Chapter 1
Introduction

The behavior of incompressible fluids is modeled with the system of the incompressible Navier–Stokes equations. These equations describe the conservation of linear momentum and the conservation of mass. In the special case of a steady-state situation and large viscosity of the fluid, the Navier–Stokes equations can be approximated by the Stokes equations. Incompressible flow problems are not only of interest by themselves, but they are part of many complex models for describing phenomena in nature or processes in engineering and industry.

Usually it is not possible to find an analytic solution of the Stokes or Navier–Stokes equations such that numerical methods have to be employed for approximating the solution. To this end, a so-called discretization has to be applied to the equations, in the general case a temporal and a spatial discretization. Concerning the spatial discretization, this monograph considers finite element methods. Finite element methods are very popular and they are understood quite well from the mathematical point of view.

First applications of finite element methods for the simulation of the Stokes and Navier–Stokes equations were performed in the 1970s. Also the finite element analysis for these equations started in this decade, e.g., by introducing in Babuška (1971) and Brezzi (1974) the inf-sup condition which is a basis of the well-posedness of the continuous as well as of the finite element problem. The early works on the finite element analysis cumulated in the monograph (Girault and Raviart 1979). The extended version of this monograph, Girault and Raviart (1986), became the classical reference work. Three decades have been passed since its publication. Of course, in this time, there were many new developments and new results have been obtained. More recent monographs that study finite element methods for incompressible flow problems, or important aspects of this topic, include Layton (2008), Boffi et al. (2008), Elman et al. (2014).

This monograph covers on the one hand a wide scope, from the derivation of the Navier–Stokes equations to the simulation of turbulent flows. On the other hand, there are many topics whose detailed presentation would amount in a monograph

V. John, *Finite Element Methods for Incompressible Flow Problems*, Springer Series in Computational Mathematics 51, DOI 10.1007/978-3-319-45750-5_1

itself and they are only sketched here. The main emphasis of the current monograph is on mathematical issues. Besides many results for finite element methods, also a few fundamental results concerning the continuous equations are presented in detail, since a basic understanding of the analysis of the continuous problem provides a better understanding of the considered problem in its entirety.

A main feature of this monograph is the detailed presentation of the mathematical tools and of most of the proofs. This feature arose from the experience in sometimes spending (wasting) a lot of time in understanding certain steps in proofs that are written in the short form which is usual in the literature. Often, such steps would have been easy to understand if there would have been just an additional hint or one more line in the estimate. Thus, the presentation is mostly self-contained in the way that no other literature has to be used for understanding the majority of the mathematical results. Altogether, the monograph is directed to a broad audience: experienced researchers on this topic, young researchers, and master students. The latter point was successfully verified. Most parts of this monograph were presented in master courses held at the Free University of Berlin, in particular from 2013–2015. As a result, several master's theses were written on topics related to these courses.

1.1 Contents of this Monograph

Chapter 2 sketches the derivation of the incompressible Navier–Stokes equations on the basis of the conservation of mass and the conservation of linear momentum. Important properties of the stress tensor are derived from physical considerations. The non-dimensionalized equations are introduced and appropriate boundary conditions are discussed.

The following structure of this monograph is based on the inherent difficulties of the incompressible Navier–Stokes equations pointed out in Chap. 2.

- First, the coupling of velocity and pressure is studied:
  - Chapter 3 presents an abstract theory and discusses the choice of appropriate finite element spaces.
  - Chapter 4 applies the abstract theory to the Stokes equations.
- Second, the issue of dominant convection is also taken into account:
  - Chapter 5 studies this topic for the Oseen equations, which are a kind of linearized Navier–Stokes equations.
- Third, the nonlinearity of the Navier–Stokes equations is considered in addition to the other two difficulties:
  - Chapter 6 studies stationary flows that occur only for small Reynolds numbers.
  - Chapter 7 considers laminar flows that arise for medium Reynolds numbers.
  - Chapter 8 studies turbulent flows that occur for large Reynolds numbers.

The coupling of velocity and pressure in incompressible flow problems does not allow the straightforward use of arbitrary pairs of finite element spaces. For obtaining a well-posed problem, the spaces have to satisfy the so-called discrete inf-sup condition. This condition is derived in Chap. 3. The derivation is based on the study of the well-posedness of an abstract linear saddle point problem. The abstract theory is applied first to a continuous linear incompressible flow problem, thereby identifying appropriate function spaces for velocity and pressure. These spaces satisfy the so-called inf-sup condition. Then, it is discussed that the satisfaction of the inf-sup condition does not automatically lead to the satisfaction of the discrete inf-sup condition. Examples of velocity and pressure finite element spaces that do not satisfy this condition are given. Next, some techniques for proving the discrete inf-sup condition are presented and important inf-sup stable pairs of finite element spaces are introduced. For some pairs, the proof of the discrete inf-sup condition is presented. In addition, a way for computing the discrete inf-sup constant is described. The final section of this chapter discusses the Helmholtz decomposition.

Chapter 4 applies the theory developed in the previous chapter to the Stokes equations. The Stokes equations, being a system of linear equations, are the simplest model of incompressible flows, modeling only the flow caused by viscous forces. First, the existence, uniqueness, and stability of a weak solution is discussed. The next section presents results from the finite element error analysis. Conforming finite element methods are considered in the first part of this section and a low order non-conforming finite element discretization is studied in the second part. Some remarks concerning the implementation of the finite element methods are given. Next, a basic introduction to a posteriori error estimation is presented and its application for adaptive mesh refinement is sketched. It follows a presentation of methods that allow to circumvent the discrete inf-sup condition. Such methods enable the usage of the same finite element spaces with respect to the piecewise polynomials for velocity and pressure, which is appealing from the practical point of view. A detailed numerical analysis of one of these methods, the PSPG method, is provided and a couple of other methods are discussed briefly. Finite element methods satisfy in general the conservation of mass only approximately. This chapter concludes with a survey of methods that reduce the violation of mass conservation or even guarantee its conservation.

The Oseen equations, i.e., a linear equation with viscous (second order), convective (first order), and reactive (zeroth order) term are the topic of Chap. 5. These equations arise in various numerical methods for solving the Navier–Stokes equations. Usually, the convective or the reactive term dominate the viscous term. A major issue in the analysis consists in tracking the dependency of the stability and error bounds on the coefficients of the problem. After having established the existence and uniqueness of a solution of the equations, the standard Galerkin finite element method is studied. It turns out that the stability and error bounds become large if convection or reaction dominates. Numerical studies support this statement. For improving the numerical solutions, stabilized methods have to be applied. The analysis of a residual-based stabilized method, the SUPG/PSPG/grad-div method, is presented in detail and some further stabilized methods are reviewed briefly.

In Chap. 6, the first nonlinear model of an incompressible flow problem is studied—the steady-state Navier–Stokes equations. At the beginning of this chapter, the nonlinear term is investigated. Different forms of this term are introduced and various properties are derived. Then, the solution of the continuous steady-state Navier–Stokes equations is studied. It turns out that a unique solution can be expected only for sufficiently small external forces, which do not depend on time, and sufficiently large viscosity. For this situation, a finite element error analysis is presented, with the emphasis on bounding the nonlinear term. Next, iterative methods for solving the nonlinear problem are discussed. The final section of this chapter presents the Dual Weighted Residual (DWR) method. This method is an approach for the a posteriori error estimation with respect to quantities of interest.

Chapter 7 starts with the investigation of the time-dependent incompressible Navier–Stokes equations. From the point of view of finite element discretizations, so-called laminar flows are considered, i.e., flows where a standard Galerkin finite element method is applicable. At the beginning of this chapter, a short introduction into the analysis concerning the existence and uniqueness of a weak solution of the time-dependent incompressible Navier–Stokes equations is given. In particular, the mathematical reason is highlighted that prevents to prove the uniqueness in the practically relevant three-dimensional case. Then, the finite element error analysis for the Galerkin finite element method in the so-called continuous-in-time case is presented, i.e., without the consideration of a discretization with respect to time. For practical simulations, a temporal discretization has to be applied. The next part of this chapter introduces a number of time stepping schemes that require the solution of a coupled velocity-pressure problem in each discrete time. In particular, θ-schemes are discussed in detail. It follows the presentation of a finite element error analysis for the fully discretized equations at the example of the backward Euler scheme. The approaches presented so far in this chapter require the solution of saddle point problems, which might be computational expensive. Projection methods, which circumvent the solution of such problems, are presented in the last section of this chapter. In these methods, only scalar equations for each component of the velocity field and for the pressure have to be solved.

The topic of Chap. 8 is the simulation of turbulent flows. There is no mathematical definition of what is a turbulent flow. Thus, this chapter starts with a description of characteristics of flow fields that are considered to be turbulent. In addition, a mathematical approach for describing turbulence is sketched. It turns out that turbulent flows possess scales that are much too small to be representable on grids with affordable fineness. The impact of these scales on the resolvable scales has to be modeled with a so-called turbulence model. The bulk of this chapter presents turbulence models that allow mathematical or numerical analysis or whose derivation is based on mathematical arguments. A very popular approach for turbulence modeling is large eddy simulation (LES). LES aims at simulating only large (resolved) scales that are defined by spatial averaging. In the first section on LES, the derivation of equations for these scales is discussed, in particular the underlying assumption of commuting differentiation and spatial averaging. It is shown that usually commutation errors occur that are not negligible. The next section presents the most popular LES model, the Smagorinsky model. For the Smagorinsky model, a well developed mathematical and numerical analysis is available. Then, LES models are described that are derived on the basis of approximations in wave number space. The final section on LES considers so-called Approximate Deconvolution models (ADM). As next turbulence model, the Leray-α model is presented. This model is based on a regularization of the velocity field. Afterward, the Navier–Stokes-α model is considered. Its derivation is performed by studying a Lagrangian functional and the corresponding trajectory. The last class of turbulence models that is discussed is the class of Variational Multiscale (VMS) methods. VMS methods define the large scales, which should be simulated, in a different way than LES models, namely by projections in appropriate function spaces. Two principal types of VMS methods can be distinguished, those based on a two-scale decomposition and those using a three-scale decomposition of the flow field. Five different realizations of VMS methods are described in detail. The final section of Chap. 8 presents a few numerical studies of turbulent flow simulations using the Smagorinsky model and one representative of the VMS models.

The linearization and discretization of the incompressible Navier–Stokes equations results for many methods in coupled algebraic systems for velocity and pressure. Chapter 9 gives a brief introduction into solvers for such equations. One can distinguish between sparse direct solvers and iterative solvers, where the latter solvers have to be used with appropriate preconditioners. Some emphasis in the presentation is on the preconditioner that was used for simulating most of the numerical examples presented in this monograph, namely a coupled multigrid method.

Appendix A provides some basic notations from functional analysis. A number of inequalities and theorems are given that are used in the analysis and numerical analysis presented in this monograph. Some basics of the finite element method are provided in Appendix B. In particular, those finite element spaces are described in some detail that are used for discretizing incompressible flow problems. The approximation of functions from Sobolev spaces with finite element functions by interpolation or projection is the topic of Appendix C. The corresponding estimates are heavily used in the finite element error analysis. Finally, Appendix D describes a number of examples for numerical simulations, which are divided into three groups:

- examples for steady-state flow problems,
- examples for laminar time-dependent flow problems,
- examples for turbulent flow problems.

The described examples were utilized for performing numerical simulations whose results are presented in this monograph.

The master courses held at the Free University of Berlin covered the following topics:

- Course 1: Chaps. 2, and 3, Sect. 4.1–4.3,
- Course 2: Sects. 4.4–4.6, Chaps. 5–7, and 9,
- Course 3: Chap. 8.

Of course, the presentation in these courses concentrated on the most important issues of each topic.