Petri Nets and Semilinear Sets (Extended Abstract)

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Abstract. Semilinear sets play a key role in many areas of computer science, in particular, in theoretical computer science, as they are characterizable by Presburger Arithmetic (a decidable theory). The reachability set of a Petri net is not semilinear in general. There are, however, a wide variety of subclasses of Petri nets enjoying semilinear reachability sets, and such results as well as analytical techniques developed around them contribute to important milestones historically in the analysis of Petri nets. In this talk, we first give a brief survey on results related to Petri nets with semilinear reachability sets. We then focus on a technique capable of unifying many existing semilinear Petri nets in a coherent way. The unified strategy also leads to various new semilinearity results for Petri nets. Finally, we shall also briefly touch upon the notion of *almost semilinear sets* which witnesses some recent advances towards the general Petri net reachability problem.

Petri nets (or, equivalently, *vector addition systems*) represent one of the most popular formalisms for specifying, modeling, and analyzing concurrent systems. In spite of their popularity, many interesting problems concerning Petri nets are either undecidable or of very high complexity. For instance, the reachability problem is known to be decidable [13] (see also [6]) and exponential-space-hard [12]. (The reader is referred to [11] for an improved upper bound.) Historically, before the work of [13], a number of attempts were made to investigate the problem for restricted classes of Petri nets, in hope of gaining more insights and developing new tools in order to conquer the general Petri net reachability problem. A common feature of those attempts is that decidability of reachability for those restricted classes of Petri nets was built upon their reachability sets being *semilinear*. As semilinear sets precisely correspond to the those characterized by Presburger Arithmetic (a decidable theory), decidability of the reachability problem follows immediately.

Formally speaking, a *Petri net* (PN, for short) is a 3-tuple (P, T, φ), where P is a finite set of *places*, T is a finite set of *transitions*, and φ is a *flow function* $\varphi : (P \times T) \cup (T \times P) \to N$. A *marking* is a mapping $\mu : P \to N$, specifying a PN's *configuration.* (μ assigns tokens to each place of the PN.) A transition $t \in T$ is *enabled* at a marking μ iff $\forall p \in P$, $\varphi(p, t) \leq \mu(p)$. If a transition t is enabled, it may *fire* by removing $\varphi(p, t)$ tokens from each input place p and

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putting $\varphi(t,p')$ tokens in each output place p'. We then write $\mu \xrightarrow{t} \mu'$, where $\mu'(p) = \mu(p) - \varphi(p,t) + \varphi(t,p)$, $\forall p \in P$. A sequence of transitions $\sigma = t_1...t_n$ is a *firing sequence* from μ_0 iff $\mu_0 \xrightarrow{t_1} \mu_1 \xrightarrow{t_2} \cdots \xrightarrow{t_n} \mu_n$ for some markings μ_1,...,μ_n. (We also write '$\mu_0 \xrightarrow{\sigma} \mu_n$' or '$\mu_0 \xrightarrow{\sigma}$' if μ_n is irrelevant.) Given a PN $\mathcal{P} = (P, T, \varphi)$, its *reachability set* w.r.t. the initial marking μ_0 is $R(\mathcal{P}, \mu_0)$=$\{\mu \mid \mu_0 \xrightarrow{\sigma} \mu,$ for some $\sigma \in T^*\}$. The *reachability relation* of $\mathcal{P}$ is $R(\mathcal{P})$=$\{(\mu_0, \mu) \mid \mu_0 \xrightarrow{\sigma} \mu,$ for some $\sigma \in T^*\}$.

A subset L of N^k is a *linear set* if there exist vectors $v_0, v_1, \ldots, v_t$ in N^k such that $L = \{v \mid v = v_0 + m_1 v_1 + \cdots + m_t v_t,\ m_i \in N\}$. The vectors v_0 (referred to as the *constant vector*) and $v_1, v_2, \ldots, v_t$ (referred to as the *periods*) are called the *generators* of the linear set L. A set $SL \subseteq N^k$ is *semilinear* if it is a finite union of linear sets, i.e., $SL = \bigcup_{1 \le i \le d} \mathcal{L}_i$, where $\mathcal{L}_i$ ($\subseteq N^k$) is a linear set. It is worthy of noting that semilinear sets are exactly those that can be expressed by *Presburger Arithmetic* (i.e., first order theory over natural numbers with addition), which is a decidable theory.

A PN is said to be *semilinear* if has a semilinear reachability set. In addition to the trivial example of finite PNs, the following are notable classes of semilinear PNs, including PNs of dimension 5 [4], conflict-free [7], persistent [7], normal [15], sinkless [15], weakly persistent [14], cyclic [1], communication-free PNs [2,5], and several others (see [16] for more). It is also known that checking whether a PN has a semilinear reachability set is decidable [3]. In view of the above, a natural question to ask is to identity, if at all possible, the key behind the exhibition of semilinear reachability sets for such a wide variety of restricted PN classes, while their restrictions are imposed on the PN model either structurally or behaviorally. We are able to answer the question affirmatively to a certain extent. In what follows, we give a sketch for the idea behind our unified strategy. The idea was originally reported in [17]. As we shall explain later, for each of considered PNs, any reachable marking is witnessed by somewhat of a *canonical computation* which will be elaborated later. Furthermore, such canonical computations can be divided into a finite number of groups, each of which has a finite number of "minimal computations" associated with a finite number of "positive loops." As one might expect, such minimal computations and positive loops exactly correspond to the constant vectors and periods, respectively, of a semilinear set. It is worth pointing out that the implication of our approach is two-fold. First, we are able to explain in a unified way a variety of semilinearity results reported in the literature. Second, perhaps more importantly, our approach yields new results in the following aspects:

(i) new semilinearity results for additional subclasses of PNs,
(ii) unified complexity and decidability results for problems including reachability, model checking, etc.

Given an $\alpha = r_1 \cdots r_{d-1} \in T^*$ and an initial marking μ_0, a computation of the form

$$\pi:\ \mu_0 \xrightarrow{\sigma_0} \mu_1 \xrightarrow{r_1} \bar{\mu}_1 \xrightarrow{\sigma_1} \mu_2 \xrightarrow{r_2} \cdots \xrightarrow{r_{d-1}} \bar{\mu}_{d-1} \xrightarrow{\sigma_{d-1}} \mu_d,$$

where $\mu_i, \bar{\mu_j} \in N^k$, and $\sigma_r \in T^*$ $(0 \le i \le d, 1 \le j \le d-1, 0 \le r \le d)$, is called an *α-computation*. We write $cv(\pi) = (\mu_1, ..., \mu_d)$. Suppose $\delta_i \in T^*, 1 \le i \le d$, are transition sequences such that $\Delta(\delta_i) \ge 0$ and $(\mu_i + \sum_{j=1}^{i-1} \Delta(\delta_j)) \xrightarrow{\delta_i}$, then following the *monotonicity* property of PNs,

$$\pi' : \mu_0 \xrightarrow{\sigma_0 \delta_1} \mu_1' \xrightarrow{r_1} \bar{\mu}_1' \xrightarrow{\sigma_1 \delta_2} \mu_2' \xrightarrow{r_2} \cdots \xrightarrow{r_{d-1}} \bar{\mu}_{d-1}' \xrightarrow{\sigma_{d-1} \delta_d} \mu_d'$$

remains a valid PN computation. In fact, we have $\mu_0 \xrightarrow{\sigma_0(\delta_1)^+ r_1 \sigma_1(\delta_2)^+ r_2 \cdots r_{d-1} \sigma_{d-1}(\delta_d)^+}$, meaning that $\delta_1, ..., \delta_d$ constitute "pumpable loops". In view of the above and if we write $cv(\pi) \overset{(\delta_1, \cdots, \delta_d)}{\Rightarrow} cv(\pi')$, clearly "$\Rightarrow$" is transitive as $v \overset{(\alpha_1, \cdots, \alpha_d)}{\Rightarrow} v'$ and $v' \overset{(\delta_1, \cdots, \delta_d)}{\Rightarrow} v''$ imply $v \overset{(\alpha_1 \delta_1, \cdots, \alpha_d \delta_d)}{\Rightarrow} v''$, where $v, v', v'' \in (N^k)^d$, $k = |P|$.

It turns out that the following properties are satisfied by several interesting subclasses of PNs all of which have semilinear reachability sets. With respect to an $\alpha \in T^d$,

(1) there is a finite set of transition sequences $F \subseteq T^*$ with nonnegative displacements (i.e., $\forall \gamma \in F$, $\Delta(\gamma) \ge 0$) such that if $(\mu_1, \cdots, \mu_d) \overset{(\delta_1, \cdots, \delta_d)}{\Rightarrow} (\mu_1', \cdots, \mu_d')$ in some α-computations, then $\delta_i = \gamma_1^i \cdots \gamma_{h_i}^i$, for some h_i where $\gamma_j^i \in F$ (i.e., δ_i can be decomposed into $\gamma_1^i \cdots \gamma_{h_i}^i$), and
(2) the number of "minimal" α-computations is finite.

Intuitively, (2) ensures the availability of a finite set of constant vectors of a semilinear set, while (1) allows us to construct a finite set of periods based on those $\Delta(\gamma), \gamma \in F$.

A PN $\mathcal{P} = (P, T, \varphi)$ with initial marking μ_0 is said to be *computationally decomposable* (or simply *decomposable*) if every reachable marking $\mu \in R(\mathcal{P}, \mu_0)$ is witnessed by an α-computation $(\alpha \in T^*)$ which meets Conditions (1) and (2) above. $\mathcal{P}$ is called *globally decomposable* if $\mathcal{P}$ is decomposable for every initial marking $\mu_0 \in N^k$. Let $RR_\alpha(\mathcal{P}, \mu_0) = \{cv(\pi) \mid \pi$ is an α-computation from μ_0 for some $\alpha \in T^*\}$, and $RR_\alpha(\mathcal{P}) = \{(\mu_0, cv(\pi)) \mid \pi$ is an α-computation from μ_0 for some $\alpha \in T^*\}$. We are able to show that if a PN is decomposable (resp., globally decomposable) then $RR_\alpha(\mathcal{P}, \mu_0)$ (resp., $RR_\alpha(\mathcal{P})$) is semilinear.

Among various subclasses of PNs, conflict-free, persistent, normal, sinkless, weakly persistent, cyclic, and communication-free PNs can be shown to be decomposable. Furthermore, each of the above classes of PNs also enjoys a nice property that

(3) there exists a finite set $\{\alpha_1, ..., \alpha_r\} \subseteq T^*$ (for some r) such that every reachable marking of the PN is witnessed by an α_i-computation, for some $1 \le i \le r$.

As a result, our unified strategy shows $R(\mathcal{P}, \mu_0)$ of a PN $\mathcal{P}$ with initial marking μ_0 for each of the above subclasses to be semilinear. Furthermore, a stronger result shows that conflict-free and normal PNs are globally decomposable; hence, their reachability relations $R(\mathcal{P})$ are always semilinear.

For semilinear PNs, a deeper question to ask is: *What is the size of its semilinear representation?* An answer to the above question is key to the complexity analysis of various problems concerning such semilinear PNs. To this end, we are able to incorporate another ingredient into our unified strategy, yielding size bounds for the semilinear representations of the reachability sets. Consider a computation $\mu \xrightarrow{\sigma} \mu'$. Suppose $T = \{t_1, ..., t_h\}$. For a transition sequence $\sigma \in T^*$, let $PK(\sigma) = (\#_\sigma(t_1), ..., \#_\sigma(t_h))$ be an h-dimensional vector of nonnegative integers, representing the so-called *Parikh map* of σ. The i-th coordinate denotes the number of occurrences of t_i in σ. In addition to Conditions (1)-(3) above, if the following is also known for a PN:

(4) a function $f(\mu)$ which bounds the size of each of the minimal elements of $ER(\mu) = \{(PK(\sigma), \mu') \mid \mu \xrightarrow{\sigma} \mu'\}$ (i.e., the so-called *extended reachability set*),

then we are able to come up with a bound for the size of the semilinear representation of a PN's reachability set.

Semilinearity for PNs is also related to the concept of the so-called *flatness*. A PN is said to be *flat* if there exist some words $\sigma_1, ..., \sigma_r \in T^*$ such that every reachable marking μ is witnessed by a computation $\mu_0 \xrightarrow{\sigma} \mu$ with $\sigma \in \sigma_1^* \cdots \sigma_r^*$, i.e., it has a witnessing sequence of transitions belonging to a bounded language. It is not hard to see the reachability set of a flat PN to be semilinear, and as shown in [10], a variety of known PN classes are indeed flat. In a recent article [9], flatness is shown to be not only sufficient but also necessary for a PN to be semilinear. We shall compare flat PNs with the aforementioned decomposable PNs.

Finally, we also briefly touch upon recent advances for the general PN reachability problem in which the notion of *(almost) semilinearity* is essential in yielding a simpler decidability proof [8] in comparison with that of [6,13].

References

1. Araki, T., Kasami, T.: Decidability problems on the strong connectivity of Petri net reachability sets. Theor. Comput. Sci. **4**, 99–119 (1977)
2. Esparza, J.: Petri nets, commutative grammars and basic parallel processes. Fundamenta Informaticae **30**, 24–41 (1997)
3. Hauschildt, D.: Semilinearity of the Reachability Set is Decidable for Petri Nets. Technical report FBI-HH-B-146/90, University of Hamburg (1990)
4. Hopcroft, J., Pansiot, J.: On the reachability problem for 5-dimensional vector addition systems. Theor. Comput. Sci. **8**, 135–159 (1979)
5. Huynh, D.: Commutative grammars: the complexity of uniform word problems. Inf. Control **57**, 21–39 (1983)
6. Kosaraju, R.: Decidability of reachability in vector addition systems. In: 14th ACM Symposium on Theory of Computing, pp. 267–280 (1982)
7. Landweber, L., Robertson, E.: Properties of conflict-free and persistent Petri nets. JACM **25**, 352–364 (1978)
8. Leroux, J.: The general vector addition system reachability problem by presburger inductive invariants. In: LICS 2009, pp. 4–13. IEEE Computer Society (2009)

9. Leroux, J.: Presburger vector addition systems. In: LICS 2013, pp. 23–32. IEEE Computer Society (2013)
10. Leroux, J., Sutre, G.: Flat counter automata almost everywhere!. In: Peled, D.A., Tsay, Y.-K. (eds.) ATVA 2005. LNCS, vol. 3707, pp. 489–503. Springer, Heidelberg (2005). doi:10.1007/11562948_36
11. Leroux, J., Schmitz, S.: Demystifying reachability in vector addition systems. In: LICS 2015, pp. 56–67. IEEE Computer Society (2015)
12. Lipton, R.: The Reachability Problem Requires Exponential Space. Technical report 62, Yale University, Dept. of CS, January 1976
13. Mayr, E.: An algorithm for the general Petri net reachability problem. In: 13th ACM Symposium on Theory of Computing, pp. 238–246 (1981)
14. Yamasaki, H.: On weak persistency of Petri nets. Inf. Process. Lett. **13**, 94–97 (1981)
15. Yamasaki, H.: Normal Petri nets. Theor. Comput. Sci. **31**, 307–315 (1984)
16. Yen, H.: Introduction to Petri net theory. In: Esik, Z., Martin-Vide, C., Mitrana, V. (eds.) Recent Advances in Formal Languages and Applications. Studies in Computational Intelligence, vol. 25, pp. 343–373. Springer, Heidelberg (2006)
17. Yen, H.: Path decomposition and semilinearity of Petri nets. Int. J. Found. Comput. Sci. **20**(4), 581–596 (2009)