A Survey of Applications of the Discrete Fourier Transform in Music Theory

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Abstract Discrete Fourier Transform may well be the most promising track in recent music theory. Though it dates back to David Lewin's first paper (Lewin, J. Music Theory (3), 1959) [33], it was but recently revived by Quinn in his PhD dissertation in 2005 (Quinn, Perspectives of New Music 44(2)–45(1), 2006–2007) [35], with a previous mention in (Vuza, Persp. of New Music, nos. 29(2) pp. 22–49; 30(1), pp. 184–207; 30(2), pp. 102–125; 31(1), pp. 270–305, 1991–1992) [40], and numerous further developments by (Andreatta, Agon, (guest eds), JMM 2009, vol. 3(2). Taylor and Francis, Milton Park) [5], (Amiot, Music Theory Online, 2, 2009) [8], (Amiot, Rahn, (eds.), Perspectives of New Music, special issue 49 (2) on Tiling Rhythmic Canons) [9], (Amiot, Proceedings of SMCM, Montreal. Springer, Berlin, 2013) [10], (Amiot, Sethares, JMM 5, vol. 3. Taylor and Francis, Milton Park (2011) [16], (Callender, J. Music Theory 51(2), 2007) [17], (Hoffman, JMT 52(2), 2008) [29] (Tymoczko, JMT 52(2), 251–272, 2008) [38], (Tymoczko, Proceedings of SMCM, Yale, pp. 258–272. Springer, Berlin, 2009) [39], (Yust, J. Music Theory 59(1) (2015) [42]. I chose to broach this subject because I have had a finger in most, or all, of the pies involved (even using Discrete Fourier Transform without consciously knowing it, in the study of rhythmic tilings).

1 Introduction

Historically Discrete Fourier Transform (hereafter DFT for short) appeared in [33], though its mention in the very end of the paper was as discrete as possible (no pun intended), considering the probable outraged reaction of JMT's readers to the introduction of 'high-level' mathematics in a Music Journal in these benighted times. The paper was devoted to the interesting new notion of Intervallic Relationship between

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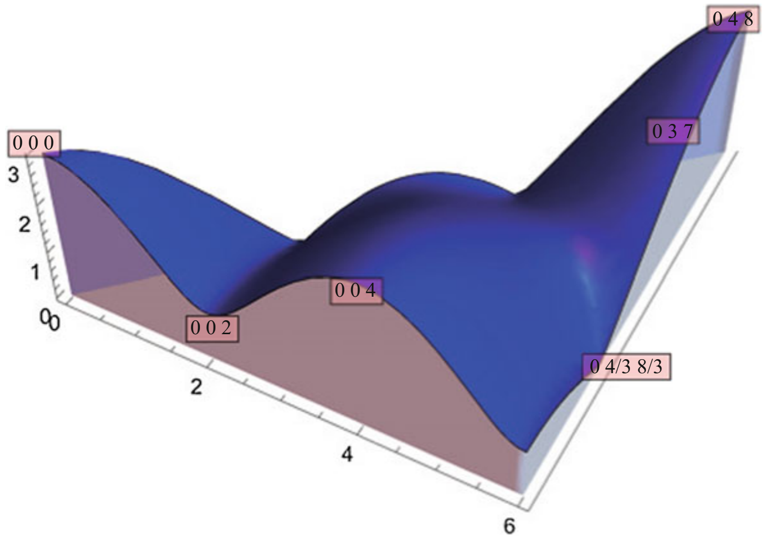

Fig. 1 The continuous landscape of 3-chords

two pc-sets,[1] and its main result was that retrieval of A knowing a fixed set B and IFunc(A, B) was possible provided B did not fall into a hodgepodge of 'special cases' —actually simply those cases when at least one of the Fourier coefficients of B (defined below) is 0.

Lewin himself returned to this notion in some of his latest papers, which may have influenced the brilliant PhD research of Ian Quinn, who encountered DFT and especially large Fourier coefficients as characteristic features of the prominent points of his 'landscape of chords' [35], see Fig. 1 below. Since he had voluntarily left aside for JMT readers the 'stultifying' mathematical work involved in the proof of one of his nicer results, connecting Maximally Even Sets and large Fourier coefficients, I did it in [14], along with a complete discussion of all maximas of Fourier coefficients of all pc-sets.

Interest in DFT having been raised, several researchers commented on it, trying to extend it to continuous pitch-classes [17] or to connect its values to voice-leadings [38, 39]. Another very original development is the study of all Fourier coefficients with a given index of all pc-sets in [29], also oriented towards questions of voice-leadings.

Meanwhile, two completely foreign topics involved a number of researchers in using the very same notion of DFT: homometry (see the state of the art in [2, 34]) and Rhythmic Canons —which are[2] really algebraic decompositions of cyclic groups as direct sums of subsets, and can be used either in the domain of periodic rhythms or

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[1] I use the modern terms.

[2] In the case of mosaic tilings by translation.

pitches modulo some 'octave' —first extensively studied by [40],[3] then connected to the general theory of tiling by [4, 6] and developed in numerous publications [5, 9, 13] which managed to interest some leading mathematician theorists in the field (Matolcsi, Kolountzakis, Szabo) in musical notions such as Vuza canons.

There were also cross-overs like [16] looking for algebraic decompositions of pc-collections (is a minor scale a sum and difference of major scales?) or an incursion in paleo-musicology, quantifying a quality of temperaments in the search for the tuning favoured by J.S. Bach [8]. The last and quite recent development of Fourier Transform takes up the dimension that Quinn had left aside, the phase (or direction) of Fourier coefficients. The position of pairs of phases (angles) on a torus was only introduced in [10] but has known tremendously interesting developments since, especially for early romantic music analysis [42].

NB: the present survey is per force much abbreviated. Details can be found in an abundant bibliography and will be more lavishly explained in a forthcoming book in Springer's CMS collection [3].

2 Basics

2.1 What is DFT?

The DFT of a pc-set (or multiset) $A \subset \mathbb{Z}_n$ is simply the Fourier transform of its characteristic function, i.e.

$$\mathcal{F}_A = \widehat{\mathbf{1}}_A : x \mapsto \sum_{k \in A} e^{-2i\pi kx/n}$$

$\mathcal{F}_A$ is a map on $\mathbb{Z}_n$ whose values $\mathcal{F}_A(0) \ldots \mathcal{F}_A(n-1) \in \mathbb{C}$ are called *Fourier coefficients*. *Inverse Fourier transform* retrieves $\mathbf{1}_A$ from $\mathcal{F}_A$ with a similar formula. For those unfamiliar with Harmonic Analysis (in the mathematical sense!) I suggest reading the illuminating introduction in [17].

Among a number of interesting features that I omit here for lack of space, it should be mentioned that the *magnitude* of $\mathcal{F}_A$ is invariant by transposition, inversion, and even complementation.[4] This is also an immediate consequence of the most important effect of DFT on convolution products, and explains the import of DFT in Sect. 3 among other implications.

[3]At the time, probably the only theorist to mention Lewin's use of DFT.

[4]Except for $\mathcal{F}_A(0)$, which is equal to the cardinality of A.

2.2 Convolution and Lewin's Problem

Convolution is familiar to engineers in signal processing and other areas, but many music theorists may not have heard of it. If however I mention Boulezian's "multiplication d'accords"or Cohn's Transpositional Combination [21], it may ring a louder bell: the convolution of chords (0, 1) and (0, 3, 6, 9) is simply the octatonic (0, 1, 3, 4, 6, 7, 9, 10) in $\mathbb{Z}_{12}$. This operation is instrumental in defining rhythmic canons as we will recall *infra*. It also serves in music-theoretic IFunc, IC functions since

$$\text{IFunc}(A, B) = \mathbf{1}_{-A} * \mathbf{1}_B \qquad \text{IC}_A = \mathbf{1}_{-A} * \mathbf{1}_A$$

where the symbol * denotes the convolution product[5] and $\mathbf{1}_A$ is the characteristic function of pc-set A.

Lewin's problem consists in finding A when B and IFunc(A, B) are given. His paper states when this is possible, not how it may be done: for instance if IFunc$(A, B) = (0, 0, 1, 1, 1, 1, 0, 1, 1, 0, 0, 0)$ and $B = \{1, 3, 6\}$ how does one find $A = \{10, 11\}$?

As Lewin had obviously noticed, solving this is much simpler if the DFT is computed, because

Proposition 1 *The Fourier transform of a convolution product is the termwise product of Fourier transforms.*

In other words, IFunc$(A, B) = f \iff \overline{\mathcal{F}_A} \times \mathcal{F}_B = \mathcal{F}_f$. This enables to compute the Fourier coefficients $\mathcal{F}_A(k) = \overline{\mathcal{F}_f(k)/\mathcal{F}_B(k)}$ and thus retrieve A, *except when* $\mathcal{F}_B(k)$ *vanishes*. The pc-sets with at least one nil Fourier coefficient are none other than the 1,502 "Lewin's special cases" which have been so difficult to describe, from [33] to later descriptions by the same author or even the ingenious 'balances' in [35].

Actually, Lewin's problem is easily solved along with many other convolution-related problems by using the matricial formalism that we introduced with Bill Sethares.

2.3 Circulating Matrices

As developed in [16], if one fills the first column of a matrix with the characteristic function of a pc-set, and the other columns are circular permutations of the first one, then the obtained circulating matrix is a very effective representation of pc-sets, since

[5] The general definition of $f * g$ is the map $t \mapsto \sum_{k \in \mathbb{Z}_n} f(k) g(t - k)$.

- The eigenvalues of the matrix are the Fourier coefficients of the set, and
- The matrix product corresponds with the convolution product of (the characteristic functions of) pc-sets.

For instance, one computes the Interval Content of a diatonic collection matricially by putting

$$M_A = \begin{pmatrix} 1&1&0&1&0&1&0&1&1&0&1&0\\ 0&1&1&0&1&0&1&0&1&1&0&1\\ 1&0&1&1&0&1&0&1&0&1&1&0\\ 0&1&0&1&1&0&1&0&1&0&1&1\\ 1&0&1&0&1&1&0&1&0&1&0&1\\ 1&1&0&1&0&1&1&0&1&0&1&0\\ 0&1&1&0&1&0&1&1&0&1&0&1\\ 1&0&1&1&0&1&0&1&1&0&1&0\\ 0&1&0&1&1&0&1&0&1&1&0&1\\ 1&0&1&0&1&1&0&1&0&1&1&0\\ 0&1&0&1&0&1&1&0&1&0&1&1\\ 1&0&1&0&1&0&1&1&0&1&0&1 \end{pmatrix} \quad \text{then} \quad M_{\mathrm{IC}(A)} = {}^T A \times A = \begin{pmatrix} 7&2&5&4&3&6&2&6&3&4&5&2\\ 2&7&2&5&4&3&6&2&6&3&4&5\\ 5&2&7&2&5&4&3&6&2&6&3&4\\ 4&5&2&7&2&5&4&3&6&2&6&3\\ 3&4&5&2&7&2&5&4&3&6&2&6\\ 6&3&4&5&2&7&2&5&4&3&6&2\\ 2&6&3&4&5&2&7&2&5&4&3&6\\ 6&2&6&3&4&5&2&7&2&5&4&3\\ 3&6&2&6&3&4&5&2&7&2&5&4\\ 4&3&6&2&6&3&4&5&2&7&2&5\\ 5&4&3&6&2&6&3&4&5&2&7&2\\ 2&5&4&3&6&2&6&3&4&5&2&7 \end{pmatrix}.$$

and one reads in the first column the 7 primes, 2 semi-tones, etc…featured in the collection. The solution of Lewin's problem (and also the more general question of Sethares, wishing to decompose a collection in an algebraic combination of translates of another, given one) is then given by solving the simple matricial equation ${}^T A \times B = M_{\mathrm{IFunc}(A,B)}$, thus by-passing the computation of DFT and inverse DFT which is the real reason why this works.

This is also a promising aspect of the study of homometric sets which we will develop in the next section.

3 Homometry and Spectral Units

Homometry is the true name [36] of Z-relation: two pc-sets are homometric whenever they share the same interval content. Since $\mathrm{IC}(A) = \mathbf{1}_A * \mathbf{1}_{-A}$ it follows fairly easily that

Proposition 2 *A and B are homometric* $\iff |\mathcal{F}_A| = |\mathcal{F}_B|$ *(the magnitudes of Fourier coefficients are equal).*

This explains and generalizes the invariance of the magnitude of Fourier coefficients under T/I operations (and complementation, i.e. the hexachordal theorem).

Among other developments, this definition by DFT induces the notion of spectral unit: setting $\mathcal{F}_u = \mathcal{F}_A / \mathcal{F}_B$ one gets by inverse Fourier transform $\mathbf{1}_A = u * \mathbf{1}_B$ where u has unit length Fourier coefficients, i.e. u is a spectral unit.[6] It is perhaps better seen with the matrices of the last section: the matrix of a spectral unit u is a unitary circulating matrix U i.e. ${}^T\overline{U}U = I_n$ i.e. the eigenvalues have magnitude one. Hence the group of all spectral units has a simple structure, it is a product of n circles.

[6] For instance $j = (0, 1, 0, \ldots 0)$ is the spectral unit that turns any pc-set A into its translate $A + 1$. Its Fourier coefficients are all n^{th} roots of unity.

This presentation enables to solve the question of homometry…in continuous space! Unfortunately it is still unknown how one could restrict the orbits (all continuous distributions homometric to one given pc-set) to pc-sets only, i.e. distributions with values 0 or 1 exclusively. A first difficult step is the classification of all spectral units with rational values and finite order, which I achieved in a constructive way, allowing in principle to apply all such spectral units to all pc-sets and select the pc-sets in the resulting orbits.[7]

Details can be found in [2, 34] and compositional applications in [30].

4 Tilings

A rhythmic canon in the sense of [40] is really a tiling of the integers with translates of one finite tile, and boils down to a direct sum decomposition of some cyclic group:

$$A \oplus B = \mathbb{Z}_n$$

where A is the motif, or inner voice, and B the list of offsets, or outer voice. For instance $\{0, 1, 3, 6\} \oplus \{0, 4\} = \mathbb{Z}_8$. This has been the subject of intense scrutiny from music theorists [1, 5–7, 9, 11–13, 23, 27, 28, 31, 41] which in turn focused the interest of some 'pure maths' specialists of tiling problems, which led eventually to a fruitful collaboration (see [32] for instance).

For the present survey, DFT appears in the definition of tiling that is fashionable today, i.e. A tiles with $B \iff$ for all $k \in \mathbb{Z}_n, k \neq 0$, either $\mathcal{F}_A(k)$ or $\mathcal{F}_B(k)$ is 0 (or equivalently the **zero sets** of $\mathcal{F}_A, \mathcal{F}_B$ cover $\mathbb{Z}_n$, 0 excepted).[8] This stems from $\mathbf{1}_A * \mathbf{1}_B = \mathbf{1}_{\mathbb{Z}_n}$.

Moreover, the zero set $Z(A)$ of Fourier coefficients of a pc-set A has remarkable structure:

Proposition 3 *$Z(A)$ is stable by the automorphisms of $\mathbb{Z}_n$, i.e. if $k \in \mathbb{Z}(A)$ then all multiples of k by any α coprime with n are also in $Z(A)$.*

In other words, $Z(A)$ is a reunion of orbits of elements sharing the same order in the group $(\mathbb{Z}_n, +)$.[9] Following [22],[10] we set R_A for the collection of the orders of elements in $Z(A)$ and let S_A be the subset of R_A of elements which are prime powers. Then it is possible to give simple sufficient, or necessary, conditions on these two rather abstract but eminently computable sets, for A to tile.

[7] There are 6,192 such spectral units for $n = 12$.

[8] With the added technical condition $\mathcal{F}_A(0)\mathcal{F}_B(0) = \#A\#B = n$.

[9] In layman's terms, this means that if motif A tiles, then so does $\alpha \times A \mod n$, for any α coprime with n. This is actually a deep algebraic property, but nonetheless it was rediscovered independently by several music composers.

[10] At the time the authors made use of polynomials, not Fourier coefficients, but this is an isomorphic point of view. We translated their definitions accordingly.

These conditions also reflect on the famous *spectral conjecture* [26, 37] and consideration of the musical notion of *Vuza canons* (originating in wondering what is actually heard while listening to a rhythmic canon) enabled some progress on this still unsolved question [13]. Moreover, new algorithms were developed, based on a classification of possible sets R_A and enhancing the exhaustive search for Vuza canons, see [32]. I skip many fascinating aspects of this beautiful question, which already gave birth to special issues of PNM and JMM [5, 9].

5 Saliency

In this section we look at Fourier coefficients which are large instead of nil.

5.1 Measuring "fifthishness"

In [35], Ian Quinn pursued the quest for a 'landscape of chords' (for some given cardinality k) and realized that most authors agreed on a prevalence of maximally even sets,[11] and that furthermore, these sets could be characterized by a high value of their k^{th} Fourier coefficient:

Theorem 1 *The highest value of $|\mathcal{F}_A(k)|$ is reached among k-pcsets for Maximally Even sets and only for them.*

The rigorous mathematical study of this characterization was done in [14]. More generally Quinn links the size of this coefficient, the *saliency* (which is both closeness to an even division in k parts of the chromatic circle, and the quality of being generated by some interval d) to the prevalence of this generating interval.[12] For instance, the magnitude of $\mathcal{F}_A(3)$ can be construed as 'major thirdness' (this coefficient being maximal for augmented triads) and that of $\mathcal{F}_A(5)$ is the 'fifthishness', maximal for pentatonic (or diatonic) collections. In a continuous setting, of course the actual maximums happen for exact divisions of the circle or subsets thereof.

5.2 A Better Approximation of Peaks

Tymoczko [39] improves on remarks by Strauss and others in laying down a connection between voice-leading distances and Fourier saliency: intuitively, since the peaks for saliency culminate for even distributions of the (continuous) circle of pcs,

[11] Such as defined in [18–20] and others.

[12] There is a good correlation between this saliency and the saturation of the collection in interval d (Aline Honing, personal communication).

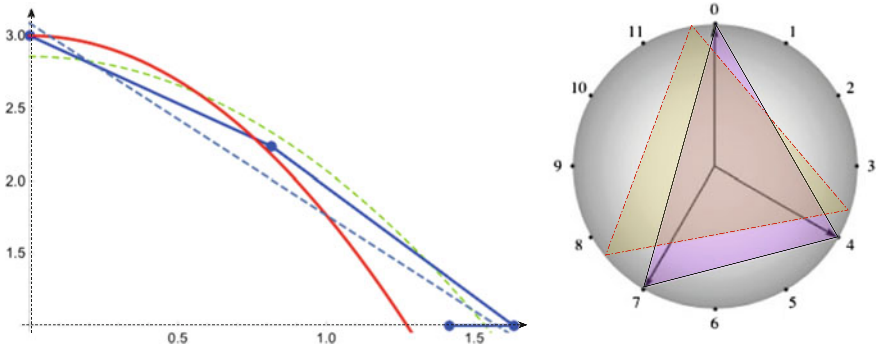

Fig. 2 Linear and quadratic correlation for 3-sets

the closest to one such peak, the largest the Fourier coefficient will be. Acting on this flimsy connection, Tymoczko computed the correlation between this closeness, measured as the standard Euclidean Voice-Leading distance between pc-sets, and was rewarded by extremely good correlation coefficients (between −0.99 and −0.95).

Being dissatisfied both with the heuristicness of the argument and with the result (near a maximum, one expects a curve to be flat, i.e. a 0 slope and not a negative one) I decided to tackle the analytic computation of the saliency of a neighbour of a peak. Not surprisingly the formulas are different,[13] and the true correlation is quadratic, not linear, as expected near a maximum (see Fig. 2 where VL is the Euclidean distance between a 3-set and the closest equilateral triangle). Still this vindicates the use of Euclidean distance for voice-leading instead of taxi-cab metric for instance [39].

6 A Torus of Phases

Another new development of DFT in Music Theory takes up the gauntlet that Ian Quinn had thrown (or rather left aground) in [35], "letting aside the direction component" i.e. focusing on magnitude and leaving aside the phase, or direction, of Fourier coefficients. [29] was probably the first to tackle the whole complex value of a given Fourier coefficient for different pc-sets (with a given cardinality), providing intriguing pictures with almost complete symmetries, see Fig. 3. His paper shows a clear understanding of the meaning of the missing phase component, stating that

> The direction of a vector indicates which of the transpositions of the even chord associated with a space predominates within the set under analysis.

[13] For 3-sets, $|\mathcal{F}_A(3)| = 3 - \dfrac{\pi^2}{8} VL^2 + o(VL^4)$, best near the maximum, whereas the linear regression yields $|\mathcal{F}_A(3)| \approx 3.39 - 1.57 \times VL$. The formula is different from the one in [39] because of a different convention in the definition of DFT.

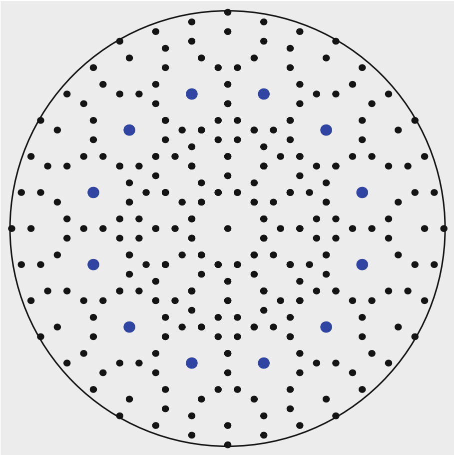

Fig. 3 a_5 coefficient for all 3-sets in $\mathbb{Z}_{12}$

It is perhaps even clearer to measure the phase of a coefficient by how much it changes under basic operations:

Lemma 1 *Transposition of a pc-set by t semitones rotates its k^{th} Fourier coefficient a_k by a $-2kt\pi/n$ angle, i.e. $\theta_k \mapsto \theta_k - 2kt\pi/n$.*

Any inversion of a pc-set similarly rotates the conjugates *of the Fourier coefficients.*

For instance, moving a diatonic collection by a fifth changes the direction of its fifth coefficient by $\pi/6$. Hoffman's pictures are particularly useful in considering close neighbours and parsimonious voice-leadings. But since they do not allow, for instance, to distinguish between all 24 major/minor triads, the following space deserves a closer look.

In [10] I introduced a 2D-space, torus shaped, defined by the pair of phases of two Fourier coefficients.[14] This space enables to project (almost) all pc-sets and is not limited to a given cardinality, this major drawback of most existing models. As it was since developed by J. Yust, it is most advantageous to feature simultaneously on the same simple 2D-model triads, dyads, single notes, diatonic collections, and whatever chords are necessary for the analysis of a given piece of music of even musical style (see [42] for a convincing utilisation of the Torus of Phases in early romantic music). Another striking advantage appears when one focuses on triads, which are disposed in this space with the same topology as the classical Tonnetz, see Fig. 4.[15]

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[14]The $3^r d$ and 5^{th} were chosen for stringent reasons. It was also the choice independently made by [42]).

[15]Please remember that this picture is a torus, i.e. opposite sides should be construed as glued together.

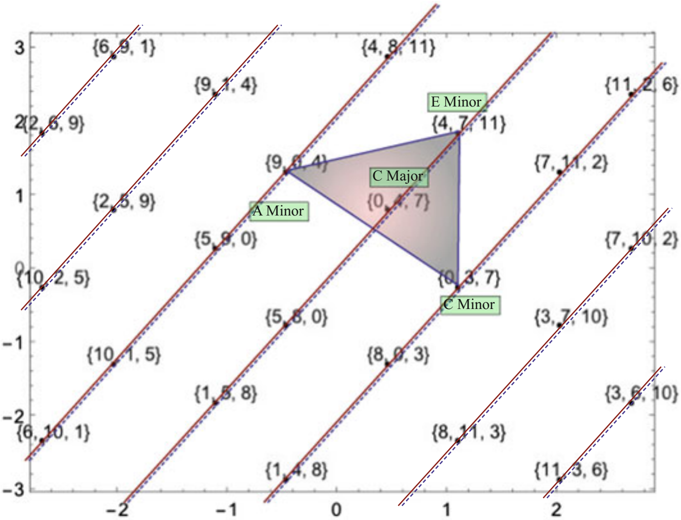

Fig. 4 The neighbours of a triad are its images by L, P and R

A particularly seductive feature of this model discovered by Yust is that central symmetries around a single pc or around dyads appears just like that, as a central symmetry on the planar representation of the torus: the T/I group and its induced action on pc-sets embeds itself in the Euclidean (quotient) group on the torus. For instance the dyad $(0, 4)$ would appear as the middle point of triads $(0, 4, 9$ and $(0, 4, 7)$ on Fig. 4. More specifically,

Proposition 4 *If A and B are symmetrical around a center c (resp. a dyad (a, b)), then their torus projections are symmetrical around the torus image of c (resp. the image of the dyad).*

This makes for especially concise and convincing representations of movements between chords, see again [42] for examples. Among other things, it enabled to explain the strange closeness of the lines connecting chromatically major and minor triads respectively (part of it in red and blue on Fig. 4) that I had presented as a baffling mystery in [10] barely a year before.

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