About the Use of Admissible Order for Defining Implication Operators

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Abstract. Implication functions are crucial operators for many fuzzy logic applications. In this work, we consider the definition of implication functions in the interval-valued setting using admissible orders and we use this interval-valued implications for building comparison measures.

Keywords: Interval-valued implication operator · Admissible order · Similarity measure

1 Introduction

Implication operators are crucial for many applications of fuzzy logic, including approximate reasoning or image processing. Many works have been devoted to

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the analysis of these operators, both in the case of fuzzy sets [1,2,14,15] and in the case of extensions [3–5,13,16]. A key problem in order to define these operators is that of monotonicity. When implication operators are extended to fuzzy extensions, this problem is not trivial, since for most of the fuzzy extensions do not exist a linear order, whereas for some applications, as it is the case of fuzzy rules-based classification systems, it is necessary to have the possibility of comparing any two elements [12].

In this work, we propose the definition of implication operators in the interval-valued setting defining its monotonicity in terms of the so-called admissible orders [11]. This is a class of linear orders which extends the usual order between intervals and which include the most widely used examples of linear orders between intervals, as lexicographical and Xu and Yager ones.

As a first step in a deeper study of these interval-valued implications with admissible orders, we show how implications which are defined in terms of admissible orders can be used to build comparison measures which are of interest from the point of view of applications.

The structure of the present work is as follows. In Sect. 2 we present some preliminary definitions and results. In Sect. 3 we present the definition of interval-valued implication function with respect to an admissible order. Section 4 is devoted to obtaining equivalence and restricted equivalence functions with respect to linear orders. In Sect. 5 we use our previous results to build comparison measures. We finish with some conclusions and references.

2 Preliminaries

In this section we introduce several well known notions and results which will be useful for our subsequent developments.

We are going to work with closed subintervals of the unit interval. For this reason, we define:

$$L([0,1]) = \{[\underline{X}, \overline{X}] \mid 0 \leq \underline{X} \leq \overline{X} \leq 1\}.$$

By $\leq_L$ we denote an arbitrary order relation on $L([0,1])$ with $0_L = [0,0]$ as its minimal element and $1_L = [1,1]$ as maximal element. This order relation can be partial or total. If we must consider an arbitrary total order, we will denote it by $\leq_{TL}$.

Example 1. The partial order relation on $L([0,1])$ induced by the usual partial order in $\mathbb{R}^2$ is:

$$[\underline{X}, \overline{X}] \precsim_L [\underline{Y}, \overline{Y}] \text{ if } \underline{X} \leq \underline{Y} \text{ and } \overline{X} \leq \overline{Y}. \tag{1}$$

As an example of total order in $L([0,1])$ we have Xu and Yager's order (see [17]):

$$[\underline{X}, \overline{X}] \leq_{XY} [\underline{Y}, \overline{Y}] \text{ if } \begin{cases} \underline{X} + \overline{X} < \underline{Y} + \overline{Y} \text{ or} \\ \underline{X} + \overline{X} = \underline{Y} + \overline{Y} \text{ and } \overline{X} - \underline{X} \leq \overline{Y} - \underline{Y}. \end{cases} \tag{2}$$

Definition 1. *An admissible order in* $L([0,1])$ *is a total order* $\leq_{TL}$ *which extends the partial order* $\precsim_L$.

In the following, whenever we speak of a total order we assume it is an admissible order.

Definition 2. *Let* $\leq_L$ *be an order relation in* $L([0,1])$. *A function* $N\colon L([0,1]) \to L([0,1])$ *is an interval-valued negation function (IV negation) if it is a decreasing function with respect to the order* $\leq_L$ *such that* $N(0_L) = 1_L$ *and* $N(1_L) = 0_L$. *A negation* N *is called strong negation if* $N(N(X)) = X$ *for every* $X \in L([0,1])$. *A negation* N *is called non-filling if* $N(X) = 1_L$ *iff* $X = 0_L$, *while* N *is called non-vanishing if* $N(X) = 0_L$ *iff* $X = 1_L$.

We recall now the definition of interval-valued aggregation function.

Definition 3. *Let* $n \geq 2$. *An (n-dimensional) interval-valued (IV) aggregation function in* $(L([0,1]), \leq_L, 0_L, 1_L)$ *is a mapping* $M : (L([0,1]))^n \to L([0,1])$ *which verifies:*

(i) $M(0_L, \cdots, 0_L) = 0_L$.
(ii) $M(1_L, \cdots, 1_L) = 1_L$.
(iii) M *is an increasing function with respect to* $\leq_L$.

Example 2. Fix $\alpha \in [0,1]$. With the order $\leq_{XY}$, the function

$$M_\alpha\colon L([0,1])^2 \to L([0,1])$$

defined by

$$M_\alpha([\underline{X}, \overline{X}], [\underline{Y}, \overline{Y}]) = [\alpha\underline{X} + (1-\alpha)\underline{Y}, \alpha\overline{X} + (1-\alpha)\overline{Y}]$$

is an IV aggregation function.

3 Interval-Valued Implication Functions

Definition 4 (cf. [5] and [2]**).** *An interval-valued (IV) implication function in* $(L([0,1]), \leq_L, 0_L, 1_L)$ *is a function* $I\colon (L([0,1]))^2 \to L([0,1])$ *which verifies the following properties:*

(i) I *is a decreasing function in the first component and an increasing function in the second component with respect to the order* $\leq_L$.
(ii) $I(0_L, 0_L) = I(0_L, 1_L) = I(1_L, 1_L) = 1_L$.
(iii) $I(1_L, 0_L) = 0_L$.

Some properties that can be demanded to an IV implication function are the following [10]:

$I4$: $I(X,Y) = 0_L \Leftrightarrow X = 1_L$ and $Y = 0_L$.
$I5$: $I(X,Y) = 1_L \Leftrightarrow X = 0_L$ or $Y = 1_L$.

NP: $I(1_L, Y) = Y$ for all $Y \in L([0,1])$.
EP: $I(X, I(Y, Z)) = I(Y, I(X, Z))$ for all $X, Y, Z \in L([0,1])$.
OP: $I(X, Y) = 1_L \Leftrightarrow X \leq_L Y$.
SN: $N(X) = I(X, 0_L)$ is a strong IV negation.
$I10$: $I(X, Y) \geq_L Y$ for all $X, Y \in L([0,1])$.
IP: $I(X, X) = 1_L$ for all $X \in L([0,1])$.
CP: $I(X, Y) = I(N(Y), N(X))$ for all $X, Y \in L([0,1])$, where N is an IV negation.
$I14$: $I(X, N(X)) = N(X)$ for all $X \in L([0,1])$, where N is an IV negation.

We can obtain IV implication functions from IV aggregation functions as follows.

Proposition 1. *Let M be an IV aggregation function such that*

$$M(1_L, 0_L) = M(0_L, 1_L) = 0_L$$

and let N be an IV negation in $L([0,1])$, both with respect to the same order $\leq_L$.
Then the function $I_M : L([0,1])^2 \to L([0,1])$ given by

$$I_M(X, Y) = N(M(X, N(Y)))$$

is an IV implication function.

Proof. It follows from a straight calculation. □

However, in this work we are going to focus on a different construction method for IV implication functions.

Proposition 2. *Let $\leq_{TL}$ be a total order in $L([0,1])$, and let N be an IV negation function with respect to that order. The function $I : L([0,1])^2 \to L([0,1])$ defined by*

$$I(X, Y) = \begin{cases} 1_L, & if\ X \leq_{TL} Y, \\ \vee(N(X), Y), & if\ X >_{TL} Y. \end{cases}$$

is an IV implication function.

Proof. It is clear that the function I is an increasing function in the second component and a decreasing function in the first component. Moreover

$$I(0_L, 0_L) = I(0_L, 1_L) = I(1_L, 1_L) = 1_L$$

and $I(1_L, 0_L) = 0_L$. □

This result can be further generalized as follows [15]:

Proposition 3. *Let $\leq_{TL}$ be a total order in $L([0,1])$, and let N be an IV negation function with respect to that order. If $M : L([0,1])^2 \to L([0,1])$ is an IV aggregation function, then the function $I : L([0,1])^2 \to L([0,1])$ defined by*

$$I(X, Y) = \begin{cases} 1_L, & if\ X \leq_{TL} Y, \\ M(N(X), Y), & if\ X >_{TL} Y, \end{cases}$$

is an IV implication function.

4 Equivalence and Restricted Equivalence Functions in $L([0,1])$ with Respect to a Total Order

Along this section only total orders are considered.

The equivalence functions [6–8] are a fundamental tool in order to build measures of similarity between fuzzy sets. In this section we construct interval-valued equivalence functions from IV aggregation and negation functions.

Definition 5. *A map $F\colon L([0,1])^2 \to L([0,1])$ is called an interval-valued (IV) equivalence function in $(L([0,1]), \leq_{TL})$ if F verifies:*

(1) $F(X,Y) = F(Y,X)$ for every $X, Y \in L([0,1])$.
(2) $F(0_L, 1_L) = F(1_L, 0_L) = 0_L$.
(3) $F(X,X) = 1_L$ for all $X \in L([0,1])$.
(4) If $X \leq_{TL} X' \leq_{TL} Y' \leq_{TL} Y$, then $F(X,Y) \leq_{TL} F(X',Y')$.

Theorem 1. *Let $M_1\colon L([0,1])^2 \to L([0,1])$ be an IV aggregation function such that $M_1(X,Y) = M_1(Y,X)$ for every $X,Y \in L([0,1])$, $M_1(X,Y) = 1_L$ if and only if $X = Y = 1_L$ and $M_1(X,Y) = 0_L$ if and only if $X = 0_L$ or $Y = 0_L$. Let $M_2\colon L([0,1])^2 \to L([0,1])$ be an IV aggregation function such that $M_2(X,Y) = 1_L$ if and only if $X = 1_L$ or $Y = 1_L$ and $M_2(X,Y) = 0_L$ if and only if $X = Y = 0_L$. Then the function $F\colon L([0,1])^2 \to L([0,1])$ defined by*

$$F(X,Y) = M_1(I(X,Y), I(Y,X)),$$

with I the IV implication function defined in the Proposition 3 taking $M = M_2$, is an IV equivalence function.

Proof. Since

$$F(X,Y) = \begin{cases} 1_L, & \text{if } X = Y, \\ M_1(M_2(N(Y),X),1_L), & \text{if } X <_{TL} Y, \\ M_1(M_2(N(X),Y),1_L), & \text{if } Y <_{TL} X, \end{cases}$$

then F verifies the four properties in Definition 5. □

In [8] the definition of equivalence function (in the real case) was modified in order to define the so-called restricted equivalence function. Now we develop a similar study for the case of IV equivalence functions.

Definition 6. *Let N be an IV negation. A map $F\colon L([0,1])^2 \to L([0,1])$ is called an interval valued (IV) restricted equivalence function (in $(L([0,1]), \leq_{TL})$) if F verifies the following properties:*

1. *$F(X,Y) = F(Y,X)$ for all $X,Y \in L([0,1])$.*
2. *$F(X,Y) = 1_L$ if and only if $X = Y$.*
3. *$F(X,Y) = 0_L$ if and only if $X = 0_L$ and $Y = 1_L$, or, $X = 1_L$ and $Y = 0_L$.*
4. *$F(X,Y) = F(N(X), N(Y))$ for all $X,Y \in L([0,1])$.*
5. *If $X \leq_{TL} Y \leq_{TL} Z$, then $F(X,Z) \leq_{TL} F(X,Y)$ and $F(X,Z) \leq_{TL} F(Y,Z)$.*

Theorem 2. *Let N be an IV negation function. Let $M_1 \colon L([0,1])^2 \to L([0,1])$ be an IV aggregation function such that $M_1(X,Y) = M_1(Y,X)$ for every $X, Y \in L([0,1])$, $M_1(X,Y) = 1_L$ if and only if $X = Y = 1_L$ and $M_1(X,Y) = 0_L$ if and only if $X = 0_L$ or $Y = 0_L$. Let $M_2 \colon L([0,1])^2 \to L([0,1])$ be an IV aggregation function such that $M_2(X,Y) = 1_L$ if and only if $X = 1_L$ or $Y = 1_L$ and $M_2(X,Y) = 0_L$ if and only if $X = Y = 0_L$. Then the function $F \colon L([0,1])^2 \to L([0,1])$ defined by*

$$F(X,Y) = M_1(I(X,Y), I(Y,X))$$

with I an IV implication function defined by

$$I(X,Y) = \begin{cases} 1_L & if\ X \leq_{TL} Y \\ M_2(N(X),Y) & otherwise, \end{cases}$$

verifies the properties (1) and (5) of Definition 6. Moreover, it satisfies property (2) if N is non-filling and property (3) if N is non-vanishing.

Proof. Since

$$F(X,Y) = \begin{cases} 1_L, & \text{if } X = Y \\ M_1(M_2(N(Y),X), 1_L), & \text{if } X <_{TL} Y \\ M_1(M_2(N(X),Y), 1_L), & \text{if } Y <_{TL} X \end{cases}$$

then F verifies:

(1) $F(X,Y) = F(Y,X)$ trivially.
(5) If $X \leq_{TL} Y \leq_{TL} Z$, then $N(Z) \leq_{TL} N(Y) \leq_{TL} N(X)$. Since M_1 is an increasing function then $F(X,Z) \leq_{TL} F(X,Y)$ and $F(X,Z) \leq_{TL} F(Y,Z)$.

Since $M_1(X,Y) = 1_L$ if and only if $X = Y = 1_L$, then, if N is non-filling, $F(X,Y) = 1_L$ if and only if $X = Y$ because

$$\begin{cases} M_2(N(Y),X) \neq 1_L, & \text{if } X <_{TL} Y \\ M_2(N(X),Y) \neq 1_L, & \text{if } X >_{TL} Y. \end{cases}$$

Moreover, $F(X,Y) = 0_L$ if and only if $X >_{TL} Y$ and $M_2(N(X),Y) = 0_L$ or $X <_{TL} Y$ and $M_2(N(Y),X) = 0_L$. Therefore, as N is non-vanishing, $F(X,Y) = 0_L$ if and only if

$$\begin{cases} X = 0_L \ \text{ or } \ Y = 1_L \ \text{ or} \\ Y = 0_L \ \text{ or } \ X = 1_L. \end{cases}$$

with $X \neq Y$. □

5 Similarity Measures, Distances and Entropy Measures in $L([0,1])$ with Respect to a Total Order

Our constructions in the previous section can be used to build comparison measures between interval-valued fuzzy sets, and, more specifically, to obtain similarity measures, distances in the sense of Fang and entropy measures. Along this section, we only deal with a total order $\leq_{TL}$.

To start, let us consider a finite referential set of n elements, $U = \{u_1, \ldots, u_n\}$. We denote by $IVFS(U)$ the set of all interval-valued fuzzy sets over U. Recall that an interval-valued fuzzy set A over U is a mapping $A : U \to L([0,1])$ [9]. Note that the order $\leq_{TL}$ induces a partial order $\leq_{TL}$ in $IVFS(U)$ given, for $A, B \in IVFS(U)$, by

$$A \leq_{TL} B \text{ if } A(u_i) \leq_{TL} B(u_i) \text{ for every } u_i \in U.$$

First of all, we show how we can build a similarity between interval-valued fuzzy sets defined over the same referential U. We start recalling the definition.

Definition 7 [8]. *An interval-valued (IV) similarity measure on $IVFS(U)$ is a mapping $SM : IVFS(U) \times IVFS(U) \to L([0,1])$ such that, for every $A, B, A', B' \in IVFS(U)$,*

(SM1) SM is symmetric.
(SM2) $SM(A,B) = 1_L$ if and only if $A = B$.
(SM3) $SM(A,B) = 0_L$ if and only if $\{A(u_i), B(u_i)\} = \{0_L, 1_L\}$ for every $u_i \in U$.
(SM4) If $A \leq_{TL} A' \leq_{TL} B' \leq_{TL} B$, then $SM(A,B) \leq_{TL} SM(A',B')$.

Then we have the following result.

Theorem 3. *Let $M : L([0,1])^n \to L([0,1])$ be an IV aggregation function with respect to the total order $\leq_{TL}$ and such that $M(X_1, \ldots, X_n) = 1_L$ if and only if $X_1 = \cdots = X_n = 1_L$ and $M(X_1, \ldots, X_n) = 0_L$ if and only if $X_1 = \cdots = X_n = 0_L$. Then, the function $SM : IVFS(U) \times IVFS(U) \to L([0,1])$ given by*

$$SM(A,B) = M(F(A(u_1), B(u_1)), \ldots, F(A(u_n), B(u_n)))$$

where F is defined as in Theorem 2 with non-filling and non-vanishing negation, is an IV similarity measure.

Proof. It follows from a straightforward calculation. □

We can make use of this construction method to recover both distances and entropy measures. First of all, let's recall the definition of both concepts.

Definition 8 [6]. *A function $D : IVFS(U) \times IVFS(U) \to L([0,1])$ is called an IV distance measure on $IVFS(U)$ if, for every $A, B, A', B' \in IVFS(U)$, D satisfies the following properties:*

(D1) $D(A,B) = D(B,A)$;
(D2) $D(A,B) = 0_L$ if and only if $A = B$;
(D3) $D(A,B) = 1_L$ if and only if A and B are complementary crisp sets;
(D4) If $A \leq_{TL} A' \leq_{TL} B' \leq_{TL} B$, then $D(A,B) \geq_{TL} D(A',B')$.

Definition 9 [6]. *A function $E : IVFS(U) \to L([0,1])$ is called an entropy on $IVFS(U)$ with respect to a strong IV negation N (with respect to $\leq_{TL}$ such that there exists $\varepsilon \in L([0,1])$ with $N(\varepsilon) = \varepsilon$ if E has the following properties:*

(E1) $E(A) = 0_L$ *if and only if* A *is crisp;*
(E2) $E(A) = 1_L$ *if and only if* $A = \{(u_i, A(u_i) = \varepsilon) | u_i \in U\}$*;*
(E3) $E(A) \leq_{TL} E(B)$ *if* A *refines* B*; that is,* $A(u_i) \leq_{TL} B(u_i) \leq_{TL} \varepsilon$ *or* $A(u_i) \geq_{TL} B(u_i) \geq_{TL} \varepsilon$*;*
(E4) $E(A) = E(N(A))$*.*

Then the following two results are straight from Theorem 3.

Corollary 1. *Let* $M : L([0,1])^n \rightarrow L([0,1])$ *be an IV aggregation function with respect to the total order* $\leq_{TL}$ *such that* $M(X_1, \ldots, X_n) = 1_L$ *if and only if* $X_1 = \cdots = X_n = 1_L$ *and* $M(X_1, \ldots, X_n) = 0_L$ *if and only if* $X_1 = \cdots = X_n = 0_L$ *and let* N *be an IV negation with respect to the order* $\leq_{TL}$ *which is non filling and non-vanishing. Then, the function* $D : IVFS(U) \times IVFS(U) \rightarrow L([0,1])$ *given by*

$$D(A, B) = N(M(F(A(u_1), B(u_1)), \ldots, F(A(u_n), B(u_n))))$$

where F *is defined as in Theorem 2, is an IV distance measure.*

Proof. It is straight from Theorem 3, since a similarity measure defines a distance in a straightforward way. □

Theorem 4. *Let* N *be a strong IV negation (with respect to* $\leq_{TL}$*) and such that there exists* $\varepsilon \in L([0,1])$ *with* $N(\varepsilon) = \varepsilon$*. Let* $M : L([0,1])^n \rightarrow L([0,1])$ *be an IV aggregation function with respect to the total order* $\leq_{TL}$ *and such that* $M(X_1, \ldots, X_n) = 1_L$ *if and only if* $X_1 = \cdots = X_n = 1_L$ *and* $M(X_1, \ldots, X_n) = 0_L$ *if and only if* $X_1 = \cdots = X_n = 0_L$*. Then, the function* $E : IVFS(U) \rightarrow L([0,1])$ *given by*

$$E(A) = M(F(A(u_1), N(A(u_1))), \ldots, F(A(u_n), N(A(u_n))))$$

where F *is defined as in Theorem 2 with non-filling and non-vanishing negation, is an IV entropy measure.*

Proof. It follows from the well known fact that, for a given IV similarity SM, the function $E(A) = SM(A, N(A))$ is an IV entropy measure [6]. □

6 Conclusions

In this paper we have considered the problem of defining interval-valued implications when the order relation is a total order. In particular, we have considered the case of admissible orders. We have also studied the construction of interval-valued equivalence and similarity functions constructed with appropriate interval-valued implication functions. Finally we have shown how our constructions can be used to get IV similarity measures, distances and entropy measures with respect to total orders. In future works we will consider the use of these functions in different image processing, classification or decision making problems.

References

1. Baczyński, M., Beliakov, G., Bustince, H., Pradera, A.: Advances in Fuzzy Implication Functions. Studies in Fuzziness and Soft Computing, vol. 300. Springer, Berlin (2013)
2. Baczyński, M., Jayaram, B.: Fuzzy Implications. Studies in Fuzziness and Soft Computing, vol. 231. Springer, Berlin (2008)
3. Bedregal, B., Dimuro, G., Santiago, R., Reiser, R.: On interval fuzzy S-implications. Inf. Sci. **180**(8), 1373–1389 (2010)
4. Burillo, P., Bustince, H.: Construction theorems for intuitionistic fuzzy sets. Fuzzy Sets Syst. **84**, 271–281 (1996)
5. Bustince, H., Barrenechea, E., Mohedano, V.: Intuitionistic fuzzy implication operators. An expression and main properties. Int. J. Uncertain. Fuzziness Knowl.-Based Syst. **12**(3), 387–406 (2004)
6. Bustince, H., Barrenechea, E., Pagola, M.: Relationship between restricted dissimilarity functions, restricted equivalence functions and normal E_N-functions: image thresholding invariant. Pattern Recogn. Lett. **29**(4), 525–536 (2008)
7. Bustince, H., Barrenechea, E., Pagola, M.: Image thresholding using restricted equivalence functions and maximizing the measure of similarity. Fuzzy Sets Syst. **128**(5), 496–516 (2007)
8. Bustince, H., Barrenechea, E., Pagola, M.: Restricted equivalence functions. Fuzzy Sets Syst. **157**(17), 2333–2346 (2006)
9. Bustince, H., Barrenechea, E., Pagola, M., Fernandez, J., Xu, Z., Bedregal, B., Montero, J., Hagras, H., Herrera, F., De Baets, B.: A historical account of types of fuzzy sets and their relationship. IEEE Trans. Fuzzy Syst. **24**(1), 179–194 (2016)
10. Bustince, H., Burillo, P., Soria, F.: Automorphisms, negations and implication operators. Fuzzy Sets Syst. **134**, 209–229 (2003)
11. Bustince, H., Fernández, J., Kolesárová, A., Mesiar, R.: Generation of linear orders for intervals by means of aggregation functions. Fuzzy Sets Syst **220**, 69–77 (2013)
12. Bustince, H., Galar, M., Bedregal, B., Kolesárová, A., Mesiar, R.: A new approach to interval-valued Choquet integrals and the problem of ordering in interval-valued fuzzy set applications. IEEE Trans. Fuzzy Syst. **21**(6), 1150–1162 (2013)
13. Cornelis, C., Deschrijver, G., Kerre, E.E.: Implication in intuitionistic fuzzy and interval-valued fuzzy set theory: construction, classification, application. Int. J. Approx. Reason. **35**(1), 55–95 (2004)
14. Massanet, S., Mayor, G., Mesiar, R., Torrens, J.: On fuzzy implication: an axiomatic approach. Int. J. Approx. Reason. **54**, 1471–1482 (2013)
15. Pradera, A., Beliakov, G., Bustince, H., De Baets, B.: A review of the relationship between implication, negation and aggregation functions from the point of view of material implication. Inf. Sci. **329**, 357–380 (2016)
16. Riera, J.V., Torrens, J.: Residual implications on the set of discrete fuzzy numbers. Inf. Sci. **247**, 131–143 (2013)
17. Xu, Z., Yager, R.R.: Some geometric aggregation operators based on intuitionistic fuzzy sets. Int. J. Gen. Syst. **35**, 417–433 (2006)