x^* + b - \Delta_w, x^* + b + \Delta_w] \times [\xi^* + \omega - \Delta_{\widehat{w}}, \xi^* + \omega + \Delta_{\widehat{w}}],$$

which is illustrated in Fig. 6 (left).

(ii) *$\mathcal{D}_a w$ and its Fourier transform are window functions, and the time-frequency window of $\mathcal{D}_a w$ is*

$$[ax^* - a\Delta_w, ax^* + a\Delta_w] \times [\xi^*/a - \Delta_{\widehat{w}}/a\,, \xi^*/a + \Delta_{\widehat{w}}/a\,],$$

which is illustrated in Fig. 6 (right).

Lemma 3.4 *Let $\psi(x)$ be a wavelet function of one variable $x \in \mathbb{R}$ and $(x^*, \xi^*) \in \mathbb{R}^2$ be the center of ψ with $\xi^* \neq 0$. For a given $(x_0, \xi_0) \in \mathbb{R}^2$ with $\xi_0 \neq 0$, put*

$$b_0 = x_0 - x^* \xi^*/\xi_0, \quad a_0 = \xi^*/\xi_0. \tag{5}$$

Then, the center of $\mathcal{T}_{b_0} \mathcal{D}_{a_0} \psi$ is (x_0, ξ_0).

Proof By Lemma 3.3, the time-frequency window of $\mathcal{M}_0 \mathcal{T}_b \mathcal{D}_a w$ is

$$[ax^* + b - a\Delta_w, ax^* + b + a\Delta_w] \times [\xi^*/a - \Delta_{\widehat{w}}/a\,, \xi^*/a + \Delta_{\widehat{w}}/a\,],$$

the center of which is $(ax^* + b, \xi^*/a)$. Solving $(ax^* + b, \xi^*/a) = (x_0, \xi_0)$ with respect to (b, a), we have (5). □

Our aim is to align centers of time-frequency windows of several wavelet functions. Let us fix the center $(x_0, \xi_0) \in \mathbb{R}^2$ with $\xi_0 \neq 0$. For every wavelet function

ψ of one variable $x \in \mathbb{R}$ with the center $(x^*, \xi^*) \in \mathbb{R}^2$, where $\xi^* \neq 0$, $\mathcal{T}_{b_0}\mathcal{D}_{a_0}\psi$ is a wavelet function with the center $(x_0, \xi_0) \in \mathbb{R}^2$. As far as the scale parameter a remains to be one dimensional, direct generalization to the multi-dimensional case is difficult except the tensor product of one-dimensional wavelet functions.

Remark 3.1 By Lemma 2.3, the continuous wavelet transform $(W_{\mathcal{T}_{b_0}\mathcal{D}_{a_0}\psi} f)(b, a)$ with respect to $\mathcal{T}_{b_0}\mathcal{D}_{a_0}\psi$ can be represented as

$$\begin{aligned}(W_{\mathcal{T}_{b_0}\mathcal{D}_{a_0}\psi} f)(b, a) &= \langle f, \mathcal{T}_b\mathcal{D}_a\mathcal{T}_{b_0}\mathcal{D}_{a_0}\psi\rangle \\ &= \langle f, \mathcal{T}_{ab_0+b}\mathcal{D}_{aa_0}\psi\rangle = (W_\psi f)(ab_0 + b, aa_0),\end{aligned}$$

which can be calculated by existing continuous wavelet transform programs for $(W_\psi f)(b, a)$, such as the wavelet toolbox in MATLAB.

4 Time-Frequency Information for Blind Signal Separation

Let ψ^p, $p = 1, \ldots, L$ be real wavelet functions. Define the time-frequency information of s_k and x_j with respect to the wavelet function ψ^p by

$$\begin{aligned}S_k^p(t, \omega) &= W_{\psi^p} s_k(t, c_{\widehat{\psi^p},1}/\omega), \\ X_j^p(t, \omega) &= W_{\psi^p} x_j(t, c_{\widehat{\psi^p},1}/\omega), \qquad \omega \in \mathbb{R}_+,\end{aligned}$$

where $c_{\widehat{\psi^p},1}$ is the center of $\widehat{\psi^p}$. Note that the continuous wavelet transform of (1) with respect to ψ^p is

$$X_j^p(t, \omega) = \sum_{k=1}^{N} a_{j,k}\, S_k^p(t, \omega). \tag{6}$$

Each continuous wavelet transform of (1) with respect to ψ^p gives candidates for a set of points explained in §1.2. We anticipate that the intersection of candidates chosen by all the continuous wavelet transforms gives more precise estimation. Figure 7 illustrates this idea with the MATLAB sample data cuspamax. In fact, the intersection makes our estimations of N and A more precise. The details can be found in [6].

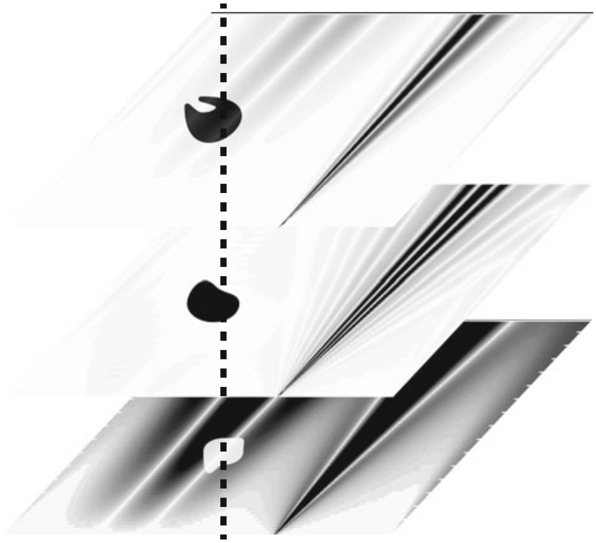

Fig. 7 Intersection of candidates given by all the continuous wavelet transforms

5 Continuous Multiwavelet Transform

In this section, we present essentials of the continuous multiwavelet transform according to [5] without proofs. We omit the design of multi-dimensional multi-wavelets, which is given in [5, §4].

In practical applications, one of the main difficulties to perform such wavelet analysis with several wavelet functions ψ^p, $p = 1, \dots, P$ is their choice. Since localization of $\mathcal{T}_b\mathcal{D}_a\psi^p$ in the time space and localization of its Fourier transform $\mathcal{F}(\mathcal{T}_b\mathcal{D}_a\psi^p)$ in the frequency space are different in general, we need to choose a space-scale parameter (b, a) properly to access to information at a given time-frequency point.

Definition 5.1 The *continuous multiwavelet transform* of $f \in L^2(\mathbb{R}^n)$ with respect to a multiwavelet function $\Psi = (\psi^p)_{p=1}^P \in L^2(\mathbb{R}^n)^P$, which is considered to be a column vector, is defined by

$$(W_\Psi f)(b, a) := \Big((W_{\psi^p} f)(b, a)\Big)_{p=1}^P, \quad a \in \mathbb{R}_+,\ b \in \mathbb{R}^n.$$

For an essentially bounded function $m \in L^\infty(\mathbb{R}^n)$ called *multiplier* or *mask*, define a *Fourier multiplier operator* $m(D)$ as a bounded linear operator on $L^2(\mathbb{R}^n)$

such that

$$(m(D)f)(x) = \mathcal{F}^{-1}\big(m(\xi)\widehat{f}(\xi)\big), \qquad f \in L^2(\mathbb{R}^n).$$

Here,

$$D = (D_1, \ldots, D_n), \qquad D_j = -i\partial/\partial x_j.$$

The continuous wavelet transform has strong compatibility with Fourier multiplier operators defined by homogeneous multipliers. For $m \in L^\infty(\mathbb{R}^n)$, we define $m(D)(\psi^p)_{p=1}^P := (m(D)\psi^p)_{p=1}^P$.

Proposition 5.2 ([5], Proposition 3) *Let $m \in L^\infty(\mathbb{R}^n)$ be positively homogeneous of degree 0, that is, $m(a\xi) = m(\xi)$ for $a \in \mathbb{R}_+$. Then, for $\Psi \in L^2(\mathbb{R}^n)^P$, we have*

$$m(D)W_\Psi f = W_\Psi m(D)f = W_{\overline{m}(D)\Psi} f.$$

Here, $m(D)W_\Psi f$ means $m(D)\big((W_\Psi f)(\cdot, a)\big)$, that is, $m(D)$ acts on the function $W_\Psi f(x, a)$ of x for each fixed $a \in \mathbb{R}_+$.

We give an extended version with two multiwavelet functions Ψ_1 and Ψ_2 like in [16, Chapt.10], although we will use the case when $\Psi_1 = \Psi_2$. Let us introduce the condition (A_{Ψ_1,Ψ_2}) for $\Psi_1, \Psi_2 \in L^2(\mathbb{R}^n)^P$ as follows:

Condition 5.3 (A_{Ψ_1,Ψ_2}) *There exists a constant M independent of ξ such that*

$$\int_{\mathbb{R}_+} |\widehat{\Psi_1}(a\xi)^* \widehat{\Psi_2}(a\xi)| \, \frac{da}{a} \le M, \quad a.e. \xi \in \mathbb{R}^n \setminus \{0\}.$$

Denote by $F^* := \overline{F^{\mathrm{T}}}$, the complex conjugate of the transpose of a vector $F \in \mathbb{C}^P$, and by G^*F, the inner product of $F, G \in \mathbb{C}^P$. For $\Psi_1, \Psi_2 \in L^2(\mathbb{R}^n)^P$ satisfying (A_{Ψ_1,Ψ_2}), define

$$C_{\Psi_1,\Psi_2}(\xi) := \int_{\mathbb{R}_+} \widehat{\Psi_1}(a\xi)^* \widehat{\Psi_2}(a\xi) \, \frac{da}{a}. \tag{7}$$

Since $C_{\Psi_1,\Psi_2}(\xi)$ is positively homogeneous of degree zero, $C_{\Psi_1,\Psi_2} \in L^\infty(\mathbb{R}^n)$. We abbreviate the condition $(A_{\Psi,\Psi})$ as (A_Ψ) and also $C_\Psi(\xi) = C_{\Psi,\Psi}(\xi)$ if Ψ satisfies (A_Ψ). Note that if Ψ_1 and Ψ_2 satisfy (A_{Ψ_1}) and (A_{Ψ_2}), then they satisfy (A_{Ψ_1,Ψ_2}). A multiwavelet function $\Psi \in L^2(\mathbb{R}^n)^P$ is called an *analyzing multiwavelet* if Ψ satisfies (A_Ψ) and $C_\Psi(\xi)$ is a nonzero constant independent of ξ, which is called the *admissibility condition*.

The *Riesz transforms*, denoted by $\mathcal{R}_j$, $j = 1, \ldots, n$, are the Fourier multiplier operators $-i D_j/|D|$, that is,

$$\mathcal{R}_j f(x) = \mathcal{F}^{-1}\left(-i\,\frac{\xi_j}{|\xi|}\widehat{f}(\xi)\right).$$

The Riesz transforms $\mathcal{R}_j f$, $j = 1, \ldots, n$ of real-valued function f are also real-valued.

It is easy to see the following Lemma 5.4.

Lemma 5.4 ([5], Lemma 4) *Let a multiwavelet function Ψ satisfies the admissibility condition.*

(i) *Each of the two multiwavelet functions*

$$(\mathcal{R}_1\Psi; \ldots; \mathcal{R}_n\Psi) \quad \textit{and} \quad (\Psi; \mathcal{R}_1\Psi; \ldots; \mathcal{R}_n\Psi)$$

also satisfies the admissibility condition, where we use the conventional notation of a semicolon to represent the termination of each row, namely,

$$(F_1; \ldots; F_m) = (F_1^{\mathrm{T}}, \ldots, F_m^{\mathrm{T}})^{\mathrm{T}}.$$

(ii) *If* $\operatorname{supp}\widehat{\psi^p} \cap \operatorname{supp}\widehat{\psi^p}(-\cdot) = \emptyset$ *for each p, then each of the three multiwavelet functions $\Re\Psi$, $\Im\Psi$, and $(\Re\Psi; \Im\Psi)$ also satisfies the admissibility condition. Here, $\Re$ and $\Im$ denote the real and imaginary parts, respectively.*

We have the following orthogonality relation and inversion formula for the multiwavelet transform.

Theorem 5.5 ([5], Theorem 5)

(i) *If $\Psi_1, \Psi_2 \in L^2(\mathbb{R}^n)^P$ satisfy (A_{Ψ_1,Ψ_2}), then for $f, g \in L^2(\mathbb{R}^n)$, we have*

$$\langle C_{\Psi_1,\Psi_2}(D)f, g\rangle = \int_{\mathbb{R}_+}\left(\int_{\mathbb{R}^n}(W_{\Psi_2}g)(b,a)^*(W_{\Psi_1}f)(b,a)\,db\right)\frac{da}{a^{n+1}}. \tag{8}$$

(ii) *If $\Psi \in L^2(\mathbb{R}^n)^P$ satisfies (A_Ψ), then W_Ψ is a bounded linear operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n \times \mathbb{R}_+, dbda/a^{n+1})^P$.*

(iii) *If $\Psi_1, \Psi_2 \in L^2(\mathbb{R}^n)^P$ satisfy (A_{Ψ_1,Ψ_2}) and C_{Ψ_1,Ψ_2} is a nonzero constant independent of ξ, then any function $f \in L^2(\mathbb{R}^n)$ is reconstructed from its multiwavelet transform by*

$$f = \frac{1}{C_{\Psi_1,\Psi_2}}\int_{\mathbb{R}^n\times\mathbb{R}_+}(\mathcal{T}_b\mathcal{D}_a\Psi_2)^{\mathrm{T}}\,(W_{\Psi_1}f)(b,a)\,\frac{db\,da}{a^{n+1}}. \tag{9}$$

In particular, if Ψ is an analysing multiwavelet, then we have the inversion formula

$$f = \frac{1}{C_\Psi} \int_{\mathbb{R}^n \times \mathbb{R}_+} (\mathcal{T}_b \mathcal{D}_a \Psi)^{\mathrm{T}} \, (W_\Psi f)(b,a) \, \frac{db\, da}{a^{n+1}}. \tag{10}$$

The right-hand side of the above inversion formula

$$W_\Psi^{-1} F := \frac{1}{C_\Psi} \int_{\mathbb{R}^n \times \mathbb{R}_+} (\mathcal{T}_b \mathcal{D}_a \Psi)^{\mathrm{T}} \, F(b,a) \, \frac{db\, da}{a^{n+1}} \in L^2(\mathbb{R}^n) \tag{11}$$

is called the *inverse multiwavelet transform* of $F \in L^2(\mathbb{R}^n \times \mathbb{R}_+, dbda/a^{n+1})^P$. The integrals in (9), (10) and (11) are interpreted in the weak sense as in [16, Corollary 10.3].

For the discrete multiwavelet, for example see [22], the scaling functions are important. In most cases, there are several scaling functions, each corresponds to $(2^n - 1)$ wavelet functions. On the other hand, in [24, §4.3], Mallat considered a scaling function for continuous wavelet transform, which aggregates the part of large a. We can also consider a *multiscaling function* Φ as follows.

Theorem 5.6 ([5], Theorem 7) *Suppose that $\Psi_1, \Psi_2 \in L^2(\mathbb{R}^n)^P$ and $\Phi_1, \Phi_2 \in L^2(\mathbb{R}^n)^Q$. Also suppose that there exists a constant M such that*

$$\int_0^1 |\widehat{\Psi_1}(a\xi)^* \widehat{\Psi_2}(a\xi)| \, \frac{da}{a} + |\widehat{\Phi_1}(\xi)^* \widehat{\Phi_2}(\xi)| \le M \quad a.e.\ in\ \mathbb{R}^n. \tag{12}$$

Set

$$C_{\Psi_1,\Psi_2;\Phi_1,\Phi_2}(\xi) := \int_0^1 \widehat{\Psi_1}(a\xi)^* \widehat{\Psi_2}(a\xi) \, \frac{da}{a} + \widehat{\Phi_1}(\xi)^* \widehat{\Phi_2}(\xi) \in L^\infty(\mathbb{R}^n). \tag{13}$$

Then, we have the following.

(i) *For $f, g \in L^2(\mathbb{R}^n)$ and $a_0 \in \mathbb{R}_+$, we have*

$$\begin{aligned}\langle C_{\Psi_1,\Psi_2;\Phi_1,\Phi_2}(a_0 D) f, g\rangle &= \int_0^{a_0} \left(\int_{\mathbb{R}^n} (W_{\Psi_2} g)(b,a)^* (W_{\Psi_1} f)(b,a) \, db \right) \frac{da}{a^{n+1}} \\ &\quad + \frac{1}{a_0^n} \int_{\mathbb{R}^n} (W_{\Phi_2} g)(b,a_0)^* (W_{\Phi_1} f)(b,a_0) \, db. \end{aligned} \tag{14}$$

(ii) *If* $C_{\Psi_1,\Psi_2;\Phi_1,\Phi_2}$ *is a nonzero constant, then any function* $f \in L^2(\mathbb{R}^n)$ *is reconstructed by*

$$f = \frac{1}{C_{\Psi_1,\Psi_2;\Phi_1,\Phi_2}} \left\{ \int_{\mathbb{R}^n \times (0,a_0)} (\mathcal{T}_b \mathcal{D}_a \Psi_2)^{\mathrm{T}} (W_{\Psi_1} f)(b,a) \frac{db\, da}{a^{n+1}} \right.$$
$$\left. + \frac{1}{a_0^n} \int_{\mathbb{R}^n} (\mathcal{T}_b \mathcal{D}_{a_0} \Phi_2)^{\mathrm{T}} (W_{\Phi_1} f)(b,a_0)\, db \right\}. \qquad (15)$$

Remark 5.7 Let $\Psi_1, \Psi_2 \in L^2(\mathbb{R}^n)^P$ satisfy that C_{Ψ_1,Ψ_2} is a constant. If $\Phi_1, \Phi_2 \in L^2(\mathbb{R}^n)^Q$ satisfies

$$\widehat{\Phi_1}(\xi)^* \widehat{\Phi_2}(\xi) = \int_1^{\infty} \widehat{\Psi_1}(a\xi)^* \widehat{\Psi_2}(a\xi) \frac{da}{a}, \quad \text{a.e. } \xi \in \mathbb{R}^n,$$

then $C_{\Psi_1,\Psi_2;\Phi_1,\Phi_2}(\xi) = C_{\Psi_1,\Psi_2}$ a.e. This enables us to construct Φ's from Ψ's.

6 Image Separation

The simplest strategy to apply the one-dimensional blind source separation method to images is reshaping matrices as vectors using the line by line scan, and applying the one-dimensional blind source separation method. However, this simplest strategy does not use any two-dimensional information of images. Natural images have pixels where there is a sharp contrast in intensity. The set of such pixels is called *edge*. Discontinuities in natural images are generated by edges and most edges are composed of piecewise continuous curves. In other words, edges of natural images are essentially one-dimensional objects. By this reason, the intersection of edges of original images is usually zero-dimensional object, which is mainly composed of intersections of piecewise continuous curves. As is well known, two-dimensional wavelet transform can extract edges and the extracted edges can be represented by wavelet coefficients. Therefore, we work with edges using multiwavelets [1, 13, 22]. For more detail, please refer to [5, §6] and [23, 25, 26].

Let us present a numerical simulation for image separation. In this simulation, we use the annular sector multiwavelets given in [5, §4] for time-scale informations. Note that the time-frequency windows of multiwavelets $\Re\psi_p$ and $\Im\psi_p$ are the same for each p. Density plots of annular sector multiwavelets $\Re\psi_p$ and $\Im\psi_p$ are shown in the right column of Fig. 8.

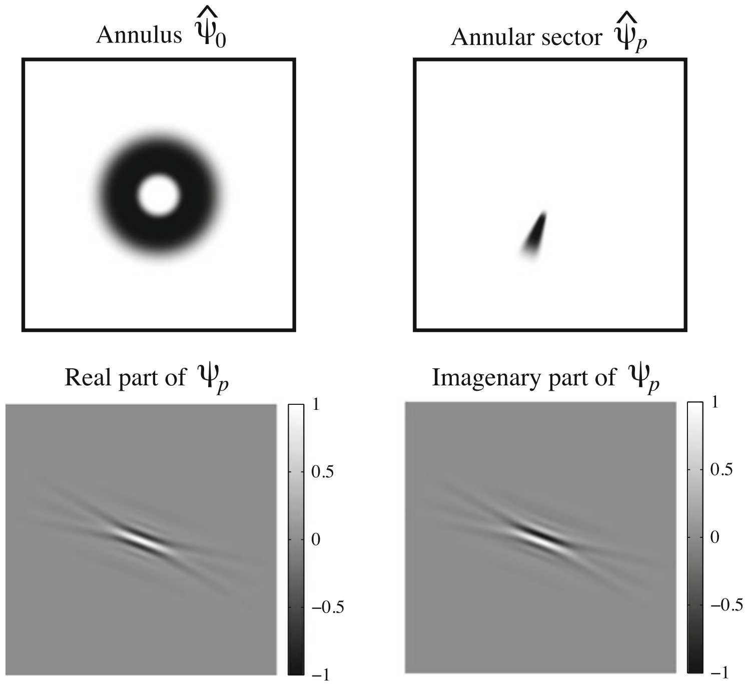

Fig. 8 One of annular sector multiwavelets

The mixing matrix $A \in \mathbb{R}^{3\times 3}$ is a random matrix uniformly distributed in $[0.2, 0.8]$ as follows:

$$A = \begin{pmatrix} 0.4776 & 0.4128 & 0.5193 \\ 0.2977 & 0.7899 & 0.7354 \\ 0.3235 & 0.5626 & 0.5734 \end{pmatrix}.$$

Since the condition number of A is $\mathrm{cond}(A) = 880.6528$, it implies that A is a good matrix for inversion. Three 512×512 standard gray scale images: Mandril, Boat, Building are used as the original source images s_1, s_2, s_3 shown in the first row of Fig. 9. Three mixed images x_1, x_2, x_3 shown in the second row of Fig. 9 are produced by (1) with the above mixing matrix A. By applying the source reduction method proposed in [6, §4], we have our estimated images shown in the third row of Fig. 9. The ordering of estimation images is an unavoidable ambiguity. The correct ordering should be $(s_1, s_2, s_3) \longleftrightarrow (\sigma_2, \sigma_1, \sigma_3)$. Our source reduction method sometimes needs to apply the black & white conversion if necessary by human decision as in the case the estimated boat image in Fig. 9. This black & white conversion is also an unavoidable ambiguity.

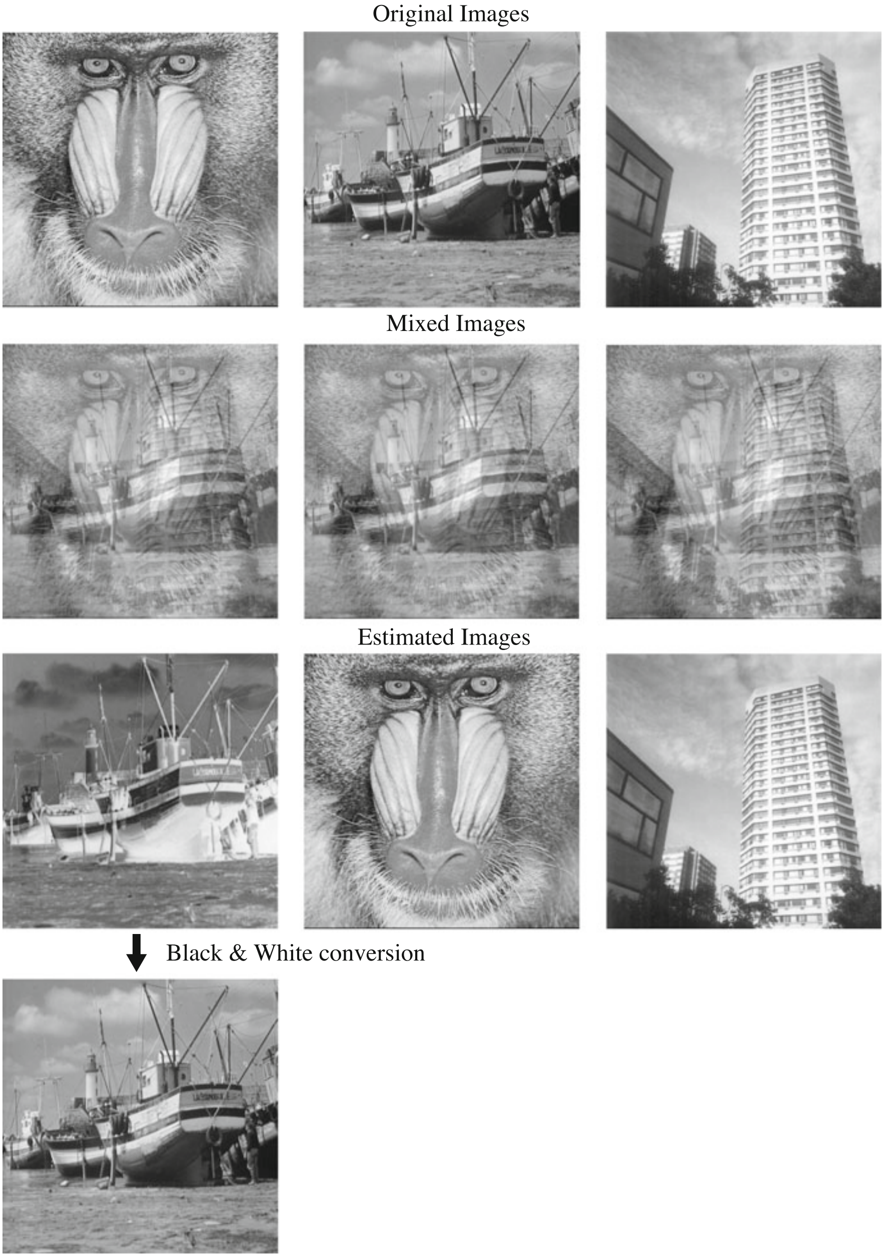

Fig. 9 Image separation

Table 1 Various errors of the estimation

Original	Estimated	Max error [%]	ℓ^1-error [%]	ℓ^2-error [%]	SNR [dB]
s_1	σ_2	0.43688	0.35445	0.32128	49.8624
s_2	σ_1	0.24901	0.18251	0.17515	55.1318
s_3	σ_3	0.53606	0.4435	0.42354	47.4621

The performance of our estimation σ_1, σ_2, σ_3 are very accurate as shown in Table 1.

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