Structural Properties of Generalized Exchanged Hypercubes

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Abstract It has been shown that, when a linear number of vertices are removed from a Generalized Exchanged Hypercube (GEH), a generalized version of the interesting exchanged hypercube, its surviving graph consists of a large connected component and smaller component(s) containing altogether a rather limited number of vertices. In this chapter, we further apply the above connectivity result to derive several fault-tolerance related structural parameters for GEH, including its restricted connectivity, cyclic vertex-connectivity, component connectivity, and its conditional diagnosability in terms of the comparison diagnosis model.

1 Introduction

It is certainly unavoidable that some of the processing nodes within a multi-processor system become faulty, leading to a faulty system. To have an effective system to work with, we are naturally interested in the fault tolerance properties of these systems, seeking answers to such questions as how many faulty nodes will disrupt such a system, or disconnect its associated graph in graph theoretical terms; and how disrupted the *surviving system(graph)* will become when a certain number of nodes and/or links become faulty, thus effectively removed. For example, will the surviving graph completely break apart, or are most of its nodes still connected in a component? We might also be interested in knowing more about the details, e.g.,

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the relationship between the maximum number of the faulty nodes and the minimum number of components in such a surviving graph.

A related issue is that, once processing nodes become faulty, could we know exactly which ones are faulty so that the fault-free status of the system can be restored? The number of such detectable faulty nodes in a system certainly depends on its topology, the restriction placed on such a faulty set, as well as the modeling assumptions, and the maximum number of detectable faulty nodes in such a system is called its *diagnosability*. One major modeling approach to this regard is called the *comparison diagnosis model* [14, 26, 27, 33], where each processing node performs a diagnosis by sending the same input to each and every pair of its distinct neighbors, and then comparing their responses. Based on such comparison results made by all the nodes, the faulty status of the whole system can be determined. Various efficient algorithms to detect such faulty sets have also been proposed, e.g., [33, 35, 47].

To address the unlikelihood that all the neighbors of a certain node in such a system will fail at the same time, the notion of the *conditional diagnosability* of a graph G was introduced in [18], defined as the maximum number of detectable faulty nodes in G, assuming that no faulty set contains all the neighbors of any node in G. Such a faulty set is henceforth referred to as a *conditional faulty set*. This more realistic notion leads to an improved measurement of the fault tolerance capability of network structures and is thus of great interest [1, 4, 5, 7, 18].

Answers to the aforementioned fault tolerance related questions are often expressed in terms of connectivity related properties of a graph underlying such a surviving structure [1, 13, 15, 24, 41–43]. In particular, a general connectivity result has been demonstrated in [9] for the *generalized exchanged hypercube* structure that, when a linear number of vertices are removed from such a structure, the surviving graph is either connected or consists of a large connected component and small components containing a small number of vertices. The results as reported in this chapter can be seen as a companion work of [9]: We apply the above general result to further derive for this topological structure several fault tolerance related measurements, including its (i) restricted connectivity, i.e., the size of a minimum vertex cut such that the degree of every vertex in the surviving graph will have a guaranteed lower bound; (ii) cyclic vertex-connectivity, i.e., the size of a minimum vertex cut such that at least two components in the surviving graph contain a cycle; (iii) component connectivity, i.e., the size of a minimum vertex cut whose removal leads to multiple components in its surviving graph; as well as (iv) conditional diagnosability in terms of the comparison diagnostic model.

The rest of this chapter proceeds as follows: We briefly review the exchanged hypercube [3, 22] and the class of generalized exchanged hypercubes [9] in the next section; our exposition is based on [9]. We state the general result obtained in [9] in Sect. 3. We then apply the aforementioned general connectivity property as associated with the generalized exchanged hypercube to derive various parameters that generalize the concept of connectivity, namely, restricted connectivity and cyclic vertex-connectivity in Sect. 4, component connectivity and conditional diagnosability in Sects. 5 and 6, respectively. We conclude this chapter with some final remarks in Sect. 7.

2 The Exchanged Hypercube and its Generalization

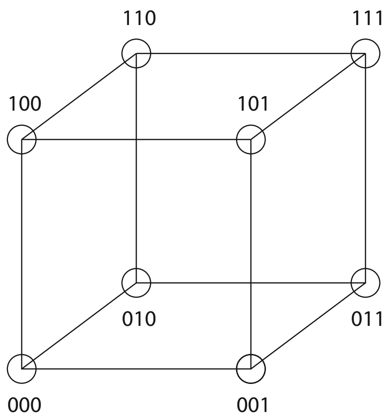
The n-dimensional hypercube [13], often referred to as the n-cube and denoted by Q_n, is perhaps one of the most studied and utilized interconnection structures, as it possesses many desirable properties such as vertex and edge symmetry, high connectivity, and small diameter thus lower communication cost, as well as the existence of a simple routing algorithm. More specifically, an n-cube has 2^n nodes 0, 1, 2, ..., $2^n - 1$ where (u, v) is an edge (arc) if u's and v's binary representations differ in exactly one position, i.e., $u = u_{n-1}u_{n-2}\cdots u_{i+1}u_iu_{i-1}\cdots u_1u_0$ and $v = u_{n-1}u_{n-2}\cdots u_{i+1}\overline{u_i}u_{i-1}\cdots u_1u_0$, $0 \le i \le n-1$. Figure 1 shows a 3-cube.

Several hypercube variants have since been suggested, including augmented cubes, crossed cubes, enhanced cubes, folded hypercube, Möbius cubes, and twisted cubes.

The exchanged hypercube was proposed in [3, 22] as another edge removal variant of the hypercube, where about half of the edges are systematically removed [3, Theorem 2]. With such a significantly reduced complexity, besides addressing a scaling issue as associated with the hypercube structure, the exchanged hypercube still manages to inherit several attractive properties of the hypercube such as incremental expandability [3], bipancyclicity [23], connectivity and super connectivity [25], and existence of a fault tolerant routing algorithm [22]. With essentially the same diameter and eccentricity, but reduced maximum degree and Wiener index [17], the bounds of its domination number, as well its surface area and average distance, have also been established in [16, 17], respectively. We will further study some of its fault tolerance related connectivity properties in this chapter.

The exchanged hypercube, denoted by $EH(s, t)$, where $s, t \ge 1$, is defined as an undirected graph (V, E), where V is the collection of all the binary strings of length $s + t + 1$. Hence, $|V(EH(s, t))| = 2^{s+t+1}$. A vertex u of an exchanged hypercube $EH(s, t)$ is denoted by $A(u)B(u)C(u)$, where $A(u) = a_{s-1}\cdots a_0$, $B(u) = b_{t-1}\cdots b_0$, and $C(u) = c$. $C(u)$ is sometimes referred to as the C *bit* of

Fig. 1 A 3-Cube



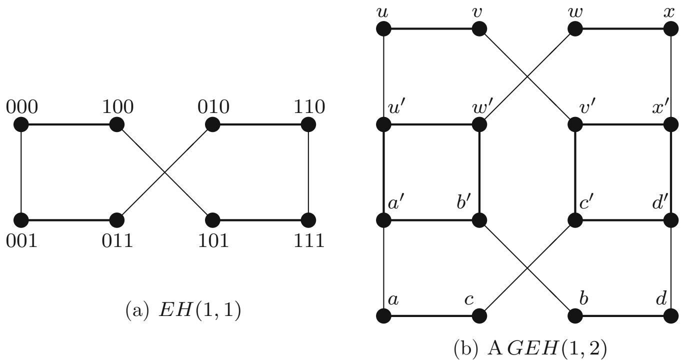


Fig. 2 Simple exchanged hypercubes

u henceforth. Let $u, v \in V(EH(s,t))$, $(u,v) \in E$ if and only if it falls into one of the following three mutually exclusive cases: E_1: $C(u) \neq C(v)$, but $A(u) = A(v)$ and $B(u) = B(v)$; E_2: $C(u) = C(v) = 0$, $A(u)$ and $A(v)$ differ in exactly one bit in position $p \in [0, s)$, while $B(u) = B(v)$; and E_3: $C(u) = C(v) = 1$, $A(u) = A(v)$, but $B(u)$ and $B(v)$ differ in exactly one bit in position $p \in [0, t)$.

Figure 2a shows $EH(1,1)$, where (000, 001), (000, 100), and (001, 011) are examples of E_1, E_2, and E_3 edges, respectively.

Each collection of 2^s vertices, sharing the same B segment and 0 as their common C bit, forms a Q_s, referred to as a *Class-0 cluster*, via the associated E_2 edges. Clearly, there are a total of 2^t such hypercubes in $EH(s,t)$. Similarly, each collection of 2^t vertices, sharing the same A segment and 1 as their common C bit, forms a Q_t, a *Class-1 cluster*, via the associated E_3 edges. There are a total of 2^s such hypercubes in $EH(s,t)$. We thus refer to both E_2 and E_3 edges collectively as *cube edges.*

Class-0 clusters and Class-1 clusters are referred to as *clusters of opposite class* of each other. Clearly, each vertex u in a cluster is adjacent to a unique vertex in a cluster of opposite class via an E_1 edge, denoted by u' in the rest of this chapter. By definition, $A(u)B(u) = A(u')B(u')$ but $C(u') = \overline{C(u)}$, namely, the complement of the *C bit* of u. Since these E_1 edges connect vertices belonging to different clusters, we refer to them as *cross edges.*

A key structural property of the exchanged hypercube is that, let u, v be two vertices of the same cluster $\mathcal{C}$ in $EH(s,t)$, $s, t \geq 1$, then u' and v' belong to two different clusters of a class opposite to that of $\mathcal{C}$, via cross edges. Here the set of cross edges are chosen specifically. One may wonder the role of the specific set of cross edges chosen among all possible sets. In terms of shortest path, it plays an important role in ensuring the resulting graph has a small diameter. However, in terms of connectivity type properties, there may be no differences among different set of cross edges. Indeed, in the recursive definition of the hypercube, one can replace the specific matching between the two smaller hypercubes by any perfect matching. This leads to a wider class of networks, which leads to the even more general matching composition networks. We can apply the same type of generalization to the exchanged hypercube, that is, although the existence of a perfect matching between vertices via the cross edges is structurally essential, the specifics of such a matching, i.e., details such as which vertices are matched with each other, is not. We will show that such generalized networks also have strong connectivity type results. Of course, the proof will be more involved due to the generality.

We thus generalized this class of exchanged hypercubes in [9] as follows: A generalized exchanged hypercube, denoted by $GEH(s, t, f), s, t \geq 1$, consists of two classes of hypercubes: One class contains 2^t s-cubes, each labeled with the shared B segment, and referred to collectively as the *Class-0 clusters*; and the other contains 2^s t-cubes, each labeled with the shared A segment, and referred to collectively as the *Class-1 clusters*. Class-0 and Class-1 clusters will be referred to as clusters of *opposite class* of each other, *same class* otherwise, and collectively as *clusters*, $C_i, i \in [0, 2^s + 2^t)$, when their categories are irrelevant to the issue. When $s = t$, we simply refer to one of the classes of hypercubes as Class-0 clusters, and the other as Class-1 clusters. Set E_h, the *cube edges,* collects all the usual $(s + t)2^{s+t-1}$ edges in the hypercubes of both classes.

The function f is a bijection between vertices of Class-0 clusters and those of Class-1 clusters such that, for u, v, two vertices of the same cluster, $f(u)$ and $f(v)$ belong to two different clusters, as observed in the aforementioned structural property of the exchanged hypercubes. We naturally refer to such an edge $(u, f(u))$ as a *cross edge.* Set E_c collects all the 2^{s+t} cross edges in between the clusters of opposite classes. Such a bijection f ensures the existence, but ignores the specifics, of a perfect matching between vertices of Class-0 clusters and those in the Class-1 clusters.

By its definition, in a generalized exchanged hypercube, all of the 2^s distinct vertices in a specific Class-0 cluster, a Q_s, out of 2^t of them, are adjacent, via cross edges, to 2^s vertices, each of which is located in a unique Class-1 cluster, a Q_t; and all of the 2^t distinct vertices in a specific Class-1 cluster, out of 2^s of them, are adjacent to 2^t vertices, each of which is located in a unique Class-0 cluster. As an example, Fig. 2b shows one example of $GEH(1, 2)$, where there are four Class-0 clusters, (u, v), (w, x), (a, c), and (b, d), each being an edge, technically a Q_1; and two Class-1 clusters, (a', b', u', w') and (c', d', v', x'), both being Q_2. Each of the two vertices in an edge is adjacent to a unique vertex in a Q_2, and each of the four vertices in a Q_2 is adjacent to a unique vertex in an edge.

The above observation motivates us to further define a labeled *structure graph*, $G(s, t, \omega)$, *associated with* $GEH(s, t, f)$, where $V(G(s, t, \omega))$ collects all the clusters in $GEH(s, t, f)$. Each vertex in this structure graph, sometimes also referred to as a *cluster,* corresponding to a Class-0 cluster, is adjacent to 2^s vertices, each corresponding to a Class-1 cluster; and conversely, each vertex corresponding to a Class-1 cluster, is adjacent to 2^t vertices, each corresponding to a Class-0 cluster. Each edge, e, in $G(S, t, \omega)$, corresponding to a cross edge $(u, f(u))$ in $GEH(s, t, f)$, is labeled with $\omega(e)$ $(= (u, f(u)))$. It is clear that such a structure graph, $G(s, t, \omega)$, is isomorphic to a complete bipartite graph $K_{2^s,2^t}$. When f and/or ω are irrelevant to the issue in the discussion, we may choose to exclude them in the notation. In particular, by $GEH(s, t)$, we mean $GEH(s, t, f)$ for some appropriate perfect matching f;

and, by $G(s, t)$, we mean a structure graph $G(s, t, \omega)$ associated with a generalized exchanged hypercube $GEH(s, t, f)$, where ω is induced by f.

An exchanged hypercube is certainly a generalized exchanged hypercube, where the cross edges are specified with E_1; while the class of generalized exchanged hypercubes is strictly more general than that of the exchanged hypercubes since we have a lot more freedom in choosing the cross edges between the clusters of opposite classes: Any perfect matching between the vertices of clusters of opposite classes will do.

Obviously, some topological properties (such as the distance between a specific pair of vertices) may vary wildly depending on the specifics of such a matching, but others do not. For example, as shown in [22, Theorem 1], $EH(s, t)$ is isomorphic to $EH(t, s)$. This property also holds for a generalized exchanged hypercube since, in the above definition of $GEH(s, t)$, the roles as played by Class-0 clusters and Class-1 clusters are symmetric to each other. As a result, we assume $1 \leq s \leq t$, when addressing $GEH(s, t)$, in the rest of this chapter. Furthermore, as we will expose in the rest of this chapter, several other structure properties, and fault tolerance related measurements, are also independent of this perfect matching between the vertices of opposite clusters. Such an observation reveals the naturalness and robustness of the generalized exchanged hypercube.

3 A Connectivity Result Associated with Linearly Many Faults

Let G be a graph, and let $S \subset V(G)$, we use $N_G(S)$ to refer to the *open neighbors of all the vertices of S in G*, excluding those in S. (We often omit the subscript G from this notation, and others, when the context is clear.) Such a graph G is *r-regular* if the degree of every vertex in $V(G)$ is r.

The *vertex connectivity* of a non-complete graph G, denoted by $\kappa(G)$, refers to the minimum size of a vertex cut F, $F \subset V(G)$, such that the surviving graph $G - F$ is disconnected, which is obtained from G by deleting all the vertices in F from G, together with edges incident to at least one vertex in F. By convention, the vertex connectivity of a complete graph K_n is $n - 1$. On the other hand, the *edge connectivity* of a graph G, denoted by $\kappa'(G)$, refers to the minimum size of an edge cut D, $D \subset E(G)$, such that the surviving graph $G - D$ is disconnected, which is obtained from G by removing all the edges as contained in D. Let $\delta(G)$ be the minimum degree among those of all the vertices in a graph G, clearly, $\delta(GEH(s, t)) = s + 1$, when $1 \leq s \leq t$. Indeed the following well-known result relates the vertex connectivity, the edge connectivity, and the minimum degree of a *simple graph G*, where there is at most one edge between any two vertices.

Lemma 1 *[37, Theorem 4.1.9] Let G be a simple graph, then* $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.

Naturally, it is desirable for a graph G to have the property that $\kappa(G) = \delta$. A non-complete graph G with at least $r + 1$ vertices is *r-connected* if deleting any set of at most $r - 1$ vertices results in a connected graph. A complete graph with $r + 1$ vertices, denoted by K_{r+1}, is k-connected for all $k \leq r$. An r-regular graph is *maximally connected* if it is r-connected. A maximally connected r-regular graph is also *tightly super-connected* if, for every $F \subset V(G)$ with $|F| = r$, the graph $G - F$ is either connected or it consists of two components, one being a singleton. Clearly, in a tightly super-connected graph, all the neighbors of the aforementioned singleton fall in such a set F. When used as an interconnection network, an r-regular tightly super-connected structure is more preferable than an r-regular maximally connected graph, as when up to r vertices become faulty, the surviving graph of such a tightly super-connected graph, except one vertex, is still connected, thus functioning. We observe that a maximally connected graph does not need to be tightly super-connected. For example, in a given $K_{3,3}$ $(= (V_1, V_2, E))$, $K_{3,3} - V_1 = V_2$, i.e., three singletons, thus $K_{3,3}$ is not tightly super-connected, although it is maximally connected. On the other hand, it is well known that Q_n is tightly super-connected [41, Theorem 3.3], thus maximally connected.

Noticing that the generalized exchanged hypercube $GEH(s, t)$, $1 \leq s \leq t$, is not regular, except when $s = t$, we thus slightly generalize the above notions as follows: We say that G is *δ-maximally connected* if, for all $F \subset V(G)$, $|F| < \delta(G)$, $G - F$ is connected; and G is *δ-tightly super-connected* if it is δ-maximally connected, and, for all $F \subset V(G)$, $|F| \leq \delta(G)$, $G - F$ is either connected or it consists of one large (connected) component plus one singleton. Clearly, $K_{m,n}$, $1 \leq m \leq n$, is δ-maximally connected [37, Example 4.1.2], although it is not δ-tightly super-connected, while Q_n is. For $GEH(s, t)$ to be useful as an interconnection network, it should be δ-tightly super-connected. In fact, an even stronger statement is true.

Theorem 1 *Let* $s \in [1, t]$, *and let* $k \in [1, s]$, *then*

1. *there is* $F \subset V(GEH(s, t))$, $|F| = ks - \frac{k(k-1)}{2} + 1$, *such that* $GEH(s, t) - F$ *contains a component of size* k*; and*
2. *for all* $F \subset V(GEH(s, t))$, $|F| \leq ks - \frac{k(k-1)}{2}$, $GEH(s, t) - F$ *is either connected or it consists of a large component and small components containing at most* $k - 1$ *vertices.*

The proof of Theorem 1 is given in [9]. (We note that the proof for the case of $s = 3$ for Part 2 was omitted due to space constraint. In the appendix, we give a proof for this case.) For example, if we set $k = 1$ in Part 2 of Theorem 1, we have that, for all F, $|F| \leq s$, $GEH(s, t) - F$ is connected, that is, it is maximally connected. On the other hand, if we set $k = 1$ in Part 1 of Theorem 1, we have that for some F, $|F| = s + 1$, $GEH(s, t) - F$ contains a singleton. Furthermore, if we then set $k = 2$ in Part 2 of Theorem 1, we have that, when $|F| \leq 2s - 1$, $GEH(s, t) - F$ is either connected or it contains a large component plus a singleton.

The following result is immediate by Theorem 1, and will be made use of in the next section, when we address the component connectivity of $GEH(s, t)$.

Corollary 1 *Let* $F \subset V(GEH(s,t)), s \in [1,t]$. *If* $GEH(s,t) - F$ *consists of a large component and other components that contain at least* k $(\in [1,s])$ *vertices, then*

$$|F| \geq ks - \frac{(k-1)k}{2} + 1.$$

On the other hand, if we set k to 3, where $k \in [1,s]$, in Part 2 of Theorem 1, the following result plays a critical role when we derive the conditional diagnosability in Sect. 6.

Corollary 2 *Let* $F \subset V(GEH(s,t)), s \in [3,t]$. *If* $|F| \leq 3s - 3$, *then* $GEH(s,t) - F$ *is either connected or it consists of a large component and small components that contain at most two vertices altogether.*

4 The Restricted and Cyclic Vertex-Connectivity

Given a non-complete graph $G(V,E)$, $F \subset V$ is a g-disconnecting set of G if $G - F$ is disconnected and every vertex in $G - F$ has degree at least g (≥ 0). The restricted connectivity of order g of G, denoted as $\kappa_g(G)$, is defined as the size of a minimum g-disconnecting set of G [10, 11].

While $\kappa_0(G)$ coincides with the traditional vertex connectivity $\kappa(G)$, $\kappa_g(G)$, $g \geq 1$, is often used to characterize other fault tolerance properties, such as the g-good-neighbor conditional diagnosability, of various network structures, including the hypercube [30, 31, 39], the m-ary n-dimensional hypercube [38, 45]. In particular, the following general result is derived in [20, Theorem 3.3].

Theorem 2 *For* $1 \leq s \leq t$, *and* $g \in [0,s]$, $\kappa_g(EH(s,t)) = (s-g+1)2^g$.

We now initiate the study of this measurement for $GEH(s,t)$.

Theorem 3 *Let* $3 \leq s \leq t$, $\kappa_1(GEH(s,t)) = 2s$.

Proof Let $k = 2$ in Theorem 1, we have that if $|F| \leq 2s - 1$, $GEH(s,t) - F$ is either connected or it has two components, one of which is a singleton. Thus $\kappa_1(GEH(s,t)) \geq 2s$.

Let u and v be two adjacent vertices in a Class-0 cluster in $GEH(s,t)$. Clearly, $|N(\{u,v\})| = 2s$ as $GEH(s,t)$ is triangle-free by definition. Now let $k = 3$ in Theorem 1, we have that if $|F| \leq 3s - 3$, $GEH(s,t) - F$ has a large component and small components with at most two vertices in total. (This includes the case when $GEH(s,t) - F$ is connected.) Since $2s \leq 3s - 3$, when $s \geq 3$, $GEH(s,t) - N(\{u,v\})$ has two components, one of which is a K_2, i.e., (u,v), while none of the vertices in the large component is isolated, thus each having a degree at least 1. Hence, $\kappa_1(GEH(s,t)) = 2s$. □

We comment that $\kappa_1(G)$ is referred to as the super connectivity of G in [25, 39], i.e., the survival graph contains no isolated vertex when such a minimum vertex cut is removed. Theorem 3 immediately leads to the super connectivity of $EH(s,t)$, $3 \le s \le t$, one of the main results in [25].

The proof for the following observation is straightforward.

Lemma 2 *Let $n \ge 4$ and let C_4 be a 4-cycle, then the degree of every vertex in $Q_n - N(C_4)$ is at least 2.*

We are now ready to prove the following result.

Theorem 4 $\kappa_2(GEH(s,t)) = 4s - 4$, $s \in [6, t]$.

Proof Let $k = 4$ in Theorem 1, we have that, if $|F| \le 4s - 6$, $GEH(s,t) - F$ has a large component and small components with at most three vertices in total. Since $GEH(s,t)$ is triangle-free, the three vertices in small components cannot form a triangle. Thus $\kappa_2(GEH(s,t)) \ge 4s - 5$. We now claim that this number is at least $4s - 4$. Suppose the size of a minimum 2-disconnecting set of $GEH(s,t)$ is $4s - 5$ and let S be such a set. Let $k = 5$ in Theorem 1, we have that, if $|F| \le 5s - 10$, $GEH(s,t) - F$ has a large component and small components with at most four vertices in total. Since $4s - 5 \le 5s - 10$, for $s \ge 5$, the statement holds for S. Furthermore, as S is a 2-disconnecting set, and the graph is triangle-free, the small component of $GEH(s,t) - S$ must contain exactly four vertices, which form a 4-cycle. To isolate this 4-cycle, we need to delete at least $4(s-1)$ $(= 4s - 4)$ vertices, a contradiction.

To show that $4s - 4$ suffices, let A be the vertex-set of a 4-cycle in a Class-0 cluster $\mathcal{C}$ of $GEH(s,t)$. It is clear that $|N(A)| = 4s - 4$. Now apply Theorem 1 with $k = 5$ again, we can conclude that $GEH(s,t) - N(A)$ contains a large component and small components with at most four vertices, as $4s - 4 \le 5s - 10$, when $s \ge 6$. Since the large component contains at least $2^{s+t+1} - 4s$ vertices, and $2^{s+t+1} - 4s \ge 2^{2s+1} - 4s > 4$, $s \ge 6$, we conclude that the surviving graph contains one large component and the prescribed 4-cycle. We claim that every vertex u in this large component of $GEH(s,t) - N(A)$ is of degree at least 2. If u is a vertex of $\mathcal{C}$, then it has degree at least 2 by Lemma 2; otherwise, the degree of u in $GEH(s,t) - N(A)$ is at least the degree of u in $GEH(s,t)$-1, thus at least 2. Therefore $N(A)$ is a 2-disconnecting set. □

It is clear that both Theorems 3 and 4 agree with Theorem 2 when setting g to 1, and 2, respectively.

Let G be a graph, we refer to F $(\subset V(G))$ a *cyclic vertex-cut* of G if $G - F$ is disconnected and at least two components in $G - F$ contain a cycle. The *cyclic vertex-connectivity* of a graph G is then defined as the size of a minimum cyclic vertex-cut in G. This notion was originally introduced to study the Four Color problem [36], and has since been applied to study other graph theory problems, including that of the Integer Flow Conjectures [46]. Recently, the cyclic vertex-connectivity results of several interconnection networks have also been reported in literature, e.g., [6, 44].

By following the arguments as we made in proving Theorem 4, we can similarly show that the cyclic vertex-connectivity of $GEH(s,t)$ is $4s - 4$.

It is pointed out in [16, pp. 159] that DC_n, the dual-cube-like network [1], which generalizes the dual-cube structure [21], is isomorphic to $EH(n-1, n-1)$, a special case of $GEH(n-1, n-1)$. Hence, we immediately have the following result:

Corollary 3 *For* $n \geq 3$, $\kappa_1(DC_n) = 2n - 2$. *For* $n \geq 7$, $\kappa_2(DC_n) = 4n - 8$, *which is also the value of its cyclic vertex-connectivity.*

5 The Component Connectivity

Component connectivity of a graph characterizes the size of a minimum vertex cut whose removal leaves its surviving graph in a certain number of components. This notion, as introduced in [2, 32] and further addressed in, e.g., [19, 28, 29], is to overcome the deficiency of the ordinary notion of vertex connectivity when used to measure the fault tolerance of interconnection networks. Indeed, with two graphs of same vertex connectivity, when a corresponding vertex cut is removed, their respective surviving graphs could have quite different number of components. For example, as pointed out in [19], the vertex connectivity of both $K_{1,n}$ and the path graph $P_{n+1}, n \geq 2$, is 1, but, when a cut vertex is removed, the surviving graph of $K_{1,n}$ consists of n singletons, while that of the path graph consists of just two components.

It is worth pointing out that there exists yet another alternative generalization of this vertex connectivity concept as proposed in [12]. The *k-tree connectivity* of a graph G is defined as the minimum k such that internally disjoint Steiner trees exist on all the k-subsets of $V(G)$. For a connection between the component connectivity and this latter tree based generalization, readers are referred to [19] and the references cited within.

Let G be a non-complete graph, an *r-component cut* of G, $r \geq 2$, refers to a set of vertices whose removal results in a surviving graph with at least r components. The *r-component connectivity,* or simply r-connectivity [19], denoted by $\bar{\kappa}_r(G)$, of G refers to the size of a minimum r-component cut of G (If there is no r-component cut of G, we simply define $\bar{\kappa}_r(G)$ to be ∞.). Clearly, $\bar{\kappa}_2(G)$ is just the usual vertex connectivity of G. It is also easy to see, by definition, that $\bar{\kappa}_m(G) \leq \bar{\kappa}_{m+1}(G), m \geq 2$.

As mentioned earlier, $\bar{\kappa}_n(K_{1,n}) = 1$. For P_{2n+1}, if we remove every other vertex, n vertices in total, the surviving graph consists of $n+1$ singletons. Thus, $\bar{\kappa}_{n+1}(P_{2n+1}) \leq n$. Clearly, $\bar{\kappa}_2(P_3) \geq 1$, and an inductive argument shows that $\bar{\kappa}_{n+1}(P_{2n+1}) \geq n$. Hence, $\bar{\kappa}_{n+1}(P_{2n+1}) = n$. The same idea also applies to P_{2n}, except that removing the last vertex will not increase the number of singletons. Thus, we only need to remove the first $n-1$ vertices and the surviving graph ends up with $n-1$ singletons and one component of size 2, n components altogether. We thus have $\bar{\kappa}_n(P_{2n}) = n - 1$.

The above analysis shows that, although both $K_{1,n}$ and P_{n+1} share the same vertex connectivity , it just takes out one cut vertex to break a $K_{1,n}$ into n pieces, but it has to take out about half of the vertices to achieve the same effect in a path graph that contains twice as many vertices. As a result, we may conclude that a path graph is more resilient as compared with a star graph from this perspective. Hence, this measure of component connectivity characterizes more faithfully the degree of an interconnection network to stay intact, when a number of processing nodes become faulty.

The following result on the $(r+1)$-component connectivity of the hypercube Q_n, $n \geq 2$, has been derived in [15, Theorem 2.1].

Theorem 5 *For all* $n \geq 3$, $k \in [1, n]$, $\bar{\kappa}_{r+1}(Q_n) = rn - \frac{r(r+1)}{2} + 1$.

We now derive $\bar{\kappa}_{r+1}(GEG(s,t))$, the component connectivity of a generalized exchanged hypercube $GEH(s,t)$, $1 \leq s \leq t$, for $r \in [2, s]$.

Theorem 6 *Let* $1 \leq s \leq t$. *For* $r \in [1, s]$, $\bar{\kappa}_{r+1}(GEH(s,t)) = rs - \frac{r(r-1)}{2} + 1$.

Proof Let u be a vertex in C_0, a Class-0 cluster of $GEH(s,t)$, $1 \leq s \leq t$, S be a collection of r $(\in [1, s+1])$ neighbors of u in $GEH(s,t)$, and let u' be the unique neighbor of u in a Class-1 cluster, C_1', via a cross edge. Depending on whether $u' \in S$, we can construct the open neighbor set $N_r(S)$ of S, where $|S| = r$, in two ways, referred to as $N_r^1(S)$ $(N_r^2(S))$, respectively.

- Assume that $u' \notin S$. Then, for those r $(\in [1, s])$ neighbors of u in C_1, each has $s+1$ neighbors in $GEH(s,t)$, a total of $r(s+1)$ vertices. But, for each of them, u is counted once as its neighbor, although it should be counted just once in $N(S)$. Moreover, every common neighbor shared by any two of these neighbors of u is counted twice, while each of them should also be counted only once in $N(S)$. As a result, we have

$$|N_r^1(S)| = r(s+1) - (r-1) - \binom{r}{2} = rs - \binom{r}{2} + 1$$

as a hypercube has no $K_{2,3}$ as a subgraph. We notice that $N_r^1(S)$ is only defined when $r \in [1, s]$. As an example, in $GEH(1, 2)$, as shown in Fig. 2b, we have that $s = 1, t = 2$, thus $r = 1$. If we pick $S = \{v\}$, then, $N_1^1(S) = \{u, v'\}$. On the other hand, the above result gives us $|N_1^1(S_1)| = 2$.
- Alternatively, when $u' \in S$, then each of the $r-1$ $(\in [0, s])$ neighbors of u in C_0 has s neighbors, plus another one via a cross edge; and u' has $t+1$ neighbors in C_0', a total of $(r-1)(s+1) + (t+1)$ vertices. Similar to the previous case, u is counted once for each of these r neighbors of u, a total of r times, while it should be included just once in $N_r^2(S)$; and, each vertex adjacent to any two of these $r-1$ neighbors of u in C_0 is counted twice, but it should be counted just once. (Notice that only u is adjacent to both u' and those $r-1$ neighbors in C_0. Just assume there is another vertex v adjacent to both u' and u_1, a neighbor of u in C_0. By definition, v cannot be in a cluster of Class 0, as then it won't be adjacent to u, a vertex in a Class-0 cluster. Thus, v is either located in C_0 or in a Class-1 cluster C'. Assume that v occurs in C_0, then, because cross edges form a perfect matching, there is only one cross edge in between C_0 and C_0', since (u, u')

is already part of the matching, v ($\neq u$) cannot be adjacent to u' via another cross edge. By the same token, because of the existence of the edge (u, u'), v cannot be located in a Class-1 cluster, either.) Removing all these redundancies, the size of this *alternative* construction can be calculated as follows:

$$\begin{aligned}|N_r(S_2)| &= (r-1)(s+1) + (t+1) - (r-1) - \binom{r-1}{2} \\ &= rs - \binom{r-1}{2} + (t-s+1) \\ &= |N_r(S_1)| + (t-s+r-1) \geq |N_r(S_1)|, \end{aligned} \tag{1}$$

when $r \in [1, s+1]$, as $1 \leq s \leq t$, by assumption. Clearly, $N_r(S_1) = N_r(S_2)$ when $s = t$ and $r = 1$.

To continue with our previous example, for $r = 1$, if we now pick $S = \{u'\}$, we have $N_1^2(S) = \{u, a', w'\}$, while Eq. 1 gives $|N_1^2(S)| = 3$. In this case, $|N_1(S_1)| < |N_1(S_2)|$, since $s \neq t$, although $r = 1$. Moreover, for $r = 2$, although $N_2^1(S)$ is not defined, when we set $S = \{v, u'\}$, the alternative construction gives us that, $N_2^2(S) = \{u, a', w', v'\}$, while $|N_2^2(S)| = 4$ by Eq. 1.

It is easy to see that, once $N_r^1(S)$ (respectively, $N_r^2(S)$) is removed, all the neighbors of vertices in S are removed and none of these vertices in S are adjacent since $GEH(s,t)$ is bipartite. Hence, $GEH(s,t) - N_r^1(S)$ (respectively, $GEH(s,t) - N_r^2(S)$) contains at least $r+1$ components, including at least r singletons. Thus, $N_r^1(S)$ (respectively, $N_r^2(S)$) is a $(r+1)$-component cut. As a result, when $r \in [1, s]$,

$$\bar{\kappa}_{r+1}(GEH(s,t)) \leq \min\{N_r(S_1), N_r(S_2)\} = N_r(S_1) = rs - \binom{r}{2} + 1.$$

Let F be a minimum $(r+1)$-component cut. Then $GEH(s,t) - F$ has at least $r+1$ components. Thus, it has one large components and r "smaller" components. Clearly, these "smaller" components collectively has at least r vertices. By Corollary 1, for $1 \leq s \leq t$, $r \in [1, s]$, $\bar{\kappa}_{r+1}(GEH(s,t)) \geq |F| \geq rs - \frac{r(r-1)}{2} + 1$.

Thus, for $1 \leq s \leq t$, $r \in [1, s]$, $\bar{\kappa}_{r+1}(GEH(s,t)) = rs - \frac{r(r-1)}{2} + 1$. □

The following result is based on the relationship between the dual-cube-like network and that of the generalized exchanged hypercube, as pointed out in the last section.

Corollary 4 *For all* $n \geq 3$, *if* $r \in [1, n-1]$, $\bar{\kappa}_{r+1}(DC_n) = r(n-1) - \frac{r(r-1)}{2} + 1$.

6 The Conditional Diagnosability

The conditional diagnosability of interconnection networks has been studied by using a number of ad-hoc methods [18, 47]. Recently, gathering various ad-hoc methods developed in the last decade, an unified approach was developed [4, 14], which has been applied to find the conditional diagnosability of many interconnection networks, e.g., [4, 5, 7]. We give a brief overview here and refer readers to the aforementioned literature for further details.

According to the comparison diagnosis model [26, 27, 33], a comparator, $w \in G$, sends the same input to each and every pair of its neighbors, v and x in G, and generates a result, which tells if v and x are faulty, assuming w is not. A collection of all such results is called a syndrome of the diagnosis. Since a faulty comparator can lead to unreliable results, a set of faulty vertices may also produce different syndromes. Two distinct faulty sets F_1 and F_2 are indistinguishable if and only if they are compatible with at least one syndrome, distinguishable otherwise. Hence, $t_c(G)$, the conditional diagnosability of G, equals the maximum number d such that for all distinct pairs of conditional faulty sets, (F_1, F_2), $|F_1| \leq d$, $|F_2| \leq d$, F_1 and F_2 are distinguishable.

We notice that the central structure of the above comparison diagnosis model is a length two path, $p_2(v, w, x)$, centered at a vertex w. Clearly, any vertex in a viable interconnection network should have at least one neighbor outside the neighborhood of such a length two path centered at w, an arbitrary but fixed vertex. Otherwise, this length two path will immediately turn into a bottleneck, and make the network fault-intolerant. This observation motivates the following notion of a good length two path [5]: Let G be a graph, we call $p_2(v, w, x)$, a path of length 2 in G, a *good path* if, for every vertex $z \notin N(\{w\}) \cup \{w\}$, $N(\{z\}) \not\subseteq N(\{v, w, x\}) \cup \{v, x\}$.

By definition, to show that, for a given graph G, $t_c(G) \leq d$, we only need to construct a pair of distinct conditional faulty sets (F_1, F_2), $|F_1| \leq d+1$, $F_2 \leq d+1$, such that (F_1, F_2) is indistinguishable. The following result [34] provides such an upper bound of $t_c(G)$.

Proposition 1 *Let G be a graph where $p_2(v, w, x)$ forms a good path of length two in G. Then $t_c(G) \leq |N_G(\{v, w, x\})|$.*

It seems that, to get an upper bound for $t_c(G)$ by applying Proposition 1, we have to minimize $|N_G(\{v, w, x\})|$ over all good paths of length two in G, which may not be easy. On the other hand, as we will show, there is often a good candidate for a minimizer. We should also point out that the above result does not imply that a conditional faulty set obtained via a length two path is always a minimizer of such an upper bound. In fact, such an upper bound is sometimes obtained through a four cycle [40].

Given $GEH(s, t)$, $2 \leq s \leq t$, we select a four cycle $C_4 = (v, w, x, u, v)$ in C_0, a Class-0 cluster, and consider $p_2(v, w, x)$. Let any vertex $z \notin N(\{w\}) \cup \{w\}$. Such a z

must exist since C_0 ($\equiv Q_s$) contains at least four vertices when $s \geq 2$. By definition, z is adjacent to vertex z' in a cluster uniquely associated with C_0, which is also different from those corresponding to either v, w or x by definition. Thus, z' cannot be either v or x, and z' cannot be adjacent to either v, w or x. In other words, $p_2(v, w, x)$ is a good path.

Both v and x have $s-1$ neighbors that are not on $p_2(v, w, x)$, while w has only $s-2$ of them. Moreover, u is a neighbor of both v and x thus gets over counted once. Finally, all three of v, w, and x are adjacent to a unique vertex in their respectively associated cluster. Hence, $|N_{GEH(s,t)}(\{v, w, x\})| = 3s - 2$.

By Proposition 1, we have achieved the following upper bound result.

Lemma 3 *For all* $2 \leq s \leq t,\ t_c(GEH(s,t)) \leq 3s - 2$.

The issue now becomes how to verify this upper bound is also a lower bound, thus an exact bound, of $t_c(GEH(s,t))$. In general, this is quite challenging since we have to show that, for all conditional faulty set pairs (F_1, F_2), $|F_1| \leq d$, $|F_2| \leq d$, they are distinguishable. Fortunately, as previously mentioned, several general results to this regard have recently emerged, one of which is the following [4].

Theorem 7 *Let* G *be a graph,* $\delta(G) \geq 3$, *such that (1) for any* $T \subset V(G)$, $|T| \leq d$, $G - T$ *contains a large component and smaller components which contain at most two vertices in total; and (2)* $|V(G)| > (\Delta(G) + 2)d + 4$, *where* $\Delta(G)$ *refers to the maximum degree of vertices in* G. *Then* $t_c(G) \geq d + 1$.

When $2 \leq s \leq t$, $\delta(GEH(s,t)) = s + 1 \geq 3$, $\Delta(GEH(s,t)) = t + 1$, and $|V(GEH(s,t))| = 2^{s+t+1}$. Condition 1 of Theorem 7, for the generalized exchanged hypercube, immediately follows from Corollary 2, when $3 \leq s \leq t$. What is left for us to do is to check Condition 2 of Theorem 7, when $d = 3s - 3$, namely,

$$2^{s+t+1} = |V(G)| > (\Delta(G) + 2)d + 4 = 3(t+3)(s-1) + 4. \tag{2}$$

We only need to show that $2^{s+t-1} > (t+3)(s-1) + 1$, which holds when $2^{s+t-1} > (t+4)(s-1)$, since $s \geq 2$. This last inequality holds if $2^{s-1} > s - 1$ and $2^t \geq t + 4$. The first part certainly holds when $s \geq 2$, while the second part holds for $t \geq 3$. Finally, setting $t = 2$ in Eq. 2, we have $2^{s+3} > 15(s-1) + 4 = 15s - 11$, which certainly holds for all $s \geq 2$.

Hence, Condition 2 holds for all $s, t \geq 2$. By Corollary 2, Lemma 3, and Theorem 7, we have obtained the following result.

Theorem 8 *For* $3 \leq s \leq t$, $t_c(GEH(s,t)) = 3s - 2$.

Last but not least, the diagnosability of the dual-cube-like network DC_n has been derived in [5, Theorem 7.2] to be $3n - 5$, $n \geq 4$, that certainly coincides with Theorem 8, when setting $s = t = n - 1 \geq 3$.

7 Concluding Remarks

In this chapter, we applied a general connectivity result to further derive several fault tolerance measurements for the generalized exchanged hypercube, including its restricted connectivity, cyclic vertex-connectivity, component connectivity, and its conditional diagnosability, in terms of the comparison diagnosis model.

These results show that the generalized exchanged hypercube is a natural and robust interconnection topology and the general connectivity result is truly general and useful, which might be applied to derive other interesting connectivity related results.

We comment that similar connectivity results have also been reported in the literature [8] for the complete cubic networks with its underlying structure graph being a complete graph.

Appendix

In this section, we give a proof of $s = 3$ for Part 2 of Theorem 1. We first state a number of preliminary results from [9]. (The proof of Lemma 5 was omitted but it is similar to the one for Lemma 4.)

Lemma 4 *[9, Lemma 3.3]* $GEH(s, t), 1 \leq s \leq t$, *is* δ*-maximally connected.*

Lemma 5 *[9, Lemma 3.4]* $GEH(s, t), 2 \leq s \leq t$, *is* δ*-tightly super-connected.*

Lemma 6 *[9, Lemma 4.1] Let* $F \subset V(GEH(s, t)), s \in [2, t], |F| \leq ks - \frac{k(k-1)}{2}$, *there exists* Y, *a connected component of* $GEH(s, t) - F$, *such that, for all* $i \in [0, 2^s + 2^t)$, *if* $C_i - F_i$ *is connected, it is a subgraph of* Y.

We note that Lemma 5 does not hold for $s = 1$ as $GEH(1, t)$ contains 2^t Class-0 clusters, each of which is an edge, and two Class-1 clusters, each isomorphic to a Q_t. (Cf. Fig. 2b). Let (u, v) be one of these edges. When $\{u', v'\} \subseteq F$, $GEH(1, t) - F$ contains (u, v) and other components containing a total of $2^{t+2} - 2 \geq 6$ vertices.

We are now ready to prove $s = 3$ for Part 2 of Theorem 1. When $s = 3$, $k \in [1, 3]$. We notice that, when $k = 1$, $|F| \leq s$, $GEH(s, t) - F$ is then connected, by Lemma 4. We thus only need to consider the cases of $k = 2$ and $k = 3$.

For the case of $k = 2$, thus $|F| \leq 5$ by Part 2, we need to show that $GEH(3, t) - F, t \geq 3$, is either connected or contains a large component together with a singleton. By Lemma 5, when $|F| \leq 4$, $GEH(3, t) - F$ is either connected or it consists of a large component and one singleton. Thus, we only need to consider the case of $|F| = 5$.

Let $F_i = F \cap V(C_i), i \in [0, 2^s + 2^t)$. If, for some l, $|F_l| = 5$, then all the other clusters contain no faulty vertices, thus they are all connected. Clearly $GEH(s, t) - F_l$ will be connected, as well, since every vertex in $C_l - F_l$ is adjacent, via a cross edge, to a vertex located in a connected cluster. If for some l, $|F_l| = 4$, and the remaining faulty vertex f falls into another cluster, then all the clusters, other than C_l, are connected. $GEH(s,t) - F$ is then either connected or contains a large component and a singleton u $(\in V(C_l) \setminus F_l)$, when C_l is isomorphic to Q_3, u is adjacent to f via a cross edge, and all the three neighbors of u in C_l fall into F_l. We now assume that $|F_l| = 3$, when the other clusters collectively hold two faulty vertices, thus all connected by the maximum connectivity of hypercubes, as $s = 3$. If $C_l - F_l$ is connected, so is $GEH(s,t) - F$ by Lemma 6. Otherwise, if $C_l - F_l$ is disconnected, then C_l is isomorphic to Q_3, and $C_l - F_l$ contains a $K_{1,3}$ and a singleton f. Since the other clusters jointly hold two faulty vertices, this $K_{1,3}$ must be part of the large component of $GEH(3,t) - F$, as at least one of its four vertices is adjacent to a non-faulty vertex in this large component. Then, $GEH(3,t), 3 \le t$, is either connected or contains a large component and one singleton u when u is adjacent to one of the two faulty vertices, while all its three neighbors in C_l form F_l. The other cases are symmetric to the above.

We now turn to the case of $k = 3$, i.e., $|F| \le 6$, when we have to show that $GEH(3,t) - F, t \ge 3$, is either connected or contains a large component and small components altogether with at most two vertices. In light of the previous case, we only need to consider the case of $|F| = 6$.

If for some l, $|F_l| \ge 4$, then other clusters, sharing at most two faulty vertices, must be individually connected in the resulting graph by the assumption of $s = 3$ and Lemma 4, and belong to the same component, say Y, in the resulting graph by Lemma 6. By definition, those non-faculty vertices of C_l are part of Y. Hence, $GEH(s,t) - F$ is either connected, or contains a large component and smaller ones with at most two vertices, when the remaining up to two vertices in $V(C_l) \setminus C_l$ are adjacent to the faulty vertices in $F \setminus F_l$ via cross edges, while sharing their faulty neighbors in F_l.

We now consider the case when, for all l, F_l contains at most three of these vertices. Since for all l, C_l is isomorphic to a cube $Q_m, m \ge s$ $(= 3)$, when $m \ge 4$, all such $C_l - F_l$'s are connected by Lemma 4, and so is $GEH(s,t) - F$, by Lemma 6. We thus only need to consider the case when C_l is isomorphic to Q_3, where $|F_l| = 3$.

If for some l, $|F_l| = 3$, and for $j \ne l$, $|F_j| < 3$, then $C_j - F_j$, $j \ne l$, will all be connected by Lemma 6. If $C_l - F_l$ is connected, so is $GEH(s,t) - F$ by Lemma 6. Now assume that $C_l - F_l$ is not connected. Notice that C_l is isomorphic to a Q_3, and its surviving graph contains a singleton u and a $K_{1,3}$. Since there are only three faulty vertices located outside C_l, and $K_{1,3}$ contains four vertices, it must be part of a large connected component. Thus, in this case, $GEH(3,t), t \ge 3$, is either connected or contains a large component and a singleton u, when u is adjacent to one of these remaining faulty vertices in $F \setminus F_l$, and all its three neighbors are contained in F_l.

We finally consider the subcase that $|F_l| = |F_{l'}| = 3$, where both C_l and $C_{l'}$ are isomorphic to Q_3, when, for $j \notin \{l, l'\}$, F_j is empty. If both $C_l - F_l$ and $C_{l'} - F_{l'}$ are connected, then $GEH(s,t)$ is also connected by Lemma 6. We now assume, without loss of generality, $C_l - F_l$ is connected, but $C_{l'} - F_{l'}$ is not, when it contains a singleton u' and a $K_{1,3}$. $GEH(s,t) - F$, in this case, is either connected or contains a large component and a singleton u' when it is adjacent to a vertex in F_l and all its three neighbors in C_l' constitute F_l'. For the remaining case, when neither of them is connected, namely, $C_l - F_l$ (respectively, $C_{l'} - F_{l'}$) contains a singleton u (respectively, u') and a $K_{1,3}$. By the same token, $GEH(s,t) - F$ is either connected or it contains a large component and smaller component(s) with at most two faulty vertices u and u', when u' (respectively, u) is adjacent to a vertex in F_l (respectively, $F_{l'}$) and all its three neighbors in C_l' (respectively, C_l) fall into F_l' (respectively, F_l).

References

1. Angjeli, A., Cheng, E., Lipták, L.: Linearly many faults in dual-cube-like networks. Theor. Comput. Sci. **472**, 1–8 (2013)
2. Chartrand, G., Kapoor, S.F., Lesniak, L., Lick, D.R.: Generalized connectivity in graphs. Bull. Bombay Math. Colloq. **2**, 1–6 (1984)
3. Chen, Y.-W.: A comment on "The exchanged hypercube". IEEE Trans. Parallel Dist. Syst. **18**, 576 (2007)
4. Cheng, E., Lipták, L.L., Qiu, K., Shen, Z.: On deriving conditional diagnosability of interconnection networks. Inf. Process. Lett. **112**, 674–677 (2012)
5. Cheng, E., Lipták, L.L., Qiu, K., Shen, Z.: A unified approach to the conditional diagnosability of interconnection networks. J. Interconnect. Netw. **13**, 1250007 (19 pages) (2012)
6. Cheng, E., Lipták, L.L., Qiu, K., Shen, Z.: Cyclic vertex connectivity of Cayley graphs generated by transposition trees. Graphs Comb. **29**(4), 835–841 (2013)
7. Cheng, E., Qiu, K., Shen, Z.: On the conditional diagnosability of matching composition networks. Theor. Comput. Sci. **557**, 101–114 (2014)
8. Cheng, E., Qiu, K., Shen, Z.: Connectivity results of complete cubic network as associated with linearly many faults. J. Interconnect. Netw. **15**(1 & 2), 1550007 (23 pages) (2015)
9. Cheng, E., Qiu, K., Shen, Z.: A strong connectivity property of the generalized exchanged hypercube. Discrete Appl. Math. http://dx.doi.org/10.1016/j.dam.2015.11.014
10. Esfahanian, A.H.: Generalized measures of fault tolerance with application to n-cube networks. IEEE Trans. Comput. **38**(11), 1586–1591 (1989)
11. Esfahanian, A.H., Hakimi, S.L.: On computing a conditional edge-connectivity of a graph. Inf. Process. Lett. **27**, 195–199 (1988)
12. Hager, M.: Pendant tree-connectivity. J. Comb. Theory **38**, 179–189 (1985)
13. Harary, F., Hayes, J.P., Wu, H.-J.: A survey of the theory of hypercube. Comput. Math. Appl. **15**(4), 277–289 (1988)
14. Hong, W.-S., Hsieh, S.-Y.: Strong diagnosability and conditional diagnosability of augmented cubes under the comparison diagnosis model. IEEE Trans. Reliab. **61**, 140–148 (2012)
15. Hsu, L.-H., Cheng, E., Lipták, L., Tan, J.J.M., Lin, C.-K., Ho, T.-Y.: Component connectivity of the hypercubes. Int. J. Comput. Math. **89**(2), 137–145 (2012)
16. Klavžar, S., Ma, M.: The domination number of exchanged hypercubes. Inf. Process. Lett. **114**, 159–162 (2014)
17. Klavžar, S., Ma, M.: Average distance, surface area, and other structural properties of exchanged hypercubes. J. Supercomput. **69**, 306–317 (2014)
18. Lai, P.-L., Tan, J.J.M., Chang, C.-P., Hsu, L.-H.: Conditional diagnosability measures for large multiprocessor systems. IEEE Trans. Comput. **54**, 165–175 (2005)
19. Li, X., Mao, Y.: The generalized 3-connectivity of lexicographic product graphs. Discrete Math. Theor. Comput. Sci. **16**(1), 339–354 (2014)
20. Li, X.-J., Xu, J.-M.: Generalized measures of fault tolerance in exchanged hypercubes. Inf. Process. Lett. **113**, 533–537 (2013)
21. Li, Y., Peng, S., Chu, W.: Efficient collective communications in dual-cube. J. Supercomput. **28**, 71–90 (2004)

22. Loh, P.K.K., Hsu, W.J., Pan, Y.: The exchanged hypercube. IEEE Trans. Parallel Dist. Syst. **16**, 866–874 (2005)
23. Ma, M., Liu, B.: Cycles embedding in exchanged hypercubes. Inf. Process. Lett. **110**, 71–76 (2009)
24. Ma, M.: The connectivity of exchanged hypercubes. Discrete Math. Algorithms Appl. **2**, 213–220 (2010)
25. Ma, M., Zhu, L.: The super connectivity of exchanged hypercubes. Inf. Process. Lett. **111**, 360–364 (2011)
26. Maeng, J., Malek, M.: A comparison connection assignment for self-diagnosis of multiprocessor systems. In: Proceedings of 11th International Symposium on Fault-Tolerant Computing, pp. 173–175 (1981)
27. Malek, M.: A comparison connection assignment for diagnosis of multiprocessor systems. In: Proceedings of 7th International Symposium on Computer Architecture, pp. 31–35 (1980)
28. Oellermann, O.R.: On the l-connectivity of a graph. Graphs Comb. **3**, 285–299 (1987)
29. Oellermann, O.R.: A note on the l-connectivity function of a graph. Congessus Numerantium **60**, 181–188 (1987)
30. Oh, A.D., Choi, H.-A.: Generalized measures of fault tolerance in n-cube networks. IEEE Trans. Parallel Dist. Syst. **4**, 702–703 (1993)
31. Peng, S.L., Lin, C.K., Tan, J.J.M., Hsu, L.H.: The g-good-neighbor conditional diagnosability of hypercube under the PMC model. Appl. Math. Comput. **218**(21), 10406–10412 (2012)
32. Sampathkumar, E.: Connectivity of a graph— A generalization. J. Comb. Inform. Syst. Sci. **9**, 71–78 (1984)
33. Sengupta, A., Dahbura, A.T.: On self-diagnosable multiprocessor systems: diagnosis by the comparison approach. IEEE Trans. Comput. **41**, 1386–1396 (1992)
34. Stewart, I.: A general technique to establish the asymptotic conditional diagnosability of interconnection networks. Theor. Comput. Sci. **452**, 132–147 (2012)
35. Sullivan, G.F.: A polynomial time algorithm for fault diagnosability. In: Proceedings of 25th Annual Symposium Foundations Computer Science, IEEE Computer Society, pp. 148–156 (1984)
36. Tai, P.G.: Remarks on the coloring of maps. Proc. R. Soc. Edinb. **10**, 501–503 (1880)
37. West, D.B.: Introduction to Graph Theory, 2nd edn. Prentice Hall, Upper Saddle River, NJ (2001)
38. Wu, J., Guo, G.: Fault tolerance measures for m-ary n-dimensional hypercubes based on forbidden faulty sets. IEEE Trans. Comput. **47**, 888–893 (1998)
39. Xu, J.-M., Wang, J.-W., Wang, W.-W.: On super and restricted connectivity of some interconnection networks. Ars Combinatoria **94**, 25–32 (2010)
40. Yang, M.-C.: Conditional diagnosability of balanced hypercubes under the MM^* model. J. Supercomput. **65**, 1264–1278 (2013)
41. Yang, X., Evans, D.J., Megson, G.M.: On the maximal connected component of a hypercube with faulty vertices. Int. J. Comput. Math. **81**(5), 515–525 (2004)
42. Yang, X., Evans, D.J., Megson, G.M.: On the maximal connected component of a hypercube with faulty vertices II. Int. J. Comput. Math. **81**(10), 1175–1185 (2004)
43. Yang, X., Evans, D.J., Megson, G.M.: On the maximal connected component of a hypercube with faulty vertices III. Int. J. Comput. Math. **83**(1), 27–37 (2006)
44. Yu, Z., Liu, Q., Zhang, Z.: Cyclic vertex connectivity of star graphs. In: Proceedings of Fourth Annual International Conference on Combinatorial Optimization and Applications (COCOA'2010) Dec. 18–20, 2010, Kailua-Kona, HI, USA, Springer LNCS 6508(Part I), pp. 212–221 (2010)
45. Yuan, J., Liu, A., Ma, X., Liu, X., Qin, X., Zhang, J.: The g-good-neighbor conditional diagnosability of k-ary n-cubes under the PMC model and MM* model. Int. J. Parallel, Emergent Distrib. Syst. **26**(4), 1165–1177 (2015)
46. Zhang, C.Q.: Integer Flows and Cycle Covers of Graphs. Marcel Dekker Inc., New York (1997)
47. Zhu, Q.: On conditional diagnosability and reliability of the BC networks. J. Supercomput. **45**, 173–1184 (2008)