Learning to Notice Student Thinking About the Equal Sign: K-8 Preservice Teachers' Experiences in a Teacher Preparation Program

Leigh A. van den Kieboom, Marta T. Magiera and John C. Moyer

Abstract In this chapter, we present our work and research related to preservice teacher (PST) noticing, describing how we provide PSTs with opportunities to notice student thinking about the equal sign and equality. We designed an instructional intervention in an integrated mathematics content and pedagogy course (with a field experience) to support PSTs in (1) learning about key mathematical ideas related to the equal sign and equality, and (2) rehearsing teacher noticing skills. Our PSTs rehearsed and reflected on their noticing skills by conducting two one-on-one clinical interviews with elementary students and participating in debriefing interviews with course instructors. Using this context, we examined (1) the extent to which PSTs attended to and further explored student understanding of the equal sign and equality, and (2) what PSTs perceived they learned about aspects of their teacher professional noticing skills and student thinking about the equal sign and equality. Our results indicate that the PSTs predominantly noticed the strategies students used to solve a task without focusing on student thinking about the equal sign and equality. In addition, our PSTs perceived that they strengthened either their own knowledge or student knowledge of the equal sign and equality while conducting their diagnostic clinical interviews.

Keywords Preservice teacher noticing · Student relational thinking about equality · Clinical interviews · Teacher preparation

L.A. van den Kieboom (✉) · M.T. Magiera · J.C. Moyer
Marquette University, Milwaukee, WI, USA
e-mail: leigh.vandenkieboom@marquette.edu

M.T. Magiera
e-mail: marta.magiera@marquette.edu

J.C. Moyer
e-mail: johnm@mscs.mu.edu

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Introduction

Mathematics education reform initiatives emphasize the need for teachers to support students in developing conceptual understanding and sense making of mathematical concepts and procedures (Kilpatrick, Swafford, & Findell, 2001; National Council of Teachers of Mathematics, [NCTM], 2000). Teachers who use student thinking to guide instruction and whose instruction supports sense making by connecting various concepts and procedures across the discipline have a positive impact on student learning (Carpenter, Fennema, Franke, Levi, & Empson, 1999; Kilpatrick et al., 2001). To help teachers, including preservice teachers (PSTs), learn to focus on student thinking, researchers have begun to characterize the professional noticing skills teachers draw on when they use student thinking to guide instruction. These interrelated skills include (1) attending to student strategies, (2) interpreting student understanding, and (3) deciding how to respond based on student understanding (Jacobs, Lamb, & Philipp, 2010).

Mathematics teacher educators argue that in order to prepare PSTs to use student thinking to guide their instruction, teacher education programs must support PSTs in developing their noticing skills early in their program (Star & Strickland, 2008; van Es, 2011; van Es & Sherin, 2002). In this chapter, we report on our work and research related to PST noticing, describing how we adapted Jacobs et al's. (2010) model of teacher professional noticing to provide PST's opportunities to rehearse their noticing skills. In contrast to Jacobs et al's. model, which focuses on classroom instruction, our work provides PSTs with opportunities to attend to and further explore student mathematical thinking in the context of one-on-one diagnostic clinical interviews. We use diagnostic clinical interviews as a context for our work because clinical interviews help us to support PSTs in focusing on one student's thinking about the equal sign as they rehearse their noticing skills. Interviewing one student reduces the number of distractions PSTs may encounter while rehearsing noticing skills during classroom instruction. We also provide PSTs with opportunities to reflect on their teacher professional noticing skills, thus helping to optimize their growth as effective teachers. Motivated by the need to provide PSTs the opportunity to develop their teacher professional noticing skills and to prepare PSTs to engage their future students in relational thinking about equality, then, this study examined the following questions:

(1) To what extent do PSTs attend to and further explore mathematically important aspects of student thinking about the equal sign during clinical interviews?
(2) After conducting clinical interviews, how do PSTs perceive their learning about aspects of teacher professional noticing and student thinking about the equal sign?

Preservice Teacher Noticing

Much of the research on developing PSTs' capacity to notice relies heavily on the use of video recordings of mathematics classroom instruction (McDuffie et al., 2014; Schack et al., 2013; Star & Strickland, 2008; van Es & Sherin, 2002). For example, van Es and Sherin (2002) documented how they used video recordings of mathematics classroom instruction to improve their PSTs' noticing skills. After viewing selected video clips of mathematics classroom instruction, the PSTs in their study were able to shift their noticing skills from a simple reporting of a sequence of events to identifying and analyzing the salient features of mathematics teaching and learning.

In their study with secondary PSTs, Star and Strickland (2008) specifically focused on the first component of teacher professional noticing, attending. They found that guided viewing of video-recorded mathematics instruction improved their PSTs ability to attend to the classroom environment, the mathematics content of a lesson, and the communication between teachers and students.

McDuffie et al. (2014) engaged their PSTs in analyzing video recordings of mathematics instruction using four lenses: Teaching Lens, Learning Lens, Task Lens, and Power and Participation Lens. Like van Es and Sherin (2002), they found that this structured activity assisted their PSTs in moving from a simple description of what they observed to being able to discuss significant classroom interactions.

Schack et al. (2013) specifically focused their study of PST noticing on elementary students' early arithmetic learning. In their research, Schack et al. first asked their PSTs to observe video clips of a teacher educator conducting one-on-one diagnostic interviews of students solving arithmetic problems. Then, the PSTs conducted their own diagnostic interviews to rehearse all three components of their teacher professional noticing skills. As consistent with the other studies discussed here, Schack's PSTs improved all three components of their teacher professional noticing skills.

Our work on PST noticing is similar to each of the studies discussed above in that our goal was to engage PSTs in opportunities to rehearse and improve their noticing skills. On the other hand, our work differs significantly from the first three studies described above in that we assessed our PSTs' noticing ability during face-to-face work with students rather than during virtual work with *representations* of practice (Grossman et al., 2009). In our study, the PSTs were required to attend to student strategies, interpret student thinking, and decide how to respond in the moment, i.e., instantly. The PSTs in the other studies, however, had the luxury of time as they attended to, interpreted, and decided how to respond to video recordings of mathematic classroom instruction. Our work also differs from the first three studies in that it focused specifically on one particular aspect of student mathematical thinking rather than mathematics classroom instruction. While it is true that like us, Schack et al. emphasized only one aspect of mathematical thinking, our work differs from theirs in that our work focuses narrowly on student thinking about the equal sign, while theirs focuses more generally on early arithmetic.

Student Thinking About the Equal Sign and Teacher Professional Noticing

To provide a context for the need to support PSTs in learning to notice how students might think about the equal sign and equality, we present an overview of the research on relational thinking about equality. Carpenter, Levi, Franke, and Zeringue (2005) defined relational thinking about equality as the ability to examine relationships among quantities using the fundamental properties of equality, numbers, and operations, rather than following a series of steps or procedures to solve an equation. Mathematics education researchers argue that students who have a relational understanding of equality are positioned to make a more successful transition from the study of arithmetic to the study of algebra (Knuth, Alibali, Weinberg, McNeil, & Stephens, 2005; Knuth, Stephens, McNeil, & Alibali, 2006). Despite that relational thinking about equality has been identified as an important component of student success in algebra, researchers have found that both practicing teachers and PSTs fail to *notice* the conceptions and misconceptions students have about the equal sign and equality and are often unaware that a limited understanding of relational thinking about equality has a negative impact on student learning (Asquith, Stephens, Knuth, & Alibali, 2007; Stephens, 2006). In addition, most K-8 teachers devote little instructional time to this important topic (Knuth et al., 2006).

In their work on relational thinking, Carpenter et al. (2005), Matthews, Rittle-Johnson, McEldon, and Taylor (2013), and Knuth et al. (2006) described the conceptions and misconceptions students have about the equal sign, how student thinking develops over time, and how teachers might foster relational thinking in their students. Their work can be used to support teachers in learning to *notice* student thinking about the equal sign and equality. For example, teachers would be more effective if they learned to notice the thinking of students who view the equal sign as an invitation to perform an operation. These students do not understand that the equal sign separates equivalent quantities and often interpret number sentences that are not in the form $a + b = \square$ as impossible to solve. Teachers are also well advised to notice the thinking of those students who have moved beyond thinking about the equal sign as a symbol to operate and can recognize the properties of equality, thus accepting as true number sentences in the form $\square = a + b$. Teachers should also notice the thinking of students who understand the equal sign as a symbol of equivalence that separates same quantities on either side of the equal sign. They ought to notice that to verify that the expressions on either side of the equal sign represent the same quantities, students often calculate and compare the two sides using a prescribed set of procedures. Finally, it is important for teachers to notice the thinking of students who view the equal sign relationally as a symbol that separates equivalent quantities. Specifically, these students can determine whether or not the quantities on both sides of the equal sign are the same by analyzing the relationships between the quantities, without computing. Table 1 summarizes the literature on the different ways that students think about the equal sign (Carpenter et al., 2005; Knuth et al., 2006; Matthews et al., 2013).

Table 1
Student thinking about the equal sign

Student thinking about the equal sign	Sample tasks and student thinking
Rigid Operational Thinking Thinking about the equal sign as an indicator to perform an operation, to solve the problem, find the answer	"You can't solve ☐ = 7 + 6 because the problem is backwards. The answer goes after the equal sign" "The number 9 goes in the box because 5 plus 4 is nine. Then you add 9 and 6 to make 15" (solving 5 + 4 = ☐ + 6)
Flexible Operational Thinking about the equal sign *as an operator* but accepting the symbol of equality in sentences where operations are not necessarily directly followed by the equal sign, e.g., $c = a + b$	"The number 13 goes in the box to make the sentence true. It doesn't matter, you could also do 7 + 6 = ☐" (solving ☐ = 7 + 6)
Computational Thinking Recognizing that the equal sign represents a relation between two equal numbers; carrying out calculations on either side of the equal sign to determine whether or not the quantities are the same	"Both sides need to equal twelve so 8 goes in the box, 8 and 4 is twelve, 5 and 7 is twelve" (solving ☐ + 4 = 5 + 7)
Relational Thinking Recognizing that the equal sign represents an equivalence between two quantities; comparing the mathematical expressions on both sides of the equal sign without carrying out the calculations	"The number 3 goes in the box because 10 is two less than 12 and 8 is one less than 9 so I need 3 to make both sides the same." (solving 12 + 9 = 10 + 8 + ☐)

Elementary and middle school students who resist changing their conception of the equal sign from that of "an indicator to operate" to that of a "symbol of sameness" benefit from explicit instruction that focuses on relational thinking about equality throughout the rest of their K-8 experience (Alibali, Knuth, Hattikudur, McNeil & Stephens, 2007; Carpenter, Franke, & Levi, 2003; McNeil & Alibali, 2005a, 2005b; McNeil et al., 2006). Given that K-8 students benefit from such instruction, it is important that PSTs learn to notice student thinking about the equal sign and equality and thus become equipped to identify and capitalize on, in their future practice, opportunities that might foster relational thinking about equality in their students.

Method

Context

Our work and research on PST noticing is grounded in the literature related to enacting ambitious mathematics instruction (Ball, Sleep, Boerst, & Bass, 2009; Grossman et al., 2009; Kazemi, Franke, & Lampert, 2009), which involves supporting PSTs in "actually doing the practice of teaching" (Kazemi et al., 2009, p. 12). With this idea in mind, we designed an instructional intervention that took place simultaneously in two semester-long mathematics content courses. Each mathematics course was integrated with a corresponding pedagogy course (with field experience). We situated our PSTs' opportunity to rehearse their teacher professional noticing skills and apply their knowledge of student thinking about the equal sign in the context of two diagnostic clinical interviews, which the PSTs conducted with elementary or middle school students. Designed by course instructors, the clinical interview protocol included a series of nine tasks written as a foundation from which the PSTs could elicit student thinking about the equal sign and equality (see Appendix 1).

Mathematics content courses. The two mathematics content courses (Number Systems and Operations for Elementary Teachers, and Algebra and Geometry for Teachers) provided the PSTs the opportunity to examine key mathematical concepts found in the elementary and middle school mathematics curriculum (e.g., relational thinking about equality). In both mathematics courses, the PSTs learned about the conceptions and misconceptions elementary and middle school students might have about the equal sign and equality (Table 1). They discussed specific problems that could be used to uncover these conceptions and misconceptions and they engaged in activities designed to help uncover them. They also analyzed the levels of mathematical thinking about the equal sign and equality that a given problem might evoke.

Pedagogy courses (with field experience). Students in each mathematics course were concurrently enrolled in a corresponding pedagogy (with field experience) course: Teaching Elementary School Mathematics and Teaching Middle School Mathematics. The instruction in the pedagogy courses was designed to engage the PSTs in rehearsing effective pedagogical practices and to prepare them to conduct diagnostic clinical interviews of an elementary or middle school student. In their respective pedagogy courses, the PSTs engaged in activities designed to support their understanding that the purpose of a diagnostic clinical interview is to investigate student thinking, not to teach. As a means of preparation, the PSTs also analyzed and discussed an illustrative video recording of a diagnostic clinical interview one of the authors conducted with a middle school student. The PSTs used the transcript from this illustrative interview to engage in a discussion about the potential of different types of questions to elicit student mathematical thinking during one-on-one interviews. They also used the video recording of the illustrative interview to practice their teacher professional noticing skills prior to conducting their own interviews of elementary or middle school students. To foster PSTs' reflection on their own noticing skills and thus help them to learn from their own practice (Sherin, Jacobs, & Philipp, 2011), we also engaged each PST in an individual debriefing interview conducted by course instructors following completion of each diagnostic clinical interview. The protocol for the debriefing interviews included questions that elicited the PSTs' thinking about their teacher professional noticing skills and the PSTs' interpretation of their student's thinking about the equal sign and equality.

Participants

Participants for the study were 32 PSTs, all juniors or seniors seeking Grades 1–8 teaching license, one or two semesters before their student teaching experience. Ten were concurrently enrolled in Number Systems and Operations and Teaching Elementary Mathematics. In their field placement, those 10 worked with 10 elementary students (3rd grade). Twenty-two PSTs were concurrently enrolled in Algebra and Geometry for Teachers and Teaching Middle School Mathematics. These 22, in their field placement, conducted clinical interviews with middle school students (8th grade).

Data Collection and Data Sources

Data for this study were collected via one-on-one diagnostic clinical interviews conducted by the PSTs during their field experience and via debriefing interviews conducted by the course instructors. Data sources included (1) transcripts from the 64 diagnostic clinical interviews the PSTs conducted with 3rd or 8th grade students at the middle and at the end of the semester, and (2) transcripts from the 64 debriefing interviews of PSTs, which were conducted by course instructors (one after each clinical interview).

Data Analysis and Results

After coding the data, we established validity and reliability by comparing sets of independently coded transcripts, citing specific examples, and clarifying coding themes and categories until 100% agreement was achieved. Once coded, the data were analyzed using a combination of qualitative and quantitative methods. We present the data analysis and results next, organized by research question.

Research Question #1: To What Extent Do PSTs Attend to and Further Explore Mathematically Important Aspects of Student Thinking About the Equal Sign During Clinical Interviews?

Our answer to this research question comes from analysis of the transcripts of the diagnostic clinical interview the PSTs conducted with their elementary or middle school students. The transcripts were analyzed in several steps. First, we divided each of the 64 transcripts into segments that corresponded to each of the nine interview tasks the PSTs posed for their students (i.e., 64 transcripts × 9 interview protocol tasks = 576 interview segments). We defined an interview segment as a portion of a transcript that began with a PST posing a task from the interview protocol and ending when the PST moved on to the next task.

Second, drawing on the descriptions of teacher professional noticing as a conceptual framework (Jacobs et al., 2010), we developed a rubric and analyzed each segment to examine whether the PSTs (1) did not attend to and further explore student thinking (Score 0), (2) attended to and further explored the strategy the student used to solve the task *without* an explicit focus on student thinking about the equal sign and equality (Score 1), or (3) attended to and further explored student thinking *with* an explicit focus on student thinking about the equal sign and equality (Score 2). We also further investigated each of the identified Score 2 segments to determine specifically which aspect(s) of student thinking about the equal sign and equality the PSTs attended to and further explored: (a) understanding of the equal sign as sameness of quantities, (b) use of a computational strategy to confirm sameness of quantities, or (c) use of a relational thinking strategy to confirm sameness of quantities.

In the third step of the analysis, we examined the frequency of Scores 0, 1, and 2 for each PST across both interviews, calculated the average number of Score 2's the PSTs received on interview #1 and interview #2, and conducted z-tests for proportions to explore patterns in the PSTs' noticing. To illustrate our scoring, we use segments in which PSTs #5, #9, and #2 posed the same Grade 3 interview task (Task 3 on the protocol) to their students. We further illustrate our scoring with segments in which PSTs #32, and #18 pose, the structurally equivalent Grade 8 interview Task 3.

Score 0: Did not attend to and further explore student thinking. Consistent with our rubric, we coded a PST's segment as Score 0 if, within that segment, we found no evidence that the PST attended to or further explored their student's thinking. We illustrate a Score 0 segment with the following example from PST #5:

1. PST #5: Question number 3. [13–7 = □ − 6] What number would go in the box?
2. Student: Six.
3. PST #5: Now what about this one, number four [referring to the next interview task].

As illustrated in the transcript segment, after reading the interview task, PST #5 received an incorrect answer from her 3rd grade student (line 2). In response, and despite explicit instructions to explore her student's thinking about the equal sign during the clinical interview, PST #5 simply moved on to the next interview task. Her actions provided no evidence that she attended to, or further explored, her student's thinking.

Score 1: Attended to and further explored the strategy the student used to solve the task *without* an explicit focus on student thinking about the equal sign and equality. We coded an interview segment as Score 1 if, within that segment, the PST attended to and further explored the strategy the student used to solve the task *without* an explicit focus on student thinking about the equal sign and equality. The transcript segment from PST #9's interview, shown next, illustrates this scoring category, highlighting how PST #9 focused on the counting strategy her student used to solve the task, rather than on her student's thinking about the equal sign and equality:

1. PST #9 The next question is fill in the box with the number that makes the sentence true. 13 minus seven equals box minus 6 [13–7 = □ − 6]
2. Student: 6.
3. PST #9 How did you get 6?
4. Student: I punched 7 and counted up and then I got 6.
5. PST #9: You counted up and got 6. Okay let's move on to the next question.

When PST #9's 3rd grade student provided an incorrect answer (line 2), PST #9 asked a general question "How did you get 6?" (line 3) and paraphrased the counting strategy her student used to solve the problem (line 5). Although she attended to the student's thinking about 13–7, she failed to explore the student's thinking about the equal sign. Rather than follow up with her student by asking about the box and the number on the other side of the equal sign (line 1), PST #9 posed the next task on the interview protocol. It might be that after receiving her student's response, PST #9 assumed that her student did not understand the equal sign as sameness but rather as a symbol to operate, and thus responded that 13–7 is six. However, without explicitly engaging her student in a conversation about the six on the right side of the equation, PST #9 could not be certain that this interpretation of her student's thinking is accurate. It may be, for example, that PST #9 did not even attend to the—6 on the right side.

Score 2: Attended to and further explored student thinking *with* an explicit focus on student thinking about the equal sign and equality. We coded a segment as Score 2 if, within that segment, the PST attended to and further explored an aspect of their student's thinking about the equal sign and equality. We provide examples in which PSTs attended to and further explored an aspect of their student's (a) understanding of the equal sign as sameness of quantities, (b) use of a computational strategy to confirm sameness of quantities, or (c) use of a relational thinking strategy to confirm sameness of quantities.

(a) ***Score 2: Focusing on student understanding of the equal sign as sameness of quantities***. PST #2's segment, shown next, illustrates how PST #2 attended to and further explored her student's understanding of the equal sign with an explicit focus on sameness of quantities on both sides of the equation.

1. PST #2: Question number 3 [13–7 = □ − 6] fill in the box that makes the number sentence true?
2. Student: 6.
3. PST #2: Six, so how did you get 6?
4. Student: Because I counted.
5. PST #2: You counted? How did you count? Did you count in your head or use your fingers?

6. Student: I used my fingers.
7. PST #2: Can you show me how you used your fingers?
8. Student: Thirteen, then twelve, eleven, ten, nine, eight, seven, six (says 13 then holds up one finger as she says each number).
9. PST #2: Why do you use your fingers?
10. Student: Because it's more easier.
11. PST #2: It's easier?
12. PST #2: Okay, what about this 6 over on the other side of the number sentence? Do we have to do anything with that?
13. Student: Just leave it.
14. PST #2: Just leave it? So you subtract 7 from 13 and put 6 in the box and just leave the other 6 over there?
15. Student: Yeah.
16. PST #2: Is this 6 part of the question you just solved?
17. Student: No.
18. PST #2: No? Okay, should we try the next one?

Similar to PST #9's student, PST #2's 3rd grade student provided an incorrect answer in response to Task 3 (13–7 = $\square - 6$). Like PST #9, PST #2 followed up with a general question about the strategy her student used to solve the equation (line 3), and then PST #2 proceeded to investigate her student's counting strategy (lines 4–11). After that, unlike PST #9, PST #2 further explored her student's understanding of the equal sign as sameness of quantities. In doing so PST #2 followed up with her student, asking whether she considered the number six on the other side of the equal sign when solving the equation. Once PST #2's student confirmed that she did not think that the number six on the other side of the equal sign was part of the equation (lines 17–18), PST #2 moved on to the next task.

(b) ***Score 2: Focusing on student use of a computational strategy to confirm sameness of quantities***. We also assigned an interview segment as Score 2 if, within that segment, the PST attended to and further explored whether or not their student computed on both sides of the equal sign to confirm sameness of quantities. We illustrate this situation using a transcript segment from PST #32's interview of an 8th grade student solving the equation 130–70 = $a - 60$.

1. PST #32: What value of a makes the number sentence true: 130–70 = $a - 60$
2. Student: 120?
3. PST #32: How did you get that answer?
4. Student: First I subtracted 130 from 70 and got 60. Then I added 60 to the other 60 to get 120.
5. PST #32: What other 60?
6. Student: The one here (pointing to $a - 60$).
7. PST #32: Okay….
8. Student: So I knew that if one side was 60 the other side had to be 60.
9. PST #32: How did you know the other side had to be 60?
10. Student: Because of the equal sign.

11. PST #32: What do you mean?
12. Student: The answers on both sides have to be the same, they're equal.
13. PST #32: Okay. Is there another way you could solve the problem?
14. Student: No, I don't think so.
15. PST #32: No? Okay, let's move on to number 4.

Investigating her student's computational procedure assisted PST #32 in determining that her student understood the equal sign as sameness between quantities. To find the value of *a* in the equation, PST #32's student first subtracted 70 from 130 on the left side of the equal sign and then added 60 and 60 (line 4). After operating on both sides of the equal sign, the student stated "if one side was 60 the other side had to be 60" (line 8). Attending to this line of thinking, PST #32 asked her student how he knew that both sides of the equation had to be 60 (line 9). When her student communicated that he understood the equal sign to mean that the answer on both sides has to be the same (line 12), PST #32 further investigated, asking her student if there was another way he could solve the equation (line 13). After the student stated he did not think so (line 14) PST #32 moved on to the next task.

(c) ***Score 2: Focusing on student use of a relational thinking strategy to confirm sameness of quantities.*** Finally, we also evaluated a segment as Score 2 if, within that segment, the PST attended to and further explored the relational thinking strategy their student used to confirm sameness of quantities.

1. PST #18: Okay, number 3. What value would make the number sentence true: 130–70 = $a - 60$.
2. Student: Would it be 120 or 140?
3. PST #18: Which one are you going with?
4. Student: 120.
5. PST #18: How did you figure that out?
6. Student: Well, since 60 is ten smaller than 70, I would just take 10 off 130. So yea, it would be like subtracting 10.
7. PST #18: Wait, you took 10 away from 130 right? Is that what you're saying? So you took 10 away from 130 for *a*, what was that doing to your equation? What do you think that did for your equation?
8. Student: Just evened it out.
9. PST #18: Evened it out, what do you mean?
10. Student: Since 60 is ten less than 70 I took 10 off 130 to make it even.
11. PST #18: Yea, so you evened it out. Why can you do this?
12. Student: To balance each side so they would be the same.
13. PST #18: Meaning to make each side of the equation the same?
14. Student: Yea.
15. PST #18: Is there another way you could have gone about that?
16. Student: Um…..not that I would have done.
17. PST #18: Okay, well, good job. Alright, so number 4.

As this example illustrates, rather than perform calculations to determine the value of a for the equation 130–70 = $a - 60$, PST #18's 8th grade student used a relational thinking strategy, finding the difference in the magnitude of numbers (line 6). To further investigate if her student understood why he could use this strategy, PST #18 asked "What do you think that did for your equation?" (line 7). In response, PST #18's student explained "just evened it out" (line 8). PST #18 then used this response as an opportunity to further explore her student's understanding of the equal sign as sameness of quantities, asking "Why can you do this?" (line 11). Satisfied that her student's explanation of "evening out" meant that he interpreted the equal sign as a symbol that indicated the sameness of quantities, PST #18 moved on to the next task.

PST noticing skills during clinical interviews. Table 2 provides a summary distribution of the scores our PSTs received for interviews #1 and #2. The organizing feature of this summary is the percentage of interview segments for which our PSTs received each score. For example suppose a PST had the following noticing scores on interview #1, segments 1–9, respectively, 1, 1, 1, 1, 0, 0, 1, 2, 0. This PST would have 3/9 (<50%) Score 0's, 5/9 (>50%) Score 1's, and 1/9 (<50%) Score 2's. Therefore, this PST would be one of the PSTs represented in the Score 1 entry for interview #1 below: 29/32 (91%).

Table 2
Comparison of PST noticing scores on interview #1 and interview #2

% of segments	Interview #1			Interview #2		
>50% (5/9–9/9)	Score 0 1/32 (3%)	Score 1 29/32 (91%)	Score 2 2/32 (6%)	Score 0 0/32 (0%)	Score 1 29/32 (91%)	Score 2 3/32 (9%)

Table 2 shows that the vast majority of our PSTs (91%) attended to and further explored, in the context of at least half of the interview segments, the strategies their students used to solve each task. In interview #1, 91% of the PSTs received Score 1 on more than half of the interview segments, meaning that while they attended to some aspects of the strategies their student used to solve each task, they failed to explore their student's thinking about the equal sign. The same was true of the PSTs' scores on interview #2.

Overall, the analysis revealed 55 instances of attending to and further exploring student thinking about the equal sign (Score 2) in the context of interview #1 and 71 instances of attending to and further exploring student thinking about the equal sign (Score 2) in the context of interview #2. While our group of PSTs, as a whole, did not statistically significantly improve with respect to attending to and further exploring student thinking about the equal sign, more than half of the 32 PSTs (19, 59%) showed improvement in their ability to attend to and further explore student thinking about the equal sign, as demonstrated by an overall positive change in their number of segments scored as "2" from the first to the second interview. The proportion of PSTs who showed an increase in the number of score 2's from the first to the second interview was significantly greater than the proportion of PSTs who did not demonstrate an increase ($z = 2.073$, $p < 0.05$). However, on average the increase was small. The average number of segments scored as "2" in interview #1 improved from slightly less than two (out of 9) score 2's per interview $\left(\overline{M} = 1.79\right)$ to slightly more than two (out of 9) score 2's per interview $\left(\overline{M} = 2.21\right)$.

Research Question #2: After Conducting Clinical Interviews, How Do PSTs Perceive Their Learning About Aspects of Teacher Professional Noticing and Student Thinking About the Equal Sign?

To explore what our PSTs learned about teacher professional noticing and student thinking about the equal sign, we interviewed each PST after they conducted each of their diagnostic clinical interviews. With a goal of stimulating our PSTs' reflection on their learning, we asked the following questions during these debriefing interviews: (a) Thinking about your interview experience, what did you notice about the mathematical thinking of a student? (b) Thinking about your interview experience, what did you learn about yourself as a teacher? (c) Did the interview help you to develop a better understanding of relational thinking about equality? The debriefing interviews were transcribed verbatim, uploaded to NVIVO software for analysis, and then coded for themes using the open coding technique described in Straus and Corbin (1998). The themes that emerged from this analysis are presented and discussed next.

PSTs' perceived learning. Table 3 provides a summary of what our PSTs perceived they learned as they reflected on their teacher professional noticing skills and student thinking about the equal sign. The table tabulates the first response each of the PSTs provided.

As Table 3 illustrates, 80% of our PSTs who worked with 3rd grade students stated that they learned that students have a limited understanding of the equal sign. Moreover, 50% of the PSTs who worked with 8th grade students articulated they learned that students tend to use computational rather than relational strategies to

Table 3
PSTs' perceived learning

Debriefing interview question	Themes	Gr. 3 PSTs (#, %)	Gr. 8 PSTs (#, %)	All PSTs (#, %)
a. Thinking about your interview experience what did you notice about the mathematical thinking of a student?	– Students have a limited understanding of the equal sign	8/10 (80%)	5/22 (23%)	13/32 (41%)
	– Students tend to use computational rather than relational strategies	0/10 (0%)	11/22 (50%)	11/32 (34%)
	– Students think about and solve problems in a variety of ways	2/10 (20%)	6/22 (27%)	8/32 (25%)
b. Thinking about your interview experience, what did you learn about yourself as a teacher?	– Teachers must understand how students' develop relational thinking about equality	3/10 (30%)	11/22 (50%)	14/32 (43%)
	– Teachers must push students to explain their thinking	5/10 (50%)	7/22 (32%)	12/32 (38%)
	– Good questions help to uncover student thinking	1/10 (10%)	5/22 (23%)	6/32 (19%)
c. Did the interview help you to develop a better understanding of relational thinking about equality? Explain	– Yes, strengthened my personal knowledge of relational thinking about equality	3/10 (30%)	9/22 (41%)	12/32 (38%)
	– Yes, strengthened my understanding of student knowledge of relational thinking about equality	7/10 (70%)	13/22 (59%)	20/32 (62%)

solve problems that require thinking about the equal sign. The PSTs who worked with 8th grade students particularly appreciated that the interview experience increased their awareness that teachers ought to understand how students develop relational thinking about equality. Finally, all of our PSTs indicated that the interview experience assisted them in developing their own knowledge (38%) or their understanding of student knowledge (62%) of the equal sign. We interpret these results to indicate that the majority of our PSTs are positioned to actually *notice* student conceptions and misconceptions of the equal sign when they begin their teaching practice.

Implications for Teacher Preparation

Effective teachers use their professional noticing skills to guide their instruction (Jacobs et al., 2010). This is because the ability to attend to student strategies, interpret student understanding, and decide how to respond based on student understanding supports teachers in designing instruction that meets the mathematical needs of their students. Mathematics teacher educators maintain that PSTs can improve their teacher professional noticing skills and should begin to work on developing these skills early in their teacher preparation program (Star & Strickland, 2008; van Es, 2011; van Es & Sherin, 2002).

In an effort to help our PSTs develop their teacher professional noticing skills, we engaged them in conducting diagnostic clinical interviews with elementary or middle school students. We also focused our PSTs' attention on student understanding of the equal sign to increase their awareness of the role that understanding of the equal sign plays in students' transition from early arithmetic to the study of algebra. Using the context of our work we investigated (1) the extent to which PSTs attended to and further explored student understanding of the equal sign, and (2) what PSTs perceived they learned about aspects of their teacher professional noticing skills and student thinking about the equal sign. The results from our first research question provide important insights into the extent to which our PSTs attended to, interpreted, and further explored student thinking about the equal sign while they were conducting diagnostic clinical interviews. While it is valuable to learn what PSTs notice about students when viewing video recordings of classroom instruction, it is especially valuable to learn what PSTs attend to, interpret, and further explore when working face-to-face, in the moment with students.

Asquith and colleagues (2007) and Stephens (2006) have raised concerns that in general, practicing teachers and PSTs are unaware of student thinking about the equal sign and that students who have an insufficient understanding of relational thinking about equality often struggle to make a successful transition from the study of arithmetic to the study of algebra. The results of our study reinforce these concerns and draw additional attention to just how elusive student thinking about the equal sign is. In general, we found that over the course of the two diagnostic clinical interviews, our PSTs predominantly noticed (Score 1) the strategies their students used to solve a task *without* focusing on student thinking about the equal sign. Very few of our PSTs, on the other hand, did not attend to and further explore student thinking (Score 0). Furthermore, more than half of our PSTs (19/32, 59%) increased the number of Score 2's they received between interview #1 and interview #2. However, since the average number of Score 2's the PSTs received on interview #2 was still less than 3 (out of 9), their ability to attend to and further explore student thinking about the equal sign remained marginal. Although we are encouraged that our PSTs noticed and further explored the strategies their students used to solve tasks, we are disappointed that, in general, they further explored something other than their student's thinking about the equal sign. In light of these findings, we have formulated two steps that we can take to further support our PSTs in specifically attending to and further exploring student thinking about the equal sign.

First, we plan to provide an abundance of examples and counter examples of interviewers attending to and further exploring student thinking about the equal sign. We believe this will help our PSTs to better comprehend what this specific noticing skill looks like. As mentioned previously, our PSTs attended to and further explored the strategies their student used to solve a task (Score 1) but fell short in specifically investigating their student's thinking about the equal sign. This finding indicates that our PSTs may have thought they were attending to and further exploring student thinking about the equal sign, when they were in fact exploring the strategies students were using to solve a problem (e.g., counting, derived fact, or traditional algorithm). We hypothesize that our PSTs attended to and further explored (Score 1) the strategies their students used to solve a problem because these strategies were patently observable to our PSTs. For example, the PSTs could literally watch their students use their fingers to count up or down in order to solve a problem or identify errors in computation. What a student does or does not understand about the equal sign, on the other hand, was less observable, and required further exploration via careful questioning. In retrospect, we realize that while we engaged our PSTs in a variety of activities that helped them to learn how students think about the equal sign and the concept of equality (Table 1), we provided fewer opportunities to learn what it means (and does not mean) to attend to and further explore student thinking about the equal sign and equality. We believe that asking our PSTs to examine numerous examples and counter examples will heighten their awareness of what it means to attend to and further explore student thinking about the equal sign.

Second, in our current work with PSTs, we now incorporate the idea of a "missed opportunity," which provides a context for our PSTs to reflect on and analyze their own ability to attend to and further explore student thinking about the equal sign and equality. We define a missed opportunity as an instance in which a PST should have further explored their student's thinking about the equal sign but failed to do so. We now ask our PSTs to transcribe their diagnostic clinical interviews, identify a segment on the interview as a missed opportunity, and propose the actions they could have taken to enhance their teacher professional noticing skills. This provides our PSTs' additional opportunities to mentally rehearse attending to and further exploring student thinking about the equal sign and equality.

Similar to Schack et al. (2013), whose research emphasized noticing children's early arithmetic learning, our work addressed teacher professional noticing in the context of one specific issue in K-8 mathematics, namely, student thinking about the equal sign and equality. The results of our second research question indicated that all of our PSTs perceived that they strengthened either their own knowledge or student knowledge of the equal sign while conducting their diagnostic clinical interviews. These results reinforce our conviction that it is advantageous to pair issues found in the K-8 mathematics curriculum with teacher professional noticing. Several mathematics education researchers have cited student thinking about the equal sign and equality as one such issue (Alibali et al., 2007; Carpenter & Franke, 1999; McNeil & Alibali, 2005a; McNeil & Alibali, 2005b; McNeil et al., 2006). Asking PSTs to specifically investigate student conceptions and misconceptions of the equal sign via one-on-one diagnostic clinical interviews creates a viable approach to prepare PSTs to notice student thinking about the equal sign and equality upon beginning their own practice.

In conclusion, this study provides some insight into how difficult it is for PSTs to notice and further explore student thinking about the equal sign and equality. While our study yielded modest gains in our PSTs' professional noticing skills, we believe our findings can be used to guide the future work and research of the mathematics education community in supporting PSTs' learning how to notice student thinking about the equal sign and equality.

Appendix 1: Diagnostic Clinical Interview Protocol

Task #1 **Gr. 3/Gr. 8:** The arrow points to a symbol. What is the name of that symbol? What does that symbol mean? Can it mean anything else? $5 + 3 = 8$ ↑	**Task #2** **Gr. 3:** What number goes in the box? $5 + 4 = \square + 6$ **Gr. 8:** What value of *a* makes the number sentence true? $55 + 54 = a + 56$	**Task #3** **Gr. 3:** Fill in the box with a number that makes the sentence true: $13–7 = \square - 6$ **Gr. 8:** What value of *a* makes the number sentence true? $130–70 = a - 60$
Task #4 **Gr. 3:** Fill in the box with a number that makes the sentence true: $\square + 4 = 5 + 7$ **Gr. 8:** What value of *a* makes the number sentence true: $a + 34 = 35 + 37$	**Task #5** **Gr. 3:** Fill in the box with a number that makes the sentence true: $\square = 7 + 6$ **Gr. 8:** What value of *a* makes the number sentence true? $a = 700 + 600$	**Task #6** **Gr. 3:** What value would make the number sentence true: $\square = 25 - 12$ **Gr. 8:** What value of *a* makes the number sentence true? $a = 2500–1200$
Task #7 **Gr. 3:** What value would make the number sentence true: $8 + \square = 12$ **Gr. 8:** What value of *a* makes the number sentence true? $8 + a = 12$	**Task #8** **Gr. 3:** What value would make the number sentence true: $12 + 9 = 10 + 8 + \square$ **Gr. 8:** What value of *a* makes the number sentence true? $120 + 90 = 100 + 80 + a$	**Task #9** **Gr. 3/Gr. 8:** Write your own number in each box to make the number sentence true: $\square + \square = \square + \square$

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