A Representative in Group Decision by Means of the Extended Set of Hesitant Fuzzy Linguistic Term Sets

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Abstract. Hesitant fuzzy linguistic term sets were introduced to grasp the uncertainty existing in human reasoning when expressing preferences. In this paper, an extension of the set of hesitant fuzzy linguistic term sets is presented to capture differences between non-compatible preferences. In addition, an order relation and two closed operation over this set are also introduced to provide a lattice structure to the extended set of hesitant fuzzy linguistic term sets. Based on this lattice structure a distance between hesitant fuzzy linguistic descriptions is defined. This distance enables differences between decision makers to be quantified. Finally, a representative of a decision making group is presented as the centroid of the group based on the introduced distance.

Keywords: Linguistic modeling · Group decision making · Uncertainty and fuzzy reasoning · Hesitant fuzzy linguistic term sets

Introduction

Different approaches involving linguistic assessments have been introduced in the fuzzy set literature to deal with the impreciseness and uncertainty connate with human preference reasoning [2,4,5,7,9]. Additionally, different extensions of fuzzy sets have been presented to give more realistic assessments when uncertainty increases [1,3,8]. In particular, Hesitant Fuzzy Sets were introduced in [10], to capture this kind of uncertainty and hesitance. Following this idea, Hesitant Fuzzy Linguistic Term Sets (HFLTSs) were introduced in [8] to deal with situations in which linguistic assessments involving different levels of precision are used. In addition, a lattice structure was provided to the set of HFLTSs in [6].

In this paper, we present an extension of the set of HFLTSs, $\overline{\mathcal{H}_{\mathcal{S}}}$, based on an equivalence relation on the usual set of HFLTSs. This enables us to establish differences between non-compatible HFLTSs. An order relation and two closed operation over this set are also introduced to define a new lattice structure in $\overline{\mathcal{H}_{\mathcal{S}}}$.

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In order to describe group decision situations in which Decision Makers (DMs) are evaluating different alternatives, Hesitant Fuzzy Linguistic Descriptions (HFLDs) were presented in [6]. A distance between HFLTSs is defined based on the lattice of $\overline{\mathcal{H}_\mathcal{S}}$. This allows us to present a distance between HFLDs that we can use to quantify differences among assessments of different DMs. Taking into consideration this distance, a group representative is suggested to describe the whole group assessment. Due to this representative is the HFLD that minimizes distances with the assessments of all the DMs, it is called the centroid of the group.

The rest of this paper is organized as follows: first, Sect. 1 presents a brief review of HFLTSs and its lattice structure. The lattice of the extended set of HFLTSs is introduced in Sect. 2. In Sect. 3, the distances between HFLTSs and HFLDs are defined and the centroid of the group is presented in Sect. 4. Lastly, Sect. 5 contains the main conclusions and lines of future research.

1 The Lattice of Hesitant Fuzzy Linguistic Term Sets

In this section we present a brief review of some concepts about HFLTSs already presented in the literature that are used throughout this paper [6,8].

From here on, let $\mathcal{S}$ denote a finite total ordered set of linguistic terms, $\mathcal{S} = \{a_1, \ldots, a_n\}$ with $a_1 < \cdots < a_n$.

Definition 1. [8] A *hesitant fuzzy linguistic term set (HFLTS)* over $\mathcal{S}$ is a subset of consecutive linguistic terms of S, i.e. $\{x \in \mathcal{S} \,|\, a_i \leq x \leq a_j\}$, for some $i, j \in \{1, \ldots, n\}$ with $i \leq j$.

The HFLTS $\mathcal{S}$ is called the *full HFLTS.* Moreover, the empty set $\{\} = \emptyset$ is also considered as a HFLTS and it is called the *empty HFLTS.*

For the rest of this paper, the non-empty HFLTS, $H = \{x \in \mathcal{S} \mid a_i \leq x \leq a_j\}$, is denoted by $[a_i, a_j]$. Note that, if $j = i$, the HFLTS $[a_i, a_i]$ is expressed as the singleton $\{a_i\}$.

The set of all the possible HFLTSs over $\mathcal{S}$ is denoted by $\mathcal{H}_\mathcal{S}$, being $\mathcal{H}_\mathcal{S}^* = \mathcal{H}_\mathcal{S} - \{\emptyset\}$ the set of all the non-empty HFLTSs. This set is provided with a lattice structure in [6] with the two following operations: on the one hand, the *connected union of two HFLTSs,* $\sqcup$, which is defined as the least element of $\mathcal{H}_\mathcal{S}$, based on the subset inclusion relation $\subseteq$, that contains both HFLTSs, and on the other hand, the intersection of HFLTSs, $\cap$, which is defined as the usual intersection of sets. The reason of including the empty HFLTS in $\mathcal{H}_\mathcal{S}$ is to make the intersection of HFLTSs a closed operation in $\mathcal{H}_\mathcal{S}$.

For the sake of comprehensiveness, let us introduce the following example that is used throughout all this paper to depict all the concepts defined.

Example 1. Given the set of linguistic terms $\mathcal{S} = \{a_1, a_2, a_3, a_4, a_5\}$, being a_1 = *very bad*, a_2 = *bad*, a_3 = *regular*, a_4 = *good*, a_5 = *very good*, possible linguistic assessments and their corresponding HFLTSs by means of $\mathcal{S}$ would be:

Assessments	*HFLTSs*
$A =$ *"between bad and regular"*	$H_A = [a_2, a_3]$
$B =$ *"bad"*	$H_B = \{a_2\}$
$C =$ *"above regular"*	$H_C = [a_4, a_5]$
$D =$ *"below regular"*	$H_D = [a_1, a_2]$
$E =$ *"not very good"*	$H_E = [a_1, a_4]$

2 The Extended Lattice of Hesitant Fuzzy Linguistic Term Sets

With the aim of describing differences between couples of HFLTSs with empty intersections, an extension of the intersection of HFLTSs is presented in this section, resulting their intersection if it is not empty or a new element that we will call *negative HFLTS* related to the rift, or gap, between them if their intersection is empty. In order to present said extension of the intersection between HFLTSs, we first need to introduce the mathematical structure that allows us to define it as a closed operation. To this end, we define the extended set of HFLTSs in an analogous way to how integer numbers are defined based on an equivalence relation on the natural numbers. To do so, we first present some needed concepts:

Definition 2. Given two non-empty HFLTSs, $H_1, H_2 \in \mathcal{H}_{\mathcal{S}}^*$, we define:

(a) The *gap between* H_1 *and* H_2 as:

$$gap(H_1, H_2) = (H_1 \sqcup H_2) \cap \overline{H_1} \cap \overline{H_2}.$$

(b) H_1 and H_2 are *consecutive* if and only if $H_1 \cap H_2 = \emptyset$ and $gap(H_1, H_2) = \emptyset$.

Proposition 1. Given two non-empty HFLTSs, $H_1, H_2 \in \mathcal{H}_{\mathcal{S}}^*$, the following properties are met:

1. $gap(H_1, H_2) = gap(H_2, H_1)$.
2. If $H_1 \subseteq H_2$, $gap(H_1, H_2) = \emptyset$.
3. If $H_1 \cap H_2 \neq \emptyset$, $gap(H_1, H_2) = \emptyset$.
4. If $H_1 \cap H_2 = \emptyset$, $gap(H_1, H_2) \neq \emptyset$ or H_1 and H_2 are consecutive.
5. If H_1 and H_2 are consecutive, there exist $j \in \{2, \ldots, n-1\}$, $i \in \{1, \ldots, j\}$ and $k \in \{j+1, \ldots, n\}$, such that $H_1 = [a_i, a_j]$ and $H_2 = [a_{j+1}, a_k]$ or $H_2 = [a_i, a_j]$ and $H_2 = [a_{j+1}, a_k]$.

Proof. The proof is straightforward. □

Note that neither $[a_1, a_j]$ nor $[a_i, a_n]$ can ever be the result of the *gap* between two HFLTSs for any i and for any j.

Notation. Given two consecutive HFLTSs, $H_1 = [a_i, a_j]$ and $H_2 = [a_{j+1}, a_k]$, then $\{a_j\}$ and $\{a_{j+1}\}$ are named as the linguistic terms that provide the consecutiveness of H_1, H_2.

Example 2. Following Example 1, $gap(H_B, H_C) = \{a_3\}$, while the HFLTSs H_A and H_C are consecutive and their consecutiveness is given by $\{a_3\}$ and $\{a_4\}$.

Definition 3. Given two pairs of non-empty HFLTSs, (H_1, H_2) and (H_3, H_4), the *equivalence relation* $\sim$, is defined as:

$$(H_1, H_2) \sim (H_3, H_4) \iff \begin{cases} H_1 \cap H_2 = H_3 \cap H_4 \neq \emptyset \\ \vee \\ gap(H_1, H_2) = gap(H_3, H_4) \neq \emptyset \\ \vee \\ \text{both pairs are consecutive and} \\ \text{their consecutiveness is provided} \\ \text{by the same linguistic terms} \end{cases}$$

It can be easily seen that $\sim$ relates couples of non-empty HFLTSs with the same intersection if they are compatible, with consecutiveness provided by the same linguistic terms if they are consecutive and with the same *gap* between them in the case that they are neither compatible nor consecutive.

Example 3. Following Example 1, the pairs of HFLTSs (H_A, H_B) and (H_A, H_D) are related according to $\sim$ given that they have the same intersection, $\{a_2\}$. Additionally, $(H_C, H_B) \sim (H_C, H_D)$ since they have the same *gap* between them, $\{a_3\}$.

Applying this equivalence relation over the set of all the pairs of non-empty HFLTSs, we get the quotient set $(\mathcal{H}_{\mathcal{S}}^*)^2/\sim$, whose equivalence classes can be labeled as:

- $[a_i, a_j]$ for the class of all pairs of compatible non-empty HFLTSs with intersection $[a_i, a_j]$, for all $i, j = 1, \ldots, n$ with $i \leq j$.
- $-[a_i, a_j]$ for the class of all pairs of incompatible non-empty HFLTSs whose *gap* is $[a_i, a_j]$, for all $i, j = 2, \ldots, n-1$ with $i \leq j$.
- α_i for the class of all pairs of consecutive non-empty HFLTSs whose consecutiveness is provided by $\{a_i\}$ and $\{a_{i+1}\}$, for all $i = 1, \ldots, n-1$.

For completeness and symmetry reasons, $(\mathcal{H}_{\mathcal{S}}^*)^2/\sim$ is represented as shown in Fig. 1 and stated in the next definition.

Example 4. Subsequent to this labeling, and following Example 1, the pair (H_C, H_B) belongs to the class $-\{a_3\}$ and so does the pair (H_C, H_D). The pair (H_C, H_A) belongs to the class α_3 and the pair (H_C, H_E) belongs to the class $\{a_4\}$.

Definition 4. Given a set of ordered linguistic term sets $\mathcal{S} = \{a_1, \ldots, a_n\}$, the *extended set of HFLTSs*, $\overline{\mathcal{H}_{\mathcal{S}}}$, is defined as:

$$\overline{\mathcal{H}_{\mathcal{S}}} = (-\mathcal{H}_{\mathcal{S}}^*) \cup \mathcal{A} \cup \mathcal{H}_{\mathcal{S}}^*,$$

where $-\mathcal{H}_{\mathcal{S}}^* = \{-H \mid H \in \mathcal{H}_{\mathcal{S}}^*\}$ and $\mathcal{A} = \{\alpha_0, \ldots, \alpha_n\}$.

In addition, by analogy with real numbers $-\mathcal{H}_{\mathcal{S}}^*$ is called the *set of negative HFLTSs*, $\mathcal{A}$ is called the *set of zero HFLTSs*, and, from now on, $\mathcal{H}_{\mathcal{S}}^*$ is called the *set positive HFLTSs.*

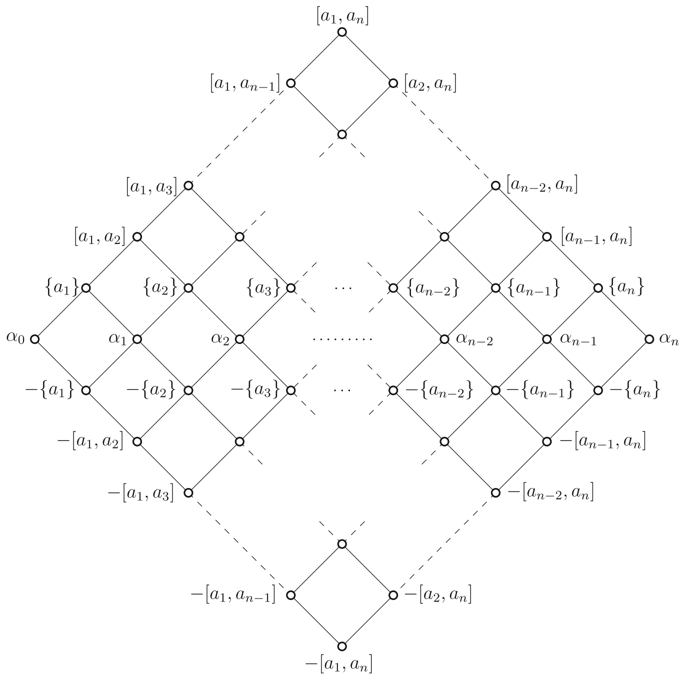


Fig. 1. Graph of the extended set of HFLTSs.

Note that HFLTSs can be characterized by couples of zero HFLTSs. This leads us to introduce a new notation for HFLTSs:

Notation. Given a HFLTS, $H \in \overline{\mathcal{H}_{\mathcal{S}}}$, it can be expressed as $H = \langle \alpha_i, \alpha_j \rangle$, where the first zero HFLTS identifies the bottom left to top right diagonal and the second one identifies the top left to bottom right diagonal. Thus, $\langle \alpha_i, \alpha_j \rangle$ corresponds with $[a_{i+1}, a_j]$ if $i < j$, with $-[a_{i+1}, a_j]$ if $i > j$ and α_i if $i = j$.

This notation is used in the following definition that we present in order to latter introduce an order relation within $\overline{\mathcal{H}_{\mathcal{S}}}$.

Definition 5. Given $H \in \overline{\mathcal{H}_{\mathcal{S}}}$ described by $\langle \alpha_i, \alpha_j \rangle$ the coverage of H is defined as:

$$cov(H) = \{\langle \alpha_{i'}, \alpha_{j'} \rangle \in \overline{\mathcal{H}_{\mathcal{S}}} \mid i' \geq i \wedge j' \leq j\}.$$

Example 5. The coverage of H_A from Example 1 can be seen in Fig. 2.

The concept of coverage of a HFLTS enables us to define the *extended inclusion relation* between elements of $\overline{\mathcal{H}_{\mathcal{S}}}$.

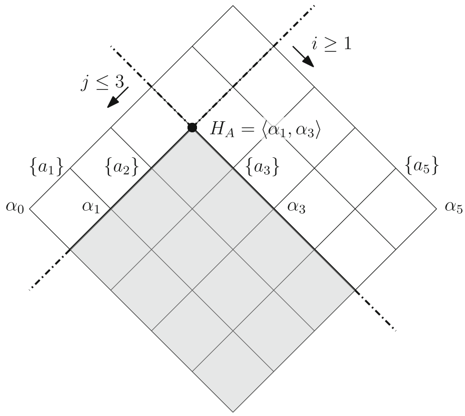


Fig. 2. Coverage of H_A.

Definition 6. The *extended inclusion relation in*$\overline{\mathcal{H}_{\mathcal{S}}}$, $\preccurlyeq$, is defined as:

$$\forall H_1, H_2 \in \overline{\mathcal{H}_{\mathcal{S}}}, \quad H_1 \preccurlyeq H_2 \iff H_1 \in cov(H_2).$$

Note that, restricting to only the positive HFLTSs, the extended inclusion relation coincides with the usual subset inclusion relation. According to this relation in $\overline{\mathcal{H}_{\mathcal{S}}}$, we can define the *extended connected union* and the *extended intersection* as closed operations within the set $\overline{\mathcal{H}_{\mathcal{S}}}$ as follows:

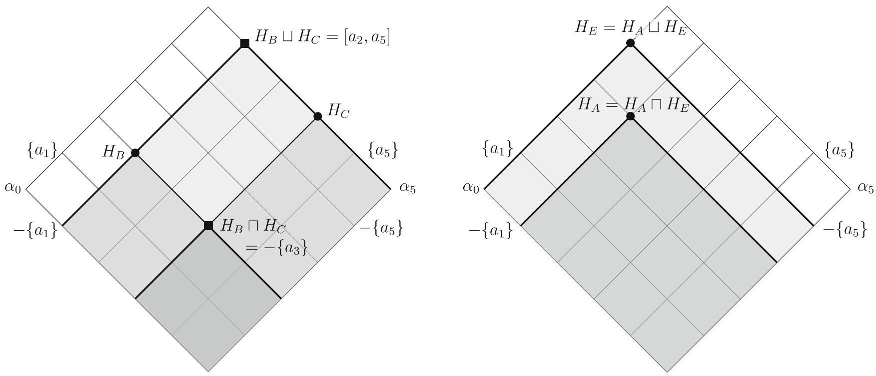
Definition 7. Given $H_1, H_2 \in \overline{\mathcal{H}_{\mathcal{S}}}$, the *extended connected union of* H_1 *and* H_2, $H_1 \sqcup H_2$, is defined as the least element that contains H_1 and H_2, according to the extended inclusion relation.

Definition 8. Given $H_1, H_2 \in \overline{\mathcal{H}_{\mathcal{S}}}$, the *extended intersection of* H_1 *and* H_2, $H_1 \sqcap H_2$, is defined as the largest element being contained in H_1 and H_2, according to the extended inclusion relation.

It is straightforward to see that the extended connected union of two positive HFLTSs coincides with the connected union presented in [6]. This justifies the use of the same symbol. About the extended intersection of two positive HFLTSs, it results the usual intersection of sets if they overlap and the *gap* between them if they do not overlap. Notice that the empty HFLTS is not needed to make the extended intersection a closed operation in $\overline{\mathcal{H}_{\mathcal{S}}}$.

Proposition 2. Given two non-empty HFLTSs, $H_1, H_2 \in \mathcal{H}^*_{\mathcal{S}}$, if $H_1 \preccurlyeq H_2$, then $H_1 \sqcup H_2 = H_2$ and $H_1 \sqcap H_2 = H_1$.

Proof. The proof is straightforward. □



(a) $H_B \sqcup H_C$ and $H_B \sqcap H_D$. (b) $H_A \sqcup H_E$ and $H_A \sqcap H_E$.

Fig. 3. $\sqcup$ and $\sqcap$ of HFLTSs.

Example 6. Figure 3 provides an example with the extended connected union and the extended intersection of H_B and H_C and of H_A and H_E from Example 1: $H_B \sqcup H_C = [a_2, a_5]$, $H_B \sqcap H_C = -\{a_3\}$, $H_A \sqcup H_E = H_E$ and $H_A \sqcap H_E = H_A$.

Proposition 3. $(\overline{\mathcal{H}_\mathcal{S}}, \sqcup, \sqcap)$ is a distributive lattice.

Proof. According to their respective definitions, both operations, $\sqcup$ and $\sqcap$, are trivially commutative and idempotent.

The associative property of $\sqcup$ is met since $(H_1 \sqcup H_2) \sqcup H_3 = H_1 \sqcup (H_2 \sqcup H_3)$ given that both parts equal the least element that contains H_1, H_2 and H_3. About the associativeness of $\sqcap$, $(H_1 \sqcap H_2) \sqcap H_3 = H_1 \sqcap (H_2 \sqcap H_3)$ given that in both cases it results the largest element contained in H_1, H_2 and H_3.

Finally, the absorption laws are satisfied given that: on the one hand $H_1 \sqcup (H_1 \sqcap H_2) = H_1$ given that $H_1 \sqcap H_2 \preccurlyeq H_1$ and on the other hand $H_1 \sqcap (H_1 \sqcup H_2) = H_1$ given that $H_1 \preccurlyeq H_1 \sqcup H_2$.

Furthermore, the lattice $(\overline{\mathcal{H}_\mathcal{S}}, \sqcup, \sqcap)$ is distributive given that none of its sublattices are isomorphic to the diamond lattice, M_3, or the pentagon lattice, N_5. □

3 A Distance Between Hesitant Fuzzy Linguistic Term Sets

In order to define a distance between HFLTSs, we introduce a generalization of the concept of cardinal of a positive HFLTS to all the elements of the extended set of HFLTSs.

Definition 9. Given $H \in \overline{\mathcal{H}_\mathcal{S}}$, the *width of* H is defined as:

$$\mathcal{W}(H) = \begin{cases} card(H) & if\ H \in \mathcal{H}_\mathcal{S}^*, \\ 0 & if\ H \in \mathcal{A}, \\ -card(-H) & if\ H \in (-\mathcal{H}_\mathcal{S}^*). \end{cases}$$

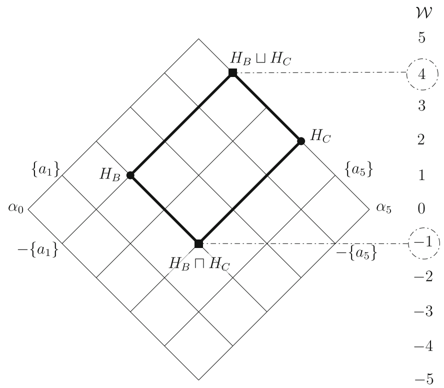


Fig. 4. Distance between HFLTSs.

Note that the width of a HFLTS could be related as well with the height on the graph of $\overline{\mathcal{H}_\mathcal{S}}$, associating the zero HFLTSs with height 0, the positive HFLTSs with positive heights and the negative HFLTSs with negative values of heights as shown in Fig. 4.

Proposition 4. $D(H_1, H_2) = \mathcal{W}(H_1 \sqcup H_2) - \mathcal{W}(H_1 \sqcap H_2)$ provides a distance in the lattice $(\overline{\mathcal{H}_\mathcal{S}}, \sqcup, \sqcap)$.

Proof. $D(H_1, H_2)$ defines a distance given that it is equivalent to the geodesic distance in the graph $\overline{\mathcal{H}_\mathcal{S}}$. The geodesic distance between H_1 and H_2 is the length of the shortest path to go from H_1 to H_2. Due to the fact that $H_1 \sqcap H_2 \preccurlyeq H_1 \sqcup H_2$, $\mathcal{W}(H_1 \sqcup H_2) - \mathcal{W}(H_1 \sqcap H_2)$ is the length of the minimum path between $H_1 \sqcup H_2$ and $H_1 \sqcap H_2$. Thus, we have to check that the length of the shortest path between $H_1 \sqcup H_2$ and $H_1 \sqcap H_2$ coincides with the length of the shortest path between H_1 and H_2.

If one of them belong to the coverage of the other one, let us suppose that $H_1 \preccurlyeq H_2$, then $H_1 \sqcup H_2 = H_2$ and $H_1 \sqcap H_2 = H_1$ and the foregoing assertion becomes obvious. If not, H_1, $H_1 \sqcup H_2$, H_2 and $H_1 \sqcap H_2$ define a parallelogram on the graph. Two consecutive sides of this parallelogram define the shortest path between $H_1 \sqcup H_2$ and $H_1 \sqcap H_2$ while two other consecutive sides of the same parallelogram define the shortest path between H_1 and H_2. Thus, the assertion becomes true as well.

Proposition 5. Given two HFLTSs, $H_1, H_2 \in \overline{\mathcal{H}_\mathcal{S}}$, then $D(H_1, H_2) \leq 2n$. If, in addition, $H_1, H_2 \in \mathcal{H}^*_\mathcal{S}$, then $D(H_1, H_2) \leq 2n - 2$.

Proof. If $H_1, H_2 \in \overline{\mathcal{H}_\mathcal{S}}$, then, the most distant pair is α_0 and α_n. Then,

$$\mathcal{W}(\alpha_0 \sqcup \alpha_n) - \mathcal{W}(\alpha_0 \sqcap \alpha_n) = \mathcal{W}([a_1, a_n]) - \mathcal{W}(-[a_1, a_n]) =$$

$$n - (-n) = 2n.$$

If $H_1, H_2 \in \mathcal{H}^*_{\mathcal{S}}$, then, the most distant pair is $\{a_1\}$ and $\{a_n\}$. Then,

$$\mathcal{W}(\{a_1\} \sqcup \{a_n\}) - \mathcal{W}(\{a_1\} \sqcap \{a_n\}) = \mathcal{W}([a_1, a_n]) - \mathcal{W}(-[a_2, a_{n-1}]) =$$

$$n - (-(n-2)) = 2n - 2.$$

□

Notice that for positive HFLTSs, $D(H_1, H_2)$ coincides with the distance $D_2(H_1, H_2)$ introduced in [6]. Additionally, in this case, the distance presented can also be calculated as $D([a_i, a_j], [a_{i'}, a_{j'}]) = |i - i'| + |j - j'|$.

Example 7. Figure 4 shows the width of the extended connected union and the extended intersection of H_B and H_C from Example 1. According to these results, $D(H_B, H_C) = \mathcal{W}(H_B \sqcup H_C) - \mathcal{W}(H_B \sqcap H_C) = 4 - (-1) = 5$.

4 A Representative of a Group Assessment

The aim of this section is to model the assessments given by a group of Decision Makers (DMs) that are evaluating a set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ by means of positive HFLTSs over $\mathcal{S} = \{a_1, \ldots, a_n\}$. To do so, we use the definition of Hesitant Fuzzy Linguistic Description (HFLD) introduced in [6].

Definition 10. A *Hesitant fuzzy linguistic description of the set* Λ *by* $\mathcal{H}_{\mathcal{S}} - \{\emptyset\}$ is a function F_H on Λ such that for all $\lambda \in \Lambda$, $F_H(\lambda)$ is a non-empty HFLTS, i.e., $F_H(\lambda) \in \mathcal{H}_{\mathcal{S}} - \{\emptyset\}$.

According to this definition, we can extend the distance between HFLTSs presented in Sect. 3 to a distance between HFLDs as follows:

Definition 11. Let us consider F^1_H and F^2_H two HFLDs of a set $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ by means of $\mathcal{H}_{\mathcal{S}}$, with $F^1_H(\lambda_i) = H^1_i$ and $F^2_H(\lambda_i) = H^2_i$, for all $i \in \{1, \ldots, r\}$. Then, the *distance* $D^{\mathcal{F}}$ *between these two HFLDs* is defined as:

$$D^{\mathcal{F}}(F^1_H, F^2_H) = \sum_{t=1}^{r} D(H^1_t, H^2_t).$$

Thus, given a set ok k DMs, we have k different HFLDs of the set of alternatives Λ. In order to summarize this k different assessments, we propose a HFLD that serves as a group representative.

Definition 12. Let Λ be a set of r alternatives, G a group of k DMs and $F^1_H, \ldots, F^k_H$ the HFLDs of Λ provided by the DMs in G, then, the*centroid of the group*is:

$$F^C_H = arg \min_{F^x_H \in (\mathcal{H}^*_{\mathcal{S}})^r} \sum_{t=1}^{k} D^{\mathcal{F}}(F^x_H, F^t_H),$$

identifying each HFLD F_H with the vector $(H_1, \ldots, H_r) \in (\mathcal{H}^*_{\mathcal{S}})^r$, where $F_H(\lambda_i) = H_i$, for all $i = 1, \ldots, r$.

Note that the HFLD of the centroid of the group does not have to coincide with any of the HFLDs given by the DMs. In addition, there can be more than one HFLDs minimizing the addition of distances to the assessments given by the DMs, so the centroid of the group is not necessarily unique. Consequently, we proceed with a further study of the possible unicity of the centroid of the group.

Proposition 6. For a specific alternative λ, let $F_H^1(\lambda), \ldots, F_H^k(\lambda)$ be the HFLTSs given as assessments of λ by a group of k DMs. Then, if $F_H^p(\lambda) = [a_{i_p}, a_{j_p}], \forall p \in \{1, \ldots, k\}$, the set of all the HFLTSs associated to the centroid of the group for λ is:

$$\{[a_i, a_j] \in \mathcal{H}_{\mathcal{S}}^* \mid i \in med(i_1, \ldots, i_k), j \in med(j_1, \ldots, j_k)\},$$

where $med(\ )$ contains the median of the values sorted from smallest to largest if k is odd or any integer number between the two central values sorted in the same order if k is even.

Proof. It is straightforward to check that the distance D between HFLTSs is equivalent to the Manhattan distance, also known as taxicab distance, because the graph of $\overline{\mathcal{H}_{\mathcal{S}}}$ can be seen as a grid. Thus, finding the HFLTSs that corresponds to the centroid of the group is reduced to finding the HFLTSs in the grid that minimizes the addition of distances to the other HFLTSs given by the DMs.

The advantage of the taxicab metric is that it works with two independent components, in this case, initial linguistic term and ending linguistic term. Therefore, we can solve the problem for each component separately. For each component, we have a list of natural numbers and we want to find the one minimizing distances. It is well known that the median is the number satisfying a minimum addition of distances to all the points, generalizing the median to all the numbers between the two central ones if there is an even amount of numbers.

Thus, all the HFLTSs satisfying a minimum addition of distances are:

$$\{[a_i, a_j] \in \overline{\mathcal{H}_{\mathcal{S}}} \mid i \in med(i_1, \ldots, i_k), j \in med(j_1, \ldots, j_k)\}.$$

Finally, we have to check that the HFLTSs associated to the centroid are positive HFLTSs for the F_H^C to be a HFLD. If $F_H^p(\lambda) = [a_{i_p}, a_{j_p}] \in \mathcal{H}_{\mathcal{S}}^*, \forall p \in \{1, \ldots, k\}$, that means $i_p \leq j_p, \forall p \in \{1, \ldots, k\}$. Therefore, if k is odd, the median of $i_1, \ldots, i_k$ is less than or equal to the median of $j_1, \ldots, j_k$, and if k is even, the minimum value of $med(i_1, \ldots, i_k)$ is less than or equal than the maximum value of $med(j_1, \ldots, j_k)$. Accordingly, there is always at least one HFLTS associated to the centroid which is a positive HFLTS. Thus,

$$\{[a_i, a_j] \in \mathcal{H}_{\mathcal{S}}^* \mid i \in med(i_1, \ldots, i_k), j \in med(j_1, \ldots, j_k)\}.$$

□

Example 8. Let us assume that H_A, H_B, H_C, H_D, H_E from Example 1 are the assessments given by 5 DMs about the same alternative. In such case, $med(2, 2, 4, 1, 1) = 2$ and $med(3, 2, 5, 2, 4) = 3$, and, therefore, the central assessment for this alternative is $[a_2, a_3]$.

Corollary 1. For a group of k DMs, if k is odd, the centroid of the group is unique.

Proof. If k is odd, both medians are from a set with an odd amount of numbers, so both medians are unique. Therefore, the corresponding HFLTS minimizing the addition of distances is also unique. □

Corollary 2. For each alternative in Λ, the set of all the HFLTSs corresponding to any centroid of the group is a connected set in the graph of $\overline{\mathcal{H}_{\mathcal{S}}}$.

Proof. If k is odd, by Corollary 1, the proof results obvious. If k is even, by the definition of $med(\ )$, the set of possible results is also connected. □

Example 9. Let G be a group of 5 DMs assessing a set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_4\}$ by means of HFLTSs over the set $\mathcal{S} = \{a_1, a_2, a_3, a_4, a_5\}$ from Example 1, and let $F_H^1, F_H^2, F_H^3, F_H^4, F_H^5$ the HFLDs describing their corresponding assessments shown in the following table together with the HFLD corresponding to the centroid of the group:

	F_H^1	F_H^2	F_H^3	F_H^4	F_H^5	$\boldsymbol{F_H^C}$
λ_1	$[a_2, a_3]$	$\{a_2\}$	$[a_4, a_5]$	$[a_1, a_2]$	$[a_1, a_4]$	$\boldsymbol{[a_2, a_3]}$
λ_2	$[a_1, a_2]$	$\{a_1\}$	$[a_2, a_3]$	$[a_1, a_2]$	$\{a_2\}$	$\boldsymbol{[a_1, a_2]}$
λ_3	$[a_3, a_5]$	$\{a_3\}$	$\{a_4\}$	$[a_1, a_4$	$[a_2, a_4]$	$\boldsymbol{[a_3, a_4]}$
λ_4	$[a_4, a_5]$	$\{a_5\}$	$\{a_5\}$	$\{a_5\}$	$[a_1, a_2]$	$\boldsymbol{\{a_5\}}$

As the last alternative shows, the centroid of the group is not sensible to outliers, due to the fact that is based on the calculation of two medians.

5 Conclusions and Future Research

This paper presents an extension of the set of Hesitant Fuzzy Linguistic Term Sets by introducing the concepts of negative and zero HFLTSs to capture differences between pair of non-compatible HFLTSs. This extension enables the introduction of a new operation studying the intersection and the gap between HFLTSs at the same time. This operation is used to define a distance between HFLTSs that allows us to analyze differences between the assessments given by a group of decision makers. Based on the study of these differences, a centroid of the group has been proposed.

Future research is focused in two main directions. First, the study of the consensus level of the total group assessments to analyze the agreement or disagreement within the group. And secondly, a real case study will be performed in the marketing research area to examine consensus and heterogeneities in consumers' preferences.

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