Transient Solution for Queue-Size Distribution in a Certain Finite-Buffer Model with Server Working Vacations

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Abstract. A finite-buffer queueing model with Poisson arrivals and exponential processing times is investigated. Every time when the system empties, the server begins a generally distributed single working vacation period, during which the service is provided with another (slower) rate. After the completion of the vacation period the processing is being continued normally, with original speed. The next working vacation period is being initialized at the next time at which the system becomes empty, and so on. The system of Volterra-type integral equations for transient queue-size distribution, conditioned by the initial level of buffer saturation, is built. The solution of the corresponding system written for Laplace transforms is given in a compact-form using the linear algebraic approach and the corresponding result obtained for the ordinary model (without working vacation regime). Numerical examples are attached as well.

Keywords: Finite buffer · Poisson process · Queue size · Transient state · Working vacation

1 Introduction

Queueing models with finite buffer capacities are widely used in the analysis of real-life systems occurring in technical and economic sciences, and in transport and logistic problems, in which the phenomena of "queueing" of items [27, 28] (packets, calls, customers, jobs, etc.) and their losses due to buffer saturation can be observed. As it seems, particularly important are models in which different-type restrictions in access to the service station are implemented additionally. In practice, these restrictions are often a kind of energy saving mechanism (e.g., cyclic succession of listening and dormant modes in wireless networks, or switching off a machine in manufacturing process in the case of the traffic with low intensity), and are associated with temporary blocking the service of items despite their presence in the accumulation buffer. The scientific literature concerning such systems is already huge and still increasing. Servi and Finn proposed in [15] for the first time the model with the so called working vacation, in which the server, instead of total service stopping, offers the processing with another speed (usually lower). This model was originally motivated by a reconfigurable WDM (Wavelength-division multiplexing) optical access network in which a

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single token cyclically visits each queue, operating at two different rates (faster and slower ones), but it can be successfully used in modelling many phenomena typical for, e.g. computer and telecommunication networks or manufacturing engineering. In particular, we can use it

- when the service station processes two types of packets with significantly different service speeds (e.g., different times of putting them in the link);
- in the case of temporary throughput reductions, due to parallel launching another application;
- in the situation of periodic reduction of the throughput of the production line (slower processing with lower power consumption).

As it is shown in [12, 29], a working vacation queueing system with two different processing rates can be successfully applied in modelling, e.g. the Ethernet Passive Optical Network (EPON), consisting of one optical line terminal (OLT), situated at the central office, and multiple optical network units (ONUs) situated at customer premises equipment (CPE), and a passive splitter/combiner. In EPON bi-directional transmissions are provided: in the downstream direction the OLT broadcasts to all ONUs and in the upstream direction (from ONUs to the OLT) the fiber channel is shared by all ONUs.

After the article [15] many papers were published on the analysis of stochastic characteristics of queueing models with working vacation mechanism. Working vacation models of GI/M/1 type are considered, e.g. in [2, 26] in the case of finite buffer capacity, and in [1, 13] for the infinite waiting room. Unfortunately, as one can see, most of the results relates only to the steady state of the system. Meanwhile, as it seems, in practice it is increasingly essential to investigate the system in the transient case. Such a study is of particular importance in the case of the observation the system shortly after its opening or applying new control mechanism. The high variability of the packet traffic (e.g., in the Internet) also can "force" the time-dependent analysis. In [18] the transient queue-size distribution for the infinite-sized M/M/1-type model with server working vacations is found. In [22] the study is extended for the multi-server case and multiple working vacation regime by using the matrix geometric method. Compact-form transient results for main stochastic characteristics of finite-buffer queues with different-type service restrictions can be found, e.g. in [7–10]. The case of infinite buffer is studied in [5, 6]. In [21] a processor-sharing model with limited total volume and probabilistic packet dropping is considered. In various data management systems (see [23]) we can find applications of queueing systems to enable faster requests management and therefore improved data mining (see e.g. [24, 25, 30, 31]).

The remaining part of the article is organized as follows. In the next Sect. 2 we give the precise description of the considered queueing model and state an auxiliary algebraic result which can be used in further analysis. Section 3 is devoted to the ordinary system (without server working vacation). The compact-form representation for the LT (=Laplace transform) of conditional transient queue-size distribution is derived there, and written by using the functional sequence recursively defined. In Sect. 4 we obtain the corresponding result for the original model with generally distributed working vacations, utilizing results from Sects. 2 and 3. In Sect. 5 numerical examples are attached and the last Sect. 6 contains a short summary and conclusions.

2 Model Description and Auxiliary Results

In the article we deal with the M/M/1/N-type queueing model with Poisson job arrivals with rate λ, exponential processing times with mean μ^{-1}, and finite capacity $N \geq 2$ ($N - 1$ places in the buffer queue and one place "in service"). Every time when the system empties the server begins a generally distributed single working vacation period, during which the processing of jobs is carried out with another (slower) rate $\mu^* < \mu$ (see Fig. 1 for the scheme of the system operation). We denote by $G(\cdot)$ the CDF (=cumulative distribution function) of the working vacation period duration. After finishing the vacation period the service process is being continued normally, with original speed. The next working vacation period is being initialized at the next time at which the queue becomes empty, and so on.

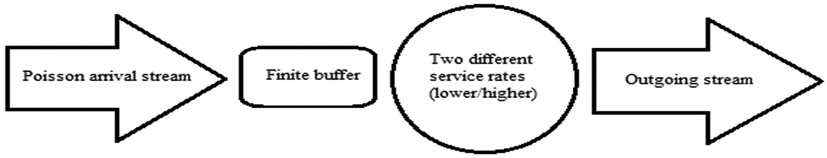

Fig. 1. Scheme of a single-server finite-buffer working vacation model with Poisson arrivals

The following theorem can be found in [11]:

Theorem 1. *Introduce two number sequences* $(\alpha_k), k \geq 0$, *and* $(\psi_k), k \geq 1$, *with the assumption* $\alpha_0 \neq 0$. *Each solution of the following system of linear equations with respect to* $x_n, n \geq 1$:

$$\sum\nolimits_{k=-1}^{n-1} \alpha_{k+1} x_{n-k} - x_n = \psi_n, \quad n \geq 1, \tag{1}$$

can be written in the form

$$x_n = CR_n + \sum\nolimits_{k=1}^{n} R_{n-k} \psi_k, \quad n \geq 1, \tag{2}$$

where C is a constant independent on n, and (R_k) *is connected with the sequence* (α_k) *by the following formula:*

$$\sum\nolimits_{k=0}^{\infty} \theta^k R_k = \frac{1}{P_\alpha(\theta) - 1}, \text{ where } P_\alpha(\theta) = \sum\nolimits_{k=-1}^{\infty} \theta^k \alpha_{k+1}, |\theta| < 1. \tag{3}$$

Moreover, in [11] it is proved that successive terms of the sequence (R_n) (called a potential) can be found recursively as follows:

$$R_0 = 0, R_1 = \alpha_0^{-1}, R_{k+1} = R_1\Big(R_k - \sum\nolimits_{i=0}^{k} \alpha_{i+1} R_{k-i}\Big), \quad k \geq 1. \tag{4}$$

As it turns out, LTs of conditional queue-size distributions in the original system and in the ordinary one, satisfy systems of equations similar to (1). Hence, in solving these systems, we will use the formula (2), where the representation for C will be found from a boundary condition.

3 Conditional Transient Queue-Size Distribution in an Ordinary System

In this section we deal with the conditional queue-size distribution in the ordinary finite-buffer M/M/1/N-type model without working vacation discipline, corresponding to the original one, with λ and μ^* being the arrival intensity and service speed, respectively, and find the representation for its LT in terms of "input" system parameters, writing it in a specific way, by using a recursively defined sequence, called a potential. Similar result was obtained in [3] for the generally-distributed service time, however it is written in another form. It should be mentioned here that transient solutions for the M/M/1/N-type queue were also obtained in [19] (see also [14]) by using the technique of eigenvalues and eigenvectors, in [16] by applying Chebyshev polynomials, in [17] by utilizing matrix technique and in [20] via LTs. Introduce the following notation:

$$P_n^O(t,m) \stackrel{\text{def}}{=} \boldsymbol{P}\big\{X^O(t) = m | X^O(0) = n\big\}, \quad 0 \leq m, n \leq N, \tag{5}$$

where $X^O(t)$ denotes the number of packets present in the ordinary system at time t. Since, due to exponential distributions of inter-arrival and service times, both arrival and service completion epochs are Markov moments, from the continuous version of the total probability formula written with respect to the first Markov moment after $t = 0$, we obtain the following system of integral equations:

$$P_0^O(t,m) = \lambda \int_0^t e^{-\lambda x} P_1^O(t-x,m)dx + e^{-\lambda t}\delta_{m,0}, \tag{6}$$

$$\begin{aligned} P_n^O(t,m) = {} & \lambda \int_0^t e^{-(\lambda+\mu^*)x} P_{n+1}^O(t-x,m)dx \\ & + \mu^* \int_0^t e^{-(\lambda+\mu^*)x} P_{n-1}^O(t-x,m)dx \\ & + e^{-(\lambda+\mu^*)t}\delta_{m,n}, \end{aligned} \tag{7}$$

where $1 \leq n \leq N-1$, and

$$\begin{aligned} P_N^O(t,m) = \lambda \int_0^t e^{-(\lambda+\mu^*)x} P_N^O(t-x,m)dx \\ + \mu^* \int_0^t e^{-(\lambda+\mu^*)x} P_{N-1}^O(t-x,m)dx \\ + e^{-(\lambda+\mu^*)t}\delta_{m,N}, \end{aligned} \tag{8}$$

where the notation $\delta_{i,j}$ stands for the Kronecker delta function. Let us comment (6)–(8) briefly. Indeed, the first summands on the right side of (7) and (8) relate to the case in which, as the first one, a jump of the arrival Poisson process occurs, while the second ones - to the situation in which the jump of the service process is observed as the first one. The last summands on the right side of (7) and (8) present the case in that there is no jump of the arrival and service processes before t. The formula (6), written for the case of the system being empty at the opening, is obvious. Defining

$$\tilde{p}_n^O(s,m) \stackrel{\text{def}}{=} \int_0^\infty e^{-st} \boldsymbol{P}\{X^O(t) = m | X^O(0) = n\} dt, \quad \Re(s) > 0, \tag{9}$$

we obtain from (6)–(8) the following equations:

$$\tilde{p}_0^O(s,m) = \frac{\lambda}{\lambda+s}\tilde{p}_1^O(s,m) + \frac{\delta_{m,0}}{\lambda+s}, \tag{10}$$

$$\tilde{p}_n^O(s,m) = \frac{\lambda}{\lambda+\mu^*+s}\tilde{p}_{n+1}^O(s,m) + \frac{\mu^*}{\lambda+\mu^*+s}\tilde{p}_{n-1}^O(s,m) + \frac{\delta_{m,n}}{\lambda+\mu^*+s}, 1 \le n \le N-1 \tag{11}$$

and

$$\tilde{p}_N^O(s,m) = \frac{\lambda}{\lambda+\mu^*+s}\tilde{p}_N^O(s,m) + \frac{\mu^*}{\lambda+\mu^*+s}\tilde{p}_{N-1}^O(s,m) + \frac{\delta_{m,N}}{\lambda+\mu^*+s}. \tag{12}$$

We will obtain the solution of the system (10)–(12) by applying the algebraic-type approach based on Theorem 1 that allows for writing the representation for $\tilde{p}_k^O(s,m)$ (at arbitrary k) via certain recursively-defined sequence (see (4)).

Let us note that, if we define

$$\alpha_0^*(s) = \frac{\lambda}{\lambda+\mu^*+s}, \alpha_1^*(s) = 0, \alpha_2^*(s) = \frac{\mu^*}{\lambda+\mu^*+s}, \alpha_k^*(s) = 0, \quad k \ge 3, \tag{13}$$

and, moreover,

$$\psi_n^*(s,m) = \phi_n^*(s,m) - D_n^*(s)\tilde{p}_0^O(s,m), \tag{14}$$

where

$$\phi_n^*(s,m) \stackrel{\text{def}}{=} -\frac{\delta_{m,n}}{\lambda+\mu^*+s}, \quad D_n^*(s) \stackrel{\text{def}}{=} \delta_{1,n}\alpha_2^*(s), \tag{15}$$

then the Eqs. (11) and (12) can be rewritten as

$$\sum\nolimits_{k=-1}^{n-1} \alpha_{k+1}^*(s)\tilde{p}_{n-k}^O(s,m) - \tilde{p}_n^O(s,m) = \psi_n^*(s,m), \quad 1 \le n \le N-1, \tag{16}$$

and

$$\tilde{p}_N^O(s,m) = \left[1-\alpha_0^*(s)\right]^{-1}\left[\alpha_2^*(s)\tilde{p}_{N-1}^O(s,m) - \phi_N^*(s,m)\right]. \tag{17}$$

Because (16) has the same form as (1) (now with α_k^* and ψ_n^* being functions of s and (s,m), respectively), then the following representation holds true (compare (2)):

$$\tilde{p}_n^O(s,m) = C^*(s,m)R_n^*(s) + \sum\nolimits_{k=1}^{n} R_{n-k}^*(s)\psi_k^*(s,m), \quad n \ge 1, \tag{18}$$

where now (see (4) and refer to (13)) for $k \ge 1$

$$R_0^*(s) = 0, R_1^*(s) = \left[\alpha_0^*(s)\right]^{-1}, R_{k+1}^*(s) = R_1^*(s)\left(R_k^*(s) - \alpha_2^*(s)R_{k-1}^*(s)\right). \tag{19}$$

In order to find the explicit representation for $C^*(s,m)$, we will use the Eq. (17), treating it as a kind of boundary condition. Indeed, implementing (18) in (17), we obtain

$$\begin{aligned}
&\left[1-\alpha_0^*(s)\right]\left[C^*(s,m)R_N^*(s) + \sum\nolimits_{k=1}^{N} R_{N-k}^*(s)\left(\phi_k^*(s,m) - D_k^*(s)\tilde{p}_0^O(s,m)\right)\right] \\
&= \alpha_2^*(s)\left[C^*(s,m)R_{N-1}^*(s) + \sum\nolimits_{k=1}^{N-1} R_{N-1-k}^*(s)\left(\phi_k^*(s,m) - D_k^*(s)\tilde{p}_0^O(s,m)\right)\right] \\
&\quad - \phi_N^*(s,m).
\end{aligned} \tag{20}$$

Observe that, taking in (18) $n = 1$, we have

$$\tilde{p}_1^O(s,m) = C^*(s,m)R_1^*(s) = C^*(s,m)\left[\alpha_0^*(s)\right]^{-1}. \tag{21}$$

Substituting now (21) into (10), we get

$$\tilde{p}_0^O(s,m) = A^*(s)C^*(s,m) + B^*(s,m), \tag{22}$$

where

$$A^*(s) \stackrel{\text{def}}{=} \frac{\lambda}{\lambda+s}\left[\alpha_0^*(s)\right]^{-1}, \quad B^*(s,m) \stackrel{\text{def}}{=} \frac{\delta_{m,0}}{\lambda+s}. \tag{23}$$

Inserting (22) into (20), we eliminate $C^*(s,m)$ in the following form:

$$C^*(s,m) = \frac{T^*(s,m)}{\Delta^*(s)}, \tag{24}$$

where we denote

$$\begin{aligned} T^*(s,m) &\stackrel{\text{def}}{=} \alpha_2^*(s)\sum\nolimits_{k=1}^{N-1} R_{N-k-1}^*(s)\left[\phi_k^*(s,m) - D_k^*(s)B^*(s,m)\right] \\ &\quad - \left[1-\alpha_0^*(s)\right]\sum\nolimits_{k=1}^{N} R_{N-k}^*(s)\left[\phi_k^*(s,m) - D_k^*(s)B^*(s,m)\right] \\ &\quad - \phi_N^*(s,m) \end{aligned} \tag{25}$$

and

$$\begin{aligned} \Delta^*(s) &\stackrel{\text{def}}{=} \left(1-\alpha_0^*(s)\right)\left[R_N^*(s) - A^*(s)\sum\nolimits_{k=1}^{N} R_{N-k}^*(s)D_k^*(s)\right] \\ &\quad + \alpha_2^*(s)\left[A^*(s)\sum\nolimits_{k=1}^{N-1} R_{N-1-k}^*(s)D_k^*(s) - R_{N-1}^*(s)\right]. \end{aligned} \tag{26}$$

Collecting the formulae (14), (18), (22) and (24), we can formulate the following:

Theorem 2. *The LT $\tilde{p}_n^O(s,m)$ of transient queue-size distribution in the ordinary M/M/1/N-type queue, conditioned by the number $0 \le n \le N$ of jobs present in the system initially, can be expressed in the following way:*

$$\tilde{p}_0^O(s,m) = A^*(s)\frac{T^*(s,m)}{\Delta^*(s)} + B^*(s,m), \tag{27}$$

$$\begin{aligned} \tilde{p}_n^O(s,m) &= \frac{T^*(s,m)}{\Delta^*(s)}R_n^*(s) \\ &\quad + \sum\nolimits_{k=1}^{N} R_{N-k}^*(s)\left[\phi_k^*(s,m) - D_k^*(s)\left(A^*(s)\frac{T^*(s,m)}{\Delta^*(s)} + B^*(s,m)\right)\right], \quad 1 \le n \le N, \end{aligned} \tag{28}$$

where $\Re(s) > 0$ and $0 \le m \le N$, and the formulae for $\phi_k^*(s,m)$, $D_k^*(s)$, $R_k^*(s)$, $A^*(s)$, $B^*(s,m)$, $T^*(s,m)$ and $\Delta^*(s)$ are given in (15), (19), (23) (25) and (26), respectively.

4 Queue-Size Distribution in a Model with Working Vacations

Let us take into consideration the original model with generally-distributed server working vacation periods (each with a CDF $G(\cdot)$) during which the processing of jobs is offered with a slower rate $\mu^* < \mu$, where μ denotes the normal-mode service rate. Introduce the following notation:

$$P_n(t,m) \stackrel{\text{def}}{=} \boldsymbol{P}\{X(t) = m|X(0) = n\}, \quad 0 \le m, n \le N, \tag{29}$$

where $X(t)$ denotes the number of packets present in the system with working vacations (original one) at time t. Assume, firstly, that the system starts its evolution being empty. So, at $t = 0$ the working vacation period begins. Observe, that the following equation is then satisfied:

$$P_0(t,m) = \sum\nolimits_{k=0}^{N} \int_0^t P_0^O(x,k) P_k^O(t-x,m) dG(x) + [1 - G(t)] P_0^O(t,m). \tag{30}$$

Indeed, the first summand on the right side of (30) presents the situation in which the working vacation period completes at time $x < t$. Hence, at time x the system starts the operation in normal mode with $0 \le k \le N$ packets with probability $P_0^O(x,k)$. The second summand in (30) relates to the case in that the time epoch t is "inside" the working vacation period. If the number of packets equals $1 \le n \le N-1$ initially, we similarly obtain

$$P_n(t,m) = \lambda \int_0^t e^{-(\lambda+\mu)x} P_{n+1}(t-x,m) dx + \mu \int_0^t e^{-(\lambda+\mu)x} P_{n-1}(t-x,m) dx + e^{-(\lambda+\mu)t} \delta_{m,n}. \tag{31}$$

Finally, $n = N$ we have

$$P_N(t,m) = \lambda \int_0^t e^{-(\lambda+\mu)x} P_N(t-x,m) dx + \mu \int_0^t e^{-(\lambda+\mu)x} P_{N-1}(t-x,m) dx + e^{-(\lambda+\mu)t} \delta_{m,N}. \tag{32}$$

The interpretation of (31) and (32) is the same as of (7) and (8). Defining

$$\tilde{p}_n(s,m) \stackrel{\text{def}}{=} \int_0^\infty e^{-st} \boldsymbol{P}\{X(t) = m|\mathrm{X}(0) = n\} dt, \quad \Re(s) > 0, \tag{33}$$

and, moreover (compare (13)–(15)),

$$\alpha_0(s) = \frac{\lambda}{\lambda+\mu+s}, \alpha_1(s) = 0, \alpha_2(s) = \frac{\mu}{\lambda+\mu+s}, \alpha_k(s) = 0, \quad k \ge 3, \tag{34}$$

and

$$\psi_n(s,m) = \phi_n(s,m) - D_n(s)\tilde{p}_0(s,m), \tag{35}$$

where

$$\phi_n(s,m) \stackrel{\text{def}}{=} -\frac{\delta_{m,n}}{\lambda+\mu+s}, D_n(s) \stackrel{\text{def}}{=} \delta_{1,n}\alpha_2(s), \tag{36}$$

we obtain from (30)–(32) the following system (see (10) ,(16) and (17)):

$$\tilde{p}_0(s,m) = \sum_{k=0}^{N} \tilde{p}_k(s,m) \int_0^\infty e^{-sx} P_0^O(x,k)dG(x) + \int_0^\infty e^{-st} P_0^O(t,m)[1-G(t)]dt, \tag{37}$$

$$\sum_{k=-1}^{n-1} \alpha_{k+1}(s)\tilde{p}_{n-k}(s,m) - \tilde{p}_n(s,m) = \psi_n(s,m), 1 \le n \le N-1, \tag{38}$$

and

$$\tilde{p}_N(s,m) = [1-\alpha_0(s)]^{-1}[\alpha_2(s)\tilde{p}_{N-1}(s,m) - \phi_N(s,m)]. \tag{39}$$

Since (38) has the same form as (1), then we can write (compare (18))

$$\tilde{p}_n(s,m) = C(s,m)R_n(s) + \sum_{k=1}^{n} R_{n-k}(s)\psi_k(s,m), \quad n \ge 1, \tag{40}$$

where here (see (19)) for $k \ge 1$

$$R_0(s) = 0, R_1(s) = \left[\alpha_0^*(s)\right]^{-1}, R_{k+1}(s) = R_1(s)(R_k(s) - \alpha_2(s)R_{k-1}(s)) \tag{41}$$

and the sequence $(\alpha_k(s))$ is defined in (34).

Let us note that, if we define

$$L_k(s) \stackrel{\text{def}}{=} \int_0^\infty e^{-st} P_0^O(t,k)dG(t) \tag{42}$$

and

$$M(s,m) \stackrel{\text{def}}{=} \int_0^\infty e^{-st} P_0^O(t,k)[1-G(t)]dt, \tag{43}$$

where $\Re(s) > 0$, then (37) can be rewritten in the following way:

$$\tilde{p}_0(s,m) = \sum_{k=0}^{N} \tilde{p}_k(s,m)L_k(s) + M(s,m). \tag{44}$$

Inserting now in (44), instead of $\tilde{p}_k(s,m)$, the right side of the representation (40), we obtain

$$\tilde{p}_0(s,m) = \tilde{p}_0(s,m)L_0(s) + \sum_{k=1}^{N} L_k(s)\Big[C(s,m)R_k(s) + \sum_{i=1}^{k} R_{k-i}(s)(\phi_i(s,m) - D_i(s)\tilde{p}_0(s,m))\Big] + M(s,m), \tag{45}$$

and hence we get the following formula:

$$\tilde{p}_0(s,m) = A(s)C(s,m) + B(s,m), \tag{46}$$

where

$$A(s) \stackrel{\text{def}}{=} \Big[1 - L_0(s) + \sum_{k=1}^{N} L_k(s)\sum_{i=1}^{k} R_{k-i}(s)D_i(s)\Big]^{-1} \sum_{k=1}^{N} L_k(s)R_k(s) \tag{47}$$

and

$$B(s,m) \stackrel{\text{def}}{=} \Big[1 - L_0(s) + \sum_{k=1}^{N} L_k(s)\sum_{i=1}^{k} R_{k-i}(s)D_i(s)\Big]^{-1} \Big[\sum_{k=1}^{N} L_k(s)\sum_{i=1}^{k} R_{k-i}(s)\phi_i(s,m) + M(s,m)\Big] \tag{48}$$

Having defined $A(s)$ and $B(s,m)$, we can execute successive steps of the procedure described for the ordinary system in (20)–(26) and formulate the following main theorem:

Theorem 3. *The LT $\tilde{p}_n(s,m)$ of time-dependent queue-size distribution in the M/M/1/N-type queue with working vacation mechanism, conditioned by the number $0 \le n \le N$ of jobs present in the system initially, can be written in the following way:*

$$\tilde{p}_0(s,m) = A(s)\frac{T(s,m)}{\Delta(s)} + B(s,m), \tag{49}$$

$$\tilde{p}_n(s,m) = \frac{T(s,m)}{\Delta(s)}R_n(s) + \sum_{k=1}^{N} R_{N-k}(s)\Big[\phi_k(s,m) - D_k(s)\Big(A(s)\frac{T(s,m)}{\Delta(s)} + B(s,m)\Big)\Big], \quad 1 \le n \le N, \tag{50}$$

where $\Re(s) > 0$ and $0 \le m \le N$, and (compare (25) and (26))

$$T(s,m) \stackrel{\text{def}}{=} \alpha_2(s)\sum_{k=1}^{N-1} R_{N-1-k}(s)[\phi_k(s,m) - D_k(s)B(s,m)] - [1 - \alpha_0(s)]\sum_{k=1}^{N} R_{N-k}(s)[\phi_k(s,m) - D_k(s)B(s,m)] - \phi_N(s,m), \tag{51}$$

$$\Delta(s) \stackrel{\text{def}}{=} (1 - \alpha_0(s)) \left[R_N(s) - A(s) \sum_{k=1}^{N} R_{N-k}(s) D_k(s) \right] + \alpha_2(s) \left[A(s) \sum_{k=1}^{N-1} R_{N-1-k}(s) D_k(s) - R_{N-1}(s) \right]. \tag{52}$$

Moreover, the formulae for $\phi_k(s,m), D_k(s), R_k(s), A(s)$ *and* $B(s,m)$ *are given in (36), (41), (47) and (48), respectively.*

Remark 1. *The formulae (49)* and *(50) allow for finding the LT of the queue-size distribution in the original system as functions of the appropriate distribution for the ordinary model (given in terms of LTs in (27)* and *(28)), by* $L_k(s)$ *and* $M(s,m)$ *defined in (42) and (43), respectively.*

Remark 2. *Having LTs of the probabilities* $\tilde{p}_n(s,m)$, *we can easily find the stationary queue-size distribution* $\pi_0, \ldots, \pi_N$, *by using the well-known Tauberian theorem, namely*

$$\pi_m = \lim_{t \to \infty} \boldsymbol{P}\{X(t) = m\} = \lim_{s \downarrow 0} s \cdot \tilde{p}_n(s,m), \quad 0 \le m \le N, \tag{53}$$

where n can be chosen arbitrarily between 0 and N.

Remark 3. *In the case of exponentially distributed working vacation period with mean* θ^{-1}, *i.e. if* $G(t) = 1 - e^{-\theta t}, t > 0$, *we can evaluate* $L_k(s)$ *and* $M(s,m)$ *explicitly. Indeed, we obtain*

$$L_k(s) = \theta \tilde{p}_0^O(s+\theta, k), M(s,m) = \tilde{p}_0^O(s+\theta, m). \tag{54}$$

5 Numerical Examples

Let us consider a node of the wireless network in which packets of sizes 200 [B] arrive according to a Poisson process with intensity 600 [kb/s]. The normal throughput equals 800 [kb/s], but every time when the buffer empties the throughput is lower (500 [kb/s]) for a random exponential time with mean 0.1 [s]. Let us note that for such parameters the traffic load equals normally 0.75 and during the working vacation period 1.20 (so, in this case the link is overloaded). In Fig. 1 transient behaviour of probabilities $P\{X(t) = m \mid X(0) = 0\}$ is presented for $m = 0, 2$ and 4, where $N = 4$. In Fig. 2 transient behaviour of $\boldsymbol{P}\{X(t) = 1 \mid X(0) = 0\}$ is visualized for different arrival rates: 200, 400 and 600 [kb/s], where the remaining system parameters are kept the same as in Fig. 1. As one can observe, for the lowest arrival rate, the time for the system stabilization is the shortest one. Figure 3 shows the behaviour of $\boldsymbol{P}\{X(t) = 4 \mid X(0) = 0\}$ at arrival rate 400 [kb/s] and normal service speed 800 [kb/s], for three different processing speeds during the WV (= working vacation) period (the case 800 [kb/s] denotes, in fact, no working vacation) (Fig. 4).

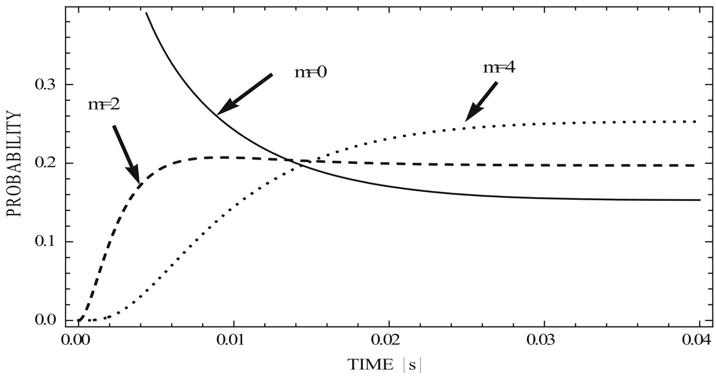

Fig. 2. Transient behaviour of $\boldsymbol{P}\{X(t) = m \,|\, X(0) = 0\}$ for different values of m

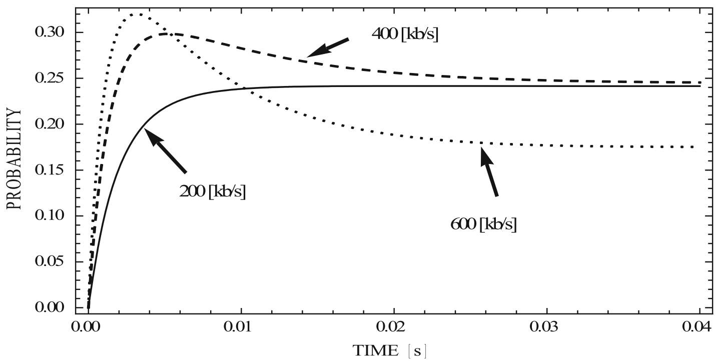

Fig. 3. Sensitivity of transient queue-size distribution on different arrival intensities

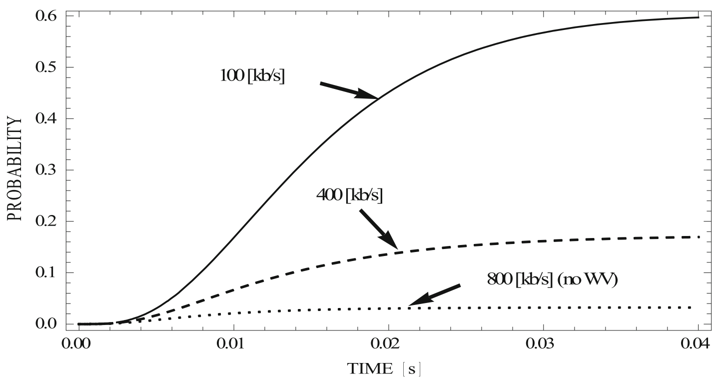

Fig. 4. Sensitivity of transient queue-size distribution on processing rates during WV

6 Conclusions and Closing Remarks

In the paper a single-server queueing model with finite buffer capacity, Poisson arrival stream and exponential processing times is investigated. The service process is governed by a FIFO discipline with working vacation algorithm, in which the service station, every time when the system empties, provide the processing with a slower rate during a generally distributed random time. The explicit formulae for the LT of transient conditional queue-size distribution are obtained. Numerical utility of the formulae is presented in examples motivated by real traffic in a node of the wireless network.

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