Axiomatisation of Discrete Fuzzy Integrals with Respect to Possibility and Necessity Measures

D. Dubois[1(✉)] and A. Rico[2]

[1] IRIT, Université Paul Sabatier, 31062 Toulouse Cedex 9, France
Didier.Dubois@irit.fr
[2] ERIC, Université Claude Bernard Lyon 1, 69100 Villeurbanne, France

Abstract. Necessity (resp. possibility) measures are very simple representations of epistemic uncertainty due to incomplete knowledge. In the present work, a characterization of discrete Choquet integrals with respect to a possibility or a necessity measure is proposed, understood as a criterion for decision under uncertainty. This kind of criterion has the merit of being very simple to define and compute. To get our characterization, it is shown that it is enough to respectively add an optimism or a pessimism axiom to the axioms of the Choquet integral with respect to a general capacity. This additional axiom enforces the maxitivity or the minitivity of the capacity and essentially assumes that the decision-maker preferences only reflect the plausibility ordering between states of nature. The obtained pessimistic (resp. optimistic) criterion is an average of the maximin (resp. maximax) criterion of Wald across cuts of a possibility distribution on the state space. The additional axiom can be also used in the axiomatic approach to Sugeno integral and generalized forms thereof. The possibility of axiomatising of these criteria for decision under uncertainty in the setting of preference relations among acts is also discussed.

Keywords: Choquet integral · Sugeno integral · Possibility theory

1 Introduction

In multiple-criteria decision making, discrete fuzzy integrals are commonly used as aggregation functions [11]. They calculate a global evaluation for objects or alternatives evaluated according to some criteria. When the evaluation scale is quantitative, Choquet integrals are often used, while in the case of qualitative scale, Sugeno integrals are more naturally considered [9]. The definition of discrete fuzzy integrals is based on a monotonic set function named capacity or fuzzy measure. Capacities are used in many areas such as uncertainty modeling [4], multicriteria aggregation or in game theory [14].

The characterization of Choquet integral on quantitative scales is based on a general capacity, for instance a lower or upper probability defined from a family of probability functions [12,15]. There are no results concerning the characterisation of the Choquet integral with respect to a possibility or a necessity measure.

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In contrast, for the qualitative setting, characterizations of Sugeno integrals with respect to possibility measures exist [3,8]. However, Sugeno integrals with respect to necessity (resp. possibility) measures are minitive (resp. maxitive) functionals, while this is not the case for the corresponding Choquet integrals.

This paper proposes a property to be added to axioms characterizing discrete Choquet integrals that may justify the use of a possibility or necessity measure representing a plausibility ordering between states. We then generalize maximin and maximax criteria of Wald. Such specific criteria are currently used in signal processing based on maxitive kernels [10] or in sequential decision [1]. We also show that the same additional property can be added to characterisations of Sugeno integrals and more general functionals, to obtain possibilistic qualitative integrals (weighted min and max). Finally we show that the additional property can be expressed in the Savage setting of preference between acts, and discuss the possibility of act-based characterizations of possibilistic Choquet and Sugeno integrals.

2 Characterization of Possibilistic Choquet Integrals

We adopt the notations used in multi-criteria decision making where some objects or alternatives are evaluated according to a common finite set $\mathcal{C} = \{1, \cdots, n\}$ of criteria. In the case of decision under uncertainty (DMU) $\mathcal{C}$ is the set of the possible states of the world. A common, totally ordered, evaluation scale V is assumed to provide ratings according to the criteria. Each object is identified with a function $f = (f_1, \cdots, f_n) \in V^n$, called a *profile*, where f_i is the evaluation of f according to the criterion i. The set of all these objects (or acts in the setting of DMU) is denoted by $\mathcal{V}$.

A capacity or fuzzy measure is a non-decreasing set function $\mu : 2^{\mathcal{C}} \to L$, a totally ordered scale with top 1 and bottom 0 such that $\mu(\emptyset) = 0$ and $\mu(\mathcal{C}) = 1$, with $L \subseteq V$. When L is equipped with a negation denoted by $1-$, the conjugate of a capacity μ is defined by $\mu^c(A) = 1 - \mu(\overline{A})$. A possibility measure Π is a capacity such that $\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$. If $\pi = (\pi_1, \ldots, \pi_n)$ is the possibility distribution associated with Π, we have $\Pi(A) = \max_{i \in A} \pi_i$, which makes it clear that $\pi_i = 1$ for some i. In multi-criteria decision making, π_i is the importance of the criterion i. In the case of decision under uncertainty, π_i represents the plausibility of the state i. A necessity measure is a capacity N such that $N(A \cap B) = \min(N(A), N(B))$; then we have $N(A) = \min_{i \notin A} 1 - \pi_i$ since functions Π and N are conjugate capacities.

2.1 Possibilistic Choquet Integrals

In this part, L is supposed to be the unit interval. The Moebius transform associated with a capacity μ is the set function $m_\mu(T) = \sum_{K \subseteq T} (-1)^{|T \setminus K|} \mu(K)$, where $\sum_{T \subseteq \mathcal{C}} m_\mu(T) = 1$. The sets T such that $m_\mu(T) \neq 0$ are called the focal sets of μ. Using m_μ, the discrete Choquet integral of a function $f : \mathcal{C} \to \mathbb{R}$ with respect to a capacity μ can be simply expressed as a generalized weighted mean:

$$C_\mu(f) = \sum_{T \subseteq \mathcal{C}} m_\mu(T) \min_{i \in T} f_i. \tag{1}$$

Suppose μ is a necessity measure N and let σ be the permutation on the criteria such that $1 = \pi_{\sigma(1)} \geq \cdots \geq \pi_{\sigma(n)} \geq \pi_{\sigma(n+1)} = 0$. The Choquet integral of f with respect to N boils down to:

$$C_N(f) = \sum_{i=1}^{n}(\pi_{\sigma(i)} - \pi_{\sigma(i+1)}) \min_{j:\pi_j \geq \pi_{\sigma(i)}} f_j = \sum_{i=1}^{n}(\pi_{\sigma(i)} - \pi_{\sigma(i+1)}) \min_{j=1}^{i} f_{\sigma(j)} \quad (2)$$

since the focal sets of N are the sets $\{\sigma(1), \cdots, \sigma(i)\}_{i=1,\cdots,n}$ and their value for the Moebius transform is $\pi_{\sigma(i)} - \pi_{\sigma(i+1)}$ respectively. Using the identity $C_\Pi(f) = 1 - C_N(1 - f)$ one obtains the Choquet integral of f with respect to the conjugate possibility measure:

$$C_\Pi(f) = \sum_{i=1}^{n}(\pi_{\sigma(i)} - \pi_{\sigma(i+1)}) \max_{j:\pi_j \geq \pi_{\sigma(i)}} f_j = \sum_{i=1}^{n}(\pi_{\sigma(i)} - \pi_{\sigma(i+1)}) \max_{j=1}^{i} f_{\sigma(j)} \quad (3)$$

Note that if $\pi_1 = \cdots \pi_n = 1$ then $C_N(f) = \min_{i=1}^n f_i$ and $C_\Pi(f) = \max_{i=1}^n f_i$ are Wald maximin and maximax criteria, respectively. Moreover if many criteria have the same importance π_i, then the expression of C_N (resp. C_Π) proves that we take into account the worst (resp. best) value of f_j according to these criteria.

It is worth noticing that the functional C_N is not minitive and C_Π is not maxitive [5] as shown by the following example.

Example 1. We consider $\mathcal{C} = \{1, 2\}$, the possibility distribution π and the following profiles f and g: $\pi_1 = 1, \pi_2 = 0.5$; $f_1 = 0.2, f_2 = 0.3$; and $g_1 = 0.4, g_2 = 0.1$. We have $C_N(f) = 0.5 \cdot 0.2 + 0.5 \cdot 0.2 = 0.2$ and $C_N(g) = 0.5 \cdot 0.4 + 0.5 \cdot 0.1 = 0.25$, but $C_N(\min(f, g)) = 0.5 \cdot 0.2 + 0.5 \cdot 0.1 = 0.15 \neq \min(C_N(f), C_N(g))$. By duality, it also proves the non-maxitivity of C_Π using acts $1 - f$ and $1 - g$.

2.2 Pessimistic and Optimistic Substitute Profiles

Using the permutation σ on the criteria associated with π, a pessimistic profile $f^{\sigma,-}$ and an optimistic profile $f^{\sigma,+}$ can be associated with each profile f:

$$f_i^{\sigma,-} = \min_{j:\pi_j \geq \pi_{\sigma(i)}} f_j = \min_{j=1}^{i} f_{\sigma(j)}; \quad f_i^{\sigma,+} = \max_{j:\pi_j \geq \pi_{\sigma(i)}} f_j = \max_{j=1}^{i} f_{\sigma(j)}. \quad (4)$$

Observe that only the ordering of elements i induced by π on $\mathcal{C}$ is useful in the definition of the pessimistic and optimistic profiles associated with f. These profiles correspond to the values of f appearing in the weighted mean expressions (2) and (3). Substituting pessimistic and optimistic profiles associated with f in these expressions, possibilistic Choquet integrals take the form of usual discrete expectations wrt a probability distribution m with $m_{\sigma(i)} = \pi_{\sigma(i)} - \pi_{\sigma(i+1)}$:

$$C_N(f) = \sum_{i=1}^{n} m_{\sigma(i)} f_i^{\sigma,-} = C_N(f^{\sigma,-}), \quad C_\Pi(f) = \sum_{i=1}^{n} m_{\sigma(i)} f_i^{\sigma,+} = C_\Pi(f^{\sigma,+}).$$

Two profiles f and g are said to be comonotone if and only if for all $i, j \in \mathcal{C}$, $f_i > f_j$ implies $g_i \geq g_j$. So f and g are comonotone if and only if there exists the permutation τ on $\mathcal{C}$ such that $f_{\tau(1)} \leq \cdots \leq f_{\tau(n)}$ and $g_{\tau(1)} \leq \cdots \leq g_{\tau(n)}$.

For any profile f, we have $f_1^{\sigma,-} \geq \cdots \geq f_n^{\sigma,-}$ and $f_1^{\sigma,+} \leq \cdots \leq f_n^{\sigma,+}$. So for any pair of profiles f and g, $f^{\sigma,-}$ and $g^{\sigma,-}$ (resp. $f^{\sigma,+}$ and $g^{\sigma,+}$) are comonotone.

We can define a sequence of progressively changing profiles $(\phi_k)_{1\leq k\leq n}$ that are equivalently evaluated by C_N. Namely, $\phi_1 = f$, $\phi_{k+1} \leq \phi_k$, $\phi_n = f^{\sigma,-}$ where $\phi_{k+1} = \phi_k$ except for one coordinate. The profiles ϕ_k are defined by

$$\phi_k(i) = \begin{cases} \min_{l:\pi_l \geq \pi_{\sigma(i)}} f_l & \text{if } i \leq k \\ f_i & \text{otherwise.} \end{cases}$$

Similarly we can define a sequence of profiles that are equivalently evaluated by C_Π. Namely, $(\phi^k)_{1\leq k\leq n}$ such that $\phi^1 = f$, $\phi^{k+1} \geq \phi^k$, $\phi^n = f^{\sigma,+}$ where $\phi^{k+1} = \phi^k$ except for one coordinate. The profiles ϕ^k are defined by

$$\phi^k(i) = \begin{cases} \max_{l:\pi_l \geq \pi_{\sigma(i)}} f_l & \text{if } i \leq k \\ f_i & \text{otherwise.} \end{cases}$$

We observe that $C_N(f) = C_N(\phi_k)$, $C_\Pi(f) = C_\Pi(\phi^k)$, for all $1 \leq k \leq n$.

2.3 Representation Theorem

Consider the case of Boolean functions, corresponding to subsets A, B of $\mathcal{C}$. Their profiles are just characteristic functions $\mathbf{1}_A, \mathbf{1}_B$. Given a permutation σ induced by π, let us find the corresponding optimistic and pessimistic Boolean profiles.

Lemma 1. *For all $A \subseteq \mathcal{C}$ non empty, $\mathbf{1}_A^{\sigma,-} = \mathbf{1}_B$ for a subset $B = A^{\sigma,-} \subseteq A$ and $\mathbf{1}_A^{\sigma,+} = \mathbf{1}_B$ for a superset $B = A^{\sigma,+} \supseteq A$.*

Proof. $\mathbf{1}_A^{\sigma,-}(i) = \min_{k=1}^{i} \mathbf{1}_A(\sigma(k)) = 1$ if $\forall k \leq i : \sigma(k) \in A$ and 0 otherwise. So $\mathbf{1}_A^{\sigma,-} = \mathbf{1}_B$ with $B \subseteq A$.

$\mathbf{1}_A^{\sigma,+}(i) = \max_{k=1}^{i} \mathbf{1}_A(\sigma(k)) = 1$ if $\exists k \leq i$ and $\sigma(k) \in A$, and 0 otherwise. So $\mathbf{1}_A^{\sigma,+} = \mathbf{1}_B$ with $A \subseteq B$. □

It is easy to realize that the set $A^{\sigma,-}$ exactly contains the largest sequence of consecutive criteria $(\sigma(1), \ldots, \sigma(k^-))$ in A, while the set $A^{\sigma,+}$ exactly contains the smallest sequence of consecutive criteria $(\sigma(1), \ldots, \sigma(k^+))$ that includes A.

Lemma 2. *A capacity μ is a necessity measure if and only if there exists a permutation σ on $\mathcal{C}$ such that for all A we have $\mu(A) = \mu(A^{\sigma,-})$.*

A capacity μ is a possibility measure if and only if there exists a permutation σ on $\mathcal{C}$ such that for all A we have $\mu(A) = \mu(A^{\sigma,+})$.

Proof. Let σ be such that $\mu(A) = \mu(A^{\sigma,-})$. Let us prove that for all $A, B \subseteq \mathcal{C}$, we have $\mu(A \cap B) = \min(\mu(A), \mu(B))$.

From Lemma 1, $(A \cap B)^{\sigma,-} \subseteq A \cap B$. So $\mu(A \cap B) = \mu((A \cap B)^{\sigma,-}) = \mu(\{\sigma(1), \cdots, \sigma(k^-)\})$. As $\sigma(k^-+1) \notin A \cap B$, then $\sigma(k^-+1) \notin A$ or $\sigma(k^-+1) \notin B$.

Suppose without loss of generality that $\sigma(k^- + 1) \notin A$. Then $A^{\sigma,-} = (A \cap B)^{\sigma,-}$ hence $\mu(A \cap B) = \mu(A) \leq \mu(B)$ so $\mu(A \cap B) = \min(\mu(A), \mu(B))$. Consequently μ is a necessity measure.

Conversely we consider a necessity measure N and the permutation such that $\pi_1 \geq \cdots \geq \pi_n$, $N(A) = 1 - \pi_{i_0}$ with $i_0 = \min\{j : j \notin A\}$. So the set $A^{\sigma,-}$ is $\{1, \cdots, i_0 - 1\}$ so $N(A^{\sigma,-}) = 1 - \pi_{i_0}$.

A similar proof can be developed for the case of possibility measures. □

Now we add suitable axioms to a known representation theorem of Choquet integral [15], and obtain a characterisation theorem for the case when the capacity is a possibility or a necessity measure.

Theorem 1. *A function $I : \mathcal{V} \to \mathbb{R}$ satisfies the following properties:*

C1 $I(1, \cdots, 1) = 1$,
C2 *Comonotonic additivity:f and g comonotone implies* $I(f+g) = I(f) + I(g)$,
C3 *Pareto-domination:*$f \geq g$ *implies* $I(f) \geq I(g)$,
*Π***4** *There exists a permutation σ on $\mathcal{C}$ such that* $\forall A$, $I(\mathbf{1}_A) = I(\mathbf{1}_{A^{\sigma,+}})$

if and only if $I = C_\Pi$, where $\Pi(A) = I(\mathbf{1}_A)$ is a possibility measure.

Proof. It is easy to check that the Choquet integral with respect to Π satisfies the properties C1-C3 and $\Pi4$ according to the permutation associated with π.

If I satisfies the properties C1-C3, then according to the results presented in [15] I is a Choquet integral with respect to the fuzzy measure μ defined by $\mu(A) = I(\mathbf{1}_A)$. The property $\Pi4$ implies $\mu(A) = I(\mathbf{1}_A^{\sigma,+}) = \mu(A^{\sigma,+})$, and using Lemma 2 this equality is equivalent to have a possibility measure. □

Note that Axiom $\Pi4$ can be replaced by: There exists a permutation σ on $\mathcal{C}$ such that $\forall f$, $I(f) = I(f^{\sigma,+})$. We have a similar result for necessity measures:

Theorem 2. *A function $I : \mathcal{V} \to \mathbb{R}$ satisfies the following properties:*

C1 $I(1, \cdots, 1) = 1$,
C2 *Comonotonic additivity: f and g comonotone implies* $I(f+g) = I(f) + I(g)$,
C3 *Pareto-domination:* $f \geq g$ *implies* $I(f) \geq I(g)$,
*N***4** *There exists a permutation σ on $\mathcal{C}$ such that* $\forall A$, $I(\mathbf{1}_A) = I(\mathbf{1}_{A^{\sigma,-}})$

if and only if $I = C_N$, where $N(A) = I(\mathbf{1}_A)$ is a necessity measure.

Axiom $N4$ can be replaced by: There exists a permutation σ on $\mathcal{C}$ such that $\forall f$, $I(f) = I(f^{\sigma,-})$. These results indicate that Choquet integrals w.r.t possibility and necessity measures are additive for a larger class of pairs of functions than usual, for instance $C_N(f+g) = C_N(f) + C_N(g)$ as soon as $(f+g)^{\sigma,-} = f^{\sigma,-} + g^{\sigma,-}$, which does not imply that f and g are comonotone.

Example 2. We consider $C = \{1, 2, 3\}$, the permutation associated with Π such that $\pi_1 \geq \pi_2 \geq \pi_3$ and the profiles $f = (1, 2, 3)$, $g = (1, 3, 2)$ which are not comonotone. It is easy to check that $(f+g)^- = f^{\sigma,-} + g^{\sigma,-}$.

The above result should be analyzed in the light of a claim by Mesiar and Šipoš [13] stating that if the capacity μ is modular on the set of cuts $\{\{i : f_i \geq \alpha\} : \alpha > 0\} \cup \{\{i : g_i \geq \alpha\} : \alpha > 0\}$ of f and g, then $C_N(f+g) = C_N(f) + C_N(g)$. For a general capacity, it holds if f and g are comonotonic. For more particular capacities, the set of pairs of acts for which modularity holds on cuts can be larger. This is what seems to happen with possibility and necessity measures.

3 A New Characterisation of Possibilistic Sugeno Integrals

In this part we suppose that $L = V$ is a finite, totally ordered set with 1 and 0 as respective top and bottom. So, $\mathcal{V} = L^{\mathcal{C}}$. Again, we assume that L is equipped with a unary order reversing involutive operation $t \to 1 - t$ called a negation. To distinguish from the numerical case, we denote by $\wedge$ and $\vee$ the minimum and the maximum on L. As we are on a qualitative scale we speak of q-integral in this section. The Sugeno q-integral [16,17], of an alternative f can be defined by means of several expressions, among which the two following normal forms [12]:

$$\int_\mu f = \bigvee_{A \subseteq C} \mu(A) \bigwedge \wedge_{i \in A} f_i = \bigvee_{A \subseteq \mathcal{C}} (1 - \mu^c(A)) \bigvee \vee_{i \in A} f_i \tag{5}$$

Sugeno q-integral can be characterized as follows:

Theorem 3 [3]. *Let $I : \mathcal{V} \to L$. There is a capacity μ such that $I(f) = \int_\mu f$ for every $f \in \mathcal{V}$ if and only if the following properties are satisfied*

1. *$I(f \vee g) = I(f) \vee I(g)$, for any comonotone $f, g \in \mathcal{V}$.*
2. *$I(a \wedge f) = a \wedge I(f)$, for every $a \in L$ and $f \in \mathcal{V}$.*
3. *$I(\mathbf{1}_{\mathcal{C}}) = 1$.*

Equivalently, conditions (1–3) can be replaced by conditions (1'–3') below.

1'. *$I(f \wedge g) = I(f) \wedge I(g)$, for any comonotone $f, g \in \mathcal{V}$.*
2'. *$I(a \vee f) = a \vee I(f)$, for every $a \in L$ and $f \in \mathcal{V}$.*
3'. *$I(\mathbf{0}_{\mathcal{C}}) = 0$.*

The existence of these two equivalent characterisations is due to the possibility of writing Sugeno q-integral in conjunctive and disjunctive forms (5) equivalently.

Moreover, for a necessity measure N, $\int_N f = \wedge_{i=1}^n (1 - \pi_i) \vee f_i$; and for a possibility measure Π, $\int_\Pi f = \vee_{i=1}^n \pi_i \wedge f_i$. The Sugeno q-integral with respect to a possibility (resp. necessity) measure is maxitive (resp. minitive), hence the following known characterization results for them:

Theorem 4. *Let $I : \mathcal{V} \to L$. There is a possibility measure Π such that $I(f) = \int_\Pi f$ for every $f \in \mathcal{V}$ if and only if the following properties are satisfied*

1. *$I(f \vee g) = I(f) \vee I(g)$, for any $f, g \in \mathcal{V}$.*
2. *$I(a \wedge f) = a \wedge I(f)$, for every $a \in L$ and $f \in \mathcal{V}$.*
3. *$I(\mathbf{1}_{\mathcal{C}}) = 1$.*

Theorem 5. *There is a necessity measure Π such that $I(f) = \int_N f$ for every $f \in \mathcal{V}$ if and only if the following properties are satisfied*

1'. $I(f \wedge g) = I(f) \wedge I(g)$, *for any* $f, g \in \mathcal{V}$.
2'. $I(a \vee f) = a \vee I(f)$, *for every* $a \in L$ *and* $f \in \mathcal{V}$.
3'. $I(\mathbf{0}_\mathcal{C}) = 0$.

However, we can alternatively characterise those simplified Sugeno q-integrals in the same style as we did for possibilistic Choquet integrals due to the following

Lemma 3.

$$\int_N f = \int_N f^{\sigma,-}, \quad \int_\Pi f = \int_\Pi f^{\sigma,+}.$$

Proof. Assume $\pi_1 \geq \cdots \geq \pi_n$ for simplicity, i.e. $\sigma(i) = i$. By definition $f^{\sigma,-} \leq f$ so $\int_N f^{\sigma,-} \leq \int_N f$ since the Sugeno q-integral is an increasing function. Let i_0 and i_1 be the indices such that $\int_N f^{\sigma,-} = \max(1 - \pi_{i_0}, \min_{j \leq i_0} f_{i_0}) = \max(1 - \pi_{i_0}, f_{i_1})$ where $i_1 \leq i_0$. Hence $\pi_{i_1} \geq \pi_{i_0}$ i.e. $1 - \pi_{i_1} \leq 1 - \pi_{i_0}$ and $\int_N f^{\sigma,-} \geq \max(1 - \pi_{i_1}, f_{i_1}) \geq \int_N f$.

By definition $f \leq f^{\sigma,+}$ so $\int_\Pi f \leq \int_\Pi f^{\sigma,+}$. Let i_0 and i_1 be the indices such that $\int_\Pi f^{\sigma,+} = \min(\pi_{i_0}, \max_{j \leq i_0} f_j) = \min(\pi_{i_0}, f_{i_1})$ where $i_1 \leq i_0$. Hence $\pi_{i_0} \leq \pi_{i_1}$ and $\int_\Pi f^{\sigma,+} \leq \min(\pi_{i_1}, f_{i_1}) \leq \int_\Pi f$. □

In particular, $\int_N f = \int_N \phi_k$, $\int_\Pi f = \int_\Pi \phi^k$, for all $1 \leq k \leq n$, as for Choquet integral. Now we can state qualitative counterparts of Theorems 1 and 2:

Theorem 6. *There is a possibility measure Π such that $I(f) = \int_\Pi f$ for every $f \in \mathcal{V}$ if and only if the following properties are satisfied*

1. $I(f \vee g) = I(f) \vee I(g)$, *for any comonotone* $f, g \in \mathcal{V}$.
2. $I(a \wedge f) = a \wedge I(f)$, *for every* $a \in L$ *and* $f \in \mathcal{V}$.
3. $I(\mathbf{1}_\mathcal{C}) = 1$.

Π4 There exists a permutation σ on $\mathcal{C}$ such that $\forall A$, $I(\mathbf{1}_A) = I({\mathbf{1}_A}^{\sigma,+})$

Theorem 7. *There is a necessity measure N such that $I(f) = \int_N f$ for every $f \in \mathcal{V}$ if and only if the following properties are satisfied*

$I(f \wedge g) = I(f) \wedge I(g)$, *for any comonotone* $f, g \in \mathcal{V}$.
$I(a \vee f) = a \vee I(f)$, *for every* $a \in L$ *and* $f \in \mathcal{V}$.
$I(0_\mathcal{C}) = 0$.

*N***4** *There exists a permutation σ on $\mathcal{C}$ such that* $\forall A$, $I(\mathbf{1}_A) = I({\mathbf{1}_A}^{\sigma,-})$

Axiom Π4 (resp., N4) can be replaced by: there exists a permutation σ on $\mathcal{C}$ such that for all f, $I(f) = I(f^{\sigma,+})$ (resp. $I(f) = I(f^{\sigma,-})$).

We can generalize Sugeno q-integrals as follows: consider a bounded complete totally ordered value scale $(L, 0, 1, \leq)$, equipped with a binary operation $\otimes$ called right-conjunction, which has the following properties:

- the top element 1 is a right-identity: $x \otimes 1 = x$,
- the bottom element 0 is a right-anihilator $x \otimes 0 = 0$,
- the maps $x \mapsto a \otimes x$, $x \mapsto x \otimes a$ are order-preserving for every $a \in L$.

Note that we have $0 \otimes x = 0$ since $0 \otimes x \leq 0 \otimes 1 = 0$. An example of such a conjunction is a semi-copula (such that $a \otimes b \leq \min(a, b)$). We can define an implication $\rightarrow$ from $\otimes$ by semi-duality: $a \rightarrow b = 1 - a \otimes (1 - b)$. Note that this implication coincides with a Boolean implication on $\{0, 1\}$, is decreasing according to its first argument and it is increasing according to the second one.

A non trivial example of semi-dual pair (implication, right-conjunction) is the contrapositive Gödel implication $a \rightarrow_{GC} b = \begin{cases} 1 - a \text{ if } a > b \\ 1 \text{ otherwise} \end{cases}$ and the associated right-conjunction $a \otimes_{GC} b = \begin{cases} a \text{ if } b > 1 - a \\ 0 \text{ otherwise} \end{cases}$ (it is not a semi-copula).

The associated q-integral is $\int_\mu^\otimes f = \vee_{A \subseteq C}((\wedge_{i \in A} f_i) \otimes \mu(A))$. This kind of q-integral is studied in [2] for semi-copulas. The associated *q-cointegral* obtained via semi-duality is of the form $\int_\mu^\rightarrow f = \wedge_{A \subseteq C}(\mu^c(A) \rightarrow (\vee_{i \in A} f_i))$. We can see that $\int_\mu^\rightarrow f = 1 - \int_{\mu^c}^\otimes (1 - f)$. But in general, $\int_\mu^\otimes f \neq \int_\mu^\rightarrow f$ even when $a \rightarrow b = 1 - a \otimes (1 - b)$ [6], contrary to the case of Sugeno q-integral, for which $\otimes$ is the minimum, and $a \rightarrow b = \max(1 - a, b)$. We have $\int_\Pi^\otimes (f) = \max_{i=1}^n f_i \otimes \pi_i$ and $\int_N^\rightarrow f = \min_{i=1}^n (1 - f_i) \rightarrow (1 - \pi_i)$ since $N^c = \Pi$.

With a proof similar as the one for Sugeno q-integral, it is easy to check that a generalized form of Lemma 3 holds: $\int_\Pi^\otimes f = \int_\Pi^\otimes f^{\sigma,+}, \quad \int_N^\rightarrow f = \int_N^\otimes f^{\sigma,-}$.

The characterisation results for Sugeno q-integral (Theorem 3) and their possibilistic specialisations (Theorems 4 and 5) can be generalised for right-conjunction-based q-integrals and q-cointegrals albeit separately for each:

Theorem 8. *A function $I : \mathcal{V} \rightarrow L$ satisfies the following properties:*

RC1 *f and g comonotone implies $I(f \vee g) = I(f) \vee I(g)$,*
RC2 $I(\mathbf{1}_A \otimes a) = I(\mathbf{1}_A) \otimes a$
RC3 $I(\mathbf{1}_\mathcal{C}) = 1$.

if and only if I is a q-integral $\int_\mu^\otimes f$ with respect to a capacity $\mu(A) = I(\mathbf{1}_A)$. Adding axiom $\Pi4$ yields an optimistic possibilistic q-integral $\int_\Pi^\otimes f$.

Theorem 9. *A function $I : \mathcal{V} \rightarrow L$ satisfies the following properties:*

IRC1 *f and g comonotone implies $I(f \wedge g) = I(f) \wedge I(g)$,*
IRC2 $I(a \rightarrow \mathbf{1}_A) = a \rightarrow I(\mathbf{1}_A)$
IRC3 $I(\mathbf{1}_\emptyset) = 0$.

if and only if I is an implicative q-integral $\int_\mu^\rightarrow f$ with respect to a capacity $\mu(A) = I(\mathbf{1}_A)$. Adding axiom $N4$ yields a pessimistic possibilistic q- cointegral $\int_N^\otimes f$.

4 Axiomatisation Based on Preference Relations

In the context of the decision under uncertainty we consider a preference relation on the profiles and we want to represent it with a Choquet integral, a Sugeno q-integral or a q-integral with respect to a possibility or a necessity.

4.1 Preference Relations Induced by Fuzzy Integrals

With the previous integrals with respect to a possibility we can define a preference relation: $f \succeq^{+}_{\oint} g$ if and only if $\oint_{\Pi}(f^{\sigma,+}) \geq \oint_{\Pi}(g^{\sigma,+})$ where $\oint$ is one of the integrals presented above. For all f, we have the indifference relation $f \sim^{+}_{\oint} f^{\sigma,+}$. An optimistic decision maker is represented using a possibility measure since the attractiveness $(f^{\sigma,+}_i)$ is never less that the greatest utility f_j among the states more plausible than i. Particularly, if the state 1 is the most plausible with $f_1 = 1$ then we have $f = (1, 0, \cdots, 0) \sim (1, \cdots, 1)$. The expected profit in a very plausible state is not affected with the expected losses in less plausible states. The Choquet integral calculates the average of the best consequences for each plausibility level.

Similarly we can define preference relations $\succeq^{-}_{\oint}$ using a necessity measure. In such a context for all f, $f \sim^{-}_{\oint} f^{\sigma,-}$. In this case, the decision maker is pessimistic since the attractiveness $(f^{\sigma,-}_i)$ is never greater that the smallest utility f_j among the states more plausible than i. Particularly, if the state 1 is the most plausible with $f_1 = 0$ then we have $f = (0, 1, \cdots, 1) \sim (0, \cdots, 0)$. The expected profits in the least plausible states cannot compensate the expected losses in more plausible states. In this case the Choquet integral calculates the average of the worst consequences for each plausibility level.

4.2 The case of Choquet integral

Let $\succeq$ be a preference relation on profiles given by the decision maker. In [4] the following axioms are proposed, in the infinite setting, where the set of criteria is replaced by a continuous set of states $\mathcal{S}$:

A1 $\succeq$ is non trivial complete preorder.
A2 Continuity according to uniform monotone convergence
  A2.1 $[f_n, f, g \in \mathcal{V},\ f_n \succeq g,\ f_n \downarrow^u f] \Rightarrow f \succeq g$;
  A2.2 $[f_n, f, g \in \mathcal{V},\ g \succeq f_n,\ f_n \uparrow^u f] \Rightarrow g \succeq f$;
A3 If $f \geq g + \epsilon$ where ϵ is a positive constant function then $f \succ g$
A4 Comonotonic independence: If f, g, h are profiles such that f and h, and g and h are comonotone, then: $f \succeq g \Leftrightarrow f + h \succeq g + h$

And we have the following result [4]:

Theorem 10. *A preference relation $\succeq$ satisfies axioms $A1 - A4$ if and only if there exists a capacity μ such that C_μ represents the preference relation. This capacity is unique.*

The notion of pessimistic and optimistic profile can be extended to the continuous case. A possibility distribution on $\mathcal{S}$ defines a complete plausibility preordering $\leq_\pi$ on $\mathcal{S}$, and given an act f, we can define its pessimistic counterpart as $f^{\leq_\pi,-}(s) = \inf_{s \leq_\pi s'} f(s')$. Let us add the pessimistic axiom:

N4 There is a complete plausibility preordering $\leq_\pi$ on $\mathcal{S}$ such that $f \sim f^{\leq_\pi,-}$.

Similarly an optimistic axiom $\Pi 4$ can be written, using optimistic counterparts of profiles $f^{\leq_\pi,+}(s) = \sup_{s \leq_\pi s'} f(s')$.

***Π*4** There is a complete plausibility preordering $\leq_\pi$ on $\mathcal{S}$ such that $f \sim f^{\leq_\pi,+}$.

We can conjecture the following result for necessity measures:

Theorem 11. *A preference relation $\succeq$ satisfies axioms $A1 - A4$ and $N4$ if and only if there exists a necessity measure N such that C_N represents the preference relation. This necessity measure is unique.*

The proof comes down to showing that the unique capacity obtained from axioms $A1 - A4$ is a necessity measure. However, it is not so easy to prove in the infinite setting. Indeed, a necessity measure then must satisfy the infinite minitivity axiom, $N(\cap_{i\in I}) = \inf_{i\in I} N(A_i)$, for any index set I, which ensures the existence of a possibility distribution underlying the capacity. But it is not clear how to extend Lemma 2 to infinite families of sets. As its stands, Lemma 2 only justifies finite minitivity. The same difficulty arises for the optimistic counterpart of the above tentative result. In a finite setting, the permutation σ that indicates the relative plausibility of states can be extracted from the preference relation on profiles, by observing special ones. More precisely, $C_N(1, \ldots, 1, 0, 1, \ldots 1) = 1 - \pi_i$ (the 0 in the case i) in the pessimistic case, and $C_\Pi(0, \ldots, 0, 1, 0, \ldots 0) = \pi_i$ in the optimistic case. This fact would still hold in the form $C_N(\mathcal{C}\backslash\{s\}) = 1 - \pi(s)$ and $C_\Pi(\{s\}) = \pi(s)$, respectively, with infinite minitivity (resp. maxitivity).

4.3 A New Characterisation for Qualitative Possibilistic Integrals

The axiomatization of Sugeno q-integrals in the style of Savage was carried out in [7]. Here, acts are just functions f from $\mathcal{C}$ to a set of consequences X. The axioms proposed are as follows, where xAf is the act such that $(xAf)_i = x$ if $i \in A$ and f_i otherwise, $x \in X$ being viewed as a constant act:

A1 $\succeq$ is a non trivial complete preorder.
WP3 $\forall A \subseteq \mathcal{C}, \forall x, y \in X, \forall f, x \geq y$ implies $xAf \succeq yAf$,
RCD: if f is constant, $f \succ h$ and $g \succ h$ imply $f \wedge g \succ h$
RDD: if f is a constant act, $h \succ f$ and $h \succ g$ imply $h \succ f \vee g$.

Act $f \vee g$ makes the best of f and g, such that $\forall s \in \mathcal{S}$, $f \vee g(s) = f(s)$ if $f(s) \succeq g(s)$ and $g(s)$ otherwise; and act $f \wedge g$, makes the worst of f and g, such that $\forall s \in \mathcal{S}$, $f \wedge g(s) = f(s)$ if $g(s) \succeq f(s)$ and $g(s)$ otherwise. We recall here the main results about this axiomatization for decision under uncertainty.

Theorem 12. *[7]: The following propositions are equivalent:*

- $(X^{\mathcal{C}}, \succeq)$ *satisfies A1, WP3, RCD, RDD.*
- *there exists a finite chain of preference levels* L*, an* L*-valued monotonic set-function* μ*, and an* L*-valued utility function* u *on* X*, such that* $f \succeq g$ *if and only if* $\int_{\mu}(u \circ f) \geq \int_{\mu}(u \circ g)$*.*

In the case of a Sugeno q-integral with respect to a possibility measure, RDD is replaced by the stronger axiom of disjunctive dominance **DD**:

Axiom DD: $\forall f, g, h, h \succ f$ and $h \succ g$ imply $h \succ f \vee g$
and we get a similar result as the above theorem, whereby $f \succeq g$ if and only if $\int_{\Pi}(u \circ f) \geq \int_{\Pi}(u \circ g)$ for a possibility measure Π [8].

In the case of a Sugeno q-integral with respect to a necessity measure, RCD is replaced by the stronger axiom of conjunctive dominance **CD**:

Axiom CD: $\forall f, g, h, f \succ h$ and $g \succ h$ imply $f \wedge g \succ h$
and we get a similar result as the above Theorem 12, whereby $f \succeq g$ if and only if $\int_{N}(u \circ f) \geq \int_{N}(u \circ g)$ for a necessity measure Π [8].

We can then replace the above representation results by adding to the characteristic axioms for Sugeno q-integrals on a preference relation between acts the same axioms based on pessimistic and optimistic profiles as the ones that, added to characteristic axioms of Choquet integrals lead to a characterisation of preference structures driven by possibilistic Choquet integrals.

Theorem 13. *The following propositions are equivalent:*

- $(X^{\mathcal{C}}, \succeq)$ *satisfies A1, WP3, RCD, RDD and* $\Pi 4$
- *there exists a finite chain of preference levels* L*, an* L*-valued possibility measure* Π*, and an* L*-valued utility function* u *on* X*, such that* $f \succeq g$ *if and only if* $\int_{\Pi}(u \circ f) \geq \int_{\Pi}(u \circ g)$*.*

Theorem 14. *The following propositions are equivalent:*

- $(X^{\mathcal{C}}, \succeq)$ *satisfies A1, WP3, RCD, RDD and N4*
- *there exists a finite chain of preference levels* L*, an* L*-valued necessity measure* N*, and an* L*-valued utility function* u *on* X*, such that* $f \succeq g$ *if and only if* $\int_{N}(u \circ f) \geq \int_{N}(u \circ g)$*.*

The reason for the validity of those theorems in the case of Sugeno q-integral is exactly the same as the reason for the validity of Theorems 1, 2, 6 and 7, adding $\Pi 4$ (resp. N4) to the representation theorem of Sugeno q-integral forces the capacity to be a possibility (resp. necessity) measure. However, this method seems to be unavoidable to axiomatize Choquet integrals for possibility and necessity measures as they are not maxitive nor minitive. In the case of possibilistics q-integrals, the maxitivity or minitivity property of the preference functional makes it possible to propose more choices of axioms. However, it is interesting to notice that the same axioms are instrumental to specialize Sugeno and Choquet integrals to possibility and necessity measures.

5 Conclusion

This paper proposes an original axiomatization of discrete Choquet integrals with respect to possibility and necessity measures, and shows that it is enough to add, to existing axiomatisations of general instances of Choquet integrals, a property of equivalence between profiles, that singles out possibility or necessity measures. Remarkably, this property, which also says that the decision-maker only considers relevant the relative importance of single criteria, is qualitative in nature and can thus be added as well to axiom systems for Sugeno integrals, to yield qualitative weighted min and max aggregation operations, as well as for their ordinal preference setting à la Savage. We suggest these results go beyond Sugeno integrals and apply to more general qualitative functionals. One may wonder if this can be done for the ordinal preference setting of the last section, by changing axioms RCD or RDD using right-conjunctions and their semi-duals.

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