Analysis of the Insurance Portfolio with an Embedded Catastrophe Bond in a Case of Uncertain Parameter of the Insurer's Share

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Abstract In this paper, a behavior of an insurer's portfolio, which consists of two layers: a classical risk process and a special financial instrument, which is known as a catastrophe bond, is analyzed. Especially, a probability of a ruin for such a portfolio is estimated using the Monte Carlo simulations. A special attention is given to a problem of an insurer's share in a whole insurance market, which associates values of the catastrophic losses with values of the claims for the considered insurer. It is also an important source of a systematic risk. Because such a share is often an uncertain parameter, then a fuzzy number is used to model its value. This approach incorporates the experts' knowledge. Based on the simulations, observed differences between a crisp and a fuzzy case are described in a more detailed way.

Keywords Risk process · Insurance portfolio · Catastrophe bond · Monte carlo simulations · Fuzzy numbers

1 Introduction

One of the main problems in an insurance industry is to evaluate a probability of an insurer's ruin. If, apart from a classical risk process, other kinds of financial and insurance instruments are taken into account, then this problem is an even more complex one and it requires a solution based on simulations. A catastrophe bond (abbreviated as a cat bond) is an example of such a special, financial instrument used by the insurers nowadays. A catastrophe bond (see, e.g., [5, 8–10]) is a part of a process, which is known as a securitization of losses, i.e. it is used to "package" the catastrophic losses (i.e., losses with extreme high values, but rather infrequent, comparing to the "standard" losses, usually considered by the insurers) into tradable

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financial assets, in the form of so-called catastrophe derivatives. A payoff received by a cat bond holder depends on an additional random variable, i.e. a triggering point. The triggering point is usually related to a cumulated value of the catastrophic losses (caused by hurricanes, tsunamis, floods etc.). If the losses, defined in a description of the cat bond, surpasses some given limit, then the payoff from this instrument is lowered, comparing to a contrary case. Then these additional funds are transferred to the insurer, which issued this cat bond.

In this paper, using the Monte Carlo simulations, we analyze a behavior of an insurer's portfolio, which consists of a classical risk process and an additional catastrophe bond, issued by the insurer. A probability of the ruin for such a portfolio and some other statistical measures are estimated. This analysis may be seen as an improvement and a reformulation of the problem stated in [10]. But in this paper, instead of a completely crisp approach, a parameter, which describes a share of the insurer in a whole insurance market, is given as a fuzzy number. It allows us to incorporate the experts' knowledge and to analyze possible differences in estimated probabilities of a ruin, if the mentioned parameter is, in some way, uncertain. This new approach requires also a completely different way of applying the Monte Carlo simulations.

This paper is organized as follows. In Sect. 2, models of processes of the aggregated losses and of an interest rate are introduced. Section 3 is devoted to a description of the insurer's portfolio, which consists of two layers: the classical risk process and the catastrophe bond. Simulated outputs for such a portfolio are numerically analyzed in Sect. 4 for two cases: if a share parameter is given as a crisp and as a fuzzy number. Then some significant differences in the obtained results, especially the probabilities of the ruin and some other statistical measures, are discussed.

2 Applied Models

Traditionally, in the insurance industry, a process N_t^* of the aggregated losses caused by the natural catastrophes is given by

$$N_t^* = \sum_{i=1}^{N_t} U_i, \tag{1}$$

where number of losses $N_t \geq 0$ is modeled by some stochastic process (e.g., a homogeneous Poisson process—HPP, or a non-homogeneous Poisson process—NHPP) and values of single claims are given by an iid random sequence $U_1, U_2, \ldots$ In our setting, we assume that N_t is given by the NHPP with a cyclic intensity function

$$\lambda_{NHPP}(t) = a + b2\pi \sin(2\pi(t - c)), \tag{2}$$

where $a = 30.875, b = 1.684, c = 0.3396$. These parameters are estimated in [3] using the data from the United States, provided by the Property Claim Services (PCS) of the ISO (Insurance Service Office Inc.). Based on the same source, the value of the single loss U_i can be modeled by the lognormal distribution with the parameters $\mu_{LN} = 17.357, \sigma_{LN} = 1.7643$.

We are interested in a present or a future value of the considered cash flow (see also [10] for a more detailed discussion), so a relevant interest rate model should be also introduced. This model reflects the value of money in time. In this paper, we apply the one-factor Vasicek model, given by

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t, \tag{3}$$

where the parameters $\kappa = 0.1179, \theta = 0.086565, \sigma^2 = 0.0004$ are fitted in [2] for the U.S. Treasury bill yield data.

As for a payment function $f\left(N_T^*\right)$ for a holder of the considered cat bond, a piecewise linear function is applied. This form of the payment function is introduced and discussed in details in [5, 8–10]. Then, we have

$$f\left(N_T^*\right) = Fv\left(1 - \sum_{i=1}^{n} \frac{\min\left(N_T^*, K_i\right) - \min\left(N_T^*, K_{i-1}\right)}{K_i - K_{i-1}} w_i\right), \tag{4}$$

where Fv is a face value of the cat bond, $w_1, \ldots, w_n > 0$ are payoff decreases (satisfying the requirement $\sum_{i=1}^{n} w_i \leq 1$), and $0 \leq K_0 \leq K_1 \leq \ldots \leq K_n$ are the triggering points. In the considered setting, we set $Fv = 1$ (one monetary unit assumption), and

$$K_0 = Q_{NHPP-LN}^{loss}(0.75), K_1 = Q_{NHPP-LN}^{loss}(0.9), \tag{5}$$

where $Q_{NHPP-LN}^{loss}(x)$ is xth quantile of a cumulated value of losses, if the number of losses is given by the NHPP and the value of the single loss is modeled by the lognormal distribution (see also [5, 9, 10] for additional details). The payoff decrease is given by $w_1 = 1$ and we apply one year time horizon, so $T = 1$. It means that, if after one year, the cumulated value of losses surpasses K_1, the bond holder receives nothing.

In order to evaluate the price of the cat bond determined by the previously mentioned parameters, the approach, considered in a more detailed way in [5, 8–10], is then applied. The cat bond pricing problem is a complex one, and requires analytical formulas, introduced in [5, 8–10], and the additional Monte Carlo simulations. Then, the price of the cat bond, which is considered here, is estimated as $I_{cb} = 0.809896$. This value will be used further on during an analysis of the insurer's portfolio.

3 Model of the Insurer's Portfolio

In this section the model of the insurer's portfolio, which consists of a few layers (i.e. an additional financial instrument) is discussed in a more detailed way.

3.1 *Risk Reserve Process*

Usually, in insurance mathematics, a risk reserve process R_t is defined as a model of evolution of the financial reserves of an insurer depending on time t, and is given by

$$R_t = u + pt - C_t^*, \tag{6}$$

where u is an initial reserve of the insurer, p is a rate of premiums paid by the insureds per unit time and C_t^* is a claim process, which is equal to $C_t^* = \sum_{i=1}^{N_t} C_i$, so that $C_1, C_2 \ldots$ are iid random values of the claims.

In the considered setting, the process of the number of claims (the same as the number of losses) is modeled by the NHPP with the intensity function (2). Therefore, similarly to the classical approach, the premium is also a constant function but directly related to $\lambda_{NHPP}(t)$, so that for the fixed moment T we have

$$p(T) = \left(1 + v_p\right) EC_i \int_0^T \lambda_{NHPP}(s) ds, \tag{7}$$

where v_p is a safety loading (a security loading) of the insurer. Usually, this loading is about 10–20 %.

In insurance mathematics, we are interested in an evaluation of a probability of an ultimate ruin (i.e. a ruin with an infinite time horizon)

$$\psi(u) = Pr(inf_{t \geq 0} R_t < 0) \tag{8}$$

or a probability of a ruin before time T (i.e. a ruin with a finite time horizon)

$$\psi(u, T) = Pr\left(inf_{t \in [0,T]} R_t < 0\right). \tag{9}$$

Similar probabilities are estimated further on, using the Monte Carlo approach (see also [10] for additional details).

3.2 Additional Layer—Catastrophe Bond

We assume, that the insurer also issues a catastrophe bond, which forms an additional layer in his portfolio. Then, apart from the risk process (6), the cash flows related to this cat bond should be taken into account. The hedger (e.g. the insurer) pays an insurance premium p_{cb} in exchange for a coverage, when the triggering point (i.e. some catastrophic event) occurs. The investors purchase an insurance-linked security for cash. The above mentioned premium and cash flows are usually directed to a SPV (Special Purpose Vehicle), which issues the catastrophe bonds. The investors hold the issued assets whose coupons and/or a principal depend on the occurrence of the triggering point. If such a catastrophic event occurs during the specified period, the SPV compensates the insurer and the cash flows for the investors are changed. Usually, these flows are lowered, i.e. there is full or partial forgiveness of the repayment of principal and/or interest. However, if the triggering point does not occur, the investors usually receive a full payment.

Let us assume, that the mentioned insurance premium p_{cb} is proportional to both a part α_{cb} of the whole price of the single catastrophic bond I_{cb}, and to a number of the issued bonds n_{cb}, so that $p_{cb} = \alpha_{cb} n_{cb} I_{cb}$.

Taking into account the previously mentioned value of money in time, the future value of the cash flows for the insurer's portfolio, if the catastrophe bond was issued, is given by

$$R_T = FV_T(u - p_{cb}) + FV_T(p(T)) - FV_T\left(C_T^*\right) + n_{cb} f_{cb}^i\left(N_T^*\right), \tag{10}$$

where $FV_T(.)$ denotes a future value of the cash flow, and $f_{cb}^i\left(N_T^*\right)$ is a payment function of the single cat bond for the insurer. Such a function is, in some way, "opposite" to the payment function $f\left(N_T^*\right)$ for the policy holder (given by (4) in our case).

3.3 Claims Versus Losses

As presented in Sects. 2 and 3.1, the process of the aggregated losses N_t^* and the process of the cumulated claims C_t^* are driven by the same process of the number of the catastrophic events N_t in our setting. However, further on, we assume that the value of the single claim C_i is only some deterministic part of the related loss U_i, i.e. $C_i = \alpha_{claim} U_i$. Such an approach models the situation, when there is no monopoly on the market, so the considered insurer has only some share of a whole insurance market. It also leads to a systematic risk, because a hedging instrument (i.e., the catastrophe bond) is issued for a process (in this case—the process of the aggregated losses) which is not exactly the same as a process which should be hedged (i.e. the process of aggregated claims).

In the following, we assume, that an exact value of the share parameter α_{claim} is unknown. Such an assumption reflects the situation, when, e.g. the level of the share of the insurer in the whole market is not exactly stated, this level varies depending on a region of a possible natural catastrophe or its source (e.g. earthquake or tsunami) etc. Therefore, the parameter α_{claim} will be given as a fuzzy number. It means, that this uncertain value is related to the experts' knowledge.

4 Simulations

Now we turn to analysis of the insurer's portfolio, which consists of the "classical" risk process with addition of the issued catastrophe bond. Based on the Monte Carlo simulations, an analysis of behavior of the process (10) will be conducted in two main cases: if the share parameter α_{claim} is given as a crisp, real value and if such a parameter is modeled by a triangular fuzzy number.

4.1 Analysis of the Crisp Case

Let us assume, that for the first layer (the risk process), we have $u = Q^{claim}_{NHPP-LN}(0.25)$, i.e. the initial reserve of the insurer is equal to 0.2 quantile of the cumulated value of the claims driven by the process C^*_t, and $\alpha_{claim} = 0.5$. Then, the share of such an insurer in the whole insurance market is equal to 50 %, so the half of the value of each catastrophic loss is turned into the claim for this insurer. We also set $v_p = 0.1$, and the process of the losses and the Vasicek interest rate model are described by the parameters introduced in Sect. 2.

The second layer consists of the catastrophe bond, which is discussed in Sects. 2 and 3.2. We assume that the number of the issued cat bonds is related to a difference between quantiles of the cumulated value of the claims, namely it is equal to $Q^{claim}_{NHPP-LN}(0.9) - Q^{claim}_{NHPP-LN}(0.75)$. It reflects the situation, when the cat bond is used as an additional source of "possible" funds, if the value of the claims is too high and there is a high probability of the insurer's ruin. Therefore, the cat bond may be seen as an alternative instrument to a reinsurance contract, which is not, in many cases, an adequate source of funds for the insurer (see also [10]). Then, if $N^*_T > K_1$ (i.e. the cumulated value of the losses surpasses the highest triggering point), the income for the insurer from the issued n_{cb} cat bonds is equal to the previously mentioned difference of the quantiles. We assume that $\alpha_{cb} = 0.1$, so this value is similar to v_p. Then we get $n_{cb} = 1481$.

Using the Monte Carlo simulations, a probability of a final ruin (i.e. $P(R_T < 0)$ for $T = 1$) is estimated as 5.3677 %, and a probability of an earlier ruin (i.e. $P(R_t < 0)$ for some $t \in (0, t])$ is equal to 18.4852 %. It means that a probability of a ruin which does not lead to the final ruin is estimated as 13.1175 %. This rather

Table 1 Statistical measures of the final value of the insurer's portfolio and the payments for the insurer in the crisp case

	Final value of the portfolio	Payments for the insurer
Minimum	–262500	0.00297548
Median	1987.47	1095.07
Mean	1739.47	957.306
Maximum	5007.92	1481
Stand. deviation	1868.66	535.415

high value is related to a necessity of an early payment for the issued bonds, even if these instruments are not used afterwards. A probability that the catastrophe bond is used (i.e. at least one triggering point is achieved) is equal to 24.8908 %. Also other statistical measures of the final value of the insurer's portfolio R_T, which are important for the practitioners, can be directly found using simulations (see Table 1 for some examples). Similar characteristics can be obtained also for non-zero values of f^i_{cb}, i.e. the non-zero payments from the catastrophe bond directed to the insurer (see Table 1).

4.2 Analysis of the Fuzzy Case

As it was previously mentioned, the share parameter α_{claim} is now given in a fuzzy form, as a fuzzy triangular number. A triangular fuzzy number $\tilde{a}$ (see, e.g., [1, 8, 9, 11] for necessary details and definitions of fuzzy sets and fuzzy numbers) is a fuzzy number with a piecewise linear membership function of the form

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x-a}{b-a}, \mathit{if}\ x \in [a,b] \\ \frac{x-c}{b-a}, \mathit{if}\ x \in [b,c] \\ 0, \mathit{otherwise} \end{cases}, \tag{11}$$

so such a number can be denoted further on as $[a,b,c]$, where a support of this number is given by the interval $[a,b]$. Using a fuzzy number, the experts' knowledge can be easily incorporated into the considered problem. Then, instead of a precise statement "the share of the insurer in the whole market is equal to 50 %" as in the previous, crisp case considered in Sect. 4.1, the experts can say "the share is about 50 %". This second statement models imprecise information, which is based on other sources than a statistical inference. Such an approach is also useful for the practitioners, if necessary information is sparse or even unavailable (see, e.g., [4] for fuzzy applications in statistics). Therefore, during our analysis we assume that $\tilde{\alpha}_{claim} = [0.4, 0.5, 0.6]$, so the support of this fuzzy number is from 40 % up to 60 % percent of the share of the whole insurance market, and for $\alpha = 1$ we have the same value, as in the case considered in Sect. 4.1. Then, for this example the expert can say “the share of the insurer is about 50 % plus/minus 10 %”.

In order to estimate the probabilities and the statistical measures similar to the previous, crisp case, the Monte Carlo simulations are also performed. Because now the evaluated output is a fuzzy number $\tilde{a}$, then it is approximated by α-level sets $\tilde{a}[\alpha]$ for a whole range of possible values of α. For the given α, if $f(x)$ is an increasing function of x, then a left end point of the considered output $\tilde{f}_L[\alpha]$ is approximated using a left end point of an α-level set of the fuzzy value $\tilde{x}$, i.e. $\widetilde{x_L}[\alpha]$. In the same way, $\widetilde{x_R}[\alpha]$ (a right end point of $\tilde{x}$) is used to find $\tilde{f}_R[\alpha]$ (a right end point of the output). This idea is related to the Zadeh’s extension principle (see, e.g., [11]) and it is also used in other areas of financial mathematics (see, e.g., [6–9] for applications in derivatives pricing and decision making problems).

During the Monte Carlo simulations, the parameter α is changed from some starting value $\alpha_0 > 0$ up to an upper bound $\alpha_1 \in (\alpha_0, 1]$ with an increment $\Delta\alpha > 0$. After the evaluation of the left and right end points of the different α-level sets of the considered function of the output, i.e. $\left[\tilde{f}_L[\alpha], \tilde{f}_R[\alpha]\right]$, the obtained intervals are put on one another, so they form an approximation of a final fuzzy outcome $\tilde{f}$.

In this way, we get fuzzy approximations of the probabilities of the final ruin (Fig. 1, left hand side) and the earlier ruin (Fig. 1, right hand side). They form L-R numbers with rather wide supports. For example, for $\alpha = 0$ and the probability of the final ruin, we get the interval $[3.8526, 6.7769]$. Then, relative differences (if we are comparing to the probability for $\alpha = 1$) are equal to –28.2236 % (the left end point of the mentioned interval) and 26.258 % (the right end point). It means that the probabilities of the ruin significantly varies, if the share of the insurer in the whole market is not given in a completely exact way (i.e. it is not stated as a crisp, real number). Then, a wrong assumption about the share can be dangerous for an estimation of the probability of the insurer’s ruin and it can lead to serious error of the whole procedure.

In the same manner, based on the output from the Monte Carlo simulations, fuzzy approximations of the statistical measures of the final value of the insurer’s

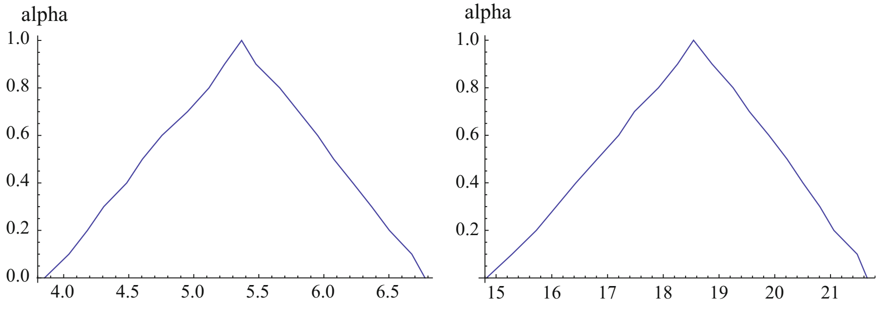


Fig. 1 Fuzzy approximations of the probability of the final probability ruin (*left*) and the earlier ruin (*right*)

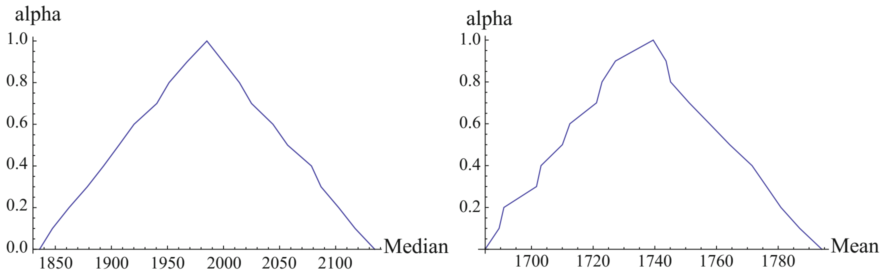
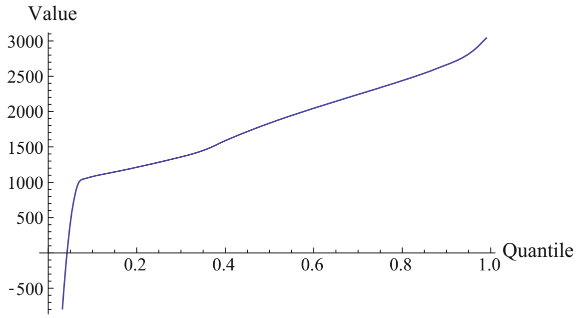


Fig. 2 Fuzzy approximations of the measures of the insurer's portfolio: median (*left*) and mean (*right*)

portfolio can be found. As relevant examples, a median and a mean are plotted in Fig. 2 (graphs on the left and right hand side, respectively). Once again, they form L-R numbers. It seems, that median is a better idea in measuring the value of the portfolio, because the obtained fuzzy approximation is much smoother. The median is also more resistant to outliers, which are very common in the generated data. This reasoning can be supported by a quantiles plot for the final value of the insurer's portfolio. An example of such a graph for $\alpha_{claim} = 0.4$ can be found in Fig. 3. As it is easily seen, the quantiles of lower ranks (especially, the quantiles of ranks < 0.1) have extremely low values, which indicates a problem with the outliers. A very similar situation exists for other values of the parameter α_{claim}. Moreover, the relative differences for median (if the relevant values of the median for $\alpha = 0$ and $\alpha = 1$ are, as previously, compared) are equal to –7.51487 % (for the left end point of the support) and 7.54409 % (for the right end point). These differences are significant, but a possible error is less dangerous than during the estimation of the probabilities.

Apart from the final ruin, the moments of the earlier ruin can be also analyzed using the output generated during the simulations. For example, the times of such events can be compared for $\alpha_{claim} = 0.4$ (which corresponds to the left end point of the α-level set for $\alpha = 0$, see Fig. 4, squares) and $\alpha_{claim} = 0.6$ (which corresponds to the right end point of the α-level set for $\alpha = 0$, see Fig. 4, circles). It is easily

Fig. 3 Graph of the quantiles of the final value of the insurer's portfolio for $\alpha_{claim} = 0.4$



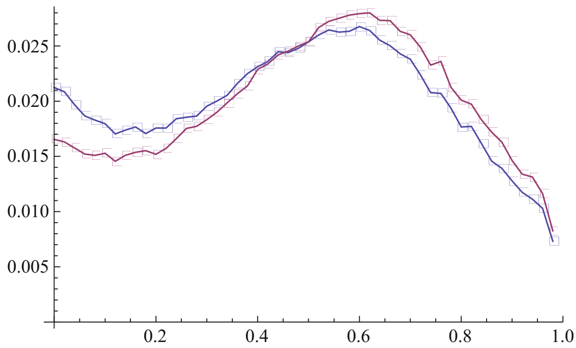


Fig. 4 Comparison of envelopes of histograms for the moments of the earlier ruin for $\alpha_{claim} = 0.4$ (squares) and $\alpha_{claim} = 0.6$ (circles)

seen, that the envelopes of the histograms of these times have similar shapes. However, up to about $t = 0.55$, a relevant frequency of the earlier ruins is higher for $\alpha_{claim} = 0.6$, afterwards the situation is quite opposite. Of course, as it was previously mentioned, the probability of the earlier ruin itself is higher for the higher value of α_{claim}.

5 Conclusions

In this paper, the behavior of the insurer's portfolio is analyzed. Such a portfolio consists of the two layers: the classical risk process and the special financial derivative, known as the catastrophe bond. A special interest is paid to the future value of this portfolio, i.e. its value for some final moment T. Therefore, the model of the interest rate, known as the one factor Vasicek model, is applied. Then, based on the Monte Carlo simulations, various probabilities and statistical measures for the portfolio are estimated. For both the model of the cumulated value of the catastrophic losses, and the interest rate model, the parameters from the real life data are applied. During the simulations, the influence of the share of the insurer in the whole insurance market on the characteristics of the portfolio is analyzed. Two main cases are considered: if the share parameters is given as a crisp value and if it is given as a fuzzy, triangular number. For these two cases, the examples of simulated output are provided and discussed in a more detailed way. The fuzzification of the share parameter introduced in this paper allows us to overcome the problem of uncertainty of data.

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