Group Theory for Pitch Sequence Representation: From the Obvious to the Emergent Complexity

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Abstract In the first two sections of this contribution we construct the groups $(S_n, +)$ and $(L(S_n), \circ)$ in order to have an intuitive way to represent musical phrases by their melodic contour. Later we derive an algorithm for composing music using a given number and the group $(L(S_n), \circ)$. Finally we offer a variation of the same algorithm to be able to translate a piece of music in a finite digit number, with analytic and deconstructive aims.

1 Introduction

Fix $n \in \mathbb{N}$. Let S_n be the set whose elements $s_j,\ j \in \{0, \ldots, n-1\}$, are sets of intervals of $\frac{12}{n} j$ semitones, including its octaves; in other words,

$$s_j = \left\{ \frac{12}{n} j + 12m \text{ semitones} \mid m \in \mathbb{Z} \right\}. \tag{1}$$

Fixing n as a divisor of 12 we have the sets S_1, S_2, S_3, S_4, S_6 and S_{12} whose elements are equivalence classes. We shall name elements in S_n using letters in ascending order starting from the letter a.

- $S_1 = \{a = [0]\}$
- $S_2 = \left\{a = [0], b = \left[\frac{12}{2}\right]\right\}$
- …
- $S_6 = \left\{a = [0], b = \left[\frac{12}{6}\right], c = \left[\frac{12}{6}2\right], d = \left[\frac{12}{6}3\right], e = \left[\frac{12}{6}4\right],\ \ f = \left[\frac{12}{6}5\right]\right\}$

Now we define the operation $+$ as the usual modular arithmetic, that is $[x] + [y] = [x + y]$. E.g. for $a, b, f \in S_6$:

$$b + f = [2 \text{ semitones} + 10 \text{ semitones}] = [12 \text{ semitones}] = a.$$

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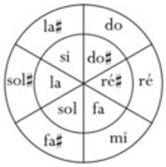
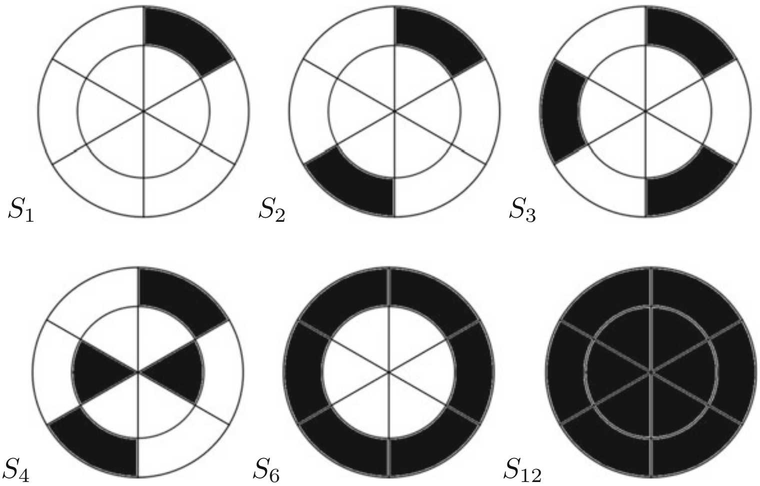
We see that $(S_n, +)$ is a group with a being the identity element. Now lets define $g : S_n \to \mathbb{Z}_n$, $g(s_j) = [j]$, $j \in \{0, \dots, n-1\}$, it is clear that g is an isomorphism from S_n to $\mathbb{Z}_n$.

As an example we show the elements in S_4 using middle *do* (i.e. do_4),[1] as reference for counting intervals:

, where each pitch corresponds to each element in S_4. Since elements in S_n are equivalence classes we have 6 partitioned sets that can be visually represented as subsets of S_{12}, being S_{12} the set of all pitches in the chromatic scale:





2 $(L(S_n), \circ)$

Let $L(S_n)$ be an infinite set of infinite strings with elements in S_n concatenated in every possible order; also, each string has an infinite string of only a to the right. That is, for S_2, $a\overline{a}$,[2] $b\overline{a}$, $ab\overline{a}$, $babbabab\overline{a}$ are in $L(S_2)$. For convenience we won't write the infinite string of a that goes with every element in $L(S_n)$, this way $babbabab\overline{a}$ will be just $babbabab$, also $a\overline{a}$ will be just a. This way we can represent pitch sequences as elements of $L(S_n)$, that is representing the movement of the melody by sequences

[1] We use the do–si pitch nomenclature in order to avoid confusion between letters here used.

[2] We use over line notation to indicate repeating and never ending a.

of musical intervals. For instance, the sequence  is seen as the element *abab* in $L\left(S_2\right)$. We note that having $\overline{a}$ to the right adds nothing to music since it is a, identity element in S_n, concatenated infinitely times and it adds no intervals.

This is how $abbbbb \in L\left(S_6\right)$ is seen in a staff:

This is the *whole-tone scale* starting at do_4, the other whole-tone scale can be generated in reference to $do\sharp_4$. Every possible sequence of sounds produced by the use of this scale can be seen as an element of $L\left(S_6\right)$. It is trivial to note that every sequence of sounds, as long as it uses some or all of the 12 pitches (disregarding enharmonics) in Western music can be seen in $L\left(S_{12}\right)$ since the latest set includes all possible sequence of intervals. Also, by fixing any $n \in \mathbb{N}$ and not just divisors of 12 we can extend S_n and later $L(S_n)$ to microtonality. Obviously we may encompass whole-tone scales using the same concepts.

Let's start with a whole-tone scale example. We take a look at Debussy's first two bars of *Prelude No.* 2, *Voiles*, from his First book of *Preludes* for piano [3]:

We can represent the upper melody in reference to do_4 as $effffaf \in L\left(S_6\right)$ and the lower melody as $cffffff \in L\left(S_6\right)$. Whole-tone elements are present in much of Debussy's repertoire. Just to mention few examples: everything from *Voiles* except 6 bars; the solo between the English horn and the cello at the end of the first movement in *La Mer*, and a number of passages in *Les Images, livre I* for piano solo.

Let $s, \grave{s} \in L\left(S_n\right)$, $s = [s_1][s_2]\ldots[s_n]\ldots$, $\grave{s} = [\grave{s}_1][\grave{s}_2]\ldots[\grave{s}_n]\ldots$ Now we define the $\circ$ operation as a coordinate-wise addition in the sense of $s \circ \grave{s} = [s_1 + \grave{s}_1][s_2 + \grave{s}_2]\ldots[s_n + \grave{s}_n]\ldots$ We note that the length of s and $\grave{s}$ does not matter since every element in $L\left(S_n\right)$ has $\overline{a}$ to the right; this means there will always be an a to operate. $(L(S_n), \circ)$ is a **group** with $\overline{a}$ being the identity element.[3] In the following example we look at the first beat, bar no. 31 of *Jeux d'eau* for solo piano from Ravel [2]  . Using $\left(L\left(S_6\right), \circ\right)$ in reference to do_4 the upper melody performed

[3]It is important to distinguish between $(L(S_n), \circ)$ and word algebra. We are using concatenated elements in S_n with a coordinate-wise addition which is a fundamentally different operation to the one used in word algebra.

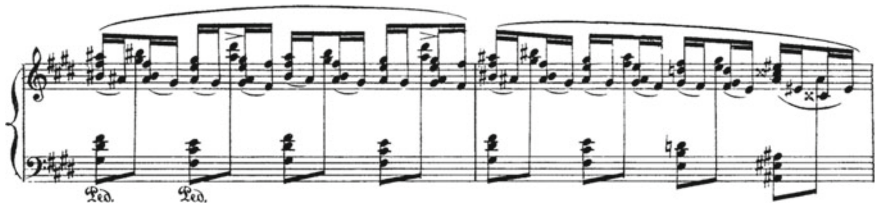


Fig. 1 *Jeux d'eau*, Ravel, bars 31 and 32

Fig. 2 Carrillo's example 25

Fig. 3 Major scale, $accbcccb \in L(S_{12})$

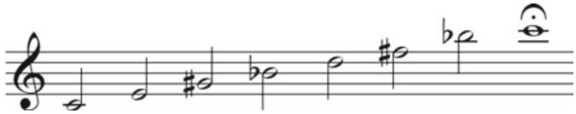
with the right hand, *la*$\sharp$ *la*$\sharp$ fa$\sharp$ *sol*$\sharp$ can be represented as the $faeb$ element in $L\ (S_6)$, now we arbitrarily select $face$ and operate $faeb \circ face$ and we obtain $eaaf$ which is the second beat: . Now we represent the upper melody in right hand from bars 31 and 32 of *Jeux d'eau* (Fig. 1) as follows: bar 31, beat 1: $faeb$ in reference to do_4; bar 1, beat 2: $faeb \circ face = eaaf$; bar 31, beat 3: $eaaf \circ baec = faeb$; bar 31, beat 4: $faeb \circ face = eaaf$; bar 32, beat 1: $eaaf \circ baac = faef$; bar 32, beat 2: $faef \circ faac = eaeb$; bar 32, beat 3: $eaeb \circ faaa = daeb$; bar 32, beat 4: $caec$ in reference to $do\sharp_4$.

Next we explain the example 25 from Julián Carrillo's treatise *Leyes de metamorfósis musicales* [Music's Metamorphosis Laws][1] using $(L(S_n), \circ)$. Here Carrillo shows a "Major scale metamorphosed to its duple" (Fig. 2).

This is the result of doubling every interval in a major scale: where there was 1 semitone now there is 2 semitones and so on. Using $(L(S_n), \circ)$ we represent every pitch sequence as a sequence of musical intervals. For a Major scale (Fig. 3) that is the element $accbcccb$ in $L(S_{12})$:

Now we do $accbcccb \circ accbcccb = aeeceeec$. Since $\circ$ operation is a coordinate-wise addition, the result of operating $accbcccb$ to itself is adding every interval in itself (see: Fig. 4).

Fig. 4 $aeeceeec \in L(S_{12})$



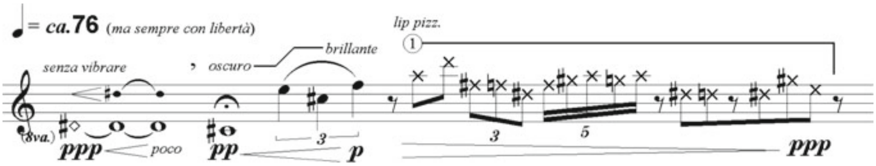
This is the ascending part of the *Major scale metamorphosed to its duple* shown above. Following this process we obtain $aeeceeeclkkkekk$ which is the whole example 25. We conclude that a *Metamorphosis to its duple* (according to Carrillo's laws of Metamorphosis) can be seen as an element in $L(S_{12})$ operated to itself.

At the beginning of this exposition we defined n as a divisor of 12 which leaded to 6 different sets, but, as mentioned before, we can extend S_n to microtonality if we choose a different $n \in \mathbb{N}$ to produce an S_n whose elements are additions of any arbitrary division of the octave. Then we use the obtained S_n and expand it to $L(S_n)$ and $(L(S_n), \circ)$. An example is given with fixed $n = 13$:

- $S_{13} = \left\{a = [0], b = \left[\frac{12}{13}\right], c = \left[\frac{12}{13}2\right], d = \left[\frac{12}{13}3\right], \quad \ldots, \quad l = \left[\frac{12}{13}11\right], m = \left[\frac{12}{13}12\right]\right\}$

3 *Piph Music* for Algorithmic Composition

For a first example on algorithmic composition using number representation, it is convenient to quote one of the first compositions systematically using irrational numbers: π *(A game within the Circle's Constant)*[4] is an awarded composition by Gabriel Pareyon, that uses the first 1000 digits of π in order to produce a solo for the bass flute. This composition associates every chromatic pitch to each digit starting by 0 as *do*, 1 as *do*♯ and so on.



In the leftmost part of this example we see the first sound: *re*♯ corresponding to 3, then *do*♯ corresponding to 1. Afterwards we find the succession 4, 1, 5, 9, 2, 6, 5, 3, 5 where each digit has its defined pitch. We see that for any 1 we will always find a *do*♯ while a 9 will always be *la*.

As a creative possibility of $(L(S_n), \circ)$ we present a different algorithm (Fig. 5) capable of reading any given finite number and returning the sequence of pitches (as equivalence classes) in order to compose music:

1. Read first digit $d \neq 0$ and define d instruments.
2. Start with the first instrument, i.e. instrument counter equals 1.
3. Next digit n defines the number of n pitches for the current bar.
4. For each of the next n digits apply[4] $g^{-1} : \mathbb{Z}_n \to S_n$ and consider the corresponding element in $L(S_n)$ for the current bar, e.g. 021 will be $acb \in L(S_n)$.
5. Check if the instrument counter is bigger than the first digit d.

5.1. If not, increase instrument counter and repeat step 3.
5.2. If yes, is this the end of the given number?
5.2.1. If not, repeat step 2.
5.2.2. If yes, end.

The use of digits in this algorithm limits the number of instruments in the score and the number of pitches to a maximum of 9. Also, due to the decimal system there is not much $(L(S_n), \circ)$ interesting options, but this "lack"can be solved using two digits instead of one for each process. Later we will see a different algorithm capable to obtain a finite number from a score. Since by now we do not consider any rhythmic, nor dynamical values, this leads, if waned, to different musical values arising from the same finite number and vice versa.

What results from using the algorithm proposed by Pareyon is different to what results using the $(L(S_n), \circ)$ algorithm. Since we understand every element in $(L(S_n), \circ)$ as a melody that results in adding intervals, it is not obvious to find a pitch with its corresponding digit, but will be easy to understand a whole melody as a sequence of digits.

As a consequent exercise we prepared a music score[5] for two treble and one bass clefs from the number π up to the digit 190 using $(L(S_6), \circ)$ and starting in *do*. Metre was assigned in equal durations $\left(\frac{1}{1}\right)$.

The first digit in π is 3, meaning 3 instruments. Next we find 14, this means 1 pitch, element 4 in $(L(S_6), \circ)$ corresponding to 8 semitones; since we start in *do* the pitch must be *sol*♯. Next there is 15, meaning 1 pitch, element 5 in $(L(S_6), \circ)$, that is *la*♯. Next 10 digits are 9265358979, meaning 9 pitches, element 265358979 in $(L(S_6), \circ)$. Below are the first four bars with a space between bars where every bold digit, the start of a new instrument, assigns how many pitches correspond for current bar: 3. **1**4**1**5**9**265358979 **3**238**4**6264**3**383 **2**79**5**02884**1**9 **7**1693993**7**5105820**9**749445923.

[4] $g^{-1} : \mathbb{Z}_n \to S_n$, $g^{-1}([j]) = s_j$, $j \in \{0, 1, 2, \ldots, 9\}$.

[5] An audio sample of this can be listen to at https://soundcloud.com/emilioerandu/pi-in-ls6.

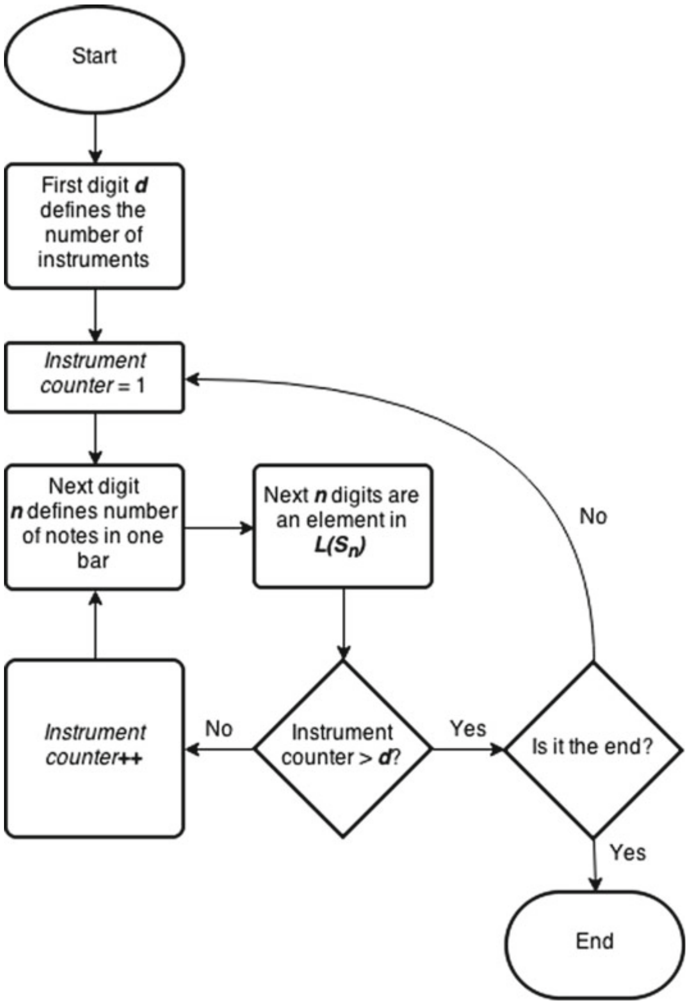


Fig. 5 $(L(S_n), \circ)$ algorithm flowchart

Five bars later there is a triple consecutive digits occurrence in the bass clef: 812848**111**74502:

Using the algorithm with more digits of π we would reach the *Feynman point* 999999 which would result in the addition of the same 9 element in given S_n.

Since we observe that any non-trivially repeated numerical sequence, like π (and typically other irrationals), contains *phrases* (i.e. sequences of ordered digits with their own sequential expressiveness), then we can extend a generalized *Piph Music* as

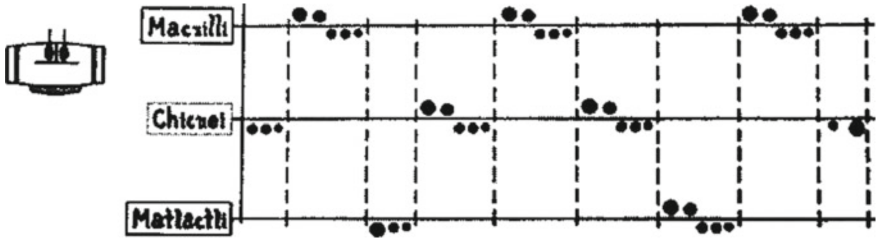


Fig. 6 Pareyon's *Xochicuicatl Cuecuechtli* (2012), excerpt from the manuscript's page 26, with three teponaztlis (wooden, carved log instruments) with the labels Macuilli, Chicuei and Matlactli (i.e. 5, 8, 10)

a branch of Group Theory. We use the term *Piph* after the given example of π as music (*Pi*), containing segments of *musical concatenation* (*ph*rase, therefore making the name $Pi + ph$ for any phrasing extracted from irrational numbers segmentation).[6]

4 Translating a Piece of Music into a Single Number

By the reverse usage of the algorithm shown above, we can translate a piece of music into a single finite number. The process we follow is:

1. Number of instruments defines first d digit.
2. Start with the first instrument, i.e. instrument counter equals 1.
3. Count the number of pitches in the current bar and define the next n digit.
4. Next n digits are obtained applying[7] $g : S_n \to \mathbb{Z}_n$ to the corresponding $L(S_n)$ element in the current bar, e.g. *abc* is 012.
5. Increase the instrument counter and check if this is bigger than the first digit d.

5.1. If not, increase instrument counter and repeat step 3.
5.2. If yes, is this the end of the piece of music?
5.2.1. If not, repeat step 2.
5.2.2. If yes, end.

For the last example (Fig. 6) we apply a variation of the proposed algorithm to the instrumental (*teponaztlis*) passage *Macuilli*, *Chicuei* and *Matlactli* (that is Five, Eight and Ten, in Nahuatl language) in the musical score *Xochicuicatl Cuecuechtli*, also composed by Pareyon [5]:

[6]Carrillo's nomenclature is somehow alluded here: we extend the name of π to other irrationals musically useful, as Carrillo employs the name of number 13 (the so called *Sonido 13*) in order to indicate pitch cardinality bigger than the traditional twelve-tone class system.

[7]$g : S_n \to \mathbb{Z}_n, g\left(s_j\right) = [j], j \in \{0, \ldots, n-1\}$.

For the *numerical translation* of this excerpt, we use $(L(S_2), \circ)$ with 1, $b \in S_2$, being the element that changes between high and low pitch and 0, $a \in S_2$ the identity element. Next we numerically represent this example, with a space between bars where every bold digit represents the start of a different instrument: 3 **03**000**0** **5**10100**00** **003**000 **05**10100**0** **5**10100**00** **05**10100**0** **005**10100 **5**10100**00** **02**00**0**.

Although this number is "mathematically useless", it may be useful to fulfil a number sequence abstraction, such as the textural-orchestrational pattern, like 030500003050500050005500020 (i.e. only taking into account bold numbers), or rather in order to abstract the *contrapuntal number* 353555552 as the key number of this segment, in turn able to be treated as a source for musical development from the same source.

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