Chapter 3
Measure Theory

If topology derives its inspiration from the qualitative features of geometry, then the subject of the present chapter, measure theory, may be thought to have its origins in the quantitative concepts of length, area, and volume. However, a careful theory of area, for example, turns out to be much more delicate than one might expect initially, as any given set may possess an irregular feature, such as having a jagged boundary or being dispersed across many subsets. Even in the setting of the real line, if one has a set E of real numbers, then in what sense can the length of the set E be defined and computed? Furthermore, to what extent can we expect the length (or area, volume) of a union $A \cup B$ of disjoint sets A and B to be the sum of the individual lengths (or areas, volumes) of A and B?

This present chapter is devoted to measure theory, which, among other things, entails a rigorous treatment of length, area, and volume. However, as with the subject of topology, the context and results of measure theory reach well beyond these basic geometric quantities.

3.1 Measurable Spaces and Functions

Definition 3.1. If X is a set, then a *σ-algebra* on X is a collection Σ of subsets of X with the following properties:

1. $X \in \Sigma$;
2. $E^c \in \Sigma$ for every $E \in \Sigma$; and
3. for every countable family $\{E_k\}_{k\in\mathbb{N}}$ of sets $E_k \in \Sigma$,

$$\bigcup_{k\in\mathbb{N}} E_k \in \Sigma\,.$$

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The pair (X,Σ) is called a *measurable space*, and the elements E of Σ are called *measurable sets*.

The smallest and largest σ-algebras on a set X are, respectively, $\Sigma = \{\emptyset, X\}$ and $\Sigma = \mathscr{P}(X)$, the power set $\mathscr{P}(X)$ of X. The following definition, while abstract in essence, allows for the determination of more interesting, intermediate examples of σ-algebras.

Definition 3.2. If $\mathscr{S}$ is any collection of subsets of X, then the intersection of all σ-algebras on X that contain $\mathscr{S}$ is called the *σ-algebra generated by $\mathscr{S}$*.

It is elementary to verify that the σ-algebra generated by a collection of $\mathscr{S}$ of subsets of X is a σ-algebra in the sense of Definition 3.1.

Definition 3.3. If $(X,\mathscr{T})$ is a topological space, then the σ-algebra generated by $\mathscr{T}$ is called the σ-algebra of *Borel sets* of X.

Let us now consider functions of interest for measure theory.

Definition 3.4. If (X,Σ) is a measurable space, then a function $f: X \to \mathbb{R}$ is *measurable* if $f^{-1}(U) \in \Sigma$, for every open set $U \subseteq \mathbb{R}$.

Proposition 3.5. *If (X,Σ) is a measurable space, then the following statements are equivalent for a function $f: X \to \mathbb{R}$:*

1. $f^{-1}((\alpha,\infty)) \in \Sigma$ *for all* $\alpha \in \mathbb{R}$;
2. $f^{-1}([\alpha,\infty)) \in \Sigma$ *for all* $\alpha \in \mathbb{R}$;
3. $f^{-1}((-\infty,\alpha)) \in \Sigma$ *for all* $\alpha \in \mathbb{R}$;
4. $f^{-1}((-\infty,\alpha]) \in \Sigma$ *for all* $\alpha \in \mathbb{R}$.

Proof. To begin, observe that (2) follows from (1), because

$$f^{-1}([\alpha,\infty)) = f^{-1}\left(\bigcap_{k\in\mathbb{N}}(\alpha - \frac{1}{k},\infty)\right) = \bigcap_{k\in\mathbb{N}} f^{-1}\left((\alpha - \frac{1}{k},\infty)\right) \in \Sigma .$$

Statement (3) follows easily from (2), since

$$f^{-1}((-\infty,\alpha)) = f^{-1}([\alpha,\infty))^c \in \Sigma .$$

Next, we see that (3) implies (4), because

$$f^{-1}((-\infty,\alpha]) = f^{-1}\left(\bigcap_{k\in\mathbb{N}}(-\infty,\alpha + \frac{1}{k})\right) = \bigcap_{k\in\mathbb{N}} f^{-1}\left((-\infty,\alpha + \frac{1}{k})\right) \in \Sigma .$$

Statement (4) implies (1), because

$$f^{-1}((\alpha,\infty)) = f^{-1}((-\infty,\alpha])^c \in \Sigma ,$$

which completes the proof. □

An additional equivalent condition for the measurability of a function is set aside, for future reference, as the following result.

Proposition 3.6 (Criterion for Measurability). *If (X,Σ) is a measurable space, then a function $f : X \to \mathbb{R}$ is measurable if and only if $f^{-1}((\alpha,\infty)) \in \Sigma$, for all $\alpha \in \mathbb{R}$.*

Proof. By definition of measurable function, $f^{-1}((\alpha,\infty)) \in \Sigma$ for all $\alpha \in \mathbb{R}$ because each (α,∞) is open in $\mathbb{R}$.

Conversely, assume that $f^{-1}((\alpha,\infty)) \in \Sigma$, for all $\alpha \in \mathbb{R}$. Let $U \subseteq \mathbb{R}$ be an open set. By Cantor's Lemma (Proposition 1.30), there is a family of pairwise disjoint open intervals $\{J_k\}_{k\in\mathbb{N}}$ such that $U = \bigcup_k J_k$. Because $f^{-1}(U) = \bigcup_k f^{-1}(J_k)$ and Σ is closed under countable unions, it is enough to prove that $f^{-1}(J) \in \Sigma$ for every open interval J. For open intervals of the form (α,∞) and $(-\infty,\alpha)$, this is handled by Proposition 3.5. If one has an open interval of the form $J = (\alpha,\beta)$, then $f^{-1}(J)$ is given by $f^{-1}(J) = f^{-1}((-\infty,\alpha])^c \cap f^{-1}([\beta,\infty))^c$, which by Proposition 3.5 is the intersection of two sets in Σ. □

Proposition 3.7. *If (X,Σ) is a measurable space, if $f,g : X \to \mathbb{R}$ are measurable functions, and if $\lambda \in \mathbb{R}$, then $f+g$, λf, $|f|$, and fg are measurable functions. If, in addition, $g(x) \neq 0$ for every $x \in X$, then f/g is measurable function.*

Proof. The equivalent criteria for measurability of Proposition 3.5 will be used in each case. We begin with a proof that $f+g$ is measurable.

Fix $\alpha \in \mathbb{R}$ and consider the set $S_\alpha = \{x \in X \,|\, f(x)+g(x) > \alpha\}$. Because f and g are measurable, for each $q \in \mathbb{Q}$ we have

$$\{x \in X \,|\, f(x) > q\} \in \Sigma \quad \text{and} \quad \{x \in X \,|\, q > \alpha - g(x)\} \in \Sigma .$$

Hence, as Σ is closed under intersections and countable unions,

$$\bigcup_{q\in\mathbb{Q}} (\{x \in X \,|\, f(x) > q\} \cap \{x \in X \,|\, q > \alpha - g(x)\}) \in \Sigma . \tag{3.1}$$

Let G denote the set in (3.1); we shall prove that $S_\alpha = G$. If $y \in S_\alpha$, then $f(y) > \alpha - g(y)$. In fact, by the density of $\mathbb{Q}$ in $\mathbb{R}$, there is a rational number $q_y \in \mathbb{Q}$ such that $f(y) > q_y > \alpha - g(y)$. Thus,

$$y \in \{x \in X \,|\, f(x) > q_y\} \cap \{x \in X \,|\, q_y > \alpha - g(x)\},$$

which shows that $S_\alpha \subseteq G$. Conversely, if $y \in G$, then there is a rational $q_y \in \mathbb{Q}$ such that $y \in \{x \in X \,|\, f(x) > q_y\} \cap \{x \in X \,|\, q_y > \alpha - g(x)\}$. Thus, $f(y) > q_y > \alpha - g(y)$ implies that $f(y)+g(y) > \alpha$, whence $y \in S_\alpha$ and, consequently, $G \subseteq S_\alpha$. This proves that $f+g$ is measurable.

The proof that λf is measurable is clear, and we move to the proof that $|f|$ is measurable. Note that if $\alpha \in \mathbb{R}$, then $|f|^{-1}((\alpha,\infty)) = X$ if $\alpha < 0$, and

$$|f|^{-1}((\alpha,\infty)) = f^{-1}((\alpha,\infty)) \cup f^{-1}((-\infty,-\alpha)), \text{ if } \alpha \geq 0 .$$

In either case, $|f|^{-1}((\alpha,\infty)) \in \Sigma$, which proves that $|f|$ is measurable.

To prove that the product fg is measurable, first assume that $h : X \to \mathbb{R}$ is a measurable function and consider h^2. If $\alpha \in \mathbb{R}$, then $\{x \in X \mid h(x)^2 > \alpha\} = X$ if $\alpha < 0$, otherwise $\{x \in X \mid h(x)^2 > \alpha\} = |h|^{-1}\left((\sqrt{\alpha}, \infty)\right)$. In either case, the sets belong to Σ. This proves that the square of a measurable function is measurable. To conclude that fg is measurable, express fg as

$$fg = \frac{1}{4}\left((f+g)^2 - (f-g)^2\right). \tag{3.2}$$

As the sums, squares, and scalar multiples of measurable functions are measurable, equation (3.2) demonstrates that fg is measurable.

If $g(x) \neq 0$ for every $x \in X$, then $1/g$ is measurable (Exercise 3.79), which implies that the function $f/g = f \cdot (1/g)$ is measurable. □

Using the algebraic features exhibited in Proposition 3.7, one deduces that the following functions are measurable as well.

Corollary 3.8. *Suppose that $f, g : X \to \mathbb{R}$ are measurable functions.*

1. *If* $\max(f,g)$ *is the function whose value at each $x \in X$ is the maximum of $f(x)$ and $g(x)$, and if* $\min(f,g)$ *is the function whose value at each $x \in X$ is the minimum of $f(x)$ and $g(x)$, then* $\max(f,g)$ *and* $\min(f,g)$ *are measurable.*
2. *f^+ is the function* $\max(f,0)$ *and f^- is the function* $-\min(f,0)$*, then f^+ and f^- are measurable.*

Proof. By Proposition 3.7, the sum, difference, and absolute value of measurable functions are measurable. Therefore, the formulae

$$\begin{aligned}
\max(f,g) &= 1/2(f+g+|f-g|),\\
\min(f,g) &= 1/2(f+g-|f-g|),\\
f^+ &= 1/2(|f|+f), \text{ and}\\
f^- &= 1/2(|f|-f)
\end{aligned}$$

imply the asserted conclusions. □

The purpose of the following result is to use sequences of measurable functions to determine new measurable functions.

Proposition 3.9. *Suppose that $f_k : X \to \mathbb{R}$ is a measurable function for each $k \in \mathbb{N}$. If*

$$\begin{aligned}
S &= \{x \in X \mid \sup_k f_k(x) \text{ exists}\},\\
LS &= \{x \in X \mid \limsup_k f_k(x) \text{ exists}\},\\
I &= \{x \in X \mid \inf_k f_k(x) \text{ exists}\},\\
LI &= \{x \in X \mid \liminf_k f_k(x) \text{ exists}\}, \text{and}\\
L &= \{x \in X \mid \lim_k f_k(x) \text{ exists}\},
\end{aligned}$$

then each of the sets S, LS, I, LI, and L is measurable. Moreover,

1. $\sup_k f_k$ *is a measurable function on S,*
2. $\lim\sup_k f_k$ *is a measurable function on LS,*
3. $\inf_k f_k$ *is a measurable function on I,*
4. $\lim\inf_k f_k$ *is a measurable function on LI, and*
5. $\lim f_k$ *is a measurable function on L.*

Proof. The set $f_k^{-1}((-\infty,q))$ is measurable for every $k \in \mathbb{N}$ and $q \in \mathbb{Q}$; therefore, so is

$$\bigcup_{q\in\mathbb{Q}} \bigcap_{k\in\mathbb{N}} f_k^{-1}((-\infty,q)) = S\,.$$

Consider now the function $\sup_k f_k$ defined on the (measurable) set S with values in $\mathbb{R}$. For every $\alpha \in \mathbb{R}$,

$$\{x \in S \,|\, \sup_k f_k(x) > \alpha\} = \bigcup_{k\in\mathbb{N}} \{x \in S \,|\, f_k(x) > \alpha\} \in \Sigma(S)\,.$$

Hence, $\sup_k f_k$ is measurable as a function $S \to \mathbb{R}$.

The proofs that I is a measurable set and that $\inf_k f_k$ is a measurable function $I \to \mathbb{R}$ are handled in a similar fashion. For example, in this case, I is given by

$$I = \bigcup_{q\in\mathbb{Q}} \bigcap_{k\in\mathbb{N}} f_k^{-1}((q,\infty))\,.$$

For each $k \in \mathbb{N}$ consider the measurable function $g_k : S \to \mathbb{N}$ defined by

$$g_k(x) = \sup_{n\geq k} f_n(x)\,, \quad x \in S\,.$$

For every $x \in LS$, $\lim\sup_k f_k$ is precisely $\inf_k g_k$. Moreover, by the discussion of the previous paragraph, $\inf g_k$ is a measurable function on the (measurable) set

$$\bigcup_{q\in\mathbb{Q}} \bigcap_{k\in\mathbb{N}} g_k^{-1}((q,\infty)) = LS\,.$$

Hence, as a function $LS \to \mathbb{R}$, $\lim\sup_k f_k$ is measurable.

The proofs that LI is a measurable set and that $\lim\inf_k f_k$ is a measurable function $LI \to \mathbb{R}$ are similarly handled.

Consider the measurable set $E = LS \cap LI$ and let $h : E \to \mathbb{R}$ be the function

$$h(x) = \lim\sup_k f_k(x) - \lim\inf_k f_k(x)\,.$$

Note that h is measurable and that

$$L = \{x \in X \,|\, \lim\sup_k f_k(x) = \lim\inf_k f_k(x)\} = h^{-1}(\{0\})\,,$$

which is a measurable set because

$$h^{-1}(\{0\}) = E\setminus\left(h^{-1}(-\infty,0)\cup h^{-1}(0,\infty)\right).$$

Finally, for every $\alpha\in\mathbb{R}$,

$$\{x\in L \mid \lim_k f_k(x) > \alpha\} = L\cap\{x\in L \mid \limsup_k f_k(x) > \alpha\} \in \Sigma(L).$$

Therefore, $\lim_k f_k$ is measurable as a function from L to $\mathbb{R}$. □

Definition 3.10. If X is a set, then the *characteristic function* of a subset $E\subseteq X$ is the function $\chi_E : X\to\mathbb{R}$ defined by

$$\chi_E(x) = 1, \text{ if } x\in E, \text{ and } \chi_E(x) = 0, \text{ if } x\notin E.$$

From the definition above, the following proposition is immediate:

Proposition 3.11. *If (X,Σ) is a measurable space and if $E\subseteq X$, then the characteristic function $\chi_E : X\to\mathbb{R}$ is a measurable function if and only if $E\in\Sigma$.*

Characteristic functions can be used to restrict or extend the domain of functions (Exercise 3.82).

Definition 3.12. If (X,Σ) is a measurable space, then a *simple function* is a measurable function $\varphi : X\to\mathbb{R}$ such that φ assumes at most a finite number of values in $\mathbb{R}$.

Suppose that φ is a simple function on a measurable space (X,Σ). If $\varphi(X) = \{\alpha_1,\dots,\alpha_n\}\subset\mathbb{R}$, then let $E_k = \varphi^{-1}(\{\alpha_k\})$ (which is a measurable set, as φ is a measurable function) so that

$$\varphi = \sum_{k=1}^{n}\alpha_k\,\chi_{E_k}$$

represents φ as a linear combination of the characteristic functions χ_{E_k}.

Definition 3.13. A sequence $\{f_k\}_{k\in\mathbb{N}}$ of real-valued functions f_k on a set X is a *monotone increasing sequence* if $f_k(x)\le f_{k+1}(x)$ for every $k\in\mathbb{N}$ and every $x\in X$.

The analysis of measurable functions depends, to a very large extent, on the following approximation theorem.

Theorem 3.14 (Approximation of Nonnegative Measurable Functions). *For every nonnegative measurable function f on a measurable space (X,Σ), there is a monotone increasing sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ of nonnegative simple functions φ_k on X such that*

$$\lim_{k\to\infty}\varphi_k(x) = f(x),$$

for every $x\in X$.

Proof. Let $\kappa_n : [0,n] \to \mathbb{Z}$ be the function whose value at t is the unique $j \in \mathbb{Z}$ for which $t \in [\frac{j}{2^n}, \frac{j+1}{2^n})$, and define $\omega_n : \mathbb{R} \to \mathbb{Q}$ by $\omega_n(t) = \kappa_n(t)/2^n$, if $t \in [0,n]$, and by $\omega_n(t) = 0$ if $t \in (n,\infty)$. The functions ω_n satisfy $\omega_n(t) \le t$ for every $t \in [0,\infty)$ and $\omega_n(t) \le \omega_{n+1}(t)$ for all $n \in \mathbb{N}$ and $t \in [0,\infty)$. Now let $\varphi_n = \omega_n \circ f$. Thus, $\{\varphi_n\}_n$ is a monotone increasing sequence of nonnegative functions, each with finite range. For each $x \in X$ there is some $n \in \mathbb{N}$ for which $f(x) \in [0,n)$.Thus, for every $k > n$,

$$f(x) - \varphi_k(x) < \frac{1}{2^k},$$

which proves that $\lim_{n\to\infty} \varphi_n(x) = f(x)$. All that remains is to verify that φ_n is measurable. To this end, select $n \in \mathbb{N}$ and let

$$E_n = f^{-1}([n,\infty)) \text{ and } E_{nj} = f^{-1}\left(\left[\frac{j-1}{2^n}, \frac{j}{2^n}\right)\right), \text{ for } 1 \le j \le 2^n n.$$

These are measurable sets and

$$\varphi_n = \sum_{j=1}^{2^n n} \frac{j-1}{2^n} \chi_{E_{nj}} + n\chi_{E_n}.$$

Hence, φ_n is a simple function. □

By decomposing a real-valued function f into a difference its positive and negative parts, namely $f = f^+ - f^-$, where

$$f^+ = \frac{|f|+f}{2} \quad \text{and} \quad f^- = \frac{|f|-f}{2}, \tag{3.3}$$

we obtain the following approximation result for arbitrary measurable functions.

Corollary 3.15. *If (X,Σ) is a measurable space and if $f : X \to \mathbb{R}$ is a measurable function, then there is a sequence $\{\psi_k\}_{k\in\mathbb{N}}$ of simple functions $\psi_k : X \to \mathbb{R}$ such that*

$$\lim_{k\to\infty} \psi_k(x) = f(x), \quad \forall x \in X.$$

3.2 Measure Spaces

Before continuing further, the values $-\infty$ and $+\infty$ will be added to the arithmetic system of $\mathbb{R}$. Formally, the *extended real number system* are the elements of the set denoted by $[-\infty,+\infty]$ and defined by $\{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$. (Here, $-\infty$ and $+\infty$ are meant to denote the "ends" of the real axis.) The arithmetic of $[-\infty,+\infty]$ is prescribed by the following laws:

1. $r \cdot s$ and $r + s$ are the usual product and sum in $\mathbb{R}$, for all $r,s \in \mathbb{R}$;
2. $0 \cdot (-\infty) = 0 \cdot (+\infty) = 0$;
3. $r \cdot (-\infty) = -\infty$ and $r \cdot (+\infty) = +\infty$, for all $r \in \mathbb{R}$ with $r > 0$ and for $r = +\infty$;

4. $r \cdot (-\infty) = +\infty$ and $r \cdot (+\infty) = -\infty$, for all $r \in \mathbb{R}$ with $r < 0$ and for $r = -\infty$;
5. $r + (-\infty) = -\infty$ and $r + (+\infty) = +\infty$, for all $r \in \mathbb{R}$.

The sum of $-\infty$ and $+\infty$ is not defined in the extended real number system, which is a small fact that will be of note in our study of signed measures in Section 3.7.

Henceforth, $[0,\infty]$ denotes the subset of the extended real numbers given by

$$[0,\infty] = [0,\infty) \cup \{+\infty\}.$$

The terminology below concerning families of sets will be used extensively, beginning with the definition of measure in Definition 3.17.

Definition 3.16. A family $\{X_\alpha\}_{\alpha\in\Lambda}$ of subsets of a given set X is a *family of pairwise disjoint sets* if $X_\alpha \cap X_\beta = \emptyset$ for all $\alpha, \beta \in \Lambda$ such that $\alpha \neq \beta$.

Definition 3.17. A *measure* on a measurable space (X, Σ) is a function $\mu : \Sigma \to [0,+\infty]$ such that $\mu(\emptyset) = 0$ and

$$\mu\left(\bigcup_{k\in\mathbb{N}} E_k\right) = \sum_{k\in\mathbb{N}} \mu(E_k)\,, \tag{3.4}$$

for every sequence $\{E_k\}_{k\in\mathbb{N}}$ of pairwise disjoint sets $E_k \in \Sigma$. Furthermore,

1. if $\mu(X) < \infty$, then μ is said to be a *finite measure*, and
2. if $\mu(X) = 1$, then μ is said to be a *probability measure*.

The (X, Σ, μ) is called a *measure space*.

Measures are not easy to construct or determine in general, but there are some very simple examples nevertheless.

Example 3.18. *Consider the measurable space (X, Σ) in which X is an uncountable infinite set and Σ is the σ-algebra of all subsets $E \subseteq X$ that have the property that E or E^c is countable* (see Exercise 3.71). *If $\mu : \Sigma \to [0,+\infty]$ is defined by*

$$\mu(E) = 0 \ \textit{ if } E \textit{ is countable, and } \mu(E) = 1 \ \textit{ if } E^c \textit{ is countable},$$

then μ is a measure on (X, Σ).

Example 3.19 (Dirac Measures). *If Σ is a σ-algebra of subsets of X in which $\{x\} \in \Sigma$ for every $x \in X$, then for each $x \in X$ the function $\delta_x : \Sigma \to [0,1]$ given by*

$$\delta_x(E) = 1 \ \textit{ if } x \in E, \ \textit{ and } \ \delta_x(E) = 0 \ \textit{ if } x \notin E,$$

is a probability measure on (X, Σ). The measures δ_x are called Dirac measures *or* point mass measures.

Example 3.20 (Counting Measure). *Consider the measurable space* $(\mathbb{N}, \mathscr{P}(\mathbb{N}))$*, where* $\mathscr{P}(\mathbb{N})$ *is the power set of* $\mathbb{N}$*. If* $\mu : \Sigma \to [0, +\infty]$ *is the function defined by*

$$\mu(E) = \textit{the cardinality of } E\,,$$

then μ *is a measure on* $(\mathbb{N}, \mathscr{P}(\mathbb{N}))$ *and is called* counting measure.

We turn now to some general properties of measures and measure spaces.

Proposition 3.21 (Monotonicity of Measure). *Let* (X, Σ, μ) *denote a measure space. Suppose that* $E, F \in \Sigma$ *are such that* $E \subseteq F$*. Then* $\mu(E) \leq \mu(F)$*. Furthermore, if* $\mu(F) < \infty$*, then* $\mu(F \setminus E) = \mu(F) - \mu(E)$*.*

Proof. Because $E \subseteq F$, we may express F as $F = E \cup (E^c \cap F)$, which is a union of disjoint sets E and $E^c \cap F$, each of which belongs to Σ. Hence, $\mu(F) = \mu(E) + \mu(E^c \cap F) \geq \mu(E)$. □

Proposition 3.22 (Continuity of Measure). *Let* (X, Σ, μ) *denote a measure space. Suppose that* $\{A_k\}_{k \in \mathbb{N}}$ *and* $\{E_k\}_{k \in \mathbb{N}}$ *are sequences of sets* $E_k \in \Sigma$*.*

1. If $A_k \subseteq A_{k+1}$*, for all* $k \in \mathbb{N}$*, then*

$$\mu\left(\bigcup_{k \in \mathbb{N}} A_k\right) = \lim_{k \to \infty} \mu(A_k)\,. \tag{3.5}$$

2. If $E_k \supseteq E_{k+1}$*, for all* $k \in \mathbb{N}$*, and if* $\mu(E_1) < \infty$*, then*

$$\mu\left(\bigcap_{k \in \mathbb{N}} E_k\right) = \lim_{k \to \infty} \mu(E_k)\,. \tag{3.6}$$

Proof. (1) Equation (3.5) plainly holds if $\mu(A_k) = \infty$ for at least one k; hence, assume that $\mu(A_k) < \infty$ for all $k \in \mathbb{N}$. The sequence $\{A_k\}_{k \in \mathbb{N}}$ is nested and ascending, and so it is simple to produce from it a sequence of pairwise disjoint sets $G_k \in \Sigma$ by taking set differences: that is, define G_1 to be A_1 and let

$$G_k = A_k \setminus A_{k-1}\,, \quad \forall\, k \geq 2\,.$$

Observe that $\mu(A_k) < \infty$ implies that $\mu(G_k) = \mu(A_k) - \mu(A_{k-1})$, by Proposition 3.21. Furthermore, the sets G_k are pairwise disjoint. Because $A_k = \bigcup_{n=1}^{k} G_n$, we have

$$\bigcup_{k \in \mathbb{N}} A_k = \bigcup_{k \in \mathbb{N}} G_k\,.$$

Thus, by the countable additivity of μ on disjoint unions,

$$\begin{aligned}\mu\left(\bigcup_{k\in\mathbb{N}} A_k\right) &= \mu\left(\bigcup_{k\in\mathbb{N}} G_k\right)\\ &= \sum_{k\in\mathbb{N}} \mu(G_k)\\ &= \mu(A_1) + \lim_{n\to\infty}\sum_{k=2}^{n}[\mu(A_k)-\mu(A_{k-1})]\\ &= \mu(A_1) + \left[\lim_{n\to\infty}\mu(A_n)\right] - \mu(A_1)\,,\end{aligned}$$

which establishes formula (3.5).

(2) The sequence $\{E_k\}_{k\in\mathbb{N}}$ is nested and descending, and so it is simple to produce from it a sequence of pairwise disjoint sets $F_k \in \Sigma$ by taking set differences: that is, let

$$F_k = E_k\backslash E_{k+1}\,,\quad \forall\, k\in\mathbb{N}\,.$$

Observe that $E_k = E_{k+1}\cup F_k$ and that $E_{k+1}\cap F_k = \emptyset$. Thus, by the countable additivity of μ on disjoint unions,

$$\mu(E_k) = \mu(E_{k+1}) + \mu(F_k)\,,\quad \forall\, k\in\mathbb{N}\,.$$

Because

$$\left(\bigcap_{k\in\mathbb{N}} E_k\right)\cap\left(\bigcup_{k\in\mathbb{N}} F_k\right) = \emptyset\,,$$

and

$$E_1 = \left(\bigcap_{k\in\mathbb{N}} E_k\right)\cup\left(\bigcup_{k\in\mathbb{N}} F_k\right),$$

the countable additivity of μ on disjoint unions yields

$$\begin{aligned}\mu(E_1) &= \mu\left(\bigcap_{k\in\mathbb{N}} E_k\right) + \mu\left(\bigcup_{k\in\mathbb{N}} F_k\right)\\ &= \mu\left(\bigcap_{k\in\mathbb{N}} E_k\right) + \sum_{k\in\mathbb{N}}\mu(F_k)\\ &= \mu\left(\bigcap_{k\in\mathbb{N}} E_k\right) + \lim_{n\to\infty}\sum_{k=1}^{n}[\mu(E_k)-\mu(E_{k+1})]\\ &= \mu\left(\bigcap_{k\in\mathbb{N}} E_k\right) + \mu(E_1) - \lim_{n\to\infty}\mu(E_{n+1})\,,\end{aligned}$$

which establishes formula (3.6). □

As an application of the continuity of measure, the following result shows that if a measurable function f on a finite measure space is unbounded, then the set on which the values of f are very large has arbitrarily small measure.

Proposition 3.23. *If* (X, Σ, μ) *is a finite measure space and if* $f : X \to \mathbb{R}$ *is measurable, then for each* $\varepsilon > 0$ *there is an* $n \in \mathbb{N}$ *such that*

$$\mu\left(\{x \in X \,|\, |f(x)| > n\}\right) < \varepsilon.$$

Proof. Let $E_n = \{x \in X \,|\, |f(x)| > n\}$, for each $n \in \mathbb{N}$. Note that $\mu(E_1) \leq \mu(X) < \infty$ and $E_{n+1} \supseteq E_n$ for every n. Hence, by Proposition 3.22, if $E = \bigcap_{n\in\mathbb{N}} E_n$, then $\mu(E) = \lim_{n\to\infty} \mu(E_n)$. Now because, in this particular case, $E = \emptyset$ and thus $\mu(E) = 0$, we deduce that for each $\varepsilon > 0$ there is an $n \in \mathbb{N}$ such that $\mu(E_n) < \varepsilon$. □

One might not have a sequence of pairwise disjoint sets at hand. Nevertheless, it is possible to obtain an estimate on the measure of their union.

Proposition 3.24 (Countable Subadditivity of Measure). *Let* (X, Σ, μ) *denote a measure space. Suppose that* $\{E_k\}_{k\in\mathbb{N}}$ *is any sequence of sets* $E_k \in \Sigma$. *Then,*

$$\mu\left(\bigcup_{k\in\mathbb{N}} E_k\right) \leq \sum_{k\in\mathbb{N}} \mu(E_k)\,. \tag{3.7}$$

Proof. For each $k \in \mathbb{N}$, let

$$F_k = E_k \setminus \left(\bigcup_{j=1}^{k-1} E_j\right).$$

Note that the sequence $\{F_k\}_{k\in\mathbb{N}}$ consists of pairwise disjoint elements of Σ and that each $F_k \subseteq E_k$. Thus, $\mu(F_k) \leq \mu(E_k)$, by Proposition 3.21. Also,

$$\bigcup_{k\in\mathbb{N}} E_k = \bigcup_{k\in\mathbb{N}} F_k\,.$$

Thus,

$$\mu\left(\bigcup_{k\in\mathbb{N}} E_k\right) = \mu\left(\bigcup_{k\in\mathbb{N}} F_k\right) = \sum_{k\in\mathbb{N}} \mu(F_k) \leq \sum_{k\in\mathbb{N}} \mu(E_k)\,,$$

which proves inequality (3.7). □

There is a rather significant difference between those measure spaces (X, Σ, μ) in which $\mu(X)$ is finite and those for which $\mu(X) = \infty$. A hybrid between these two alternatives occurs with the notion of a σ-finite space.

Definition 3.25. A measure space (X,Σ,μ) is *σ-finite* if there is a sequence $\{X_n\}_{n\in\mathbb{N}}$ of measurable sets $X_n \in \Sigma$ such that $\mu(X_n) < \infty$ for every n and $X = \bigcup_{n\in\mathbb{N}} X_n$.

3.3 Outer Measures

Having examined to this point some properties of measures, we turn now to the issue of constructing measures. This will be done by first defining an outer measure.

Definition 3.26. If X is a set, then a function $\mu^* : \mathscr{P}(X) \to [0,\infty]$ on the power set $\mathscr{P}(X)$ of X is an *outer measure* on X if

1. $\mu^*(\emptyset) = 0$,
2. $\mu^*(S_1) \le \mu^*(S_2)$, if $S_1 \subseteq S_2$, and
3. $\mu^*\left(\bigcup_{k\in\mathbb{N}} S_k\right) \le \sum_{k\in\mathbb{N}} \mu^*(S_k)$ for every sequence $\{S_k\}_{k\in\mathbb{N}}$ of subsets $S_k \subseteq X$.

An outer measure is generally not a measure. And note that the domain of an outer measure is the power set $\mathscr{P}(X)$, rather than some particular σ-algebra of subsets of X.

Definition 3.27. A *sequential cover* of X is a collection $\mathscr{O}$ of subsets of X with the properties that $\emptyset \in \mathscr{O}$ and for every $S \subseteq X$ there is a countable subcollection $\{I_k\}_{k\in\mathbb{N}} \subseteq \mathscr{O}$ such that

$$S \subseteq \bigcup_{k\in\mathbb{N}} I_k .$$

Sequential covers lead to outer measures as follows.

Proposition 3.28. *Assume that $\mathscr{O}$ is a sequential cover of a set X. If $\lambda : \mathscr{O} \to [0,\infty)$ is any function for which $\lambda(\emptyset) = 0$, then the function $\mu^* : \mathscr{P}(X) \to [0,\infty]$ defined by*

$$\mu^*(S) = \inf\left\{ \sum_{k=1}^{\infty} \lambda(I_k) \,\middle|\, \{I_k\}_{k\in\mathbb{N}} \subseteq \mathscr{O} \text{ and } S \subseteq \bigcup_{k\in\mathbb{N}} I_k \right\} \tag{3.8}$$

is an outer measure on X.

Proof. Clearly $\mu^*(\emptyset) = 0$. If $S \subseteq T$, then any $\{I_k\}_{k\in\mathbb{N}} \subseteq \mathscr{O}$ that covers the set T also covers the set S, and so $\mu^*(S) \le \mu^*(T)$. Thus, all that remains is to verify that μ^* is countable subadditive.

To this end, suppose that $\{S_k\}_{k\in\mathbb{N}}$ is a sequence of subsets $S_k \subseteq X$. Since we aim to show that

$$\mu^*\left(\bigcup_{k\in\mathbb{N}} S_k\right) \le \sum_{k\in\mathbb{N}} \mu^*(S_k) ,$$

only the case where the sum $\sum_k \mu^*(S_k)$ converges need be considered. For this case, suppose that $\varepsilon > 0$. For each $k \in \mathbb{N}$ there is a countable family $\{I_{kj}\}_{j\in\mathbb{N}} \subseteq \mathscr{O}$ such that $S_k \subseteq \bigcup_j I_{kj}$ and

$$\sum_j \lambda(I_{kj}) \leq \mu^*(S_k) + \frac{\varepsilon}{2^k}.$$

Thus, $\{I_{kj}\}_{k,j\in\mathbb{N}}$ forms a countable subcollection of sets from $\mathscr{O}$ that cover $\cup_k S_k$ and satisfies

$$\mu^*\left(\bigcup_{k\in\mathbb{N}} S_k\right) \leq \sum_k \sum_j \lambda(I_{kj}) \leq \sum_k \left(\mu^*(S_k) + \frac{\varepsilon}{2^k}\right) \leq \sum_{k\in\mathbb{N}} \mu^*(S_k) + \varepsilon.$$

As $\varepsilon > 0$ is chosen arbitrarily, μ^* is indeed countably subadditive. □

The value of an outer measure is two-fold: (i) it is frequently easier to define an outer measure on the power set of X than it is to define a measure on some σ-algebra of subsets of X (indeed, determining nontrivial σ-algebras on X is in itself a nontrivial task), and (ii) if one has an outer measure at hand, then there is a σ-algebra Σ of subsets of X for which the restriction of μ^* to Σ is a measure on (X,Σ). This latter fact is the content of the following theorem.

Theorem 3.29 (Carathéodory). *If μ^* is an outer measure on a set X, then*

1. *the collection $\mathfrak{M}_{\mu^*}(X)$ of all subsets $E \subseteq X$ for which*

$$\mu^*(S) = \mu^*(E \cap S) + \mu^*(E^c \cap S), \quad \forall S \subseteq X,$$

is a σ-algebra, and

2. *the function $\mu : \mathfrak{M}_{\mu^*}(X) \to [0,\infty]$ defined by $\mu(E) = \mu^*(E)$, $E \in \mathfrak{M}_{\mu^*}(X)$, is a measure on the measurable space $(X, \mathfrak{M}_{\mu^*}(X))$.*

The criterion (1) in Theorem 3.29 for membership in $\mathfrak{M}_{\mu^*}(X)$ is called the *Carathéodory criterion*. The proof of Theorem 3.29 requires the following lemma.

Lemma 3.30. *If $E_1,\ldots,E_n \in \mathfrak{M}_{\mu^*}(X)$, then*

$$\bigcup_{k=1}^n E_k \in \mathfrak{M}_{\mu^*}(X).$$

Moreover, if $E_1,\ldots,E_n \in \mathfrak{M}_{\mu^*}(X)$ are pairwise disjoint, then

$$\mu^*\left(S \bigcap \left[\bigcup_{k=1}^n E_k\right]\right) = \sum_{k=1}^n \mu^*(S \cap E_k) \qquad \forall S \subseteq X. \tag{3.9}$$

Proof. It is sufficient to consider the case $n = 2$, as the remaining cases follow by induction on n.

We shall prove that $E_1 \cup E_2 \in \mathfrak{M}_{\mu^*}(X)$, for all $E_1, E_2 \in \mathfrak{M}_{\mu^*}(X)$ Let $S \subseteq X$ be arbitrary and note that $S \cap (E_1 \cup E_2)$ can be written as

$$S \cap (E_1 \cup E_2) = (S \cap E_1) \cup (S \cap E_2) = (S \cap E_1) \cup ([S \cap E_1^c] \cap E_2).$$

Likewise,

$$S \cap (E_1 \cup E_2)^c = S \cap (E_1^c \cap E_2^c) = (S \cap E_1^c) \cap E_2^c.$$

Thus,

$$\begin{aligned} \mu^*(S) &\leq \mu^*\left(S \cap (E_1 \cup E_2)\right) + \mu^*\left(S \cap (E_1 \cup E_2)^c\right) \\ &= \mu^*\left((S \cap E_1) \cup ([S \cap E_1^c] \cap E_2)\right) + \mu^*\left([S \cap E_1^c] \cap E_2^c\right) \\ &\leq \mu^*(S \cap E_1) + \mu^*\left([S \cap E_1^c] \cap E_2)\right) + \mu^*\left([S \cap E_1^c] \cap E_2^c\right) \\ &= \mu^*(S \cap E_1) + \mu^*(S \cap E_1^c) \\ &= \mu^*(S), \end{aligned}$$

where the final two equalities are because of $E_2 \in \mathfrak{M}_{\mu^*}(X)$ and $E_1 \in \mathfrak{M}_{\mu^*}(X)$, respectively. Hence,

$$\mu^*(S) = \mu^*\left(S \cap (E_1 \cup E_2)\right) + \mu^*\left(S \cap (E_1 \cup E_2)^c\right), \quad \forall\, S \subseteq X.$$

This proves that $E_1 \cup E_2 \in \mathfrak{M}_{\mu^*}(X)$.

Next, let $E_1, E_2 \subseteq X$ be disjoint elements of $\mathfrak{M}_{\mu^*}(X)$. If $S \subseteq X$, then

$$\begin{aligned} [S \cap (E_1 \cup E_2)] \cap E_2 &= S \cap E_2 \quad \text{and} \\ [S \cap (E_1 \cup E_2)] \cap E_2^c &= S \cap E_1. \end{aligned} \tag{3.10}$$

Thus, by using (3.10) together with the fact that $E_2 \in \mathfrak{M}_{\mu^*}(X)$, we obtain

$$\mu^*(S \cap E_1) + \mu^*(S \cap E_2) = \mu^*\left(S \cap (E_1 \cup E_2)\right),$$

which completes the proof. □

We are now equipped to prove Theorem 3.29.

Proof. To prove (1), namely that $\mathfrak{M}_{\mu^*}(X)$ is a σ-algebra, recall that a subset $E \subseteq X$ is an element of $\mathfrak{M}_{\mu^*}(X)$ if and only if $E^c \in \mathfrak{M}_{\mu^*}(X)$. Hence, $\mathfrak{M}_{\mu^*}(X)$ is closed under complements. Further, the empty set $\emptyset$ clearly belongs to $\mathfrak{M}_{\mu^*}(X)$. Thus, all that remains is to prove that $\mathfrak{M}_{\mu^*}(X)$ is closed under countable unions.

Lemma 3.30 states that $\mathfrak{M}_{\mu^*}(X)$ is closed under finite unions. To get the same result for finite intersections, note that

$$\begin{aligned} E_1, E_2 \in \mathfrak{M}_{\mu^*}(X) &\implies E_1^c, E_2^c \in \mathfrak{M}_{\mu^*}(X) \\ &\implies E_1^c \cup E_2^c \in \mathfrak{M}_{\mu^*}(X) \\ &\implies (E_1^c \cup E_2^c)^c \in \mathfrak{M}_{\mu^*}(X) \\ &\implies (E_1^c)^c \cap (E_2^c)^c = E_1 \cap E_2 \in \mathfrak{M}_{\mu^*}(X). \end{aligned}$$

That is, $E_1 \cap E_2 \in \mathfrak{M}_{\mu^*}(X)$. By induction, $E_1 \cap \cdots \cap E_n \in \mathfrak{M}_{\mu^*}(X)$, for all $E_1, \ldots, E_n \in \mathfrak{M}_{\mu^*}(X)$.

Now let $\{A_k\}_{k\in\mathbb{N}}$ be a sequence for which $A_k \in \mathfrak{M}_{\mu^*}(X)$ for all $k \in \mathbb{N}$. Let $E_0 = \emptyset$ and

$$E_k = A_k \setminus \bigcup_{j=1}^{k-1} A_j, \quad \forall k \in \mathbb{N}.$$

As $\mathfrak{M}_{\mu^*}(X)$ is closed under finite unions and intersections, $E_k \in \mathfrak{M}_{\mu^*}(X)$ for all $k \in \mathbb{N}$. Furthermore, by Exercise 3.76, $\{E_k\}_{k\in\mathbb{N}}$ is a sequence of pairwise disjoint sets for which

$$\bigcup_{k\in\mathbb{N}} E_k = \bigcup_{k\in\mathbb{N}} A_k.$$

Let

$$E = \bigcup_{k\in\mathbb{N}} E_k \quad \text{and} \quad F_n = \bigcup_{k=1}^{n} E_k, \quad \forall n \in \mathbb{N}.$$

Because $F_n \subseteq E$, we have that $E^c \subseteq F_n^c$. The sets F_n are elements of $\mathfrak{M}_{\mu^*}(X)$; thus, for any subset $S \subseteq X$,

$$\begin{aligned} \mu^*(S) &= \mu^*(S \cap F_n) + \mu^*(S \cap F_n^c) \\ &\geq \mu^*(S \cap F_n) + \mu^*(S \cap E^c). \end{aligned}$$

Equation (3.9) of Lemma 3.30 yields

$$\mu^*(S \cap F_n) = \sum_{k=1}^{n} \mu^*(S \cap E_k).$$

Thus, this equation and the inequality $\mu^*(S) \geq \mu^*(S \cap F_n) + \mu^*(S \cap E^c)$ imply that

$$\mu^*(S) \geq \sum_{k=1}^{n} \mu^*(S \cap E_k) + \mu^*(S \cap E^c), \quad \forall n \in \mathbb{N}.$$

Therefore, by making use of the fact that μ^* is countably subadditive,

$$\mu^*(S) \geq \sum_{k=1}^{\infty} \mu^*(S \cap E_k) + \mu^*(S \cap E^c)$$

$$\geq \mu^*(S \cap E) + \mu^*(S \cap E^c)$$

$$\geq \mu^*(S).$$

Hence, $\mu^*(S) = \mu^*(S \cap E) + \mu^*(S \cap E^c)$, which proves that $E \in \mathfrak{M}_{\mu^*}(X)$.

To prove (2), namely that μ^* restricted to $\mathfrak{M}_{\mu^*}(X)$ is a measure, note first that $\mu(\emptyset) = 0$ and that the range of μ is obviously all of $[0, \infty]$.

Suppose now that $\{E_k\}_{k \in \mathbb{N}}$ is a sequence in $\mathfrak{M}_{\mu^*}(X)$ of pairwise disjoint sets and let $E = \bigcup_k E_k$. We aim to prove that $\mu(E) = \sum_k \mu(E_k)$. Outer measure is countably subadditive; thus,

$$\mu(E) = \mu^*(E) \leq \sum_{k=1}^{\infty} \mu^*(E_k) = \sum_{k=1}^{\infty} \mu(E_k).$$

Let $S \subseteq X$ be arbitrary. By Lemma 3.30,

$$\mu^*\left(S \bigcap \left[\bigcup_{k=1}^{n} E_k\right]\right) = \sum_{k=1}^{n} \mu^*(S \cap E_k) \quad \text{for every } n \in \mathbb{N}.$$

In particular, for $S = X$, this yields, for every $n \in \mathbb{N}$,

$$\mu\left(\bigcup_{k=1}^{n} E_k\right) = \mu^*\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu^*(E_k) = \sum_{k=1}^{n} \mu(E_k).$$

Thus,

$$\sum_{k=1}^{\infty} \mu(E_k) \geq \mu\left(\bigcup_{k=1}^{\infty} E_k\right) \geq \mu\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu(E_k),$$

for every $n \in \mathbb{N}$, and so $\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$. □

One useful consequence is the following simple result.

Proposition 3.31. *Suppose that $E, F \in \mathfrak{M}(X)$. If $E \subseteq F$ and if $\mu^*(F) < \infty$, then $\mu^*(F \backslash E) = \mu^*(F) - \mu^*(E)$.*

Proof. Write F as $F = (F \backslash E) \cup E$, which is a disjoint union of elements of $\mathfrak{M}(X)$. Both $\mu^*(F)$ and $\mu^*(F \backslash E)$ are finite. Thus, $\mu^*(F) = \mu^*(F \backslash E) + \mu^*(E)$, by (3.9) [with $S = X$]. □

The next definition and proposition indicate that sets that have zero outer measure are measurable.

Definition 3.32. If μ^* is an outer measure on X, then a subset $S \subset \mathbb{R}$ is μ^*-*null* if $\mu^*(S) = 0$.

Proposition 3.33. *If μ^* is an outer measure on X and if $E \subseteq X$ is μ^*-null, then $E \in \mathfrak{M}_{\mu^*}(X)$.*

Proof. Let $E \subseteq X$ be a μ^*-null set. If $S \subseteq X$, then $E \cap S \subseteq E$ and so $0 \leq \mu^*(E \cap S) \leq \mu^*(E) = 0$. Hence, by the subadditivity of outer measure,

$$\mu^*(S) \leq \mu^*(E \cap S) + \mu^*(E^c \cap S) = 0 + \mu^*(E^c \cap S) \leq \mu^*(S).$$

That is, $\mu^*(S) = \mu^*(E \cap S) + \mu^*(E^c \cap S)$ for every $S \subseteq X$. □

What other subsets $E \subseteq X$ will belong to the σ-algebra $\mathfrak{M}_{\mu^*}(X)$? The answer to this question depends, of course, on the character of the outer measure μ^*. A useful answer in the setting of metric spaces is given by Proposition 3.35 below, for which following definition will be required.

Definition 3.34. If (X,d) is a metric space and if A and B are nonempty subsets of X, then the *distance between A and B* is the quantity denoted by $\operatorname{dist}(A,B)$ and defined by

$$\operatorname{dist}(A,B) = \inf\{d(a,b) \,|\, a \in A,\, b \in B\}.$$

If, in a metric space (X,d), the distance between subsets A and B is positive, then A and B are disjoint and $\mu^*(A \cup B) \leq \mu^*(A) + \mu^*(B)$. If equality is achieved in all such cases, then the induced σ-algebra $\mathfrak{M}_{\mu^*}(X)$ will contain the Borel sets of X.

Proposition 3.35. *If an outer measure μ^* on a metric space (X,d) has the properties that $\mu^*(X) < \infty$ and that*

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B),$$

for all subsets $A, B \subseteq X$ for which $\operatorname{dist}(A,B) > 0$, then every Borel set of X belongs to the σ-algebra $\mathfrak{M}_{\mu^*}(X)$ induced by μ^*.

Proof. By the Carathéodory criterion of Theorem 3.29, our objective is to show that, for every open subset $U \subseteq X$, the equation

$$\mu^*(S) = \mu^*(S \cap U) + \mu^*(S \cap U^c)$$

holds for all $S \subseteq X$.

To this end, select a nonempty subset S of X. If $S \cap U = \emptyset$, then the equation $\mu^*(S) = \mu^*(S \cap U) + \mu^*(S \cap U^c)$ holds trivially. Thus, assume that $S \cap U \neq \emptyset$, and for each $n \in \mathbb{N}$ let

$$S_n = \left\{ x \in U \cap S \,|\, \operatorname{dist}(\{x\}, U^c) \geq \frac{1}{n} \right\}.$$

Observe that $S_n \subseteq S_{n+1}$ for all $n \in \mathbb{N}$ and that $U \cap S = \bigcup_{n\in\mathbb{N}} S_n$. By the hypothesis on μ^*, the distance inequalities $\operatorname{dist}(S_n, S \cap U^c) \geq \operatorname{dist}(S_n, U^c) \geq \frac{1}{n} > 0$ imply that

$$\mu^*(S) \geq \mu^*\left((S \cap U^c) \cup S_n\right) = \mu^*(S \cap U^c) + \mu^*(S_n).$$

Because $\mu^*(X) < \infty$ and because the sets S_n form an ascending sequence, the limit $\lim_{n\to\infty} \mu^*(S_n)$ exists and is bounded above by $\mu^*(S \cap U)$. If it were known that $\lim_{n\to\infty} \mu^*(S_n) = \mu^*(S \cap U)$, then the inequality above would lead to

$$\mu^*(S) \geq \mu^*(S \cap U^c) + \mu^*(S \cap U),$$

which, when coupled with the inequality $\mu^*(S) \leq \mu^*(S \cap U^c) + \mu^*(S \cap U)$ arising from the subadditivity of μ^*, would imply $\mu^*(S) = \mu^*(S \cap U) + \mu^*(S \cap U^c)$. Therefore, all that remains is to prove that $\lim_{n\to\infty} \mu^*(S_n) = \mu^*(S \cap U)$.

For every $n \in \mathbb{N}$, let $A_n = S_{n+1} \setminus S_n$. If $m, n \in \mathbb{N}$ satisfy $|m - n| \geq 2$, then the distance between A_m and A_n is positive, and so $\mu^*(A_m \cup A_n) = \mu^*(A_m) + \mu^*(A_n)$. Therefore, by induction,

$$\sum_{k=1}^{n} \mu^*(A_{2k}) = \mu^*\left(\bigcup_{k=1}^{n} A_{2k}\right) \leq \mu^*(S_{2n+1}) \leq \mu^*(S \cap U) < \infty.$$

Hence, the series $\sum_{k=1}^{\infty} \mu^*(A_{2k})$ converges. Likewise, $\sum_{k=1}^{\infty} \mu^*(A_{2k+1})$ converges, and so the series $\sum_{k=1}^{\infty} \mu^*(A_k)$ converges. Therefore, by the countable subadditivity of μ^*,

$$\mu^*(S_n) \leq \mu^*(S \cap U) \leq \mu^*(S_n) + \sum_{k=n+1}^{\infty} \mu^*(A_k),$$

and so

$$|\mu^*(S \cap U) - \mu^*(S_n)| \leq \sum_{k=n+1}^{\infty} \mu^*(A_k).$$

The convergence of $\sum_{k=1}^{\infty} \mu^*(A_k)$ yields $\lim_{n\to\infty} |\mu^*(S \cap U) - \mu^*(S_n)| = 0$. □

3.4 Lebesgue Measure

The original motivation for the development of measure theory was to put the notion of length, area, volume, and so forth on rigorous mathematical footing, with the understanding that the sets to be measured may not be intervals, rectangles, or boxes. The measures that captures length, area, and volume are called Lebesgue measures.

Proposition 3.36. *The collection*

$$\mathscr{O}_n = \left\{ \prod_{i=1}^{n} (a_i, b_i) \,|\, a_i, b_i \in \mathbb{R},\, a_i < b_i \right\}$$

is a sequential cover of $\mathbb{R}^n$.

Proof. Let $S \subseteq \mathbb{R}^n$. For each $x \in S$ there is a neighbourhood U_x of x of the form $U_x = \prod_{i=1}^{n}(a_i, b_i)$. Let $V = \bigcup_{x \in S} U_x$, which is an open set. By Proposition 1.26, the set $\mathscr{B}$ of all finite open intervals with rational end points is a basis for the topology of $\mathbb{R}$. Thus,

$$\mathscr{B}_n = \left\{ \prod_{i=1}^{n} (p_i, q_i) \,|\, p_i, q_i \in \mathbb{Q},\, p_i < q_i \right\}$$

is a basis for the topology of $\mathbb{R}^n$. By Proposition 1.24, every open set is a union of basic open sets. Thus, since $\mathscr{B}_n$ is countable, there is a countable family $\{I_k\}_{k \in \mathbb{N}} \subseteq \mathscr{B}_n \subseteq \mathscr{O}_n$ such that $V = \bigcup_{k \in \mathbb{N}} I_k$, whence $S \subseteq \bigcup_{k \in \mathbb{N}} I_k$. □

Definition 3.37. *Lebesgue outer measure* on $\mathbb{R}^n$ is the function m^* on $\mathscr{P}(\mathbb{R}^n)$ defined by

$$m^*(S) = \inf \left\{ \sum_{k=1}^{\infty} \lambda(I_k) \,|\, \{I_k\}_{k \in \mathbb{N}} \subseteq \mathscr{O}_n \text{ and } S \subseteq \bigcup_{k \in \mathbb{N}} I_k \right\},$$

where $\mathscr{O}_n$ is the sequential cover of $\mathbb{R}^n$ given by Proposition 3.36 and the function λ is defined by

$$\lambda\left(\prod_{i=1}^{n} (a_i, b_i) \right) = \prod_{i=1}^{n} (b_i - a_i).$$

Observe that if $E \subseteq \mathbb{R}^n$ is an open box in $\mathbb{R}^n$ (that is, $E \in \mathscr{O}_n$), then $\lambda(E)$ is the volume of E and $m^*(E) = \lambda(E)$.

The first proposition shows that, in the case $n = 1$, m^* is a length function for all finite intervals, open or otherwise.

Proposition 3.38. *If $a, b \in \mathbb{R}$ are such that $a < b$, then*

$$m^*([a,b]) = m^*((a,b]) = m^*([a,b)) = m^*((a,b)) = b - a.$$

Proof. Because m^* is an outer measure, $m^*(S_1) \leq m^*(S_2)$ if $S_1 \subseteq S_2$. Therefore, $m^*((a,b)) \leq m^*((a,b]) \leq m^*([a,b])$ and $m^*((a,b)) \leq m^*([a,b)) \leq m^*([a,b])$. Since by definition, $m^*((a,b)) = b-a$, it is enough to prove that $m^*([a,b]) = b-a$. To this end, observe that, for every $\varepsilon > 0$, $[a,b] \subset (a-\varepsilon, b+\varepsilon)$. Because this open interval covers $[a,b]$, we have that $m^*([a,b]) \leq \lambda((a-\varepsilon, b+\varepsilon)) = b-a+2\varepsilon$. As this is true for every $\varepsilon > 0$, one concludes that $m^*([a,b]) \leq b-a = m^*((a,b))$. □

Similarly, one has:

Proposition 3.39. *If $E \in \mathscr{O}_n$, then $m^*(\overline{E}) = m^*(E)$.*

Proof. Exercise 3.88.

The notion of μ^*-null set, for an outer measure μ^* on a set X, was introduced earlier. To simplify the terminology here, we shall say a subset $S \subseteq \mathbb{R}$ is a *null set* if its Lebesgue outer measure $m^*(S)$ is 0. Thus, from Proposition 3.33, every null set $S \subseteq \mathbb{R}^n$ is necessarily Lebesgue measurable.

Example 3.40 (Some Null Sets). *The following subsets of $\mathbb{R}^n$ are null sets:*

1. *every finite or countably infinite set;*
2. *every countable union of null sets;*
3. *every subset of a null set;*
4. *the Cantor ternary set in $\mathbb{R}$.*

Proof. The details of these examples are left as an exercise (Exercise 3.89), but the case of the Cantor set is described here.

The Cantor ternary set $\mathscr{C}$ is given by $\mathscr{C} = \bigcap_{n\in\mathbb{N}} \mathscr{C}_n$, where each $\mathscr{C}_n$ is a union of 2^n pairwise disjoint closed intervals $F_{n,j}$ of length $(1/3)^n$. Thus,

$$m^*(\mathscr{C}) \leq m^*(\mathscr{C}_n) = m^*\left(\bigcup_{j=1}^{2^n} F_{n,j}\right) \leq \sum_{j=1}^{2^n} m^*(F_{n,j}) = \left(\frac{2}{3}\right)^n.$$

As the inequality above holds for all $n \in \mathbb{N}$, $m^*(\mathscr{C}) = 0$. □

Proposition 3.41. *If c is the cardinality of the continuum, then the cardinality of $\mathfrak{M}(\mathbb{R})$ is 2^c (the cardinality of the power set of $\mathbb{R}$).*

Proof. The Cantor ternary set $\mathscr{C}$ has the cardinality if the continuum (Proposition 1.83) and every subset of $\mathscr{C}$ is Lebesgue measurable. Hence, the cardinality of $\mathfrak{M}(\mathbb{R})$ is the cardinality of the power set of $\mathbb{R}$. □

In addition to null sets, every open set is a Lebesgue measurable set.

Proposition 3.42. *Every open set in $\mathbb{R}^n$ is Lebesgue measurable.*

Proof. If $W = (a,b)$ and $U = (p,q)$ are open intervals, and if $a < p < b < q$, then $W \cap U$ is the interval $I = (p,b)$ with length $\lambda(I) = (b-p)$ and $W \cap U^c$ is an interval $J = (a,p)$ with length $\lambda(J) = (p-a)$. Thus,

$$b-a = m^*(W) = m^*(I) + m^*(J) = m^*(W \cap U) + m^*(W \cap U^c).$$

The equation above holds in cases where the inequalities $a < p < b < q$ are not satisfied, because either one of $W \cap U$ or $W \cap U^c$ is empty, or $W \cap U$ and $W \cap U^c$ are nonempty disjoint open intervals whose lengths sum to $b-a$.

A similar feature holds in $\mathbb{R}^n$. If $W = \prod_{j=1}^{n}(a_j,b_j)$ and $U = \prod_{j=1}^{n}(p_j,q_j)$ are elements of $\mathscr{O}_n$, then either one of $W \cap U$ or $W \cap U^c$ is empty, or $W \cap U$ and $W \cap U^c$ are nonempty disjoint elements of $\mathscr{O}_n$ whose volumes sum to $\prod_{j=1}^{n}(b_j - a_j)$. Hence,

$$m^*(W) = m^*(W \cap U) + m^*(W \cap U^c)$$

for all $W, U \in \mathscr{O}_n$.

To prove that every open set in $\mathbb{R}^n$ is Lebesgue measurable, assume that $V \subseteq \mathbb{R}^n$ is an open set. Because $\mathbb{R}^n$ has a countable basis for its topology, every open set is a countable union of open sets. Therefore, we may assume without loss of generality that V is a basic open set: $V = \prod_{j=1}^{n}(a_j,b_j)$, for some $a_j, b_j \in \mathbb{Q}$. Let $S \subseteq \mathbb{R}^n$ be arbitrary and assume that $\varepsilon > 0$. Select a covering $\{U_k\}_k \subset \mathscr{O}_n$ of S such that $\sum_k \lambda(U_k) \leq m^*(S) + \varepsilon$. Because

$$S \cap V \subseteq \bigcup_k (U_k \cap V) \text{ and } S \cap V^c \subseteq \bigcup_k (U_k \cap V^c),$$

we have that

$$\begin{aligned} m^*(S \cap V) + m^*(S \cap V^c) &\leq \textstyle\sum_k (m^*(U_k \cap V) + m^*(U_k \cap V^c)) \\ &= \textstyle\sum_k m^*(U_k) \\ &\leq m^*(S) + \varepsilon. \end{aligned}$$

As $\varepsilon > 0$ is arbitrary, we deduce that $m^*(S) = m^*(S \cap V) + m^*(S \cap V^c)$, which proves that the open set V is Lebesgue measurable. □

Corollary 3.43. *Every Borel subset of* $\mathbb{R}^n$ *is Lebesgue measurable.*

If E and F are Lebesgue-measurable sets, then it is necessarily true that

$$m(E \cup F) + m(E \cap F) = m(E) + m(F).$$

Proposition 3.44 below extends this property to outer measure of arbitrary sets, but at the expense of weakening the equality above to an inequality.

Proposition 3.44. *For any subsets $A, B \subseteq \mathbb{R}^n$,*

$$m^*(A \cup B) + m^*(A \cap B) \le m^*(A) + m^*(B).$$

Proof. Let $\varepsilon > 0$ be given, and let $\{I_k\}_k$ and $\{J_i\}_i$ be coverings of A and B, respectively, by open boxes I_k and J_i such that

$$\sum_k \ell(I_k) \le m^*(A) + \varepsilon \quad \text{and} \quad \sum_i \ell(J_i) \le m^*(B) + \varepsilon.$$

Let $U = \bigcup_k I_k$ and $V = \bigcup_i J_i$. Thus, $A \subseteq U$ and $B \subseteq V$, and $A \cup B \subseteq U \cup V$ and $A \cap B \subseteq U \cap V$. Because U and V are open sets, they are Lebesgue measurable and, hence,

$$\begin{aligned} m^*(A \cup B) + m^*(A \cap B) &\le m^*(U \cup V) + m^*(U \cap V) = m(U \cup V) + m(U \cap V) \\ &= m(U) + m(V) \le \sum_k m(I_k) + \sum_i m(J_i) \le m^*(A) + m^*(B) + 2\varepsilon. \end{aligned}$$

Because $\varepsilon > 0$ is arbitrary, we have $m^*(A \cup B) + m^*(A \cap B) \le m^*(A) + m^*(B)$. □

The notion of σ-finite measure space was introduced in Definition 3.25 as a hybrid of finite measure space and infinite measure space. Lebesgue measure on $\mathbb{R}^n$ is a concrete example of a σ-finite space.

Proposition 3.45. *The measure space* $(\mathbb{R}^n, \mathfrak{M}(\mathbb{R}^n), m)$ *is σ-finite.*

Proof. If $K_j = \prod_1^n [-j, j]$ for each $j \in \mathbb{N}$, then K_j is measurable of finite measure $m(K_j) = (2j)^n$, and $\mathbb{R}^n = \bigcup_{j \in \mathbb{N}} K_j$. □

Every Borel subset of $\mathbb{R}^n$ is Lebesgue measurable, and Borel sets are determined by open subsets. Therefore, it seems natural to expect that the measures of arbitrary Lebesgue-measurable sets can be approximated by the measures of open and/or closed sets—this is the notion of regularity. The idea of translation invariance of measure is related to the fact, for example, that if one moved an n-cube C in $\mathbb{R}^n$ to some other position in space, the volume of C would not change.

A tool in analysing the regularity and translation invariance of Lebesgue measure is the following proposition.

Proposition 3.46. *The following statements are equivalent for a subset $E \subseteq \mathbb{R}^n$.*

1. *E is a Lebesgue-measurable set.*
2. *For every $\varepsilon > 0$ there is an open set $U \subseteq \mathbb{R}^n$ such that $E \subseteq U$ and $m^*(U \backslash E) < \varepsilon$.*
3. *For every $\varepsilon > 0$ there is a closed set $F \subseteq \mathbb{R}^n$ such that $F \subseteq E$ and $m^*(E \backslash F) < \varepsilon$.*

Proof. The logic of proof is slightly unusual in that following implications will be established: (1)⇒(2) and (1)⇒(3), then followed by (3)⇒(1) and (2)⇒(1).

To prove that (1) implies (2), suppose that $E \subseteq \mathbb{R}^n$ is Lebesgue measurable and let $\varepsilon > 0$ be given. The cases where $m^*(E)$ is finite or infinite will be treated separately.

In the first case, assume that $m^*(E) < \infty$. By definition, there is countable covering $\{I_k\}_{k\in\mathbb{N}} \subset \mathscr{O}_n$ of E such that

$$\sum_{k=1}^{\infty} \lambda(I_k) < m^*(E) + \varepsilon\,.$$

Let $U = \bigcup_k I_k$, which is an open (and, hence, Lebesgue measurable) set containing E. Note that

$$m^*(E) \leq m^*(U) \leq \sum_{k=1}^{\infty} \lambda(I_k) < m^*(E) + \varepsilon\,.$$

Because $m^*(U) < \infty$ and $E \subseteq U$ is a containment of Lebesgue-measurable sets, Proposition 3.31 states that

$$m^*(U \backslash E) = m^*(U) - m^*(E) \leq \sum_{k=1}^{\infty} \lambda(I_k) - m^*(E) < \varepsilon\,,$$

which proves (2) in the case where $m^*(E) < \infty$.

Assume now that $m^*(E) = \infty$. Define $E_k = E \bigcap ([-k,k]^n)$, for each $k \in \mathbb{N}$. Hence,

$$m^*(E_k) \leq (2k)^n \text{ and } E = \bigcup_{k\in\mathbb{N}} E_k\,.$$

Because $m^*(E_k) < \infty$, the first case implies there are open sets $U_k \subseteq \mathbb{R}^n$ such that $E_k \subseteq U_k$ and $m^*(U_k \backslash E_k) < \frac{\varepsilon}{2^{k+1}}$. Let $U = \bigcup_k U_k$, which is open and contains E. Thus,

$$U \backslash E \subseteq \bigcup_{k\in\mathbb{N}} U_k \backslash E_k$$

and

$$m^*(U \backslash E) \leq m^*\left(\bigcup_{k\in\mathbb{N}} U_k \backslash E_k\right) \leq \sum_{k=1}^{\infty} m^*(U_k \backslash E_k) \leq \frac{1}{2}\sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} < \varepsilon\,,$$

which proves (2) in the case where $m^*(E) = \infty$.

For the proof of (1) implies (3), suppose that $E \subseteq \mathbb{R}^n$ is Lebesgue measurable and let $\varepsilon > 0$ be given. As E is Lebesgue measurable, so is E^c. Apply (1)⇒(2) to

E^c to conclude that there is an open set U such that $E^c \subseteq U$ and $m^*(U \backslash E^c) < \varepsilon$. Let $F = U^c$, which is a closed set contained in E. Thus,

$$m^*(E \backslash F) = m^*(E \cap F^c) = m^*(E \cap U) = m^*(U \backslash E^c) < \varepsilon,$$

thereby proving that (1) implies (3).

To prove that (3) implies (1), assume hypothesis (3) and let $\varepsilon > 0$ be given. By hypothesis, there is a closed set F such that $F \subseteq E$ and $m^*(E \backslash F) < \epsilon$. Let $S \subseteq \mathbb{R}^n$ be any set. Note that $(S \cap E) \cap F = S \cap F$ and $(S \cap E) \cap F^c \subseteq E \cap F^c$; hence,

$$\begin{aligned} m^*(S \cap E) &= m^*((S \cap E) \cap F) + m^*((S \cap E) \cap F^c) \\ &\leq m^*(S \cap F) + m^*(E \cap F^c) \\ &= m^*(S \cap F) + m^*(E \backslash F) \\ &\leq m^*(S \cap F) + \varepsilon. \end{aligned} \tag{3.11}$$

The inclusion $F \subseteq E$ implies that

$$m^*(S \cap E^c) \leq m^*(S \cap F^c). \tag{3.12}$$

Therefore, (3.11) and (3.12) combine to produce

$$\begin{aligned} m^*(S) &\leq m^*(S \cap E) + m^*(S \cap E^c) \\ &\leq \varepsilon + m^*(S \cap F) + m^*(S \cap F^c) \\ &= \varepsilon + m^*(S). \end{aligned} \tag{3.13}$$

(The final equality arises from the fact that F—being closed—is Lebesgue measurable.) As ε is arbitrary, the inequalities (3.13) imply that $m^*(S) = m^*(S \cap E) + m^*(S \cap E^c)$. That is, E is Lebesgue measurable.

Lastly, the proof of (2) implies (1) is similar to the proof of the (3)⇒(1) and is, therefore, omitted. □

Proposition 3.47 (Regularity of Lebesgue Measure). *Lebesgue measure m on $\mathbb{R}^n$ has the following properties:*

1. $m(K) < \infty$ *for every compact subset* $K \subset \mathbb{R}^n$*;*
2. $\mu(E) = \inf\{\mu(U) \,|\, U \subseteq \mathbb{R}^n$ *is open and* $E \subseteq U\}$ *for every measurable set* E*;*
3. $\mu(E) = \sup\{\mu(K) \,|\, K$ *is compact and* $K \subseteq E\}$*, for every measurable set* E*.*

Proof. Assume that $K \subset \mathbb{R}^n$ is compact. For each $x \in K$ there is an open box $W_x \in \mathscr{O}_n$ of volume 1 such that $x \in W_x$. From the open cover $\{W_x\}_{x \in K}$ of the compact set K

extract a finite subcover $\{W_{x_j}\}_{j=1}^{\ell}$ and deduce that

$$m(K) \leq \sum_{j=1}^{\ell} m(W_{x_j}) = \ell < \infty.$$

Next, assume that $E \subseteq \mathbb{R}^n$ is a Lebesgue-measurable set and that $\varepsilon > 0$. By Proposition 3.46, there is an open set $U \subseteq \mathbb{R}$ such that $E \subseteq U$ and $m(U \backslash E) < \varepsilon$. Thus,

$$m(U) = m(E) + m(U \backslash E) \leq m(E) + \varepsilon .$$

Hence, $\mu(E) = \inf\{\mu(U) \,|\, U \subseteq \mathbb{R}^n \text{ is open and } E \subseteq U\}$.

Now assume that E is a Lebesgue-measurable set such that the closure $\overline{E}$ of E is compact. Let $\varepsilon > 0$ be given. By the previous paragraph there is an open set U containing $\overline{E} \setminus E$ such that $m(U) < m(\overline{E} \setminus E) + \varepsilon$. Let $K = \overline{E} \cap U^c$, which is a closed subset of the compact set $\overline{E}$; hence, K is compact. Furthermore, if $x \in K$, then $x \in \overline{E}$ and $x \not\in \overline{E} \cap E^c$, which is to say that $x \in E$. Thus, $K \subseteq E$. Because

$$m(\overline{E}) - m(K) = m(\overline{E}) - \big(m(\overline{E}) - m(U)\big) = m(U) < m(\overline{E}) - m(E) + \varepsilon,$$

we deduce that $m(E) < m(K) + \varepsilon$ and $\mu(E) = \sup\{\mu(K) \,|\, K \text{ is compact and } K \subseteq E\}$.

For each $k \in \mathbb{N}$, the set $B_k = \prod_1^n [-k, k]$ is compact. If $E_k = E \cap B_k$, then $\{E_k\}_{k \in \mathbb{N}}$ is an ascending sequence of sets such that $E = \bigcup_{k \in \mathbb{N}} E_k$. Thus, by continuity of measure, $m(E) = \lim_{k \to \infty} m(E_k)$. Choose any positive $r \in \mathbb{R}$ such that $r < m(E)$. Thus, there is a $k \in \mathbb{N}$ such that $r < m(E_k) < m(E)$. Because $\overline{E}_k$ is compact, the previous paragraph shows that there is a compact subset K of E_k such that $r < m(K)$. Now since $E_k \subseteq E$, K is also a subset of E. As the choice of $r < m(E)$ is arbitrary, this shows that $\mu(E) = \sup\{\mu(K) \,|\, K \text{ is compact and } K \subseteq E\}$. □

If $x \in \mathbb{R}^n$ and $S \subseteq \mathbb{R}^n$, then $x + S$ denotes the subset of $\mathbb{R}^n$ defined by

$$x + S = \{x + y \,|\, y \in S\}.$$

Proposition 3.48 (Translation Invariance of Lebesgue Measure). *If $E \subset \mathbb{R}^n$ is Lebesgue measurable and if $x \in \mathbb{R}$, then $x + E$ is Lebesgue measurable and*

$$m(x + E) = m(E). \tag{3.14}$$

Proof. If $I \in \mathscr{O}_n$ is the open box $I = \prod_{j=1}^n (a_j, b_j)$, then $x + I$ and I have the same volume. Thus, for any subset $S \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$, the outer measures of S and $x + S$

coincide. Therefore, we aim to prove that $x+E$ is a Lebesgue-measurable set if E is a Lebesgue-measurable set. To this end, we shall employ Proposition 3.46.

Let $\varepsilon > 0$. Because E is measurable, Proposition 3.46 states that there is an open set $U \subseteq \mathbb{R}^n$ such that $E \subseteq U$ and $m^*(U \backslash E) < \varepsilon$. Thus, there is a countable covering of $U \backslash E$ by open boxes I_k such that

$$\sum_{k \in \mathbb{N}} \lambda(I_k) < \varepsilon .$$

For each k, $x+I_k$ is an open box of volume $\lambda(x+I_k) = \lambda(I_k)$. Furthermore, because U is a countable union of basic opens (all of which are open boxes), the set $x+U$ is open, the inclusion $x+E \subseteq r+U$ is clear, and

$$(x+U) \setminus (x+E) = \{x+y \mid y \in U \backslash E\} = x+(U \backslash E) \subseteq \bigcup_{k \in \mathbb{N}} (x+I_k) .$$

Thus,

$$m^*\left((x+U) \setminus (x+E)\right) \leq \sum_{k \in \mathbb{N}} \lambda(x+I_k) = \sum_{k \in \mathbb{N}} \lambda(I_k) < \varepsilon .$$

Hence, $x+E$ satisfies the hypothesis of Proposition 3.46, thereby completing the proof that $x+E$ is Lebesgue measurable. □

It is natural to wonder whether every subset of $\mathbb{R}$ is Lebesgue measurable. That is not the case, as the following theorem shows. Because the proof of the theorem below requires the Axiom of Choice, the result is existential rather than constructive.

Theorem 3.49 (Vitali). *There is a subset $\mathscr{V}$ of $\mathbb{R}$ such that $\mathscr{V}$ is not Lebesgue measurable.*

Proof. Consider the relation $\sim$ on $\mathbb{R}$ defined by $x \sim y$ if and only if $y-x \in \mathbb{Q}$. It is not difficult to verify that $\sim$ is an equivalence relation, and so the equivalence classes $\dot{x}$ of $x \in \mathbb{R}$ form a partition of $\mathbb{R}$. Note that $\dot{x} = x+\mathbb{Q}$, for each $x \in \mathbb{R}$.

For each $x \in (-1,1)$, let $A_x = \dot{x} \cap (-1,1)$. Of course, if $x_1, x_2 \in (-1,1)$, then either $A_{x_1} = A_{x_2}$ or $A_{x_1} \cap A_{x_2} = \emptyset$. By the Axiom of Choice, there is a set $\mathscr{V}$ such that, for every $x \in (-1,1)$, $\mathscr{V} \cap A_x$ is a singleton set.

The set $\mathbb{Q} \cap (-2,2)$ is countable; hence, we may write

$$\mathbb{Q} \cap (-2,2) = \{q_k \mid k \in \mathbb{N}\} .$$

For each $k \in \mathbb{N}$, consider $q_k + \mathscr{V}$. Suppose that $x \in (q_k+\mathscr{V}) \cap (q_m+\mathscr{V})$, for some $k,m \in \mathbb{N}$. Then there are $c_k, c_m \in \mathscr{V}$ such that $q_k + c_k = q_m + c_m$; that is, $c_k - c_m = q_m - q_k \in \mathbb{Q}$, which implies that $c_k \in A_{c_m}$. As $\mathscr{V} \cap A_{c_m}$ is a singleton set, it must be that $c_k = c_m$ and $q_k = q_m$. Hence, $\{q_k+\mathscr{V}\}_{k \in \mathbb{N}}$ is a countable family of pairwise disjoint sets, each of which is obviously contained in the open interval $(-3,3)$.

Let $x \in (-1,1)$ and consider A_x. By construction of $\mathscr{V}$, there is precisely one element $y \in (-1,1)$ that is common to both A_x and $\mathscr{V}$. Thus, x and y are equivalent, which is to say that $x-y \in \mathbb{Q}$. Because $x,y \in (-1,1)$, $x-y \in (-2,2)$; hence, $x-y = q_k$, for some $k \in \mathbb{N}$. Therefore, $x \in q_k + \mathscr{V}$.

The arguments above establish that

$$(-1,1) \subset \bigcup_{k\in\mathbb{N}} (q_k + \mathscr{V}) \subset (-3,3)\,. \tag{3.15}$$

If $\mathscr{V}$ were Lebesgue measurable, then each $q_k + \mathscr{V}$ would be Lebesgue measurable, by Proposition 3.48, and $m(q_k + \mathscr{V})$ would equal $m(\mathscr{V})$. Therefore, if $\mathscr{V}$ were Lebesgue measurable, then

$$m\left(\bigcup_{k\in\mathbb{N}} (q_k + \mathscr{V})\right) = \sum_{k\in\mathbb{N}} m(q_k + \mathscr{V}) = \sum_{k\in\mathbb{N}} m(\mathscr{V})$$

would hold. Furthermore, computation of Lebesgue measure in (3.15) would yield

$$2 < \sum_{k=1}^{\infty} m(\mathscr{V}) < 6\,. \tag{3.16}$$

But there is no real number $m(\mathscr{V})$ for which (3.16) can hold. Therefore, it cannot be that $\mathscr{V}$ is a Lebesgue-measurable set. □

Corollary 3.50. *Outer measure m^* on $\mathbb{R}$ is not countably additive. That is, there is a sequence $\{S_k\}_{k\in\mathbb{N}}$ of pairwise disjoint subsets $S_k \subseteq \mathbb{R}$ such that*

$$m^*\left(\bigcup_{k\in\mathbb{N}} S_k\right) < \sum_{k\in\mathbb{N}} m^*(S_k)\,.$$

Proof. Let $S_k = q_k + \mathscr{V}$, as in the proof of Theorem 3.49. Because m^* is countably subadditive and because $(-1,1) \subseteq \bigcup_{k\in\mathbb{N}} (q_k + \mathscr{V})$,

$$2 \le m^*\left(\bigcup_{k\in\mathbb{N}} S_k\right).$$

Therefore, because $m^*(q_k + \mathscr{V}) = m^*(\mathscr{V})$, we have that. $m^*(S_k) = m^*(S_1)$, for all $k \in \mathbb{N}$, and the inequality above shows that $m^*(S_1) \neq 0$. Thus,

$$\sum_{k\in\mathbb{N}} m^*(S_k) = \infty\,.$$

On the other hand,

$$\bigcup_{k\in\mathbb{N}} S_k \subset (-3,3) \quad \Longrightarrow \quad m^*\left(\bigcup_{k\in\mathbb{N}} S_k\right) \le 6\,.$$

Hence,

$$m^*\left(\bigcup_{k\in\mathbb{N}} S_k\right) < \sum_{k\in\mathbb{N}} m^*(S_k),$$

as claimed. □

Vitali's Theorem produces a nonmeasurable subset of $(-1,1)$; the argument can be modified to produce a nonmeasurable subset of any measurable set of positive measure.

Theorem 3.51. *If $E \in \mathfrak{M}(\mathbb{R})$ and if $m(E) > 0$, then there is a subset $\mathscr{V} \subset E$ such that $\mathscr{V} \notin \mathfrak{M}(\mathbb{R})$.*

Proof. Let $A_k = E \cap [-k,k]$ for each $k \in \mathbb{N}$. The sequence $\{A_k\}_{k\in\mathbb{N}}$ is an ascending sequence in $\mathfrak{M}(\mathbb{R})$ with union E. Hence, by continuity of measure,

$$0 < m(E) = \lim_{k\to\infty} m(A_k),$$

and so $m(A_{k_0}) > 0$ for some $k_0 \in \mathbb{N}$. Now apply the argument of Theorem 3.49 using $E \cap [-k_0,k_0]$ in place of $(-1,1)$ to determine a nonmeasurable subset $\mathscr{V}$ of $E \cap [-k_0,k_0]$. □

The Borel sets and null sets determine the structure of Lebesgue-measurable sets.

Proposition 3.52. *The following statements are equivalent for a subset $E \subseteq \mathbb{R}$:*

1. *E is a Lebesgue-measurable set;*
2. *there exist $B, E_0 \subseteq \mathbb{R}$ such that:*
   a. *B is a Borel set,*
   b. *E_0 is a null set,*
   c. *$E_0 \cap B = \emptyset$, and*
   d. *$E = B \cup E_0$.*

Proof. Exercise 3.91. □

Proposition 3.52 shows how Borel sets can be used to characterise Lebesgue-measurable sets. Much less obvious is the following theorem, which indicates that these two σ-algebras are in fact distinct.

Theorem 3.53 (Suslin). *There exist Lebesgue-measurable subsets of $\mathbb{R}$ that are not Borel sets. In fact, there are Lebesgue-measurable subsets of the Cantor ternary set that are not Borel sets.*

Proof. Let $\tilde{\Phi}$ denote an extension of the Cantor ternary function (see Proposition 1.86) $\Phi : [0,1] \to [0,1]$ to a function $\mathbb{R} \to [0,1]$ by setting $\tilde{\Phi} = 0$ on $(-\infty,0)$, $\tilde{\Phi} = \Phi$ on $[0,1]$, and $\tilde{\Phi} = 1$ on $(1,\infty)$. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \tilde{\Phi}(x) + x, \quad \forall\, x \in \mathbb{R}.$$

Observe that f is continuous and monotone increasing.

Define a collection Σ of subsets of $\mathbb{R}$ as follows:

$$\Sigma = \{S \subseteq \mathbb{R} \,|\, f(S) \in \mathfrak{B}(\mathbb{R})\}\,.$$

We now show that Σ is a σ-algebra of subsets of $\mathbb{R}$. Because $\mathbb{R} = f(\mathbb{R})$, $\mathbb{R} \in \Sigma$. Moreover, if $A \in \Sigma$, then $f(A^c) \cap f(A) = \emptyset$ and $\mathbb{R} = f(\mathbb{R}) = f(A) \cup f(A^c)$ imply that $f(A^c) = f(A)^c$, whence $A^c \in \Sigma$. That is, Σ is closed under complementation. Now suppose that $\{A_k\}_{k\in\mathbb{N}} \subset \Sigma$; then

$$f\left(\bigcup_{k\in\mathbb{N}} A_k\right) = \bigcup_{k\in\mathbb{N}} f(A_k) \in \mathfrak{B}(\mathbb{R})\,.$$

Hence, Σ is a σ-algebra.

If $p,q \in \mathbb{Q}$ and $p < q$, then the continuity of f and the fact that f is monotone increasing leads to $f\,((p,q)) = (f(p), f(q))$. Therefore, $(p,q) \in \Sigma$ for all $p,q \in \mathbb{Q}$. Because Σ is a σ-algebra and Σ contains the base for the topology on $\mathbb{R}$, Σ necessarily contains the Borel sets of $\mathbb{R}$. Hence,

$$f(B) \in \mathfrak{B}(\mathbb{R})\,, \quad \forall B \in \mathfrak{B}(\mathbb{R})\,. \tag{3.17}$$

In particular, if $\mathscr{C}$ is the Cantor ternary set, then $f(\mathscr{C})$ is a Borel set. We now show that $f(\mathscr{C})$ has positive measure.

To this end note that $[0,1]\backslash\mathscr{C}$ is a union of countably many pairwise disjoint intervals (a_k, b_k), where $a_k, b_k \in \mathscr{C}$ for all $k \in \mathbb{N}$. Proposition 1.86 shows that Φ is constant on each such open interval. Therefore,

$$\begin{aligned}
2 = m([0,2]) &= m(f([0,1)) \\
&= m(f(\mathscr{C} \cup ([0,1]\backslash\mathscr{C}))) \\
&= m(f(\mathscr{C})) + m(f([0,1]\backslash\mathscr{C})) \\
&= m(f(\mathscr{C})) + \sum_{k\in\mathbb{N}} m((a_k + \Phi(a_k), b_k + \Phi(b_k)) \\
&= m(f(\mathscr{C})) + \sum_{k\in\mathbb{N}} (b_k - a_k) \\
&= m(f(\mathscr{C})) + m([0,1]\backslash\mathscr{C}) \\
&= m(f(\mathscr{C})) + 1\,.
\end{aligned}$$

Thus, $m(f(\mathscr{C})) = 1 > 0$ and so, by Theorem 3.51, $f(\mathscr{C})$ contains a subset $\mathscr{V}$ that is not Lebesgue measurable. Let $Q = f^{-1}(\mathscr{V})$. Because f is an injective function, the preimage Q of $\mathscr{V}$ under f must be contained in $\mathscr{C}$. Thus, Q is a null set and, hence, is a Lebesgue-measurable set. However, Q is not a Borel set. (If Q were a Borel set, then inclusion 3.17 would imply that $f(Q) = \mathscr{V}$ would be a Borel set—but it is not.) Hence, $Q \in \mathfrak{M}(\mathbb{R})$ and $Q \notin \mathfrak{B}(\mathbb{R})$. □

Similar results hold in higher dimensions.

Theorem 3.54. *Not every subset of $\mathbb{R}^n$ is Lebesgue measurable, and there exist Lebesgue-measurable subsets of $\mathbb{R}^n$ that are not Borel measurable.*

Proof. Exercise 3.92. □

3.5 Atomic and Non-Atomic Measures

There are a variety of ways to distinguish between qualitative properties of measures, and in this section we consider atomic measures and their polar opposites, non-atomic measures.

Definition 3.55. Assume that (X, Σ, μ) is a measure space.

1. A measurable subset $E \subseteq X$ is an *atom* for μ if $\mu(E) > 0$ and one of $\mu(E \cap F)$ or $\mu(E \cap F^c)$ is 0, for every $F \in \Sigma$.
2. The measure μ on (X, Σ) is *atomic* if every measurable set of positive measure contains an atom for μ.
3. The measure μ is *non-atomic* if μ has no atoms.

Thus, counting measure on $\mathbb{N}$ is atomic, whereas Lebesgue measure on $\mathbb{R}$ is non-atomic (Exercises 3.93 and 3.94). Every measure can be decomposed uniquely as a sum of two such measures, as shown by Proposition 3.57 below. The proof will make use of the following concept of singularity.

Definition 3.56. If μ and $\tilde{\mu}$ are measures on (X, Σ), then μ is *singular with respect to $\tilde{\mu}$* for each $E \in \Sigma$ there exists a set $F \in \Sigma$ with the properties that

1. $F \subseteq E$,
2. $\mu(E) = \mu(F)$, and
3. $\tilde{\mu}(F) = 0$.

The notation $\mu \mathscr{S} \tilde{\mu}$ indicates that μ is singular with respect to $\tilde{\mu}$, and if both $\mu \mathscr{S} \tilde{\mu}$ and $\tilde{\mu} \mathscr{S} \mu$ occur, then μ and $\tilde{\mu}$ are said to be *mutually singular*.

Proposition 3.57. *Every measure μ on a measurable space (X, Σ) has the form $\mu = \mu_{\mathrm{a}} + \mu_{\mathrm{na}}$, for some mutually singular atomic measure μ_{a} and non-atomic measure μ_{na} on (X, Σ). Moreover, if $\tilde{\mu}_{\mathrm{a}}$ and $\tilde{\mu}_{\mathrm{na}}$ are atomic and non-atomic measures on (X, Σ) such that $\mu = \tilde{\mu}_{\mathrm{a}} + \tilde{\mu}_{\mathrm{na}}$, then $\tilde{\mu}_{\mathrm{a}} = \mu_{\mathrm{a}}$ and $\tilde{\mu}_{\mathrm{na}} = \mu_{\mathrm{na}}$.*

Proof. Let $\mathscr{D}$ be the family of all countable unions of sets that are atoms for μ. For each $E \in \Sigma$, define

$$\mu_{\mathrm{a}}(E) = \sup\{\mu(E \cap D) \,|\, D \in \mathscr{D}\}$$

$$\mu_{\text{na}}(E) = \sup\{\mu(E \cap N) \,|\, \mu_{\text{a}}(N) = 0\}.$$

Observe that μ_{a} and μ_{na} are measures on (X, Σ) and satisfy $\mu = \mu_{\text{a}} + \mu_{\text{na}}$. If $\mu_{\text{a}}(N) = 0$, then $\mu(N \cap D) = 0$ for all $D \in \mathscr{D}$; hence, $\mu_{\text{na}}(D) = 0$ for all $D \in \mathscr{D}$, and so $\mu_{\text{a}} \mathscr{S} \mu_{\text{na}}$. By definition of μ_{na}, $\mu_{\text{na}}(D) = \mu_{\text{na}}(D \cap N)$ and $\mu_{\text{a}}(N) = 0$ imply that $\mu_{\text{a}}(D \cap N) = 0$; hence, $\mu_{\text{na}} \mathscr{S} \mu_{\text{a}}$.

To show that μ_{a} is atomic, suppose $E \in \Sigma$ such that $\mu_{\text{a}}(E) \neq 0$. Because $\mu_{\text{a}}(E) = 0$ when $\mu(E) = 0$, we deduce that $\mu(E) \neq 0$. Furthermore, by the definition of μ_{a}, $\mu_{\text{a}}(E) \neq 0$ implies there is some $D \in \mathscr{D}$ such that $\mu(E \cap D) \neq 0$. By the definition of $\mathscr{D}$, we can write $D = \bigcup_{n \in \mathbb{N}} D_n$, where each D_n is an atom for μ, and so $\mu(E \cap D_n) \neq 0$ for some $n \in \mathbb{N}$. Since $E \cap D_n$ is an atom for μ such that $\mu_{\text{a}}(E \cap D_n) \neq 0$, and because $\mu_{\text{a}}(E) = 0$ when $\mu(E) = 0$, we see that $E \cap D_n$ is an atom for μ_{a}. Thus, μ_{a} is an atomic measure.

To show that μ_{na} is non-atomic, suppose that $\mu_{\text{na}}(E) \neq 0$. Therefore, $\mu(E \cap N) \neq 0$ for some $N \in \Sigma$ with $\mu_{\text{a}}(N) = 0$. The set $E \cap N$ is not an atom for μ, because $\mu_{\text{a}}(E \cap N) \neq 0$ if $E \cap N$ were an atom. Since $\mu(E \cap N) \neq 0$ and because $E \cap N$ is not an atom for μ, there exists $F \in \Sigma$ such that $\mu(E \cap N \cap F) \neq 0$ and $\mu((E \cap N) \setminus F) \neq 0$. Hence, $\mu_{\text{na}}(E \cap F) \neq 0$ and $\mu_{\text{na}}(E \setminus F) \neq 0$, implying that μ_{na} is non-atomic.

The proof of the uniqueness of the decomposition is left as Exercise 3.95. □

3.6 Measures on Locally Compact Hausdorff Spaces

If one considers the Borel sets of a topological space X, then it is natural to expect that certain topological features of X play a role in the measure theory of X. But for this to occur, the particular measure under consideration needs to be aware of the topology. One class of measures that is sensitive to topology is the class of regular measures.

Definition 3.58. Let $(X, \mathscr{T})$ be a topological space and consider a measurable space (X, Σ) in which Σ contains the σ-algebra $\mathfrak{B}(X)$ of Borel sets of X. A measure μ on (X, Σ) is said to be a *regular measure* if

1. $\mu(K) < \infty$ for every compact subset $K \subseteq X$,
2. $\mu(E) = \inf\{\mu(U) \,|\, U \text{ is open and } E \subseteq U\}$, for every $E \in \Sigma$, and
3. $\mu(U) = \sup\{\mu(K) \,|\, K \text{ is compact and } K \subseteq U\}$, for every open set U.

Observe that Proposition 3.47 asserts that Lebesgue measure is regular.

The third property above for the measure of an open set extends to arbitrary measurable sets of finite measure.

Proposition 3.59. *Assume that μ is a regular measure on (X, Σ), where X is a topological space and where Σ contains the Borel sets of X. If $E \in \Sigma$ satisfies $\mu(E) < \infty$, then*

$$\mu(E) = \sup\{\mu(K)\,|\,K \textit{ is compact and } K \subseteq E\}.$$

Proof. Assume that $E \in \Sigma$ has finite measure and let $\varepsilon > 0$. Because $\mu(E) = \inf\{\mu(U)\,|\,U \text{ is open and } E \subseteq U\}$, there is an open set $U \subseteq X$ such that $E \subseteq U$ and $\mu(U) < \mu(E) + \varepsilon/2$. Hence, $\mu(U) < \infty$ and $\mu(U \setminus E) < \varepsilon/2$. Because U is open and has finite measure, the same type of argument shows that there is a compact set A with $A \subseteq U$ and $\mu(U) < \mu(A) + \varepsilon/2$. Lastly, since $\mu(U \setminus E) < \varepsilon/2$, there is an open set W with $U \setminus E \subseteq W$ and $\mu(W) < \varepsilon/2$. The set W^c is closed and is contained in $U^c \cup E$. Thus, $K = A \cap W^c$ is a closed subset of a compact set and is, hence, compact. Further,

$$K = A \cap W^c \subseteq A \cap (U^c \cup E) = (A \cap U^c) \cup (A \cap E) = A \cap E \subseteq E$$

and

$$\begin{aligned} \mu(E) &\leq \mu(U) \\ &< \mu(A) + \varepsilon/2 \\ &= \mu(A \cap W) + \mu(A \cap W^c) + \varepsilon/2 \\ &< \varepsilon/2 + \mu(K) + \varepsilon/2. \end{aligned}$$

Hence, $K \subseteq E$ and $\mu(E) < \mu(K) + \varepsilon$ implies that $\mu(E)$ is the least upperbound of all real numbers $\mu(K)$ in which K is a compact subset of E. □

Proposition 3.59 admits a formulation for σ-finite spaces, which will be of use in our analysis of L^p-spaces.

Proposition 3.60. *If (X, Σ, μ) is a σ-finite measure space in which X is a topological space, Σ contains the Borel sets of X, and μ is regular, then*

$$\mu(E) = \sup\{\mu(K)\,|\,K \textit{ is compact and } K \subseteq E\}$$

for every $E \in \Sigma$.

Proof. Exercise 3.96. □

Continuous functions are, from the point of view of analysis, fairly well understood. In comparison, measurable functions appear to be harder to grasp because of the existential nature of measurability. Therefore, in this light, the following two theorems are striking, for they show that, under the appropriate conditions, measurable functions within ε of being continuous.

Theorem 3.61. *Assume that μ is a regular finite measure on (X, Σ), where X is a compact Hausdorff space and where Σ contains the Borel sets of X. If $f : X \to \mathbb{R}$ is a* *bounded measurable function, then for every* $\varepsilon > 0$ *there exist a continuous function* $g : X \to \mathbb{R}$ *and a compact set* K *such that* $g_{|K} = f_{|K}$ *and* $\mu(K^c) < \varepsilon$.

Proof. To begin with, assume f is a simple function with range $\{\alpha_1, \dots, \alpha_n\}$, where $\alpha_1, \dots, \alpha_n \in [0,1]$ are distinct real numbers. Let $E_j = f^{-1}(\{\alpha_j\})$, which is a measurable set; note that $E_j \cap E_i = \emptyset$ if $i \neq j$. Let $\varepsilon > 0$ be given. By Proposition 3.59, for each j there is a compact subset $K_j \subseteq E_j$ with $\mu(K_j) + \frac{\varepsilon}{n} > \mu(E_j)$. Thus, $\mu(E_j \setminus K_j) < \frac{\varepsilon}{n}$ for every j. Because $E_j \cap E_i = \emptyset$ for $i \neq j$, if $K = \bigcup_{j=1}^{n} K_j$, then

$$\mu(K^c) = \mu\left(\bigcup_{j=1}^{n} E_j \setminus K_j\right) = \sum_{j=1}^{n} \mu(E_j \setminus K_j) < \varepsilon.$$

The restriction $f_{|K}$ of f to the compact set K is plainly continuous. Since X is normal, the Tietze Extension Theorem asserts that $f_{|K}$ has a continuous extension $g : X \to [0,1]$.

Assume now that f is an arbitrary measurable function with $0 \leq f(x) \leq 1$ for every $x \in X$. By Proposition 3.14 there is a monotone increasing sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ of nonnegative simple functions φ_n on X such that $\lim_{n\to\infty} \varphi_n(x) = f(x)$ for every $x \in X$. In fact, because $f(X) \subseteq [0,1]$, the convergence of $\{\varphi_n\}_n$ to f is uniform on X (Exercise 3.83). By the previous paragraph, for each $n \in \mathbb{N}$ there is a compact set K_n such that $\mu(K_n^c) < \varepsilon/2^n$ and $\varphi_{n|K_n}$ is continuous. Observe that $\varphi_1(x) + \sum_{n=2}^{N}(\varphi_n(x) - \varphi_{n-1}(x)) = \varphi_N(x)$, and so $\varphi_1 + \sum_{n=2}^{\infty}(\varphi_n - \varphi_{n-1}) = f$. Because $\varphi_1 + \sum_{n=2}^{N}(\varphi_n - \varphi_{n-1})$ is continuous on $K = \bigcap_{n=1}^{\infty} K_n$, and because $\varphi_1 + \sum_{n=2}^{\infty}(\varphi_n - \varphi_{n-1})$ converges uniformly to f, the measurable function f is continuous on K. By the Tietze Extension Theorem, $f_{|K}$ has a continuous extension $g : X \to [0,1]$. Because

$$\mu(K^c) \leq \sum_{n=1}^{\infty} \mu(K_n^c) < \varepsilon,$$

this completes the proof of the theorem in the case where $f(X) \subseteq [0,1]$.

For the case of general f, select $\alpha \in \mathbb{R}$ such that $\alpha f(X) \subseteq [-1,1]$, and decompose αf as $(\alpha f)^+ - (\alpha f)^-$, where $(\alpha f)^+$ and $(\alpha f)^-$ are measurable functions with ranges contained in $[0,1]$. Thus, the case of general f follows readily from the case of f with $f(X) \subseteq [0,1]$. □

Theorem 3.62 (Lusin). *Assume that* μ *is a regular measure on* (X, Σ)*, where* X *is a locally compact Hausdorff space and where* Σ *contains the Borel sets of* X*. If* $f : X \to \mathbb{R}$ *is a measurable function with the property that* $f_{|E^c} = 0$ *for some* $E \in \Sigma$ *with finite measure, then for every* $\varepsilon > 0$ *there exists a continuous and bounded* *function* $g : X \to \mathbb{R}$ *such that*

$$\mu(\{x \in X \,|\, f(x) \neq g(x)\}) < \varepsilon.$$

Proof. By hypothesis, $f_{|E^c} = 0$; thus, $E_n = \{x \in X \,|\, |f(x)| > n\}$ is a subset of E for every $n \in \mathbb{N}$. Because $\mu(E) < \infty$, Proposition 3.23 implies that $\mu(E_n) < \varepsilon/6$ for some $n \in \mathbb{N}$. Hence, if $F = E \cap E_n^c$, which is a set of finite measure, then $f_{|F}$ is bounded.

By Proposition 3.59, there is a compact subset $Y \subseteq F$ such that $\mu(F \setminus Y) < \varepsilon/6$. Consider the bounded measurable function $f_{|Y}$. By Theorem 3.61, there is a compact subset $K \subseteq Y$ and a continuous function $g_0 : Y \to \mathbb{R}$ such that $g_{0|K} = f_{|K}$ and $\mu(Y \setminus K) < \varepsilon/6$. Thus,

$$E \setminus K = E_n \cup (F \setminus Y) \cup (Y \setminus K)$$

yields $\mu(E \setminus K) < \varepsilon/2$.

The regularity of μ again implies the existence of an open set $U \subseteq X$ for which $E \subseteq U$ and $\mu(U) < \mu(E) + \varepsilon/2$. Hence, $\mu(U)$ is finite and $\mu(U \setminus E) < \varepsilon/2$. Because K is compact and $K \subseteq U$, Theorem 2.43 asserts that g_0 admits a continuous and bounded extension $g : X \to \mathbb{R}$ such that $g(x) = 0$ for all $x \notin U$. Therefore, $0 = g_{|U^c} = f_{|U^c}$ and, hence,

$$\begin{aligned}\mu(\{x \in X \,|\, f(x) \neq g(x)\}) &= \mu(\{x \in E \,|\, f(x) \neq g(x)\}) + \mu(\{x \in E^c \,|\, f(x) \neq g(x)\}) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon,\end{aligned}$$

which completes the proof. □

3.7 Signed and Complex Measures

Extending the notions of length, area, volume, and other arbitrary measures to real- and complex-valued quantities results in the concepts of signed measure and complex measure.

Definition 3.63. A function $\omega : \Sigma \to [-\infty, +\infty]$ on a measurable space (X, Σ) is called a *signed measure* if $\omega(\emptyset) = 0$ and

$$\omega\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \omega(E_k),$$

for every sequence $\{E_k\}_{k \in \mathbb{N}}$ of pairwise disjoint sets $E_k \in \Sigma$.

The definition of signed measure entails some subtleties. First of all, arithmetic in the extended real number system $[-\infty,+\infty]$ does not admit sums of the form $(-\infty)+(+\infty)$ or $(+\infty)+(-\infty)$, which implies that for each $E\in\Sigma$ at most one of $\omega(E)$ or $\omega(E^c)$ can have an infinite value (because $\omega(X)=\omega(E)+\omega(E^c)$). In particular, this means that if there exists a measurable set E with $\omega(E)=+\infty$, then necessarily $\omega(X)=+\infty$; and if there exists a measurable set E with $\omega(E)=-\infty$, then $\omega(X)=-\infty$ necessarily. Therefore, $\omega(X)$ can achieve at most one of the values $-\infty$ or $+\infty$. If μ does not achieve either of these infinite values, then ω is said to be a *finite signed measure*. The triple (X,Σ,ω) is called a *signed measure space*.

Definition 3.64. If (X,Σ,ω) is a signed measure space, and if $P,N\in\Sigma$, then

1. P is said to be *positive* with respect to ω if $\omega(E\cap P)\geq 0$ for every $E\in\Sigma$, and
2. N is said to be *negative* with respect to ω if $\omega(E\cap N)\leq 0$ for every $E\in\Sigma$.

Interestingly, a signed measure partitions a signed measure space into a positive part and a negative part, as shown by the Hahn Decomposition Theorem below.

Theorem 3.65 (Hahn Decomposition of Signed Measures). *If (X,Σ,ω) is a signed measure space, then there exist $P,N\in\Sigma$ such that*

1. *P is positive with respect to ω and N is negative with respect to ω,*
2. *$P\cap N=\emptyset$, and*
3. *$X=P\cup N$.*

Proof. We may assume without loss of generality that $-\infty$ is not one of the values assumed by ω. Let $\alpha=\inf\{\omega(E)\,|\,E\in\Sigma$ is a negative set$\}$. (Because $\emptyset$ is a negative set, the infimum is defined.) Let $\{E_k\}_{k\in\mathbb{N}}$ be a sequence of measurable sets for which $\alpha=\lim_k\omega(E_k)$. For each k let $N_k=E_k\setminus\left(\bigcup_{j=1}^{k-1}E_k\right)$ so that $\{N_k\}_{k\in\mathbb{N}}$ is a sequence of pairwise disjoint negative sets such that $\alpha=\inf_k\omega(N_k)$. Thus, with $N=\bigcup_{k=1}^{\infty}N_k$, we have for every $j\in\mathbb{N}$ that $\omega(N)=\sum_{k=1}^{\infty}\omega(N_k)\leq\omega(N_j)$. Hence, $\omega(N)=\alpha$ and N is a negative set. Because $-\infty$ is not in the range of ω, it must be that $\omega(N)\in\mathbb{R}$. Hence, α is the minimum measure of all negative subsets of X.

Let $P=N^c$. Assume, contrary to what we aim to prove, that P is not a positive set. Thus, there exists a measurable subset $E\subset P$ such that $\omega(E)<0$. The set E is not negative because, if it were, then $N\cup E$ would also be a negative set of measure $\omega(N\cup E)=\alpha+\omega(E)<\alpha$, which contradicts the fact that α is the minimum measure of all negative subsets of X. Hence, E must possess a measurable subset F of positive measure. Let $n_1\in\mathbb{N}$ denote the smallest positive integer for which there exists a measurable subset $F_1\subset E$ of measure $\omega(F_1)\geq 1/n_1$. Since $E\setminus F_1$ and F_1 are disjoint and have union E, $\omega(E)=\omega(E\setminus F_1)+\omega(F_1)$. That is, $\omega(E\setminus F_1)=\omega(E)-\omega(F_1)\leq\omega(E)-n_1{}^{-1}<\omega(E)$. For the very same reasons given earlier, the set $E\setminus F_1$ cannot be negative; thus, $E\setminus F_1$ contains a measurable subset of positive measure. Let $n_2\in\mathbb{N}$

denote the smallest positive integer for which there exists a measurable subset $F_2 \subset (E \setminus F_1)$ of measure $\omega(F_2) \geq 1/n_2$. Repeating this argument inductively produces a subset $\{n_k\}_{k\in\mathbb{N}} \subseteq \mathbb{N}$ and a sequence $\{F_k\}_{k\in\mathbb{N}}$ of pairwise disjoint measurable subsets $F_k \subset E$ such that the set $F = \bigcup_{k=1}^{\infty} F_k$ satisfies

$$\omega(F) = \sum_{k=1}^{\infty} \omega(F_k) \geq \sum_{k=1}^{\infty} \frac{1}{n_k} > 0.$$

Therefore, the subset $G = E \setminus F$ of E satisfies $\omega(G) \leq \omega(G) + \omega(F) = \omega(E) < 0$. Since $-\infty$ is not in the range of ω, $\omega(G)$ is a negative real number, and so

$$0 < \sum_{k=1}^{\infty} \frac{1}{n_k} \leq \sum_{k=1}^{\infty} \omega(F_k) = \omega(F) = \omega(E) - \omega(G) < |\omega(G)| < \infty$$

implies that $\lim_k n_k^{-1} = \lim_k \omega(F_k) = 0$. Therefore, if Q is a measurable subset of G, then

$$Q \subseteq G = E \cap F^c = E \cap \left(\bigcap_{k=1}^{\infty} F_k^c \right) = \bigcap_{k=1}^{\infty} (E \setminus F_k)$$

implies that $Q \subseteq E \setminus F_k$ for every $k \in \mathbb{N}$. If it were true that $\omega(Q) > 0$, then for some $j \in \mathbb{N}$ we would have $\omega(Q) > \frac{1}{n_j - 1}$, which is to say that Q is a subset of $E \setminus F_j$ of measure $\omega(Q) > \frac{1}{n_j - 1} > \frac{1}{n_j}$, in contradiction to the property of n_j being the smallest positive integer for which $E \setminus F_j$ has a subset A of measure $\omega(A) > \frac{1}{n_j}$. Hence, $\omega(Q) \leq 0$ and the fact that Q is an arbitrary measurable subset of G implies that G is a negative set. But $G \subseteq P$ implies that $G \cap N = \emptyset$ and so the negative subset $G \cup N$ satisfies $\omega(G \cup N) < \alpha$, which is in contradiction to the fact that α is the minimum measure of all negative subsets of X. Therefore, it must be that P is a positive set.

□

The sets P and N that arise in Theorem 3.65 are said to be a *Hahn decomposition* of the signed measure space (X, Σ, ω). While this decomposition need not be unique, Exercise 3.99 shows that if (P_1, N_1) and (P_2, N_2) are two Hahn decompositions of a signed measure space (X, Σ, ω), then

$$\omega(E \cap P_1) = \omega(E \cap P_2) \quad \text{and} \quad \omega(E \cap N_1) = \omega(E \cap N_2)$$

for all $E \in \Sigma$. Therefore, the functions $\omega_+, \omega_- : \Sigma \to [0, +\infty]$ defined by

$$\omega_+(E) = \omega(E \cap P) \quad \text{and} \quad \omega_-(E) = -\omega(E \cap N) \tag{3.18}$$

are measures on (X, Σ) and are independent of the choice of Hahn decomposition (P, N) of (X, Σ, ω). Note, also, that at least one of ω_+ and ω_- is a finite measure. These observations give rise to the next theorem.

Theorem 3.66 (Jordan Decomposition Theorem). *For every signed measure ω on a measurable space (X,Σ), there exist measures ω_+ and ω_- on (X,Σ) such that*

1. *at least one of ω_+ and ω_- is a finite measure, and*
2. *$\omega(E) = \omega_+(E) - \omega_-(E)$, for every $E \in \Sigma$.*

Furthermore, if γ, δ are measures on (X,Σ), where at least one of which is finite, and if $\omega(E) = \gamma(E) - \delta(E)$ for every $E \in \Sigma$, then $\omega_+(E) \leq \gamma(E)$ and $\omega_-(E) \leq \delta(E)$, for all $E \in \Sigma$.

Proof. Exercise 3.100. □

Turning now to complex measures, the definition below departs from the definitions of measure and signed measure in that it is assumed from the outset that the measure is finite.

Definition 3.67. A function $\nu : \Sigma \to \mathbb{C}$ on a measurable space (X,Σ) is called a *complex measure* if $\nu(\emptyset) = 0$ and

$$\nu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \nu(E_k),$$

for every sequence $\{E_k\}_{k\in\mathbb{N}}$ of pairwise disjoint sets $E_k \in \Sigma$.

By decomposing a complex measure ν into its real and imaginary parts $\Re\nu$ and $\Im\nu$, two finite signed measures are obtained, each of which is a difference of finite measures. Hence, there are finite measures μ_j on (X,Σ), for $j = 1,\ldots,4$, such that

$$\nu = (\mu_1 - \mu_2) + i(\mu_3 - \mu_4).$$

By considering the function $E \mapsto |\nu(E)|$, something close to a measure is obtained—but the triangle inequality makes this function countably subadditive rather than additive on sequences of pairwise disjoint sets. Therefore, to obtain a measure from a complex measure requires slightly more effort.

Definition 3.68. In a measurable space (X,Σ), a *measurable partition* of a measurable set $E \subseteq X$ is a family $\mathscr{P}_E$ of countably many subsets $A \in \Sigma$ such that $A \subseteq E$ for all $A \in \mathscr{P}_E$, $\bigcup_{A\in\mathscr{P}_E} A = E$, and $A \cap B = \emptyset$ whenever $A, B \in \mathscr{P}_E$ are distinct.

Proposition 3.69. *If ν is a complex measure on a measurable space (X,Σ) and if $|\nu| : \Sigma \to [0,\infty]$ is defined by*

$$|\nu|(A) = \sup\left\{\sum_{E\in\mathscr{P}_A} |\nu(E)| \,\middle|\, \mathscr{P}_A \text{ is a measurable partition of } A\right\},$$

then $|\nu|$ is a finite measure on (X,Σ).

Proof. Because $\mathscr{P}_\emptyset = \{A, B\}$, where $A = B = \emptyset$, is measurable partition of the empty set $\emptyset$, we have that $\nu(\emptyset) = \nu(\emptyset) + \nu(\emptyset)$ in $\mathbb{C}$ and so $\mu(\emptyset) = 0$.

To prove that $|\nu|$ is countably additive, let $\{E_k\}_{k\in\mathbb{N}}$ be a sequence of pairwise disjoint measurable sets and let $E = \bigcup_{k\in\mathbb{N}} E_k$. For each k, consider an arbitrary measurable partition $\{F_{kj}\}_{j\in\mathbb{N}}$ of E_k; thus, $\sum_j |\nu(F_{kj})| \leq |\nu|(E_k)$. Because $\{F_{kj}\}_{k,j\in\mathbb{N}}$ is an arbitrary measurable partition of E,

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} |\nu(F_{kj})| \leq |\nu|(E).$$

For each k, $|\nu|(E_k)$ is the supremum of $\sum_j |\nu(F_{kj})|$ over all measurable partitions $\{F_{kj}\}_{j\in\mathbb{N}}$ of E_k, and therefore the inequality above yields $\sum_{k=1}^{\infty} |\nu|(E_k) \leq |\nu|(E)$.

Conversely, select an arbitrary measurable partition $\{A_\ell\}_{\ell\in\mathbb{N}}$ of E. Because the sets $\{E_k\}_{k\in\mathbb{N}}$ are pairwise disjoint, $\{A_\ell \cap E_k\}_{k\in\mathbb{N}}$ is a partition of A_ℓ for every $\ell \in \mathbb{N}$, and $\{A_\ell \cap E_k\}_{\ell\in\mathbb{N}}$ is a partition of E_k for every $k \in \mathbb{N}$. Thus,

$$\sum_{\ell=1}^{\infty} |\nu(A_\ell)| \leq \sum_{\ell=1}^{\infty}\sum_{k=1}^{\infty} |\nu(A_\ell \cap E_k)| = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} |\nu(A_\ell \cap E_k)| \leq \sum_{k=1}^{\infty} |\nu(E_k)|,$$

and so $|\nu|(E) \leq \sum_{k=1}^{\infty} |\nu(E_k)|$. Hence, $|\nu|$ is countably additive.

As indicated previously, there are finite measures μ_j on (X, Σ), for $j = 1, \ldots, 4$, such that $\nu = (\mu_1 - \mu_2) + i(\mu_3 - \mu_4)$. Thus, for any measurable set $E \in \Sigma$, $|\nu(E)| \leq \sum_{j=1}^{4} \mu_j(E)$. Therefore, if $\mathscr{P}_X$ is a partition of X, then

$$\sum_{E\in\mathscr{P}_X} |\nu(E)| \leq \sum_{E\in\mathscr{P}_X}\sum_{j=1}^{4} \mu_j(E) = \sum_{j=1}^{4}\sum_{E\in\mathscr{P}_X} \mu_j(E) = \sum_{j=1}^{4} \mu_j(X) < \infty.$$

Hence, $|\nu|(X) \leq \sum_{j=1}^{4} \mu_j(X)$, which proves that $|\nu|$ is a finite measure. □

Definition 3.70. In Proposition 3.69 above, the measure $|\nu|$ on (X, Σ) induced by the complex measure ν is called the *total variation of* ν.

Problems

3.71. Show that the collection Σ of all subsets E of an infinite set X for which E or the complement E^c of E is countable is a σ-algebra.

3.72. Prove that if Λ is a family of σ-algebras on a subset X, then $\bigcap_{\Sigma\in\Lambda} \Sigma$ is a σ-algebra.

3.73. Assume that Σ is a σ-algebra of subsets of X. Show that, for each $E \in \Sigma$, the collection $\Sigma(E)$ of subsets of X defined by

$$\Sigma(E) = \{E \cap A \,|\, A \in \Sigma\}$$

is a σ-algebra on E.

3.74. Let Σ be a σ-algebra of subsets of a nonempty set X, and let $E_k \in \Sigma$ for $k \in \mathbb{N}$. Define

$$\begin{aligned} \limsup E_k &= \textstyle\bigcap_{k\geq 1} \left(\bigcup_{n\geq k} E_n\right), \\ \liminf E_k &= \textstyle\bigcup_{k\geq 1} \left(\bigcap_{n\geq k} E_n\right). \end{aligned}$$

Prove the following statements.

1. $\limsup E_k$ and $\liminf E_k$ belong to Σ.
2. If $E_1 \subseteq E_2 \subseteq E_3 \subseteq \dots$, then $\limsup E_k = \bigcup_k E_k = \liminf E_k$

3.75. Let E_k denote the closed interval $E_k = [0, 1 + \frac{(-1)^k}{k}]$. Determine the sets $\limsup E_k$ and $\liminf E_k$. (Suggestion: consider the cases k even and k odd separately.)

3.76. Let X be a nonempty set X and let $\{A_k\}_{k\in\mathbb{N}}$ be a sequence of subsets of X. Define $E_0 = \emptyset$ and, for $n, m \in \mathbb{N}$,

$$E_m = \bigcup_{k=1}^{m} A_k, \quad F_n = A_n \setminus E_{n-1}.$$

Prove the following statements.

1. $\{E_n\}_n$ is a monotone increasing sequence of sets (that is, $E_n \subseteq E_{n+1}$ for all n).
2. $\{F_n\}_n$ is a sequence of pairwise disjoint sets.
3. $\bigcup_n E_n = \bigcup_n F_n = \bigcup_n A_n$.

3.77. Prove that if a σ-algebra Σ on an infinite set X has infinitely many elements, then Σ is uncountable.

3.78. Prove that if $(X,\mathscr{T})$ is a topological space, and if $\Sigma_{\mathscr{T}}$ is the σ-algebra generated by $\mathscr{T}$, then, with respect to the measurable space $(X,\Sigma_{\mathscr{T}})$, every continuous function $f : X \to \mathbb{R}$ is a measurable function.

3.79. Suppose that (X,Σ) is a measurable space and that $h : X \to \mathbb{R}$ is a measurable function for which $h(x) \neq 0$, for all $x \in X$. Prove that the function $1/h$ is measurable.

3.80. Prove that if (X,Σ) is a measurable space and if $E \subseteq X$, then the characteristic function $\chi_E : X \to \mathbb{R}$ is a measurable function if and only if $E \in \Sigma$.

3.81. Let U be a nonempty subset of $\beta\mathbb{N}$ (see Section 2.6), and consider the characteristic function χ_U. Prove that χ_U is continuous if and only if both U and $\overline{U}$ are open in $\beta\mathbb{N}$.

3.82. Assume that (X,Σ) is a measurable space and that $E \in \Sigma$.

1. If $f : E \to \mathbb{R}$ is a measurable function relative to the measurable space $(E,\Sigma(E))$, then prove that the extension $\tilde{f} : X \to \mathbb{R}$ of f defined by $\tilde{f} = f\chi_E$ is a measurable function with respect to the measurable space (X,Σ).
2. Conversely, if $\tilde{f} : X \to \mathbb{R}$ is a measurable function with respect to the measurable space (X,Σ), and if $f = \tilde{f}_{|E}$ (the restriction of $\tilde{f}$ to E), then prove that $f : E \to R$ is a measurable function with respect to the measurable space $(E,\Sigma(E))$.

3.83. If $f : X \to [0,1]$ is a measurable function, then prove that there is a monotone-increasing sequence of nonnegative simple functions $\varphi_n : X \to [0,1]$ such that $\lim_{n\to\infty} \varphi_n(x) = f(x)$ uniformly—that is, for every $\varepsilon > 0$ there is an $N_\varepsilon \in \mathbb{N}$ such that $|f(x) - \varphi_n(x)| < \varepsilon$ for all $n \geq N_\varepsilon$ and all $x \in X$.

3.84. Let X be an infinite set and let Σ be the σ-algebra in Exercise 3.71. Define a function $\mu : \Sigma \to [0,\infty]$ by $\mu(E) = 0$ if $E \in \Sigma$ is countable and $\mu(E) = 1$ if $E \in \Sigma$ is uncountable. Show that μ is a measure on (X,Σ).

3.85. Consider the measurable space $(\mathbb{N},\mathscr{P}(\mathbb{N}))$, where $\mathscr{P}(\mathbb{N})$ is the power set of $\mathbb{N}$. Prove that the function $\mu : \Sigma \to [0,\infty]$ defined by

$$\mu(E) = \text{the cardinality of } E$$

defines a measure on $(\mathbb{N},\mathscr{P}(\mathbb{N}))$.

3.86. A function $\mu : \Sigma \to [0,\infty)$ on a measurable space (X,Σ) is *finitely additive* if, for all finite sub-collections $\{E_k\}_{k=1}^n$ of pairwise disjoint measurable sets E_k,

$$\mu\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu(E_k).$$

Prove that if a finitely additive function μ also satisfies $\lim_k \mu(A_k) = 0$, for every descending sequence $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ of sets $A_j \in \Sigma$ in which $\bigcap_{k=1}^{\infty} A_k = \emptyset$, then μ is in fact a measure on (X, Σ).

3.87. Assume that μ is a measure on a measurable space (X, Σ). Prove that

$$\mu(E \cup F) + \mu(E \cap F) = \mu(E) + \mu(F),$$

for all $E, F \in \Sigma$.

3.88. Prove that if $E \in \mathscr{O}_n$, then $m^*(\overline{E}) = m^*(E)$, where

$$\mathscr{O}_n = \{\prod_{i=1}^{n}(a_i, b_i) \,|\, a_i, b_i \in \mathbb{R},\, a_i \leq b_i\}.$$

3.89. Prove that each of the following subsets of $\mathbb{R}^n$ is a null set.

1. Every finite or countably infinite set.
2. Every countable union of null sets.
3. Every subset of a null set.

3.90. Prove that if $E \subseteq \mathbb{R}^n$ is Lebesgue measurable such that $m(E) > 0$, then E contains a nonmeasurable subset.

3.91. Prove that the following statements are equivalent for a subset $E \subseteq \mathbb{R}$:

1. E is a Lebesgue-measurable set;
2. there exist $B, E_0 \subseteq \mathbb{R}$ such that:
   a. B is a Borel set,
   b. E_0 is a null set,
   c. $E_0 \cap B = \emptyset$, and
   d. $E = B \cup E_0$.

3.92. Prove that there exist subsets S of $\mathbb{R}^n$ that are not Lebesgue measurable, and that there exist Lebesgue-measurable subsets E of $\mathbb{R}^n$ that are not Borel measurable.

3.93. Determine the atoms for counting measure on $\mathbb{N}$.

3.94. Prove that Lebesgue measure on $\mathbb{R}^n$ is non-atomic.

3.95. Suppose that $\mu = \mu_{\mathrm{a}} + \mu_{\mathrm{na}} = \tilde{\mu}_{\mathrm{a}} + \tilde{\mu}_{\mathrm{na}}$ are two decompositions of a measure μ on (X, Σ) as the sum of an atomic measure and a non-atomic measure, where $\mu_{\mathrm{a}} \mathscr{S} \mu_{\mathrm{na}}$, $\mu_{\mathrm{na}} \mathscr{S} \mu_{\mathrm{a}}$, $\tilde{\mu}_{\mathrm{a}} \mathscr{S} \tilde{\mu}_{\mathrm{na}}$, and $\tilde{\mu}_{\mathrm{na}} \mathscr{S} \tilde{\mu}_{\mathrm{a}}$.

1. Show that $\mu_{\mathrm{na}} \mathscr{S} \tilde{\mu}_{\mathrm{a}}$ $\tilde{\mu}_{\mathrm{a}} \mathscr{S} \mu_{\mathrm{na}}$
2. Show that $\tilde{\mu}_{\mathrm{na}}(E) - \mu_{\mathrm{na}}(E) \geq 0$ and $\mu_{\mathrm{a}}(E) - \tilde{\mu}_{\mathrm{a}}(E) \geq 0$ dor every $E \in \Sigma$
3. Show that $\tilde{\mu}_{\mathrm{a}} = \mu_{\mathrm{a}}$ and $\tilde{\mu}_{\mathrm{na}} = \mu_{\mathrm{na}}$.

3.96. Prove that if (X,Σ,μ) is a σ-finite measure space in which X is a topological space, Σ contains the Borel sets of X, and μ is regular, then

$$\mu(E)=\sup\{\mu(K)\,|\,K \text{ is compact and } K\subseteq E\}$$

for every $E\in\Sigma$.

3.97. Let Σ denote the Borel sets of $X=[0,1]$ and define a function $\mu:\Sigma\to[0,\infty]$ by $\mu(E)=m(E)$, if $0\notin E$, and $\mu(E)=\infty$, if $0\in E$.

1. Prove that μ is a measure on (X,Σ).
2. Prove that (X,Σ,μ) is not a σ-finite measure space.

3.98. Show that, in a signed measure space (X,Σ,ω), the union and intersection of finitely many positive sets are positive sets, and that the union and intersection of finitely many negative sets are negative sets.

3.99. Suppose that (P_1,N_1) and (P_2,N_2) are Hahn decompositions of a signed measure space (X,Σ,ω). Prove that, for every $E\in\Sigma$,

$$\omega(E\cap P_1)=\omega(E\cap P_1\cap P_2)=\omega(E\cap P_2).$$

3.100. Assume that (X,Σ,ω) is a signed measure space with Hahn decomposition (P,N). Show that the functions ω_+ and ω_- defined by

$$\omega_+(E)=\omega(E\cap P)\quad\text{and}\quad\omega_-(E)=-\omega(E\cap N),$$

for $E\in\Sigma$ are measures on (X,Σ) with the following properties:

1. at least one of ω_+ and ω_- is a finite measure;
2. $\omega(E)=\omega_+(E)-\omega_-(E)$, for every $E\in\Sigma$;
3. if γ,δ are measures on (X,Σ), where at least one of which is finite, and if $\omega(E)=\gamma(E)-\delta(E)$ for every $E\in\Sigma$, then $\omega_+(E)\leq\gamma(E)$ and $\omega_-(E)\leq\delta(E)$, for all $E\in\Sigma$.