Mazzola's Escher Theorem

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Abstract In this note we give a full proof of Mazzola's Escher Theorem (Mazzola, J Math Music, 3(1):31–58, 2009, [4]). This theorem is needed for the development of the theory that Mazzola seeks to realize, and it helps us to understand better the concept of hypergesture as used in his work (Mazzola, J Math Music 3(1):31–58, 2009, [4], Mazzola, Musical performance-A comprehensive approach: theory, analytical tools, and case studies, 2011, [5], Mazzola and Andreata, J. Math. Music, 1(1):23–4, 2007, [6], Mazzola et al., Musical creativity-strategies and tools in composition and improvisation, 2011, [7]). A *gesture* is a morphism from a digraph into a topological space, and is one of the fundamental blocks in the Mathematical Theory of Performance. A *hypergesture* is a gesture built upon another gesture, describing, in a way, the variation of the latter. The non-trivial fact that the variation of the former gesture, as described by the latter, is given by the *same* hypergesture is essentially the content of the Escher Theorem.

1 Basic Concepts

We review the graph and category theory necessary for fixing notation and deliver the concepts of gesture and hypergesture. The reader already familiar with those, may skip the following paragraphs and proceed directly to Sect. 2.

Definition 1 We consider a **digraph *D*** as an ordered pair (V_D, A_D), where V_D is a set of vertices and A_D a set of arrows, disjoint from V_D, together with an *incidence function* ψ_D that associates with each arrow of D an ordered pair of vertices (not necessarily distinct) of D. This is $\psi_D : A_D \longrightarrow V_D \times V_{D.}$ Generally speaking, if $\psi_D(a) = (u, v)$, we will call u the *tail* of a and v the *head* of a.

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Definition 2 Let D and G be digraphs. A **morphism of digraphs** $f : D \longrightarrow G$ is a pair (ϕ, θ) of functions $\phi : A_D \longrightarrow A_G$ and $\theta : V_D \longrightarrow V_G$, making the following diagram commute:

$$\begin{array}{ccc} A_D & \xrightarrow{\phi} & A_G \\ \psi_D \downarrow & \circlearrowleft & \downarrow \psi_G \\ V_D^2 & \xrightarrow{\theta^2} & V_G^2 \end{array}$$

where $V_G^2 = V_G \times V_G$ and $\theta^2 := (\theta, \theta) : V_G^2 \longrightarrow V_G^2$.

The category $\mathbf{D}$ of digraphs has as objects the collection $Obj(\mathbf{D})$ of digraphs, and for each pair of digraphs, Γ and Δ, the set $\Gamma @_{\mathbf{D}} \Delta = \mathbf{D}(\Gamma, \Delta)$ of morphism of digraphs as arrows [1, 3].

The composition of morphisms of digraphs $f = (u, v) \in \Gamma @_{\mathbf{D}} \Delta$, $g = (w, z) \in \Delta @_{\mathbf{D}} K$ with Γ, Δ and K digraphs, and where each of the morphisms $u : A_\Gamma \longrightarrow A_{\Delta,}$ $v : V_\Gamma \longrightarrow V_\Delta$, $w : A_\Delta \longrightarrow A_K$ and $z : V_\Delta \longrightarrow V_K$ makes sense, is given by pasting commutative squares. Namely

$$\begin{array}{ccccc} A_\Gamma & \xrightarrow{u} & A_\Delta & \xrightarrow{w} & A_K \\ {\scriptstyle\Gamma}\downarrow & & \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle K} \\ V_\Gamma^2 & \xrightarrow[v^2]{} & V_\Delta^2 & \xrightarrow[z^2]{} & V_K^2 \end{array}$$

that is, $g \circ f = (w \circ u, z \circ v) \in \Gamma @_{\mathbf{D}} K$.

Now consider the set

$$A_{\overrightarrow{X}} = I @_{\mathbf{Top}} X := \{c : I \longrightarrow X | c \text{ is a continuous curve}\}$$

with $X \in \mathbf{Top}$ (the category of topological spaces and continuous functions) and I a fixed closed interval in $\mathbb{R}$ with its canonical orientation [9]. Thus we define $\overrightarrow{X}$ such that $A_{\overrightarrow{X}}$ is the set of its arrows and $V_{\overrightarrow{X}} = X$ that of its vertices. It is clear that $\overrightarrow{X}$ is a digraph.

The digraph $\overrightarrow{X}$ is a very special one, since it is defined *inside* the arbitrary topological space X, and with the concepts above at hand, we may consider the subcollection (of the category $\mathbf{D}$) of *spatial digraphs*, $\mathbf{SD}$, as follows:

1. $Obj(\mathbf{SD}) = \{\overrightarrow{X} : A_{\overrightarrow{X}} \longrightarrow V_{\overrightarrow{X}}^2 | X \in \mathbf{Top}$, $\overrightarrow{X}$ the incidence function, where $A_{\overrightarrow{X}}$ are the arrows of a digraph $\overrightarrow{X}$ and $V_{\overrightarrow{X}} = X$ its vertices$\}$.
2. $\mathbf{SD}(\overrightarrow{X}, \overrightarrow{Y}) = \overrightarrow{X} @_{\mathbf{SD}} \overrightarrow{Y} = \{\overrightarrow{f} : \overrightarrow{X} \longrightarrow \overrightarrow{Y} | \overrightarrow{f}$ is a digraph morphism induced canonically by a continuous function $f : X \longrightarrow Y$ in $\mathbf{Top}\}$.

The aforementioned collection of objects is evidently contained in $Obj(\mathbf{D})$ and in the same way, the collection of arrows for every pair $\overrightarrow{X}$ and $\overrightarrow{Y}$ of spatial digraphs is evidently contained in $\overrightarrow{X}\,@_{\mathbf{D}}\overrightarrow{Y}$.

The fact that **SD** is actually a subcategory of **D** is nothing but a straightforward argument, and is left to the reader [8].

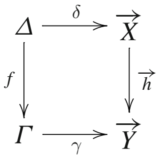
2 The Category of Gestures

This section aims at defining the category of gestures.

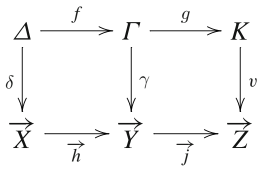
Definition 3 Let $\Gamma \in Obj(\mathbf{D})$ and $\overrightarrow{X} \in Obj(\mathbf{SD})$ be given objects. A ***Γ*− *gesture*** in a topological space X is a morphism $g : \Gamma \longrightarrow \overrightarrow{X}$ between digraphs.

In this case Γ will be called the **skeleton of the gesture,** meanwhile the topological space X will be called the **gesture space**, and the curve defined into X given by g will be called the **body** of the gesture.

Definition 4 Consider $\delta : \Delta \longrightarrow \overrightarrow{X}$ and $\gamma : \Gamma \longrightarrow \overrightarrow{Y}$ two gestures, a **gesture morphism** $\widetilde{f} : \delta \longrightarrow \gamma$ consists of a pair of morphisms $\widetilde{f} := (f, \overrightarrow{h})$, where $f : \Delta \longrightarrow \Gamma$ is a digraph morphism, such that there is a digraph morphism $\overrightarrow{h} : \overrightarrow{X} \longrightarrow \overrightarrow{Y}$, not necessarily continuous, making the following diagram commute:



In particular, note that for *gestures* $\delta : \Delta \longrightarrow \overrightarrow{X}$, $\gamma : \Gamma \longrightarrow \overrightarrow{Y}$, and $\kappa : K \longrightarrow \overrightarrow{Z}$, and the *morphisms* of gestures $\widetilde{f} : \delta \longrightarrow \gamma$ and $\widetilde{g} : \gamma \longrightarrow \kappa$, such that $\widetilde{f} = (f, \overrightarrow{h})$ and $\widetilde{g} = (g, \overrightarrow{j})$ with $f : \Delta \longrightarrow \Gamma$, $\overrightarrow{h} : \overrightarrow{X} \longrightarrow \overrightarrow{Y}$, $g : \Gamma \longrightarrow K$ and $\overrightarrow{j} : \overrightarrow{Y} \longrightarrow \overrightarrow{Z}$, the following diagram commutes



that is: $\widetilde{g} \circ \widetilde{f} = (g \circ f, \overrightarrow{j} \circ \overrightarrow{h})$.

If we now consider the collections given by:

1. $Obj(\mathbf{G}) := \{\delta : \Delta \longrightarrow \overrightarrow{X} | \Delta \in Obj(\mathbf{D}),\ \overrightarrow{X} \in Obj(\mathbf{SD})$ and δ a morphism$\}$.
2. $\mathbf{G}(\delta, \gamma) = \delta @_{\mathbf{G}} \gamma := \{\widetilde{f} : \delta \longrightarrow \gamma | \widetilde{f} = (f, \overrightarrow{h})$ are gesture morphisms with $\gamma \circ f = \overrightarrow{h} \circ \delta\}$ (for every pair of gestures δ and γ in $Obj(\mathbf{G})$),

subject to the composition of gestures morphisms with $\widetilde{f} \in \delta @_{\mathbf{G}} \gamma$, $\widetilde{g} \in \gamma @_{\mathbf{G}} \upsilon$, for all δ, γ and υ gestures, as we just mentioned above, then it is clear that we obtain a category $\mathbf{G}$, the category of gestures.

Now if we consider certain gestures as *points* in a space, it is possible to study gestures inside a *gesture space*, which will be called **hypergestures**.

To define them, we need first to know how to make the set of gestures $\Delta @_{\mathbf{D}} \overrightarrow{X}$ into a topological space. This we will show below.

3 Hypergestures with an Approach to Escher's Theorem

First consider the very particular case $\Delta := \uparrow$, that is, a digraph with a single arrow. It is well known how to get a topological space $\uparrow @_{\mathbf{D}} \overrightarrow{X} \cong I @_{\mathbf{Top}} X$ by using the compact-open topology. This, along with the following proposition, is the basis for all that follows.

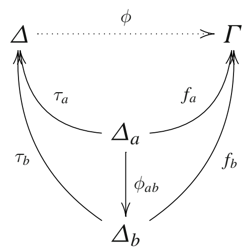
Proposition 1 *Let Δ be a digraph, then it is the direct limit of a direct system.*

Proof Let $\langle A_\Delta, = \rangle$ be a preordered set. We can give the direct system $\{(\Delta_a)_{a \in A_\Delta}, (\varphi_{ab})_{a=b}\}$ where $\Delta_a := \uparrow_a \longrightarrow (t(a), h(a))$ for every $a \in A_\Delta$ and $(\varphi_{ab} : \Delta_a \longrightarrow \Delta_b)_{a=b}$ is a family of isomorphisms of digraphs such that:

$$\varphi_{ab} = (\overline{\varphi}_{ab}, Id),$$

where $\overline{\varphi}_{ab} : \uparrow_a \longrightarrow \uparrow_b$.

Now suppose there is a digraph Γ and a corresponding family of morphisms in $\mathbf{D}$ $(f_\alpha : \Delta_\alpha \longrightarrow \Gamma)_{\alpha \in A_\Delta}$ making the following diagram a commutative one:



Consider $\phi = f$ such that $f|_{\Delta_a} = f_a$ for each $a \in \Lambda$. Then

$$(\phi \circ \tau_a)(\Delta_a) = \phi(\tau_a(\Delta_a)) = \phi(\Delta_a) = f_a(\Delta_a).$$

Therefore $\Delta \cong \varinjlim_{a \in A_\Delta} \Delta_\alpha$. ♦

Now that we can regard a digraph as a direct limit, the following results from category theory are at hand. For the proofs, the interested reader may consult [2].

Proposition 2 *If $\{M_i, \psi_{ji}\}$ is an inverse system of digraphs, then there is an isomorphism*

$$\omega : \mathbf{D}(\Delta, \varprojlim M_i) \longrightarrow \varprojlim \mathbf{D}(\Delta, M_i)$$

for every digraph Δ. i.e., $\Delta @ \varprojlim M_i \cong \mathbf{D}(\Delta, \varprojlim M_i) \cong \varprojlim \mathbf{D}(\Delta, M_i) \cong \varprojlim (\Delta @ M_i)$.

Proposition 3 *If $\{M_i, \psi_{ij}\}$ is a direct system of digraphs, then there is an isomorphism*

$$\theta : \mathbf{D}(\varinjlim M_i, \overrightarrow{X}) \longrightarrow \varprojlim \mathbf{D}(M_i, \overrightarrow{X})$$

for every digraph $\overrightarrow{X}$. i.e., $(\varinjlim M_i) @ X \cong \mathbf{D}(\varinjlim M_i, X) \cong \varprojlim \mathbf{D}(M_i, X) \cong \varprojlim (M_i @ X)$.

Proposition 4 *Let Δ, Γ be given digraphs, and $\{(\uparrow_i)_{i \in A_\Gamma} (\psi_{ij})_{i \preceq j}\}$ and $\{(\uparrow_c)_{c \in A_\Delta}, (\psi_{cd})_{c \preceq d}\}$ direct systems of digraphs, then there is an isomorphism*

$$\eta : \varprojlim_{b \in A_\Delta} (\varprojlim_{a \in A_\Gamma} (\uparrow_b @ (\uparrow_a @ X))) \rightarrow \varprojlim_{a \in A_\Gamma} (\varprojlim_{b \in A_\Delta} (\uparrow_b @ (\uparrow_a @ X)))$$

Proposition 5 *If Γ, Δ are digraphs and X is a topological space, then there is a canonical homeomorphism*

$$\Gamma \overrightarrow{@} \Delta \overrightarrow{@} X \cong \Delta \overrightarrow{@} \Gamma \overrightarrow{@} X$$

This last proposition is nothing but a weaker version of Escher Theorem as the reader will find out in Sect. 4 below.

4 Topological Categories and Mazzola's Escher Theorem

The last ingredient needed for the formulation of Mazzola's Escher Theorem is that of a topological category.

Definition 5 Let K be a category endowed with the property that its set of maps is a topological space, and in which both functions, domain and codomain, and the composition of morphisms as well are continuous.

In this case K is called a **topological category.**

Example 1 The *simplex categoy* ∇ associated with the unit interval I.

In this case the set of maps is $\nabla = \{(x, y)|x, y \in I \text{ and } x \leq y\}$ and the functions domain and codomain are given by $d(x, y) := (x, x)$, $c(x, y) := (y, y)$ respectively. The composition of morphisms is $(x, y) \circ (y, z) = (x, z)$, and the topology on ∇ is the relative topology of the product inherited on $I \times I \subset \mathbb{R} \times \mathbb{R}$.

Definition 6 Let K, L be two topological categories. A **topological functor** $F : K \longrightarrow L$ is a functor which in turn is a continuous function between sets of morphisms.

The definitions above give rise to the category **TopCat** of topological categories, whose objects are topological categories and has as morphisms the topological functors between topological categories. We denote this collection of morphisms by K $©L := \mathbf{TopCat}(K, L)$.

Mimicking the construction of a spatial digraph, we may consider two continuous functors, tail and head, respectively by $t, h : \nabla ©K \longrightarrow K$.

Now if $\nu : f \longrightarrow g$ is a natural transformation between morphisms (or, which is the same, topological functors) $f, g : \nabla \longrightarrow K$, then $t(\nu) = \nu(0) : f(0) \longrightarrow g(0)$ and $h(\nu) = \nu(1) : f(1) \longrightarrow g(1)$.

The resulting topological diagram of categories and continuous functors is called a **categorical digraph** $\overrightarrow{K}$ of K.

Thus if Γ is a digraph, then the set of morphism $\Gamma @_{\mathbf{D}} \overrightarrow{K}$ is the set of digraph morphism in the underlying spatial digraph K. In other words, each morphism assigns an object of K for every vertex in Γ, and a continuous curve (a topological functor) $\nabla \longrightarrow K$ for every arrow of Γ. Then the digraph morphism $g : \Gamma \longrightarrow \overrightarrow{K}$ will be called **a gesture with skeleton in and body in K**.

Proposition 6 *Let $\Gamma @ \overrightarrow{K}$ be the set of gestures with skeleton in Γ and body K, with K a topological category. Then $\Gamma @ \overrightarrow{K}$ is a topological category.*

Proof Recalling that $\Gamma \cong \varinjlim_{a \in A_\Gamma} \Gamma_\alpha$, in particular we have $(\Gamma_a)_{a \in A_\Gamma} \cong (\uparrow_a)_{a \in A_\Gamma}$.

On the other hand, we know that $\uparrow @ \overrightarrow{K} \cong \nabla ©K \in \mathbf{TopCat}$.

Thus

$$\Gamma @ \overrightarrow{K} \cong (\varinjlim_{a \in A_\Gamma} \Gamma_\alpha) @ \overrightarrow{K} \cong (\varinjlim_{a \in A_\Gamma} \uparrow_a) @ \overrightarrow{K} \cong \varprojlim_{a \in A_\Delta} (\uparrow_a @ \overrightarrow{K}),$$

since each $\uparrow_a @ \overrightarrow{K} \cong \nabla ©K$ is a topological category, then $\varprojlim_{a \in A_\Delta} (\uparrow_a @ \overrightarrow{K}) \cong \Gamma @ \overrightarrow{K}$ is also a topological category, because of the properties of inverse limits. ♦

Theorem (**Mazzola's Escher theorem** [4]) *Let Γ, Δ be digraphs and K a topological category, then we have a canonical isomorphism of topological categories.*

$$\Gamma \overrightarrow{@} \Delta \overrightarrow{@} K \cong \Delta \overrightarrow{@} \Gamma \overrightarrow{@} K.$$

Proof On one hand, this implies that the space of hypergesture $\Gamma \overrightarrow{@} \Delta \overrightarrow{@} K$ is the limit $(\varinjlim_{a \in A_\Gamma} \Gamma_\alpha)@(\Delta \overrightarrow{@} K)$, but in particular we can say that $(\Gamma_a)_{a \in A_\Gamma} \cong (\uparrow_a)_{a \in A_\Gamma}$.

Then

$$\Gamma \overrightarrow{@} \Delta \overrightarrow{@} K \cong (\varinjlim_{a \in A_\Gamma} \uparrow_a)@(\Delta \overrightarrow{@} K).$$

Even more, being $(\_@(\Delta \overrightarrow{@} K))$ a contravariant functor which converts direct limits on inverses limits, this must be isomorphic to:

$$(\varinjlim_{a \in A_\Gamma} \uparrow_a)@(\Delta \overrightarrow{@} K) \cong \varprojlim_{a \in A_\Gamma}(\uparrow_a @(\Delta \overrightarrow{@} K)).$$

Proceeding similarly we get:

$$\varprojlim_{a \in A_\Gamma}(\uparrow_a @(\Delta \overrightarrow{@} K)) \cong \varprojlim_{a \in A_\Gamma}(\uparrow_a @(\varinjlim_{b \in A_\Delta} \Delta_b @K)) \cong \varprojlim_{a \in A_\Gamma}(\uparrow_a @(\varinjlim_{b \in A_\Delta} \uparrow_b @K)).$$

Because $(\_@K)$ is a contravariant functor and converts direct limits in inverse limits

$$\varprojlim_{a \in A_\Gamma}(\uparrow_a @(\varinjlim_{b \in A_\Delta} \uparrow_b @K)) \cong \varprojlim_{a \in A_\Gamma}(\uparrow_a @(\varprojlim_{b \in A_\Delta}(\uparrow_b @K))$$

Thus, since $(\uparrow_a @\_)$ is a covariant functor preserving inverse limits

$$\varprojlim_{a \in A_\Gamma}(\uparrow_a @(\varprojlim_{b \in A_\Delta}(\uparrow_b @K)) \cong \varprojlim_{a \in A_\Gamma}(\varprojlim_{b \in A_\Delta}(\uparrow_a @(\uparrow_b @K))) \cong \varprojlim_{a \in A_\Gamma}(\varprojlim_{b \in A_\Delta}(\uparrow_a @ \uparrow_b @K)).$$

Then,

$$\varprojlim_{a \in A_\Gamma}(\varprojlim_{b \in A_\Delta}(\uparrow_a @(\uparrow_b @K))) \cong \varprojlim_{a \in A_\Gamma}(\varprojlim_{b \in A_\Delta}(\uparrow_a @ \uparrow_b @K)).$$

By proposition 4

$$\varprojlim_{a \in A_\Gamma}(\varprojlim_{b \in A_\Delta}(\uparrow_a @ \uparrow_b @K)) \cong \varprojlim_{b \in A_\Delta}(\varprojlim_{a \in A_\Gamma}(\uparrow_a @ \uparrow_b @K)),$$

and considering that $(\uparrow_a @ \uparrow_b) \cong (\uparrow_b @ \uparrow_a) \cong I^2$, then

$$\varprojlim_{a \in A_\Gamma}(\varprojlim_{b \in A_\Delta}(\uparrow_a @ \uparrow_b @K)) \cong \varprojlim_{b \in A_\Delta}(\varprojlim_{a \in A_\Gamma}(\uparrow_b @ \uparrow_a @K)).$$

So

$$\varprojlim_{b \in A_\Delta}(\varprojlim_{a \in A_\Gamma}(\uparrow_b @ \uparrow_a @K)) \cong \varprojlim_{b \in A_\Delta}(\varprojlim_{a \in A_\Gamma}(\uparrow_b @(\uparrow_a @K)).$$

Now, since $(\uparrow_b @\_)$ is a covariant functor preserving inverse limits,

$$\varprojlim_{b\in A_\Delta}(\varprojlim_{a\in A_\Gamma}(\uparrow_b @(\uparrow_a @K)) \cong \varprojlim_{b\in A_\Delta}(\uparrow_b @\varprojlim_{a\in A_\Gamma}(\uparrow_a @K)),$$

and $(\_@K)$ is a contravariant functor which turns direct limits into inverse limits

$$\varprojlim_{b\in A_\Delta}(\uparrow_b @\varprojlim_{a\in A_\Gamma}(\uparrow_a @K)) \cong \varprojlim_{b\in A_\Delta}(\uparrow_b @(\varinjlim_{a\in A_\Gamma}\uparrow_a @K)).$$

Finally, $(\_@(\Gamma\overrightarrow{@}K))$ being a contravariant functor converting direct limits on inverse limits, we have

$$\varprojlim_{b\in A_\Delta}(\uparrow_b @(\varinjlim_{a\in A_\Gamma}\uparrow_a @K)) \cong \varprojlim_{b\in A_\Delta}(\uparrow_b @(\varinjlim_{a\in A_\Gamma}\Gamma_a @K)) \cong \varprojlim_{b\in A_\Delta}(\uparrow_b @(\Gamma\overrightarrow{@}K))$$

$$\cong (\varinjlim_{b\in A_\Delta}\Delta_b @(\Gamma\overrightarrow{@}K)) \cong (\Delta\overrightarrow{@}(\Gamma\overrightarrow{@}K)) \cong \Delta\overrightarrow{@}\Gamma\overrightarrow{@}K.$$

Therefore $\Gamma\overrightarrow{@}\Delta\overrightarrow{@}K \cong \Delta\overrightarrow{@}\Gamma\overrightarrow{@}K$. ♦

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