Stable Homotopy Groups of Moore Spaces

Inès Saihi

Abstract We determine explicitly the stable homotopy groups of Moore spaces up to the range 7, using an equivalence of categories which allows to consider each Moore space as an exact couple of $\mathbb{Z}$-modules.

Keywords Moore spaces · Stable homotopy groups · Equivalence of categories

1991 Mathematics Subject Classification Primary 55Q10 · Secondary 55U20 · 18G99

1 Introduction

Moore spaces and their stable homotopy groups were widely studied and a complete reference on this subject is the book of Baues [1].

In this paper, we propose a new approach allowing to see Moore spaces as exact couples of $\mathbb{Z}$-modules by means of an equivalence of categories. Even though a similar result is proven in [1], the approach given here is of independent interest, since it is used to determine explicitly the stable homotopy groups of Moore spaces up to the range 7.

Let G be an abelian group and n an integer greater than 1. A Moore space $M(G, n)$ is a simply connected CW-complex X such that $H_n(X) \simeq G$ and $\widetilde{H}_i(X) = 0$ for $i \neq n$. The homotopy type of $M(G,n)$ is uniquely determined by the pair (G,n) (see [6]).

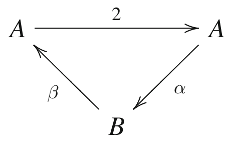
I. Saihi (✉)
Université de Tunis, École nationale supérieure d'ingénieurs de Tunis,
Tunis, Tunisia
e-mail: ines.saihi@esstt.rnu.tn

I. Saihi
Laboratoire LATAO, faculté des sciences de Tunis, université de Tunis-El Manar,
Tunis, Tunisia

T. Abualrub et al. (eds.), *Mathematics Across Contemporary Sciences*,
Springer Proceedings in Mathematics & Statistics 190,
DOI 10.1007/978-3-319-46310-0_11

Let $\mathscr{M}_n$ be the category whose objects are Moore spaces $M(A,n)$, where A is a $\mathbb{Z}$-module, and whose morphisms are homotopy classes of pointed maps between such Moore spaces. Notice that, unlike the Eilenberg-MacLane, the set of homotopy classes of pointed maps $[M(A,n)\ ,\ M(B,n)]$ between two Moore spaces is different from $\mathrm{Hom}(A,B)$ (see proposition 2.1).

Let $\mathcal{M}od$ be the category of $\mathbb{Z}$-modules and let $\mathscr{D}_e$ be the category of exact couples in $\mathcal{M}od$



such that $\alpha\beta = 2$.

There are two exact functors Φ_1 and Φ_2 from $\mathscr{D}_e$ to $\mathcal{M}od$ assigning to a diagram the $\mathbb{Z}$-module A or B respectively.

The aim of Sect. 2 is to construct, for $n \geq 3$, an equivalence of categories $\mathcal{E}$ between $\mathscr{M}_n$ and $\mathcal{D}_e$. In [1] and in a different context, Baues gave a similar result using the properties of the Whitehead Γ-functor.

In Sect. 3, the stable homotopy groups $\pi_i^S(X)$ $(0 \leq i \leq 7)$ of a Moore space X will be expressed in term of $\mathcal{E}(X)$. The same techniques can be used to determine $\pi_i^S(M(A,n))$ for $i \geq 8$, but calculations become complicated.

2 Equivalence of Categories Between Moore Spaces and Diagrams

2.1 Category of Diagrams

In this section, we propose an equivalence of categories that allows to consider Moore spaces as diagrams of $\mathbb{Z}$-modules.

Recall that the suspension functor from $\mathscr{M}_n$ to $\mathscr{M}_{n+1}$ is an equivalence of categories for $n \geq 3$, so next results are independent of n.

Consider two modules A et B. Let X be the Moore space $X = M(A,n)$, Y the Moore space $Y = M(B,n)$ and $[X,Y]$ the set of homotopy classes of pointed maps from X to Y; this set is an abelian group (see [2]). Moreover:

Proposition 2.1 *([1], [2]) There is a natural exact sequence:*

$$0 \longrightarrow \mathrm{Ext}(A,B/2) \longrightarrow [X,Y] \longrightarrow \mathrm{Hom}(A,B) \longrightarrow 0. \tag{2.1}$$

Set $S = M(\mathbb{Z},n)$ and $P = M(\mathbb{Z}/2,n)$. Applying the exact sequence (2.1) to S and X, we obtain the exact sequence:

$$0 \longrightarrow \mathrm{Ext}(\mathbb{Z}, A/2) \longrightarrow [S,X] \longrightarrow \mathrm{Hom}(\mathbb{Z}, A) \longrightarrow 0$$

and $[S,X]$ is isomorphic to A. Similarly, applied to P and X, (2.1) becomes:

$$0 \longrightarrow \mathrm{Ext}(\mathbb{Z}/2, A/2) \longrightarrow [P,X] \longrightarrow \mathrm{Hom}(\mathbb{Z}/2, A) \longrightarrow 0.$$

Since $\mathrm{Ext}(\mathbb{Z}/2, A/2)$ is naturally isomorphic to $A/2$ (see Proposition 2.7), we have the exact sequence:

$$0 \longrightarrow A/2 \longrightarrow A' \longrightarrow A_2 \longrightarrow 0 \tag{2.2}$$

where A' is the module $[P,X]$ and A_2 is the set of order 2 elements in A. In other words, we have the long exact sequence:

$$A \xrightarrow{2} A \xrightarrow{\alpha_X} A' \xrightarrow{\beta_X} A \xrightarrow{2} A \tag{2.3}$$

or equivalently, the exact couple denoted by D_X:

$$\begin{array}{ccc} A & \xrightarrow{\quad 2 \quad} & A \\ {\scriptstyle \beta_X}\nwarrow & & \swarrow{\scriptstyle \alpha_X} \\ & A' & \end{array}$$

Moreover, if f is a map between two Moore spaces $X = M(A,n)$ and $Y = M(B,n)$, then we can deduce a map $\bar{f} : D_X \longrightarrow D_Y$ as follows: $\bar{f} = (f_1, f_2)$ where $f_1 : A \simeq [S,X] \longrightarrow B \simeq [S,Y]$ and $f_2 : A' = [P,X] \longrightarrow B' = [P,Y]$ are the natural maps induced by f. The following diagrams commute:

$$\begin{array}{ccccccccc} A & \xrightarrow{2} & A & \xrightarrow{\alpha_X} & A' & \xrightarrow{\beta_X} & A & \xrightarrow{2} & A \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_1} \\ B & \xrightarrow{2} & B & \xrightarrow{\alpha_Y} & B' & \xrightarrow{\beta_Y} & B & \xrightarrow{2} & B \end{array}$$

2.1.1 Particular Case of P

When $X = P$, we have the next results:

Proposition 2.2 $[P,P] \simeq \mathbb{Z}/4$.

The proof of this result can be found in [2] or [7].

Lemma 2.3 *The composition* $\alpha_P \beta_P$ *is multiplication by* 2 *on* $\mathbb{Z}/4$.

Proof When $X = P$, the exact sequence (2.3) becomes:

$$\mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/2 \xrightarrow{\alpha_P} \mathbb{Z}/4 \xrightarrow{\beta_P} \mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/2$$

i.e.:

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{\alpha_P} \mathbb{Z}/4 \xrightarrow{\beta_P} \mathbb{Z}/2 \longrightarrow 0$$

then $\alpha_P = 2$ and β_P is the canonical surjection. Hence $\alpha_P\beta_P$ is the multiplication by 2 on $\mathbb{Z}/4$.

2.1.2 General Case

Lemma 2.4 *For any Moore space* $X = M(A,n)$*, the composition* $\alpha_X\beta_X : A' \longrightarrow A'$ *is multiplication by* 2*.*

Proof Let $u \in A' = [P, X]$ and $f : P \longrightarrow X$ a representative of u. We have two maps f_1 and f_2 and the commutative diagram:

$$\begin{array}{ccccccccc}
\mathbb{Z}/2 & \xrightarrow{2=0} & \mathbb{Z}/2 & \xrightarrow{\alpha_P} & \mathbb{Z}/4 & \xrightarrow{\beta_P} & \mathbb{Z}/2 & \xrightarrow{2=0} & \mathbb{Z}/2 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_1} \\
A & \xrightarrow{2} & A & \xrightarrow{\alpha_X} & A' & \xrightarrow{\beta_X} & A & \xrightarrow{2} & A
\end{array}$$

If u_0 denotes the class of the identity map in $[P, P]$, then $f_2(u_0) = u$. The result is an immediate consequence of Lemma 2.3.

2.1.3 Category of Diagrams

Definition 2.5 Let $\mathscr{D}_e$ be the category of exact couples in the category $\mathcal{M}od$ of $\mathbb{Z}$-modules

$$\begin{array}{ccc}
A & \xrightarrow{2} & A \\
{\scriptstyle \beta}\nwarrow & & \swarrow{\scriptstyle \alpha} \\
 & B &
\end{array} \tag{2.4}$$

such that $\alpha\beta = 2$.

A morphism f between two objects D and D' is a couple $f = (f_1, f_2)$ such that the following diagrams commute:

$$
\begin{array}{ccccccccc}
A & \xrightarrow{2} & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & A & \xrightarrow{2} & A \\
{\scriptstyle f_1}\downarrow & & {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_1} \\
A' & \xrightarrow{2} & A' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & A' & \xrightarrow{2} & A'
\end{array}
$$

Notations: For ease, an object of $\mathscr{D}_e$ will be denoted by

$$A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B$$

and a morphism between two objects D and D' will be denoted by

$$
\begin{array}{ccc}
A & \underset{\beta_X}{\overset{\alpha_X}{\rightleftarrows}} & A' \\
{\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\
B & \underset{\beta_Y}{\overset{\alpha_Y}{\rightleftarrows}} & B'
\end{array}
$$

The previous constructions can be summarized in the following statement:

Proposition 2.6 *There is a functor $\mathcal{E} : \mathscr{M}_n \longrightarrow \mathscr{D}_e$ assigning to each Moore space X the diagram D_X, and to each homotopy class f of pointed maps between two Moore spaces X and Y the map $\bar{f} : D_X \longrightarrow D_Y$.*

In the remaining of this section, we will prove that the functor $\mathcal{E}$ is an equivalence of categories.

Notations: Let Φ_1 and Φ_2 denote the two functors from $\mathscr{D}_e$ to $\mathcal{M}od$ defined as follows: if D is an object of$\mathscr{D}$ given by:

$$A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B \tag{2.5}$$

then $\Phi_1(D) = A$ and $\Phi_2(D) = B$.

Notice that there is a natural transformation between functors Φ_1 and Φ_2 obtained by associating to a diagram D given by (2.5), the morphism α. By associating to the diagram D the morphism β, we get a natural transformation from Φ_2 to Φ_1.

2.2 *Equivalence of Categories Between $\mathscr{M}_n$ and $\mathscr{D}_e$*

2.2.1 Some Algebraic Results

This section is devoted to prove some general algebraic results needed to obtain the equivalence of categories announced above.

Proposition 2.7 *For every $\mathbb{Z}/2$-modules A and B, there is an isomorphism $\lambda_{(A,B)}$, natural in A and in B:*

$$\lambda_{(A,B)} : \mathrm{Ext}(A,B) \xrightarrow{\sim} \mathrm{Hom}(A,B).$$

Proof An element e of $\mathrm{Ext}(A,B)$ is represented by an extension:

$$0 \longrightarrow B \xrightarrow{f} E \xrightarrow{g} A \longrightarrow 0.$$

Each element $x \in A$ is of order 2 and g is surjective, so there is $y \in E$ such that $g(y) = x$ and $2y \in \ker g = \mathrm{Im} f$. Since f is injective, there exists a unique $z \in B$ such that $f(z) = 2y$. The map assigning to x the element z is well defined; then we obtain a morphism:

$$\lambda_{(A,B)} : \mathrm{Ext}(A,B) \longrightarrow \mathrm{Hom}(A,B).$$

Since A is free, there is a natural isomorphism $\mathrm{Ext}(A,B) \longrightarrow \mathrm{Hom}(A,\mathrm{Ext}(\mathbb{Z}/2,B))$ obtained by restriction. (Each $a \in A$ defines a map $\mathbb{Z}/2 \longrightarrow A$ which induces an extension of $\mathbb{Z}/2$ by B using a pull-back.) But $\mathrm{Ext}(\mathbb{Z}/2,B)$ is naturaly isomorphic to B, so we get an isomorphism from $\mathrm{Ext}(A,B)$ to $\mathrm{Hom}(A,B)$, which is $\lambda_{(A,B)}$.

Remark 2.8 If A and B are two $\mathbb{Z}$-modules, we construct similarly a natural morphism

$$\lambda_{(A,B)} : \mathrm{Ext}(A,B) \longrightarrow \mathrm{Hom}(A_2,B/2)$$

obtained by the composition:

$$\lambda_{(A,B)} : \mathrm{Ext}(A,B) \longrightarrow \mathrm{Ext}(A_2,B/2) \xrightarrow{\lambda_{(A_2,B/2)}} \mathrm{Hom}(A_2,B/2),$$

where the first morphism is induced by restriction to order 2 elements in A and the projection of B on $B/2$.

Corollary 2.9 *If A is a $\mathbb{Z}/2$-module and B a $\mathbb{Z}$-module, then* $\mathrm{Ext}(A,B)$ *is isomorphic to* $\mathrm{Hom}(A,B/2)$.

Proof The morphism $\lambda_{(A,B)}$ is the composition

$$\lambda_{(A,B)} : \mathrm{Ext}(A,B) \xrightarrow{pr} \mathrm{Ext}(A,B/2) \xrightarrow{\lambda_{(A,B/2)}} \mathrm{Hom}(A,B/2)$$

where pr is the morphism induced by the projection of B on $B/2$. By (2.7), $\lambda_{(A,B/2)}$ is an isomorphism; it suffices to show that $pr : \mathrm{Ext}(A,B) \longrightarrow \mathrm{Ext}(A,B/2)$ is bijective. But A is a $\mathbb{Z}/2$-module, so A is free and then can be written $A = \oplus\mathbb{Z}/2$. Since $\mathrm{Ext}(\oplus\mathbb{Z}/2,B) = \prod \mathrm{Ext}(\mathbb{Z}/2,B)$ we can show the result for $A = \mathbb{Z}/2$. Using the resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0,$$

we get the diagram:

$$\begin{array}{ccccccccc} \mathrm{Hom}(\mathbb{Z}/2, B) & \longrightarrow & B & \xrightarrow{2} & B & \longrightarrow & \mathrm{Ext}(\mathbb{Z}/2, B) & \longrightarrow & 0 \\ \simeq \downarrow & & pr \downarrow & & pr \downarrow & & \downarrow \simeq & & \\ B_2 & \longrightarrow & B/2 & \xrightarrow{2=0} & B/2 & \longrightarrow & \mathrm{Ext}(\mathbb{Z}/2, B/2) & \longrightarrow & 0 \end{array}$$

Corollary 2.10 *If A is a $\mathbb{Z}$-module and B a $\mathbb{Z}/2$-module, then* $\mathrm{Ext}(A, B) \simeq \mathrm{Hom}(A_2, B)$.

Proof Since the morphism $\lambda_{(A,B)}$ is the composition

$$\lambda_{(A,B)} : \mathrm{Ext}(A, B) \xrightarrow{\mathrm{R}} \mathrm{Ext}(A_2, B) \xrightarrow{\lambda_{(A_2,B)}} \mathrm{Hom}(A_2, B),$$

where R is the morphism induced by the restriction to A_2, and $\lambda_{(A_2,B)}$ is an isomorphism, we have just to show that R is bijective.

But B is free, so $B = \oplus\mathbb{Z}/2$; consider the injective module $I = \oplus(\mathbb{Q}/\mathbb{Z})$; then we have the exact sequence:

$$0 \longrightarrow B \longrightarrow I \xrightarrow{2} I \longrightarrow 0.$$

Applying the functor $\mathrm{Hom}(A, \cdot)$, we obtain the following diagram:

$$\begin{array}{ccccccc} \mathrm{Hom}(A, I) & \xrightarrow{2} & \mathrm{Hom}(A, I) & \longrightarrow & \mathrm{Ext}(A, B) & \longrightarrow & 0 \\ R \downarrow & & R \downarrow & & \downarrow R & & \\ \mathrm{Hom}(A_2, I) & \xrightarrow{2=0} & \mathrm{Hom}(A_2, I) & \longrightarrow & \mathrm{Ext}(A_2, B) & \longrightarrow & 0 \end{array}$$

where R denotes the morphism induced by the restriction to A_2.

On the other hand, we have the exact sequence:

$$0 \longrightarrow A_2 \longrightarrow A \xrightarrow{2} A$$

Applying the functor $\mathrm{Hom}(\cdot, I)$, we get an isomorphism between $\mathrm{Hom}(A_2, I)$ and $\mathrm{Hom}(A, I)/2$, so $R : \mathrm{Ext}(A, B) \longrightarrow \mathrm{Ext}(A_2, B)$ is bijective.

2.2.2 Equivalence of Categories

Theorem 2.11 *The functor $\mathcal{E}$ is an equivalence of categories between $\mathscr{M}_n$ and $\mathscr{D}_e$.*

To prove this theorem, we need the next two lemmas.

Lemma 2.12 *For each diagram D in $\mathcal{D}_e$, there exists a Moore space X in $\mathcal{M}_n$ such $\mathcal{E}(X) = D$.*

Proof Let D be an object of $\mathcal{D}_e$ given by:

$$A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B$$

Set X the Moore space $X = M(A,n)$. The diagram associated to X is given by:

$$A \underset{\beta_X}{\overset{\alpha_X}{\rightleftarrows}} A'$$

Then, we have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A/2 & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & A_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Id} & & & & \downarrow{\scriptstyle Id} & & \\ 0 & \longrightarrow & A/2 & \xrightarrow{\alpha_X} & A' & \xrightarrow{\beta_X} & A_2 & \longrightarrow & 0 \end{array}$$

where the lines are exact. Each horizontal exact sequence defines an element in $\mathrm{Ext}(A_2,A/2) \simeq \mathrm{Hom}(A_2,A/2)$. Since $\beta\alpha = 2$ on B and $\beta_X\alpha_X = 2$ on A', the two extensions give the same element in $\mathrm{Hom}(A_2,A/2)$ and then the two extensions are isomorphic.

Lemma 2.13 *If X and Y are two Moore spaces, then $[X,Y]$ is isomorphic to* $\mathrm{Hom}(D_X,D_Y) = \mathrm{Hom}(\mathcal{E}(X),\mathcal{E}(Y))$.

Proof Let $X = M(A,n)$ and $Y = M(B,n)$, then there is an exact sequence:

$$0 \longrightarrow \mathrm{Ext}(A,B) \longrightarrow [X,Y] \longrightarrow \mathrm{Hom}(A,B) \longrightarrow 0$$

But we have: $\mathrm{Ext}(A,B/2) \simeq \mathrm{Ext}(A_2,B/2) \simeq \mathrm{Hom}(A_2,B/2)$, so we obtain the exact sequence:

$$0 \longrightarrow \mathrm{Hom}(A_2,B/2) \longrightarrow [X,Y] \longrightarrow \mathrm{Hom}(A,B) \longrightarrow 0.$$

On the other side, the forgetful morphism $Fr : \mathrm{Hom}(D_X,D_Y) \longrightarrow \mathrm{Hom}(A,B)$ is surjective. Recall that an element $g \in \mathrm{Hom}(D_X,D_Y)$ is given by two maps g_1 and g_2 such that:

$$\begin{array}{ccccccccc} A & \xrightarrow{2} & A & \xrightarrow{\alpha_X} & A' & \xrightarrow{\beta_X} & A & \xrightarrow{2} & A \\ \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_1} \\ B & \xrightarrow{2} & B & \xrightarrow{\alpha_Y} & B' & \xrightarrow{\beta_Y} & B & \xrightarrow{2} & B \end{array}$$

so an element $g \in \mathrm{Hom}(D_X, D_Y)$ is in the kernel of the forgetful morphism if $g_1 = 0$ and then we obtain a morphism $A_2 \longrightarrow B/2$. Hence, we get the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(A_2, B/2) & \longrightarrow & [X\ ,\ Y] & \longrightarrow & \mathrm{Hom}(A,B) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle f_{X,Y}} & & \Big\downarrow & & \Big\downarrow{\scriptstyle Id} & & \\ 0 & \longrightarrow & \mathrm{Hom}(A_2, B/2) & \longrightarrow & \mathrm{Hom}(D_X, D_Y) & \xrightarrow{Fr} & \mathrm{Hom}(A,B) & \longrightarrow & 0 \end{array}$$

To prove the isomorphism between $[X,Y]$ and $\mathrm{Hom}(D_X, D_Y)$, it suffices to verify that $f_{X,Y}$ is the identity map. Notice that $f_{X,Y}$ is a bifunctor, covariant in B and contravariant in A.

When $X = P$ and $Y = S$, the diagram becomes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & [P\ ,\ S] \simeq \mathbb{Z}/2 & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle f_{P,S}} & & \Big\downarrow & & & & \\ 0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & \mathrm{Hom}(D_P, D_S) \simeq \mathbb{Z}/2 & \xrightarrow{Fr} & 0 & \longrightarrow & 0 \end{array}$$

so $f_{P,S}$ is necessarily the identity map. When $X = P$ and $Y = M(B,n)$: an element $y \in B$ defines a morphism $\mathbb{Z} \longrightarrow B$ that can be realized by a map between Moore spaces $S \longrightarrow Y$ and then a map $\bar{y} : \mathbb{Z}/2 \longrightarrow B/2$. By assigning $\bar{y}$ to the generator of $\mathbb{Z}/2$, we get the commutative diagram:

$$\begin{array}{ccc} \mathbb{Z}/2 & \longrightarrow & \mathrm{Hom}(\mathbb{Z}/2, B/2) \simeq B/2 \\ \Big\downarrow{\scriptstyle f_{P,S}=Id} & & \Big\downarrow{\scriptstyle f_{P,Y}} \\ \mathbb{Z}/2 & \longrightarrow & \mathrm{Hom}(\mathbb{Z}/2, B/2) \simeq B/2 \end{array}$$

Since $\mathrm{Hom}(\mathbb{Z}/2, B/2)$ is naturally isomorphic to $B/2$, even in this case $f_{P,Y} = Id$.

Given $x \in A_2$, it defines a map $\mathbb{Z}/2 \longrightarrow A_2 \subset A$ which can be realized by a map of Moore spaces $P \longrightarrow X$. This map allows to have the following commutative diagram, using the functoriality of $f_{X,Y}$:

$$\begin{array}{ccc} \mathrm{Hom}(A_2, B/2) & \longrightarrow & \mathrm{Hom}(\mathbb{Z}/2, B/2) \simeq B/2 \\ \Big\downarrow{\scriptstyle f_{X,Y}} & & \Big\downarrow{\scriptstyle f_{P,Y}=Id} \\ \mathrm{Hom}(A_2, B/2) & \longrightarrow & \mathrm{Hom}(\mathbb{Z}/2, B/2) \simeq B/2 \end{array}$$

the horizontal maps assign to a morphism $\varphi : A_2 \longrightarrow B/2$ its evaluation $\varphi(x) \in B/2$. To conclude that $f_{X,Y}$ is the identity map on $\mathrm{Hom}(A_2, B/2)$, it suffices to notice that the module A_2 is $\mathbb{Z}/2$-free, and if $\{u_i\}_{i \in I}$ is a basis of A_2 then $\mathrm{Hom}(A_2, B/2) \simeq$

$\prod \mathrm{Hom}(\mathbb{Z}/2, B/2) \simeq \prod B/2$. Using the evaluation on each generator u_i, we deduce the desired result.

Remark 2.14 With Lemmas 2.12 et 2.13, we get the proof of Theorem 2.11.

3 Stable Homotopy Groups of Moore Spaces

Let $X = M(A,n)$ and consider the Atiyah-Hirzebruch spectral sequence in homology with coefficients in the stable homotopy groups:

$$H_p(X; \pi_q^S) \Rightarrow \pi_{p+q}^S(X).$$

This spectral sequence contains just two non trivial columns and induces the following exact sequence:

$$0 \longrightarrow A \otimes \pi_q^S \xrightarrow{\nu^X} \pi_{n+q}^S(X) \xrightarrow{\mu^X} \mathrm{Tor}(A, \pi_{q-1}^S) \longrightarrow 0 \qquad (3.1)$$

Moreover, this exact sequence is natural in X.

Notice that, if $\underline{X}$ denotes the spectrum associated to the Moore space X, then $\pi_{n+i}^S(X) = \pi_i^S(\underline{X})$. In the following, the spectrum associated to a space X will also be denoted by X.

Recall the first stable homotopy groups (see [3]):

$$\pi_0^S = \mathbb{Z}, \pi_1^S = \mathbb{Z}/2, \pi_2^S = \mathbb{Z}/2, \pi_3^S = \mathbb{Z}/24, \pi_4^S = \pi_5^S = 0, \pi_6^S = \mathbb{Z}/2, \pi_7^S = \mathbb{Z}/240,$$

so the exact sequence (3.1) allows to obtain, for any Moore space X:

$$\pi_0^S(X) = A, \qquad \pi_1^S(X) \simeq A \otimes \mathbb{Z}/2 = A/2, \qquad \pi_4^S(X) \simeq \mathrm{Tor}(A, \mathbb{Z}/24) = A_{24},$$

$$\pi_5^S(X) = 0, \qquad \pi_6^S(X) \simeq A \otimes \mathbb{Z}/2 = A/2$$

but we can't determine explicitly $\pi_2^S(X)$, $\pi_3^S(X)$ and $\pi_7^S(X)$.

To compute $\pi_i^S(X)$, for $i = 2, 3, 7$, we need the following lemma:

Lemma 3.1 $\pi_2^S(P) = \mathbb{Z}/4$, $\pi_3^S(P) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$, $\pi_7^S(P) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Proof These groups are given in [7], but we propose an easier proof of these results using the arguments of Sect. 2.

For $q = 2, 3, 7$, the exact sequence (3.1) becomes:

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \pi_q^S(P) \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

then $\pi_q^S(P) \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/2$ or $\pi_q^S(P) \simeq \mathbb{Z}/4$.

There is a cofibration sequence:

$$\xrightarrow{\theta} P \xrightarrow{\delta} S \xrightarrow{2} S \xrightarrow{\theta} P \xrightarrow{\delta} \tag{3.2}$$

where θ is of degree 0 and δ of degree -1. If λ denotes the composition of δ by the Hopf map from S to S, then we get the Moore spectra diagram

$$\begin{array}{ccc} S & \xrightarrow{\quad 2 \quad} & S \\ {\scriptstyle\lambda}\nwarrow & & \swarrow{\scriptstyle\theta} \\ & P & \end{array} \tag{3.3}$$

verifying $2\theta = 0$, $2\lambda = 0$, $\lambda\theta = 0$ et $\theta\lambda = 2$.

Applying the functor π_2^S to (3.3), we obtain the following diagram:

$$\begin{array}{ccc} \mathbb{Z}/2 & \xrightarrow{\quad 2=0 \quad} & \mathbb{Z}/2 \\ {\scriptstyle\lambda_*}\nwarrow & & \swarrow{\scriptstyle\theta_*} \\ & \pi_2^S(P) & \end{array}$$

where $2\lambda_* = 0$, $2\theta_* = 0$ and $\theta_*\lambda_* = 2$. This diagram is not necessarily exact, but, the exact sequence of stable homotopy groups applied to the cofibration (3.2) gives:

$$\ker(\theta_* : \mathbb{Z}/2 \longrightarrow \pi_2^S(P)) = \operatorname{Im}(2 : \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2).$$

Then it suffices to find an element $u \in \pi_2^S(P)$ such that $\lambda_*(u) = 1$. For this purpose, we can choose $n = 2$ so $S = S^2$ and $P = P_2 = \Sigma\mathbb{R}P_2$. We have the cofibration:

$$S^2 \longrightarrow P_2 \longrightarrow S^3.$$

Applying the stable homotopy functor, we get:

$$\begin{array}{ccccccccccc}
\pi_4^S(S^2) & \longrightarrow & \pi_4^S(P_2) & \longrightarrow & \pi_4^S(S^3) & \longrightarrow & \pi_3^S(S^2) & \longrightarrow & \pi_3^S(P_2) & \longrightarrow & \pi_3^S(S^3) \\
\| & & \| & & \| & & \| & & \| & & \| \\
\pi_2^S & \longrightarrow & \pi_2^S(P) & \longrightarrow & \pi_1^S & \longrightarrow & \pi_1^S & \longrightarrow & \pi_1^S(P) & \longrightarrow & \pi_0^S \\
\| & & \| & & \| & & \| & & \| & & \| \\
\mathbb{Z}/2 & \longrightarrow & \pi_2^S(P) & \longrightarrow & \mathbb{Z}/2 & \xrightarrow{0} & \mathbb{Z}/2 & \xrightarrow{Id} & \mathbb{Z}/2 & \xrightarrow{0} & \mathbb{Z} \\
 & & & \searrow & \simeq\downarrow{\scriptstyle \text{Hopf}} & & & & & & \\
 & & & & \mathbb{Z}/2 & & & & & &
\end{array}$$

Then $\pi_2^S(P)$ is surjected on $\mathbb{Z}/2 = \pi_1^S$ which is sent by the Hopf map on $\mathbb{Z}/2 = \pi_2^S$ by assigning to the generator η of $\pi_1^S = \mathbb{Z}/2$ the generator η^2 of $\pi_2^S = \mathbb{Z}/2$.

Now, applying the functor π_3^S to (3.3), we get:

$$\begin{array}{ccc} \mathbb{Z}/24 & \xrightarrow{\;2\;} & \mathbb{Z}/24 \\ {\scriptstyle \lambda_*}\nwarrow & & \swarrow{\scriptstyle \theta_*} \\ & \pi_3^S(P) & \end{array}$$

with $2\theta_* = 0$, $2\lambda_* = 0$ and $\theta_*\lambda_* = 2$. This diagram is not necessarily exact, but

$$\ker(\theta_* : \mathbb{Z}/24 \longrightarrow \pi_3^S(P)) = \mathrm{Im}(2 : \mathbb{Z}/24 \longrightarrow \mathbb{Z}/24).$$

Let $x \in \pi_3^S(P)$, then $2\lambda_*(x) = 0$. There exists $u \in \mathbb{Z}/24$ such that $\lambda_*(x) = 12u$. So $2x = \theta_*(\lambda_*(x)) = \theta_*(12u) = 0$ since $2\theta_* = 0$. This implies that elements of $\pi_3^S(P)$ vanish when multiplied by 2 and then $\pi_3^S(P) \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

The same argument shows that $\pi_7^S(P) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Consider a Moore space $X = M(A,n)$. The next theorems compute $\pi_i^S(X)$, for $i = 2,3,7$, in terms of the modules $\Phi_1(D_X)$ and $\Phi_2(D_X)$.

Theorem 3.2 *For each generator $\gamma \in \pi_2^S(P)$, there is a natural isomorphism $\pi_2^S(X) \simeq \Phi_2(D_X) = [P,X]$.*

Proof Consider the exact sequence (2.2) and the exact sequence (3.1) for $q = 2$:

$$0 \longrightarrow A/2 \xrightarrow{\;\alpha\;} A' \xrightarrow{\;\beta\;} A_2 \longrightarrow 0$$

$$0 \longrightarrow A/2 \xrightarrow{\;\nu^X\;} \pi_2^S(X) \xrightarrow{\;\mu^X\;} A_2 \longrightarrow 0$$

We construct a map $A' \longrightarrow \pi_2^S(X)$ as follows: choose γ a generator of $\pi_2^S(P) \simeq \mathbb{Z}/4$. Let $u \in A'$ and consider f representing the class $u \in A' = [P,X]$.Then f induces a map $f_* : \pi_2^S(P) \longrightarrow \pi_2^S(X)$ and we define $\varphi_\gamma(u) = f_*(\gamma)$. the map φ_γ relies the two exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A/2 & \xrightarrow{\;\alpha\;} & A' & \xrightarrow{\;\beta\;} & A_2 & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \varphi_\gamma} & & & & \\ 0 & \longrightarrow & A/2 & \xrightarrow{\;\nu^X\;} & \pi_2^S(X) & \xrightarrow{\;\mu^X\;} & A_2 & \longrightarrow & 0 \end{array}$$

Now, we may prove that the composite map

$$A' \xrightarrow{\varphi_\gamma} \pi_2^S(X) \xrightarrow{\mu^X} A_2$$

is $\beta : A' \longrightarrow A_2$. Using the functoriality, it suffices to prove the result when $X = P$. In this case, he diagram becomes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}/2 & \xrightarrow{\alpha} & \mathbb{Z}/4 & \xrightarrow{\beta} & \mathbb{Z}/2 & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \varphi_\gamma} & & \downarrow{\scriptstyle Id} & & \\
0 & \longrightarrow & \mathbb{Z}/2 & \xrightarrow{\nu^P} & \pi_2^S(P) \simeq \mathbb{Z}/4 & \xrightarrow{\mu^P} & \mathbb{Z}/2 & \longrightarrow & 0
\end{array}$$

where we see clearly that $\mu^P \circ \varphi_\gamma = \beta$.

By functoriality, for each Moore space $X = M(A,n)$ we get the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A/2 & \xrightarrow{\alpha} & A' & \xrightarrow{\beta} & A_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \varphi_\gamma} & & \downarrow{\scriptstyle Id} & & \\
0 & \longrightarrow & A/2 & \xrightarrow{\nu^X} & \pi_2^S(X) & \xrightarrow{\mu^X} & A_2 & \longrightarrow & 0
\end{array}$$

here h is the natural map making the diagram commute. Notice that h is functorial in X. Then, to determine $h : A/2 \longrightarrow A/2$, it suffices to study the case $X = S$. In that case, the diagram becomes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}/2 & \xrightarrow{\alpha} & \mathbb{Z}/2 & \xrightarrow{\beta} & 0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \varphi_\gamma} & & \downarrow{\scriptstyle Id} & & \\
0 & \longrightarrow & \mathbb{Z}/2 & \xrightarrow{\nu^S} & \pi_2^S = \mathbb{Z}/2 & \xrightarrow{\mu^S} & 0 & \longrightarrow & 0
\end{array}$$

and then h is necessarily the identity map.

Let $X = M(A,n)$ be a Moore space. Each element $x \in A$ defines a maps $f : \mathbb{Z} \longrightarrow A$ given by $f(1) = x$. This map is realized by a map between Moore spaces $f : S \longrightarrow X$ and induces, by naturality of h, the following commutative diagram:

$$\begin{array}{ccc}
\mathbb{Z}/2 & \xrightarrow{\bar{f}} & A/2 \\
{\scriptstyle h=Id}\downarrow & & \downarrow{\scriptstyle h} \\
\mathbb{Z}/2 & \xrightarrow{\bar{f}} & A/2
\end{array}$$

so $h : A/2 \longrightarrow A/2$ is still the identity map.

Remark 3.3 The isomorphism $\pi_2^S(X) \simeq A'$ depends on the choice of the generator $\gamma \in \pi_2^S(P) = \mathbb{Z}/4$. Choosing the generator $-\gamma$ multiplies the isomorphisme by -1.

Theorem 3.4 *For each $\gamma \in \pi_3^S(P)$ such that $\mu^P(\gamma) = 1 \in \mathbb{Z}/2$, there is a natural isomorphism $\pi_3^S(X) \simeq A' \oplus_{A/2} A/24$ obtained by the pushout*

$$\begin{array}{ccc} A/2 & \xrightarrow{\alpha} & A' \\ {\scriptstyle \times 12}\downarrow & & \downarrow \\ A/24 & \longrightarrow & \pi_3^S(X) \end{array}$$

where $A = \Phi_1(D_X)$ and $A' = \Phi_2(D_X)$.

Proof When $q = 3$, the exact sequence (3.1) becomes:

$$0 \longrightarrow A/24 \xrightarrow{\nu^X} \pi_3^S(X) \xrightarrow{\mu^X} A_2 \longrightarrow 0$$

For $X = P$ we get

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{\nu^P} \pi_3^S(P) \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow{\mu^P} \mathbb{Z}/2 \longrightarrow 0$$

Choose $\gamma \in \pi_3^S(P)$ such that $\mu^P(\gamma)$ is the generator of $\mathbb{Z}/2$. We construct a map $\varphi_\gamma : A' \longrightarrow \pi_3^S(X)$ as follows:

Let $u \in A' = [P, X]$ and let $f : P \longrightarrow X$ representing the class u. Then $\varphi_\gamma(u) = f_*(\gamma)$.

As in the proof of Theorem 3.2 we show that the composition of $\mu^X : \pi_3^S(X) \longrightarrow A_2$ by φ_γ is $\beta : A' \longrightarrow A_2$.

We obtain the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A/2 & \xrightarrow{\alpha} & A' & \xrightarrow{\beta} & A_2 & \longrightarrow & 0 \\ & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \varphi_\gamma} & & \downarrow{\scriptstyle Id} & & \\ 0 & \longrightarrow & A/24 & \xrightarrow{\nu^X} & \pi_3^S(X) & \xrightarrow{\mu^X} & A_2 & \longrightarrow & 0 \end{array}$$

Since h is natural, we need just to determine it for $X = S$. In that case, the map $h : \mathbb{Z}/2 \longrightarrow \mathbb{Z}/24$ assigns to the generator of $\mathbb{Z}/2$ an element of $\mathbb{Z}/24$ vanishing when multiplied by 2, that means 0 or 12. Then $h = 0$ or $h = \times 12$. To prove that $h = \times 12$, we consider the cofibration

$$S \xrightarrow{2} S \longrightarrow P$$

which induces the long exact sequence:

$$\cdots \longrightarrow \pi_n^S \xrightarrow{2} \pi_n^S \longrightarrow \pi_n^S(P) \longrightarrow \pi_{n-1}^S \xrightarrow{2} \pi_{n-1}^S \longrightarrow \cdots \quad (3.4)$$

For $n = 3$, we have:

$$\pi_3^S = \mathbb{Z}/24 \xrightarrow{2} \pi_3^S = \mathbb{Z}/24 \longrightarrow \pi_3^S(P) \longrightarrow \pi_2^S = \mathbb{Z}/2 \xrightarrow{2=0} \pi_2^S = \mathbb{Z}/2$$

This proves that $\pi_3^S(P) \longrightarrow \pi_2^S$ is surjective, so every map $S \longrightarrow S$ of degree 2 can be lifted to a map $S \longrightarrow P$ of degree 3.

If $\gamma \in \pi_3^S(P)$ is represented by a map, denoted also $\gamma : S^5 \longrightarrow P_2$, and since $\mu^P(\gamma) = 1 \in \mathbb{Z}/2 = \mathrm{Tor}(\mathbb{Z}/2, \pi_2^S)$, then the map $\pi_3^S(P) \longrightarrow \pi_2^S$ takes γ to the generator $1_{\mathbb{Z}/2} \in \pi_2^S = \mathbb{Z}/2$.

Let

$$u : P_2 \xrightarrow{\delta_2} S^3 \xrightarrow{Hopf} S^2$$

be a representative of the nonzero element of $[P, S] = \mathbb{Z}/2$ and $a : S^5 \longrightarrow S^3$ a representative of the generator of π_2^S. Then $\varphi_\gamma([u]) = u_*(\gamma) = \eta \times (\delta_2)_*(\gamma) = \eta \times [a]$ where η denotes the multiplication by the class of the Hopf map. But the multiplication by the Hopf map class takes the generator of π_2^S to product by 12 of the generator of π_3^S (see [3]). This allows to deduce that $\varphi_\gamma([u]) = 12 \in \mathbb{Z}/24$ and that h is multiplication by 12.

Remark 3.5 The isomorphism $\pi_3^S(X) \simeq A' \oplus_{A/2} A/24$ depends on the choice of $\gamma \in \pi_3^S(P) \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/2$ verifying $\mu^P(\gamma) = 1$. There are two possible choices.

If we choose γ' such that $\mu^P(\gamma') = 1$, then $\nu^P(1_{\mathbb{Z}/2}) = \gamma - \gamma'$. We can show that

$$\varphi_{\gamma'} = \varphi_\gamma + \tilde{\lambda} \circ \beta$$

where $\tilde{\lambda} : A_2 \longrightarrow \pi_3^S(X)$ is defined as follows: if $a \in A_2$, we can represent it by a map $a : S \longrightarrow X$ such that $2a = 0$. This map induces $a_* : \pi_3^S \longrightarrow \pi_3^S(X)$ taking all generators of π_3^S to the same element $a_*(1_{\mathbb{Z}/24}) \in \pi_3^S(X)$ since $2a_* = 0$. Then we define $\tilde{\lambda}$ by $\tilde{\lambda}(a) = a_*(1_{\mathbb{Z}/24})$.

Theorem 3.6 *For each $\gamma \in \pi_7^S(P)$ such that $\mu^P(\gamma) = 1 \in \mathbb{Z}/2$, there is a natural isomorphism $\pi_7^S(X) \simeq A/240 \oplus A_2$, where $A = \Phi_1(D_X)$.*

Proof Using the same construction of the case of $\pi_3^S(X)$, we get the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A/2 & \xrightarrow{\alpha} & A' & \xrightarrow{\beta} & A_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \varphi_\gamma} & & \downarrow{\scriptstyle Id} & & \\
0 & \longrightarrow & A/240 & \xrightarrow{\nu^X} & \pi_7^S(X) & \xrightarrow{\mu^X} & A_2 & \longrightarrow & 0
\end{array}$$

To determine h, it suffices to consider the case of $X = S$, since it is natural on X. In that case $h : \mathbb{Z}/2 \longrightarrow \mathbb{Z}/240$ is the multiplication by 0 or 120.

For $n = 7$, the long exact sequence (3.4) becomes:

$$\pi_7^S = \mathbb{Z}/240 \xrightarrow{2} \pi_7^S = \mathbb{Z}/240 \longrightarrow \pi_7^S(P) \longrightarrow \pi_6^S = \mathbb{Z}/2 \xrightarrow{2=0} \pi_6^S = \mathbb{Z}/2$$

showing that $\pi_7^S(P) \longrightarrow \pi_6^S$ is surjective. We use the same techniques of the previous theorem proof, and the fact that the product by the Hopf class on π_6^S is zero (see [3]), we deduce that $h = 0$

Remark 3.7 Using the new universal coefficient exact sequence of [4], we can represent the functor π_i^S on $\mathscr{M}_n$ as a tensor product by particular objects of an abelian category $\mathscr{D}$ containing $\mathscr{D}_e$.

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