An Equivalent Definition of Pan-Integral

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Abstract. In this note, we introduce the concepts of support disjointness super-$\oplus$-additivity and positively super-$\otimes$-homogeneity of a functional (with respect to pan-addition $\oplus$ and pan-multiplication $\otimes$, respectively). By means of these two properties of functionals, we discuss the characteristics of pan-integrals and present an equivalent definition of the pan-integral. As special cases, we obtain the equivalent definitions of the Shilkret integral, the $+, \cdot$-based pan-integral, and the Sugeno integral.

Keywords: Pan-integral · Sugeno integral · Shilkret integral · Support disjointness super-$\oplus$-additivity · Positively super-$\otimes$-homogeneity

1 Introduction

In non-additive measure theory, several prominent nonlinear integrals, for example, the Choquet integral [3] and the Sugeno [12] integral, have been defined and discussed in detail [4,10,16].

As a generalization of the Legesgue integral and Sugeno integral, Yang [17] introduced the pan-integral with respect to a monotone measure and a commutative isotonic semiring $(\overline{R}_+, \oplus, \otimes)$, where $\oplus$ is a pan-addition and $\otimes$ a pan-multiplication [16,17]. The researches on this topic can be also found in [1,5,8–10,13,18].

On the other hand, Lehrer introduced a new kind of nonlinear integral — the concave integral with respect to a capacity, see [6,7,14]. Let $(X, \mathcal{A})$ be a measurable space and $\mathcal{F}_+$ denote the class of all finite nonnegative real-valued measurable functions on $(X, \mathcal{A})$. For fixed capacity ν, the concave integral with respect to ν is a concave and positively homogeneous nonnegative functional on $\mathcal{F}_+$. Observe that such integral was defined as the infimum taken over all concave and positively homogeneous nonnegative functionals H defined on $\mathcal{F}_+$ with the condition: $\forall A \in \mathcal{A}, H(\chi_A) \geq \mu(A)$.

Inspiration received from the definition of concave integral, we try to characterize the pan-integrals via functionals over $\mathcal{F}_+$ (with some additional restricts). We introduce the concepts of *support disjointness super-$\oplus$-additivity* and *positively super-$\otimes$-homogeneity* of a functional on $\mathcal{F}_+$ (with respect to pan-addition $\oplus$ and pan-multiplication $\otimes$, respectively). We will show the pan-integral, as

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a functional defined on $\mathcal{F}_+$, is support disjointness super-$\oplus$-additive and positively super-$\otimes$-homogeneous. We shall present an equivalent definition of the pan-integral by using monotone, support disjointness super-$\oplus$-additive and positively super-$\otimes$-homogeneous functionals on $\mathcal{F}_+$.

2 Preliminaries

Let X be a nonempty set and $\mathcal{A}$ a σ-algebra of subsets of X, $R_+ = [0, +\infty)$, $\overline{R}_+ = [0, +\infty]$. Recall that a set function $\mu : \mathcal{A} \to \overline{R}_+$ is a monotone measure, if it satisfies the following conditions:

(1) $\mu(\emptyset) = 0$ and $\mu(X) > 0$;

(2) $\mu(A) \le \mu(B)$ whenever $A \subset B$ and $A, B \in \mathcal{A}$.

In this paper we restrict our discussion on a fixed measurable space $(X, \mathcal{A})$. Unless stated otherwise all the subsets mentioned are supposed to belong to $\mathcal{A}$. Let $\mathcal{M}$ be the set of all monotone measures defined on $(X, \mathcal{A})$. When μ is a monotone measure, the triple $(X, \mathcal{A}, \mu)$ is called a monotone measure space [10,16].

The concept of a pan-integral involves two binary operations, the pan-addition $\oplus$ and pan-multiplication $\otimes$ of real numbers [16,17].

Definition 1. *An binary operation $\oplus$ on $\overline{R}_+$ is called a pan-addition if it satisfies the following requirements:*

(PA1) $a \oplus b = b \oplus a = a$ (commutativity);

(PA2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (associativity);

(PA3) $a \le c$ and $b \le d$ imply that $a \oplus b \le c \oplus d$ (monotonicity);

(PA4) $a \oplus 0 = a$ (neutral element);

(PA5) $a_n \to a$ and $b_n \to b$ imply that $a_n \oplus b_n \to a \oplus b$ (continuity).

Definition 2. *Let $\oplus$ be a given pan-addition on $\overline{R}_+$. A binary operation $\otimes$ on $\overline{R}_+$ is said to be a pan-multiplication corresponding to $\oplus$ if it satisfies the following properties:*

(PM1) $a \otimes b = b \otimes a$ (commutativity);

(PM2) $(a \otimes b) \otimes c = a \otimes (b \otimes c)$ (associativity);

(PM3) $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$ (distributive law);

(PM4) $a \le b$ implies $(a \otimes c) \le (b \otimes c)$ for any c (monotonicity);

(PM5) $a \otimes b = 0 \Leftrightarrow a = 0$ or $b = 0$ (annihilator);

(PM6) there exists $e \in [0, \infty]$ such that $e \otimes a = a$ for any $a \in [0, \infty]$ (neutral element);

(PM7) $a_n \to a \in [0, \infty)$ and $b_n \to b \in [0, \infty)$ imply $(a_n \otimes b_n) \to (a \otimes b)$ (continuity).

When $\oplus$ is a pseudo-addition on $\overline{R}_+$ and $\otimes$ is a pseudo-multiplication (with respect to $\oplus$) on $\overline{R}_+$, the triple $(\overline{R}_+, \oplus, \otimes)$ is called a *commutative isotonic semiring* (with respect to $\oplus$ and $\otimes$) [16].

Notice that similar operations called pseudo-addition and pseudo-multiplication can be found in the literature [1,2,5,8–10,13,15,18].

In the following, we recall the concept of *pan-integral* [16,17].

Definition 3. Consider a commutative isotonic semiring $(\overline{R}_+, \oplus, \otimes)$. Let $\mu \in \mathcal{M}$ and $f \in \mathcal{F}_+$. *The pan-integral of* f on X with respect to μ is defined via

$$\mathbf{I}_{pan}^{(\oplus,\otimes)}(\mu, f) = \sup\left\{\bigoplus_{i=1}^{n}\Big(\lambda_i \otimes \mu(A_i)\Big) : \bigoplus_{i=1}^{n}\Big(\lambda_i \otimes \chi_{A_i}\Big) \le f, \{A_i\}_{i=1}^{n} \in \mathcal{P}\right\},$$

where χ_A is the characteristic function of A which takes value e on A and 0 elsewhere, and $\mathcal{P}$ is the set of all finite partitions of X.

For $A \in \mathcal{A}$, the pan-integral of f on A is defined by $\mathbf{I}_{pan}^{(\oplus,\otimes)}(\mu, f \otimes \chi_A)$.

Note: A finite partition of X is a finite disjoint system of sets $\{A_i\}_{i=1}^{n} \subset \mathcal{A}$ such that $A_i \cap A_j = \emptyset$ for $i \ne j$ and $\cup_{i=1}^{n} A_i = X$.

Note that in the case of commutative isotonic semiring $(\overline{R}_+, \vee, \wedge)$, Sugeno integral [12] is recovered, while for $(\overline{R}_+, \vee, \cdot)$, Shilkret integral [11] is covered by the pan-integral in Definition 3.

Proposition 1. *Consider a commutative isotonic semiring* $(\overline{R}_+, \oplus, \otimes)$ *and fixed* $\mu \in \mathcal{M}$. *Then* $\mathbf{I}_{pan}^{(\oplus,\otimes)}(\mu, \cdot)$, *as a functional on* $\mathcal{F}_+$, *is monotone, i.e., for any* $f, g \in \mathcal{F}_+$,

$$f \le g \implies \mathbf{I}_{pan}^{(\oplus,\otimes)}(\mu, f) \le \mathbf{I}_{pan}^{(\oplus,\otimes)}(\mu, g).$$

Proposition 2. *For any* $A \in \mathcal{F}$, $\mathbf{I}_{pan}^{(\oplus,\otimes)}(\mu, \chi_A) \ge \mu(A)$.

3 Main Results

In this section we present an equivalent definition of pan-integral. In order to do it, we first introduce two new concepts and show two lemmas.

Definition 4. Consider a commutative isotonic semiring $(\overline{R}_+, \oplus, \otimes)$. A functional $F : \mathcal{F}_+ \to \overline{R}_+$ is said to be

(i) *positively super-*$\otimes$*-homogeneous,* if for any $f \in \mathcal{F}_+$ and any $a > 0$, we have

$$F(a \otimes f) \ge a \otimes F(f). \tag{1}$$

(ii) *support disjointness super-*$\oplus$*-additive,*if for any $f, g \in \mathcal{F}_+$, $supp(f) \cap supp(g) = \emptyset$, we have

$$F(f \oplus g) \ge F(f) \oplus F(g), \tag{2}$$

here $supp(f) = \{x \in X : f(x) > 0\}$ since we do not concern the topology.

Lemma 1. *Consider a commutative isotonic semiring* $(\overline{R}_+, \oplus, \otimes)$ *and fixed* $\mu \in \mathcal{M}$. *Then* $\mathbf{I}_{pan}^{(\oplus,\otimes)}(\mu, \cdot)$, *as a functional on* $\mathcal{F}_+$, *is positively super-*$\otimes$*-homogeneous, i.e., for any* $f \in \mathcal{F}_+$ *and any* $a > 0$, *we have*

$$\mathbf{I}_{pan}^{(\oplus,\otimes)}(\mu, a \otimes f) \geq a \otimes \mathbf{I}_{pan}^{(\oplus,\otimes)}(\mu, f). \tag{3}$$

Proof. For any finite partition $\{A_1, \ldots, A_n\}$ of X and $\{\lambda_1, \ldots, \lambda_n\} \subset R_+$ with $\bigoplus_{i=1}^n (\lambda_i \otimes \chi_{A_i}) \leq f$, we have that $\bigoplus_{i=1}^n \Big((a \otimes \lambda_i) \otimes \chi_{A_i}\Big) \leq a \otimes f$. Thus,

$$\begin{aligned}
&\mathbf{I}_{pan}^{(\oplus,\otimes)}(\mu, a \otimes f)\\
&= \sup\left\{\bigoplus_{j=1}^m \Big(\beta_j \otimes \mu(B_j)\Big) : \bigoplus_{j=1}^m \Big(\beta_j \otimes \chi_{B_j}\Big) \leq a \otimes f, \{B_j\}_{j=1}^m \in \mathcal{P}\right\}\\
&\geq \sup\left\{\bigoplus_{i=1}^n \Big((a \otimes \lambda_i) \otimes \mu(A_i)\Big) : \bigoplus_{i=1}^n \Big((a \otimes \lambda_i) \otimes \chi_{A_i}\Big) \leq a \otimes f, \{A_i\}_{i=1}^n \in \mathcal{P}\right\}\\
&= \sup\left\{a \otimes \bigoplus_{i=1}^n \Big(\lambda_i \otimes \mu(A_i)\Big) : a \otimes \bigoplus_{i=1}^n \Big(\lambda_i \otimes \chi_{A_i}\Big) \leq a \otimes f, \{A_i\}_{i=1}^n \in \mathcal{P}\right\}\\
&\geq a \otimes \sup\left\{\bigoplus_{i=1}^n \Big(\lambda_i \otimes \mu(A_i)\Big) : \bigoplus_{i=1}^n \Big(\lambda_i \otimes \chi_{A_i}\Big) \leq f, \{A_i\}_{i=1}^n \in \mathcal{P}\right\}\\
&= a \otimes \mathbf{I}_{pan}^{(\oplus,\otimes)}(\mu, f).
\end{aligned}$$

□

Remark 1. Notice that for the commutative isotonic semiring $(\overline{R}_+, \oplus, \cdot)$, i.e., $\otimes$ is the usual multiplication, then the related pan-integral is *positively homogeneous*, i.e.,

$$\mathbf{I}_{pan}^{(\oplus,\cdot)}(\mu, af) = a \cdot \mathbf{I}_{pan}^{(\oplus,\cdot)}(\mu, f).$$

In fact, by Lemma 1, $\mathbf{I}_{pan}^{(\oplus,\cdot)}(\mu, af) \geq a\ \mathbf{I}_{pan}^{(\oplus,\cdot)}(\mu, f)$. On the other hand, $\mathbf{I}_{pan}^{(\oplus,\cdot)}(\mu, f) = \mathbf{I}_{pan}^{(\oplus,\cdot)}(\mu, \frac{1}{a}(af)) \geq \frac{1}{a}\mathbf{I}_{pan}^{(\oplus,\cdot)}(\mu, af)$, which implies the reverse inequality and hence the equality holds.

Lemma 2. *Consider a commutative isotonic semiring* $(\overline{R}_+, \oplus, \otimes)$ *and fixed* $\mu \in \mathcal{M}$. *Then* $\mathbf{I}_{pan}^{(\oplus,\otimes)}(\mu, \cdot)$, *as a functional on* $\mathcal{F}_+$, *is support disjointness super-*$\oplus$*-additive, i.e., for any* $f, g \in \mathcal{F}_+$ *such that* $supp(f) \cap supp(g) = \emptyset$, *we have*

$$\mathbf{I}_{pan}^{(\oplus,\otimes)}(\mu, f \oplus g) \geq \mathbf{I}_{pan}^{(\oplus,\otimes)}(\mu, f) \oplus \mathbf{I}_{pan}^{(\oplus,\otimes)}(\mu, g). \tag{4}$$

Proof. If one of the two integrals on the right-hand side of Ineq. (4) is infinite then, by the monotonicity of the pan-integral, $\mathbf{I}_{pan}^{(\oplus,\otimes)}(\mu, f \oplus g)$ also equals to infinity, which implies that (4) holds.

So, without loss of generality, we can suppose that both $\mathbf{I}_{pan}^{(\oplus,\otimes)}(\mu, f)$ and $\mathbf{I}_{pan}^{(\oplus,\otimes)}(\mu, g)$ are finite. Let $l_n \nearrow \mathbf{I}_{pan}^{(\oplus,\otimes)}(\mu, f)$ and $r_n \nearrow \mathbf{I}_{pan}^{(\oplus,\otimes)}(\mu, f)$ be two sequences of real number. Then, for each n, there is a partition $\{A_i^{(n)}\}_{i=1}^{k_n}$ of

$supp(f)$, a partition $\{B_j^{(n)}\}_{j=1}^{m_n}$ of $supp(g)$, and two sequences of positive number $\{\alpha_i^{(n)}\}_{i=1}^{k_n}$ and $\{\beta_j^{(n)}\}_{j=1}^{m_n}$ such that $\bigoplus_{i=1}^{k_n}(\alpha_i^{(n)} \otimes \chi_{A_i^{(n)}}) \leq f$, $\bigoplus_{j=1}^{m_n}(\beta_j^{(n)} \otimes \chi_{B_j^{(n)}}) \leq g$ and both the following two inequalities hold

$$\bigoplus_{i=1}^{k_n}\left(\alpha_i^{(n)} \otimes \mu(A_i^{(n)})\right) \geq l_n, \qquad \bigoplus_{j=1}^{m_n}\left(\beta_j^{(n)} \otimes \mu(B_j^{(n)})\right) \geq r_n.$$

By the fact of $supp(f) \cap supp(g) = \emptyset$, we know that $\{A_i^{(n)}\}_{i=1}^{k_n} \cup \{B_j^{(n)}\}_{j=1}^{m_n}$ is a partition of $supp(f \oplus g)$. Moreover, we have that

$$\left(\bigoplus_{i=1}^{k_n}(\alpha_i^{(n)} \otimes \chi_{A_i^{(n)}})\right) \bigoplus \left(\bigoplus_{j=1}^{m_n}(\beta_j^{(n)} \otimes \chi_{B_j^{(n)}})\right) \leq f \oplus g,$$

and

$$\begin{aligned}\mathbf{I}_{pan}^{(\oplus,\otimes)}(\mu, f \oplus g) &\geq \left(\bigoplus_{i=1}^{k_n}\left(\alpha_i^{(n)} \otimes \mu(A_i^{(n)})\right)\right) \bigoplus \left(\bigoplus_{j=1}^{m_n}\left(\beta_j^{(n)} \otimes \mu(B_j^{(n)})\right)\right) \\ &\geq l_n \oplus r_n.\end{aligned}$$

Letting $n \to \infty$, by the continuity of pan-addition, we get that

$$\mathbf{I}_{pan}^{(\oplus,\otimes)}(\mu, f \oplus g) \geq \mathbf{I}_{pan}^{(\oplus,\otimes)}(\mu, f) \oplus \mathbf{I}_{pan}^{(\oplus,\otimes)}(\mu, g).$$

The proof is complete. □

Consider a commutative isotonic semiring $(\overline{R}_+, \oplus, \otimes)$. Let $\mathcal{C}_{\oplus,\otimes}$ be the set of all nonnegative, monotone, positively super-$\otimes$-homogeneous and support disjointness super-$\oplus$-additive functionals on $\mathcal{F}_+$.

The following is our main result which provides an equivalent definition of the pan-integral.

Theorem 1. *Consider a commutative isotonic semiring $(\overline{R}_+, \oplus, \otimes)$ and fixed $\mu \in \mathcal{M}$. Then for any $f \in \mathcal{F}_+$,*

$$\mathbf{I}_{pan}^{(\oplus,\otimes)}(\mu, f) = \inf\left\{F(f) : F \in \mathcal{C}_{\oplus,\otimes}, \forall A \in \mathcal{A}, F(\chi_A) \geq \mu(A)\right\}.$$

Proof. By Propositions 1 and 2, Lemmas 1 and 2, we know that $\mathbf{I}_{pan}^{(\oplus,\otimes)}(\mu, \cdot) : \mathcal{F}^+ \to [0, \infty]$ is monotone, positively super-$\otimes$-homogeneous, support disjointness super-$\oplus$-additive, i.e., $\mathbf{I}_{pan}^{(\oplus,\otimes)}(\mu, \cdot) \in \mathcal{C}_{\oplus,\otimes}$ and $\mathbf{I}_{pan}^{(\oplus,\otimes)}(\mu, \chi_A) \geq \mu(A)$ for any $A \in \mathcal{A}$. Therefore,

$$\mathbf{I}_{pan}^{(\oplus,\otimes)}(\mu, f) \geq \inf\left\{F(f) : F \in \mathcal{C}_{\oplus,\otimes}, \forall A \in \mathcal{A}, F(\chi_A) \geq \mu(A)\right\}.$$

On the other hand, for any $f \in \mathcal{F}^+$, any $\bigoplus_{i=1}^{n}\big(\lambda_i \otimes \chi_{A_i}\big) \le f$ and any $F \in \mathcal{C}_{\oplus,\otimes}$ with $F(\chi_A) \ge \mu(A), \forall A \in \mathcal{A}$, we have

$$F(f) \ge F\Big(\bigoplus_{i=1}^{n}\big(\lambda_i \otimes \chi_{A_i}\big)\Big) \ge \bigoplus_{i=1}^{n} F\Big(\lambda_i \otimes \chi_{A_i}\Big)$$
$$\ge \bigoplus_{i=1}^{n}\Big(\lambda_i \otimes F(\chi_{A_i})\Big) \ge \bigoplus_{i=1}^{n}\Big(\lambda_i \otimes \mu(A_i)\Big).$$

Thus,

$$F(f) \ge \sup\Big\{\bigoplus_{i=1}^{n}\big(\lambda_i \otimes \mu(A_i)\big) : \bigoplus_{i=1}^{n}\big(\lambda_i \otimes \chi_{A_i}\big) \le f\Big\} = \mathbf{I}_{pan}^{(\oplus,\otimes)}(\mu, f).$$

By the arbitrariness of F, we infer that

$$\inf\Big\{F(f) : F \in \mathcal{C}_{\oplus,\otimes}, \forall A \in \mathcal{A}, F(\chi_A) \ge \mu(A)\Big\} \ge \mathbf{I}_{pan}^{(\oplus,\otimes)}(\mu, f),$$

which proves the conclusion. □

Let $\mathcal{C}_{\oplus,\cdot}^{(1)}$ be the set of nonnegative, monotone, positively homogeneous and support disjointness super-$\oplus$-additive functionals on $\mathcal{F}^+$. Then $\mathcal{C}_{\oplus,\cdot}^{(1)} \subset \mathcal{C}_{\oplus,\cdot}$. Noting that $\mathbf{I}_{pan}^{(\oplus,\cdot)}(\mu,\cdot)\colon \mathcal{F}^+ \to [0,\infty]$ is positively homogeneous (Remark 1), then $\mathbf{I}_{pan}^{(\oplus,\cdot)}(\mu,\cdot) \in \mathcal{C}_{\oplus,\cdot}^{(1)}$. Thus we have the following result.

Theorem 2. *Let $(X, \mathcal{F}, \mu)$ be a monotone measure space. Then for any $f \in \mathcal{F}_+$,*

$$\mathbf{I}_{pan}^{(\oplus,\cdot)}(\mu, f) = \inf\Big\{F(f) : F \in \mathcal{C}_{\oplus,\cdot}^{(1)}, \forall A \in \mathcal{A}, F(\chi_A) \ge \mu(A)\Big\}.$$

Let the commutative isotonic semiring be $(\overline{R}_+, \vee, \cdot)$. Noticing that $\mathbf{I}_{pan}^{(\vee,\cdot)}(\mu, \chi_A) = \mu(A), \forall A \in \mathcal{A}$, by Theorem 2, we get an equivalent definition for the Shilkret integral.

Corollary 1. *Let $(X, \mathcal{F}, \mu)$ be a monotone measure space. Then for any $f \in \mathcal{F}_+$,*

$$\mathbf{I}_{pan}^{(\vee,\cdot)}(\mu, f) = \inf\Big\{F(f) : F \in \mathcal{C}_{\vee,\cdot}^{(1)}, \forall A \in \mathcal{A}, F(\chi_A) = \mu(A)\Big\}.$$

If we let $\oplus = +$, then we get an equivalent definition for the usual addition and multiplication based pan-integral.

Corollary 2. *Let $(X, \mathcal{F}, \mu)$ be a monotone measure space. Then for any $f \in \mathcal{F}_+$,*

$$\mathbf{I}_{pan}^{(+,\cdot)}(\mu, f) = \inf\Big\{F(f) : F \in \mathcal{C}_{+,\cdot}^{(1)}, \forall A \in \mathcal{A}, F(\chi_A) \ge \mu(A)\Big\}.$$

Noting that the Sugeno integral is positively $\wedge$-homogeneous [16] and satisfies $\mathbf{I}_{pan}^{(\vee,\wedge)}(\mu, \chi_A) = \mu(A), \forall A \in \mathcal{A}$, we also have the following result.

Corollary 3. *Let $(X, \mathcal{F}, \mu)$ be a monotone measure space. Then for any $f \in \mathcal{F}_+$,*

$$\mathbf{I}_{pan}^{(\vee,\wedge)}(\mu, f) = \inf\left\{F(f) : F \in \mathcal{C}_{\vee,\wedge}^{(1)}, \forall A \in \mathcal{A}, F(\chi_A) = \mu(A)\right\}.$$

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