A Suspension Bridge Problem: Existence and Stability

Salim A. Messaoudi and Soh Edwin Mukiawa

Abstract In this work, we consider a semilinear problem describing the motion of a suspension bridge in the downward direction in the presence of its hanger restoring force $h(u)$ and a linear damping δu_t, where $\delta > 0$ is a constant. By using the semigroup theory, we establish the well posedness. We also use the multiplier method to prove a stability result.

Keywords Suspension bridge · Semigroup theory · Well posedness · Stability · Exponential decay

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1 Introduction

A simple model for a bending energy of a deformed thin plate $\Omega = (0, L) \times (-\ell, \ell)$ is given by

$$E_B(u) = \int_\Omega \left(\frac{K_1^2}{2} + \frac{K_2^2}{2} + \sigma K_1 K_2 \right) dxdy, \tag{1.1}$$

where $u = u(x, y)$ represents the downward vertical displacement of the plate and $K_1,\ K_2$ are the principal curvatures of the graph of u. The constant $\sigma = \frac{\lambda}{2\lambda+\mu}$ is the Poission ratio and $\lambda,\ \mu$ are called the Lamé moduli. For some physical reasons, $\lambda \geq 0$ and $\mu > 0$, hence $0 < \sigma < \frac{1}{2}$. For small deformation u, the following approximations hold

$$(K_1 + K_2)^2 \approx (\Delta u)^2, \quad K_1 K_2 \approx det(D^2 u) = u_{xx} u_{yy} - u_{xy}^2.$$

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As a result, we get

$$\frac{1}{2}K_1^2 + \frac{1}{2}K_2^2 + \sigma K_1 K_2 \approx \frac{1}{2}(\Delta u)^2 + (\sigma - 1)det(D^2 u).$$

Consequently, the energy functional (1.1) takes the form

$$E_B(u) = \int_\Omega \left(\frac{1}{2}(\Delta u)^2 + (\sigma - 1)det(D^2 u)\right) dxdy. \tag{1.2}$$

We note here that, for $0 < \sigma < \frac{1}{2}$, E_B is convex and is also coercive in suitable state spaces such as $H_0^2(\Omega)$ or $H^2(\Omega) \cap H_0^1(\Omega)$.
If f is an external vertical load acting on the plate Ω, then the total energy is given by

$$\begin{aligned} E_T(u) &= E_B(u) - \int_\Omega f u dxdy \\ &= \int_\Omega \left[\left(\frac{1}{2}(\Delta u)^2 + (\sigma - 1)(u_{xx}u_{yy} - u_{xy}^2)\right) - fu\right] dxdy. \end{aligned} \tag{1.3}$$

The unique minimizer u of the functional (1.3) satisfies the Euler-Lagrange equation

$$\Delta^2 u(x, y) = f(x, y), \ in \ \Omega. \tag{1.4}$$

For totally supported plate ($u = \frac{\partial u}{\partial \eta} = 0$), the problem has been first solved by Navier [17] in 1823. Since the bridge is usually simply supported on the vertical sides ($x = 0, x = L$, i.e. the $y-$ axis) only

$$u(0, y) = u_{xx}(0, y) = u(L, y) = u_{xx}(L, y) = 0,$$

then different boundary conditions should be considered for the horizontal sides ($y = -\ell, y = \ell$, i.e. $x-$axis). Various problems on a rectangular plate Ω, where only the vertical sides are simply supported, were discussed by many authors, see, for instance Mansfield [11]. Naturally, one should consider the plate Ω with free horizontal sides. In such a situation, the boundary conditions are

$$\begin{cases} u_{yy}(x, \pm\ell) + \sigma u_{xx}(x, \pm\ell) = 0, \ \ for \ x, \in \ (0, L), \\ u_{yyy}(x, \pm\ell) + (2 - \sigma)u_{xxy}(x, \pm\ell) = 0, \ \ for \ x \in \ (0, L), \end{cases} \tag{1.5}$$

see Ventsel and Krauthammer [19]. Putting all pieces together (see Ferrero and Gazzola [5]), the boundary value problem for a thin plate Ω modeling a suspension bridge is

$$\begin{cases} \Delta^2 u(x,y) = f(x,y), \quad in \;\; \Omega, \\ u(0,y) = u_{xx}(0,y) = u(L,y) = u_{xx}(L,y) = 0, \quad y \in (-\ell,\ell), \\ u_{yy}(x,\pm\ell) + \sigma u_{xx}(x,\pm\ell) = 0, \quad x \in \; (0,L), \\ u_{yyy}(x,\pm\ell) + (2-\sigma)u_{xxy}(x,\pm\ell) = 0, \quad x \in \; (0,L). \end{cases} \tag{1.6}$$

In order to describe the action of the hangers (cables), Ferrero and Gazzola [5] introduced a nonlinear function $h(x, y, u)$ which admits a potential energy given by $\int_\Omega H(x, y, u)dxdy$. As a result, the total energy (1.3) becomes

$$E_T(u) = \int_\Omega \left[\left(\frac{1}{2}(\Delta u)^2 + (\sigma - 1)(u_{xx}u_{yy} - u_{xy}^2) \right) + H(x,y,u) - fu \right] dxdy, \tag{1.7}$$

whose unique minimizer satisfies the stationary problem

$$\begin{cases} \Delta^2 u(x,y) + h(x,y,u(x,y)) = f(x,y), \quad in \;\; \Omega, \\ u(0,y) = u_{xx}(0,y) = u(L,y) = u_{xx}(L,y) = 0, \quad y \in (-\ell,\ell), \\ u_{yy}(x,\pm\ell) + \sigma u_{xx}(x,\pm\ell) = 0, \quad x \in \; (0,L), \\ u_{yyy}(x,\pm\ell) + (2-\sigma)u_{xxy}(x,\pm\ell) = 0, \quad x \in \; (0,L). \end{cases} \tag{1.8}$$

If the external force f depends on time, $f = f(x, y, t)$, then the kinetic energy $\frac{1}{2}\int_\Omega u_t^2 dxdy$ has to be added to the static total energy (1.7). Thus, the total energy becomes

$$\begin{aligned} E_T(u) =& \frac{1}{2}\int_\Omega u_t^2 dxdy + \int_\Omega \left[\left(\frac{1}{2}(\Delta u)^2 + (\sigma - 1)(u_{xx}u_{yy} - u_{xy}^2) \right) \right. \\ & \left. + \; H(x,y,u) - fu \right] dxdy. \end{aligned} \tag{1.9}$$

Also, the equation of motion becomes

$$u_{tt}(x,y,t) + \Delta^2 u(x,y,t) + h(x,y,u(x.y,t)) = f(x,y,t). \tag{1.10}$$

Finally, we might add a damping term due to some internal friction or viscosity. In this case, Eq. (1.10) takes the form

$$u_{tt}(x,y,t) + \delta u_t(x,y,t) + \Delta^2 u(x,y,t) + h(x,y,u(x.y,t)) = f(x,y,t), \tag{1.11}$$

where $\delta > 0$ is called the friction constant. Equation (1.11) together with the boundary conditions of (1.8) and initial data has been discussed by Ferrero and Gazzola [5], for a general nonlinear restoring force h. They proved the existence of a unique solution, using the Galerkin method. In addition, they discussed several stationary problems. Recent results by Wang [20] and Al-Gwaiz et al. [2] have also made use of the above mention boundary conditions.

Early results concerning suspension bridges go back to McKenna and collaborators. For instance, Glover et al. [8] considered the damped couple system

$$\begin{cases} u_{tt} + u_t + u_{xxxx} + \gamma_1 u_t + k(u-v)^+ = f, \\ \epsilon v_{tt} - v_{xx} + \gamma_2 v_t - k(u-v)^+ = g, \end{cases} \tag{1.12}$$

where,

$$u, v : [0, L] \times \mathbb{R}^+ \longrightarrow \mathbb{R}$$

represent the downward deflection and the vertical displacement of the string. For rigid suspension bridges, Lazer and Mckenna [12] reduced the system (1.12) to the following fourth-order equation

$$u_{tt} + u_{xxxx} + u_t + k^2 u^+ = f, x \in (0, 1), \quad t > 0, \tag{1.13}$$

and established existence of periodic solutions by assuming the suspension bridge as a bending beam. Equation (1.13) has been studied by a few authors (see [1, 4]). Mckenna and Walter, [14, 15] also investigated the nonlinear oscillations of suspension bridges and the existence of travelling wave solutions have been established. To achieve this, they considered the suspension bridge as a vibrating beam. Bochicchio et al. [3] considered

$$u_{tt} + u_t + u_{xxxx} + (p - \|u_x\|^2_{L^2((0,1))}) u_{xx} + ku^2 = f, \tag{1.14}$$

where p is a force that acts directly on the central axis of the bridge (axial force) and f a general external source term. They established a well-posedness as well as existence of global attractor. For more literature concerning the suspension bridges, we refer the reader to Mckenna [13], Mckenna et al. [16], Filippo et al. [7], Imhof [9], and Gazzola [6].

In this work, we consider the following fourth order semilinear plate problem

$$\begin{cases} u_{tt}(x,y,t) + \delta u_t(x,y,t) + \Delta^2 u(x,y,t) + h(u(x.y,t)) = 0, \quad in \ \ \Omega \times (0, +\infty), \\ u(0,y,t) = u_{xx}(0,y,t) = u(\pi,y,t) = u_{xx}(\pi,y,t) = 0, \quad (y,t) \in (-\ell,\ell) \times (0,+\infty), \\ u_{yy}(x,\pm\ell,t) + \sigma u_{xx}(x,\pm\ell,t) = 0, \quad (x,t) \in (0,\pi) \times (0,+\infty), \\ u_{yyy}(x,\pm\ell,t) + (2-\sigma) u_{xxy}(x,\pm\ell,t) = 0, \quad (x,t) \in (0,\pi) \times (0,+\infty), \\ u(x,y,0) = u_0(x,y), \ u_t(x,y,0) = u_1(x,y), \quad in \ \ \Omega. \end{cases} \tag{1.15}$$

The aim of this work is to reformulate (1.15) into a semigroup setting and then make use of the semigroup theory (see Pazy [18]) to establish the well-posedness. We also use the multiplier method (see Komornik [10]) to prove a stability result for problem (1.15). The rest of this work is organized as follows. In Sect. 2, we present some basic and fundamental materials needed to establish our main results. In Sect. 3, we establish a well-posedness result for problem (1.15). In Sect. 4, we state and prove our stability result.

2 Preliminaries

In this section we present some basic and fundamental results which will be used in proving our main results. For this, we impose the following assumptions on the function h

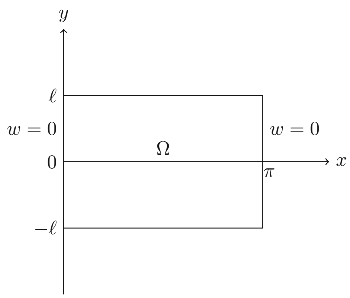
$$\begin{cases} h:\mathbb{R}\longrightarrow\mathbb{R} \quad is \quad lipschitz \quad such \quad that \quad h(0)=0, \\ H(s)=\displaystyle\int_0^s h(\tau)d\tau \quad is \quad positive, \\ sh(s)-H(s)\geq 0, \quad \forall s\in\mathbb{R}. \end{cases} \tag{2.1}$$

Example 2.1 An example of a function satisfying (2.1) is

$$h(s)=a|s|^{p-1}s,\ a\geq 0,\ p\geq 1.$$

As in [5], we introduce the space

$$H_*^2(\Omega)=\left\{w\in H^2(\Omega): w=0 \text{ on } \{0,\pi\}\times(-\ell,\ell)\right\}, \tag{2.2}$$



together with the inner product

$$(u, v)_{H^2_*} = \int_\Omega [(\Delta u \Delta v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})]dxdy. \tag{2.3}$$

For the completeness of $H^2_*(\Omega)$, we have the following results by Ferrero and Gazzola [5].

Lemma 2.1 *[5] Assume* $0 < \sigma < \frac{1}{2}$. *Then, the norm* $\|.\|_{H^2_*(\Omega)}$ *given by* $\|u\|^2_{H^2_*(\Omega)} = (u, u)_{H^2_*}$ *is equivalent to the usual* $H^2(\Omega)$*-norm. Moreover,* $H^2_*(\Omega)$ *is a Hilbert space when endowed with the scalar product* $(., .)_{H^2_*}$. □

Lemma 2.2 *[5] Assume* $0 < \sigma < \frac{1}{2}$ *and* $f \in L^2(\Omega)$. *Then there exists a unique* $u \in H^2_*(\Omega)$ *such that*

$$\int_\Omega [\Delta u \Delta v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})]dxdy = \int_\Omega fv, \quad \forall v \in H^2_*(\Omega). \tag{2.4}$$

□

Remark 2.1 The function $u \in H^2_*(\Omega)$ satisfying (2.4) is called the weak solution of the stationary problem (1.6).

Lemma 2.3 *[5] The weak solution* $u \in H^2_*(\Omega)$, *of* (2.4), *is in* $H^4(\Omega)$ *and there exists a* $C = C(l, \sigma) > 0$ *such that*

$$\|u\|_{H^4(\Omega)} \le C\|f\|_{L^2(\Omega)}. \tag{2.5}$$

In addition if $u \in C^4(\bar{\Omega})$, *then u is called a classical solution of* (1.6). □

Lemma 2.4 *[20] Let* $u \in H^2_*(\Omega)$ *and suppose* $1 \le p < +\infty$. *Then, there exists a positive constant* $C_e = C_e(\Omega, p)$ *such that*

$$\|u\|^p_p \le C_e\|u\|^p_{H^2_*(\Omega)}.$$

□

Lemma 2.5 *[10] Let* $E : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *be a non-increasing function. Assume that there exists* $C > 0$ *such that*

$$\int_s^\infty E(t)dt \le CE(s), \ 0 < s < \infty.$$

Then, there exists $\lambda > 0$ *a constant such that*

$$E(t) \le E(0)e^{-\lambda t}, \ \forall t \ge 0. \tag{2.6}$$

3 Well-Posedness

In this section we establish the well-posedness of problem (1.15) using the semigroup theory. For this, we set $u_t = v$, then problem (1.15) becomes

$$(P)\begin{cases} U_t + AU = F \\ \\ U(0) = U_0, \end{cases}$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad AU = \begin{pmatrix} -v \\ \Delta^2 u + \delta v \end{pmatrix}, \quad F(U) = \begin{pmatrix} 0 \\ -h(u) \end{pmatrix}, \quad U_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

We define the Hilbert space

$$\mathcal{H} = H_*^2(\Omega) \times L^2(\Omega)$$

equipped with the inner product

$$(U, V)_{\mathcal{H}} = (u, \tilde{u})_{H_*^2(\Omega)} + (v, \tilde{v})_{L^2(\Omega)}, \tag{3.1}$$

where

$$U = (u, v)^T, \ V = (\tilde{u}, \tilde{v})^T \in \mathcal{H}.$$

Next, we introduce the following notation

$$\begin{cases} u_{xx}(0, y) = u_{xx}(\pi, y) = 0 \\ u_{yy}(x, \pm\ell) + \sigma u_{xx}(x, \pm\ell) = 0 \\ u_{yyy}(x, \pm\ell) + (2-\sigma)u_{xxy}(x, \pm\ell) = 0. \end{cases} \tag{3.2}$$

The domain of the operator A is defined as

$$D(A) = \left\{(u, v) \in \mathcal{H}/u \in H^4(\Omega)\ satisfying\ (3.2),\ v \in H_*^2(\Omega)\right\}.$$

Lemma 3.1 *We have*

$$(\Delta^2 u, v)_{L^2(\Omega)} = (u, v)_{H_*^2}, \qquad \forall u, v \in D(A). \tag{3.3}$$

Proof Using Green's formula we obtain that

$$\int_\Omega v\Delta^2 u = \int_\Omega \Delta u \Delta v + \int_{\partial\Omega} [v\frac{\partial \Delta u}{\partial \eta} - \Delta u \frac{\partial v}{\partial \eta}]. \tag{3.4}$$

Integration in (3.4) leads to

$$\begin{aligned}\int_\Omega v\Delta^2 u = \int_\Omega \Delta u \Delta v &- \int_0^\pi v(x,-\ell)[u_{xxy}(x,-\ell)+u_{yyy}(x,-\ell)]dx \\ &+ \int_0^\pi v(x,\ell)[u_{xxy}(x,\ell)+u_{yyy}(x,\ell)]dx \\ &+ \int_0^\pi v_y(x,-\ell)[u_{xx}(x,-\ell)+u_{yy}(x,-\ell)]dx \\ &- \int_{-\ell}^{\ell} v_x(\pi,y)[\cancelto{0}{u_{xx}(\pi,y)}+\cancelto{0}{u_{yy}(\pi,y)}]dy \\ &- \int_0^\pi v_y(x,\ell)[u_{xx}(x,\ell)+u_{yy}(x,\ell)]dx \\ &+ \int_{-\ell}^{\ell} v_x(0,y)[\cancelto{0}{u_{xx}(0,y)}+\cancelto{0}{u_{yy}(0,y)}]dy].\end{aligned}$$

This gives

$$\begin{aligned}\int_\Omega v\Delta^2 u = \int_\Omega \Delta u \Delta v &- \int_0^\pi v(x,-\ell)[u_{xxy}(x,-\ell)+u_{yyy}(x,-\ell)]dx \\ &+ \int_0^\pi v(x,\ell)[u_{xxy}(x,\ell)+u_{yyy}(x,\ell)]dx \\ &+ \int_0^\pi v_y(x,-\ell)[u_{xx}(x,-\ell)+u_{yy}(x,-\ell)]dx \\ &- \int_0^\pi v_y(x,\ell)[u_{xx}(x,\ell)+u_{yy}(x,\ell)]dx. \qquad (3.5)\end{aligned}$$

By using (3.2), we obtain

$$\begin{aligned}\int_\Omega v\Delta^2 u = \int_\Omega \Delta u \Delta v &+ (1-\sigma)\int_0^\pi [v(x,-\ell)u_{xxy}(x,-\ell) - v(x,\ell)u_{xxy}(x,\ell)]dx \\ &+ (1-\sigma)\int_0^\pi [v_y(x,-\ell)u_{xx}(x,-\ell) - v_y(x,\ell)u_{xx}(x,\ell)]dx. \qquad (3.6)\end{aligned}$$

By performing similar integration by part on the right hand side of (2.3), we obtain (3.6). Hence the result. □

Lemma 3.2 *The operator $A : D(A) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is monotone.*

Proof Exploiting Lemma 3.1, we obtain, for all $U = \begin{pmatrix} u \\ v \end{pmatrix} \in D(A)$,

$$
\begin{aligned}
(AU, U)_{\mathcal{H}} &= \left(\begin{pmatrix} -v \\ \Delta^2 u + \delta_v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right)_{\mathcal{H}} \\
&= -(u, v)_{H^2_*(\Omega)} + \left(\Delta^2 u + \delta v, v\right)_{L^2(\Omega)} \\
&= -(u, v)_{H^2_*(\Omega)} + \left(\Delta^2 u, v\right)_{L^2(\Omega)} + \delta \|v\|^2_{L^2(\Omega)} = \delta \|v\|^2_{L^2(\Omega)} \geq 0. \quad (3.7)
\end{aligned}
$$

Thus, A is a monotone operator. □

Lemma 3.3 *The operator $A : D(A) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is maximal, that is $R(I + A) = H$.*

Proof Let $G = (k, l) \in \mathcal{H}$ and consider the stationary problem

$$U + AU = G, \quad (3.8)$$

where $U = \begin{pmatrix} u \\ v \end{pmatrix}$. From (3.8) we obtain

$$
\begin{cases}
u - v = k, \\
v + \Delta^2 u + \delta v = l.
\end{cases} \quad (3.9)
$$

Combining $(3.9)_1$ and $(3.9)_2$ gives, for $\delta_0 = \delta + 1$,

$$\delta_0 u + \Delta^2 u = l + \delta_0 k. \quad (3.10)$$

The weak formulation of (3.10) is then

$$\delta_0 \int_\Omega u\phi + (u, \phi)_{H^2_*(\Omega)} = \int_\Omega (l + \delta_0 k)\phi, \quad \forall \phi \in H^2_*(\Omega). \quad (3.11)$$

We define the following bilinear and linear forms on $H^2_*(\Omega)$

$$B(u, \phi) = \delta_0 \int_\Omega u\phi + (u, \phi)_{H^2_*(\Omega)}, \quad (3.12)$$

$$\mathcal{F}(\phi) = \int_\Omega (l + \delta_0 k)\phi. \quad (3.13)$$

By using Lemmas 2.1 and 2.4, we show that B is bounded and coercive, and $\mathcal{F}$ is bounded. For this, we can easily see that

$$|B(u, \phi)| \leq C \|u\|_{H^2_*} \|\phi\|_{H^2_*}.$$

Furthermore, we have that

$$B(u,u) = \delta_0 \|u\|^2_{L^2} + \|u\|^2_{H^2_*} \geq \|u\|^2_{H^2_*}. \tag{3.14}$$

Therefore B is bounded and coercive.
Also,

$$|\mathcal{F}(\phi)| \leq \|l\|_{L^2}\|\phi\|_{L^2} + \delta_0\|k\|_{L^2}\|\phi\|_{L^2} \leq C(\|l\|_{L^2} + \delta_0\|k\|_{H^2_*})\|\phi\|_{H^2_*}.$$

This implies that $\mathcal{F}$ is bounded. Thus, Lax- Milgram Theorem guarantees the existence of a unique $u \in H^2_*(\Omega)$ satisfying (3.11), which yields

$$(u,\phi)_{H^{2*}(\Omega)} = \int_\Omega [l + \delta_0 k - \delta_0 u]\phi, \ \ \forall \phi \in H^2_*(\Omega). \tag{3.15}$$

Since $l + \delta_0 k - \delta_0 u \in L^2(\Omega)$, it follows from Lemma 2.3 that $u \in H^4(\Omega)$. Thus, we get $u \in H^2_*(\Omega) \cap H^4(\Omega)$. By performing similar integration by parts as in Lemma 3.1 to Eq. (3.11), we obtain

$$\begin{aligned}
&\int_\Omega [\delta_0 u + \Delta^2 u - l + \delta_0 k]\phi + \int_{-\ell}^{\ell} [u_{xx}(\pi,y)\phi_x(\pi,y) - u_{xx}(0,y)\phi_x(0,y)]dy \\
&+ \int_0^\pi \{[u_{yy}(x,\ell) + \sigma u_{xx}(x,\ell)]\phi_y(x,\ell) - [u_{yy}(x,-\ell) + \sigma u_{xx}(x,-\ell)]\phi_y(x,-\ell)\}dx \\
&+ \int_0^\pi [u_{yyy}(x,-\ell) + (2-\sigma)u_{xxy}(x,-\ell)]\phi(x,l)dx \\
&- \int_0^\pi [u_{yyy}(x,\ell) + (2-\sigma)u_{xxy}(x,\ell)]\phi(x,l)dx = 0, \ \ \forall \phi \in H^2_*(\Omega).
\end{aligned} \tag{3.16}$$

Now, by considering $\phi \in C_0^\infty(\Omega)$ $(hence \ \ \phi \in H^2_*(\Omega))$, then all the boundary terms of (3.16) vanish and we obtain

$$\int_\Omega [\delta_0 u + \Delta^2 u - l + \delta_0 k]\phi = 0, \ \ \forall \phi \in C_0^\infty(\Omega). \tag{3.17}$$

Hence (by density) we have

$$\int_\Omega [\delta_0 u + \Delta^2 u - l + \delta_0 k]\phi = 0, \ \ \forall \phi \in L^2(\Omega). \tag{3.18}$$

This implies

$$\delta_0 u + \Delta^2 u = l + \delta_0 k, \ \ in \ \ L^2(\Omega). \tag{3.19}$$

We take

$$v = u - k \ \ in \ \ H^2_*(\Omega)$$

and obtain

$$v + \Delta^2 u + \delta u = l, \ \ in \ \ L^2(\Omega).$$

Thus, $u \in H_*^2(\Omega) \cap H^4(\Omega)$ and $v \in H_*^2(\Omega)$ solves (3.9). Again, by choosing $\phi \in C^\infty(\bar{\Omega}) \cap H_*^2(\Omega)$ in (3.16) and using (3.19), we get

$$\begin{aligned}
&\int_\Omega [\delta_0 u + \Delta^2 u - w]\phi \;(\to 0) + \int_{-\ell}^{\ell} [u_{xx}(\pi, y)\phi_x(\pi, y) - u_{xx}(0, y)\phi_x(0, y)]dy \\
&+ \int_0^\pi \{[u_{yy}(x, \ell) + \sigma u_{xx}(x, \ell)]\phi_y(x, \ell) - [u_{yy}(x, -\ell) + \sigma u_{xx}(x, -\ell)]\phi_y(x, -\ell)\}dx \\
&+ \int_0^\pi [u_{yyy}(x, -\ell) + (2 - \sigma)u_{xxy}(x, -\ell)]\phi(x, l)dx \\
&- \int_0^\pi [u_{yyy}(x, \ell) + (2 - \sigma)u_{xxy}(x, \ell)]\phi(x, l)dx = 0.
\end{aligned} \tag{3.20}$$

By the arbitrary choice of $\phi \in C^\infty(\bar{\Omega}) \cap H_*^2(\Omega)$, we obtain from (3.20) the boundary conditions (3.2). Therefore there exists a unique

$$U = \begin{pmatrix} u \\ v \end{pmatrix} \in D(A)$$

satisfying (3.9). Thus, A is a maximal operator. □

Lemma 3.4 *The function F is Lipschtz.*

Proof Let $U, V \in \mathcal{H}$ and recall assumption $(2.1)_1$ to have

$$\begin{aligned}
\|F(U) - F(V)\|_{\mathcal{H}} &= \| \begin{pmatrix} 0 \\ -h(u) \end{pmatrix} - \begin{pmatrix} 0 \\ -h(\tilde{u}) \end{pmatrix} \|_{\mathcal{H}} \\
&= \| \begin{pmatrix} 0 \\ h(\tilde{u}) - h(u) \end{pmatrix} \|_{\mathcal{H}} = \|h(\tilde{u}) - h(u)\|_{L^2(\Omega)} \\
&\leq C\|u - \tilde{u}\|_{L^2(\Omega)} \leq C\|U - V\|_{\mathcal{H}}.
\end{aligned}$$

So, F is lipsctitz. □

Thus, by the semigroup theory [18], we have the following existence result.

Theorem 3.1 *Assume that (2.1) hold. Let $U_0 \in \mathcal{H}$ be given. Then the problem (P) has a unique weak solution*

$$U \in C\left([0, +\infty), \mathcal{H}\right).$$

Moreover, if h is linear and $U_0 \in D(A)$, then (P) has a unique strong solution

$$U \in C\left([0, +\infty), D(A)\right) \cap C^1([0, +\infty), \mathcal{H}).$$

Proof Follows from Lemmas 3.2, 3.3 and 3.4. □

4 Stability

In this section, we use the multiplier method (see Komornik [10]) to establish a stability result for the energy functional associated to problem (1.15).

Corollary 4.1 *We have*

$$\int_{\Omega} u\Delta^2 u = \|u\|^2_{H^2_*}, \quad \forall u \in D(A). \tag{4.1}$$

Proof Let $v = u$ in Lemma 3.1. □

The energy functional associated to problem (1.15) is defined by

$$E(t) = \frac{1}{2}\|u_t(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|u(t)\|^2_{H^2_*} + \int_{\Omega} H(u(t)). \tag{4.2}$$

Lemma 4.1 *Let* $(u_0, u_1) \in D(A)$ *be given and assume that* (2.1) *hold. Then the energy functional* (4.2) *satisfies*

$$\frac{dE(t)}{dt} = -\delta \int_{\Omega} u_t^2 \leq 0. \tag{4.3}$$

Proof Multiply $(1.15)_1$ by u_t and integrate over Ω to get

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\Omega} u_t^2 + \frac{1}{2}\|u\|^2_{H^2_*} + \int_{\Omega} H(u)\right) + \delta \int_{\Omega} u_t^2 = 0. \tag{4.4}$$

Hence, the result. The inequality in (4.3) remains true for weak solution by simple density argument. Moreover, we get that E is a non-increasing functional. □

Theorem 4.1 *Let* $(u_0, u_1) \in D(A)$ *be given and assume (2.1) holds. Then, there exist constants* $K > 0$, $\lambda > 0$ *such that the energy functional* (4.2) *satisfies*

$$E(t) \leq Ke^{-\lambda t}, \quad \forall t \geq 0. \tag{4.5}$$

Proof We multiply $(1.15)_1$ by u and integrate over $\Omega \times (s, T)$, for $0 < s < T$ to get

$$\int_s^T \int_{\Omega} \left(u_{tt}u + u\Delta^2 u + uh(u) + \delta u u_t\right) = 0. \tag{4.6}$$

By using Corollary 4.1 we obtain

$$\int_s^T \int_\Omega (u_t u)_t - \int_s^T \int_\Omega u_t^2 + \int_s^T \|u\|_{H_*^2}^2 + \int_s^T \int_\Omega H(u) + \int_s^T \int_\Omega (uh(u) - H(u)) \\ + \delta \int_s^T \int_\Omega u u_t = 0.$$

This gives

$$\int_s^T E(t)dt + \int_s^T \int_\Omega (u_t u)_t - \frac{3}{2}\int_s^T \int_\Omega u_t^2 + \frac{1}{2}\int_s^T \|u\|_{H_*^2}^2 + \int_s^T \int_\Omega (uh(u) \\ - H(u)) + \delta \int_s^T \int_\Omega u u_t = 0.$$

By exploiting assumption (2.1), we obtain

$$\int_s^T E(t)dt \leq -\int_s^T \int_\Omega (u_t u)_t + \frac{3}{2}\int_s^T \int_\Omega u_t^2 - \delta \int_s^T \int_\Omega u u_t. \tag{4.7}$$

Now, we estimate the terms on the right-hand side of (4.7). By using Lemma 2.4 and Young's inequality, the first term can be estimated as follows

$$\begin{aligned} |-\int_\Omega \int_s^T (u_t u)_t| &\leq |\int_\Omega u_t(s)u(s)| + |\int_\Omega u_t(T)u(T)| \\ &\leq \frac{1}{2}\int_\Omega u_t^2(s) + \frac{1}{2}\int_\Omega u^2(s) + \frac{1}{2}\int_\Omega u_t^2(T) + \frac{1}{2}\int_\Omega u^2(T) \\ &\leq E(s) + C\|u(s)\|_{H_*^2}^2 + E(T) + C\|u(T)\|_{H_*^2}^2 \\ &\leq CE(s) + CE(T) \leq CE(s). \end{aligned} \tag{4.8}$$

For the second term, we have

$$\frac{3}{2}\int_s^T \int_\Omega u_t^2 = \frac{3}{2\delta}\int_s^T (-E'(t))dt = \frac{3}{2\delta}E(s) - \frac{3}{2\delta}E(T) \leq \frac{3}{2\delta}E(s). \tag{4.9}$$

For the third term, we have for any $\epsilon > 0$ to be specified later

$$
\begin{aligned}
|-\delta\int_s^T\int_\Omega uu_t| &\le C_\epsilon\delta\int_s^T\int_\Omega u_t^2+\delta\frac{\epsilon}{2}\int_s^T\int_\Omega u^2\\
&\le C_\epsilon\delta\int_s^T(-E'(t))dt+\delta C_e\frac{\epsilon}{2}\int_s^T\|u\|_{H_*^2}^2\\
&\le C_\epsilon\delta E(s)+\delta C_e\frac{\epsilon}{2}\int_s^T E(t)dt. \qquad (4.10)
\end{aligned}
$$

Combining (4.8)–(4.10), we obtain

$$
\left(1-C_e\delta\frac{\epsilon}{2}\right)\int_s^T E(t)dt\le\left(C+\frac{3}{2\delta}+\delta C_\epsilon\right)E(s). \qquad (4.11)
$$

We then choose $\epsilon>0$ small enough so that $\left(1-C_e\delta\frac{\epsilon}{2}\right)>0$ and obtain

$$
\int_s^T E(t)dt\le CE(s),\ \forall\, s>0. \qquad (4.12)
$$

Letting T go to infinity and applying Lemma 2.5, we conclude from (4.12) the existence of two constants $K,\lambda>0$ such that the energy of the solution of (1.15) satisfies

$$
E(t)\le Ke^{-\lambda t},\quad \forall t\ge 0. \qquad (4.13)
$$

This complete the proof. □

Remark 4.1 The decay estimate (4.5) remains valid for weak solutions by virtue of the density of $D(A)$ in $\mathcal{H}$.

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