Classical Moduli Spaces and Rationality

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Abstract Moduli spaces and rational parametrizations of algebraic varieties have common roots. A rich album of moduli of special varieties was indeed collected by classical algebraic geometers and their (uni)rationality was studied. These were the origins for the study of a wider series of moduli spaces one could define as classical. These moduli spaces are parametrize several type of varieties which are often interacting: curves, abelian varieties, K3 surfaces. The course will focus on rational parametrizations of classical moduli spaces, building on concrete constructions and examples.

1 Introduction

These lectures aim to study, even in a historical perspective, the interplay and the several links between two notions which are at the core of Algebraic Geometry, from its origins to modern times. The key words to recognize these notions are *Rationality* and *Moduli*.

Rationality is a classical notion. As we all know, an algebraic variety X over a field k is rational if there exists a birational map

$$f : k^d \to X$$

with $d = dim\ X$. The notion reflects the original attempt to study some algebraic varieties, the rational ones, via invertible parametric equations defined by rational functions.

Given an algebraic variety X the *rationality problem for* X, when it is meaningful, is the problem of understanding wether X is rational or not. In most of the relevant cases this is an outstanding problem coming from classical heritage. Therefore the rationality problems, for some important classes of algebraic varieties, often are

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guiding themes, marking the history of Algebraic Geometry. As a consequence of this fact many related notions came to light and are of comparable importance.

Let us recall the basic ones and some relevant properties. We will often consider the notion of *unirational variety*. This originates from the most natural instance to weaken the notion of rationality. Indeed we may want simply to construct a dominant separable rational map

$$f : k^n \to X$$

not necessarily birational. We say that f is a *rational parametrization of* X.

Definition 1.1 An algebraic variety X is unirational if there exists a rational parametrization $f : k^n \to X$.

In spite of their simple definitions, these classical notions are intrinsically hard. Studying the rationality, or unirationality, of algebraic varieties has proven to be a very nasty problem, even for the tools of contemporary Algebraic Geometry. This was often remarked in the modern times by many authors, see Kollar's article [K] about. In order to have a more doable notion, that of *rational connectedness* became central in the last three decades.

Definition 1.2 X is rationally connected if there exists a non empty open subset U of X such that: $\forall\, x, y \in U\ \exists f_{x,y} : k \to X \,/\, x, y \in f(k)$.

An important example where rational connectedness appears as more doable is the following. Assume k is the complex field and consider an irreducible family $f : \mathcal{X} \to B$ of smooth irreducible complex projective varieties. Then the next statement is well known: the property of a fibre of being rationally connected extends from one to all the fibres of f.

The same statement is instead unproven if we replace the word rationally connected by the words rational or unirational. Actually such a statement is expected to be false. If so this would be a crucial difference between the notions considered. For completeness let us also recall that[1]

Definition 1.3 X is uniruled if there exists a non empty open subset U of X such that: $\forall\, x \in U,\ \exists$ a non constant $f_x : k \to X \,/\, x \in f(k)$.

In view of the contents of these lectures we will work from now on over the complex field. It is easy to see that

$$\textit{Rational} \overset{(1)}{\Rightarrow} \textit{unirational} \overset{(2)}{\Rightarrow} \textit{rationally connected} \overset{(3)}{\Rightarrow} \textit{uniruled.} \overset{(4)}{\Rightarrow} \textit{kod}(X) = -\infty$$

[1] A first counterexample to this outstanding problem has been finally produced by B. Hassett, A. Pirutka and Y. Tschinkel in 2016. See: *Stable rationality of quadric surfaces bundles over surfaces*, arXiv1603.09262

where $kod(X)$ is the *Kodaira dimension* of X.[2] Remarking that the inverse of implication (3) is false is a very easy exercise: just consider as a counterexample $X = Y \times \mathbf{P}^1$, where Y is not rationally connected.

On the other hand the remaining problems, concerning the inversion of the other implications, have a more than centennial history and a prominent place in Algebraic Geometry, cfr. [EM, Chaps. 6–10]. Inverting (1) is the classical Lüroth problem, which is false in dimension ≥ 3. As is well known this is due to the famous results of Artin and Mumford, Clemens and Griffiths, Manin and Iskovskih in the Seventies of last century. It is reconsidered elsewhere in this volume. Wether (2) is invertible or not is a completely open question:

Does rational connectedness imply unirationality?

This could be considered, in some sense, the contemporary version of the classical Lüroth problem. That (4) is invertible is a well known conjecture due to Mumford. It is time to consider the other topic of these lectures.

Moduli, in the sense of moduli space, is a classical notion as well. Riemann used the word Moduli, for algebraic curves, with the same meaning of today. Under some circumstances the moduli space $\mathcal{M}$ of a family of algebraic varieties X is dominated by the space of coefficients of a *general* system of polynomial equations of fixed type. Hence $\mathcal{M}$ is unirational in this case.

For instance every hyperelliptic curve H of genus $g \geq 2$ is uniquely defined as a finite double cover $\pi : H \to \mathbf{P}^1$ and π is uniquely reconstructed from its branch locus $b \in \mathbf{P}^{2g+2} := |\mathcal{O}_{\mathbf{P}^1}(2g+2)|$. Therefore the assignment $b \longrightarrow H$ induces a dominant rational map $m : \mathbf{P}^{2g+2} \to \mathcal{H}_g$, where $\mathcal{H}_g$ is the moduli space of hyperelliptic curves of genus g. This implies that $\mathcal{H}_g$ is unirational for every genus g. Actually, a modern result in Invariant Theory due to Katsylo says that $\mathcal{H}_g$ is even rational [K5].

The latter example highlights the so many possible relations between questions related to rationality and the natural search of suitable parametrizations for some moduli spaces. As remarked, these relations closely follow the evolution of Algebraic Geometry from nineteenth century to present times. This is specially true for the moduli space $\mathcal{M}_g$, of curves of genus g, and for other moduli spaces, quite related to curves, more recently appeared in this long history. In some sense we could consider these spaces as the classical ones. Here is a list, according to author's preferences:

- $\mathcal{M}_g$, *moduli of curves of genus* g;
- $Pic_{d,g}$, *universal Picard variety over* $\mathcal{M}_g$;
- $\mathcal{R}_g$, *moduli of Prym curves of genus* g;

[2] We recall that $kod(X) := \min\{\dim f_m(X'),\ m \geq 1\}$. Here X' is a complete, smooth birational model of X. Moreover f_m is the map defined by the linear system of pluricanonical divisors $P_m := \mathbf{P}H^0(det(\Omega^1_{X'})^{\otimes m})$. If P_m is empty for each $m \geq 1$ one puts $kod(X) := -\infty$.

- $\mathcal{S}_g^+/\mathcal{S}_g^-$, *moduli of even/odd spin curves of genus* g;
- $\mathcal{A}_g$, *moduli of p.p. abelian varieties of dimension* g;
- $\mathcal{F}_g$, *moduli of polarized K3 surfaces of genus* g.

In these lectures we review a variety of results on some of these spaces and their interplay with rationality, for this reason we specially concentrate on the cases of low genus g. We focus in particular on the next themes:

- *How much the rationality of these spaces is extended*,
- *Uniruledness/ Unirationality/ Rationality*,
- *Transition to non negative Kodaira dimension as g grows up*.

Classical geometric constructions grow in abundance around the previous spaces, revealing their shape and their intricate connections. The contents of the forthcoming lectures are organized as follows.

In Sect. 2 of these notes we study the families $\mathcal{V}_{d,g}$ of nodal plane curves of degree d and genus g. Starting from Severi's conjecture on the (uni)rationality of $\mathcal{M}_g$ for any g, and coming to the results of Harris and Mumford which disprove it, we describe results and attempts in order to parametrize $\mathcal{M}_g$ by a rational family of linear systems of nodal plane curves.

Then we study, by examples and in low genus, the effective realizations of non isotrivial pencils of curves with general moduli on various types of surfaces. In particular we use these examples to revisit the slope conjecture, for the cone of effective divisors of the moduli of stable curves $\overline{\mathcal{M}}_g$, and its counterexamples.

In Sect. 3 we study the moduli spaces $\mathcal{R}_g$ of Prym curves and the Prym map $P_g : \mathcal{R}_g \to \mathcal{A}_{g-1}$. Then we prove the unirationality of $\mathcal{R}_g$ for $g \leq 6$. We outline a new method of proof used in the very recent papers [FV4, FV5]. We use families of nodal conic bundles over the plane, instead of families of nodal plane curves, to construct a rational parametrization of $\mathcal{R}_g$. We also discuss the known results on the rationality of $\mathcal{R}_g$ for $g \leq 4$.

In Sect. 4 we profit of these results to discuss the unirationality of $\mathcal{A}_p$. The unirationality of $\mathcal{A}_p$ is granted by the unirationality of $\mathcal{R}_{p+1}$ and the dominance of the Prym map P_{p+1} for $p \leq 5$. An interesting object here is the universal Prym, that is the pull back by P_{p+1} of the universal principally polarized abelian variety over $\mathcal{A}_p$. We construct an effective rational parametrization of the universal Prym $\mathcal{P}_5$.

Finally we discuss the slope of a toroidal compactification of $\mathcal{A}_p$, $p \leq 6$. Let $\overline{\mathcal{A}}_p$ be the perfect cone compactification of $\mathcal{A}_p$, it turns out that, for $p = 6$, the boundary divisor D of $\overline{\mathcal{A}}_6$ is dominated by the universal Prym $\mathcal{P}_5$. We use its effective parametrization, and a sweeping family of rational curves on it, to compute a lower bound for the slope of $\overline{\mathcal{A}}_6$, outlining the main technical details from [FV4].

In Sect. 5 we deal with the theme of curves and K3 surfaces. Mukai constructions of general K3 surfaces of genus $g \leq 11$, and hence of general canonical curves for $g \neq 10$, are considered. We apply these to prove the unirationality of the universal Picard variety $Pic_{d,g}$ for $g \leq 9$, we will use it to obtain further unirationality results for $\mathcal{M}_g$. The transition of the Kodaira dimension of $Pic_{d,g}$, from $-\infty$ to the maximal one, is also discussed.

Then we concentrate on the family of special K3 surfaces S, having Picard number at least two, embedded in $\mathbf{P}^{g-2}$ by a genus $g-2$ polarization and containing a smooth paracanonical curve C of genus g. This is the starting point for going back to Prym curves (C, η) of genus g and discuss when $\mathcal{O}_C(1)$ is the Prym canonical line bundle $\omega_C \otimes \eta$.

In the family of K3 surfaces S we have the family of Nikulin surfaces, where the entire linear system $|C|$ consists of curves D such that $\mathcal{O}_D(1)$ is Prym canonical. We discuss this special situation and some interesting analogies between the families of curves C of genus $g \leq 7$ contained in a general Nikulin surface S and the families of curves of genus $g \leq 11$ contained in a general K3 surface.

An outcome of the discussion is the proof that the Prym moduli space $\mathcal{R}_7$ is unirational, cfr. [FV6]. Finally, in Sect. 6, we go back to $\mathcal{M}_g$. Relying on the previous results, we outline from [Ve1] the proof that $\mathcal{M}_g$ is unirational for $g \leq 14$.

2 $\mathcal{M}_g$ and the Rationality

2.1 Nodal Plane Sextics

We begin our tour of classical moduli spaces and related geometric constructions from moduli of curves. If we want to go back to the classical point of view, adding some historical perspectives, it is natural to consider at first plane curves. Any algebraic variety V of dimension d is birational to a hypersurface in $\mathbf{P}^{d+1}$ and the study of these birational models of V was considered a natural option to understand V: curves in the plane, surfaces in the space and so on.

In 1915 Severi publishes the paper "Sulla classificazione delle curve algebriche e sul teorema di esistenza di Riemann," an exposition of his recent results on the birational classification of algebraic curves. The next sentence from it is the starting point of a long history:

> Ritengo probabile che la varietá H sia razionale o quanto meno che sia riferibile ad un' involuzione di gruppi di punti in uno spazio lineare S_{3p-3}; o, in altri termini, che nell' equazione di una curva piana di genere p (e per esempio dell' ordine $p+1$) i moduli si possano far comparire razionalmente.

See [S], in the text H is the moduli space of curves $\mathcal{M}_g$. The conjecture of Severi is that $\mathcal{M}_g$ is "probably rational" or at least unirational. The idea is that there exists an irreducible family of plane curves, to be written in affine coordinates as

$$\sum_{0 \leq i,j \leq d} f_{ij} X^i Y^j = 0,$$

so that:

- *its general element is birational to a curve of genus g*,
- *the* f_{ij}*'s are rational functions of* $3g-3$ *parameters*,
- *the corresponding natural map* $f : \mathbf{C}^{3g-3} \to \mathcal{M}_g$ *is dominant.*[3]

In other words the claim is that there exists a unirational variety

$$\mathbb{P} \subset |\mathcal{O}_{\mathbb{P}^2}(d)|$$

of possibly singular plane curves of degree d such that f is dominant. In the same paper a proof is given that such a family exists for $g \leq 10$.

Now we postpone for a while the discussion of this result to concentrate on the simplest possible case, namely when $\mathbb{P}$ is a linear space. In this case $\mathbb{P}$ is a linear system of plane curves, possibly singular at some of the base points. The first question we want to consider, even in a more general framework than $\mathbf{P}^2$, is the following.

Question 2.1 *Let* $\mathbb{P}$ *be a linear system of curves of geometric genus g on a smooth surface P, when the natural map* $m : \mathbb{P} \to \mathcal{M}_g$ *is dominant?*

The complete answer follows from a theorem of Castelnuovo, with the contribution of Beniamino Segre and later of Arbarello [Ar, Se1]. By resolution of indeterminacy there exists a birational morphism

$$\sigma : S \to P$$

such that S is smooth and the strict transform of $\mathbb{P}$ is a base point free linear system $|C|$ of smooth, integral curves of genus g. Then the answer to Question 2.1 is provided by the next theorem.

Theorem 2.2 *Let* $|C|$ *be as above. Assume that the natural map*

$$m : |C| \to \mathcal{M}_g$$

is dominant, then S is rational and $g \leq 6$.

Proof It is not restrictive to assume that $|C|$ is base point free. Since $|C| \to \mathcal{M}_g$ is dominant we have $\dim\ |C| \geq 3g-3$. This implies that $\mathcal{O}_C(C)$ is not special. Then, by Riemann Roch, it follows $C^2 \geq 4g-4$ and we have $CK_S \leq -2g+2$ by adjunction formula. Since $|C|$ has no fixed components, it follows that $|mK_S| = \emptyset$ for $m \geq 1$. Hence S is ruled and birational to $R \times \mathbf{P}^1$. In particular C admits a finite map $C \to R$. Since the curves of $|C|$ have general moduli, this is impossible unless R is rational. Hence S is rational. Now let $g \geq 10$, we observe that $\dim |C| \geq 3g-3 \geq 2g+7$. But then a well known theorem of Castelnuovo, on linear systems on a rational surface,

[3] Unless differently stated, we assume $g \geq 2$ to simplify the exposition.

implies that the elements of $|C|$ are hyperelliptic [C1]. This is a contradiction, hence the theorem follows for $g \geq 10$. The cases $g = 7, 8, 9$ were excluded by Beniamino Segre in [Se1]. □

Let $g \leq 6$, one can easily show that $\mathbb{P}$ can be chosen so that $d = 6$ and its base points are ordinary nodes for a general $\Gamma \in \mathbb{P}$. For $g \geq 2$ these are also in general position in $\mathbf{P}^2$. In some sense the case of plane sextics of genus $g \leq 6$ will become a guiding example for these lectures.

2.2 *Rationality of $\mathcal{M}_6$*

In spite of the centennial history of moduli of curves, the rationality of $\mathcal{M}_g$ remains unsettled, as an outstanding problem, for most of the values of g where $\mathcal{M}_g$ is known to be unirational or uniruled. Presently this means $g \leq 16$. The rationality of $\mathcal{M}_g$ is actually established for $g \leq 6$.

The known rationality results are mainly related to Invariant Theory, starting from the classical description

$$\mathcal{M}_1 = \mathfrak{h}/SL(2, \mathbb{Z}).$$

Here the action of $SL(2, \mathbb{Z})$ on the Siegel upper half plane $\mathfrak{h}$ is

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad \forall \tau \in \mathfrak{h}\ ,\ \forall \begin{pmatrix} a\, b \\ c\, d \end{pmatrix} \in SL(2, \mathbb{Z})$$

and $\mathcal{M}_1$ is actually the affine line. As it is well known this rationality result goes back to Weierstrass and to the Weierstrass form of the equation of a plane cubic. The rationality of $\mathcal{M}_2$ was proven in 1960 by Igusa [I]. Later the proof of the rationality of $\mathcal{M}_4$ and $\mathcal{M}_6$ came around 1985. This is due to Shepherd-Barron [SB] and [SB1]. Finally the rationality of $\mathcal{M}_5$ and of $\mathcal{M}_3$ is due to Katsylo [K1] and [K3]. See also [Bo, K4].

The rationality of $\mathcal{M}_6$ is related to plane sextics with four-nodes. In turn, as we are going to see in these lectures, the family of nodal plane sextics of genus $g \leq 6$ is also related to the unirationality of the Prym moduli spaces $\mathcal{R}_g$ for $g \leq 6$ [FV4].

Let $\mathbb{P}$ be the linear system of plane sextics having multiplicity ≥ 2 at the fundamental points $F_1 = (1 : 0 : 0)$, $F_2 = (0 : 1 : 0)$, $F_3 = (0 : 0 : 1)$, $F_4 = (1 : 1 : 1)$ of $\mathbf{P}^2$. Then a general $\Gamma \in \mathbb{P}$ is birational to a genus six curve. We know from the previous section that the moduli map

$$m : \mathbb{P} \to \mathcal{M}_6$$

is dominant. Hence m is generically finite, since $\dim \mathbb{P} = \dim \mathcal{M}_6$. What is the degree of m? To answer this question consider the normalization

$$\nu : C \to \Gamma$$

of a general $\Gamma \in \mathbb{P}$. At first we notice that $\nu^*\mathcal{O}_\Gamma(1)$ is an element of

$$W^2_6(C) := \{L \in Pic^6(C) \ / \ h^0(L) \geq 3\} \subset Pic^6 C.$$

It is well known that the cardinality of $W^2_6(C)$ is five and we can easily recover all the elements of this set from Γ. One is $\nu^*\mathcal{O}_\Gamma(1)$. Moreover let

$$Z \subset \{F_1 \dots F_4\}$$

be a set of three points, then $L_Z = \nu^*\mathcal{O}_\Gamma(2) \otimes \mathcal{O}_C(-\nu^* Z)$ is also an element of $W^2_6(C)$. In particular the linear system $|L_Z|$ is just obtained by taking the strict transform by ν of the net of conics through Z.

Next we remark that two elements $\Gamma_1, \Gamma_2 \in \mathbb{P}$ are birational to the same C and defined by the same $L \in W^2_6(C)$ iff there exists $\alpha \in \mathfrak{s}_4 \subset Aut\, \mathbb{P}^2$ such that $\alpha(\Gamma_1) = \Gamma_2$.

Here we denote by $\mathfrak{s}_4$ the stabilizer of $\{F_1 \dots F_4\}$ in $Aut\, \mathbb{P}^2$, which is a copy of the symmetric group in four letters. Since the cardinality of $W^2_6(C)$ is five we conclude that:

Proposition 2.3 *m has degree* 120.

Is this degree related to an action of the symmetric group $\mathfrak{s}_5$ on $\mathbb{P}$ so that m is the quotient map of this action?

The answer is positive: in the Cremona group of $\mathbf{P}^2$ we can consider the subgroup generated by $\mathfrak{s}_4$ and by the quadratic transformations

$$q_Z : \mathbf{P}^2 \to \mathbf{P}^2$$

centered at the subsets of three points $Z \subset \{F_1 \dots F_4\}$. This is actually a copy of $\mathfrak{s}_5$ and we will denote it in the same way. Notice that the strict transform of $\mathbb{P}$ by q_Z is $\mathbb{P}$, hence $\mathfrak{s}_5$ is exactly the subgroup of the Cremona group of $\mathbf{P}^2$ leaving $\mathbb{P}$ invariant.

Equivalently we can rephrase the previous construction in terms of the surface obtained by blowing up $\mathbf{P}^2$ along $\{F_1 \dots F_4\}$. Let

$$\sigma : S \to \mathbf{P}^2$$

be such a blowing up. Then S is a quintic Del Pezzo surface and, moreover, the strict transform of $\mathbb{P}$ by σ is exactly the linear system

$$|-2K_S|,$$

where K_S is a canonical divisor. As is well known, the anticanonical linear system $|-K_S|$ defines an embedding $S \subset \mathbf{P}^5$ as a quintic Del Pezzo surface and S is the unique smooth quintic Del Pezzo up to projective equivalence.

The action of $\mathfrak{s}_5$ on $\mathbf{P}^2$ lifts to an action of $\mathfrak{s}_5$ on S as a group of biregular automorphisms and it is known, and easy to see, that

$$\mathfrak{s}_5 = Aut\, S.$$

The conclusion is now immediate:

Theorem 2.4 $\mathcal{M}_6 \cong \mathbb{P}/\mathfrak{s}_5 \cong |-2K_S|/Aut\, S.$

This situation is considered in [SB]. Building on representation theory of $\mathfrak{s}_5$, the author then shows the following result.

Theorem 2.5 *$|-2K_S|/Aut\, S$, and hence $\mathcal{M}_6$, is rational.*

Let us sketch the proof very briefly. The symmetric group $\mathfrak{s}_5 = Aut\, S$ acts on the vector space $V := H^0(\mathcal{O}_S(-K_S))$ so that V is a representation of degree 6 of $\mathfrak{s}_5$. This is the unique irreducible representation of degree 6 of $\mathfrak{s}_5$. Passing to $Sym^2 V$ one has

$$Sym^2 V = I_2 \oplus H^0(\mathcal{O}_S(-2K_S))$$

where I_2 is Kernel of the multiplication map $Sym^2 V \to H^0(\mathcal{O}_S(-2K_S))$. Clearly the summands are representations of $\mathfrak{s}_5$. For $H^0(\mathcal{O}_S(-2K_S))$ the point is to show that it is isomorphic to

$$\mathbf{1} \oplus \phi \oplus \chi \oplus \chi' \oplus \sigma.$$

Here $\mathbf{1}$ and σ denote the trivial and sign representations of degree 1 and ϕ is the standard representation of degree 4. Moreover χ, χ' are irreducible of degree 5 and either $\chi = \chi'$ or $\chi' = \chi \otimes \sigma$.

Blowing up the Kernel of the projection $\mathbf{1} \oplus \phi \oplus \chi \oplus \chi' \oplus \sigma \to \phi$ one obtains a vector bundle $\mathcal{E} \to \phi$ of rank 12. Notice that $\mathfrak{s}_5$ acts freely on an invariant open set $U \subset \phi$ and that the action of $\mathfrak{s}_5$ is linearized on $\mathcal{E}$. Therefore $\mathcal{E}$ descends to a vector bundle $\overline{\mathcal{E}} \to U/\mathfrak{s}_5$ and, moreover, it follows that

$$\mathbf{P}(\overline{\mathcal{E}}) \;\cong\; \mathbf{P}^{11} \times U/\mathfrak{s}_5 \;\cong |-2K_S|/\mathfrak{s}_5 \;\cong \mathcal{M}_6.$$

It remains to show that $U/\mathfrak{s}_5$ is rational, but this just follows from the theorem of symmetric functions.

2.3 Nodal Plane Curves

As a second step, of our investigation on linear systems of plane curves dominating $\mathcal{M}_g$, we replace a single linear system by a family. We say that

$$\mathbb{P}_T \subset T \times |\mathcal{O}_{\mathbb{P}^2}(d)|$$

is a family of linear systems of curves of degree d and genus g if:

(1) *T is a integral variety in the Grassmannian of r-spaces of $|\mathcal{O}_{\mathbb{P}^2}(d)|$,*
(2) *the first projection $p_1 : \mathbb{P}_T \to T$ is the universal r-space over T,*
(3) *a general $\Gamma \in p_2(\mathbb{P}_T)$ is an integral curve of genus g.*

Here p_2 denotes the second projection $\mathbb{P}_T \to |\mathcal{O}_{\mathbf{P}^2}(d)|$, we will set

$$\mathbb{P} := p_2(\mathbb{P}_T) \subset |\mathcal{O}_{\mathbb{P}^2}(d)|.$$

Clearly we have

$$\mathbb{P} = \bigcup_{t \in T} \mathbb{P}_t,$$

where we denote by $\mathbb{P}_t$ the fibre of p_1 at t. $\mathbb{P}_t$ is an r-dimensional linear system of plane curves of degree d and geometric genus g.

Definition 2.6 If $r \geq 1$ we say that $\mathbb{P}$ is a scroll of plane curves of degree d and genus g.

Assume that $\mathbb{P}$ is such a scroll and that the natural moduli map

$$m : \mathbb{P} \to \mathcal{M}_g$$

is dominant. Then it turns out that $m/\mathbb{P}_t$ is not constant for a general t, see Lemma 2.9, and hence a general point of $\mathcal{M}_g$ belongs to a unirational variety $m(\mathbb{P}_t)$. This implies that $\mathcal{M}_g$ is uniruled. Not so differently: let T be unirational and m dominant. Then $\mathbb{P}_T$, hence $\mathcal{M}_g$, are unirational.

The classical attempts to prove the unirationality or the uniruledness of $\mathcal{M}_g$ are in this spirit. To summarize it, we raise the following:

Question 2.7 *Does there exist a scroll $\mathbb{P}$ of plane curves of degree d and genus g such that the moduli map $m : \mathbb{P} \to \mathcal{M}_g$ is dominant?*

We know nowadays that no such a $\mathbb{P}$ exists at least from $g \geq 22$. Indeed $\mathcal{M}_g$ is not uniruled for $g \geq 22$ because its Kodaira dimension is not $-\infty$. We also point out that the existence of $\mathbb{P}$ is a sufficient condition for the uniruledness. But it is not at all necessary, as we will see in Example 2.32. Let's start a sketch of the classical approaches of Severi and Beniamino Segre.

Let $t \in T$, we consider the base scheme B_t of $\mathbb{P}_t$ and denote its reduced scheme as $Z_t := B_{t,red}$. Up to replacing T by a non empty open subset, we will assume that T is smooth and that the family

$$Z := \{(x, t) \in \mathbf{P}^2 \times T \ / \ x \in Z_t\}$$

is a flat family of smooth 0-dimensional schemes of length b.

Lemma 2.8 *There exists a commutative diagram*

$$\begin{array}{ccc} T \times \mathbf{P}^2 & \xrightarrow{F} & T \times \mathbf{P}^2 \\ p\downarrow & & p\downarrow \\ T & \xrightarrow{id_T} & T \end{array}$$

such that F is birational and, for a general $t \in T$, the strict transform of $\mathbb{P}_t$ by $F/\{t\} \times \mathbf{P}^2$ is a linear system of curves with ordinary multiple points.

The proof follows in a standard way from Noether's theorem on reduction of the singularities of a plane curve to ordinary ones by a Cremona transformation, we omit it. We assume from now on that a general $\Gamma \in \mathbb{P}$ has ordinary singularities. For general $t \in T$ and $\Gamma \in \mathbb{P}_t$ we then have

$$g = \binom{d-1}{2} - \sum_{i=1\dots b} \binom{m_i}{2},$$

where m_i is the multiplicity of Γ at the base point $x_i \in Z_t$. Let

$$\sigma : S \to \mathbf{P}^2$$

be the blowing up of Z_t. Then the strict transform of Γ is a smooth, integral curve C of genus g on the smooth rational surface S. From the standard exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_C(C) \to 0$$

it follows $h^1(\mathcal{O}_S(C)) = h^1(\mathcal{O}_C(C))$, moreover we have

$$\dim |C| = h^0(\mathcal{O}_C(C)).$$

More in general let $C \subset S$, where S is any smooth surface not birational to $C \times \mathbf{P}^1$. Consider the moduli map $m : |C| \to \mathcal{M}_g$, then we have:

Lemma 2.9 *If* $\dim |C| \geq 1$ *and C is general then m is not constant.*

Proof Assume $g \geq 3$ and that C moves in a pencil $P \subset |C|$ whose general member has constant moduli. Since C is general we can assume $Aut\ C = 1$. Hence we can define the birational map $\phi : S \to C \times P$ such that $\phi(x) = (x, z)$, where $x \in C_z = C$ and C_z is the unique element of P passing through x. This implies that S is birational to $C \times \mathbf{P}^1$: a contradiction. The easy extension of this argument to the case $g = 2$ is left to the reader. □

We can finally start our search for a scroll $\mathbb{P}$, outlining the classical approach. Let $\mathbb{P}'_t \subset |C|$ be the strict transform of $\mathbb{P}_t$ by $\sigma : S \to \mathbf{P}^2$. Since a general $\Gamma \in \mathbb{P}_t$ has ordinary multiple points it follows that

$$C^2 = d^2 - m_1^2 - \cdots - m_b^2.$$

Since C is integral and $\dim |C| \geq 1$, it is elementary but crucial that

$$C^2 \geq 0.$$

From now on we denote by δ the number of singular points of a general $\Gamma \in \mathbb{P}$. Clearly we have $\delta \leq b$, hence it follows

$$d^2 - m_1^2 - \cdots - m_\delta^2 \geq 0,$$

where m_i is the multiplicity of the i-th singular point of Γ. The starting point of Severi is the case where Γ is a *nodal plane curve*, that is, the case

$$m_1 = \cdots = m_\delta = 2.$$

Analyzing this case, Severi obtains the following result:

Theorem 2.10 *$\mathcal{M}_g$ is unirational for $g \leq 10$.*

Severi's arguments rely on the irreducibility of $\mathcal{M}_g$ and on Brill-Noether theory, two established results for the usually accepted standards of that time. Further precision was however needed, as we will see. Let us discuss this matter, after a few reminders on Brill-Noether theory [ACGH, Chap. IV].

Definition 2.11 The Brill-Noether loci of a curve C are the loci

$$W^r_d(C) := \{L \in Pic^d(C) \; / \; h^0(L) \geq r + 1\}.$$

$W^r_d(C)$ has a natural structure of determinantal scheme. The tangent space at the element $L \in W^r_d(C) - W^{r+1}_d(C)$ is determined by the Petri map

$$\mu : H^0(L) \otimes H^0(\omega_C \otimes L^{-1}) \to H^0(\omega_C).$$

Indeed it is the orthogonal space $Im\ \mu^\perp \subset H^0(\omega_C)^*$. If μ is injective one computes that $dim\ Im\ \mu^\perp$ is the Brill-Noether number

$$\rho(g, r, d) := g - (r + 1)(r + g - d),$$

where g is the genus of C. The next theorems are nowadays well known.

Theorem 2.12 *Let C be a general curve of genus g, then:*

(1) *$W^r_d(C)$ is not empty iff $\rho(g, r, d) \geq 0$,*
(2) *let $\rho(g, r, d) \geq 0$ then $dim\ W^r_d(C) = \rho(g, r, d)$.*

Theorem 2.13 *Let C be a general curve and L general in $W^r_d(C)$, then:*

(1) *if $r \geq 3$ then the line bundle L is very ample,*
(2) *if $r = 2$ then $|L|$ defines a generically injective map in $\mathbf{P}^2$ whose image is a nodal curve.*

Let $\Gamma \subset \mathbf{P}^2$ be a plane curve of degree d and geometric genus g. Let $\nu : C \to \Gamma$ be its normalization map. Applying Theorem 2.12 (1) it follows:

Proposition 2.14 *If the curve C has general moduli then*

$$g \leq \frac{3}{2}d - 3.$$

To describe the geometry of $\mathcal{M}_g$, Severi considers the families

$$\mathcal{V}_{d,g} \subset |\mathcal{O}_{\mathbf{P}^2}(d)|$$

of all integral and nodal plane curves of degree d and genus g. Let

$$H_\delta$$

be the open set, in the Hilbert scheme of δ points of $\mathbf{P}^2$, whose elements are smooth. Then $\mathcal{V}_{d,g}$ is endowed with the morphism

$$h : \mathcal{V}_{d,g} \to H_\delta$$

sending Γ to $h(\Gamma) := Sing\ \Gamma$, where $\delta = \binom{d-1}{2} - g$. The families of curves $\mathcal{V}_{d,g}$ are well known as the *Severi varieties* of nodal plane curves.

Now assume that a scroll $\mathbb{P}$, of plane curves of degree d and genus g, dominates $\mathcal{M}_g$ via the moduli map. Then we have

$$\mathbb{P} \subseteq \overline{\mathcal{V}}_{d,g}$$

where $\overline{\mathcal{V}}_{d,g}$ denotes the closure of $\mathcal{V}_{d,g}$. Notice also that the fibres of the map $f : h/\mathbb{P} : \mathbb{P} \to H_\delta$ are the linear systems $\mathbb{P}_t$ already considered. Therefore we can replace T by $h(\mathbb{P})$ and $\mathbb{P}$ by $\overline{\mathcal{V}}_{d,g}$ and directly study the latter one.

However the study of $\mathcal{V}_{d,g}$ involves some delicate questions, which were left unsettled for long time after Severi. These are related to his unirationality result for $\mathcal{M}_g$, $g \leq 10$. A main question, now solved, concerns the irreducibility of $\mathcal{V}_{d,g}$, claimed in [S1] and finally proven by Harris in [H2]:

Theorem 2.15 *Let $\mathcal{V}_{d,g}$ be the Severi variety of integral nodal curves of degree d and genus $g = \binom{d-1}{2} - \delta$. Then $\mathcal{V}_{d,g}$ is integral, smooth and of codimension δ in $|\mathcal{O}_{\mathbf{P}^2}(d)|$.*

Once this theorem is granted, a first very natural issue is to consider the case where both the maps

$$h : \overline{\mathcal{V}}_{d,g} \to H_\delta \text{ and } m : \overline{\mathcal{V}}_{d,g} \to \mathcal{M}_g$$

are dominant. A first reason for doing this is that H_δ is rational. If h is dominant then h defines a $\mathbf{P}^r$-bundle structure over an open subset of H_δ. Then, since it is irreducible, $\mathcal{V}_{d,g}$ is birational to $H_\delta \times \mathbf{P}^r$. Hence $\mathcal{M}_g$ is unirational if m is dominant.

We know from Brill-Noether theory that in this case $g \leq \frac{3}{2}d - 3$. So we have to compare this inequality and the condition that h is dominant. For a general $\Gamma \in \mathcal{V}_{d,g}$, let $Z := Sing\ \Gamma$ and $\mathcal{I}_Z$ its ideal sheaf. We can use deformation theory for the family of nodal plane curves $\mathcal{V}_{d,g}$ as it is given in [Ser2, 4.7]. In particular we have that $h^0(\mathcal{I}_Z(d) - h^0(\mathcal{I}_Z^2(d))$ is the rank of the tangent map dh at Γ. Now assume that h is dominant. Then Z is general in H_δ and hence $h^0(\mathcal{I}_Z(d)) = h^0(\mathcal{O}_{\mathbf{P}^2}(d)) - \delta$. Furthermore dh_Γ is surjective and $h^0(\mathcal{I}_Z^2(d)) \geq 1$ because $\Gamma \in |\mathcal{I}_Z^2(d)|$. Hence we have

$$h^0(\mathcal{I}_Z^2(d)) = h^0(\mathcal{O}_{\mathbf{P}^2}(d)) - 3\delta = \binom{d+2}{2} - 3\delta \geq 1.$$

Assume h and m are both dominant. Since $\delta = \binom{d-1}{2} - g$ we deduce

$$g \leq \frac{3}{2}d - 3 \text{ and } \frac{d^2}{3} - 2d + 1 \leq g.$$

Then, relying on the previous theorems, one easily concludes that

Theorem 2.16 *m and h are dominant iff* $g \leq 10$, $d \leq 9$ *and* $g \leq \frac{3}{2}d - 3$.

This is the situation considered by Severi: moving Z in a non empty open set $U \subset H_\delta$, and fixing d, g in the previous range, one can finally construct a unirational variety

$$\mathbb{P} := \bigcup_{Z \in U} |\mathcal{I}_Z^2(d)|$$

dominating $\mathcal{M}_g$. Still a subtlety is missed: the construction implicitly relies on a positive answer to the following question. Let Z be general in H_δ and let $h^0(\mathcal{O}_{\mathbf{P}^2}(d)) - 3\delta \geq 1$, so that $|\mathcal{I}_Z^2(d)|$ is not empty:

Question 2.17 *Is a general* $\Gamma \in |\mathcal{I}_Z^2(d)|$ *integral and nodal of genus g?*

A simple counterexample actually exists. Let $\delta = 9$, $d = 6$ and Z general. Then $|\mathcal{I}_Z^2(6)|$ consists of a unique element Γ and $\Gamma = 2E$, where E is the unique plane cubic containing Z. Fortunately this is the unique exception, as is shown by

Arbarello and Sernesi in [AS]. This completes the description of the proof of the unirationality of $\mathcal{M}_g$, $g \leq 10$, via nodal plane curves.

For $g \geq 11$ one cannot go further with scrolls $\mathbb{P}$ in $\overline{\mathcal{V}}_{d,g}$ whose general element is a nodal curve. This remark is also proven by E. Sernesi (Unpublished note, 2010).

Theorem 2.18 *In $\overline{\mathcal{V}}_{d,g}$ there is no scroll $\mathbb{P}$ as above such that $m : \mathbb{P} \to \mathcal{M}_g$ is dominant and $g \geq 11$.*

Proof Assume that $\mathbb{P}$ exists. Let $\Gamma \in \mathbb{P}_t \subset \mathbb{P}$ be general and let $\sigma : S \to \mathbf{P}^2$ be the blowing up of Z_t. Since $\dim \mathbb{P}_t \geq 1$ the strict transform C of Γ is a curve of genus g such that $\dim |C| \geq 1$. Since Γ is nodal it follows that $C^2 \geq d^2 - 4\delta = -d^2 + 6d - 4 + 4g \geq 0$. Since C has general moduli we have also $g \leq \frac{3}{2}d - 3$. This implies that $d \leq 10$ and $g \leq 12$. The cases $g = 10, 11, 12$ can be excluded by an ad hoc analysis. □

Remark 2.19 Apparently, an intrinsic limit of the results we have outlined is the use of nodal plane curves. As remarked above $\overline{\mathcal{V}}_{d,g}$ is not ruled by linear spaces if it dominates $\mathcal{M}_g$ and $g \geq 11$. Equivalently let $C \subset S$ be a smooth, integral curve of genus $g \geq 11$ with general moduli in a rational surface S, then $\dim |C| = 0$. Of course this does not exclude the uniruledness of $\mathcal{V}_{d,g}$ for other reasons. This is for instance the case for $d = 10$ and $g = 11, 12$.

Leaving nodal plane curves, Beniamino Segre made a thorough attempt to construct scrolls $\mathbb{P} \subset |\mathcal{O}_{\mathbf{P}^2}(d)|$ such that

- *a general $\Gamma \in \mathbb{P}$ has genus g and ordinary singularities*,
- *Sing Γ is a set of points of $\mathbf{P}^2$ in general position*,
- *$m : \mathbb{P} \to \mathcal{M}_g$ is dominant.*

See [Se1]. This search gave negative answers:

Theorem 2.20 (B. Segre) *There is no scroll $\mathbb{P}$ for $g \geq 37$.*

A possible extension to $g \geq 11$ is also suggested by Segre. Let $\Gamma \in \mathbb{P}_t \subset \mathbb{P}$ and let $p_1 \dots p_s$ be the base points of $\mathbb{P}_t$, assumed to be general in $\mathbf{P}^2$. Consider the zero dimensional subscheme Z supported on them and locally defined at p_i by $\mathcal{I}_{p_i}^{m_i}$, where $\mathcal{I}_{p_i}$ is the ideal sheaf of p_i and m_i its multiplicity in Γ. We say that the linear system $|\mathcal{I}_Z(d)|$ is *regular* if it is not empty and $h^1(\mathcal{I}_Z(d)) = 0$. Segre shows that:

Theorem 2.21 *No scroll $\mathbb{P}$ exists for $g \geq 11$ if some $|\mathcal{I}_Z(d)|$ is regular.*

He says that the hypothesis regularity of $|\mathcal{I}_Z(d)|$ should follow from an unproved claim of intuitive evidence. This is probably the remote origin of Segre's conjecture, formulated much later in 1961, and the origin to many related conjectures:

Conjecture 2.22 If there exists a reduced curve $\Gamma \in |\mathcal{I}_Z(d)|$ then the linear system $|\mathcal{I}_Z(d)|$ is regular.

See [Se2] and [Ci1].

2.4 Rational Curves on $\overline{\mathcal{M}_g}$

The epilogue of the history we have described is well known to all algebraic geometers: Severi's conjecture was, somehow surprisingly, disproved. In 1982 Harris and Mumford proved that $\mathcal{M}_g$ is of general type as soon as g is odd and $g > 23$ [HM]. In 1984 Harris proved the same result in even genus [H1]. It was also proved that $\mathcal{M}_{23}$ has Kodaira dimension ≥ 0. The present updated picture, for every genus g, is as follows:

1. *$\mathcal{M}_g$ is rational for $g \leq 6$.*
2. *$\mathcal{M}_g$ is unirational for $g \leq 14$.*
3. *$\mathcal{M}_{15}$ is rationally connected, $\mathcal{M}_{16}$ is uniruled.*
4. *kod$(\mathcal{M}_g)$ is unknown for $g = 17, \ldots, 21$.*
5. *$\mathcal{M}_g$ is of general type for $g = 22$ and $g \geq 24$.*
6. *$\mathcal{M}_{23}$ has Kodaira dimension ≥ 2.*

See [HMo1] or [Ve] for more details on several different contributions. It is time to quit the world of curves on rational surfaces and to discuss more in general, as far as the value of g makes it possible, the next

Question 2.23 *When does a general point of $\mathcal{M}_g$ lie in a rational curve?*

This is of course equivalent to ask wether $\mathcal{M}_g$ is uniruled. It is very easy to realize that this is also equivalent to discuss the next

Theme I *When does a general curve C embed in a smooth integral surface S so that the moduli map $m : |\mathcal{O}_S(C)| \to \mathcal{M}_g$ is not constant?*

The discussion made in the previous section also concerns the moduli of pairs (C, L) such that $L \in W^2_d(C)$, where d and the genus g of C are fixed. The coarse moduli space of pairs (C, L) such that $L \in W^r_d(C)$ will be

$$\mathcal{W}^r_{d,g}$$

and we will say that these spaces are the universal Brill-Noether loci. Passing from $\mathcal{M}_g$ to $\mathcal{W}^r_{d,g}$ a second theme is then natural here:

Theme II *Discuss the uniruledness problem for $\mathcal{W}^r_{d,g}$.*

Notice that Severi's method to prove the unirationality of $\mathcal{M}_g$, $g \leq 10$, immediately implies the unirationality of $\mathcal{W}^2_{d,g}$ in the same range, that is for $d \leq 9$ and $g \leq \frac{3}{2}d - 3$ On the other hand the minimal degree of a map $f : C \to \mathbf{P}^1$ is $k = [\frac{g+3}{2}]$ for a general curve C of genus g. Of special interest for these notes will be the case of $\mathcal{W}^1_{k,g}$.

We close this section by some concrete tests and examples on these themes, discussing curves with general moduli moving on non rational surfaces. This is going to involve K3 surfaces but also elliptic surfaces and some canonical surfaces in $\mathbf{P}^{g-1}$. As a byproduct in genus 10, we retrieve by a different method a counterexample to the slope conjecture given in [FP]. We use pencils of curves on some elliptic surfaces studied in [FV5].

Preliminarily we recall some basic facts on the compactification of $\mathcal{M}_g$ by the moduli of stable curves $\overline{\mathcal{M}}_g$, see e.g. [ACG, Chaps. 12 and 13], [HMo, F].

$\overline{\mathcal{M}}_g$ is an integral projective variety with canonical singularities. Denoting by $[D]$ the class of the divisor D, the Picard group of $\overline{\mathcal{M}}_g$ is generated over $\mathbb{Q}$ by the following divisorial classes:

- $\lambda = [det\ \Lambda]$, Λ being the Hodge bundle with fibre $H^0(\omega_C)$ at $[C]$.
- $\delta_0 = [\Delta_0]$. For a general $[C] \in \Delta_0$ C is 1-nodal and integral.
- $\delta_i = [\Delta_i]$. For a general $[C] \in \Delta_0$ C is 1-nodal and $C = C_i \cup C_{g-i}$ where C_i and C_{g-i} are smooth, integral of genus i and $g - i$ respectively.

For the canonical class we have

$$[K_{\overline{\mathcal{M}}_g}] = 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \cdots - 2\delta_{[\frac{g}{2}]} := k_g.$$

Putting as usual $\delta := \sum_{i=0\ldots[\frac{g}{2}]} \delta_i$ we have for the canonical class

$$k_g = 13\lambda - 2\delta - \delta_1.$$

More in general we can consider divisors D such that

$$[D] = a\lambda - b\delta - \sum c_i\delta_i,$$

with $a, b > 0$ and $c_i \geq 0$, and define the slope of any divisor as follows [HMo].

Definition 2.24 Let D be any divisor: if $[D] = a\lambda - b\delta - \sum c_i\delta_i$ as above, then the slope of D is the number $s(D) := \frac{a}{b}$. If not we put $s(D) := \infty$.

Definition 2.25 The slope of $\overline{\mathcal{M}}_g$ is

$$s_g := \inf\{s(E),\ [E] \in NS(\overline{\mathcal{M}}_g) \otimes \mathbb{R}\ E \text{ effective divisor}\}.$$

As we can see one has $s(K_{\overline{\mathcal{M}}_g}) = \frac{13}{2}$ for the slope of $K_{\overline{\mathcal{M}}_g}$.

Can we deduce something about the effectiveness of $K_{\overline{\mathcal{M}}_g}$? It is natural to answer quoting the theorem of Boucksom et al. [BDPP]. The theorem says that an integral projective variety with canonical singularities is non uniruled iff its canonical divisor is limit of effective divisors (pseudoeffective). A uniruledness criterion for $\overline{\mathcal{M}}_g$ is therefore:

$$s_g > s(k_g) = \frac{13}{2}.$$

How to apply this criterion? A typical way involves the study of the irreducible flat families of curves in $\overline{\mathcal{M}}_g$ having the property that a general point of $\overline{\mathcal{M}}_g$ belongs to some members of the family. We say that a family like this is a *family of curves sweeping* $\overline{\mathcal{M}}_g$ *or a sweeping family*. Let

$$m : B \to \overline{\mathcal{M}}_g$$

be a non constant morphism where B is an integral curve. Up to base change and stable reduction we can assume that B is smooth of genus p and endowed with a flat family $f : S \to B$ of stable curves such that

$$m(b) = \text{the moduli point of } f^*(b).$$

Consider any effective divisor $E \subset \overline{\mathcal{M}}_g$ and a non constant morphism

$$m : B \to \overline{\mathcal{M}}_g$$

such that $m(B)$ is a sweeping curve. Then we can assume $m(B) \not\subset E$, so that $\deg m^*E \geq 0$. The next lemma is well known and summarizes some properties of the numerical characters of a fibration of a surface onto a curve.

Lemma 2.26

- *deg* $m^*E \geq 0$.
- $s(E) \geq \frac{m^*\delta}{m^*\lambda}$.
- deg $m^*\delta = c_2(S) + (2-2p)(2g-2)$
- $\deg m^*\lambda = \deg\ \omega_{S/B}$.

The next theorem is an elementary but crucial corollary:

Theorem 2.27 *Let* $m : B \to \overline{\mathcal{M}}_g$ *be a morphism such that* $m(B)$ *moves in a sweeping family of curves. Then* $\frac{\deg m^*\delta}{\deg m^*\lambda} > \frac{13}{2}$ *implies that* $\overline{\mathcal{M}}_g$ *is uniruled.*

In the next examples we only deal with the case $B = \mathbf{P}^1$ that is with rational curves $m(B)$ in the moduli space $\overline{\mathcal{M}}_g$.

Here is a recipe to construct examples in this case: in a regular surface S construct a smooth integral curve $C \subset S$, having genus g and such that $h^0(\mathcal{O}_C(C)) \geq 1$. Then the standard exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_C(C) \to 0$$

implies $\dim |C| \geq 1$. Take, possibly, a Lefschetz pencil $B \subset |C|$. Then the map $m : B \to \overline{\mathcal{M}}_g$ provides a rational curve $m(B)$ in $\overline{\mathcal{M}}_g$. Let $\sigma : S' \to S$ be the blow up of the base locus of B, it is standard to compute that

$$deg\ m^*\delta = c_2(S') + 4g - 4\ ,\ deg\ m^*\lambda = \chi(\mathcal{O}_S) + g - 1.$$

*Example 2.28 (*K3 SURFACES*)* This is a very important case, to be discussed later as well. We will consider pairs (S, L) where S is a K3 surface and L a big and nef line bundle such that $c_1(L)^2 = 2g - 2$. If $g \geq 3$ and (S, L) is general L is very ample. Then L defines an embedding $S \subset \mathbf{P}^g$ with canonical curves of genus g as hyperplane sections. Let B be a general pencil of hyperplane sections of S, applying the previous formulae to B we obtain:

$$\frac{deg\ m^*\delta}{deg\ m^*\lambda} = 6 + \frac{12}{g+1}.$$

This equality fits very well with the first results of Mumford-Harris on $\mathcal{M}_g$, which say that $\mathcal{M}_g$ is of general type for $g > 23$. Indeed we have

$$6 + \frac{12}{g+1} > \frac{13}{2}$$

exactly for $g > 23$. Though $m(B)$ does not move in a sweeping family, unless $g \leq 11$ and $g \neq 10$, these numbers lend some plausibility to the following

Conjecture 2.29 The slope of $\overline{\mathcal{M}}_g$ is $s_g = 6 + \frac{12}{g+1}$

See [HMo1] for further precision on this statement, which is known as *slope conjecture*. It was however disproved by Farkas and Popa in [FP] for infinite values of g, starting from the very interesting case $g = 10$.

Example 2.30 (Elliptic Surfaces) Here we give another proof of the counterexample to the slope conjecture for $g = 10$. We use elliptic surfaces with $p_g = 2$ and $q = 0$, endowed with a very ample linear system of curves C of genus 10 and such that $C^2 = 12$.

Actually we describe and apply the results in [FV5] on curves of even genus g on some classes of elliptic surfaces. Let $g = 2k$, in [FV5] an integral family $\mathcal{S}_k$ of projective surfaces

$$S \subset \mathbf{P}^k$$

is constructed. They are elliptic surfaces with $p_g = 0$ and $q = 0$. We assume $k \geq 5$ to have that a general S is smooth. Let C be a smooth hyperplane section of S. The notable properties of this construction are as follows:

- *C has genus $g = 2k$,*
- *$|K_S|$ is a pencil of elliptic curves of degree $k + 1$,*
- *$M := \mathcal{O}_C(K_S) \in W^1_{k+1}(C)$,*
- *$\mathcal{O}_C(1) \cong \omega_C(-M)$.*

Notice that $W^1_{k+1}(C)$ is finite if C is general and no pencil of divisors of degree $\leq k$ exists on C. Furthermore let S be general in $\mathcal{S}_k$, then:

- S is projectively normal,
- C has general moduli for $g \leq 12$.

Coming to genus 10, the family of curves of this genus which can be embedded in a K3 surface play a special role for $\overline{\mathcal{M}}_{10}$. As is well known the locus of these curves is an irreducible divisor $D_{K3} \subset \overline{\mathcal{M}}_{10}$. Let us sketch a proof that its slope is 7 so contradicting the slope conjecture $s_{10} = 6 + \frac{12}{11}$.

Putting $k = 5$ fix a general pencil $B \subset |C|$ on a general elliptic surface $S \subset \mathbf{P}^5$ as above. One can show that B is a Lefschetz pencil, then every $C \in B$ is integral and nodal with at most one node. Since S is projectively normal, the multiplication map

$$\mu : Sym^2 H^0(\mathcal{O}_C(1)) \to H^0(\mathcal{O}_C(2))$$

is of maximal rank, in our case it is an isomorphism, for every $C \in B$. Equivalently C is not embedded in a K3 surface of degree 6 of the hyperplane $< C >= \mathbf{P}^4$. On the other hand $\mathcal{O}_C(1)$ belongs to $W^4_{12}(C)$. Then the results of Voisin in [V], cfr. 3.2 and 4.13 (b), imply that the multiplication map $\mu_A : Sym^2 H^0(A) \to H^0(A^{\otimes 2})$ is isomorphic for any $M \in W^4_{12}(C)$ and hence that no $C \in B$ embeds in a K3 surface. This implies that

Lemma 2.31 $m^* D_{K3} = 0$.

One can also show that $[D_{K3}] = a\lambda - b\delta - \sum c_i \delta_i$, where $i > 0, a > 0$ and $b, c_i \geq 0$. Since B is a Lefschetz pencil, it follows that $m^*\delta_i = 0$ for $i > 0$. Moreover, by the previous formulae, we compute

$$\deg\ m^*\delta = 84\ ,\ \deg\ m^*\lambda = \chi(\mathcal{O}_S) + (g-1) = 12.$$

Hence, by Lemma 2.31, it follows $\deg m^* D_{K3} = 12a - 84b = 0$ so that $s(D_{K3}) = \frac{a}{b} = 7$. This contradicts the slope conjecture.

Example 2.32 (Canonical Surfaces) Let $d = 10$ and $g = 11, 12$ we give a hint to prove that $\mathcal{W}^2_{10,g}$ is uniruled.

Assume that C of genus g has general moduli and that $L \in W^2_{10}(C)$. Consider the embedding $C \subset \mathbf{P}^2 \times \mathbf{P}^n$ defined by the product of the maps associated to L and $\omega_C(-L)$, $n = g - 9$.

Let $g = 11$: since $n = 2$, C is embedded in $S \subset \mathbf{P}^2 \times \mathbf{P}^2$, S is a smooth complete intersection S of two hypersurfaces of bidegree $(2, 2)$. Then S is a regular canonical surface in the Segre embedding of $\mathbf{P}^2 \times \mathbf{P}^2$, and $\mathcal{O}_C(1, 1)$ is ω_C. Hence $\mathcal{O}_C(C)$ is trivial by adjunction formula and $\dim |C| = 1$.

Let $D \in |C|$ then D is endowed with $L_D := \mathcal{O}_D(1, 0) \in W^2_{10}(D)$. The image of the corresponding moduli map $m : |C| \to \mathcal{W}^2_{10,1}$ is a rational curve passing through a general point of $\mathcal{W}^2_{10,11}$.

Let $g = 12$: since $n = 3$, C is embedded in $S \subset \mathbf{P}^2 \times \mathbf{P}^3$, S is a complete intersection of three hypersurfaces of bidegrees (1,2), (1,2), (2,1). Then the argument of the proof is the same.

3 $\mathcal{R}_g$ in Genus at Most 6

3.1 Prym Curves and Their Moduli

A smooth *Prym curve* of genus g is a pair (C, η) such that C is a smooth, connected curve of genus g and η is a non zero 2-torsion element of $Pic^0(C)$.

It is useful to recall the following characterizations of a pair (C, η) and to fix consequently the notation. Keeping C fixed let us consider the sets:

$T :=$ {non trivial line bundles $\eta \in Pic(C) \mid \eta^{\otimes 2} \cong \mathcal{O}_C,$ },
$E :=$ {non split étale double coverings $\pi : \tilde{C} \to C$ }.
$I :=$ {fixed point free involutions i on a connected curve $\tilde{C} | C \cong \tilde{C}/ < i >$ }.

Then the next property is standard and well known.

Theorem 3.1 *The sets T, I, E are naturally bijective.*

Indeed let $\eta \in T$, then there exists a unique isomorphism $\beta : \mathcal{O}_C \to \eta^{\otimes 2}$ modulo $\mathbf{C}^*$. From it one uniquely defines the projective curve

$$\tilde{C} := \{(1 : v) \in \mathbb{P}(\mathcal{O}_C \oplus \eta)_x \ / \ v \otimes v = \beta_x(1), \ x \in C\} \subset \mathbf{P}(\mathcal{O}_C \oplus \eta).$$

$\tilde{C}$ is a smooth, integral curve of genus $2g - 1$. It is endowed with:

- the fixed point free involution $i : \tilde{C} \to \tilde{C}$ such that $i(u : v) = (u : -v)$,
- the étale 2:1 cover $\pi : \tilde{C} \to C$ induced by the projection $\mathbf{P}(\mathcal{O}_C \oplus \eta) \to C$.

The assignments $\eta \to (\tilde{C}, i)$ and $\eta \to \pi$ define the required bijections $T \leftrightarrow I$ and $T \leftrightarrow E$. π is the quotient map of i. We omit more details.

Let (C, η) be a Prym curve, throughout all this exposition we will keep the notation $\pi : \tilde{C} \to C$ and $i : \tilde{C} \to \tilde{C}$ for the maps constructed as above from (C, η). We can now begin with the following:

Definition 3.2 The Prym moduli space of genus g is the moduli space of smooth Prym curves of genus g. It will be denoted as $\mathcal{R}_g$.

As it is well known $\mathcal{R}_g$ is an integral quasi projective variety of dimension $3g-3$. Actually the forgetful map $f : \mathcal{R}_g \to \mathcal{M}_g$ is a finite morphism of degree $2^{2g} - 1$, with fibre $T \setminus \{\mathcal{O}_C\}$ over the moduli point of C. The notion of smooth Prym curve needs to be extended, in order to construct suitable compactifications of $\mathcal{R}_g$. The moduli space of quasi stable Prym curves provides a useful compactification, that we now describe. See [FL].

A component E of a semistable curve C is said to be *exceptional* if E is a copy of $\mathbf{P}^1$ and $E \cap \overline{C - E}$ is a set of two points.

Definition 3.3 A semistable curve C is quasi stable if its exceptional components are two by two disjoint.

We recall that a vector bundle on C has degree d if d is the sum of the degrees of its restrictions to the irreducible components of C.

Definition 3.4 (C, η, β) is a quasi stable Prym curve of genus g if:

- C is quasi stable of genus g and η is a line bundle on C of degree 0,
- $\beta : \mathcal{O}_C \to \eta^{\otimes 2}$ is a morphism of sheaves,
- let D be the union of the exceptional components of C, then β is an isomorphism on $C - D$.

For short we will say that (C, η, β) is a Prym curve of genus g. The moduli space of these triples is denoted as $\overline{\mathcal{R}}_g$. This is a projective normal variety with finite quotient singularities and $\mathcal{R}_g$ is a dense open subset of it.

$\overline{\mathcal{R}}_g$ has good properties in order to study the Kodaira dimension, as it is done by Farkas and Ludwig in [FL]. They show that any resolution of singularities $X \to \overline{\mathcal{R}}_g$ induces an isomorphism between the spaces of pluricanonical forms $H^0(\omega_X^{\otimes m})$ and $H^0(\omega_{\overline{\mathcal{R}}_g}^{\otimes m})$, $m \geq 1$. Then the key point is the study of the forgetful map

$$f : \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g$$

sending $[C, \eta, \beta]$ to $[st(C)]$, where $st(C)$ denotes the stable model of C. It turns out that f is finite and ramifies, with simple ramification, precisely on

$$D_0^{ram} := \{[C, \eta, \beta] \ / \ C \ contains \ exceptional \ components\},$$

which is an irreducible divisor. In particular f offers a useful description of the canonical class of $\overline{\mathcal{R}}_g$ in terms of D_0^{ram} and of the pull back of the canonical class of $\overline{\mathcal{M}}_g$. An outcome of this approach is the following:

Theorem 3.5 *$\overline{\mathcal{R}}_g$ is of general type for $g \geq 14$ and $g \neq 15$. The Kodaira dimension of $\overline{\mathcal{R}}_{15}$ is bigger or equal than 1.*

Though needed later, let us introduce now the picture of the boundary of $\overline{\mathcal{R}}_g$. Its irreducible components are divisors which are obtained from the boundary divisors $\Delta_i, \ i = 0 \dots [\frac{g}{2}]$, of $\overline{\mathcal{M}}_g$ as follows. Let $\overline{C} := st(C)$, consider the standard exact sequence

$$0 \to \mathbf{C}^{*k} \to Pic^0 \ \overline{C} \stackrel{\nu^*}{\to} Pic^0 \ N \to 0$$

where $\nu : N \to \overline{C}$ is the normalization map and $k = |Sing \ \overline{C}|$. We have:

$$f^*(\Delta_0) = D_0' + D_0'' + 2D_0^{ram}$$

so that D_0' and D_0'' are the following irreducible divisors

- $D_0' := \{[C, \eta, \beta] \in \overline{\mathcal{R}}_g \ / \ [\overline{C}] \in \Delta_0 \ , \ \nu^*\eta \ is \ non \ trivial\}$
- $D_0'' := \{[C, \eta, \beta] \in \overline{\mathcal{R}}_g \ / \ [\overline{C}] \in \Delta_0 \ , \ \nu^*\eta \ is \ trivial\}$

Remark 3.6 Let $p := [C, \eta, \beta] \in f^*(\Delta_0)$ be a general point then $\overline{C}$ is integral. Let $\pi : \tilde{C} \to C$ be the morphism defined by η. If $p \in D_0'$ then $C = \overline{C}$ and π is étale. If $p \in D_0''$ then $C = \overline{C}$ and π is a Wirtinger covering, cfr. [FL]. If $p \in D_0^{ram}$ then π is obtained from a 2 : 1 cover of $\overline{C}$ branched on the tangent directions of the nodes.

Moreover we have

$$f^*(\Delta_i) = D_i + D_{g-i} + D_{i:g-i}$$

for $i = 1 \leq i \leq [\frac{g}{2}]$. Here a general point $p = [C, \eta, \beta] \in f^*(\Delta_i)$ is such that

$$C = C_i \cup C_{g-i},$$

where C_i and C_{g-i} are smooth, integral curve of genus i and $g - i$. By definition $p \in D_{i:g-i}$ if η is non trivial on both C_i and C_{g-i}. Moreover let $x = i, g - i$ then $p \in D_x$ if η is non trivial on C_x and trivial on C_{g-x}. As usual we will denote the classes of the previous divisors as follows:

$$[\Delta_0'] := \delta_0' \ , \ [\Delta_0''] := \delta_0'' \ , \ [\Delta_0^{ram}] := \delta_0^{ram} \ , \ [D_i] = \delta_i \ , \ [D_{i:g-i}] := \delta_{i:g-i}.$$

Considering f it turns out that the canonical class of $\overline{\mathcal{R}}_g$ is

$$K_{\overline{\mathcal{R}}_g} \equiv 13\lambda - 2(\delta_0' + \delta_0'') - 3\delta_0^{ram} - 2 \sum_{i=1\ldots\frac{g}{2}} (\delta_i + \delta_{g-i} + \delta_{i:g-i}) - (\delta_1 + \delta_{g-1} + \delta_{1:g-1}).$$

3.2 Unirational Approaches to $\mathcal{R}_g$

Rationality problems are connected to Prym curves, their moduli and related topics by very important links and this fact is also well visible in the historical evolution of the two fields. It is the moment for opening some perspectives about.

The most relevant link between rationality problems and Prym curves appears to be the notion of Prym variety $P(C, \eta)$ associated to a smooth Prym curve (C, η) of genus g. $P(C, \eta)$ is a $g - 1$-dimensional principally polarized abelian variety we more precisely define later.

Prym varieties appear as intermediate Jacobians of several unirational threefolds. Notably we have among them any smooth cubic threefold X. They play a crucial role to prove that X is not rational, hence to produce counterexamples to Lüroth problem. This was a strong motivation to modern studies on Prym varieties and Prym curves.

Secondly the assignment $(C, \eta) \longrightarrow P(C, \eta)$ defines the *Prym map*

$$P_g : \mathcal{R}_g \to \mathcal{A}_{g-1}.$$

The Prym map is a fundamental tool, essentially the unique one, to prove the unirationality of $\mathcal{A}_{g-1}$ for $g \leq 6$. In the same range it is also useful for understanding more of the birational geometry of $\mathcal{A}_{g-1}$. For instance one can study the slope of a suitable compactification via the parametrization offered by the Prym map. This is possible in low genus thanks to a long history of results on the unirationality of $\mathcal{R}_g$ to be outlined here.

We highlight some old and some new approaches to the problem. These are often induced by analogous questions, or themes, we considered for $\mathcal{M}_g$ in the previous section. Let us recall the following ones:

Theme I *When does a Severi variety $\mathcal{V}_{d,g}$ dominate $\mathcal{M}_g$ and contain an open subset covered by rational curves with non constant moduli?*

Theme II *When does a curve C with general moduli embed in a smooth surface S so that the moduli map $m : |\mathcal{O}_S(C)| \to \mathcal{M}_g$ is not constant?*

We introduce some versions for Prym curves of these themes.

Let $p : P \to \mathbf{P}^2$ be a $\mathbf{P}^2$-bundle endowed with a very ample tautological bundle $\mathcal{O}_P(1)$. Then a general $Q \in |\mathcal{O}_P(2)|$ is a smooth conic bundle

$$p/Q : \ Q \to \mathbf{P}^2.$$

Let $\Gamma \subset \mathbf{P}^2$ be the discriminant curve of Q. It is well known that Γ is smooth for a general smooth Q and endowed with a non split étale double cover $\pi_\Gamma : \tilde{\Gamma} \to \Gamma$. In particular π_Γ is defined by a non trivial $\eta_\Gamma \in Pic^0(\Gamma)$ such that $\eta_\Gamma^{\otimes 2} \cong \mathcal{O}_\Gamma$. Hence p/Q defines the Prym curve (Γ, η_Γ). The same happens choosing Q general in the Severi variety of nodal conic bundles

$$\mathcal{V}_{P,\delta} := \{Q \in |\mathcal{O}_P(2)| \ \mid \textit{Sing } Q \textit{ consists of } \delta \textit{ ordinary double points}\}.$$

In this case Γ is integral with δ nodes. Let d be the degree of Γ and C its normalization. Then p/Q induces a non split étale double cover $\pi : \tilde{C} \to C$, hence a smooth Prym curve (C, η) of genus $g = \frac{1}{2}(d-1)(d-2) - \delta$. The assignment $Q \to (C, \eta)$ defines the moduli map

$$m : \mathcal{V}_{P,\delta} \to \mathcal{R}_g.$$

Theme I [Nodal conic bundles] *When is m dominant and does a general curve $\Gamma \in \mathcal{V}_{P,\delta}$ move in a rational family with non constant moduli?*

Let (C, η) be a Prym curve of genus g with general moduli, going back to surfaces the second theme is:

Theme II [Surfaces] *Does it exist an embedding $C \subset S$ and $E \in Pic\, S$ so that S is a smooth surface, $\eta \cong \mathcal{O}_S(E)$ and $m : | C | \to \mathcal{R}_g$ is not constant?*

What about unirationality and rationality of $\mathcal{R}_g$ in low genus? Answering this question we can revisit, by the way, some nice classical constructions.

The unirationality of $\mathcal{R}_g$ is known for $g \leq 7$. In genus 7 this is a recent result due to Farkas and the author [FV6].

In genus 6 the unirationality was independently proven by Mori and Mukai in [MM], by Donagi in [Do1] and in [Ve2]. Donagi's proof uses nets of quadrics of $\mathbf{P}^6$ whose discriminant curve splits in the union of a line and a 4-nodal sextic. The other two proofs rely on Enriques surfaces.

The case of genus 5 was more recently proved, see [ILS] and [Ve3]. The work of Clemens, on rational parametrizations of $\mathcal{A}_p$, $p \leq 4$, via nodal quartic double solids, is strictly related to these results [Cl].

A very recent quick proof of the unirationality of $\mathcal{R}_g$, $g \leq 6$, is given in [FV4] and presented here, see Sect. 3.4. It uses linear systems of nodal conic bundles in the spirit of Theme I.

3.3 Rationality Constructions

The rationality of $\mathcal{R}_g$ is known for $g \leq 4$. For $g = 2, 3$ it follows from the results of various authors: Catanese [Ca], Katsylo [K2] and Dolgachev [D1], starting from the first published result due to Katsylo.

For $\mathcal{R}_4$ the rationality result is due to Catanese. This is perfectly in the spirit of theme II and we revisit it. For $\mathcal{R}_4$ we are back, as in the case of $\mathcal{M}_6$, to a linear system of nodal plane sextics. Let $(x_1 : x_2 : x_3)$ be projective coordinates on $\mathbf{P}^2$ and let $u = x_1 + x_2 + x_3$. The linear system we want to consider can be written very explicitly as follows:

$$(x_1x_2x_3u)q + b_1(x_2x_3u)^2 + b_2(x_1x_3u)^2 + b_3(x_1x_2u)^2 + b_4(x_1x_2x_3)^2 = 0,$$

where $q := \sum_{1 \leq i \leq j \leq 3} a_{ij}x_ix_j$. Clearly such a linear system is 9-dimensional. We denote it as $\mathbf{P}^9$ and the coefficients $(a : b) := (a_{ij} : b_1 : \cdots : b_4)$ are projective coordinates on $\mathbf{P}^9$. We point out that $\mathbf{P}^9$ is precisely the linear system of sextics which are singular along the nodes of the *quadrilateral*

$$T = \{x_1x_2x_3u = 0\}.$$

It is easy to prove that a general $\Gamma \in \mathbf{P}^9$ is an integral nodal curve. In particular let $\nu : C \to \Gamma$ be the normalization map, then C has genus 4. Now we consider the effective divisors $h \in |\nu^*\mathcal{O}_\Gamma(1)|$ and $n = \nu^* Sing\ \Gamma$. Note that n is just the sum of the points over the six nodes of Γ.

Lemma 3.7 *Let* $\eta := \mathcal{O}_C(n - 2h)$ *then:*

- η *is non trivial*
- $\eta^{\otimes 2} \cong \mathcal{O}_C$.
- $\mathcal{O}_C(h) \cong \omega_C \otimes \eta$.

Proof One has $2n = \nu^*T \sim 4h$ so that $\eta^{\otimes 2} \cong \mathcal{O}_C(4h - 2n) = \mathcal{O}_C$. Assume η itself is trivial, then $2h \sim n$ and hence $3h - n \sim h$. Since $3h - n$ is the canonical class K_C it follows that Γ is the image of a map defined by a net $N \subset |K_C|$. This implies that a line section of Γ pulls back by ν to a canonical divisor and, by adjunction theory for plane curves, that the six nodes of Γ are on a conic. This is a contradiction because $Sing\ \Gamma = Sing\ T$. Finally $K_C \sim 3h - n$ and $\eta \cong \mathcal{O}_C(n - 2h) \Rightarrow \mathcal{O}_C(h) \cong \omega_C \otimes \eta$.□

It follows that (C, η) is a Prym curve and this defines a moduli map

$$m : \mathbf{P}^9 \to \mathcal{R}_4$$

as usual. Conversely, starting from (C, η), we can reconstruct Γ modulo projective automorphisms of $\mathbf{P}^2$. Indeed $\dim |\omega_C \otimes \eta| = 2$ and the image of the map defined by $\omega_C \otimes \eta$ is projectively equivalent to Γ. This implies that Γ is uniquely reconstructed from (C, η), modulo the group of projective automorphisms which are leaving invariant the linear system $\mathbf{P}^9$. This is precisely the stabilizer of the quadrilateral T, hence it is the symmetric group $\mathfrak{s}_4$ and m has degree 24. Since $\mathbf{P}^9$ and $\mathcal{R}_4$ are integral of the same dimension, it follows that m is dominant. Therefore we can conclude that

$$\mathbf{P}^9/\mathfrak{s}_4 \cong \mathcal{R}_4.$$

Analyzing the action of $\mathfrak{s}_4$ it follows that the quotient $\mathbf{P}^9/\mathfrak{s}_4$ is rational [Ca]. This shows that

Theorem 3.8 *The Prym moduli space $\mathcal{R}_4$ is rational.*

Some special Del Pezzo surfaces arise as an interesting complement of the geometry of $\mathcal{R}_4$. Let $\mathcal{I}_{Sing\ T}$ be the ideal sheaf of the six singular points of T. Then the linear system of plane cubics $|\mathcal{I}_{Sing\ T}(3)|$ defines a generically injective rational map $f : \mathbf{P}^2 \to \mathbf{P}^3$ whose image $\overline{S} := f(\mathbf{P}^2)$ is a 4-nodal cubic surface. It is worth mentioning some more geometry related to the Prym curve (C, η) and to the sextic Γ. We have a commutative diagram

$$\begin{array}{ccc} \tilde{S}' & \xrightarrow{f'} & \tilde{S} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \overline{\pi}} \\ \mathbf{P}^2 & \xrightarrow{f} & \overline{S} \end{array}$$

which is described as follows:

- f contracts the 4 lines of T to $Sing\ \overline{S}$ and blows up the 6 points of $Sing\ T$ to 6 lines in $\overline{S}$. These are the edges of the tetrahedron $Sing\ \overline{S}$.
- $\pi : \tilde{S}' \to \mathbf{P}^2$ is the finite 2:1 cover branched on T and $\tilde{S}'$ is a singular Del Pezzo surface of degree 2 with six nodes, $Sing\ \tilde{S}' = \pi^{-1}(Sing\ T)$.

- After blowing up $Sing\ \tilde{S}'$, the strict transforms of the irreducible components of $\pi^{-1}(T)$ become -1 lines.
- f' is the contraction of these lines and $\tilde{S}$ is a smooth Del Pezzo surface of degree 6. We leave the completion of many details to the reader.

Also, we just mention that:

- the strict transform of Γ by f is the canonical model $\overline{C}$ of C.
- $\tilde{C} := \overline{\pi}^{-1}(\overline{C})$ is the étale double cover of C defined by η.
- $\tilde{\Gamma} := \pi^{-1}(\Gamma)$ is birational to $\tilde{C}$.

3.4 Nodal Conic Bundles for $g \leq 6$

Following the approach suggested in Theme I we construct here rational families of nodal conic bundles dominating $\mathcal{R}_g$ for $g \leq 6$. This will offer a unique and quick method for proving all unirationality results in this range.

We fix a $\mathbb{P}^2$-bundle $p : \mathbb{P} \to S$ over a smooth rational surface S admitting a very ample tautological line bundle $\mathcal{O}_{\mathbb{P}}(1)$. As already remarked a general $Q \in |\mathcal{O}_{\mathbb{P}}(2)|$ is an integral threefold such that the projection

$$p/Q : Q \to S$$

is flat. This means in our situation that every fibre $Q_x := (p/Q)^*x$ is a conic in the plane $\mathbb{P}_x := p^*x$. We will assume that either Q is smooth or that $Sing\ Q$ consist of finitely many ordinary double points. Then it follows that the branch locus of p/Q is either empty or the curve

$$\Gamma := \{x \in S \ / \ rk\ Q_x \leq 2\}.$$

We assume that Γ is not empty and say that Γ is the *discriminant curve* of p/Q. The next lemma follows from [Be1] I.

Lemma 3.9 *Let $x \in S - p(Sing\ Q)$, then $x \in Sing\ \Gamma$ iff Q_x has rank one.*

As it is well known Γ is endowed with a finite double cover

$$\pi_\Gamma : \tilde{\Gamma} \to \Gamma,$$

where $\tilde{\Gamma}$ parametrizes the lines which are components of Q_x, $x \in \Gamma$. Let us fix from now on the next assumption, which is generically satisfied by a general $Q \in |\mathcal{O}_{\mathbb{P}}(2)|$ in all the cases to be considered:

- *Q_x has rank two at each point $x \in p(Sing\ Q)$.*

It is not difficult to see that $p(Sing\ Q) \subseteq Sing\ \Gamma$ and that Γ is nodal. Moreover, under the previous assumptions the next lemma follows.

Lemma 3.10 *Let $Sing\ \Gamma = p(Sing\ Q)$, then the morphism*

$$\pi_\Gamma : \tilde{\Gamma} \to \Gamma$$

is an étale double covering.

Let $Sing\ Q = Sing\ \Gamma$, then to give π_Γ is equivalent to give a line bundle η_Γ on Γ whose square is $\mathcal{O}_\Gamma$. It is well known that $\pi_{\Gamma *}\mathcal{O}_{\tilde{\Gamma}} \cong \mathcal{O}_\Gamma \oplus \eta_\Gamma$ and, moreover, π_Γ is uniquely reconstructed from η_Γ. In particular η_Γ is an element of $Pic^0_2\ \Gamma$. Let $\nu : C \to \Gamma$ be the normalization map, then we have the standard exact sequence of 2-torsion groups

$$0 \to (\mathbf{C}^*)^\delta_2 \to Pic^0_2\ \Gamma \xrightarrow{\nu^*} Pic^0_2\ C \to 0,$$

$\delta = |Sing\ \Gamma|$. It will be not restrictive to assume $\eta_\Gamma \neq \mathcal{O}_\Gamma$ so that π_Γ *is not split*. For a non trivial $\eta_\Gamma \in Ker\ \nu^*$ we say that π_Γ is a *Wirtinger covering*. Notice that, since $\nu^*\eta_\Gamma$ is trivial, π_Γ is Wirtinger or split iff the induced morphism $\pi : \tilde{C} \to C$ of the normalized curves is split.

We will say that $Q \in |\mathcal{O}_{\mathbb{P}}(2)|$ is a *conic bundle* if *Sing Q* consists of finitely many ordinary double points and all the assumptions we made are satisfied. $\forall\ \delta \geq 0$ we consider the Severi varieties of nodal conic bundles

$$V_{\mathbb{P},\delta} = \{Q \in |\mathcal{O}_{\mathbb{P}}(2)|\ /\ Q \text{ as above and } |Sing\ Q| = \delta\}.$$

In view of our application in genus $g \leq 6$, it will be sufficient to consider the trivial projective bundle, so we put $\mathbb{P} := \mathbf{P}^2 \times \mathbf{P}^2$ and $\mathcal{O}_{\mathbb{P}}(1) := \mathcal{O}_{\mathbf{P}^2\times\mathbf{P}^2}(1,1)$. Fixing coordinates $(x,y) = (x_1 : x_2 : x_3) \times (y_1 : y_2 : y_3)$ on $\mathbf{P}^2 \times \mathbf{P}^2$, the equation of an element $Q \in |\mathcal{O}_{\mathbf{P}^2\times\mathbf{P}^2}(2,2)|$ is

$$\sum a_{ij}(x)y_iy_j = 0.$$

For Q general the discriminant curve Γ is a smooth sextic. Its equation is $det\ (a_{ij}) = 0$. The 2:1 cover $\pi_\Gamma : \tilde{\Gamma} \to \Gamma$ is étale and defined by a non trivial element $\eta_\Gamma \in Pic^0_2\ \Gamma$. To give η_Γ is equivalent to give the exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^2}(-4)^3 \xrightarrow{A} \mathcal{O}_{\mathbf{P}^2}(-2)^3 \to \eta_\Gamma \to 0,$$

where $A = (a_{ij})$ is a symmetric matrix of quadratic forms, see [Be2]. The construction extends to nodal plane sextics Γ with δ nodes, see [FV4] for the details. We consider the most important case, namely the case $\delta = 4$. It will be enough to fix the four nodes of our conic bundle as follows:

$$(o_1, o_1) \dots (o_4, o_4) \in \mathbf{P}^2 \times \mathbf{P}^2 \subset \mathbf{P}^8.$$

Let O be the set of these points and $\mathcal{I}_O$ its ideal sheaf, we then have the linear system of conic bundles

$$\mathbf{P}^{15} := |\mathcal{I}_O^2(2,2)| \subset |\mathcal{O}_{\mathbf{P}^2 \times \mathbf{P}^2}(2,2)|.$$

The general $Q \in \mathbf{P}^{15}$ is a 4-nodal conic bundle satisfying our previous assumptions. Furthermore we have a natural map

$$m : \mathbf{P}^{15} \to \mathcal{R}_6$$

sending Q to $[C, \eta]$, where $\nu : C \to \Gamma$ is the normalization map and $\eta := \nu^* \eta_\Gamma$. This map is actually dominant. To see this take a general $[C, \eta]$ and any sextic model of C as a 4-nodal plane sextic Γ. Fixing any η_Γ such that $\nu^* \eta_\Gamma \cong \eta$ one can reconstruct as above a symmetric matrix of quadratic forms $A = (a_{ij})$ such that $det\ A = 0$ is the equation of Γ. Moreover A defines the conic bundle Q of equation $\sum a_{ij} y_i y_j = 0$. Finally, up to biregular automorphisms of $\mathbb{P}$, Q belongs to $|\mathcal{O}_\mathbb{P}(2)|$ and clearly $m(Q) = [C, \eta]$.

It is not difficult to extend this argument to the case $\delta \geq 5$ in order to construct a rational family of linear systems of δ-nodal conic bundles which dominates $\mathcal{R}_g$, $g \leq 5$. This concludes a very quick proof of the next

Theorem 3.11 *$\mathcal{R}_g$ is unirational for $g \leq 6$.*

Remark 3.12 A further use of families of nodal conic bundles, in higher genus and in some $\mathbf{P}^2$-bundle over a rational surface S, could be a priori not excluded. However this approach seems difficult and we are aware of a very small number of possible applications. As we will see, the unirationality of $\mathcal{R}_7$ is better reached via K3 surfaces.

Now we want to desingularize, so to say, the previous family of 4-nodal conic bundles in order to see more geometry of it. More precisely we want to pass, by suitable birational transformations, from $\mathbf{P}^2 \times \mathbf{P}^2$ to the $\mathbf{P}^2$-bundle $p : \mathbb{P} \to S$, where S is a smooth quintic Del Pezzo surface embedded in the Grassmannian $G(3,5)$ and $\mathbb{P}$ is the universal plane over it. We start from the Segre embedding $\mathbf{P}^2 \times \mathbf{P}^2 \subset \mathbf{P}^8$ and consider the linear projection

$$h : \mathbf{P}^2 \times \mathbf{P}^2 \dashrightarrow \mathbf{P}^4.$$

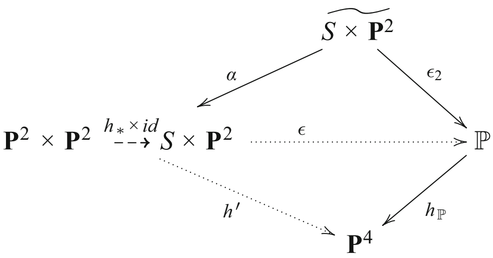
of center the set of four points O. Since the degree of the Segre embedding is six, then h is a rational dominant map of degree two. The map h induces a generically injective rational map

$$h_* : \mathbf{P}^2 \dashrightarrow S \subset G(3,5).$$

h_* defines a *congruence of planes* S of $\mathbf{P}^4$ that is a surface in the Grassmann variety $G(3,5)$. It is easy to check that S is a smooth quintic Del Pezzo surface and a linear section of $G(3,5)$. Let $\sigma : S \to \mathbf{P}^2$ be the blowing up of the set of four points o_i, $i = 1 \dots 4$, where $(o_i, o_i) \in O$, one can also check that σ is precisely the inverse of h_*. Now let $\mathcal{M}$ be the universal bundle of $G(3,5)$ restricted to S and let

$$\mathbb{P} := \mathbf{P}\mathcal{M}.$$

Then we have the commutative diagram



where $h = h' \circ (h_* \times id)$, $h_{\mathbb{P}}$ is the tautological map of $\mathbb{P}$ and ϵ is birational. $h_{\mathbb{P}} : \mathbb{P} \to \mathbf{P}^4$ is a morphism of degree two, branched on a very interesting singular quartic threefold considered in [SR] and $\mathfrak{s}_5$-invariant.

Let $E_i = \sigma^{-1}(o_i)$, α is the blow up of $\cup_{i=1\dots4} E_i \times \{o_i\}$ and ϵ_2 is a divisorial contraction. A main point is that the strict transform by $\epsilon \circ (h_* \times d)$ of $|\mathcal{I}_O(1,1)|$ is the tautological linear system $|\mathcal{O}_{\mathbb{P}}(1)|$ [FV4]. This implies that

Proposition 3.13 *$|\mathcal{O}_{\mathbb{P}}(2)|$ is the strict transform by the map $\epsilon \circ (h_* \times d)$ of the 15-dimensional linear system of nodal conic bundles $|\mathcal{I}^2_O(2,2)|$.*

Let $Q \in |\mathcal{O}_{\mathbb{P}}(2)|$ be general, then its discriminant C_Q is a general element of $|-2K_S|$. It is endowed with the étale 2:1 cover $\pi_Q : \tilde{C}_Q \to C_Q$ due to the conic bundle structure. Thus a commutative diagram follows:

$$\begin{array}{ccc} |\mathcal{O}_{\mathbb{P}}(2)| & \xrightarrow{d} & |-2K_S| \\ r\downarrow & & m\downarrow \\ \mathcal{R}_6 & \xrightarrow{f} & \mathcal{M}_6 \end{array}$$

where d is the discriminant map $Q \to C_Q$, f is the forgetful map, m is the moduli map and r is the map sending Q to the moduli point of π_Q. This is of independent interest and, hopefully, it will be reconsidered elsewhere.

4 $\mathcal{A}_p$ and the Prym Map

4.1 Prym Varieties and the Prym Map

Let (C, η) be a smooth Prym curve and $\pi : \tilde{C} \to C$ be the étale double cover defined by η, then $\tilde{C}$ has genus $2g-1$ but depends on $3g-3$ moduli. Hence $\tilde{C}$ is not general and, in particular, Brill-Noether theory does not apply a priori to it. In this section we recall the basic results of this theory, modified for the case of $\tilde{C}$, and define the Prym variety of (C, η). Let

$$Nm : Pic^d\, \tilde{C} \to Pic^d\, C,$$

be the *Norm map*, which is defined as follows $Nm(\mathcal{O}_{\tilde{C}}(a)) := \mathcal{O}_C(\pi_* a)$. Nm is clearly surjective and, for $d = 0$, is a morphism of abelian varieties

$$Nm : Pic^0 \tilde{C} \to Pic^0 C.$$

Then dim $Ker\ Nm = g - 1$ and its connected component of zero

$$P(C, \eta) := (KerNm)^0$$

is a $g-1$-dimensional abelian variety. $P(C, \eta)$ is known as *the Prym variety of* (C, η). Moreover it is endowed with a natural principal polarization.

Following Mumford's foundation of the theory of Prym varieties we want now to see a different, and very convenient, construction for this abelian variety and its principal polarization.

How many connected components do we have for $Ker\ Nm$? The answer follows from the exact sequence of 2-torsion groups

$$0 \to <\eta> \to Pic^0_2\, C \xrightarrow{\pi^*} Pic^0_2\, \tilde{C} \xrightarrow{Nm} Pic^0_2\, C \to 0.$$

This implies that $(Ker\ Nm)_2$ has order 2^{2g-1}. Since $\dim P(C, \eta) = g - 1$ then $P(C, \eta)_2$ has order 2^{2g-2} and hence index 2 in $KerNm$. Therefore the connected components are two and every fibre of $Nm : Pic^d\, \tilde{C} \to Pic^d\, C$ is the disjoint union of two copies of $P(C, \eta)$.

Following [M8] it is convenient to fix $d = 2g - 2$ and to study the map

$$Nm : Pic^{2g-2} \tilde{C} \to Pic^{2g-2} C$$

and, in particular, its fibre over the canonical class $o \in Pic^{2g-2} C$. This is

$$Nm^{-1}(o) = \{\tilde{L} \in Pic^{2g-2} \tilde{C} \mid Nm\, \tilde{L} \cong \omega_C\}.$$

In this case the splitting of $Nm^{-1}(o)$ in two copies of $P(C, \eta)$ is ruled by the parity of $h^0(\tilde{L})$. Denoting these copies by P and P^- we have:

- $P := \{\tilde{L} \in Nm^{-1}(o) \ / \ h^0(\tilde{L}) \ is \ even\}$,
- $P^- := \{\tilde{L} \in Nm^{-1}(o) \ / \ h^0(\tilde{L}) \ is \ odd\}$.

Now let $\tilde{g} = 2g - 1$ be the genus of $\tilde{C}$. Since $\tilde{g} - 1 = 2g - 2$ a natural copy of the theta divisor of $Pic^{2g-2}\tilde{C}$ is provided by the Brill Noether-locus

$$W^0_{2g-2}(\tilde{C}) := \tilde{\Theta}.$$

By Riemann singularity theorem and the definition of P, the intersection $P \cap W^0_{2g-2}$ is entirely contained in the singular locus $W^1_{2g-2}(\tilde{C})$ of $\tilde{\Theta}$. Building on this remark it turns out that the scheme

$$\Xi := P \cdot W^1_{2g-2}(\tilde{C})$$

is a principal polarization on P [M8, ACGH]. Notice also that $P \cdot \tilde{\Theta} = 2\Xi$. From a smooth Prym curve (C, η) we can therefore define a principally polarized abelian variety (P, Ξ) of dimension $g - 1$.

Definition 4.1 The pair (P, Ξ) is the Prym variety of (C, η).

With a slight abuse $P(C, \eta)$ we will also denote (P, Ξ). The loci we have considered so far in $Nm^{-1}(o)$ are examples of *Prym Brill-Noether loci*.

Definition 4.2 The r-th Prym Brill-Noether scheme $P^r(C, \eta)$ is

- $P \cdot W^r_{2g-2}(\tilde{C})$ if $r + 1$ is even,
- $P^- \cdot W^r_{2g-2}(\tilde{C})$ if $r + 1$ is odd.

The following properties, in analogy to Brill-Noether theory, are satisfied by the Prym Brill-Noether loci $P^r(C, \eta) \subset P \cup P^-$, see [We]

Theorem 4.3

(1) Let (C, η) be any smooth Prym curve: if $\binom{r+1}{2} \leq g - 1$ then $P^r(C, \eta)$ is not empty of codimension $\leq \binom{r+1}{2}$.

(2) Let (C, η) be a general smooth Prym curve and let $\binom{r+1}{2} \leq g - 1$ then

- *$P^r(C, \eta)$ has codimension $\binom{r+1}{2}$,*
- *it is irreducible if* $\dim P(C, \eta) > 0$,
- *Sing $P^r(C, \eta) = P^{r+2}(C, \eta)$.*

After the family of Jacobians of curves of genus g, the family of Prym varieties of dimension $g - 1$ is of special interest in the family of all principally polarized abelian varieties of the same dimension. The reason is that Prym varieties, due to the way they are constructed, can be investigated by means of the theory of curves.

For instance the theta divisor Ξ of a general Prym $P(C, \eta)$ is singular if $\dim P(C, \eta) \geq 6$. Its singular locus is precisely the Prym Brill-Noether locus $P^3(C, \eta) \subset \Xi$. By the way a general point of it is a quadratic singularity with quadratic tangent cone of rank 6.

Prym varieties are specially important in low dimension p. Let $\mathcal{A}_p$ be the moduli space of principally polarized abelian varieties of dimension p. Indeed most knowledge on $\mathcal{A}_p$ for $p \leq 5$ is due to the next property:

Theorem 4.4 *A general ppav of dimension ≤ 5 is a Prym variety.*

This theorem follows considering the Prym map:

Definition 4.5 The Prym map $P_g : \mathcal{R}_g \to \mathcal{A}_{g-1}$ is the map sending the moduli point of (C, η) to the moduli point of its associated Prym $P(C, \eta)$.

One can actually show that P_g is dominant for $g \leq 6$. This follows by showing that the tangent map of P_g is generically surjective for $g \leq 6$. To sketch the proof of this latter fact let us briefly recall the following.

For $i^* : H^0(\omega_{\tilde{C}}) \to H^0(\omega_{\tilde{C}})$ the eigenspaces decomposition is

$$H^0(\omega_{\tilde{C}}) = H^+ \oplus H^-,$$

with $H^+ = \pi^* H^0(\omega_C)$ and $H^- = \pi^* H^0(\omega_C \otimes \eta)$.

As is well known the Prym variety P is the image of the map

$$1 - i^* : Pic^0 \tilde{C} \to Pic^0 \ \tilde{C}$$

so that its cotangent bundle is canonically isomorphic to $\mathcal{O}_P \otimes H^-$.

For the cotangent space to $\mathcal{R}_g$ at $x := [C, \eta]$ we have the canonical identifications $T^*_{\mathcal{R}_g,x} = T^*_{\mathcal{M}_g,[C]} = H^0(\omega_C^{\otimes 2})$. On the other hand we have $T^*_{\mathcal{A}_{g-1},y} = Sym^2 H^0(\omega_C \otimes \eta)$ for the cotangent space to $\mathcal{A}_{g-1}$ at $y = [P(C, \eta)]$.

Finally it turns out that the multiplication map

$$\mu : Sym^2 H^0(\omega_C \otimes \eta) \to H^0(\omega_C^{\otimes 2})$$

is the cotangent map of P_g at x, see for instance [Be3]. Therefore P_g is dominant at x iff μ is injective. It is well known that, for a general Prym curve (C, η) of genus g, μ has maximal rank, that is either it is injective or surjective. Counting dimensions we conclude that

Theorem 4.6 *μ is injective for a general (C, η) iff $g \leq 6$.*

For $g \geq 7$ the map μ is surjective for a general (C, η), that is, the tangent map of P_g is injective at a general point. The Prym-Torelli theorem says that P_g is generically injective for $g \leq 7$. The description of the loci where P_g fails to be injective is in many respects an open problem.

Revisiting the fibres of the Prym map P_g for $g \leq 6$ provides a remarkable sequence of beautiful geometric constructions. For brevity we only mention, for $3 \leq g \leq 6$, a geometric description of a general fibre. Also, we omit to discuss the extension of P_g to $\overline{\mathcal{R}}_g$. The fibre over $[P, \Xi] \in \mathcal{A}_{g-1}$ is birationally described as follows, see [Ve2, Re, Do2]

- $g = 3$. *The Siegel modular quartic threefold* $|2\Xi|/\mathbb{Z}_2^4$.
- $g = 4$. *The Kummer variety* $P/ < -1 >$.
- $g = 5$. *The double cover of the Fano surface of a cubic threefold.*

By a famous result of Donagi and Smith, the Prym map in genus 6 has degree 27 and its monodromy group is the Weyl group of the lattice E_6. The configuration of a general fibre of

$$p_6 : \mathcal{R}_6 \to \mathcal{A}_5$$

is the one of 27 lines on a smooth cubic surface [DS, Do2]. The ramification divisor $\mathcal{D}$ is the locus of elements $[C, \eta] \in \mathcal{R}_6$ such that μ is not an isomorphism. Equivalently the image of C in $\mathbf{P}^4$ under the map defined by $|\omega_C \otimes \eta|$ is contained in a quadric. A general fibre of p_6 over the branch divisor $p_6(\mathcal{D})$ has the configuration of the lines of a cubic surface with an ordinary double point: 6 points of simple ramification and 15 unramified points.

4.2 Unirationality of universal Pryms

Since $\mathcal{R}_g$ is unirational and the Prym map is dominant for $g \leq 6$, we have

Theorem 4.7 *$\mathcal{A}_p$ is unirational for $p \leq 5$.*

Let us review more in detail what is known for the moduli space $\mathcal{A}_p$:

- *$\mathcal{A}_p$ is rational for $p \leq 3$ (since it is birational to $\mathcal{M}_p$),*
- *$\mathcal{A}_4$ and $\mathcal{A}_5$ are unirational (via the Prym map),*
- *$\mathcal{A}_6$ is a remarkable open problem,*
- *$\mathcal{A}_p$ is of general type for $p \geq 7$ (Mumford, Tai).*

We can also study the universal principally polarized abelian. See [M7, T] for p bigger or equal than 7.

$$v_g : \mathcal{X}_p \to \mathcal{A}_p$$

and its pull-back by the Prym map. We define it via the fibre product

$$\begin{array}{ccc} \mathcal{P}_{g-1} & \xrightarrow{\chi_{g-1}} & \mathcal{X}_{g-1} \\ u_g \downarrow & & v_g \downarrow \\ \mathcal{R}_g & \xrightarrow{P_g} & \mathcal{A}_{g-1} \end{array}$$

Definition 4.8 $\mathcal{P}_{g-1}$ is the universal Prym over $\mathcal{R}_g$.

Let $Pic^{inv}_{0,2g-1}$ be the universal Picard variety over the moduli of curves of genus $2g-1$ with a fixed point free involution. To give an equivalent definition let us consider the universal Norm map

$$Nm : Pic^{inv}_{0,2g-1} \to Pic^0_{0,g}.$$

Since the zero section $o : \mathcal{A}_{g-1} \to \mathcal{X}_{g-1}$ is fixed, the universal Prym $\mathcal{P}_{g-1}$ can be also defined as the connected component of o in the Kernel of the universal Norm map. The *unirationality* of $\mathcal{P}_p$ and of $\mathcal{X}_p$ is known for:

- $\mathcal{P}_p$, $p \leq 3$ *(implicit to several constructions)*,
- $\mathcal{P}_4$ *(via Prym Brill-Noether theory* [Ve4]),
- $\mathcal{P}_5$ *(via nodal conic bundles,* [FV4]*)*.

After our previous use of conic bundles over $\mathbf{P}^2$ having a 4-nodal sextic as discriminant curve, we insist in using them to show that

Theorem 4.9 $\mathcal{P}_5$ *is unirational.*

Preliminarily we recall that the n-th *Abel-Prym map* of (C, η) is the map

$$a_n^- : \tilde{C}^n \to Ker\ Nm$$

obtained from the composition of the maps

$$\tilde{C}^n \xrightarrow{a_n} Pic^n\tilde{C} \xrightarrow{1-i^*} KerNm \subset Pic^0\tilde{C},$$

where a_n is the Abel map, that is, $a_n(x_1, \ldots, x_n) = \mathcal{O}_{\tilde{C}}(x_1 + \cdots + x_n)$ and $(1 - i^*)(L) = L \otimes i^*L^{-1}$. It is known that the image of a^- is in the connected component of zero $(Ker\ Nm)^0$ iff n is even. Furthermore a_n^- is generically injective for $n \leq g-1$ and dominant for $n \geq g-1$, see [Be1].

Let $\tilde{\mathcal{C}}^n \to \mathcal{R}_g$ be the universal product with fibre $\tilde{C}^n$ at $[C, \eta]$ and let

$$\mathfrak{a}_n^- : \tilde{\mathcal{C}}^n \to \mathcal{P}_{g-1}$$

be the universal Abel-Prym map defined as follows:

- n even then $\mathfrak{a}_n^-(x_1, \ldots, x_n) = \mathcal{O}_{\tilde{C}}(x_1 - i(x_1) + \cdots + x_n - i(x_n))$,
- n odd then $\mathfrak{a}_n^-(x_1, \ldots, x_n) = \mathcal{O}_{\tilde{C}}(2x_1 - 2i(x_1) + \cdots + x_n - i(x_n))$.

We can now prove that the universal Prym $\mathcal{P}_5$ is unirational. In the spirit of the previous method it will be enough to use the convenient linear system $\mathbf{P}^{15}$ of singular $(2, 2)$ hypersurfaces in $\mathbf{P}^2 \times \mathbf{P}^2$ considered in Sect. 3.4. We will show that $\mathbf{P}^{15}$ dominates $\mathcal{P}_5$. Putting $n = 5$ and $g = 6$ it is clear that $\mathfrak{a}_5^- : \tilde{\mathcal{C}}^5 \to \mathcal{P}_5$ is dominant. Therefore we are left to show that

Theorem 4.10 *$\tilde{\mathcal{C}}^5$ is unirational.*

Proof Let $[\tilde{C}; x_1, \dots, x_5]$ be an element of $\tilde{\mathcal{C}}^5$. We know that $\tilde{C}$ parametrizes the family of lines which are components of the singular conics of a conic bundle $T \in \mathbf{P}^{15}$. Therefore x_i corresponds to a line of this type, say $l_i \subset T$, $i = 1 \dots 5$. Since three conditions are needed so that some $T \in \mathbf{P}^{15}$ contains l_i, it follows that $\tilde{\mathcal{C}}^5$ is dominated by the family of 5-tuples of lines as above. On the other hand it is clear that the Hilbert scheme of these lines is $\mathbf{P}^2 \times \mathbf{P}^{2*}$. Hence it follows that there exists a dominant rational map

$$(\mathbf{P}^2 \times \mathbf{P}^{2*})^5 \to \tilde{\mathcal{C}}^5.$$

This completes the proof of the unirationality of $\mathcal{P}_5$. □

4.3 Testing the slope of $\overline{\mathcal{A}}_p$ in low Genus

It is time to compactify the Prym map and use the parametrizations of $\mathcal{R}_g$ we have constructed, in order to study the slope of $\overline{\mathcal{A}}_p$ and, possibly, deduce further properties in low genus. In what follows $\overline{\mathcal{A}}_p$ denotes one of the toroidal compactifications in use, namely the first Voronoi or perfect cone compactification. See [AMRT] and [SB1] for a fundamental account.

The study of the slope performed here includes the case of $\overline{\mathcal{A}}_6$. We reach a lower bound of its slope using families of rational curves which are sweeping the boundary divisor. This just means that the union of these curves contains a dense open subset of the boundary divisor. The latter is actually dominated by the universal Prym $\mathcal{P}_5$ and hence by the rational parametrization of $\mathcal{P}_5$ we have constructed. Let us consider the extended Prym map

$$\overline{P}_g : \overline{\mathcal{R}}_g \to \overline{\mathcal{A}}_{g-1}$$

induced by P_g. We have already considered the moduli of quasi stable Prym curves $\overline{\mathcal{R}}_g$. On the other hand it turns out that $\overline{\mathcal{A}}_p$ is the blowing up of the Satake compactification

$$\mathcal{A}_p^s = \mathcal{A}_p \sqcup \mathcal{A}_{p-1} \sqcup \dots \sqcup \mathcal{A}_0$$

along its boundary, cfr. [SB1]. It is well known that the exceptional divisor

$$D_p \subset \overline{\mathcal{A}}_p$$

of such a blowing up is integral, moreover

$$CH^1(\overline{\mathcal{A}}_p) = \mathbb{Z}\lambda_p \oplus \mathbb{Z}\delta_p$$

where λ_p is the Hodge class and δ_p is the class of the boundary D_p. Let $E \subset \overline{\mathcal{A}}_p$ be an effective divisor of class $a\lambda_p - b\delta_p$, such that $a, b > 0$. We define its slope as $s(E) := \frac{a}{b}$.

Definition 4.11 The slope of $\overline{\mathcal{A}}_p$ is $s(\overline{\mathcal{A}}_p) := min\{s(E) \ / \ E \ as \ above\}$.

We recall that $s(\overline{\mathcal{A}})_p$ governs the pseudoeffectiveness of the canonical class of $\overline{\mathcal{A}}_p$ and that, as a consequence of the results in [BDPP], it follows:

- $s(\overline{\mathcal{A}}_p) > p + 1$ *implies that* $\overline{\mathcal{A}}_p$ *is uniruled*,
- $s(\overline{\mathcal{A}}_p) < p + 1$ *implies that* $\overline{\mathcal{A}}_p$ *is of general type*.

For $p \leq 6$ this is the situation, to be partially presented here:

- $s(\overline{\mathcal{A}}_4) = 8$ [GSM]
- $s(\overline{\mathcal{A}}_5) = \frac{54}{7}$ [FGSMV]
- $s(\overline{\mathcal{A}}_6) \geq 5, 3$ [FV4]

To sketch a description of D_p let us fix a ppav P of dimension $p-1$. As is well known the family of algebraic groups defined by the extensions

$$0 \to \mathbf{C}^* \to A \to P \to 0$$

defines a map $j_P : P \to D_p$ whose image is birational to the Kummer variety $P/ < -1 >$, cfr. [AMRT]. Since P is a fibre of $u_{p-1} : \mathcal{X}_{p-1} \to \mathcal{A}_{p-1}$, the constructions yields a dominant map

$$j : \mathcal{X}_{p-1} \to D_p.$$

It is therefore natural to test the slope of $\overline{\mathcal{A}}_p$ using families of curves R sweeping $\overline{\mathcal{X}}_{p-1}$. From curves like R one can try to compute a lower bound of $s(\overline{\mathcal{A}}_p)$, using the inequality $R \cdot j^*E \geq 0$ for E effective of class $a\lambda_p - b\delta_p$ with $a, b > 0$. Let ρ be the class of R. Of course this implies:

$$s(\overline{\mathcal{A}}_p) \geq \frac{\deg \rho \cdot j^*\delta_p}{\deg \rho \cdot j^*\lambda_p}.$$

In order to describe some intersection classes for $p = 6$, we conclude this section with a few technical remarks. Let $\tilde{\mathcal{R}}_g$ be the complement in $\overline{\mathcal{R}}_g$ to $\Delta_i \cup \Delta_{i:g-i}$, $i \geq 1$. It will be sufficient to work in $\tilde{\mathcal{R}}_g$. Let $\tilde{\mathcal{P}}_g$ be the pull-back of $\tilde{\mathcal{R}}_g$ by $u_g : \mathcal{P}_g \to \overline{\mathcal{R}}_g$ and let $\tilde{\mathcal{A}}_g := \overline{\mathcal{A}}_g - D_g$. Then the diagram

$$\begin{array}{ccccc}
\tilde{\mathcal{P}}_{g-1} & \xrightarrow{\chi} & \tilde{\mathcal{X}}_{g-1} & \xrightarrow{j} & \overline{\mathcal{A}}_g \\
{\scriptstyle u_g}\downarrow & & {\scriptstyle v_g}\downarrow & & \\
\tilde{\mathcal{R}}_g & \xrightarrow{P_g} & \tilde{\mathcal{A}}_{g-1} & &
\end{array}$$

is commutative. The next formulae are known. Let $\theta \in CH^1(\widetilde{\mathcal{X}}_{g-1})$ be the class of the universal theta divisor, trivialized along the zero section. Let $\theta_{pr} := \chi^*(\theta) \in CH^1(\tilde{\mathcal{P}}_g)$, then the next formulae hold for g even:

- $j^*([D_g]) = -2\theta + v_{g-1}^*([D_{g-1}]) \in CH^1(\widetilde{\mathcal{X}}_{g-1})$.
- $(j \circ \chi)^*(\lambda_g) = u_g^*\big(\lambda - \frac{1}{4}\delta_0^{\mathrm{ram}}\big) \in CH^1(\tilde{\mathcal{P}}_{g-1})$.
- $(j \circ \chi)^*([D_g]) = -2\theta_{pr} + u_g^*(\delta_0') \in CH^1(\tilde{\mathcal{P}}_{g-1})$,

see [GZ, FV4]. We can also put into play the Abel-Prym map

$$\mathfrak{a}_{g-1}^- : \tilde{\mathcal{C}}^{g-1} \to \tilde{\mathcal{P}}_{g-1}.$$

In $CH^1(\tilde{\mathcal{C}}^n)$ we have the ψ-classes $\psi_{x_1} \dots \psi_{x_n}$, defined by the cotangent spaces at $x_1, \dots, x_n$ in the pointed curve $(\tilde{C}; x_1, \dots, x_n)$. One can compute the class of $(\mathfrak{a}_{g-1}^-)^*\theta_{pr}$ in $CH^1(\tilde{\mathcal{C}}^{g-1})$ as in [FV4]:

$$\mathfrak{a}_{g-1}^{-}{}^*\theta_{pr} = \frac{1}{2}\sum_{j=1}^{g-2} \psi_{x_j} + 2\psi_{x_{g-1}} + 0 \cdot \Big(\lambda + (\mathfrak{a}_{\mathfrak{g}-1}^- \circ u_g)^*(\delta_0' + \delta_0'' + \delta_0^{\mathrm{ram}})\Big) + \dots$$

We want only to point out that $\lambda, \delta_0', \delta_0'', \delta_0^{\mathrm{ram}}$ have zero coefficient.

4.4 On the slope and boundary of $\overline{\mathcal{A}}_6$

The previous analysis of divisorial classes can be in principle used to study the slope of $\overline{\mathcal{A}}_p$ or other properties. This is our program for $p = 6$. A geometric basis for it also exists. This is provided by the unirationality of $\mathcal{P}_5$ and by the linear system of conic bundles

$$\mathbf{P}^{15} = |\mathcal{O}_{\mathbb{P}}(2)| = |\mathcal{I}_O^2(2,2)|$$

that we already considered in Sect. 3. We keep the previous notation: in particular $\mathbb{P}$ is the $\mathbf{P}^2$-bundle we know over the quintic Del Pezzo surface S. Let $\mathbb{L}$ be the Hilbert scheme of lines which are in the fibres of $\mathbb{P} \to S$. Then $\mathbb{L}$ is just $\mathbb{P}^*$ and we can define a dominant rational map as follows

$$q : \mathbb{L}^5 \to \tilde{\mathcal{C}}^5.$$

For a general $x := (l_1, \dots, l_5) \in \mathbb{L}^5$ a unique $Q \in |\mathcal{O}_{\mathbb{P}}(2)|$ contains $l_1 \dots l_5$. Hence x defines a point $y \in \tilde{C}^5$, where $\tilde{C}$ is the family of the lines in the singular fibres of Q. By definition $q(x) := y$. Therefore we have a sequence of dominant rational maps

$$\mathbb{L}^5 \xrightarrow{q} \tilde{\mathcal{C}}^5 \xrightarrow{\mathfrak{a}_5^-} \mathcal{P}_5 \xrightarrow{j \circ P_5} \mathcal{D}_6 \subset \overline{\mathcal{A}}_6.$$

Let $\phi := j \circ \mathfrak{a}_5^- \circ q$, then ϕ is dominant. Now we choose a family of maps

$$f : \mathbf{P}^1 \to D_6$$

such that the curve $f_*(\mathbf{P}^1)$ moves in a family sweeping D_6. Previously, it will be useful to know some numerical characters of the linear system $|\mathcal{O}_{\mathbb{P}}(2)|$, see [FV4]:

Lemma 4.12 *For a smooth $Q \in \left|\mathcal{O}_{\mathbf{P}}(2)\right|$, we have that $\chi_{top}(Q) = 4$, whereas $\chi_{top}(Q) = 5$ if Sing Q is an ordinary double point.*

Lemma 4.13 *In a Lefschetz pencil $P \subset |\mathcal{O}_{\mathbf{P}}(2)|$ there are precisely 77 singular conic bundles and 32 conic bundles with a double line.*

A family of maps f is constructed as follows: fix a general configuration (l_1, l_2, l_3, l_4, o) of four lines $l_1 \dots l_4$ in the fibres of $\mathbb{P}$ and a point $o \in \mathbb{P}$. Let $\mathbf{P}^1$ be the pencil of lines through o in the fibre of $\mathbb{P}$ containing o. Each line $l \in \mathbf{P}^1$ defines the element $\phi(l_1, l_2, l_3, l_4, l) \in D_6$. This defines a map

$$f : \mathbf{P}^1 \to D_6 \subset \overline{\mathcal{A}}_6$$

and the family of curves $f(\mathbf{P}^1)$ is sweeping D_6. Such a family and the characters of a Lefschetz pencil P of conic bundles are the geometric support for the intersection classes count we are going to outline. For the complete set of these computations see [FV4, Sects. 3 and 4] .

Let $m : \mathbf{P}^1 \to \overline{\mathcal{R}}_6$ be the moduli map sending (l_1, l_2, l_3, l_4, l) to the Prym curve (C, η) which is the discriminant of the conic bundle Q. Moreover consider also the natural map $q : \mathbf{P}^1 \to \tilde{\mathcal{C}}^5$. After more work we obtain:

$$m^*\lambda = 9 \times 6\ ,\ m^*\delta_0' = 3 \times 77\ ,\ m^*\delta_0^{\mathrm{ram}} = 3 \times 32\ ,\ m^*\delta_0'' = 0\ ,\ q^*\psi_{l_j} = 9.$$ [4]

We use these data to bound the slope of $\overline{\mathcal{A}}_6$: at first consider the effective class $\rho := [f_*(\mathbf{P}^1)] \in NE_1(\overline{\mathcal{A}}_6)$, one can compute that

$$\rho \cdot \lambda_6 = q_*(\mathbf{P}^1) \cdot (\mathfrak{a}_{\mathfrak{g}-1}^- \circ u_g)^*\Big(\lambda - \frac{1}{4}\delta_0^{\mathrm{ram}}\Big) = 6 \times 9 - \frac{3 \times 32}{4} = 30, \quad \text{and}$$

$$\begin{aligned}\rho \cdot [D_6] &= -q_*(\mathbf{P}^1) \cdot \Big(\sum_{j=1}^{4} \psi_{x_j} + 4\psi_{x_5}\Big) + i_*(\mathbf{P}^1) \cdot (\mathfrak{a}_{\mathfrak{g}-1}^- \circ u_g)^*(\delta_0') \\ &= -8 \times 9 + 3 \times 77 = 159.\end{aligned}$$

[4] To simplify the notation we identify $Pic(\mathbf{P}^1)$ to $\mathbb{Z}$ via the degree map.

Since ρ is the class of a family of curves sweeping D_6, it follows that $\rho \cdot E \geq 0$ for every effective divisor E. This implies the bound

$$s(\overline{\mathcal{A}}_6) \geq \frac{\rho \cdot [D_6]}{\rho \cdot \lambda_6} = 5,3.$$

5 Prym Curves and K3 Surfaces

5.1 Mukai Constructions and universal Jacobians

In order to discuss the uniruledness of some moduli spaces as $\mathcal{M}_g$, $\mathcal{R}_g$ or $\mathcal{A}_g$, we have made a large use of constructions involving rational families of curves on rational surfaces. On the other hand we have often remarked that further constructions definitely come into play in the historical and scientific evolution of this subject. Now the next constructions to be considered are Mukai constructions for K3 surfaces and canonical curves in low genus.

As it is well known, these can be very well used to deduce the unirationality of $\mathcal{M}_g$ for $7 \leq g \leq 9$ and $g = 11$. Furthermore they represent a very natural motivation for this result. To add a dubious speculation, it is possibly not excluded that the rationality problem for $\mathcal{M}_g$ could be approached via these constructions and Geometric Invariant Theory, in the range $7 \leq g \leq 9$. For $g \leq 9$, the unirationality of the universal Picard variety $Pic_{d,g}$ also follows from these constructions, cfr. [Mu1, Mu2, Ve1]

However we concentrate in this part on some related but different constructions which are natural and useful to study the Prym moduli space $\mathcal{R}_g$ in low genus. For every g we consider a special family of K3 surfaces of genus g, namely the family of *Nikulin surfaces*.

To improve our study of $\mathcal{R}_g$ we rely on these surfaces, their hyperplane sections and their moduli. The geometric description of Nikulin surfaces in low genus g presents some unexpected and surprising analogies to Mukai constructions, cfr. [FV3, FV6, Ve5].

Definition 5.1 A K3 surface of genus g is a pair (S, L) such that S is a K3 surface and $L \in Pic\ S$ is primitive, big and nef, and $c_1(L)^2 = 2g - 2$.

The *moduli space of* (S, L) is denoted by $\mathcal{F}_g$, it is quasi projective and integral of dimension 19. Let $g \geq 3$ and (S, L) general. Then L is very ample and defines an embedding $S \subset \mathbf{P}^g$ as a surface of degree $2g - 2$, cfr. [Huy]. The next result is due to Mukai and Mori [MM]:

Theorem 5.2 *A general canonical curve C of genus g is a hyperplane section of a K3 surface $S \subset \mathbf{P}^g$ iff $g \leq 11$ and $g \neq 10$.*

Since the moduli map $m : |\mathcal{O}_S(1)| \to \mathcal{M}_g$ turns out to be not constant, the theorem implies the *uniruledness* of $\mathcal{M}_g$ for $g \leq 11$, $g \neq 10$. Actually the theorem fits in the nice series of geometric constructions discovered by Mukai. They relate K3 surfaces, canonical curves in low genus and homogenous spaces. Here is the list of homogeneous spaces in use. A space in the list is denoted by $\mathbb{S}_g$ and it is embedded in $\mathbf{P}^{N_g}$ by the ample generator $\mathcal{O}_{\mathbb{S}_g}(1)$ of $Pic\ \mathbb{S}_g$:

- $\mathbb{S}_6$ *is the Grassmannian* $G(2,5)$ *embedded in* $\mathbf{P}^9$,
- $\mathbb{S}_7$ *is the orthogonal Grassmannian* $OG(5,10)$ *embedded in* $\mathbf{P}^{15}$,
- $\mathbb{S}_8$ *is the Grassmannian* $G(2,6)$ *embedded in* $\mathbf{P}^{14}$,
- $\mathbb{S}_9$ *is the symplectic Grassmannian* $SO(3,6)$ *embedded in* $\mathbf{P}^{13}$,
- $\mathbb{S}_{10}$ *is the* $\mathbb{G}_2$*-homogenous space* G_2 *embedded in* $\mathbf{P}^{13}$.

Mukai constructions are developed in [Mu1, Mu2, Mu3, Mu4]. In particular they imply the next two theorems:

Theorem 5.3 *Let* (S,L) *be a general K3 surface of genus* $g \in [7,10]$*, then* S *is biregular to a 2-dimensional linear section of* $\mathbb{S}_g \subset \mathbf{P}^{N_g}$ *and* $L \cong \mathcal{O}_S(1)$.

This is often called *Mukai linear section theorem*. We also include:

Theorem 5.4 *Let* $g = 6$*, then a general* S *is a 2-dimensional linear section of a quadratic complex of the Grassmannian* $\mathbb{S}_6 \subset \mathbf{P}^9$ *and* $L \cong \mathcal{O}_S(1)$.

What is the link between Mukai constructions and the unirationality of $\mathcal{M}_g$ in low genus? A direct consequence of these constructions is that

Theorem 5.5 *The universal Picard variety* $Pic_{d,g}$ *is unirational for* $g \leq 9$.

Proof See [Ve1] for details, here we will give a sketch the proof in view of some applications. Let $U \subset \mathbb{S}_g^g$ be the open set of g-tuples of points $x := (x_1, \dots, x_g)$ spanning a space $< x >$ transversal to $\mathbb{S}_g$. Then $C_x := \mathbb{S}_g \cap < x >$ is a smooth, g-pointed canonical curve. Given the non zero integers $d_1, \dots, d_g$ such that $d_1 + \cdots + d_n := d$, let $a : C_x^g \to Pic^d\ C$ be the map sending $(y_1, \dots, y_g)$ to $\mathcal{O}_{C_x}(\sum d_i y_i)$. It is well known that a is surjective. Now consider the map $A_g : U \to Pic_{d,g}$ defined as follows: $A_g(x) := [C_x, L_x] \in Pic_{d,g}$, where $L_x := \mathcal{O}_{C_x}(\sum d_i x_i)$. By Mukai results and the latter remark A_g is dominant for $g \leq 9$. Since U is rational it follows that $Pic_{d,g}$ is unirational. □

For $g \geq 10$ the transition of $Pic_{d,g}$ from negative Kodaira dimension to general type has remarkable and nice aspects:

- $Pic_{d,10}$ *has Kodaira dimension* 0,
- $Pic_{d,11}$ *has Kodaira dimension* 19,
- $Pic_{d,g}$ *is of general type for* $g \geq 12$.

This picture summarizes several results, see [BFV, FV1, FV3]. The presence of the number $19 = dim\ \mathcal{F}_{11}$ is not a coincidence. It reflects the Mukai construction for $g = 11$, which implies that a birational model of $\mathcal{M}_{11}$ is a $\mathbf{P}^{11}$-bundle over $\mathcal{F}_{11}$. Building in an appropriate way on this information, it follows that $kod(Pic_{d,11}) = 19$.

It is now useful to summarize some peculiar aspects of the family of hyperplane sections of K3 surfaces of genus g, made visible after Mukai. Let $C \subset S$ be a smooth, integral curve of genus g in a *general* $K3$ surface:

- Parameter count: *C cannot be general for $g \geq 12$. One expects the opposite for $g \leq 11$.*
- Genus 10 is unexpected: *if C is a curvilinear section of $\mathbb{S}_{10}$, then C is not general and the parameter count is misleading.*
- Genus $g \in [6, 9]$ and $g = 11$ is as expected: *C is a curvilinear section of $\mathbb{S}_g$ and general in moduli.*
- Genus 11: *C embeds in a unique K3. Birationally $\mathcal{M}_{11}$ is a $\mathbf{P}^{11}$-bundle on $\mathcal{F}_{11}$.*
- Genus 10: *Special syzygies: the Koszul cohomology group $K_2(C, \omega_C)$ is non zero.*

With respect to the above properties some unexpected analogies appear, as we will see, when considering the family of hyperplane sections of the so called Nikulin surfaces. Nikulin surfaces are K3 surfaces of special type to be reconsidered in the next sections.

5.2 Paracanonical Curves on K3 Surfaces

A smooth integral curve C of genus g is said to be *paracanonical* if it is embedded in $\mathbf{P}^{g-2}$ so that $\mathcal{O}_C(1) \in Pic^{2g-2}(C)$. Let $g \geq 5$, we are now interested to study the following situation

$$C \subset S \subset \mathbf{P}^{g-2},$$

where S is a smooth K3 surface of genus $g-2$ and C is paracanonical. Then $h := c_1(\mathcal{O}_S(1))$ is a very ample polarization of genus $g-2$ and $|C|$ is a linear system of paracanonical curves on S. In particular each $D \in |C|$ is endowed with a degree zero line bundle

$$\alpha_D := \omega_D(-h) \in Pic^0(D).$$

The Picard number $\rho(S)$ of S is at least two. At first we deal with the case $\rho(S) = 2$. Then we specialize to $\rho(S) \geq 2$ in order to construct Nikulin surfaces and see the predicted analogies with Mukai constructions. Let

$$c := c_1(\mathcal{O}_S(C)) \ , \ n := c_1(\mathcal{O}_S(C-H))$$

where $H \in |\mathcal{O}_S(1)|$, then we have

$$\begin{pmatrix} c^2 & 0 \\ 0 & n^2 \end{pmatrix} = \begin{pmatrix} 2g-2 & 0 \\ 0 & -4 \end{pmatrix}$$

The matrix defines an abstract rank 2 lattice embedded in *Pic S*. Notice that h, c generate the same lattice and that $hc = c^2 = 2g - 2$ and $h^2 = 2g - 6$. Consistently with definition 5.1, it is perhaps useful to adopt the following

Definition 5.6 A K3 surface (S, L) of genus g is d-nodal if *Pic S* contains a primitive vector n orthogonal to $c_1(L)$ and such that $n^2 = -2d \leq -2$.

If $d = 1$ then n or $-n$ is an effective class, for good reasons this is said to be a nodal class. For $d \geq 2$ the classes n and $-n$ are in general not effective. The locus, in the moduli space $\mathcal{F}_g$, of d-nodal K3 surfaces is an integral divisor whose general element is a K3 surface of genus g and Picard lattice as above, cfr. [Huy]. It will be denoted as

$$\mathcal{F}_{g,d}.$$

We are only interested to the case $n^2 = -4$, so we assume the latter equality from now on. Let (S, L) be general 2-nodal and $c = c_1(L)$, then

$$Pic(S) = \mathbb{Z}c \oplus \mathbb{Z}n.$$

We notice from [BaV] the following properties: $h^i(\mathcal{O}_S(n)) = 0$, $i = 0, 1, 2$, and $h^0(\mathcal{O}_D(n)) = 0$, $\forall D \in |C|$. Moreover every $D \in |C|$ is an integral curve. Then we consider the compactified universal Picard variety

$$j : \mathcal{J} \to |C|,$$

whose fibre at a general element $D \in |C|$ is $Pic^0(D)$. Interestingly the map sending D to $\alpha_D := \omega_D(-h) \in Pic^0(D)$ defines a regular section

$$s : |C| \to \mathcal{J}.$$

Now we have in $\mathcal{J}$ the locus

$$\mathcal{J}_m := \{(D, A) \in \mathcal{J} \ / \ h^0(A^{\otimes m}) \geq 1\}$$

If D is smooth, the restriction of $\mathcal{J}_m$ to the fibre $Pic^0(D)$ of j is precisely the m-torsion subgroup $Pic^0_m(D)$. What is the scheme $s^*(\mathcal{J}_m)$?

Theorem 5.7

- *Let s be transversal to $\mathcal{J}_m$, then $s^*\mathcal{J}_m$ is smooth of length $\binom{2m^2-2}{g}$.*
- *Assume (S, L) is general then s is transversal to $\mathcal{J}_m$ and all the elements $D \in s^*\mathcal{J}_m$ are smooth curves.*

Proof We sketch the proof of the first statement, see [BaV]. The nice proof of the second statement is much more recent, see [FK]. Let $D \in |C|$ and let $N \in Div(S)$ be a divisor of class n. Consider the standard exact sequence

$$0 \to \mathcal{O}_S(mN - C) \to \mathcal{O}_S(mN) \to \mathcal{O}_D(mN) \to 0$$

and its associated long exact sequence. Then consider the cup product

$$\mu : H^1(\mathcal{O}_S(mN - C)) \otimes H^0(\mathcal{O}_S(C)) \to H^1(\mathcal{O}_S(mN)).$$

Let us set $\mathbf{P}^a := \mathbf{P}H^1(\mathcal{O}_S(mN - C))$, $\mathbf{P}^b := \mathbf{P}H^0(\mathcal{O}_S(C))$ and $\mathbf{P}^{ab+a+b} := \mathbf{P}(H^1(\mathcal{O}_S(mN - C)) \otimes H^0(\mathcal{O}_S(C)))$. We have the Segre embedding

$$\mathbf{P}^a \times \mathbf{P}^b \subset \mathbf{P}^{ab+a+b}.$$

It turns out that $s^*(\mathcal{J}_m)$ can be viewed as the intersection scheme

$$\mathbf{P}(Ker\ \mu) \cdot (\mathbf{P}^a \times \mathbf{P}^b).$$

One can check that $a = \max\{0, 2m^2 - 2 - g\}$, $b = g$ and $codim\, \mathbf{P}(Ker\ \mu) = a + b$. Hence the length of $s^*\mathcal{J}_m$ is the degree of $\mathbf{P}^a \times \mathbf{P}^b$, that is $\binom{2m^2-2}{g}$. □

Remark 5.8 We point out that the *most non transversal* situation is possible: $|C| = s^*\mathcal{J}_m$. See the next discussion.

5.3 Mukai Constructions and Nikulin Surfaces

We want to discuss the case $m = 2$, that is to study the set $s^{-1}(\mathcal{J}_2)$ and some related topics. Preliminarily we remark that a smooth $C \in s^{-1}(\mathcal{J}_2)$ is embedded with S in $\mathbf{P}^{g-2}$ as a Prym canonical curve of genus g. This just means that $\mathcal{O}_C(1)$ is isomorphic to $\omega_C \otimes \eta$, with $\eta \ncong \mathcal{O}_C$ and $\eta^{\otimes 2} \cong \mathcal{O}_C$.

By the previous formula the scheme $s^*\mathcal{J}_2$ has length $\binom{6}{g}$, provided it is 0-dimensional. This immediately implies that

Lemma 5.9 *The set $s^{-1}(\mathcal{J}_2)$ is either empty or not finite for $g \geq 7$.*

Let $m_2 : Sym^2 H^0(\mathcal{O}_C(H)) \to H^0(\mathcal{O}_C(2H))$ be the multiplication map, where $H \in |\mathcal{O}_S(1)|$. We notice the next lemma, whose proof is standard.

Lemma 5.10 *m_2 is surjective iff $h^1(\mathcal{O}_S(2H - C)) = 0$.*

Notice also that, since C is embedded by a non special line bundle, the surjectivity of m_2 is equivalent to the surjectivity of the multiplication

$$m_k : Sym^k H^0(\mathcal{O}_C(H)) \to H^0(\mathcal{O}_C(kH))$$

for $k \geq 1$ [ACGH] ex. D-5 p. 140. In other words a smooth C is projectively normal iff $h^1(\mathcal{O}_S(2H - C)) = 0$. Consider the standard exact sequence

$$0 \to \mathcal{O}_S(2N - C) \to \mathcal{O}_S(2N) \to \mathcal{O}_C(2N) \to 0$$

where $N = C - H$. Then observe that the divisor $2N - C$ is not effective. Indeed H is very ample and we have $(2N - C) \cdot H = 10 - 2g \leq 0$ under our assumption that $g \geq 5$. From this the non effectiveness of $2N - C$ follows. Moreover we have $H^1(\mathcal{O}_S(2N - C)) \cong H^1(\mathcal{O}_S(2H - C))$ by Serre duality. Hence, passing to the associated long exact sequence, we have

$$0 \to H^0(\mathcal{O}_S(2N)) \to H^0(\mathcal{O}_C(2N)) \to H^1(\mathcal{O}_S(2H - C)).$$

This implies the next statement:

Theorem 5.11 *Assume $h^1(\mathcal{O}_S(2H - C)) = 0$, then it follows that*

$$h^0(\mathcal{O}_S(2N)) = 1 \Longleftrightarrow s^{-1}(\mathcal{J}_2) = |C|$$

We can start our discussion from this statement. It is obvious that the condition $h^0(\mathcal{O}_S(2N)) = 1$ implies $s^{-1}(\mathcal{J}_2) = |C|$, without any assumption. For the purposes of this section, and to avoid technical details, it will be enough to discuss the existence and the features of the families of K3 surfaces as above which satisfy the following condition:

- $h^0(\mathcal{O}_S(2N)) = 1$ *and the unique curve $E \in |2N|$ is smooth.*

Then the curve E immediately highlights some beautiful geometry of the surface S. From $E^2 = -16$ and $HE = 8$ one can easily deduce that:

Proposition 5.12 *E is the union of eight disjoint lines $E_1 \dots E_8$.*

Actually these surfaces exist and their families have been studied by many authors, in particular by Nikulin in [N], by van Geemen and Sarti in [SvG] and by Garbagnati and Sarti in [GS]. Using the curve E we can introduce them as follows. Since $E \sim 2N$ we have the 2:1 covering $\pi : \tilde{S}' \to S$ branched on E. From π_S we have the commutative diagram

$$\begin{array}{ccc} \tilde{S}' & \xrightarrow{\sigma'} & \tilde{S} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \overline{\pi}} \\ S & \xrightarrow{\sigma} & \overline{S} \end{array}$$

Here σ is the contraction of the -2-lines $E_1 \dots E_8$ to eight nodes. Moreover σ' is the contraction of the exceptional lines $\pi^{-1}(E_i)$, $i = 1 \dots 8$ and finally $\overline{\pi}$ is the $2:1$ cover branched over the even set of nodes $Sing\ \overline{S}$.

It is easy to see that $\tilde{S}$ is a smooth and minimal K3 surface. Actually $\pi_{\overline{S}}$ is the quotient map of a symplectic involution $i : \tilde{S} \to \tilde{S}$ on $\tilde{S}$. The fixed points of i are eight, by Lefschetz fixed point theorem. See [N, SvG, GS].

Definition 5.13 Let S be a K3 surface and $n \in Pic\ S$. n is a Nikulin class if $2n \sim E_1 + \cdots + E_8$, where $E_1, \dots, E_8$ are disjoint copies of $\mathbf{P}^1$.

Let $\mathbb{L}_n$ be the lattice generated by n and by the classes of $E_1 \dots E_8$. As an abstract lattice $\mathbb{L}_n$ is known as the *Nikulin lattice*.

Definition 5.14 A K3 surface (S, L) of genus g is a K3 quotient by a symplectic involution if it is endowed with a Nikulin class n such that:

- $i : \mathbb{L}_n \to Pic\ S$ is a primitive embedding,
- n is orthogonal to $c = c_1(L)$.

The moduli spaces of K3 quotients by a symplectic involution are known. Their irreducible components have dimension 11 and are classified for every g, see [SvG]. In these notes we are interested to the most natural irreducible component, which is birationally defined for every g as follows.

Definition 5.15 $\mathcal{F}_g^N$ is the closure in $\mathcal{F}_g$ of the moduli of pairs (S, L) such that $Pic(S) = \mathbb{Z}c \oplus \mathbb{L}_n$, where $c = c_1(L)$, n is a Nikulin class and $c \cdot n = 0$.

It is well known that $\mathcal{F}^N$ is integral of dimension 11. It is an irreducible component of the moduli of K3 surfaces of genus g which are K3 quotients by a symplectic involution. In these notes we fix the following definition.

Definition 5.16 A Nikulin surface of genus g is a K3 surface (S, L) of genus g with moduli point in $\mathcal{F}_g^N$.

For a general Nikulin surface (S, L) the morphism $f : S \to \mathbf{P}^g$, defined by $L = \mathcal{O}_S(C)$, factors as $f = \overline{f} \circ \sigma$. Here $\sigma : S \to \overline{S}$ is the contraction of $E_1, \dots, E_8$ already considered and $\overline{f} : \overline{S} \to \mathbf{P}^g$ is an embedding.

In what follows we assume $\overline{S} \subset \mathbf{P}^g$ via the embedding $\overline{f}$. In particular the hyperplane sections of $\overline{S}$ are the canonical models $\overline{C}$ of the previously considered Prym canonical curves $C \subset S \subset \mathbf{P}^{g-2}$.

The next table shows some analogies which are unexpected. They occur between the sequence of families of hyperplane sections $\overline{C}$ of Nikulin surfaces $\overline{S}$ of genus g and the sequence of the families of hyperplane sections of general K3 surfaces of genus g. See the next section and [FV3].

- Parameter count. *C cannot be general for $g \geq 8$. On the other hand one expects the opposite for $g \leq 7$.*
- Genus 6 is unexpected. *$\overline{C}$ is a linear section of a particular quasi homogeneous space. This makes it not general.*
- Genus $g \leq 5$ and $g = 7$. *As expected $\overline{C}$ has general moduli.*
- $g = 7$. *A general Prym canonical curve C admits a unique embedding in a Nikulin surface S.*

- $g = 7$. *This defines a morphism* $\mathcal{R}_7 \to \mathcal{F}_7^N$ *which is a projective bundle over a non empty open subset of* $\mathcal{F}_7^N$.
- $g = 6$. *C has special syzygies, indeed it is not quadratically normal.*

5.4 Nikulin Surfaces and $\mathcal{R}_7$

To discuss some of the predicted analogies we start from genus 7. This, by dimension count, is the biggest value of g where one can have embeddings $C \subset S \subset \mathbf{P}^{g-2}$ so that C is a general Prym canonical curve and S is a Nikulin surface. The analogy here is with the family of general K3 surfaces of genus 11, the last value of g such that the family of hyperplane sections of K3 surfaces of genus g dominates $\mathcal{M}_g$. In genus 11 we have a map

$$f : \mathcal{M}_{11} \to \mathcal{F}_{11},$$

constructed by Mukai. In genus 7, we will show that there exists a quite simple analogous map

$$f^N : \mathcal{R}_7 \to \mathcal{F}_7^N.$$

In both cases the fibre of the map at the moduli point of $(S, \mathcal{O}_S(C))$ is the image in $\mathcal{R}_7$ of the open subset of $|C|$ parametrizing smooth curves.

Let us define f^N. Assume (C, η) is a general Prym curve of genus $g = 7$. We recall that then $\omega_C \otimes \eta$ is very ample and that the Prym canonical embedding $C \subset \mathbb{P}^5$ defined by $\omega_C \otimes \eta$ is projectively normal.

Note that we have $h^0(\mathcal{I}_C(2)) = 3$ and $C \subset S \subset \mathbf{P}^5$, where S is the base scheme of the net of quadrics $|\mathcal{I}_C(2)|$.

Lemma 5.17 *S is a smooth complete intersection of three quadrics.*

The proof of this lemma is shown in [FV3], as well as the next arguments reported here. The lemma implies that S is a smooth K3 surface. Due to Theorem 5.11, it will be not surprising the following result:

Lemma 5.18 $(S, \mathcal{O}_S(C))$ *is a Nikulin surface of genus* 7.

The construction uniquely associates a Nikulin surface S to the Prym curve (C, η). Therefore it defines a rational map

$$f^N : \mathcal{R}_7 \to \mathcal{F}_7^N.$$

Recall that $\mathcal{F}_g^N$ is integral of dimension 11 and notice that f^N is dominant. Moreover the fibre of f^N, at the moduli point of a general Nikulin surface S as above, is the family of smooth elements of $|C|$. Then it is not difficult to conclude that a birational model of $\overline{\mathcal{R}}_7$ is a $\mathbf{P}^7$-bundle over $\mathcal{F}_7^N$. This argument immediately implies the unirationality of $\mathcal{R}_7$, once we have shown that $\mathcal{F}_7^N$ is unirational. Let us sketch the proof of this property, see [FV3].

Theorem 5.19 *$\mathcal{F}_7^N$ is unirational.*

Proof With the previous notation let $C \subset S \subset \mathbf{P}^5$, where C is Prym canonical and S a general Nikulin surface. Let $H \in |\mathcal{O}_S(1)|$, as we have seen in proposition 5.13 S contains 8 disjoint lines $E_1 \dots E_8$ such that $E_1 + \dots + E_8 \sim 2C - 2H$ and $E_1 C = \dots = E_8 C = 0$. Fixing the line E_8 we consider the curve

$$R \in | C - E_1 - \dots - E_7 | .$$

Note that $R^2 = -2$ and $HR = 5 \geq 0$, then R exists and it is an isolated -2 curve of degree 5. Since S is general R is a rational normal quintic in $\mathbf{P}^5$. Now fix a rational normal quintic $R \subset \mathbf{P}^5$ and consider a general point $x := (x_1, \dots, x_{14}) \in R^{14}$. We can consider the curve

$$C_0 := R \cup \overline{x_1 x_8} \cup \dots \cup \overline{x_7 x_{14}} \subset \mathbf{P}^5.$$

C_0 is union of R and seven disjoint lines $\overline{x_i x_{i+7}}$, $i = 1 \dots 7$. One can show that C_0 is contained in a smooth complete intersection S of three quadrics. S is actually a Nikulin surface. To see this consider in S the seven lines $E_i := \overline{x_i x_{i+7}}$, $i = 1 \dots 7$ and $H \in |\mathcal{O}_S(1)|$. Observe that there exists one line more, namely

$$E_8 \sim 2C_0 - 2H - E_1 - \dots - E_7 = 2R + E_1 + \dots + E_7 - 2H.$$

Hence x uniquely defines a Nikulin surface S and we have constructed a dominant rational map $R^{14} \to \mathcal{F}_7^N$. □

The previous theorem implies that

Theorem 5.20 *$\mathcal{R}_7$ is unirational.*

One can do better, see [FV6, Theorem 1.3]: let $\tilde{\mathcal{F}}_g^N$ be the moduli space of Nikulin surfaces of genus 7 endowed with one of the lines $E_1, \dots, E_8$. Then:

Theorem 5.21 *$\tilde{\mathcal{F}}_7^N$ is rational.*

6 Unirationality and $\mathcal{M}_g$

6.1 *The Program for $g \leq 14$*

We started these notes with a discussion on the rationality problem for $\mathcal{M}_g$ in very low genus. In this last section we complement it by a short discussion on the known unirationality / uniruledness results for $\mathcal{M}_g$.

Up to now this means $g \leq 16$: the uniruledness of $\mathcal{M}_g$ is known for $g = 16$, after [CR3] and [BDPP]: see [F]. The rational connectedness is known for $g = 15$ [BrV], and the unirationality for $g \leq 14$, [S, CR1, Ser1, Ve1] for $g < 14$. We will mainly describe the constructions given in [Ve1] in order to prove the unirationality for $g \leq 14$. See also Schreyer's paper [Sc], where the methods in use are improved by the support of computational packages.

Let us review the program we intend to follow, even if it is not so surprising after reading the previous sections.

- Actually we are going to sketch the unirationality of some universal Brill Noether loci $\mathcal{W}^r_{d,g}$ dominating $\mathcal{M}_g$.
- We will use families of *canonical complete intersection surfaces* $S \subset \mathbf{P}^r$. The list of their types is short: (5), (3,3), (2,4), (2,2,3), (2,2,2,2).
- Let $C \subset S \subset \mathbf{P}^r$ be a smooth integral curve of genus g. We will study the possible cases where there exists *a linkage of C to a curve B*:

  $$S \cdot F = C \cup B,$$

  so that F is a hypersurface and B is a smooth integral of genus $p < g$. This will be useful to parametrize $\mathcal{M}_g$ by a family of curves of lower genus.

Now let $V \subset H^0(\mathcal{O}_B(1))$ be the space defining the embedding $B \subset \mathbf{P}^r$ and let $\mathcal{G}$ be the moduli space of fourtuples (B, V, S, F). Then the assignment $(B, V, S, F) \longrightarrow (C, \mathcal{O}_C(1))$, determined by $S \cdot F$, defines a rational map

$$\psi : \mathcal{G} \to \mathcal{W}^r_{d,g}.$$

On the other hand let $b = \deg B$, then we have the rational map

$$\phi : \mathcal{G} \to Pic_{b,p}$$

induced by the assignment $(B, V, S, F) \to (B' \mathcal{O}_B 1))$.

In the effective situations to be considered ψ is dominant and $\mathcal{G}$ birational to $Pic_{b,p} \times \mathbf{P}^n$ for some n. Moreover, for $p \leq 9$, $Pic_{b,p}$ is unirational, therefore $\mathcal{G}$ and also $\mathcal{W}^r_{d,g}$ are unirational. An outcome of this program is represented by the next theorem [Ve1].

Theorem 6.1

(1) *genus 14:* $\mathcal{W}^1_{8,14}$ *is birational to* $Pic_{14,8} \times \mathbf{P}^{10}$,
(2) *genus 13:* $\mathcal{W}^2_{11,13}$ *is dominated by* $Pic_{12,8} \times \mathbf{P}^8$,
(3) *genus 12:* $\mathcal{W}^0_{5,12}$ *is birational to* $Pic_{15,9} \times \mathbf{P}^5$,
(4) *genus 11:* $\mathcal{W}^0_{6,11}$ *is birational to* $Pic_{13,9} \times \mathbf{P}^3$.

The previous Brill-Noether loci dominate their corresponding spaces $\mathcal{M}_g$ via the forgetful map. So the next corollary is immediate.

Corollary 6.2 *$\mathcal{M}_g$ is unirational for $g = 11, 12, 13, 14$.*

In the next sections we briefly outline the required ad hoc constructions for the proofs of these results.

6.2 The Case of Genus 14

Let (C, L) be a general pair such that $g = 14$ and $L \in W^1_8(C)$. It is easy to see that $\omega_C(-L)$ is very ample and defines an embedding $C \subset \mathbf{P}^6$.

Proposition 6.3 *Let $\mathcal{I}_C$ be the ideal sheaf of C:*

- *C is projectively normal, in particular $h^0(\mathcal{I}_C(2)) = 5$.*
- *A smooth complete intersection of 4 quadrics contains C.*
- *C is linked to a projectively normal integral curve B of genus 8 by a complete intersection of 5 quadrics.*
- *$B \cup C$ is a nodal curve.*

See [Ve1, Sect. 4]. In particular B is smooth of degree 14. Let $\mathcal{H}$ be the Hilbert scheme of B in $\mathbf{P}^6$. The previous properties are satisfied in an irreducible neighborhood $\mathcal{U} \subset \mathcal{H}$ of B which is $Aut\ \mathbf{P}^6$-invariant. Here we have that $\mathcal{U}/Aut\ \mathbf{P}^6$ is birational to $Pic_{14,8}$ via the natural moduli map.

Let $D \in \mathcal{U}$ and let $\mathcal{I}_D$ be its ideal sheaf. We can assume that D is projectively normal so that $h^0(\mathcal{I}_D(2)) = 7$. Then, over a non empty open set of $Pic_{14,8}$, let us consider the Grassmann bundle

$$\phi : \mathcal{G} \to Pic_{14,8}$$

with fibre $G(5, H^0(\mathcal{I}_D(2)))$ at the moduli point of $(D, \mathcal{O}_D(1))$. Note that, counting dimensions, $\dim\ \mathcal{G} = \dim\ \mathcal{W}^1_{8,14} = \dim\ \mathcal{M}_{14}$. Using the linkage of C and B we can finally define a rational map

$$\psi : \mathcal{G} \to \mathcal{W}^1_{8,14}.$$

Let $(D, \mathcal{O}_D(1), V)$ be a triple defining a general point $x \in \mathcal{G}$. Then V is a general subspace of dimension 5 in $H^0(\mathcal{I}_D(2))$. Let $|V|$ be the linear system of quadrics defined by V. Then its base scheme is a nodal curve $D \cup C'$ such that the pair $(C', \omega_{C'}(-1))$ defines a point $y \in \mathcal{W}^1_{8,14}$. This follows because $\mathcal{G}$ contains the moduli point of the triple $(B, \mathcal{O}_B(1), H^0(\mathcal{I}_C(2)))$ and the same open property holds for the pair $(C, \omega_C(-1))$ defined by this triple. Then we set by definition $\psi(x) = y$. The map ψ is clearly invertible: to construct ψ^{-1} just observe that $V = H^0(\mathcal{I}_{C'}(2))$. We conclude that

$$\mathcal{W}^1_{8,14} \cong Pic_{14,8} \times G(5,7).$$

Remark 6.4 (Plane Octics of Genus 8 with Two Triple Points) To guarantee the construction one has to prove that a general curve B is generated by quadrics and projectively normal. The proof works well by a computational package or else geometrically, [Sc] and [Ve1]. The geometric way in [Ve1] relies on a family of rational surfaces $X \subset \mathbf{P}^6$ containing smooth curves B of genus 8 with the required properties. Studying more properties of the curves B of this family has some interest. We study such a family here, so going back to singular plane curves. Indeed these curves B admit a simple plane model of degree 8 of special type and are general in moduli.

A general X is a projectively normal surface of degree 10 and sectional genus 5. Moreover it is the blowing up $\sigma : X \to \mathbf{P}^2$ of 11 general points. Let $L_1 \dots L_5, E_1 \dots E_6$ be the exceptional lines of σ, $P \in |\sigma^* \mathcal{O}_{\mathbf{P}^2}(1)|$, $H \in |\mathcal{O}_X(1)|$. It turns out that

$$|H| := |6P - 2(L_1 + \cdots + L_5) - (E_1 + \cdots + E_6)|$$

and that $|2P - L_1 - L_2 - E_1 - E_2|$ is a base point free pencil of rational normal sextics R. The curves B are then elements of $|2H - R|$ that is

$$|10P - 3(L_1 + L_2) - 4(L_3 + L_4 + L_5) - (E_1 + E_2) - 2(E_3 + \cdots + E_6)|,$$

see [Ve1]. Notice that in $\mathbf{P}^2$ the curve $\sigma(B)$ has three 4-tuples points $f_i = \sigma(L_i)$, $(i = 3, 4, 5)$, two triple points $t_j = \sigma(L_j)$, $(j = 1, 2)$, and four nodes $n_k = \sigma(E_k)$, $(k = 3, 4, 5, 6)$. Then consider the quadratic transformation

$$q : \mathbf{P}^2 \to \mathbf{P}^2,$$

centered at f_3, f_4, f_5 and the strict transform Γ of $\sigma(B)$. Then Γ is a general octic curve with seven nodes and two triple points. Let us see that the family of curves Γ dominates $\mathcal{M}_8$ and admits the following description.

In the quadric $\mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^3$ consider the Severi variety $\mathcal{V}$ of nodal integral curves Γ' of type $(5, 5)$ and genus 8. Note that a general curve Γ is obtained by projecting some Γ' from one of its nodes. Moreover a general curve D of genus 8 is birational to some Γ': to see this just take two distinct $L_1, L_2 \in W^1_5(D)$. These define a generically injective morphism $D \to \mathbf{P}^1 \times \mathbf{P}^1$ whose image belongs to $\mathcal{V}$.

6.3 Genus 11, 12, 13

We give a brief description of the constructions used in genus 11, 12 and 13, see [Sc, Ve1] for details.

- *Genus 11.*

  Let (C, L) be a pair defining a general point of $\mathcal{W}^0_{6,11}$, then $\omega_C(-L)$ is a very ample line bundle. It defines an embedding $C \subset \mathbf{P}^4$ as a projectively normal curve of degree 14. The analogy to the situation described in genus 14 is complete. Let us summarize:

C is linked to a smooth integral curve B of degree 13 and genus 9 by a complete intersection of 3 cubics. B is projectively normal and $B \cup C$ is nodal. Let $\mathcal{H}$ be the Hilbert scheme of B. Then B has an irreducible neighborhood $\mathcal{U} \subset \mathcal{H}$ such that $\mathcal{U}/Aut\, \mathbf{P}^4$ is birational to $Pic_{13,9}$.

Let $p \in Pic_{13,9}$ be a general point representing the pair (D, L), then D is embedded in $\mathbf{P}^4$ by L. Let $\mathcal{I}_D$ be its ideal sheaf then $h^0(\mathcal{I}_D(3)) = 4$. Therefore we have a $\mathbf{P}^3$-bundle $\phi : \mathcal{G} \to Pic_{13,9}$ with fibre $|\mathcal{I}_D(3)|^*$ at p. As in genus 14 we can use linkage of curves to define a map

$$\psi : \mathcal{G} \to \mathcal{W}^0_{6,11}.$$

With the same arguments as in genus 14, we show that ψ is birational.

- *Genus 12*

In this case we can go back to nodal curves $C \cup B$ which are complete intersection of five quadrics in $\mathbf{P}^6$. The situation does not change:

C is general of degree 17 and genus 12 in $\mathbf{P}^6$ and the pair $(C, \omega_C(-1))$ defines a point of $\mathcal{W}^0_{5,12}$. B is general of degree 15 and genus 9 in $\mathbf{P}^6$ and $(B, \mathcal{O}_B(1))$ defines a point $p \in Pic_{15,9}$. We have $h^0(\mathcal{I}_B(2)) = 6$. Then we have a $\mathbf{P}^5$-bundle $\phi : \mathcal{G} \to Pic_{15,9}$ with fibre $|\mathcal{I}_B(2)|^*$ at p. As in the previous cases the linkage of B and C defines a birational map $\psi : \mathcal{G} \to \mathcal{W}^0_{5,12}$.

- *Genus 13*

Here we need a different construction. We consider the Severi variety $\mathcal{V}_{11,13}$ of plane curves of degree 11 and genus 13. $\mathcal{V}_{11,13}$ dominates the universal Brill-Noether locus $\mathcal{W}^2_{11,13}$ and this dominates $\mathcal{M}_{13}$. Let

$$\tilde{\mathcal{W}}^2_{11,13}$$

be the moduli space of triples (C, L, n) such that $[C, L] \in \mathcal{W}^2_{11,13}$, $\Gamma := f_L(C)$ is nodal and $f_L(n) \in Sing\, \Gamma$. Now we construct a birational map

$$\psi : Pic_{12,8} \times \mathbf{P}^{12} \to \tilde{\mathcal{W}}^2_{11,13}.$$

The existence of ψ implies the unirationality of $\mathcal{W}_{11,13}$ and of $\mathcal{M}_{13}$.

We start with triples (B, M, o) such that (B, M) defines a general point of $Pic_{12,8}$ and $o \in \mathbf{P}^4 := \mathbf{P}H^0(M)^*$ is general. We denote their moduli space by $\mathcal{P}$. Clearly $\mathcal{P}$ is birational to a $\mathbf{P}^4$-bundle on $Pic_{12,8}$. Let x be the moduli point of (B, M, o), if x is general we can assume that B is embedded by M in $\mathbf{P}^4$ as a projectively normal curve and that B is generated by cubics.

We have $h^0(\mathcal{I}_B(3)) = 6$. Then, for a general triple (B, M, o), one can show that there exists a unique cubic $F_o \in |\mathcal{I}_B(3)|$ such that $Sing\, F_o = \{o\}$ and o is a node for it. Let $\mathcal{J}_x$ be the ideal sheaf of $B \cup \{o\}$ in F_o, we consider the Grassmannian $\mathcal{G}_x := G(2, H^0(\mathcal{J}_x(3))$ and then the Grassmann bundle

$$\phi : \mathcal{G} \to \mathcal{P}$$

with fibre $\mathcal{G}_x$ at x. Since $\dim \mathcal{G}_x = 8$ and $\mathcal{P} \cong Pic_{12,8} \times \mathbf{P}^4$, it follows that $\mathcal{G}$ is birational to $Pic_{12,8} \times \mathbf{P}^{12}$. One defines a rational map

$$\psi : \mathcal{G} \to \tilde{\mathcal{W}}^2_{11,13}$$

as follows. A general $p \in \mathcal{G}_x \subset \mathcal{G}$ defines a general pencil P of cubic sections of F_o through $B \cup \{o\}$. One can show that the base scheme of P is a nodal curve $B \cup C_n$ where C_n is integral of genus 13 and degree 15, singular exactly at o. Let $\nu : C \to C_n$ be the normalization map and let $n = \nu^* o$. Putting

$$L := \omega_C(n) \otimes \nu^* \mathcal{O}_{C_n}(-1)$$

one can check that $L \in W^2_{11}(C)$ and that $h^0(L(-n)) = 2$. Hence the triple (C, L, n) defines a point $y \in \tilde{\mathcal{W}}_{11,13}$ and we set $\psi(p) = y$. Let us just mention how to invert ψ: starting from (C, L, n) one reconstructs the curve $C_n \subset \mathbf{P}^4$ from the line bundle $\omega_C(n) \otimes L^{-1}$. Let $\mathcal{I}_{C_n}$ be its ideal sheaf, it turns out that $h^0(\mathcal{I}_{C_n}(3)) = 3$. The base scheme of $|\mathcal{I}_{C_n}(3)|$ is $B \cup C_n$.

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