On Carathéodory Quasilinear Functionals for BV Functions and Their Time Flows for a Dual H^1 Penalty Model for Image Restoration

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Abstract We extend the theory of functionals defined on BV space by including certain Carathéodory functions $\varphi(x, \mathbf{p})$ for functionals of the form $\int_\Omega \varphi(x, Du)$, $u \in BV(\Omega)$, so that φ is only measurable in x without the usual continuity assumption in x, and prove lower semicontinuity in L^1 of $\int_\Omega \varphi(x, Du)$ as well as compactness with an extra with an L^1 condition on φ. We also consider the case of the dual H^1 penalty model with integral constraint introduced in Osher-Solé-Vese [38] for image restoration, with the more general energy term $\int_\Omega \varphi(x, Du)$, analyze the time flow of the dual H^1 model in BV, and derive an integral property for the flow in the case of one space dimension.

Keywords Bounded variation · Image restoration · Gradient flows · Dual of h^1 · Anisotropic diffusion

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1 Introduction

In this paper we present some results of gradient time flows in $L^2(\Omega)$ corresponding to minimization problems of functionals of the form

$$\mathcal{F}(u) := \int_\Omega \varphi(x, Du) + \lambda \int_\Omega |\nabla(\Delta^{-1})(I-u)|^2 \, dx$$

with dual H^1 penalty term $\lambda \int_\Omega |\nabla(\Delta^{-1})(I-u)|^2 \, dx$ defined for $u \in BV(\Omega) \cap L^2(\Omega)$, $\Omega \subset \mathbb{R}^n$ open and bounded and constant $\lambda > 0$. Here we assume the

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Carathéodory function $\varphi(x, \mathbf{p})$, $\varphi : \Omega \times \mathbb{R}^n \to [0, \infty)$, is for a.e. x both convex and has a linear growth assumption in $\mathbf{p}$, and also has an additional integrability assumption to insure compactness. We make no assumption of continuity in x. As described later in this section, the minimization problem was originally proposed for image restoration applications in Osher et al. [38] for the case of pure total variation term $\varphi(Du) = |Du|$ with dual H^1 penalty.

Existence, uniqueness, and qualitative properties for solutions for flows in L^1 and L^2 with pure total variation term and different boundary conditions were obtained in [9–13, 18] with no penalty term for the L^1 case, and simple L^2 penalty for the L^2 case. For the purpose of the study of entropy solutions, they also consider flows in L^1 with quasilinear term $\phi(x, Du)$ for $u \in BV$ where ϕ has a strong continuity assumption in x. For our case, in addition to the dual H^1 penalty, $\varphi(x, Du)$ includes certain Carathéodory functions that are only measurable in x with no continuity assumption in x. The flow considered in this paper is

$$\frac{\partial u}{\partial t} = div\left(\nabla_{\mathbf{p}}\varphi(x, Du)\right) - 2\lambda\Delta^{-1}(I - u) \text{ for } t > 0, \text{ on } \Omega$$

with constraint $\int_\Omega u\,dx = \int_\Omega I\,dx$, initial condition $u(0, x) = I(0)$, Neumann boundary condition $\frac{\partial u}{\partial \mathbf{n}} = 0$ on $\partial\Omega$, for open bounded $\Omega \subset \mathbb{R}^1$ or $\mathbb{R}^2$ with Lipschitz boundary, and $\varphi(x, Du)$ as mentioned above.

One of the objectives of image processing is to restore corrupted images while retaining important features of the image, such as edges. One of the first models for this purpose using total variation was the Rudin-Osher-Fatemi (ROF) model [40, 41]. The ROF model consists of finding a minimizer $u_m \in L^2(\Omega)$ of the functional

$$\mathcal{R}(u) := \int_\Omega |\nabla u| + \frac{\lambda}{2}\int_\Omega (u - I)^2\,dx \tag{1.1}$$

where $I : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ bounded and open, represents the noisy or corrupted image and u_m represents the restored or cleaned image. For these types of minimization models, the images are represented by functions $u : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^2$ is typically a rectangle, and $u(x)$ the image intensity at x. The first term on the right in the above functional is the total variation of u:

$$\begin{aligned} TV(u) &:= \int_\Omega |\nabla u| \\ &:= \sup\left\{\int_\Omega u\nabla\cdot\varphi\,dx \;:\; \varphi \in C_c^1(\Omega; \mathbb{R}^n), |\varphi(x)| \le 1 \text{ for all } x \in \Omega\right\}. \end{aligned}$$

The space of all such $u \in L^1(\Omega)$ with $TV(u) < \infty$ is known as the space of functions of bounded variation, or $BV(\Omega)$, with the norm $\|u\|_{BV} =: \|u\|_{L^1(\Omega)} + \int_\Omega |\nabla u|$. Any minimizer of $\mathcal{R}$ will be in $BV(\Omega)$. It is common to use the Lebesgue decomposition to write any $u \in BV$ as

$$\int_\Omega |Du| = \int_\Omega |\nabla u|\, dx + \int_\Omega |D^s u|$$

where we decompose the total variation measure Du,with $|Du| =: \int_\Omega |\nabla u|$ into the absolutely continuous part with respect to Lebesgue measure $\nabla u\, dx$ and the singular part $D^s u$ as

$$Du = \nabla u\, dx + D^s u.$$

In [15] the above integral for $u \in BV$ is extended to $\int_\Omega \varphi(x, Du)$ for functions $\varphi(x, \mathbf{p})$, $x \in \Omega$, $\mathbf{p} \in \mathbb{R}^n$, continuous on $\Omega \times \mathbb{R}^n$, and convex and of linear growth in $\mathbf{p}$ (see Theorem 2 in the next section). We also refer the reader to [27] for results concerning certain functionals of the form $\int_\Omega \sqrt{1 + |Du|}dx + \int_\Omega G(x, u)\, dx$.

The use of BV space with the TV term is that minimizers of $\mathcal{R}$ may still be discontinuous with jumps corresponding to edges, unlike images restricted to the Sobolev space $W^{1,1}$. The second term of $\mathcal{R}$ is the penalty which ensures that the restored image u does not deviate too far from the input image I. One way to solve this is to solve the gradient flow of the Euler-Lagrange equation

$$0 = div\left(\frac{\nabla u}{|\nabla u|}\right) - \lambda(u - I)$$

and let $t \to \infty$ for the solution $u(x, t)$. See also [45] for the time flow for applications to plasticity. The gradient flow is then

$$\frac{\partial u}{\partial t} = div\left(\frac{\nabla u}{|\nabla u|}\right) - \lambda(u - I) \text{ on } \Omega \times [0, \infty) \text{ with } \frac{\partial u}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega$$
$$u(x, 0) = I(x) \text{ on } \Omega.$$

We should also mention the use of primal dual methods, instead of the gradient time flow, for minimizing functionals such as (1.1). These are especially used for models with pure TV term $\int_\Omega |Du|$ due its non differentiability. See, for example, [23] and [31].

In general, the above model works very well for image denoising while retaining edges. Modifications of the ROF model have also been introduced in other works to provide better restoration of noisy images due to such unwanted effects such as the stair casing effect, which may occur in solving (1.1) numerically. See [1–6, 16, 21, 22, 28, 29, 44] for further discussion and models.

Certain details, such a oscillatory textures are not well preserved with the above L^2 norm penalty $\frac{\lambda}{2}\int_\Omega (u - I)^2\, dx$. In [35], Meyer introduced a new penalty designed to overcome this, by replacing the L^2 penalty with a weaker norm that can retain oscillatory textures. In [35], the new model problem is to find a minimizer of

$$\mathcal{M}(u) := \int_\Omega |\nabla u| + \lambda \left\| I - u \right\|_* .$$

The new penalty norm $\|f\|_*$ is defined there for all $f \in G$ by

$$\|f\|_* = \inf \left\{ \sqrt{g_1^2 + ... + g_n^2} : g = (g_{1,...,} g_n),\ g_i \in L^\infty(\Omega) \text{ each } i, \text{and} f = \operatorname{div} g \right\},$$

where G is the Banach space of all generalized functions f that can be written as $f = \operatorname{div} g$ on Ω for some $g = (g_{1,...,} g_n)$, $g_i \in L^\infty(\Omega)$ each i, open $\Omega \subset \mathbb{R}^n$.

To simplify the Euler-Lagrange equation for the $n = 2$ case, the authors in [42] replaced the minimization of $\mathcal{M}$ with finding

$$\inf_{u,g_1,g_2} \left\{ G_p(u, g_1, g_2) = \int_\Omega |\nabla u| + \lambda \int_\Omega |I - (u + \partial_x g_1 + \partial_y g_2)|^2 \, dxdy \right.$$
$$\left. + \mu \left[\int_\Omega \left(\sqrt{g_1^2 + g_2^2} \right)^p dxdy \right]^{1/p} \right\}$$

where λ, μ are parameters and $p \to \infty$. Due to the three variable functions u, g_1, g_2 this yields three coupled equations as a result of the Euler-Lagrange equations.

This approach is further simplified in [38] by dropping the last term in the above functional, by writing $I - u = \operatorname{div} g$ for $g \in L^2(\Omega)^2$, and by formally using the Hodge decomposition of g:

$$g = \nabla P + q$$

where q is a divergence free vector field, thus giving $u - I = -\operatorname{div} g = -\Delta P$. The inverse Laplace operator Δ^{-1} is then defined by $P =: -\Delta^{-1}(u - I)$. In fact we have (see for example, [26])

Theorem 1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open region with Lipschitz boundary $\partial\Omega$ and $V_0 = \left\{ u \in H^1(\Omega) : \int_\Omega u \, dx = 0 \right\}$. If $v \in L^2(\Omega)$ with $\int_\Omega v \, dx = 0$, then the problem*

$$-\Delta P = v, \quad \frac{\partial P}{\partial n}|_{\partial\Omega} = 0,$$

has a unique solution P in V_0.

Consequently, the OSV model proposed in [38] is to instead find a minimizer of

$$\mathcal{E}(u) := \int_\Omega |\nabla u| + \lambda \int_\Omega |\nabla(\Delta^{-1})(I - u)|^2 \, dx = \int_\Omega |\nabla u| + \lambda \, \|I - u\|^2_{H^{-1}(\Omega)} \quad (1.2)$$

over the space $L^2(\Omega)$ with the constraint $\int_\Omega u \, dx = \int_\Omega I \, dx$. For the last term on the right side it is shown in [38] that for functions $v \in L^2(\Omega)$ with $\int_\Omega v \, dx = 0$, $\|v\|^2_{H^{-1}(\Omega)} = \int_\Omega |\nabla(\Delta^{-1})v|^2 \, dx$. The Euler-Lagrange equation for this is formally

$$0 = div\left(\frac{\nabla u}{|\nabla u|}\right) - 2\lambda\Delta^{-1}(I - u) \text{ on } \Omega \tag{1.3}$$
$$\frac{\partial u}{\partial \mathbf{n}}|_{\partial\Omega} = 0$$

with constraint $\int_\Omega u\,dx = \int_\Omega I\,dx$. This is solved there numerically on a rectangle $\Omega \subset \mathbb{R}^2$ by applying $-\Delta$ to both sides of (1.3) and solving the following time flow for $u(x, y, t)$

$$\frac{\partial u}{\partial t} = -\Delta\,\text{div}\left(\frac{\nabla u}{|\nabla u|}\right) + 2\lambda(I - u)$$
$$0 = \frac{\partial u}{\partial \mathbf{n}}|_{\partial\Omega} = \frac{\partial\,\text{div}\left(\frac{\nabla u}{|\nabla u|}\right)}{\partial \mathbf{n}}|_{\partial\Omega},\ u(x, y, 0) = I(x, y) \text{ on } \Omega,$$

$\int_\Omega u\,dx = \int_\Omega I\,dx$, and letting $t \to \infty$ to drive to the steady state solution of (1.3). Clearly, the first term on the right of the equation is not defined for all functions u in BV or even $W^{1,1}$. We thus need to define a weak solution to the time flow to (1.3).

We will expand the functional $\mathcal{E}$ to include a class of Carathéodory functions for the energy term $\varphi(x, \mathbf{p})$ that are convex and of linear growth in $\mathbf{p}$. By definition, a Carathéodory function, $\varphi : \Omega \times \mathbb{R}^n \to \mathbb{R}$, satisfies the following conditions:

(1) for each $\mathbf{p} \in \mathbb{R}^n$, $\varphi(\cdot, \mathbf{p}) : \Omega \to \mathbb{R}$ is a measurable function defined on Ω and

(2) for a.e. $x \in \Omega$, $\varphi(x, \cdot) : \mathbb{R}^n \to \mathbb{R}$ is continuous in the $\mathbf{p}$ variable.

The functional is now

$$\mathcal{F}(u) := \int_\Omega \varphi(x, Du) + \lambda \int_\Omega |\nabla(\Delta^{-1})(I - u)|^2\,dx$$
$$\text{such that} \int_\Omega u\,dx = \int_\Omega I\,dx.$$

For example we have the variable exponent case,

$$\varphi(x, \mathbf{p}) = \begin{cases} \frac{1}{q(x)}|\mathbf{p}|^{q(x)} & \text{if } |\mathbf{p}| \le 1 \\ |\mathbf{p}| - \frac{q(x)-1}{q(x)} & \text{if } |\mathbf{p}| > 1 \end{cases} \tag{1.4}$$

where $q(x) \in L^\infty(\Omega)$, $1 < \alpha \le q(x) \le 2$ a.e. See [36] and [24] for an application of a functional using the anisotropic diffusion term (1.4) with simple L^2 penalty term $\int_\Omega (u - I)^2\,dx$. We also refer the reader to [39] PDE problems with variable exponent. The time flow of the Euler-Lagrange equations for $\mathcal{F}$ becomes

$$\begin{aligned} \frac{\partial u}{\partial t} &= div\left(\nabla_{\mathbf{p}}\varphi(x, \nabla u)\right) - 2\lambda\Delta^{-1}(I - u) \\ \frac{\partial u}{\partial \mathbf{n}} &= 0 \text{ on } \partial\Omega \\ u(x, 0) &= I(x) \\ \int_\Omega u\,dx &= \int_\Omega I\,dx \text{ for all } t. \end{aligned} \tag{1.5}$$

The rest of the paper is organized as follows. We extend the definition of functionals $\int_\Omega \varphi(x, Du)$ defined for $u \in BV(\Omega)$ by including certain Carathéodory functions $\varphi(x, \mathbf{p})$ where we directly use the convex dual function φ^* of φ rather than the theory of convex functionals of measures as in [8, 14, 15]. We only assume measurability in x for φ whereas previous work uses a continuity condition in x to prove lower semicontinuity of $\int_\Omega \varphi(x, Du)$ in L^1. In addition we prove compactness in L^1 with an extra L^1 integrability condition on φ. This then allows for a greater class of functionals to be considered for minimization problems that use both the $\int_\Omega \varphi(x, Du)$ term for smoothing and the dual H^1 penalty for retaining oscillatory features of images. For example we may use a more robust selective smoothing term $\int_\Omega \varphi(x, Du)$ in place of the simple total variation term $\int_\Omega |Du|$ that is used in the OSV model. We thus consider the OSV dual H^1 penalty model from [38] with general energy term $\int_\Omega \varphi(x, Du)$ and the corresponding gradient time flow (1.5). We then use the semigroup method to prove existence, L^2 stability, and asymptotic convergence for the weak solution to the time flow (1.5). It should be noted that the semigroup method is used in [9–13], where, as previously mentioned, they proved existence of a strong solution of the total variation flow

$$\begin{aligned} \frac{\partial u}{\partial t} &= div\left(\frac{\nabla u}{|\nabla u|}\right) \\ u(x, 0) &= I(x) \end{aligned}$$

with both Neumann or Dirichlet boundary conditions in L^1 and L^2. They also considered flows with a quasilinear term $\mathrm{div}(\nabla_{\mathbf{p}}\phi(x, \nabla u))$ with a modulus of continuity assumption for ϕ in the x variable. Since the flow for our case is in the Hilbert space L^2, we apply the theory of semigroups based on classical maximal monotone theory of Brezis [17]. Finally, we derive an integral property for solutions to the gradient flow for the case of space dimension $n = 1$ with pure TV term $\int_\Omega |Du|$. As we note in the Conclusion, it is hoped to extend this integral property, or possibly derive other properties, to our general case with the $\int_\Omega \varphi(x, Du)$ term.

2 The Stationary Problem and Important Results for BV Functions

We will first state some important theorems concerning functions in BV space. The following theorem is from [43]. Also see [7].

Theorem 2 *Let $\Omega \subset \mathbb{R}^n$ be bounded and open, $\varphi(x, \mathbf{p})$ be C^1 on $\Omega \times \mathbb{R}^n$, convex in $\mathbf{p}$, with linear growth for $|\mathbf{p}| \geq \beta > 0$, that is $c_1|\mathbf{p}| \leq \varphi(x, \mathbf{p}) \leq c_2(|\mathbf{p}| + 1)$ for $|\mathbf{p}| \geq \beta$ with constants $c_1, c_2, \beta > 0$, and where $\lim_{t\to\infty} \varphi(x, t\frac{\mathbf{p}}{|\mathbf{p}|})/t = \varphi^\infty(x)$. Then $\int_\Omega \varphi(x, Du)$ is lower semicontinuous in $L^1(\Omega)$.*

From [15] we also have a formula for $\int_\Omega \varphi(x, Du)$ for $u \in BV(\Omega)$ where φ is the C^1 function as stated in Theorem 2, in fact

$$\int_\Omega \varphi(x, Du) = \int_\Omega \varphi(x, \nabla u)\, dx + \int_\Omega \varphi^\infty(x)|D^s u|. \tag{2.1}$$

The approximation of BV functions by smooth functions for the anisotropic functional with (1.4) is given by:

Theorem 3 *If $\Omega \subset \mathbb{R}^n$ is bounded, open, $u \in BV(\Omega) \cap L^2(\Omega)$, and φ given by (1.4), then (1) there exists a sequence $\{u_j\} \subset C^\infty(\Omega) \cap H^1(\Omega)$ such that*

$$u_j \to u \text{ in } L^2(\Omega) \text{ and}$$
$$\lim_{j\to\infty} \int_\Omega \varphi(x, Du_j) = \int_\Omega \varphi(x, Du);$$

(2) if $\displaystyle\int_\Omega u\, dx = c$, we may take the sequence above to also satisfy $\displaystyle\int_\Omega u_j\, dx = c$; (3) if $u \in L^\infty(\Omega)$, and φ is independent of x, then we may also take the sequence to satisfy $\|u_j\|_{L^\infty} \leq C(\Omega)\|u\|_{L^\infty}$, and if Ω has Lipschitz boundary $\partial\Omega$, we may also take the sequence to satisfy $u_j \in C^\infty(\overline{\Omega})$.

Proof With simple modifications, the first part is proved as in [24] (in their case for u with trace value $Tu_{|\partial\Omega}$) using $\int_\Omega \varphi(x, Du) = \sup_{\phi\in\mathcal{V}} \left\{-\int_\Omega u div\phi + \varphi^*(x, \phi)\, dx\right\}$. In fact it is only assumed that $q(x) \in L^\infty(\Omega)$, $1 < \alpha \leq q(x) \leq 2$ a.e. For the second part we note that $u_j \to u$ in $L^2(\Omega)$ implies that $\displaystyle\int_\Omega u_j\, dx \to \int_\Omega u\, dx = c$. We then let $\widetilde{u}_j = u_j - \frac{1}{|\Omega|}\displaystyle\int_\Omega (u_j - c)dx$ giving

$$\widetilde{u}_j \to u \text{ in } L^2(\Omega) \text{ and}$$
$$\lim_{n\to\infty}\int_\Omega \varphi(x, D\widetilde{u}_j) = \int_\Omega \varphi(x, Du)$$
$$\int_\Omega \widetilde{u}_j\, dx = 0 \text{ for all } j.$$

For (3), note the remark in [25]. □

Remark 1 For the case of pure total variation $\varphi(\mathbf{p}) = |\mathbf{p}|$, from [32] Theorems 2 and 3 hold with the proof of (2) the same as above.

For similar approximation results with a different proof for functions φ defined on $\Omega \times \mathbb{R}^n$, with certain continuity conditions on both x and $\mathbf{p}$, see for example [12].

We now extend the definition of $\int_\Omega \varphi(x, Du)$ for a Carath éodory function which is continuous in $\mathbf{p}$ and no continuity assumption in x, by using the Legendre transform. For a convex function g on $\mathbb{R}^n$, the convex dual or Legendre transform $g^*$of g, is defined as $g^*(\mathbf{q}) = \sup_{\mathbf{p}\in\mathbb{R}^n}\{\mathbf{p}\cdot\mathbf{q}-g(\mathbf{p})\}$. If g is continuous then by the Fenchel-Moreau theorem [19] we in fact have $g^{**}(\mathbf{p}) = g(\mathbf{p}) = \sup_{\mathbf{q}\in\mathbb{R}^n}\{\mathbf{p}\cdot\mathbf{q}-g^*(\mathbf{q})\}$.

Proposition 1 *Let $\Omega \subset \mathbb{R}^n$ be open, $\varphi(x, \mathbf{p})$ a Carathéodory function on $\Omega \times \mathbb{R}^n$, continuous and convex in $\mathbf{p}$, of linear growth in $\mathbf{p}$ with $c_1|\mathbf{p}| - c_2 \le \varphi(x, \mathbf{p}) \le c_1(|\mathbf{p}| + 1)$ for $|\mathbf{p}| \ge \beta$, for constants $c_1 > 0$, $\beta, c_2 \ge 0$. Then (1) for a.e. x, $\varphi^*(x, \mathbf{q}) = \sup_{\{\mathbf{p}\in\mathbb{R}^n, |\mathbf{p}|\le\beta\}}\{\mathbf{q}\cdot\mathbf{p} - \varphi(x, \mathbf{p})\} = \max_{\{\mathbf{p}\in\mathbb{R}^n, |\mathbf{p}|\le\beta\}}\{\mathbf{q}\cdot\mathbf{p} - \varphi(x, \mathbf{p})\}$ and (2) $\varphi^*(x, \mathbf{q})$ is a Carathéodory function on $\Omega \times \{|\mathbf{q}| \le c_1\}$. Furthermore $\varphi^*(x, \mathbf{q}) = \infty$ for a.e. x, $|\mathbf{q}| > c_1$.*

Proof By the linear growth condition $\varphi(x, \mathbf{p}) \le c_1(|\mathbf{p}| + 1)$, we have $\varphi^*(x, \mathbf{q}) < \infty$ if and only if $|\mathbf{q}| \le c_1$ and this occurs for $|\mathbf{p}| \le \beta$ from the assumption $c_1|\mathbf{p}| - c_2 \le \varphi(x, \mathbf{p})$. The fact that the supremum is a maximum follows by continuity. This proves (1). To prove (2) we fix a.e. x and first assume that $\varphi(x, \mathbf{p})$ is strictly convex for $|\mathbf{p}| \le \beta$. The case where $\beta = 0$ gives $\varphi^*(x, \mathbf{q}) = \max_{\{\mathbf{p}\in\mathbb{R}^n, |\mathbf{p}|=0\}}\{\mathbf{q}\cdot\mathbf{p} - \varphi(x, \mathbf{p})\} = -\varphi(x, \mathbf{0})$ if $|\mathbf{q}| \le c_1$. Now assume $\beta > 0$. Then by strict convexity there is a unique $\mathbf{p}^*(\mathbf{q})$ with $|\mathbf{p}^*(\mathbf{q})| \le \beta$ so that $\varphi^*(x, \mathbf{q}) = \mathbf{q}\cdot\mathbf{p}^*(\mathbf{q}) - \varphi(x, \mathbf{p}^*(\mathbf{q}))$. To show that $\mathbf{p}^*$ is continuous we let $\mathbf{q}_n \to \mathbf{q}$. Thus there is a subsequence $\mathbf{q}_{n_k}$ such that $\mathbf{p}^*(\mathbf{q}_{n_k}) \to \mathbf{p}'$ for some $|\mathbf{p}'| \le \beta$. Hence for each $\mathbf{q}_{n_k}$, $\varphi^*(x, \mathbf{q}_{n_k}) = \mathbf{q}_{n_k}\cdot\mathbf{p}^*(\mathbf{q}_{n_k}) - \varphi(x, \mathbf{p}^*(\mathbf{q}_{n_k})) \ge \mathbf{q}_{n_k}\cdot\mathbf{p} - \varphi(x, \mathbf{p})$ for all $|\mathbf{p}| \le \beta$. Thus for each $|\mathbf{p}| \le \beta$ $\mathbf{q}\cdot\mathbf{p} - \varphi(x, \mathbf{p}) \le \lim_{k\to\infty}\varphi^*(x, \mathbf{q}_{n_k}) = \lim_{k\to\infty}(\mathbf{q}_{n_k}\cdot\mathbf{p}^*(\mathbf{q}_{n_k}) - \varphi(x, \mathbf{p}^*(\mathbf{q}_{n_k}))) = \mathbf{q}\cdot\mathbf{p}' - \varphi(x, \mathbf{p}')$. Therefore $\mathbf{p}' = \mathbf{p}^*(\mathbf{q})$. To show that the full sequence $\mathbf{p}^*(\mathbf{q}_n)$ converges to $\mathbf{p}^*(\mathbf{q})$ we assume that there is another subsequence $\mathbf{q}_{n_i}$ and $\varepsilon > 0$ such that $\mathbf{q}_{n_i} \to \mathbf{q}$ but $\left|\mathbf{p}^*(\mathbf{q}_{n_i}) - \mathbf{p}^*(\mathbf{q})\right| \ge \varepsilon$ for all n_i. We extract a further subsequence $\mathbf{q}_{n_{i_j}}$ with $\mathbf{q}_{n_{i_j}} \to \mathbf{q}$ and $\mathbf{p}^*(\mathbf{q}_{n_{i_j}}) \to \mathbf{p}''$. Repeating the above argument we have $\mathbf{p}'' = \mathbf{p}^*(\mathbf{q})$ but $\left|\mathbf{p}''-\mathbf{p}^*(\mathbf{q})\right| \ge \varepsilon$, a contradiction. Since $\mathbf{p}^*(\mathbf{q})$ is continuous, so is $\varphi^*(x, \mathbf{q})$. Without the strict convex assumption on $\varphi(x, \mathbf{q})$ we consider $\varphi_\varepsilon(x, \mathbf{p}) := \varphi(x, \mathbf{p}) + \varepsilon\,|\mathbf{p}|^2$ for $|\mathbf{p}| \le \beta$. As $\varepsilon \ge \varphi^*(x, \mathbf{p}) - \varphi^*_\varepsilon(x, \mathbf{p})$ and

$\varphi^*(x,\mathbf{p}) \geq \varphi_\varepsilon^*(x,\mathbf{p})$ we have $\varepsilon \geq |\varphi^*(x,\mathbf{p}) - \varphi_\varepsilon^*(x,\mathbf{p})|$ and thus $\varphi_\varepsilon^* \to \varphi^*$ uniformly on $|\mathbf{p}| \leq \beta$ as $\varepsilon \to 0$. Since $\varphi_\varepsilon(x,\mathbf{p})$ is strictly convex for $|\mathbf{p}| \leq \beta$, $\varphi_\varepsilon^*(x,\mathbf{p})$ is continuous in $\mathbf{p}$, hence it follows that $\varphi^*(x,\mathbf{p})$ is continuous for $|\mathbf{p}| \leq \beta$. Finally, φ^* being for fixed $\mathbf{q}$ the pointwise maximum of measurable functions in x, is measurable in x. Item (2) is proved. □

This proposition then allows us to define the following:

Definition 1 For open $\Omega \subset \mathbb{R}^n$ and $\varphi(x,\mathbf{p})$ a Carathéodory function on $\Omega \times \mathbb{R}^n$, continuous and convex in $\mathbf{p}$, of linear growth in $\mathbf{p}$ with $c_1|\mathbf{p}| - c_2 \leq \varphi(x,\mathbf{p}) \leq c_1(|\mathbf{p}|+1)$ for $|\mathbf{p}| \geq \beta$, for constants $c_1 > 0$, $\beta, c_2 \geq 0$. Define

$$\int_\Omega \varphi(x,Du) = \sup_{\phi\in\mathcal{V}} \left\{ -\int_\Omega u div\phi + \varphi^*(x,\phi(x))\, dx \right\}$$

where $\varphi^*(x,\mathbf{q}) = \sup_{\{\mathbf{p}\in\mathbb{R}^n, |\mathbf{p}|\leq\beta\}}\{\mathbf{q}\cdot\mathbf{p} - \varphi(x,\mathbf{p})\}$ for each $\mathbf{q} \in \mathbb{R}^n$ with $|\mathbf{q}| \leq c_1$ and

$$\mathcal{V} = \left\{ \phi \in C_c^1(\Omega,\mathbb{R}^n) : |\phi(x)| \leq c_1 \text{ for all } x \in \Omega \right\}.$$

Note that the supremum is only taken for $\phi \in \mathcal{V}$ since from the proposition $\varphi^*(x,\mathbf{q}) = \infty$ if $|\mathbf{q}| > c_1$.

We remark that this is the definition used in [24] for the specific case of the anisotropic functional $\int_\Omega \varphi(x,Du)$ where φ, given by (1.4), satisfies the conditions of Definition 1, and φ^* is directly calculated. Also for the total variation case $\varphi(\mathbf{p}) = |\mathbf{p}|$ we have $c_1 = 1$ and φ^* is the usual

$$\varphi^*(\mathbf{q}) = \begin{cases} 0 & \text{if } |\mathbf{q}| \leq 1 \\ \infty & \text{otherwise} \end{cases}$$

From Definition 1, lower semicontinuity in $L^1(\Omega)$ follows immediately as in [32].

Theorem 4 *If Ω and φ satisfy the conditions of Definition 1, $\int_\Omega \varphi(x,Du)$ is lower semicontinuous in $L^1(\Omega)$.*

Proof Let $u_n \to u$ in $L^1(\Omega)$. Then for fixed $\phi \in \mathcal{V}$ we have $-\int_\Omega u div\phi + \varphi^*(x,\phi)\, dx = \lim_{n\to\infty} \left(-\int_\Omega u_n div\phi + \varphi^*(x,\phi)\, dx\right) \leq \liminf_{n\to\infty} \int_\Omega \varphi(x,Du_n)$. Taking the supremum on the left gives $\int_\Omega \varphi(x,Du) \leq \liminf_{n\to\infty} \int_\Omega \varphi(x,Du_n)$. □

With an added L^1 condition on φ we have

Theorem 5 *If $\Omega \subset \mathbb{R}^n$ is open and bounded, φ satisfies the conditions of Definition 1 and in addition $\int_\Omega \sup_{|\mathbf{p}|\leq\beta} |\varphi(x,\mathbf{p})|\, dx \leq c_3$ for some $c_3 > 0$, then $\int_\Omega \varphi(x,Du) < \infty$ if and only if $u \in BV(\Omega)$. In fact we have $c_1 \int_\Omega |Du| \leq \int_\Omega \varphi(x,Du) + C(c_1,c_3,\beta,\Omega)$ and $\int_\Omega \varphi(x,Du) \leq c_1 \int_\Omega |Du| + C(c_1,c_3,\beta,\Omega)$ for some constant $C(c_1,c_3,\beta,\Omega) \geq 0$.*

Proof From the definition of φ^* we have $\varphi^*(x, \phi(x)) \leq |\phi(x)|\beta + \sup_{|\mathbf{p}|\leq\beta} |\varphi(x, \mathbf{p})|$ and thus

$$
\begin{aligned}
c_1 \int_\Omega |Du| &= \sup_{\phi\in\mathcal{V}} \left\{ -\int_\Omega u div\phi \, dx \right\} \\
&\leq \sup_{\phi\in\mathcal{V}} \left\{ -\int_\Omega u div\phi + \varphi^*(x, \phi) \, dx \right\} \\
&\quad + \sup_{\phi\in\mathcal{V}} \left| \int_\Omega \varphi^*(x, \phi) \, dx \right| \\
&\leq \int_\Omega \varphi(x, Du) + C(c_1, c_3, \beta, \Omega)
\end{aligned}
$$

where $C(c_1, c_3, \beta, \Omega) \geq 0$; and also

$$
\begin{aligned}
\int_\Omega \varphi(x, Du) &= \sup_{\phi\in\mathcal{V}} \left\{ -\int_\Omega u div\phi + \varphi^*(x, \phi) \, dx \right\} \\
&\leq c_1 \int_\Omega |Du| + C(c_1, c_3, \beta, \Omega).
\end{aligned}
$$

□

We then have the compactness theorem:

Theorem 6 *Let φ satisfy the conditions of Theorem 5. Let u_j be a sequence in $BV\ (\Omega)$ with $\int_\Omega \varphi(x, Du_j)$ bounded, where $\Omega \subset \mathbb{R}^n$ is bounded with Lipschitz boundary $\partial\Omega$. Then there is a subsequence of u_j, also denoted by u_j, and $u \in L^p\ (\Omega)$ such that $u_j \to u$ strongly in $L^p\ (\Omega)$ for all $1 \leq p < n/(n-1)$ and weakly in $L^{n/(n-1)}\ (\Omega)$.*

Proof From Theorem 5, u_j is a sequence bounded in $BV\ (\Omega)$. The theorem then follows from Giusti [32]. □

Remark 2 We assumed that $c_1|\mathbf{p}| - c_2 \leq \varphi(x, \mathbf{p}) \leq c_1(|\mathbf{p}| + 1)$ for $|\mathbf{p}| \geq \beta$ for ease of proof. However, we may replace this with the more general linear growth condition $k_1|\mathbf{p}| - c \leq \varphi(x, \mathbf{p}) \leq k_2(|\mathbf{p}| + 1)$ for $|\mathbf{p}| \geq \beta$ for $k_2 > k_1 > 0$, $\beta, c \geq 0$, with the same convex and Carathéodory condition on φ. In this case we still have $\varphi^*(x, \mathbf{q}) < \infty$ if and only if $|\mathbf{q}| \leq k_2$. If $\varphi^*(x, \mathbf{q})$ achieves is supremum on a bounded set $|\mathbf{p}| \leq K$ where K is independent of $\mathbf{q}$, then Proposition 1, Definition 1, and Theorems 4–7 hold with the respective L^1 integral condition on φ.

We return to the minimization problem from [37] using the OSV model. We extend this model to include any φ as stated in Theorem 5. This assumption will hold in the sequel unless stated otherwise. As stated in the introduction the minimization model is

$$\min_{u \in BV \cap V_I} \mathcal{F}(u) := \int_\Omega \varphi(x, Du) + \lambda \|u - I\|^2_{H^{-1}(\Omega)} = \int_\Omega \varphi(x, Du) + \lambda \int_\Omega |\nabla(\Delta^{-1})(I - u)|^2 \, dx \tag{2.2}$$

where $V_I =: \left\{u \in L^2(\Omega) \mid \int_\Omega u \, dx = \int_\Omega I \, dx\right\}$, $\Omega \subset \mathbb{R}^n$.

Theorem 7 *For $n = 1$ or 2, the functional F is convex, lower semicontinuous and thus the stationary problem (2.2) has a unique solution.*

Proof This is proved in [38] for the original problem of minimizing (1.2). We just note that for φ we still have lower semicontinuity and compactness from Theorems 4 and 5. Existence and uniqueness then follows from standard theory. □

For the rest of the paper we assume $n = 1$ or 2.

3 Time Flow of the Weak Solution

We return to the gradient time flow corresponding to the stationary problem (2.2) using the Euler-Lagrange equation of (2.2), $div\,(\varphi_P(x, \nabla u)) - 2\lambda\Delta^{-1}(u - I) = 0$. Without loss of generality we will assume $\varphi(x, \mathbf{0}) = 0$.

Definition 2 The time flow of (2.2) is defined by

$$\frac{\partial u}{\partial t} = div\left(\nabla_{\mathbf{p}}\varphi(x, \nabla u)\right) - 2\lambda\Delta^{-1}(I - u) \tag{3.1}$$

$$\frac{\partial u}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega \tag{3.2}$$

$$u(x, 0) = I(x) \tag{3.3}$$

$$\int_\Omega u \, dx = \int_\Omega I \, dx \text{ for all } t. \tag{3.4}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded region with Lipschitz boundary $\partial\Omega$, and $I \in L^2(\Omega) \cap BV(\Omega)$.

Since u is assumed to be only in BV, this must be defined as a weak solution as will be given below. In the sequel, Ω satisfies the conditions stated in Definition 2. Following, for example, [25, 45] we motivate the definition of a weak solution to (3.1)–(3.4) by assuming sufficient smoothness of u and v satisfying the constraint

$$\int_\Omega u \, dx = \int_\Omega v \, dx = \int_\Omega I \, dx$$

for a.e. t, multiplying (3.1) by $v-u$, integrating by parts, using convexity of φ, namely $\varphi(x,\mathbf{p})-\varphi(x,\mathbf{q})\geq\nabla\varphi_P(x,\mathbf{q})\cdot(\mathbf{p}-\mathbf{q})$, noting that

$$\int_\Omega \Delta^{-1}(I-u)(v-u)\,dx = -\int_\Omega \Delta^{-1}(I-u)\Delta\Delta^{-1}(v-u)\,dx$$
$$= \int_\Omega \nabla\Delta^{-1}(I-u)\cdot\nabla\Delta^{-1}(v-I+I-u)\,dx,$$

and finally expanding and using Young's inequality to get for a.e.t

$$\int_\Omega u_t(v-u)\,dx + \int_\Omega \varphi(x,\nabla v) + \lambda\int_\Omega |\nabla(\Delta^{-1})(I-v)|^2\,dx \tag{3.5}$$
$$\geq \int_\Omega \varphi(x,\nabla u) + \lambda\int_\Omega |\nabla(\Delta^{-1})(I-u)|^2\,dx.$$

By Theorem 3 we see that (3.5) holds for u, v in $BV(\Omega)$ satisfying the above constraint for a.e. t. We therefore define a *weak solution* $u\in L^2((0,T);L^2(\Omega)\cap V_I)\cap L^1((0,T);BV(\Omega))$, $u_t\in L^2((0,T),L^2(\Omega))$ of (3.1)–(3.4) to satisfy (3.5) for all

$$v\in L^2((0,T);L^2(\Omega)\cap V_I)\cap L^1((0,T);BV(\Omega))$$

where

$$V_I=\left\{v\in L^2(\Omega):\ \int_\Omega v\,dx=\int_\Omega I\,dx\right\}.$$

In what follows, let H_0 be the Hilbert space

$$H_0=\left\{v\in L^2(\Omega):\ \int_\Omega v\,dx=0\right\}.$$

Theorem 8 *Let φ satisfy the conditions of Theorem 5 and $I\in L^2(\Omega)\cap BV(\Omega)$. There exists a unique weak solution $u(t)$ to (3.1)–(3.4). That is, for a.e. $t>0$, $u(t)\in L^2(\Omega)$ with $u(t)-I\in BV(\Omega)\cap H_0$, $u_t\in L^\infty((0,\infty);H_0)$*

$$\int_\Omega u_t(v-u)\,dx + \int_\Omega \varphi(x,Dv) + \lambda\int_\Omega |\nabla(\Delta^{-1})(I-v)|^2\,dx \tag{3.6}$$
$$\geq \int_\Omega \varphi(x,Du) + \lambda\int_\Omega |\nabla(\Delta^{-1})(I-u)|^2\,dx$$

for each $v-I\in BV(\Omega)\cap H_0$. Hence for the case with constraint $\int_\Omega I\,dx=0$ we have for a.e.$t>0$

$$\frac{1}{2}\int_{\Omega} u^2\,dx + \int_0^t\int_{\Omega}\varphi(x,Du)\,ds + \lambda\int_0^t\int_{\Omega}|\nabla(\Delta^{-1})(I-u)|^2\,dx\,ds \tag{3.7}$$
$$\leq \lambda\int_0^t\int_{\Omega}|\nabla\Delta^{-1}I|^2\,dxds$$

and

$$\frac{1}{2}\int_{\Omega}(u-I)^2\,dx + \int_0^t\int_{\Omega}\varphi(x,Du)\,ds \tag{3.8}$$
$$+\lambda\int_0^t\int_{\Omega}|\nabla(\Delta^{-1})(I-u)|^2\,dxds \leq \int_0^t\int_{\Omega}\varphi(x,DI)\,ds;$$

and for the general case $\int_{\Omega} I\,dx = c$, *we have (3.8) for a.e.* $t>0$

$$\frac{1}{2}\int_{\Omega}(u-\frac{c}{|\Omega|})^2\,dx + \int_0^t\int_{\Omega}\varphi(x,Du)\,ds + \lambda\int_0^t\int_{\Omega}|\nabla(\Delta^{-1})(I-u)|^2\,dxds \tag{3.9}$$
$$\leq \lambda\int_0^t\int_{\Omega}|\nabla(\Delta^{-1})(I-\frac{c}{|\Omega|})|^2\,dxds.$$

Also for initial conditions $I_1, I_2 \in L^2(\Omega)\cap BV(\Omega)$ *with corresponding solutions* u_1, u_2,

$$\|u_1-u_2\|_{L^2(\Omega)} \leq \|I_1-I_2\|_{L^2(\Omega)}$$

for a.e. $t>0$. *Finally, The solution* u *to (3.1)–(3.4) converges weakly in* $L^2(\Omega)$ *and strongly in* $L^1(\Omega)$ *to the minimizer of* u_∞ *of 2.2 as* $t\to\infty$.

Proof We first assume $\int_{\Omega} I\,dx = 0$. The functional

$$F(u) =: \begin{cases} \int_{\Omega}\varphi(x,Du) + \lambda\int_{\Omega}|\nabla(\Delta^{-1})(I-u)|^2\,dx & \text{if } u\in BV(\Omega)\cap H_0 \\ \infty & \text{if } u\in H_0\backslash BV(\Omega) \end{cases}$$

on H_0 is proper, convex, and lower semicontinuous from Theorem 4. Consequently from the theory from maximal monotone operators and semigroups [17], the subdifferential $\partial F(u)$ is a maximal monotone operator with a unique, absolutely continuous solution $u(t)\in[0,\infty)\to H_0$, $u(0)=I$, $u_t\in L^\infty((0,\infty);H_0)$, to

$$-u_t \in \partial F(u(t)).$$

Thus by the definition of ∂F, the first inequality (3.6) holds. Also from [17]

$$\|u_1-u_2\|_{L^2(\Omega)} \leq \|I_1-I_2\|_{L^2(\Omega)}$$

for solutions u_1, u_2 with corresponding initial conditions $I_1, I_2\in L^2(\Omega)\cap BV(\Omega)$. The inequalities (3.7) and (3.8) are obtained by letting $v=0$ and $v=I$ respectively and integrating with respect to t. For the general constraint $\int_{\Omega} I\,dx = c$, we replace

u in (3.6) with $\widetilde{u} = u - \frac{c}{|\Omega|}$ so that $\int_\Omega \widetilde{u}\, dx = 0$. Letting $v = \frac{c}{|\Omega|}$ gives (3.9), noting $\varphi(x, \mathbf{0}) = 0$.

We now consider the asymptotic limit of the solution $u(t)$ as $t \to \infty$. Let u be the solution to (3.1)–(3.3). Since $-\frac{du}{dt} \in \partial F(u)$ the theorem from [20] proves that $u(t) \rightharpoonup u_\infty$ in $L^2(\Omega)$ weakly as $t \to \infty$. To prove strong convergence in $L^1(\Omega)$ we use Theorem A.33 in [12], which implies that, after adjusting by a constant if necessary, $\int_\Omega \varphi(x, Du(t)) + \lambda \int_\Omega |\nabla(\Delta^{-1})(I - u(t))|^2\, dx$ is a decreasing function of t with

$$\int_\Omega \varphi(x, Du(t)) + \lambda \int_\Omega |\nabla(\Delta^{-1})(I - u(t))|^2\, dx \le \int_\Omega \varphi(x, DI).$$

From Poincaré's inequality for BV functions, $\int_\Omega |u - u_\Omega| dx \le C(\Omega) \int_\Omega |Du|$ where $u_\Omega := \dfrac{\int_\Omega u dx}{|\Omega|} = 0$ for a.e. t. Thus by Theorem 5 $u(t)$ is bounded in $BV(\Omega)$ and by compactness (Theorem 6) and uniqueness of limits, there exists a subsequence $u(t_n) \to u_\infty$ in $L^1(\Omega)$ as $t_n \to \infty$. Hence $u(t) \to u_\infty$ in $L^1(\Omega)$ as $t \to \infty$. Again adjusting by a constant, we again have $u(t) \to u_\infty$ in $L^1(\Omega)$ for the general case of $\int_\Omega u dx = c$. □

For φ satisfying the conditions of Theorem 2, formula (2.1) holds. Now replacing v in (3.6) with $\eta(v - u) + u$ for $\eta > 0$ dividing by η and letting $\eta \to 0^+$ we obtain as in [33, 34] for any $v \in BV(\Omega) \cap H_0(\Omega)$ with $D^s v << |D^s u|$

$$\begin{aligned}\int_\Omega (v - u) u_t\, dx \le -\int_\Omega \nabla_{\mathbf{p}} \varphi(x, \nabla u) \cdot (\nabla v - \nabla u)\, dx + \int_\Omega \varphi^\infty(x) \frac{D^s u}{|D^s u|} \cdot (D^s v - D^s u) \\ - \int_\Omega \left[2\lambda \Delta^{-1}(I - u)\right] (v - u)\, dx\end{aligned}$$

where $\frac{D^s u}{|D^s u|}$ denotes the Radon-Nikodym derivative of $D^s u$ with respect to $|D^s u|$. Note that $\left|\frac{D^s u}{|D^s u|}\right| = 1$, $|D^s u|$-a.e. Repeating for $\eta < 0$ we have equality:

$$\begin{aligned}\int_\Omega (v - u) u_t\, dx = -\int_\Omega \nabla_{\mathbf{p}} \varphi(x, \nabla u) \cdot (\nabla v - \nabla u)\, dx \\ + \int_\Omega \varphi^\infty(x) \frac{D^s u}{|D^s u|} \cdot (D^s v - D^s u) - \int_\Omega \left[2\lambda \Delta^{-1}(I - u)\right] (v - u)\, dx\end{aligned} \tag{3.10}$$

for a.e. $t \ge 0$, for all $v \in BV(\Omega) \cap H_0$ with $D^s v << |D^s u|$. Now letting $v = u + \phi$ for any $\phi \in C_0^\infty(\Omega) \cap H_0$

$$\frac{\partial u}{\partial t} = div\,(\varphi_P(x, \nabla u)) - 2\lambda \Delta^{-1}(I - u)\ \mathcal{D}'(\Omega) \cap H_0$$

as $D^s \phi = 0$. This gives

Corollary 1 *For the case* $\varphi \in C^1(\Omega \times \mathbb{R}^n)$ *satisfying the conditions of Theorem 2, the weak solution* $u(t)$ *to (3.1)–(3.4), satisfies for a.e.* $t \geq 0$,

$$\frac{\partial u}{\partial t} = div\,(\nabla \varphi_P(x, \nabla u)) - 2\lambda \Delta^{-1}(I-u)\ \mathcal{D}'(\Omega) \cap H_0$$

(in the distributional sense), and for fixed a.e. $t \geq 0$, *(3.10) holds for all* $v \in BV(\Omega) \cap H_0$ *with* $D^s v << |D^s u|$.

In the following theorem we note a property of the weak solution u to (3.1)–(3.3), inspired by a result for the stationary case in [38]. Additionally we extend this to an integral result for the case of $n = 1$.

Theorem 9 *Let* $w =: -2\lambda\Delta^{-1}(I-u)$. *If* u *is a weak solution to (3.1)-(3.3) for* $n = 1$ *and* $\varphi(\mathbf{p}) = \mathbf{p}$ *then there exists a* $g \in L^\infty(\Omega)$ *with* $\|g\|_\infty \leq 1$ *such that* $g' = w - u_t =: -2\lambda\Delta^{-1}(I-u) - u_t$.

Proof By assumption we have for a.e. $t \in [0, T]$

$$-u_t \in \partial J(u) + 2\lambda\Delta^{-1}(I-u)$$

where

$$J(u) =: \int_\Omega |\nabla u|.$$

Thus

$$-u_t - 2\lambda\Delta^{-1}(I-u) \in \partial J(u)$$

and hence by duality (see [30])

$$u \in \partial J^*(-2\lambda\Delta^{-1}(I-u) - u_t)$$

for a.e. t. where

$$J^*(u) = \sup_{u \in L^2(\Omega)} \int_\Omega (uv - J(u))dx = \begin{cases} 0 \text{ if } u \in K \\ +\infty \text{ otherwise} \end{cases}$$

and

$$K =: \left\{ div\,\mathbf{g} |\ \mathbf{g} \in (L^2(\Omega))^2 \text{ and } \|\mathbf{g}\|_\infty \leq 1 \right\}.$$

Therefore

$$\begin{aligned} 0 &\in -2\lambda u + 2\lambda\partial J^*(-2\lambda\Delta^{-1}(I-u) - u_t) \\ &= 2\lambda(I-u) - 2\lambda I + 2\lambda\partial J^*(-2\lambda\Delta^{-1}(I-u) - u_t). \end{aligned}$$

Hence for $w =: -2\lambda\Delta^{-1}(I-u)$

$$0 \in \partial\frac{\left\|\nabla(w+2\lambda\Delta^{-1}I)\right\|_{L^2}^2}{2} + 2\lambda\partial J^*(w-u_t)$$
$$= \partial\frac{\left\|\nabla(w+2\lambda\Delta^{-1}I)\right\|_{L^2}^2}{2} + 2\lambda\partial\bar{J}(w)$$

where $\bar{J}(v) =: J^*(v-u_t)$ and ∂ denotes the subdifferential. Thus w is in fact a minimizer of

$$G(\widehat{w}) =: \frac{\left\|\nabla(\widehat{w}+2\lambda\Delta^{-1}I)\right\|_{L^2}^2}{2} + 2\lambda\bar{J}(\widehat{w})$$

over all $\widehat{w} \in H^1(\Omega) \cap V_0$. For $n=1$, Ω an open interval, by choosing a $\widehat{w} \in H^1(\Omega) \cap V_0$ with $\|\widehat{w}-u_t\|_{L^2} \le |\Omega|^{-1/2}$ we can find g on Ω with $g' = \widehat{w}-u_t$ and $\|g\|_\infty \le 1$, namely $g(x) = \int_a^x(\widehat{w}-u_t)\,dx$, some $a \in \Omega$. Thus the functional $\bar{J}$ (and hence G) is proper, that is, $\bar{J}$ is finite for some $\widehat{w}$. Therefore $\widehat{w} \in K$ and the theorem is proved. □

Corollary 2 *If u is a weak solution to (3.1)–(3.3) for $n=1$, $\varphi(\mathbf{p}) = |\mathbf{p}|$, with Ω an open interval, then for each subinterval $[z,z'] \subset \Omega$,*

$$ess\sup_{t\ge 0}\left|\int_z^{z'} \lambda\Delta^{-1}(I-u) + \frac{1}{2}u_t\,dx\right| \le 1.$$

Proof On each subinterval $[z,z']$ of Ω we have for a.e. $t \ge 0$

$$\int_z^{z'} g'\,dx = \int_z^{z'} (-2\lambda\Delta^{-1}(I-u)-u_t)\,dx.$$

Hence as $\|g\|_\infty \le 1$,

$$\left|\int_z^{z'} (-2\lambda\Delta^{-1}(I-u)-u_t)\,dx\right| \le 2$$

for a.e. $t \ge 0$. □

4 Conclusion

We have defined $\int_\Omega \varphi(x,Du)$ for a class Carath éodory functions $\varphi(x,\mathbf{p})$ that are convex and of linear growth in $\mathbf{p}$, with the use of the convex dual φ^* of φ. With this definition, lower semicontinuity in L^1 immediately follows without any continuity assumption in x as was assumed in previous work. We then used these results to prove the existence of the flow in $BV \cap L^2$ of the dual H^1 penalty image restoration model, with our general energy term $\int_\Omega \varphi(x, Du)$, rather than just $\int_\Omega |Du|$ as in [44].

For further study, we note that for functions $u \in W^{1,1}(\Omega)$, integration by parts and the Fenchel-Moreau theorem gives

$$
\begin{aligned}
-\int_\Omega u \operatorname{div} \phi + \varphi^*(x, \phi(x))\, dx &= \int_\Omega \nabla u \cdot \phi - \varphi^*(x, \phi(x))\, dx \\
&\le \int_\Omega \sup_{\phi \in \mathcal{V}} \left\{ \nabla u \cdot \phi - \varphi^*(x, \phi(x)) \right\} dx \\
&= \int_\Omega \varphi^{**}(x, \nabla u)\, dx = \int_\Omega \varphi(x, \nabla u) dx.
\end{aligned}
$$

Thus $\int_\Omega \varphi(x, Du) \le \int_\Omega \varphi(x, \nabla u) dx$. To show the reverse inequality, we require a sequence of functions ϕ_j in $C_c^1(\Omega)$ such that

$$
\sup_j \int_\Omega \nabla u \cdot \phi_j - \varphi^*(x, \phi_j(x))\, dx \ge \int_\Omega \sup_{\phi \in \mathcal{V}} \left\{ \nabla u \cdot \phi - \varphi^*(x, \phi(x)) \right\} dx.
$$

For $\varphi \in C^1(\Omega \times \mathbb{R}^n)$ we may use the implicit function theorem as was done in [43], whereas in this case we only have φ^* measurable in x. Using Proposition 1, it is hoped we can extend Theorem 3 to our class of φ as was done for the anisotropic model in [24], as well as extend formula 2.1 and hence Corollary 1, if φ is C^1 in $\mathbf{p}$. We may also consider extensions of Theorem 9 and Corollary 2 for this class of φ, noting the use of the dual J^*.

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