

Chapter 5

Finite Volume Method— Convection-Diffusion Problems

Abstract This chapter is an extension of the previous one on diffusion-convection. The treatment is again using one dimensional finite volume method and closed form solutions.

Keywords Convection-diffusion • Finite volume method

Fluid flow plays significant role in transporting heat and diffusing the same in the flow (also to the surrounding solid—heat transfer in both the liquid and solid simultaneously, called conjugate heat transfer; not considered here). The steady convection-diffusion problem (neglecting time dependent terms) can be obtained from the transport equation (4.7) for a general property ϕ

$$\text{div}(\rho\phi\mathbf{u}) = \text{div}(\Gamma \text{grad } \phi) + S_\phi \quad (5.1)$$

5.1 Steady State One-Dimensional Convection and Diffusion

Let us consider the case with no source terms and confining to one-dimensional problems as in Chap. 4, Eq. (5.1) is

$$\frac{d}{dx}(\rho u \phi) = \frac{d}{dx}\left(\Gamma \frac{d\phi}{dx}\right) \quad (5.2)$$

The continuity equation from (3.8) should also be satisfied, which for the one-dimensional steady flow reduces to

$$\frac{d(\rho u)}{dx} = 0 \quad (5.3)$$

Fig. 5.1 One dimensional control volume around node P

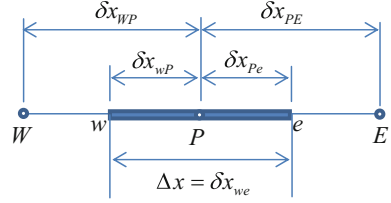


Figure 4.2 gives the one-dimensional control volume around node P , with the neighboring nodes W to the west and E to the east as shown in Fig. 5.1.

The right-hand side of Eq. (5.2) represents diffusive terms and the left-hand side the convective terms. Upon integration over the control volume, Eqs. (5.2) and (5.3) give

$$(\rho u A \phi)_e - (\rho u A \phi)_w = \left(\Gamma A \frac{d\phi}{dx} \right)_e - \left(\Gamma A \frac{d\phi}{dx} \right)_w \quad (5.4)$$

$$(\rho u A)_e - (\rho u A)_w = 0 \quad (5.5)$$

In the above ρu is the convective mass flux per unit area F and $\frac{\Gamma}{\delta x}$ is the diffusion conductance D at the faces of the cell, i.e.,

$$\begin{aligned} F &= \rho u \\ D &= \frac{\Gamma}{\delta x} \end{aligned} \quad (5.6)$$

The values of F and D at the west face w and east face e are

$$\begin{aligned} F_w &= (\rho u)_w \\ D_w &= \left(\frac{\Gamma}{\delta x} \right)_w \\ F_e &= (\rho u)_e \\ D_e &= \left(\frac{\Gamma}{\delta x} \right)_e \end{aligned} \quad (5.7)$$

For a uniform cell along the length, $A_e = A_w = A$, Eqs. (5.4) and (5.5) reduce to

$$\begin{aligned} (\rho u \phi)_e - (\rho u \phi)_w &= \left(\Gamma \frac{d\phi}{dx} \right)_e - \left(\Gamma \frac{d\phi}{dx} \right)_w \Rightarrow \\ F_e \phi_e - F_w \phi_w &= D_e (\phi_E - \phi_P) - D_w (\phi_P - \phi_W) \end{aligned} \quad (5.8)$$

$$\begin{aligned}(\rho u)_e - (\rho u)_w = 0 \Rightarrow \\ F_e - F_w = 0\end{aligned}\quad (5.9)$$

The diffusion problem in Eq. (4.9) for a uniform bar is given by an ordinary second order differential equation and is solvable to determine the two constants of integration by using the boundary values. In the case of convection it is a coupled problem with the velocity u and the transport property ϕ as given in Eqs. (5.2) and (5.3). If we are able to determine u in some manner we still need to calculate the transport property ϕ_e and ϕ_w at the faces e and w , as required in (5.8). The simplest thing is to adapt a linear variation of the transport property between W and E of the cell in Fig. 5.1. We can then write the transport property values at e and w in terms of the nodal values at W , P and E as

$$\begin{aligned}\phi_e &= \frac{1}{2}(\phi_P + \phi_E) \\ \phi_w &= \frac{1}{2}(\phi_W + \phi_P)\end{aligned}\quad (5.10)$$

Equation (5.8) is now written for the nodal values of the transport property as

$$\frac{1}{2}F_e(\phi_P + \phi_E) - \frac{1}{2}F_w(\phi_W + \phi_P) = D_e(\phi_E - \phi_P) - D_w(\phi_P - \phi_W) \quad (5.11)$$

Rearranging the above in terms of nodal values of transport property

$$\begin{aligned}\phi_P \left[(D_w - \frac{1}{2}F_w) + (D_e + \frac{1}{2}F_e) \right] &= \phi_w (D_w + \frac{1}{2}F_w) + \phi_E (D_e - \frac{1}{2}F_e) \Rightarrow \\ \phi_P \left[(D_w + \frac{1}{2}F_w) + (D_e - \frac{1}{2}F_e) + (F_e - F_w) \right] &= \phi_w (D_w + \frac{1}{2}F_w) + \phi_E (D_e - \frac{1}{2}F_e)\end{aligned}\quad (5.12)$$

Rewriting

$$\begin{aligned}a_P \phi_P &= a_W \phi_W + a_E \phi_E \\ \text{where} \\ a_W &= (D_w + \frac{1}{2}F_w) \\ a_E &= (D_e - \frac{1}{2}F_e) \\ a_P &= a_W + a_E + (F_e - F_w)\end{aligned}\quad (5.13)$$

The difference between the pure diffusion problem given in (4.17) and the above convection-diffusion problem of (5.13) is the presence of additional terms containing the convective mass flux per unit area $F = \rho u$.

5.1.1 Exact Solution for Convection-Diffusion Problem

To satisfy Eq. (5.3), we notice u is constant, therefore Eq. (5.2) becomes

$$\begin{aligned} \frac{d}{dx}(\rho u \phi) &= \frac{d}{dx}(\Gamma \frac{d\phi}{dx}) \Rightarrow \\ \frac{d^2\phi}{dx^2} - \frac{\rho u}{\Gamma} \frac{d\phi}{dx} &= 0 \end{aligned} \quad (5.14)$$

The auxiliary equation is

$$\left(D^2 - \frac{\rho u}{\Gamma} D\right)\phi = 0 \quad (5.15)$$

Then

$$\begin{aligned} (m^2 - \frac{\rho u}{\Gamma} m) &= 0 \Rightarrow \\ m = 0 \quad \text{and} \quad m &= \frac{\rho u}{\Gamma} \end{aligned} \quad (5.16)$$

$$\therefore \phi = A + B e^{\frac{\rho u}{\Gamma} x} \quad (5.17)$$

Let ϕ_0 and ϕ_L be prescribed at $x = 0$ and $x = L$, then

$$\begin{aligned} A + B &= \phi_0 \\ A + B e^{\frac{\rho u}{\Gamma} L} &= \phi_L \end{aligned} \quad (5.18)$$

i.e.,

$$\begin{aligned} A + (\phi_0 - A) e^{\frac{\rho u}{\Gamma} L} &= \phi_L \Rightarrow A(1 - e^{\frac{\rho u}{\Gamma} L}) = \phi_L - \phi_0 e^{\frac{\rho u}{\Gamma} L} \\ \therefore A &= \frac{\phi_L - \phi_0 e^{\frac{\rho u}{\Gamma} L}}{(1 - e^{\frac{\rho u}{\Gamma} L})} \end{aligned} \quad (5.19)$$

$$B = \phi_0 - \frac{\phi_L - \phi_0 e^{\frac{\rho u}{\Gamma} L}}{(1 - e^{\frac{\rho u}{\Gamma} L})} \Rightarrow \frac{\phi_0 - \phi_L}{(1 - e^{\frac{\rho u}{\Gamma} L})} \quad (5.20)$$

Therefore

$$\begin{aligned} \phi &= \frac{\phi_L - \phi_0 e^{\frac{\rho u}{\Gamma} L}}{(1 - e^{\frac{\rho u}{\Gamma} L})} + \frac{\phi_0 - \phi_L}{(1 - e^{\frac{\rho u}{\Gamma} L})} e^{\frac{\rho u}{\Gamma} x} \Rightarrow \\ \phi(1 - e^{\frac{\rho u}{\Gamma} L}) &= \phi_0(-e^{\frac{\rho u}{\Gamma} L} + e^{\frac{\rho u}{\Gamma} x}) + \phi_L(1 - e^{\frac{\rho u}{\Gamma} x}) \end{aligned} \quad (5.21)$$

$$\begin{aligned} \phi(1 - e^{\frac{\rho u}{\Gamma} L}) &= \phi_0(-e^{\frac{\rho u}{\Gamma} L} - 1 + 1 + e^{\frac{\rho u}{\Gamma} x}) + \phi_L(1 - e^{\frac{\rho u}{\Gamma} x}) \\ \frac{(\phi - \phi_0)}{(\phi_L - \phi_0)} &= \frac{(e^{\frac{\rho u}{\Gamma} x} - 1)}{(e^{\frac{\rho u}{\Gamma} L} - 1)} \end{aligned} \quad (5.22)$$

5.1.2 Finite Volume Method for Convection-Diffusion Problem

Consider the one dimensional domain in Fig. 5.2 80 cm long in which the property ϕ is transported. $\phi_0 = 1$ at $x = 0$ and $\phi_L = 0$ at $x = 0.8$ m. $\rho = 1 \text{ kg/m}^3$, $u = 0.1 \text{ m/s}$ and $\Gamma = 0.1 \text{ kg/m/s}$; i.e., $\frac{\rho u}{\Gamma} = 1 \text{ m}^{-1}$.

The domain is discretized into 4 cells as shown with 4 nodes 1, 2, 3 and 4 with $\delta x = 0.2 \text{ m}$. From Eq. (5.7)

$$F = \rho u = 0.1$$

$$D = \left(\frac{\Gamma}{\delta x}\right) = \frac{0.1}{0.2} = 0.5$$

First, we notice that Eq. (5.13) is valid for mid nodes, 2 and 3. Therefore

$$a_{W2,3} = \left(D + \frac{1}{2}F\right) = 0.55$$

$$a_{E2,3} = \left(D - \frac{1}{2}F\right) = 0.45 \tag{5.23}$$

$$a_{P2,3} = a_W + a_E + (F_e - F_w) = 1.0$$

Therefore for cells 2 and 3

$$a_P \phi_P = a_W \phi_W + a_E \phi_E$$

$$\phi_2 = 0.55\phi_1 + 0.45\phi_3 \tag{5.24}$$

$$\phi_3 = 0.55\phi_2 + 0.45\phi_4$$

For end nodes 1 and 4, we have to develop appropriate relations. For cell 1, $\phi_w = \phi_A = 1$, we make an approximation in the convective flux term with $D_A = \frac{2\Gamma}{\delta x} = 2D$ at this boundary in Eq. (5.11) as

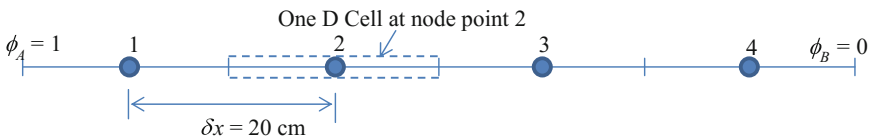


Fig. 5.2 One dimensional convection-diffusion problem

$$\begin{aligned}
\frac{1}{2}F_e(\phi_P + \phi_E) - \frac{1}{2}F_w(\phi_W + \phi_P) &= D_e(\phi_E - \phi_P) - D_w(\phi_P - \phi_W) \Rightarrow \\
\frac{1}{2}F_e(\phi_P + \phi_E) - F_A\phi_A &= D_e(\phi_E - \phi_P) - D_A(\phi_P - \phi_A) \\
0.05(\phi_1 + \phi_2) - 0.1 &= 0.5(\phi_2 - \phi_1) - (\phi_1 - 1) = 0.5\phi_2 - 1.5\phi_1 + 1 \\
1.55\phi_1 &= 0.45\phi_2 + 1.1
\end{aligned} \tag{5.25}$$

Similarly for cell 4 with $D_B = \frac{2\Gamma}{\delta x} = 2D$

$$\begin{aligned}
\frac{1}{2}F_e(\phi_P + \phi_E) - \frac{1}{2}F_w(\phi_W + \phi_P) &= D_e(\phi_E - \phi_P) - D_w(\phi_P - \phi_W) \Rightarrow \\
F_B\phi_B - \frac{1}{2}F_w(\phi_W + \phi_P) &= D_B(\phi_B - \phi_P) - D_w(\phi_P - \phi_W) \\
0 - 0.05(\phi_3 + \phi_4) &= -\phi_4 - 0.5(\phi_4 - \phi_3) = -1.5\phi_4 + 0.5\phi_3 \\
1.45\phi_4 &= 0.55\phi_3 \Rightarrow \phi_4 = 0.37931\phi_3
\end{aligned} \tag{5.26}$$

Using the above result in the third equation of (5.24)

$$\phi_3 = 0.6632\phi_2$$

Substituting the above in the second equation of (5.24)

$$\phi_2 = 0.784\phi_1$$

Now substitute the above in (5.25)

$$1.55\phi_1 = 0.45\phi_2 + 1.1 \Rightarrow \phi_1 = 0.9188$$

Then

$$\begin{aligned}
\phi_1 &= 0.9188 \\
\phi_2 &= 0.7203 \\
\phi_3 &= 0.4777 \\
\phi_4 &= 0.1812
\end{aligned} \tag{5.27}$$

The exact solution in (5.22) is

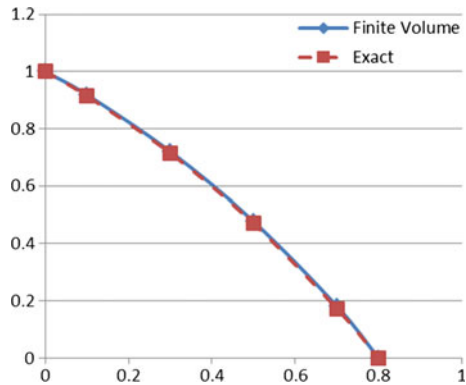
$$\phi = 1 - \frac{(e^x - 1)}{(e^{0.8} - 1)} \Rightarrow 1.815966 - 0.815966e^x \tag{5.28}$$

Finite Volume Method solution Eq. (5.27) is compared with the Exact Solution in (5.28) in Table 5.1 and Fig. 5.3. We note that the finite volume method agrees closely with the exact solution.

Table 5.1 Comparison of finite volume solution with exact values

Distance m	FV solution	Exact
0	1	1
0.1	0.9188	0.9141841
0.3	0.7203	0.7145271
0.5	0.4777	0.4706655
0.7	0.1812	0.1728123
0.8	0	0

Fig. 5.3 Comparison of finite volume method with exact solution



Exercises 5

- 5.1 Consider Eqs. (4.5) and (4.6) to derive a single general transport equation.
- 5.2 Simplify the general transport equation for the steady state case without any viscous effects and without any source.
- 5.3 Explain convection and diffusion by imagining a ϕ to represent some dye made up of little particles suspended in the fluid. Discuss the convective term as transport of this ϕ , say temperature, due to the fluid motion and derive the corresponding governing Eq.
- 5.4 Derive a numerical solution using one dimensional finite volume method for the problem of diffusion (temperature) in a flow. You can also give a closed for solution for a problem with just one cell.
- 5.5 In the diffusion equation if ϕ represents a dye that is transported while diffusing show that the coefficient of diffusion Γ is $\frac{\text{kg}}{\text{m}\cdot\text{s}}$. A 1 m long pipe carries water at 10 cm/s. One unit of dye at $x = 0$ that vanishes at the end of 1 m. The coefficient of diffusivity can be taken as 0.1 kg/m²/s. Make a plot of the transported dye as a function of length using three cells of the pipe.
- 5.6 Compare the result obtained in 5.5 from an exact solution.