Chapter 1
Introduction

Lieb–Robinson bounds are upper-bounds on time-dependent commutators and were originally used to estimate propagation velocities of information in quantum spin systems. They have first been derived in 1972 by Lieb and Robinson [LR]. Nowadays, they are widely used in quantum information and condensed matter physics. Phenomenological consequences of Lieb–Robinson bounds have been experimentally observed in recent years, see [Ch].

For the reader's convenience and completeness, we start by deriving such bounds for fermions on the lattice with (possibly non-autonomous) interactions. As explained in [NS] in the context of quantum spin systems, Lieb–Robinson bounds are only expected to hold true for systems with short-range interactions. We thus define Banach spaces $\mathcal{W}$ of short-range interactions and prove Lieb–Robinson bounds for the corresponding fermion systems. The spaces $\mathcal{W}$ include density–density interactions resulting from the second quantization of two-body interactions defined via a real-valued and integrable interaction kernel $v(r) : [0, \infty) \to \mathbb{R}$. Considering fermions with spin $1/2$, our setting includes, for instance, the celebrated Hubbard model (and any other system with finite-range interactions) or models with Yukawa-type potentials. Two-body interactions decaying polynomially fast in space with sufficiently large degree are also allowed, but the Coulomb potential is excluded because it is not summable at large distances. The method of proof we use to get Lieb–Robinson bounds for non-autonomous C^*-dynamical systems related to lattice fermions is, up to simple adaptations, the one used in [NS] for (autonomous) quantum spin systems. Compare Theorem 4.3, Lemma 4.4, Theorem 5.1 and Corollary 5.2 with [NS, Theorems 2.3. and 3.1.]. See also [BMNS] where (usual) Lieb–Robinson bounds for non-autonomous quantum spin systems have already been derived [BMNS, Theorems 4.6].

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Once the Lieb–Robinson bounds for commutators are established, we combine them with results of the theory of strongly continuous semigroups to derive properties of the infinite-volume dynamics. These allow us to extend Lieb–Robinson bounds to time-dependent *multi*-commutators, see Theorems 4.10, 4.11 and 5.4. The new bounds on multi-commutators make possible rigorous studies of dynamical properties that are relevant for response theory of interacting fermion systems. For instance, they yield tree-decay bounds in the sense of [BPH1, Sect. 4] if interactions decay sufficiently fast in space (typically some polynomial decay with large enough degree is needed). In fact, by using the Lieb–Robinson bounds for multi-commutators, we extend in [BP5, BP6] our results [BPH1, BPH2, BPH3, BPH4] on free fermions to interacting particles with short-range interactions. This is an important application of such new bounds: The rigorous microscopic derivation of Ohm and Joule's laws for *interacting* fermions, in the AC-regime. See Chap. 6 and [BP4] for a historical perspective on this subject.

Via Theorems 6.1 and 6.5, we show, for example, how Lieb–Robinson bounds for multi-commutators can be applied to derive decay properties of the so-called *AC-conductivity measure* at high frequencies. This result is new and is obtained in Chap. 6. Cf. [BP5, BP6]. Lieb–Robinson bounds for multi-commutators have, moreover, further applications which go beyond the use on linear response theory presented in Chap. 6. For instance, as explained in Sects. 4.5 and 5.3, they also make possible the study of *non-linear* corrections to linear responses to external perturbations.

The new bounds can also be applied to *non-autonomous systems*. Indeed, the existence of a fundamental solution for the non-autonomous initial value problem related to infinite systems of fermions with time-dependent interactions is usually a non-trivial problem because the corresponding generators are time-dependent unbounded operators. The time-dependency cannot, in general, be isolated into a bounded perturbation around some unbounded time-constant generator and usual perturbation theory cannot be applied. In many important cases, the time-dependent part of the generator is not even relatively bounded with respect to (w.r.t.) the constant part. In fact, no unified theory of non-autonomous evolution equations that gives a complete characterization of the existence of fundamental solutions in terms of properties of generators, analogously to the Hille–Yosida generation theorems for the autonomous case, is available. See, e.g., [K4, C, S, P, BB] and references therein. Note that the existence of a fundamental solution implies the well-posedness of the initial value problem related to states or observables of interacting lattice fermions, provided the corresponding evolution equation has a unique solution for any initial condition.

The Lieb–Robinson bounds on multi-commutators we derive here yield the existence of fundamental solutions as well as other general results on non-autonomous initial value problems related to fermion systems on lattices with interactions which are non-vanishing in the whole space and time-dependent. This is done in a rather constructive way, by considering the large volume limit of finite-volume dynamics, without using standard sufficient conditions for existence of fundamental solutions of non-autonomous linear evolution equations. If interactions decay exponentially fast in space, then we moreover show, also by using Lieb–Robinson bounds on multi-commutators, that the *non-autonomous* dynamics is smooth w.r.t. its generator on the dense set of local observables. See Theorem 5.6. Note that the generator of the (non-autonomous) dynamics generally has, in our case, a time-dependent domain, and the existence of a dense set of smooth vectors is a priori not at all clear.

Observe that the evolution equations for lattice fermions are not of parabolic type, in the precise sense formulated in [AT], because the corresponding generators do not generate analytic semigroups. They seem to be rather related to Kato's hyperbolic case [K2, K3, K4]. Indeed, by structural reasons – more precisely, the fact that the generators are derivations on a C^*-algebra – the time-dependent generator defines a stable family of operators in the sense of Kato. Moreover, this family always possesses a common core. In some specific situations one can directly show that the completion of this core w.r.t. a conveniently chosen norm defines a so-called admissible Banach space $\mathcal{Y}$ of the generator at any time, which satisfies further technical conditions leading to Kato's hyperbolic conditions [K2, K3, K4]. See also [BB, Sect. 5.3.] and [P, Sect.VII.1]. Nevertheless, the existence of such a Banach space $\mathcal{Y}$ is a priori unclear in the general case treated here (Theorem 5.5).

Our central results are Theorems 4.10, 4.11 and 5.4. Other important assertions are Corollary 4.12 and Theorems 5.5, 5.6, 5.8, 5.9, 6.1 and 6.5. The manuscript is organized as follows:

- In order to make our results accessible to a wide audience, in particular to students in Mathematics with little Physics background, Chap. 2 presents basics of Quantum Mechanics, keeping in mind its algebraic formulation.
- Chapter 3 introduces the algebraic setting for fermions, in particular the CAR C^*-algebra. Other standard objects (like fermions, bosons, Fock space, CAR, etc.) of quantum theory are also presented, for pedagogical reasons.
- Chapter 4 is devoted to Lieb–Robinson bounds, which are generalized to multi-commutators. We also give a proof of the existence of the infinite-volume dynamics as well as some applications of such bounds. The tree-decay bounds on time-dependent multi-commutators (Corollary 4.12) are proven here. However, only the autonomous dynamics is considered in this section.
- Chapter 5 extends results of Chap. 4 to the non-autonomous case. We prove, in particular, the existence of a fundamental solution for the non-autonomous initial value problems related to infinite interacting systems of fermions on lattices with time-dependent interactions (Theorem 5.5). This implies well-posedness of the corresponding initial value problems for states and observables, provided their solutions are unique for any initial condition. Applications in (possibly non-linear) response theory (Theorems 5.8, 5.9) are discussed as well.
- Finally, Chap. 6 explains how Lieb–Robinson bounds for multi-commutators can be applied to study (quantum) charged transport properties within the AC-regime. This analysis yields, in particular, the asymptotics at high frequencies of the so-called AC-conductivity measure. See Theorems 6.1 and 6.5.

Notation 1.1

(i) *We denote by D any positive and finite generic constant. These constants do not need to be the same from one statement to another.*

(ii) *A norm on the generic vector space* $\mathcal{X}$ *is denoted by* $\|\cdot\|_{\mathcal{X}}$ *and the identity map of* $\mathcal{X}$ *by* $\mathbf{1}_{\mathcal{X}}$. *The* C^**-algebra of all bounded linear operators on* $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ *is denoted by* $\mathcal{B}(\mathcal{X})$. *The scalar product on a Hilbert space* $\mathcal{X}$ *is denoted by* $\langle\cdot,\cdot\rangle_{\mathcal{X}}$.
(iii) *If* O *is an operator,* $\|\cdot\|_O$ *stands for the graph norm on its domain.*
(iv) *By a slight abuse of notation, we denote in the sequel elements* $X_i \in Y$ *depending on the index* $i \in I$ *by expressions of the form* $\{X_i\}_{i\in I} \subset Y$ *(instead of* $(X_i)_{i\in I} \subset I \times Y$*).*