3

Nil Fourier Coefficients and Tilings

Summary. Originally, vanishing Fourier coefficients appeared as an obstruction: they impede phase retrieval and prevent, for instance, the solution of Lewin's problem (find A knowing B and $\mathrm{IFunc}(A,B)$). But recent research and problems shed a more positive light: for instance the set $Z(A)$ of indexes k such that $a_k = \widehat{\mathbf{1}_A}(k) = 0$, a highly organised subset of $\mathbb{Z}_n$, is now the fashionable introduction to a definition of tilings. The theory of tilings is a crossroad of geometry, algebra, combinatorics, topology; and one of those privileged domains where musical ideas enable us to make some headway in non-trivial mathematics. Here the notion of Vuza canon together with transformational techniques (often introduced by composers) allowed some progress on difficult conjectures. More generally, tiling situations provide rich compositional material as we will see later in this book, cf. Section 4.3.3. In that respect, I included in this chapter Section 3.3 on algorithms (for practical purposes, though there are some interesting theoretical implications in there too.

We will need some additional algebraic material on polynomials, which is introduced in the preliminary section. A few more technical results of Galois theory are recalled and admitted without proof.

Cyclotomic polynomials

We will require the notion of cyclotomic polynomial. The etymology is telling: much of our work relates to 'splitting the circle', and this notion is the most powerful tool to do it.

Lemma 3.1. *Let $\Phi_m(X) = \prod(X-\xi)$ where ξ runs over the set of roots of unity with order exactly m, i.e. $\xi^m = 1$ but $\xi^p \neq 1$ for $0 < p < m$. In other words,*

$$\Phi_m(X) = \prod_{k\in\mathbb{Z}_m^*} (X - \mathrm{e}^{2\mathrm{i}\pi k/m}).$$

Then $\Phi_m \in \mathbb{Z}[X]$ (it has integer coefficients) and Φ_m is irreducible in the ring $\mathbb{Q}[X]$ (or $\mathbb{Z}[X]$): any divisor of Φ_m is a constant or Φ_m itself.

Proof. The non-obvious point is the irreducibility in $\mathbb{Q}[X]$, we refer the curious reader to textbooks or the Internet. The integral character of the coefficients derives

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from the following formula, each polynomial being monic. It is also an effective way of computing these polynomials by Euclidean division:

$$X^n - 1 = \prod_{d|n} \Phi_d(X) \tag{3.1}$$

For instance for $n = p$ prime, we get $\Phi_p(X) = \dfrac{X^p - 1}{X - 1} = 1 + X + \ldots X^{p-1}$.

The meaning of this is that any rational polynomial which vanishes in some root of unity must be divisible by Φ_m, i.e. it also features all other roots with the same order. Actually this is one way to prove the irreducibility, using the Galois automorphisms of the cyclotomic field which permutes roots with the same order so that any polynomial featuring the factor $(X - \xi)$ in $\mathbb{C}[X]$ also features $(X - \xi')$ if ξ' has the same order.

By induction one derives the following from formula 3.1:

Proposition 3.2. $\Phi_n(1)$ *is equal to* $\begin{cases} p & \text{if } n \text{ is a prime power } p^\alpha \\ 1 & \text{else} \end{cases}$.

3.1 The Fourier nil set of a subset of $\mathbb{Z}_n$

3.1.1 The original caveat

It is now clear that when Lewin wrote his first paper [62] wherein he considered the question of identifying A from the knowledge of another pc-set B and $\text{IFunc}(A, B)$, he had in mind the formula

$$\mathbf{1}_A * \mathbf{1}_{-B} = \text{IFunc}(A, B) \iff \mathscr{F}_A \times \overline{\mathscr{F}_B} = \widehat{\text{IFunc}(A, B)}.$$

However he could only allude to Fourier transform (and even that earned him outraged reactions from readers of the *Journal of Music Theory*). So perhaps he was right in stating the condition that $\mathscr{F}_B$ vanished in less mathematical terms. However, ‘Lewin’s conditions’ are far from convenient. Let us enumerate these cases[1] which prevent[2] recuperation of one pc-set from its intervallic relationship with another:

1. **the whole-tone scale property**
   A chord has this property if it “has the same number of notes in one whole-tone set, as it has in the other [whole-tone set].”
2. **the diminished-seventh chord property**
   A chord has this property if it “has the same number of notes in common with each of the three diminished-seventh chord sets.”
3. **the augmented triad property**
   A chord has this property if, “for any augmented-triad set A, [it] has the same number of notes in common with $T_6(A)$,[3] as it has in common with A.”

[1] We use a more synthetic presentation [63] than the original one [62] which is frankly unreadable.

[2] See however the new method in Section 2.2.2 above.

[3] As usual in music theory, $T_k(A) = A + k$ denotes the transposition by k semitones.

4. **the tritone property**
   A chord has this property if "for any (0167)-set K, [it] has the same number of notes in common with $T_3(K)$, as it has in common with K." This is equivalent to keeping the difference of notes between the intersection with a tritone $\mathscr{T}$ and its translate $T_3(\mathscr{T})$ a constant, hence the original name.
5. **the exceptional property**
   A chord has this property if it "can be expressed as a disjoint union of tritone sets and/or augmented-triad sets" (the original definition enumerated no less than 10 sub-cases).

The least one can say about these properties (especially the last two) is that they are not exactly straightforward, especially when compared to the concise '$\mathscr{F}_A(k) = 0$' (respectively for $k = 6, 4, 3, 2, 1$ as we will develop below). More precisely, they originate in the nullity of several specific Fourier coefficients, respectively (at least)

1. The 6^{th} for the whole-tone property;
2. The 4^{th} and 8^{th} for the diminished-seventh property;
3. The 3^{rd} and 9^{th} for the augmented triad property;
4. The 2^{nd} and 10^{th} for the "tritone" property;
5. The $1^{st}, 5^{th}, 7^{th}$ and 11^{th} for the exceptional property.

See below the discussion around Theorem 3.11 for an explanation of this multiplication of nil Fourier coefficients. In Fig. 3.1, one can see an example for each situation, following the order in which they are enumerated in the text.

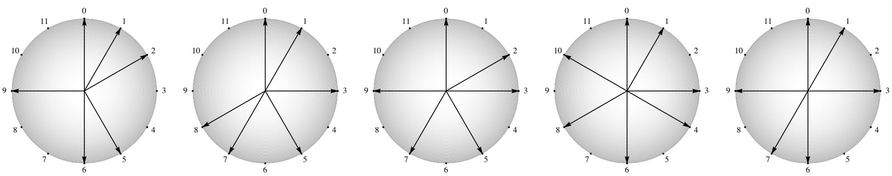

Fig. 3.1. The five special cases enumerated by Lewin

In his dissertation, Ian Quinn introduced a wonderfully telling implementation of these conditions, in terms of 'balances' (the word here is taken in the non-musical meaning of [weighing] 'scale', this word being admittedly misleading in the context). For instance, the third one is expressed by the balance of four pans, each containing the intersection of A with one of the four augmented triads, see Fig. 3.2. Though the expression of the five conditions with Quinn's balances has an aesthetic charm of its own, it is still cumbersome to check whether a given pc-set will fail one of them. We can provide a more synthetic characterisation of the 'bad cases' of Lewin's problem:

Theorem 3.3. *A distribution s has at least one nil Fourier coefficient iff the associated circulating matrix $\mathscr{S}$ is singular, which can be checked for instance with its determinant (or rank).*

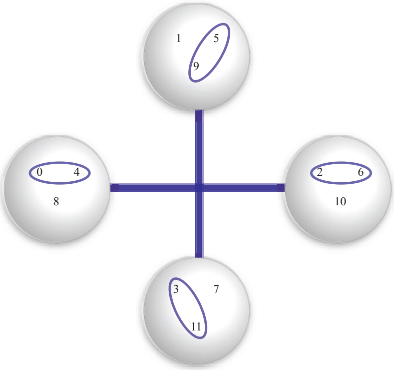

Fig. 3.2. Condition $\mathscr{F}_A(3) = 0$ is checked by pc-set $\{0, 2, 3, 4, 5, 6, 9, 11\}$.

Example 3.4. One can check whether the melodic A minor $\{0, 2, 4, 6, 8, 9, 11\}$ is a 'bad case' by computing the following determinant, which is straightforward for most pocket calculators and does not involve the complex numbers and exponentials featured in the definition of the DFT:

$$\det(\mathscr{S}) = \begin{vmatrix} 1&1&0&1&1&0&1&0&1&0&1&0 \\ 0&1&1&0&1&1&0&1&0&1&0&1 \\ 1&0&1&1&0&1&1&0&1&0&1&0 \\ 0&1&0&1&1&0&1&1&0&1&0&1 \\ 1&0&1&0&1&1&0&1&1&0&1&0 \\ 0&1&0&1&0&1&1&0&1&1&0&1 \\ 1&0&1&0&1&0&1&1&0&1&1&0 \\ 0&1&0&1&0&1&0&1&1&0&1&1 \\ 1&0&1&0&1&0&1&0&1&1&0&1 \\ 1&1&0&1&0&1&0&1&0&1&1&0 \\ 0&1&1&0&1&0&1&0&1&0&1&1 \\ 1&0&1&1&0&1&0&1&0&1&0&1 \end{vmatrix} = 0$$

Remark 3.5. Another way to check that the matrix is singular consists of noticing that the sums of columns $1, 6, 7, 12$ is the same as that of columns $3, 4, 9, 10$, namely $(3, 2, 2, 3, 2, 2, 3, 2, 2, 3, 2, 2)^T$.

In our opinion it is high time that a spade be called a spade, and 'Lewin's special cases' should be computed in the way they were discovered, i.e. by checking the nullity of Fourier coefficients.

Usually a clock diagram of the *multiset* $(kA)_{mult}$ (all multiples of elements of A, times k mod n, counted with their multiplicities) will enable one to see at a glance whether $\mathscr{F}_A(k) = 0$. In Fig. 3.3 one can see the diagrams for $\mathscr{F}_A(1)$ and $\mathscr{F}_A(2)$ where A is the melodic minor above. For the first coefficient, the clock represents just A and one cancels out 0-6 and 2-8; the remainder 4-9-11 obviously does not sum to 0. On the next clock, $(2A)_{mult} = \{0, 4, 8, 0, 4, 6, 10\}$ is a multiset with 0-4 redoubled. Gathering 0-4 together with 8 as a subset with sum 0 leaves 0-4-6-10 which also sums to nil. All cases of nil coefficients for $n = 12$ are similarly reducible to obvious cases (see Conjecture 3.16 and Fig. 3.5 below though), the 'special case' being actually the simplest, since no multiplication of A into a multiset is necessary.

A complete table of the 134 pc-sets classes (up to transposition) with some nil Fourier coefficient is provided on Table 8.2.[4]

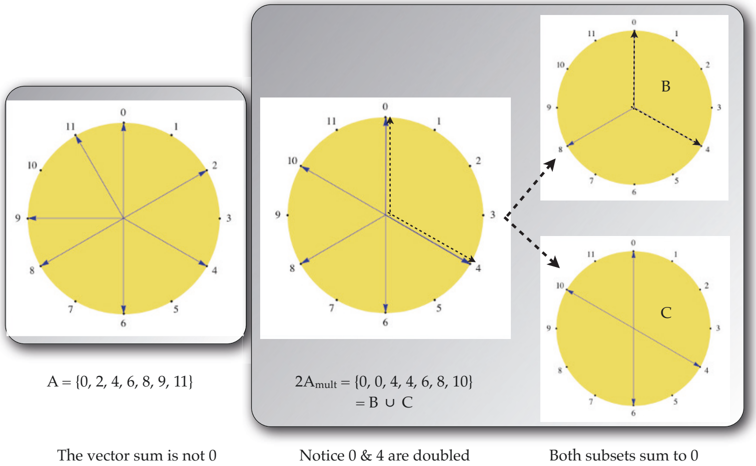

Fig. 3.3. Checking nullity of some Fourier coefficients

3.1.2 Singular circulating matrixes

According to Theorem 3.3, the vanishing of some Fourier coefficients can be checked by computing a determinant. We introduce the corresponding matricial vocabulary for convenience:

Definition 3.6. *A distribution* $s \in K^n$ *is* singular $\iff \det \mathscr{S} = 0$, *i.e. when at least one of its Fourier coefficients is nil (*$\mathscr{S}$ *is the circulating matrix associated with* s*). Otherwise it is* invertible.

From the characterisation of singular matrixes by the linear dependency of their columns we get the useful

Proposition 3.7. *A subset* A *of* $\mathbb{Z}_n$ *is singular iff the subset is a linear combination of its translates* $A+k, k \neq 0$.

For instance, the whole-tone scale is equal to every one of its translates by an even number of semitones. Less trivially, a minor third is a combination of other minor thirds, as for instance

[4] There are 1,502 special cases out of 4,094 subsets of $\mathbb{Z}_{12}$, a fairly common occurrence.

$$(C, E\flat) - (E\flat, F\sharp) + (F\sharp, A) = (A, C)$$

This might appear to be a consequence of the minor third dividing the octave equally, but this is wrong, since the scale matrix of the major third is invertible, and the scale matrix of the fifth is singular.[5]

Since these singular cases are troublesome for reconstruction problems, [13] explored the simplest cases of singular subsets: dyads.

Theorem 3.8. *The pair $(0,d)$ in $\mathbb{Z}_n$ is never singular if n is odd. If $n = 2^v q$ with q odd, it is invertible iff 2^v divides d, the span of the dyad. Otherwise, the rank of the matrix associated with $(0,d)$ is equal to $n - \gcd(d,n)$; it is minimal for $d = n/2$, the equal division of the octave (the generalised tritone). In that case, it is equal to $n/2$.*

For instance, when $n = 12$ the only 'invertible dyad' is the major third.

Proof. The matrix $\mathscr{S}$ of the dyad $(0,d)$ is equal to identity plus the matrix $\mathscr{D}$ of the permutation $i \mapsto i + d \mod n$. Hence the kernel (or nullspace) of $\mathscr{S}$ is the eigenspace of $\mathscr{D}$ for eigenvalue -1. Let us reason geometrically, considering the vectors $e_0 \dots e_{n-1}$ of the canonical basis of $\mathbb{R}^n$. A vector $x = \sum_{i=0}^{n-1} x_i e_i$ lies in this eigenspace iff

$$\sum_{i=0}^{n-1} x_i e_i = \sum_{i=0}^{n-1} -x_i e_{i+d} \iff \forall i = 0, \dots n-1 \quad x_{i+d} = -x_i$$

(all indexes are computed modulo n).

From this we get $x_{i+kd \mod n} = (-1)^k x_i$. Hence $x_{i+nd \mod n} = x_i = (-1)^n x_i$: if n is odd then the only solution is $x = 0$, i.e. $\mathscr{S}$ is invertible.

Say now that n is even, $n = 2^v q$ where q is odd. Let k be the smallest integer such that $kd = 0 \mod n$, e.g. $k = n/\gcd(d,n) = n/g$ (we put throughout $g = \gcd(d,n)$ for concision). If k is odd, for instance 2^v divides d, then we have the same impossibility, and S is invertible. We have proved that if 2^v divides d, then $\mathscr{S}$ is invertible.

Assume now that 2^v does not divide d, i.e. $d = 2^u d'$ with d' odd and $u < v$. We can produce the eigenvectors, i.e. elements of the kernel of S, in the following way:

- Fix one coordinate – say $x_0 = 1$.
- From the equation above, $x_d = x_{0+d} = -x_0 = -1$.
- Iterate until back to x_0: $x_{2d} = +1, x_{3d} = -1, \dots x_0 = x_n = x_{n/g \times d} = +1$. The last value is indeed +1 because n/g is an even number.

So the value of one coordinate determines the value of n/g coordinates. We have thus $n/\big(n/g\big) = g$ arbitrary coordinates $x_0, x_1 \dots x_{g-1}$, that is to say g degrees of freedom, and hence the dimension of the kernel of S is exactly g. Its largest possible value (apart from $d = 0$ which is no more a dyad) is for $d = n/2$. In general, we get the rank of matrix $\mathscr{S}$ by way of the rank-nullity theorem: rank($\mathscr{S}$) = $n - g$, remembering though that rank$(\mathscr{S}) = n$ when 2^v divides d.

[5] The sum of all fifths beginning on one whole-tone scale is equal to the whole aggregate, as is the similar sum starting on the other whole-tone scale. Hence any single fifth is a linear combination of all the others.

The special case of the tritone (= half-octave) is worth a deeper analysis. Its matrix has the lowest possible rank, and more precisely all Fourier coefficients with odd index are nil. We can see for $n = 12$ how the codomain is generated by the first six columns, and the computation next to it shows the nullity of the odd Fourier coefficients.

$$\mathscr{T} = \begin{pmatrix} 1&0&0&0&0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0&0&0&0&1\\ 1&0&0&0&0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0&0&0&0&1 \end{pmatrix} \qquad \widehat{t}(2p+1) = 1 + \mathrm{e}^{\frac{(2p+1)6\times 2\mathrm{i}\pi}{12}} = 0.$$

This was actually noted by Yust in [96], who proved the following statement.

Lemma 3.9 (The tritone lemma). *Adding a tritone to a pc-set does not change its third and fifth Fourier coefficients.*

This follows directly from the linearity of the Fourier transform, and is also true for all other odd indexed coefficients. For instance, a fifth and the associated dominant seventh (GD and GBDF) have identical odd coefficients. So do a pentatonic (non hemitonic) scale and the associated diatonic (CDEGA and CDEFGAB), or even a single note and the diminished triad it divides (D and BDF). Conversely, one can remove a tritone from a melodic minor and get a singular hemitonic pentatonic with the same Fourier coefficients (ABCDEF♯G♯ → ABCEF♯). More impressive still, a minor triad has the same odd coefficients as the whole (harmonic minor) scale since they differ by two tritones. The most striking case I have found is the initial figure of Alban Berg's Sonata op. 1, which despite its spectacularly atonal character reduces to the single pc B when the tritones are removed, cf. Fig. 3.4.

There is a partial reciprocal, more technical, which involves Lemma 3.1.

Proposition 3.10. *Let A be a pc-set for which the Fourier coefficients $\mathscr{F}_A(3)$ and $\mathscr{F}_A(5)$ are nil. Then A is a tritone or a reunion of tritones.*

Proof. Consider the characteristic polynomial $\mathbf{A}(X) = \sum_{a\in A} X^a$.

Since the k^{th} Fourier coefficient of A is simply $\mathscr{F}_A(k) = \mathbf{A}(\mathrm{e}^{-2\mathrm{i}k\pi/12})$ by Proposition 1.32, we are assuming that

$$\mathrm{e}^{-2\mathrm{i}3\pi/12} = -i \text{ and } \mathrm{e}^{-2\mathrm{i}5\pi/12} = \mathrm{e}^{-5\mathrm{i}\pi/6} = -\frac{\sqrt{3}}{2} + \frac{\mathrm{i}}{2}$$

are roots of the polynomial $\mathbf{A}(X)$.

Fig. 3.4. The initial motif and B have identical odd Fourier coefficients

$\mathbf{A}(X)$ has integer coefficients, the minimal polynomials of these roots in $\mathbb{Z}[X]$ are the cyclotomic polynomials $\Phi_4(X) = X^2+1$ and $\Phi_{12}(X) = X^4 - X^2 + 1$. Both being irreducible, $\mathbf{A}(X)$ must be a multiple of their product $\Phi_4(X) \times \Phi_{12}(X) = X^6+1$, which is the characteristic polynomial of a tritone.

Let $\mathbf{B}(X) = \dfrac{\mathbf{A}(X)}{X^6+1} = \sum\limits_{0 \leqslant k \leqslant 5} b_k X^k$ be the exact quotient, with degree at most 5 since $\mathbf{A}(X)$ has degree at most 11.

It must have integer coefficients since X^6+1 is unitary, which must be 0's or 1's because they are coefficients of $\mathbf{A}(X)$:

$$\mathbf{A}(X) = (1+X^6) \times \sum_{0 \leqslant k \leqslant 5} b_k X^k = b_0 + \dots b_5 X^5 + b_0 X^6 + \dots b_5 X^{11}.$$

Hence $\mathbf{A}(X)$ is the characteristic polynomial of a union of tritones, for example

$$(X^6+1) \times (X + X^2 + X^4) = X + X^7 + X^2 + X^8 + X^4 + X^{10}.$$

We leave as an exercise the generalisation to $\mathbb{Z}_n$ with even n.

One must beware that this does not exhaust all possible cases of non injectivity. For instance, as we will see in discussing the torus of phases, since dyads $\{0,11\}$ and $\{4,7\}$ have the same *phase* coordinates, so does their reunion, the major seventh $\{0,4,7,11\}$.[6]

3.1.3 Structure of the zero set of the DFT of a pc-set

Lynx-eyed readers may have noticed that Lewin's conditions only consider the nullity of five Fourier coefficients. Perhaps this is sufficient because of the symmetry property $\mathscr{F}_A(n-k) = \overline{\mathscr{F}_A(k)}$, true for any real-valued distribution. Or is it? We left in the dark the values of $\mathscr{F}_A(5), \mathscr{F}_A(7)$. But actually it is enough to compute the $\mathscr{F}_A(k)$ when k is a divisor of n (in the set $\mathbb{N}$ of integers) because of the deep result below:

[6] I am indebted to J. Yust for this example.

Theorem 3.11. *For any rational-valued distribution f (*a fortiori *for any pc-set) we have*

$$\forall \alpha \in \mathbb{Z}_n^* \quad \widehat{f}(k) = 0 \iff \widehat{f}(\alpha k) = 0.$$

Remember that $\mathbb{Z}_n^*$ denotes the invertible elements of $\mathbb{Z}_n$. Other equivalent formulations involve *associated elements*:[7]

Definition 3.12. *k is* associated *with ℓ in $\mathbb{Z}_n \iff \exists \alpha \in \mathbb{Z}_n^*, \ell = \alpha k$.*

Actually the transformations $k \mapsto \alpha k$ for invertible α's are the automorphisms of the additive group $(\mathbb{Z}_n, +)$. Hence

Proposition 3.13.
- *Two elements of $\mathbb{Z}_n$ are associated iff they have the same order in the additive group $(\mathbb{Z}_n, +)$.*
- *Any element of $\mathbb{Z}_n$ is associated with (the class modulo n of) exactly one divisor of n.*
- *The classes of the relation 'being associated with' are the orbits of homotheties in $\mathbb{Z}_n$.*

For instance these classes in $\mathbb{Z}_{12}$ are $(0), (\mathbf{1}, 5, 7, 11), (\mathbf{2}, 10), (\mathbf{3}, 9), (\mathbf{4}, 8), (\mathbf{6})$. Thus Theorem 3.11 states that when the DFT vanishes in k it vanishes *for all classes modulo n associated with k*. Finally, this vindicates the exhaustiveness of the five Lewin's conditions, indexed by divisors 6, 4, 3, 2 and 1. The proof of the theorem involves cyclotomic polynomials again.

Proof. Let f be any integer-valued distribution[8] and $\mathbf{F} \in \mathbb{Z}[X]$ the associated polynomial: $\mathbf{F}(X) = \sum f(p) X^p$.

Say $\widehat{f}(k) = 0$. Since $\widehat{f}(k) = \mathbf{F}(\mathrm{e}^{-2ik\pi/n})$ by Proposition 1.32, it means that $\mathrm{e}^{-2ik\pi/n}$ is a root of $\mathbf{F}$. The order of $\mathrm{e}^{-2ik\pi/n}$ in the group $(\mathbb{C}^*, \times)$ is $m = n/\gcd(n,k)$. By lemma 3.1, Φ_m must divide $\mathbf{F}$, hence all roots of unity with order m are roots of $\mathbf{F}$, i.e. all elements in $\mathbb{Z}_n$ associated with n/m are zeroes of the DFT, which is the result of the theorem.

It is high time we defined and considered the zero-set of a DFT:

Definition 3.14. *For a distribution $f \in \mathbb{C}^n$ (resp. a subset $A \in \mathbb{Z}_n$) the zero-set of its DFT is the set $Z(f)$ (resp. $Z(A)$) of the indexes k, satisfying $\widehat{f}(k) = 0$ (resp. $\mathscr{F}_A(k) = 0$).*

Theorem 3.11 proves that (for rational-valued distributions) $Z(A)$ is **structured as a reunion of classes** $d\,\mathbb{Z}_n^*$, orbits of associated elements, indexed by the set of divisors of n. Another way to put it is the invariance of $Z(A)$ under multiplication (by invertible elements). This is a strong feature: there are for instance $2^{20} - 1 = 1{,}048{,}575$

[7] Already met in the proof of Theorem 2.10.

[8] Actually this result is true for rational-valued coefficients, which is trivial in a way – any rational polynomial being an integer-coefficient polynomial divided by some integer – and deep too, because of the topological density of rational polynomials in $\mathbb{R}[X]$.

subsets of $\mathbb{Z}_{20}$, but only $64 = 2^6$ of them can be zero-sets, pieced together from six orbits which partition the whole group. This will provide access to a method of classification and exhaustive search for tiling canons as we will see in Section 3.3.

As we will develop soon, coverings with zero-sets is the condition for tiling by translation, and the relationships between the diverse classes constituting $Z(A)$ may give clues to abstract conditions for tiling and help lead to solutions of baffling open problems, such as the spectral conjecture.

Example 3.15.
1. For a tritone $T \subset \mathbb{Z}_{12}, Z(T) = \{1,3,5,7,9,11\}$.
2. For a melodic minor scale **mms** such as (A B C D E F♯ G♯) alias $\{0,2,4,6,8,9\}$, $Z(\mathbf{mms}) = \{2,10\}$.
3. Remember that in the example of 3−homometry in $\mathbb{Z}_{32}$, one subset was
$$A = \{0,7,8,9,12,15,17,18,19,20,21,22,26,27,29,30\}.$$
Here $Z(A)$ is the set of even classes, which can be decomposed as
$$Z(A) = 2\mathbb{Z}_{32} \setminus \{0\} = \{2,4,\ldots,30\} = 2\,\mathbb{Z}_{32}^* \cup 4\,\mathbb{Z}_{32}^* \cup 8\,\mathbb{Z}_{32}^* \cup 16\,\mathbb{Z}_{32}^*.$$
4. Anticipating the next section, the subset $A = \{0,6,8,14\}$ tiles $\mathbb{Z}_{16}$, and
$$\begin{aligned} Z(A) &= \{1,3,\mathbf{4},5,7,9,11,\mathbf{12},13,15\} \\ &= \{1,3,5,7,9,11,13,15\} \cup \{\mathbf{4},\mathbf{12}\} = 1\,\mathbb{Z}_{16}^* \cup 4\,\mathbb{Z}_{16}^*. \end{aligned}$$
Here the odd numbers are the invertibles (whose order is 16), and $\{4,12\}$ are the elements with order 4 in $\mathbb{Z}_{16}$ ($4 \times 4 = 12 \times 4 = 0$).

This is an algebraic constraint. One can well wonder how a Fourier coefficient manages to be equal to 0 in the first place. In the examples that we have detailed so far, it derived from Lemma 1.6, that it to say the exponentials involved in the sum are placed on the vertices of a regular polygon (for instance $1 + \mathrm{i} + (-1) + (-\mathrm{i}) = 0$ expresses the sum of the complex numbers on the vertices of a square). It seems natural to conjecture that, at least in the case of a subset distribution, a nil sum of exponentials can be decomposed into such regular subsums, a geometric constraint.

Conjecture 3.16. Let $A \in \mathbb{Z}_n$ such that $\sum_{a\in A} \mathrm{e}^{2\mathrm{i}\pi a/n} = 0$. Then A can be partitioned as a disjoint reunion of regular polygons.

However, this conjecture is false as seen in Fig. 3.5.[9] The smallest counter-example that I found is $A = \{0,1,7,11,17,18,24\} \subset \mathbb{Z}_{30}$. Checking that the sum is exactly 0 involves finding the factor Φ_{30} in the characteristic polynomial $\mathbf{A}(X)$, see [14], which is equivalent to saying that $\mathbf{A}(\mathrm{e}^{2\mathrm{i}\pi/30}) = 0$. This sobering result warns that the study of nil Fourier coefficients is trickier than it seems.[10]

9 Apparently it was first noticed in the 1950s but I could not find a precise reference. More about the algorithmic search for this counter-example in [14].

10 A very recent paper [67] studies precisely those ‘perfectly balanced sets’ and hints that they can always be expressed as *algebraic linear combinations* of perfect polygons, in the

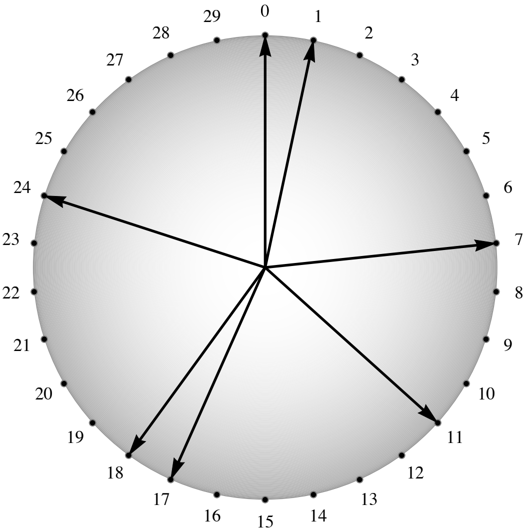

Fig. 3.5. Exponentials summing to 0 without regular subsums

3.2 Tilings of $\mathbb{Z}_n$ by translation

3.2.1 Rhythmic canons in general

The notion of musical canon is as old as the hills and remains popular even to our day in kindergarten songs. Informally, a canon is made of several voices playing the same tune, or pattern, or motif, at different times, i.e. starting with different offbeats. Often the canon is repeated in a loop and called a 'round', which expresses well its social function. Well-known examples in Anglo-Saxon culture are 'Brother John, Are You Sleeping?', 'Row, Row, Row Your Boat' or 'Three Blind Mice'. On the other hand, Ockeghem and Bach are known for brilliant intellectual constructions which played some part much later in the development of serial techniques.

Here we focus on just one musical dimension, usually considered as rhythm (though it could be any quantified musical quantity, and indeed there exist multi-dimensional canons tiling the spaces of rhythm and pitch for instance). Furthermore, in accordance with the topic of the book, we will mostly focus on canons by translations. It is of course possible to build canons with retrogradation, augmentation or any transformation of the motif, or to allow several notes to occur on the same beat (say an odd number of notes, see [27] for a recent study of canons mod p), but very little is known about these cases mathematically speaking (see [11] for a recent

spirit of linear combination of scales in [13]. For instance, my example can be decomposed as three pentagons: $\{0,6,12,18,24\} \oplus \{0,1,4\}$ united with two dyads {or digons, or diameters} $\{2,17\},\{8,23\}$ *minus* three dyads $\{1,16\},\{4,19\},\{10,25\}$ and two equilateral triangles $\{0,10,20\} \oplus \{2,8,12\}$. This decomposition does prove the nullity of the Fourier coefficient. However it is hardly a practical method.

survey). A typical canon by translation is shown in Fig. 3.6 and was composed by George Bloch as a birthday greeting card (each voice sings ‘Happy birthday’).

Fig. 3.6. A birthday greeting periodic canon

The mathematical model of this canon is very simple: counting beats in sixteenth notes and setting the origin 0 at the start of the repeated bar, the four rhythmic voices are

$$\{0,4,5,9\},\{1,8,12,13\},\{2,3,7,14\},\{6,10,11,15\}$$

which are all copies of the initial $\{0,4,5,9\}$ with offsets of 0, 8, -2, and 6 respectively, the computation being made modulo 16 which expresses the repetition of a bar. A notion emerges: the tiling of a cyclic group with translates of one subset. Already we can see that the musical feature of repeating the bar models modular arithmetic. As we will see below, musical concepts are a great help in the mathematical study of rhythmic tilings.

Another essential feature of this canon is its perfect packing of the bar: each beat is played once and only once, which is a substantial difference from common musical canons where overlappings and silences are the rule rather than the exception. For musical treatment we will need this constraint (which still allows for billions of canons).

If only translations of the motif are allowed, it has been shown in the 1950s that a tiling of $\mathbb{Z}$ with a finite tile always has a period:

Theorem 3.17 (Hajòs, de Bruijn 1950). *Let A be a finite subset of $\mathbb{Z}$ and B such that $A \oplus B = \mathbb{Z}$. Then $\exists n \in \mathbb{N}^*, C \subset \mathbb{Z}$ such that $B = n\mathbb{Z} \oplus C$, i.e. $A \oplus C = \mathbb{Z}_n$ (reducing A, C modulo n).*

Hence the limitation to tilings of a cyclic group, which will be the only ones studied in this chapter. It has been recently shown by Kolountzakis and others [55] that the width of the motif does not really limit the period of the canon, refuting the long-standing conjecture that the latter was limited to twice the former (see again [11]).[11]

The study of tilings of cyclic groups (and more generally of abelian groups) was initiated in the 1950s, mostly by East-European mathematicians. The musical approach was single-handedly tackled by Dan Tudor Vuza ([94]) who rediscovered on

[11] The initial idea of Kolountzakis involves unfolding a cyclic group in 3 dimensions using its decomposition as a group product and geometric constructions. A similar vision probably presided over the creation of Szabó’s counterexamples in [81], see Section 3.3.

his own the results of Hajòs, Redei, de Bruijn, Sands and others. The notion of 'Vuza canons' provided new impetus for these researches, especially since [6] connected them to difficult conjectures on tilings. Consequently, new algorithms have been devised for their enumeration ([57]), and these will be detailed below (section 3.3) for the sake of their relationship with DFT.

3.2.2 Characterisation of tiling sets

Definition 3.18. *A rhythmic canon*[12] *is a tiling of a cyclic group by translates of one tile, called motif.*

The motif A is called 'inner voice', and the set of its offsets is the 'outer voice' B. They form a rhythmic canon iff $A \oplus B = \mathbb{Z}_n$.

Example 3.19. In Fig. 3.6 one has $A = \{0,4,5,9\}, B = \{0,6,8,14\}$ and $A \oplus B = \{0,1,\dots 15\} = \mathbb{Z}_{16}$.

Proposition 3.20.

$$A \oplus B = \mathbb{Z}_n \iff 1_A * 1_B = 1_{\mathbb{Z}_n} = \mathbf{1}$$

(the constant map equal to 1 for any element of $\mathbb{Z}_n$*)*

As we have seen in Chapter 1, the convolution product of characteristic functions turns into ordinary product of characteristic polynomials:

Proposition 3.21.

$$A \oplus B = \mathbb{Z}_n \iff \mathbf{A}(X) \times \mathbf{B}(X) = 1 + X + X^2 + \dots X^{n-1} \mod (X^n - 1)$$

Either taking the DFT or plugging in $X = \mathrm{e}^{-2i\pi k/n}$ in the last equation, we get

Proposition 3.22.

$$A \oplus B = \mathbb{Z}_n \iff \widehat{1_A} \times \widehat{1_B} = n\,\widehat{1_{\mathbb{Z}_n}} = n\,\delta = \left(x \mapsto \begin{cases} n & \text{for } x = 0 \\ 0 & \text{else} \end{cases} \right)$$

Essentially, setting apart the case of 0, the product of the Fourier transforms of the characteristic maps of the inner and outer voices must be nil. This vindicates again the definition of $Z(A) = \{k \in \mathbb{Z}_n, \widehat{1_A}(k) = 0\}$, the set of zeroes of the Fourier transform of (the characteristic map of) A already given above, and firmly grounds the question of tiling (by translation) in Fourier space:

Proposition 3.23. *Motif A tiles with outer voice B if and only if*

$$Z(A) \cup Z(B) = \mathbb{Z}_n \setminus \{0\} \quad \text{and} \quad \#A \times \#B = n.$$

The zeroes of the Fourier transforms of A and B must *cover* $\mathbb{Z}_n$ (minus 0), allowing overlaps. For instance, with $A \oplus B = \{0,4,5,9\} \oplus \{0,6,8,14\} = \mathbb{Z}_{16}$ (the factors in Fig. 3.6) we have

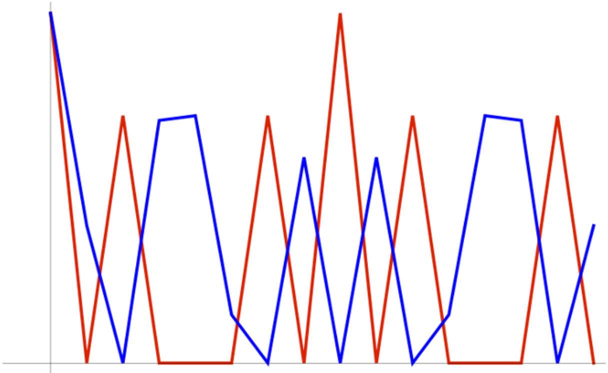

Fig. 3.7. $Z(A)$ and $Z(B)$ cover $\mathbb{Z}_{16} \setminus \{0\}$

$$Z(A) = \{1,3,4,5,7,9,11,12,13,15\} \quad \text{and} \quad Z(B) = \{2,6,8,10,14\}$$

as can be seen on the graphs of $|\mathscr{F}_A|$ and $|\mathscr{F}_B|$ featured in Fig. 3.7. Again, a complex phenomenon in musical space is seen at a glance in Fourier space, cf. Theorem 1.11.

At this point, the question of building all rhythmic canons with period n (i.e. all tilings of $\mathbb{Z}_n$ by translation, i.e. all factorisations $\mathbb{Z}_n = A \oplus B$), or the subproblem of 'completing' a given motif A with its counterpart B, appears as an extension of the phase retrieval problem: given a pair of zero sets covering $\mathbb{Z}_n$ – a very limited choice since these sets must be unions of a few orbits, according to Theorem 3.11 – is it possible to find corresponding subsets? But knowing only where $\mathscr{F}_A = 0$ is even less informative than knowing $|\mathscr{F}_A|$ (which is what we know in homometry questions) since the magnitude of the DFT has yet to be chosen where it is not (necessarily) nil; the problem is hence even more formidable. Precisely,

Proposition 3.24. *If A tiles $\mathbb{Z}_n$ with B (i.e. $A \oplus B = \mathbb{Z}_n$) then any A' homometric with A also tiles with B : $A' \oplus B = \mathbb{Z}_n$.*

This includes all the transforms of A under the dihedral group T/I, of course.[13] Less trivial cases are possible: for instance[14]

$$\text{both } A = \{0,1,6,10,12,13,15,19\},\ A' = \{0,2,5,6,11,12,15,17\}$$
$$\text{tile } \mathbb{Z}_{24} \text{ with } B = \{0,8,16\},$$

though A, A' are homometric but not at all isometric (they both cover all residues modulo 8, however) as can be seen in Fig. 3.8.

[12] Properly speaking, a 'mosaic rhythmic canon by translation'.

[13] And there are scarcely any other sets homometric with a given A as seen in Chapter 2. This will be extended to affine transforms of A in Section 3.2.5.

[14] I am indebted to M. Andreatta who urged me to research these cases, probably the simplest subsets which tile and admit a non-trivially homometric twin.

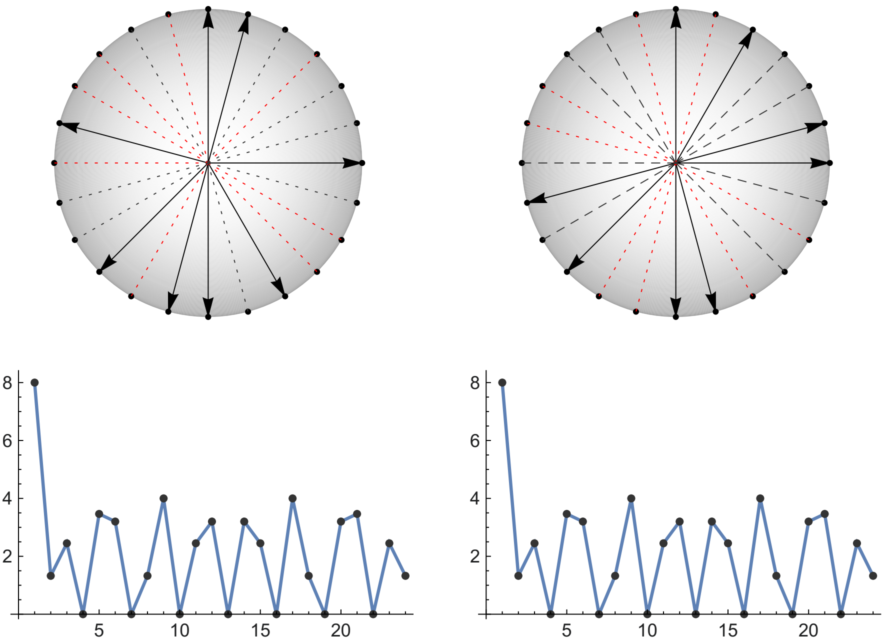

Fig. 3.8. Non-trivial homometric tiles, illustrating Proposition 3.24

However, some choices of $Z(A), Z(B)$ are impossible:[15] for instance a set like $A = \{0, 2, 4, 5, 6, 7\}$ cannot possibly tile, as is easily gathered from trial and error (no way to fill the gaps 1, 3 with the 'lumpy' 4567 obstructing the process), and it can also be seen on $Z(A)$ as we will see in the next subsection. Notice that A tiles with its inversion $(3 - A)$ though. Some reasons for such obstructions are known, and are our next topic.

3.2.3 The Coven-Meyerowitz conditions

[35] was the first paper enumerating general sufficient and (sometimes) necessary conditions for a finite motif to tile some cyclic group. Considering that the study of factorisations originated around 1948, this was long overdue. How does one check, for instance, whether $\{0, 1, 2, 5, 22, 2415\}$ does tile[16], other than by finding a complement (which would be a long and arduous search considering the diameter of A)? Coven and Meyerowitz discovered that the cyclotomic factors of the characteristic polynomial are the key, and indeed provide something very close to a sufficient and necessary condition. As we have already explained, this prevalence of cyclotomic polynomials is another way of expressing the rigid structure of Fourier zero sets.

[15] For genuine pc-sets at least.
[16] It does. See below.

In [35] they introduced, for $A \subset \mathbb{Z}_n$,

Definition 3.25.
$R_A = \{d, d \mid n \text{ and } \Phi_d \mid A(X)\}$ *and* $S_A = \{p^\alpha \in R_A, p \text{ prime}, \alpha \geq 1\}$.

The elements of R_A are exactly the orders in $(\mathbb{Z}_n, +)$ of the elements of $Z(A)$, see Theorem 3.11:

$$Z(A) = \bigcup_{d \in R_A} \{x \in \mathbb{Z}_n \mid ord(x) = d\} = \bigcup_{d \in R_A} \frac{n}{d}.\mathbb{Z}_n^*$$

For instance with $A = \{0,3,6,12,23,27,36,42,47,48,51,71\}$ one gets $R_A = \{2,8,9,18,72\}, S_A = \{2,8,9\}$.[17]
The presence of all factors $\Phi_d, d \mid n$, in $A(X) \times B(X)$ entails that

- $S_A \cup S_B$ is the set[18] of all prime powers dividing n, and
- $R_A \cup R_B$ is the set of all divisors of n (1 excepted).

Coven and Meyerowitz then proceeded to prove the following statements, the last of which is quite difficult.

Theorem 3.26. *Defining conditions*

(T_1): $\prod_{p^\alpha \in S_A} p = \#A$;

(T_2): $p^\alpha, q^\beta, r^\gamma \dots \in S_A \Rightarrow p^\alpha q^\beta r^\gamma \dots \in R_A$ *(products of powers of distinct primes belonging to S_A are in R_A);*

one has

1. *If A tiles, then (T_1) is true.*
2. *If both $(T_1), (T_2)$ are true, then A tiles.*
3. *If #A has at most two different prime factors, and A tiles, then both $(T_1), (T_2)$ are true.*

As of today, it is not known whether condition (T_2) is always necessary for tiling. With the example above we can check (T_1) : $\#A = 12 = 2 \times 2 \times 3$ since $S_A = \{2^1, 2^3, 3^2\}$, and $(T_2) : 2 \times 9 \in R_A$ and $8 \times 9 \in R_A$.

With the unreasonable tile given before, $A' = \{0,1,2,5,22,2415\}$, with $\#A' = 6$ it is soon verified[19] that $S_{A'} = \{2,3\}$ and $6 \in R_{A'}$, hence A' tiles quite trivially (it tiles $\mathbb{Z}_6$ and hence any $\mathbb{Z}_{6n}$).

17 Actually the definition of [35] stands for $A \subset \mathbb{Z}$; we simplify slightly their exposition, since for any other polynomial congruent with $A(X) \mod (X^n - 1)$, the subset of the divisors of n in R_A, which are the indexes of the relevant cyclotomic factors, does not change. We choose this as our definition for R_A. Anyhow, S_A is always made of divisors of n.

18 They show that corresponding cyclotomic polynomials occur only once, so this is a partition of the set of all prime powers dividing n. On the other hand, sometimes $R_A \cap R_B \neq \varnothing$.

19 By computing $\mathbf{A}(e^{2i\pi/3}) = 0 = \mathbf{A}(-1)$.

Part of the proof of Theorem 3.26 is the useful

Lemma 3.27. *If A tiles some cyclic group, then it tiles $\mathbb{Z}_n$ where $n = \text{lcm}(S_A)$ (reducing A modulo n).*

The link with Fourier transforms is straightforward: recall the organisation of $Z(A)$ in subsets of elements with equal multiplicative orders, these orders are precisely the elements of R_A.

3.2.4 Inner periodicities

Recall that $A \subset \mathbb{Z}_n$ is periodic, meaning $A + \tau = A$ for some $0 < \tau < n$, if and only if[20] $\mathscr{F}_A(t) = 0$ except when t belongs to some subgroup of $\mathbb{Z}_n$. This comes from $\mathscr{F}_{A+\tau}(t) = \mathscr{F}_A(t)e^{-2i\pi\tau t/n}$, hence $\mathscr{F}_{A+\tau}(t) = \mathscr{F}_A(t) \Rightarrow \mathscr{F}_A(t) = 0$ except when $\tau t \in n\mathbb{Z}$, i.e. $t \in \frac{n}{\gcd(\tau,n)}\mathbb{Z}$.

It turns out to be quite an effective way to check *a priori* periodicity, especially when one considers the *complement set of* $Z(A)$. The following theorem expresses the above in terms of $Z(A)$:

Theorem 3.28. *A is periodic in $\mathbb{Z}_n$ if and only if the complement set of $Z(A)$ is part of some subgroup of $\mathbb{Z}_n$. In practice, since any such subgroup is part of a maximal proper subgroup $p\mathbb{Z}_n$ with p a prime factor of n, it is sufficient to check whether there exists such a p which divides all elements* not *in $Z(A)$ in order to know whether A is n/p-periodic.*

This can be checked almost visually.

For $A' = \{0,5,8,13\}$, which tiles $\mathbb{Z}_{16}$, $R_{A'} = S_{A'} = \{2,(10),16\}$[21] and (keeping $n = 16$) the complement of $Z(A') = \{1,3,5,7,8,9,11,13,15\}$ is contained in the subgroup $2\mathbb{Z}_{16}$, meaning that A' is $16/2 = 8$-periodic. The non zeroes of the DFT are clearly members of the even subgroup materialised by big dots in Fig. 3.9 (though 8 is also a zero, inherited from $A = \{0,5\}$ from which A' is concatenated, see below).

In this example, $A' = A \oplus \{0,8\}$ where $A = \{0,5\}$, and we recognize the kinship between their respective Fourier transforms in Fig. 3.10. It is a 'multiplication d'accords' but in $\mathbb{Z}_{16}$, though the DFT of $\{0,5\}$ is drawn in $\mathbb{Z}_8$.

Some motifs can be completed by either periodic or aperiodic outer voices:

$A = \{0,8,16,18,26,34\}$ tiles $\mathbb{Z}_{72}$ with

$B = \{0,9,12,21,24,33,\dots 60,69\} = \{0,9\} \oplus \{0,12,24,36,48,60\}$, 12-periodic, but also with $B' = \{0,3,12,23,27,36,42,47,48,51,71\}$. Comparison of zero sets is illuminating:

$$R_B = \{2,6,8,9,18,24,36,72\} \supset R_{B'} = \{2,8,9,18,72\}$$

[20] Notice that without loss of generality one may replace τ with $\gcd(\tau,n)$ and assume that τ is a divisor of n.

[21] Φ_{10} divides $\mathbf{A}(X)$ but is discounted since 10 is not a divisor of 16, according to Def. 3.25: this factor disappears if one changes any element of A' by a multiple of 16, see Footnote 17.

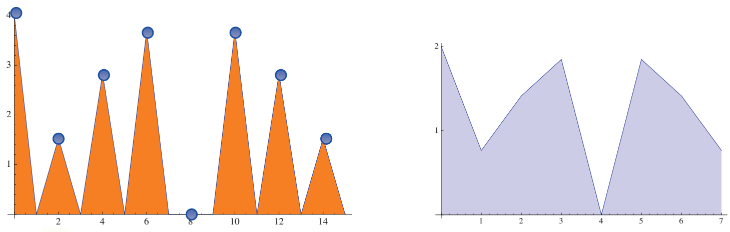

Fig. 3.9. The complement of $Z(A')$ is in $2\mathbb{Z}_{16}$, as seen on the graph of $|\mathscr{F}_{A'}|$. On the right, graph of $|\mathscr{F}_A|$.

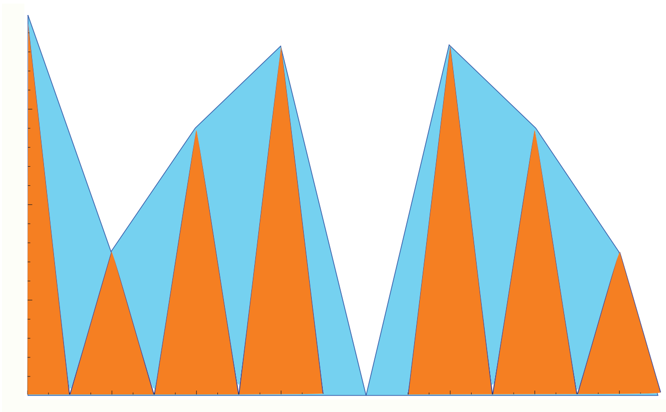

Fig. 3.10. The Fourier transforms of $A = \{0,5\} \subset \mathbb{Z}_8$ and of $A' = \{0,5\} \oplus \{0,8\} \subset \mathbb{Z}_{16}$.

while $R_A = \{3,4,6,12,24,36\}$.

It is time to introduce

Definition 3.29. *A Vuza canon*[22] *is a counterexample to Hajós's 1950 conjecture, i.e. a rhythmic canon* $\mathbb{Z}_n = A \oplus B$ *where neither A nor B is periodic.*

I would like to point out that the notion of Vuza canons is musical, inasmuch as a canon with (say) a periodic outer voice is heard as the repetition of a shorter canon (with a shorter outer voice). This leads to a useful decomposition process, as we will see later. It took three decades for several top-notch mathematicians to establish the following theorem, which was rediscovered independently by D.T. Vuza in the 1980s ([77, 94]).

Theorem 3.30.

1. There exist Vuza canons.

[22] In some older papers, this term specifies those canons provided by Vuza's algorithm; this is no longer the case and we call 'Vuza canons' what he himself called 'Rhythmic Canons of Maximal Category'.

2. *Vuza canons exist in* $\mathbb{Z}_n \iff n$ *is* not *of the form*

$$n = p^\alpha, n = p^\alpha q, n = p^2 q^2, n = p^2 qr, n = pqrs$$

where p, q, r, s *are different primes and* $\alpha \geqslant 1$.

A cyclic group $\mathbb{Z}_n$ with n having any of the 5 forms above is often called, after Hajós, a 'good group'; the other cyclic groups are 'bad' (meaning that Hajòs's conjecture fails in them). The smallest bad group is $\mathbb{Z}_{72}$, the next ones occur for $n = 108, 120, 144, 168, 180\ldots$[23] Classification of Vuza canons based on the zero sets of the factors is also a way of computing them exhaustively, which has been achieved for the values of n just stated. Some of the algorithms involved are mentioned in Section 3.2.3, the condition in Theorem 3.28 enabling the pruning of many cases where the only factors available would be periodic. The simplest construction of a Vuza canon in $\mathbb{Z}_n$ uses the recipe provided by Jedrzejewski: let p_1, p_2 be prime numbers and n_1, n_2, n_3 integers such that $\gcd(n_1 p_1, n_2 p_2) = 1$. Then have

$$\begin{aligned} A &= n_2 n_3 \times \{0, \ldots p_2 - 1\} \oplus p_2 n_1 n_2 n_3 \times \{0, \ldots p_1 - 1\} \\ B &= n_1 n_3 \times \{0, \ldots p_1 - 1\} \oplus p_1 n_1 n_2 n_3 \times \{0, \ldots p_2 - 1\} \\ S &= p_2 n_2 n_3 \times \{0, \ldots n_1 - 1\} \oplus p_1 n_1 n_3 \times \{0, \ldots n_2 - 1\} \\ R &= \big(\{1, \ldots n_3 - 1\} \oplus B\big) \bigcup A. \end{aligned}$$

Then $R \oplus S = \mathbb{Z}_n$ yields a Vuza canon.

3.2.5 Transformations

Transformation of an existing canon has two obvious aims: the production of new canons, and their classification and taxonomy. For instance, $\{0, 4, 5, 9\}$ and its translate $\{0, 1, 5, 12\}$ tile identically $\mathbb{Z}_{16}$ with complement $\{0, 6, 8, 14\}$, itself the same as $\{0, 2, 8, 10\}$ if the origin of time is changed. Perceptively, in a canon repeated periodically, there is no privileged starting note or starting voice. Mathematically it is thus natural to consider the factors A, B up to translation in $\mathbb{Z}_n$. But there are other transformations which unravel less obvious relationships between canons.

Definition 3.31. *The dual canon of* $A \oplus B = \mathbb{Z}_n$ *is* $B \oplus A = \mathbb{Z}_n$ *(revert the roles of inner and outer voice).*

This is useful mainly for classification purposes, though some musical applications could be imagined. One other transformation does not change the size of the tiling:

Proposition 3.32. *If* A *tiles* $\mathbb{Z}_n$ *with* B *then* mA *tiles with* B *too for any* m *coprime with* n.

Proof. This is a direct consequence of Theorem 3.11, since the zero set $Z(mA)$ must be equal to $Z(A)$. Remarkably, this non-trivial feature of tilings was (re)discovered experimentally by not one, but several composers.

[23] Sloane's sequence of integers A102562.

This allows a finer classification of rhythmic canons than orbits under T or even T/I.

For instance, for $n = 72$ there are only two different Vuza canons up to affine transformation,

$$A = \{0,3,6,12,23,27,36,42,47,48,51,71\}$$
$$\text{or } A' = \{0,4,5,11,24,28,35,41,47,48,52,71\}$$

with one outer voice $B = \{0,8,10,18,26,64\}$ – instead of six inner voices and three outer voices under T/I.

Remember also that the famous Z-related sets $\{0,1,3,7\}$ and $\{0,1,4,6\}$ are affinely related[24] in $\mathbb{Z}_{12}$, but this is a more complicated case since non-nil Fourier coefficients must be permutated according to the affine transform. In this example, all odd (resp. even) coefficients share the same size $\sqrt{2}$ (resp. 2). A neater generalisation comes with J. Wild's FLIDs, see Section 4.3.3.

Further transformations of canons change n. In order to proceed we need to overcome an apparent ambiguity here: there is no canonical way to turn a subset of $\mathbb{Z}_n$ into a subset $\mathbb{Z}_{kn}$ but this will prove to be irrelevant:

Definition 3.33. *For any B in $\mathbb{Z}_n$, we call* immersion *of B in $\mathbb{Z}_{kn}$ any subset $B' \subset \mathbb{Z}_{kn}$ such that the canonical projection $\pi_n = \mathbb{Z}_{kn} \to \mathbb{Z}_n$ maps bijectively B' to B.*

In the transformations discussed below, any choice of B' will do, elements of B' being chosen up to a multiple of n.[25] The trick is to keep in mind that $R(B) = R(B')$ but $Z(B) \neq Z(B')$ when $\mathbb{Z}_n$ changes into $\mathbb{Z}_{kn}$. The rule is a simple one, preserving the multiplicative order:

Lemma 3.34. *With the same notations, $Z(B') = k(Z(B))$.*

The most important transformation is the next one:

Definition 3.35. *Concatenation of a canon consists in replacing the motif by itself, repeated several times. In other words, $A \in \mathbb{Z}_n$ turns into*

$$\overline{A}^k = A' \oplus \{0, n, 2n, \dots (k-1)n\} \in \mathbb{Z}_{kn}$$

where A' is an immersion of A.

For instance, $A = \{0,1,4,5\} \subset \mathbb{Z}_8$ (which tiles $\mathbb{Z}_8$ with $B = \{0,2\}$) can be prolonged to $\overline{A}^3 = \{0,1,4,5, 8,9,12,13, 16,17,20,21\} \subset \mathbb{Z}_{24}$. Obviously this new motif still tiles with complement $B' = \{0,2\} \subset \mathbb{Z}_{24}$. This is general:

Proposition 3.36. *A tiles $\mathbb{Z}_n$ with B if and only if $\overline{A}^k$ tiles $\mathbb{Z}_{kn}$ with B'.*

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[24] In $\mathbb{Z}_{12}$, $5 \times \{0,1,3,7\} = \{0,3,5,11\} = \{0,1,4,6\} - 1$.

[25] In practice one uses the elements of B not caring whether they are integers, classes modulo n, or modulo kn, i.e. one replaces B with B' ruthlessly, usually choosing integers inside $[\![0, n-1]\!]$.

This property is easily checked with the geometric definition of a tiling[26], but with an eye on the next subsection, we will provide a more complicated proof involving the DFT.

Concatenation is the simplest recipe for building periodic motifs: $\overline{A}^k$ is $n-$periodic in $\mathbb{Z}_{kn}$, and conversely, any periodic motif is by nature concatenated from a shorter one. Hence as proved already, all Fourier coefficients, except those with index multiple of k, must be 0.

Lemma 3.37. *With the notations above, the elements of $Z(\overline{A}^k)$ have the same orders as those in $Z(A)$, plus those orders which are divisors of kn but not divisors of n:*

$$R(\overline{A}^k) = R(A) \cup \big(\mathrm{Div}(kn) \setminus \mathrm{Div}(n)\big).$$

This will entail Proposition 3.36, since all non-nil elements of $\mathbb{Z}_{kn}$ will fall either in $Z(\overline{A}^k)$ or $Z(B)$. Notice that elements with sthe ame orders are different because the group changes.

Proof. Using the characteristic polynomials:

$$\overline{\mathbf{A}^k}(X) = (1 + X^n + X^{2n} + \dots X^{(k-1)n}) \times \mathbf{A}(X) = \frac{X^{kn}-1}{X^n-1} \times \mathbf{A}(X).$$

The roots of $\mathbf{A}(X)$ are still roots of $\overline{\mathbf{A}^k}(X)$, keeping the same order (as roots of unity), adding only the roots of $\dfrac{X^{kn}-1}{X^n-1}$, whose orders divide kn but not n, as stated.

Concatenation is an extension (to a larger group) of ‘multiplication d’accords’, i.e. a convolution product of characteristic functions or sum of (multi)sets: $\overline{A}^k = A' \oplus n\mathbb{Z}_{kn}$, and the computation of the zero set $Z(\overline{A}^k)$ might have been derived from the following trivial corollary of Theorem 1.10 (first noticed by J. Yust):

Proposition 3.38. *If a distribution f is singular (i.e. some Fourier coefficients are nil) then so is the convolution product $f * g$ for any distribution g. In terms of pc-(multi)sets, it means that if $A \subset \mathbb{Z}_n$ is one of Lewin’s ‘special cases,’ then so is $A_{mult} + B_{mult}$ for any (multi)set B.*[27]

This is more general than the repetition/oversampling transformation that we have already considered in Chapter 1; it applies to collections of disjoint tritones or minor thirds, for instance. See Fig. 1.1 for an example of a singular set in Chopin which can be factored in a (singular) dyad $\times$ a (singular) minor triad.

Here is an example of computation of $Z(\overline{A}^k)$.

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[26] $\overline{A}^k = A' \oplus n\mathbb{Z}_{kn}, A \oplus B = \mathbb{Z}_n, \mathbb{Z}_n{}' \oplus n\mathbb{Z}_{kn} = \mathbb{Z}_{kn} \Rightarrow \overline{A}^k \oplus B' = A' \oplus n\mathbb{Z}_{kn} \oplus B' = A' \oplus B' \oplus n\mathbb{Z}_{kn} = \mathbb{Z}_n{}' \oplus n\mathbb{Z}_{kn} = \mathbb{Z}_{kn}$.

[27] The index means that we consider multisets, and count multiplicities of elements of $A_{mult} + B_{mult}$ if necessary. Beware that this is different from the common (musicological) usage in ‘multiplication d’accords’ or transpositional combination.

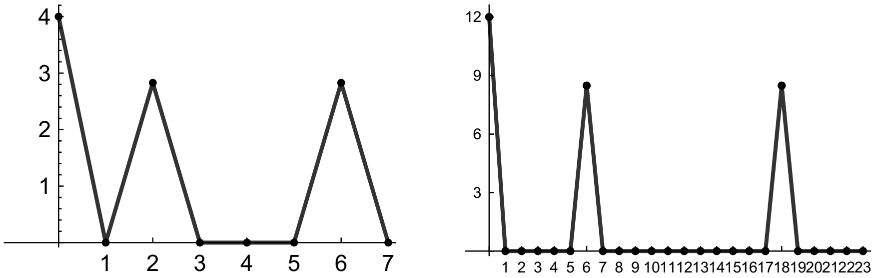

Fig. 3.11. Concatenation of $\{0,1,4,5\}$ and associated DFT.

Example 3.39. $A = \{0,1,4,5\}$ tiles $\mathbb{Z}_8$, the elements of $Z(A) \in \mathbb{Z}_8$ are $1,3,5,7$ and 4, with orders 8 and 2. For its third order repetition $\overline{A}^3 = A' \oplus \{0,8,16\} \subset \mathbb{Z}_{24}$, the elements of $Z(\overline{A}^3)$ have order $2,8$, and also the divisors of 24 which do not divide 8, i.e. $3,6,12$ and 24 – the multiples of 3 (see Fig. 3.11; the numerous new 0s are due to the additional orders, for instance 1, 5, 7... have order 24). It is perhaps even more straightforward to look at the other side: $Z(B')$ is still made up of the elements with order 4, which were $2,6$ in $\mathbb{Z}_8$ and become $6,18$ in $\mathbb{Z}_{24}$ (the same, times 3).

This statement could also be expressed in terms of sets R_A and $R_{\overline{A}^k}$ with R_A defined in Section 3.2.3 above, or alternatively with an expression of the DFT of a direct sum, a distinct possibility since a direct sum of subsets is 'une multiplication d'accords,' i.e. a convolution product of characteristic functions, i.e. a termwise product of DFTs. It could even be argued with sleight of hands that if a subset has some inner period, i.e. a smaller period than the size of the group it tiles, then fewer Fourier coefficient are required to describe the subset. The explicit description of the zero set that we have computed is a bit cumbersome but explicit.

Concatenation creates a periodic tile. Conversely, unless a canon is a Vuza canon, factor A or B (or both) is periodic, i.e. is a concatenation of smaller motifs. Iterating the process until it is no longer possible, we get the two following cases:

Proposition 3.40. *Any canon can be produced by concatenation (and duality) from either the trivial canon* $\{0\} \oplus \{0\}$, *or a Vuza canon.*

Moreover, this entails a recursive construction of *all* tilings of finite ranges $[\![0, n-1]\!]$ (i.e. without reduction modulo n), since

Theorem 3.41. *Any* **compact** *canon, i.e.* $A \oplus B = [\![0, n-1]\!]$ *(without reduction modulo n), can be reduced by concatenation and duality to the trivial canon.*

This was proved by N. G. de Bruijn in [37].

Example 3.42. $\{0,1,4,5\} \oplus \{0,2\} = [\![0,7]\!]$ is concatenated from $\{0,1\} \oplus \{0,2\} = [\![0,3]\!]$, this last from $\{0,1\} \oplus \{0\} = [\![0,1]\!]$ which is a duplication of the trivial canon $\{0\} \oplus \{0\} = [\![0,0]\!]$.

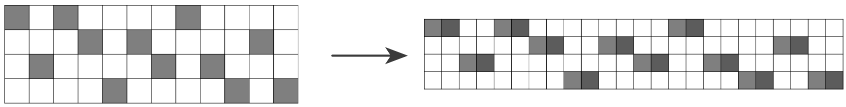

Fig. 3.12. Example of stuttering

Other cases of reducible canons include the 'assymmetric rhythms' of [48], whose study originates in ethnomusicology.

Zooming and stuttering are two dual transformations. I called *stuttering* (one could see it as 'upsampling') the act of replacing each note or rest in the motif by k repetitions of itself. Of course one must again replace $\mathbb{Z}_n$ with $\mathbb{Z}_{kn}$ in the process.

Example 3.43. From $\{0,2,7\} \oplus \{0,3,6,9\} = \mathbb{Z}_{12}$ one gets

$$\{(0,1), (4,5), (14,15)\} \oplus \{0,6,12,18\} = \mathbb{Z}_{24},$$

cf. Fig. 3.12.

Algebraically, this means turning A into[28] $\text{Stut}(A,k) = (kA)' \oplus \{0,1,2\ldots k-1\} \subset \mathbb{Z}_{kn}$ (remember that $(kA)'$ is kA seen in $\mathbb{Z}_{kn}$). This time, in order to keep a canon it is necessary to *augment*, i.e. zoom in, the outer voice B into kB', i.e.

Theorem 3.44. *A tiles $\mathbb{Z}_n$ with B if and only if $Stut(A,k) = (kA)' \oplus \{0,1,2\ldots k-1\}$ tiles $\mathbb{Z}_{kn}$ with kB'.*

In this book, we find it desirable to clarify what happens to the DFT during such transformations.

Lemma 3.45. *The transformation $B \mapsto kB'$ from $\mathbb{Z}_n$ to $\mathbb{Z}_{kn}$ turns $Z(B)$ into $Z(kB') = Z(B)' \oplus n\mathbb{Z}_{kn}$. Equivalently, $R(kB') = R(B)$.*

Proof. This is what we had already stated about oversampling. In the following line, t' is any preimage in $\mathbb{Z}_{kn}$ of $t \in \mathbb{Z}_n$, i.e. $t = \pi_n(t')$, i.e. $t' \equiv t \mod n$:

$$\mathscr{F}_{kB}(t') = \sum_{x\in kB} \mathrm{e}^{-2\mathrm{i}\pi xt'/(kn)} = \sum_{y\in B} \mathrm{e}^{-2\mathrm{i}\pi yt'/n} = \sum_{y\in B} \mathrm{e}^{-2\mathrm{i}\pi yt/n} = \mathscr{F}_B(t)$$

does not change with the choice of t', i.e. if t' is modified by some multiple of n. Hence $\mathscr{F}_{kB}$ vanishes on $Z(B) \oplus n\mathbb{Z}_{kn}$.

Example 3.46. Say $B = \{0,1,4,5\} \subset \mathbb{Z}_8$, then $Z(B) = \{1,3,4,5,7\}$ and

$$Z(3B)' = \{1,3,4,5,7,\ 9,11,12,13,15,\ 17,19,20,21,23\} = \{1,3,4,5,7\} \oplus \{0,8,16\}.$$

[28] It may be construed as a kind of tensorial product, as Franck Jedrzejewski showed in an unpublished conference at the MaMuX seminar in IRCAM (Paris). With the matricial formalism introduced in Section 1.2.3, this is equivalent to tensorial products of matrixes, which would yield the same results but in a more cumbersome way.

Lemma 3.47. *Stuttering A into $Stut(A,k) = (kA)' \oplus \{0,1,2\dots k-1\} \in \mathbb{Z}_{kn}$ from $\mathbb{Z}_n$ to $\mathbb{Z}_{kn}$ turns $Z(A)$ into $Z(Stut(A,k)) = n\mathbb{Z}_{kn} \cup \big(Z(A) \oplus n\mathbb{Z}_{kn}\big) \setminus \{0\}$.*

Proof.

$$\mathscr{F}_{\mathrm{Stut}(A,k)}(t') = \sum_{x \in kA' \oplus [\![0,k-1]\!]} \mathrm{e}^{-2\mathrm{i}\pi x t'/(kn)} = \sum_{a \in A, \ell \in [\![0,k-1]\!]} \mathrm{e}^{-2\mathrm{i}\pi(ka+\ell)t'/(kn)}$$

$$= \sum_{a \in A} \mathrm{e}^{-2\mathrm{i}\pi a t'/n} \times \sum_{\ell \in [\![0,k-1]\!]} \mathrm{e}^{-2\mathrm{i}\pi \ell t'/(kn)} = \mathscr{F}_A(t) \times \begin{cases} \dfrac{1-\mathrm{e}^{-2\mathrm{i}\pi t'/n}}{1-\mathrm{e}^{-2\mathrm{i}\pi t/(kn)}} & \text{if defined} \\ k & \text{else.} \end{cases}$$

This is 0 whenever t' is a multiple of n or when $t \in Z(A)$, i.e. $t' \in Z(A) \oplus n\mathbb{Z}_{kn}$.

Example 3.48. Let $A = \{0,2,7\} \in \mathbb{Z}_{12}$: since $\mathbf{A}(\mathrm{e}^{2\mathrm{i}\pi/3}) = 1 + j^2 + j^7 = 0$, $Z(A) = \{4,8\}$. Now for $3A' \oplus \{0,1,2\} = \{0,1,2,\ 6,7,8,\ 21,22,23\}$ we get the zero set

$$\{4,8,12,16,20,24,28,32\} = \big(\{0,4,8\} \oplus \{0,12,24\}\big) \setminus \{0\}.$$

Quite contrary to concatenation, these operations preserve the non-periodicity of either voice, and hence turn a Vuza canon into a (larger) Vuza canon. Historically, this has been used (in combination with the other transformations) in order to produce larger Vuza canons, for instance before Harald Fripertinger managed to enumerate all of them for periods 72 and 108 ([44]). Of course, it is equally possible to zoom on A and stutter with B.

Multiplexing is a generalisation of stuttering (see example in Fig. 3.13): instead of building $\{0,1,2\dots k-1\} \oplus kA$, one chooses k inner voices $A_0,\dots A_{k-1}$ which tile with the same outer voice B, i.e. $A_0 \oplus B = A_1 \oplus B = \cdots = \mathbb{Z}_n$, and the new motif with period kn is $\widetilde{A} = \bigcup_{i=0}^{k-1} \big(kA'_i + i\big)$. Again,

Theorem 3.49. $\widetilde{A} \oplus kB' = \mathbb{Z}_{kn} \iff \forall i = 0\dots k-1, A_i \oplus B = \mathbb{Z}_n$.

The easy proof is left to the reader.

It seems ambitious to look for the zero set of such a complicated construction. But all $Z(A_k)$'s have enough in common to warrant a statement:

Lemma 3.50. *$Z(\widetilde{A})$ is at least the same as the zero set obtained by stuttering,*

$$Z(\widetilde{A}) \supset Z(Stut(A,k)) = n\mathbb{Z}_{kn} \cup \big(Z(A) \oplus n\mathbb{Z}_{kn}\big) \setminus \{0\} = \big((\{0\} \cup Z(A)) \oplus n\mathbb{Z}_{kn}\big) \setminus \{0\}$$

(according to Lemma 3.47) if we define $Z(A)$ as $\cap_k Z(A_k)$, which complements $Z(B)$ in $\mathbb{Z}_n$ by hypothesis.

Example 3.51. In Fig. 3.13, both motifs $\{0,1,11\}, \{0,2,7\}$ share the same $Z(A) = \{4,8\}$ and hence the multiplexed motif satisfies $Z(\widetilde{A}) \supset \{4,8,12,16,20\}$.

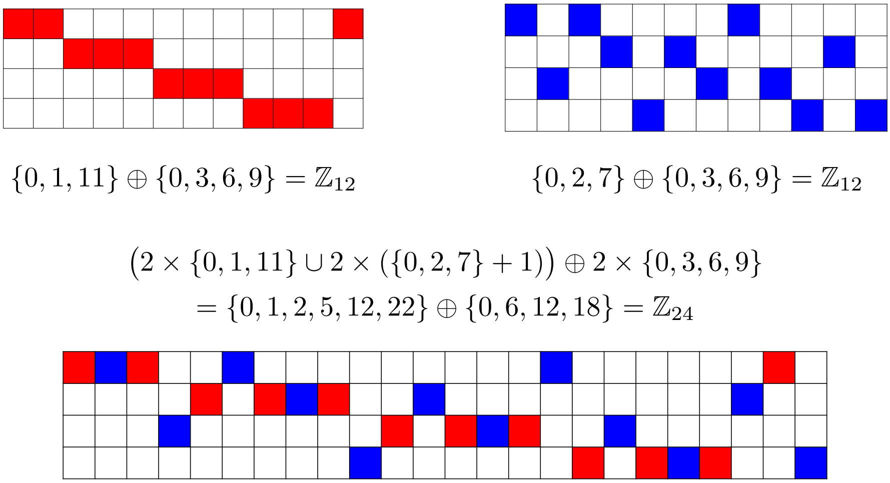

Fig. 3.13. An example of multiplexing.

This transformation opens interesting compositional possibilities, since several canons merge into a larger one while remaining audible, cf. Fig. 3.13. The dual transformation (multiplexing the outer voice) enlarges the motif and complexifies its outer voice.

An interesting theoretical point is that a kind of reciprocal stands: each canon wherein the outer voice can be written kB (i.e. up to translation, all elements of the outer voice are divisible by a common k) is multiplexed from a canon k times smaller (see in Fig. 3.13 how the smaller canons can be retrieved from the larger one). It was conjectured, in various contexts and by several authors, that essentially all canons were instances of some such multiplexing; but this is not true, as demonstrated by [81], though the smallest known counter examples have period 900, see Section 3.3. This precludes, to this day, reducing all canons to the trivial canon.

Uplifting

The last transformation we will study, uplifting (Fig. 3.14), came to the fore in recent developments of the search for Vuza canons [57], though it was probably used by composers before. It stems from the simple idea that allowed us above to immerse a subset of $\mathbb{Z}_n$ in a larger group:

Proposition 3.52. *If A tiles $\mathbb{Z}_n$ then A – or rather its immersion A' – tiles any larger cyclic overgroup $\mathbb{Z}_{kn}$; moreover, translating any individual element of A' by any multiple of n provides a new motif A'' that also tiles $\mathbb{Z}_{kn}$.*

Proof. If $A \oplus B = \mathbb{Z}_n$, let $A' = \{a_1 + k_1 n, \dots a_p + k_p n\} \subset \mathbb{Z}_{kn}$ where $A = \{a_1, \dots a_p\} \subset \mathbb{Z}_n$ and $k_1, \dots k_p \in \mathbb{Z}$. This makes sense, since applying the canonical projection π_n from $\mathbb{Z}_{kn}$ to $\mathbb{Z}_n$ yields $\pi_n(a + kn) = a$ as in the other transformations studied above. Let also $B' = \{b_i + \kappa n, b_i \in B, \kappa = 0, \dots k-1\}$; then it is straightforward to check that

$$A' \oplus B' = \mathbb{Z}_{kn},$$

considering[29] that the map $A' \times B' \ni (a,b) \mapsto a+b$ is still injective and that $\#A' \times \#B' = kn$.

Again, one can reach most of the zero set of the new motif:

Lemma 3.53. *$Z(A')$ contains at least $kZ(A)'$ (equivalently, $R(A') \supset R(A)$).*

Proof. When A is immersed in $\mathbb{Z}_{kn}$ its DFT changes and $Z(A)$ (in $\mathbb{Z}_n$) turns into $Z(A') \supset kZ(A)'$ (in $\mathbb{Z}_{kn}$), since now

$$t' \in kZ(A)' \Rightarrow \mathscr{F}_A(t') = \sum_{a' \in A'} e^{\frac{-2i\pi a' t'}{kn}} = \sum_{a' \in A'} e^{\frac{-2i\pi a' kt}{kn}} = \sum_{a \in A} e^{\frac{-2i\pi at}{n}} = \mathscr{F}_A(t) = 0$$

since $t' = kt$ for some $t \in Z(A)$.

As we see in the computation, changing any element $a \in A$ by any multiple of n does not change the result.

For instance, from $\{0,1,4,5\} \oplus \{0,2\} = \mathbb{Z}_8$ one 'uplifts' to the Bloch canon in Example 3.6, e.g.

$$\{0, 9 = 1+8, 4, 5\} \oplus \{0, 2, 8, 10\} = \mathbb{Z}_{16}.$$

The zero sets are respectively $\{1,3,4,5,7\} \subset \mathbb{Z}_8$ and $\{2,6,8,10,14\} \subset \mathbb{Z}_{16}$, the orders of their elements being in both cases 8 or 2.

This is most probably what Bloch actually did in order to produce his canon. But the main strength of this transformation is made clear when one is looking for some motif $A \in \mathbb{Z}_{kn}$ *knowing that A also tiles* $\mathbb{Z}_n$. This was instrumental in many cases in the algorithmic quest for all the smallest Vuza canons, see [57, 11] and Section 3.3 below.

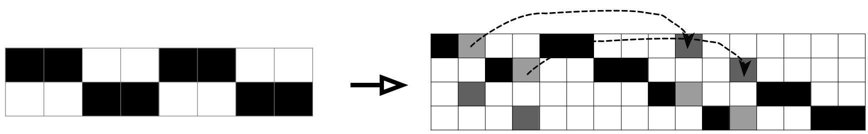

Fig. 3.14. Uplifting a canon in $\mathbb{Z}_8$ to a larger one in $\mathbb{Z}_{16}$

In all these transformations, we keep control of $Z(A)$. Hence, in order to prove most conjectures on rhythmic canons, it is enough to check only those canons who generate all other ones by those transformations, i.e. Vuza canons.

[29] Alternatively one can reason on sets, writing $B' = B \oplus n\mathbb{Z}_{kn}$.

3.2.6 Some conjectures and routes to solve them

The (T_2) conjecture

Let us recall the Coven-Meyerowitz conditions introduced in Section 3.2.3:

1. If A tiles, then (T_1) is true.
2. If both $(T_1),(T_2)$ are true, then A tiles.
3. If $\#A$ has at most two different prime factors, and A tiles, then both $(T_1),(T_2)$ are true.

[35] carefully refrained from enunciating the sometimes improperly stated 'Coven-Meyerowitz conjecture,' namely

Conjecture 3.54. A tiles $\iff$ both $(T_1),(T_2)$ are true.

The discussion of $Z(A)$ in the section above shows that (T_2) is inherited through all transformations:

Theorem 3.55. *If $A \oplus B = \mathbb{Z}_n$ is concatenated (or zoomed, or stuttered, or multiplexed) to a larger rhythmic canon, then (T_2) is true for the large canon whenever it is true for the smaller one.*

Proof. Consider for instance $\overline{A}^k$, the concatenation of the motif A k times. As we have established above, $R(\overline{A}^k) = R(A) \bigcup \big(\mathrm{Div}(kn) \setminus \mathrm{Div}(n)\big)$. Hence $S(\overline{A}^k)$ is $S(A)$, adding p^α whenever $p^\alpha \mid k$ though p does not divide n, and changing p^α to $p^{\alpha+\beta}$ if p^α, p^β are the powers of p in n,k respectively.

Remark 3.56. Checking condition (T_1) was not required, because it must be satisfied in both the short and large canons; but it would be straightforward to verify that it is true for A whenever it is true for $\overline{A}^k$.

From the equation above, clearly condition (T_2) holds in $R(\overline{A}^k)$ iff it holds in $R(A)$: apart from $R(A)$ itself, in $R(\overline{A}^k)$ we have also all terms with p^α as a factor when p divides k but not n, and when p is a factor of both k and n then the $p^{\alpha+\beta}q^\gamma\dots$ as above are in $R(\overline{A}^k)$ since they are divisors of kn **but not of n** because the exponent of p is too large.

The other factor B of the tiling does not change[30], and neither do $R(B), S(B)$ or hence condition (T_2) for B. The proof is similar for other transformations, using the results of the lemmas in last section.

Similar arguments hold for the other transformations, see [6, 46]. Since any canon can be deconcatenated down to a Vuza canon (or to the trivial canon, $\{0\} \oplus \{0\}$), it follows:

Proposition 3.57. *Conjecture 3.54 is true $\iff$ it is true for Vuza canons.*

[30] With the notations above, B changes to B' but for instance the polynomial $\mathbf{B}(X)$ stays the same.

This result revived the interest in Vuza canons when it was first published in [6], proving Conjecture 3.54 (and the spectral conjecture below too) in 'good groups', adding cases $n = p^m qr, pqrs$ (with p, q, r, s distinct primes) to [35]'s case $n = p^\alpha q^\beta$.

This deconcatenation technique also applies to all 'compact canons' (i.e. $A \subset \mathbb{Z}, B \subset \mathbb{Z}$ with $A \oplus B = \{0, 1, 2 \dots n-1\}$ without modulo n reduction), and [35] already noted that this implied the truth of Conjecture 3.54 in that case.

The spectral conjecture

Despite its name, the origin of the *spectral conjecture* is extraneous to the field of the present book, but it is still open in dimension 1 and 2, the former being our topic. It states

Conjecture 3.58. (Fuglede, 1974) A tiles some $\mathbb{Z}_n \iff A$ is spectral.[31]

Here, 'spectral' means that the tile (a measurable subset of $\mathbb{R}^n$ in the most general context) admits a Hilbert basis of exponential functions, meaning, in the seminal case, that any map in $L^2([0,1[)$ is the sum of its Fourier expansion. In dimension 1 we have a less esoteric definition involving difference sets:

Definition 3.59. *A subset $A \in \mathbb{Z}$ is spectral if there exists a* spectrum $\Lambda \subset [0,1[$, *i.e. a subset with the same cardinality as A, such that* $e^{2i\pi(\lambda_i - \lambda_j)}$ *is a root of the characteristic polynomial* $\mathbf{A}(X)$ *for all distinct* $\lambda_i, \lambda_j \in \Lambda$.

In other words, $Z(A)$ must include a (large enough) difference set.

It is still unknown whether in general the $\lambda_i - \lambda_j$ must be rational, i.e. whether the roots in question are roots of unity, though some progress was recently made in that respect. But if we consider A as a set in $\mathbb{Z}$ defined modulo $n\mathbb{Z}$, i.e. any element of A can be twiddled by any multiple of n – since this does not change the condition that A tiles $\mathbb{Z}_n$ – then *only those roots of* $A(X)$ *which are* n^{th} *roots of unity are unchanged.* Hence we may assume that $\Lambda \subset \{0, \frac{1}{n}, \frac{2}{n} \dots \frac{n-1}{n}\}$, i.e. $n\Lambda \subset \mathbb{Z}_n$, which we will do henceforth.[32]

The spectral conjecture has been proved in many cases (convex tiles for instance) but in general it is false, as first shown in high dimension by Fields medalist Terence Tao [82]. Following further work [56], the conjecture only remains open in dimensions 1 and 2. In dimension 1, which is our context for rhythmic canons, Izabella Łaba has proved [59] that $(T_1) + (T_2)$ implies 'spectral', explicitly constructing a spectrum under these conditions, just as [35] proved that $(T_1) + (T_2)$ implies 'tiling'. So the conjecture is known to be true when n has only two prime factors, by the

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[31] Originally it is a question of tiling $\mathbb{R}^n$ but in dimension 1 it can be reduced to tilings of $\mathbb{Z}$, see [45, 59].

[32] Twiddling an element by n adds $X^n - 1$ to the characteristic polynomial $\mathbf{A}(X)$, which destroys any root which is not common to both polynomials, hence this statement. [46] argues for this restricted definition of 'spectral', through characters of the group $\mathbb{Z}_n$, which also makes perfect sense and yields the same overset of Λ. Perhaps this condition should be properly labeled 'spectrality in a cyclic group'.

last result in Theorem 3.26; it is also true for motifs that tile a 'good group', because by deconcatenation such a tiling reduces to the trivial tiling and hence inherits $(T_1)+(T_2)$ (first proved in [6]). More generally, it is true for any motif that can be reduced to a tiling satisfying $(T_1)+(T_2)$, for instance the compact tilings mentioned above.[33]

Without condition (T_2) we have a direct heredity result:

Theorem 3.60. *Let $A \subset \mathbb{Z}$ be a finite motif of some tiling. We know from [35] that it tiles $\mathbb{Z}_n$ with $n = \gcd(R(A))$; then $\overline{A}^k$ is spectral if and only if A is spectral.*

This was announced in [3, 8], but first properly stated and proved in printed form in [46], which we follow below. If all Vuza canons are spectral, meaning both factors A, B are spectral sets, then by concatenation (and duality) any canon is spectral too. Hence the spectral conjecture (in the direction tiling $\Rightarrow$ spectral) is true if and only if it is true for all Vuza canons, which is another stringent motivation for their study.

Proof. Consider the concatenation of $A, \overline{A}^k \subset \mathbb{Z}_{kn}$. We have proved above that $R(\overline{A}^k) = R(A) \bigcup \big(\mathrm{Div}(kn) \setminus \mathrm{Div}(n)\big)$. Assume that we know a spectrum Λ for A, meaning that $e^{2i\pi(\lambda_i - \lambda_j)}$ is a root of the characteristic polynomial $\mathbf{A}(X)$ for all distinct $\lambda_i, \lambda_j \in \Lambda$. But in the ring of polynomials,

$$\overline{\mathbf{A}}^k(X) = (1 + X^n + X^{2n} + \dots X^{(k-1)n}) \times \mathbf{A}(X) = \frac{X^{nk}-1}{X^n-1} \times \mathbf{A}(X).$$

Hence Λ already produces some roots of $\overline{\mathbf{A}}^k(X)$. But $\#\overline{A}^k = k \times \#A$ and we need a larger spectrum. A possible solution is the sum

$$\Lambda' = \Lambda + \big\{0, \frac{1}{nk}, \frac{2}{nk} \dots, \frac{k-1}{nk}\big\}.$$

First, this spectrum has the right cardinality $k\#A$ (one has to check that the sum is direct, this follows from the fact that $\lambda_i - \lambda_j = q/n$ as assumed above).

Last, any element of Λ', i.e. $(\lambda_i - \lambda_j) \pm \frac{p}{nk}$, is equal either to $\lambda_i - \lambda_j$ (when $p = 0$), providing a root of $\mathbf{A}(X)$ as mentioned in the beginning of the proof, or to some $\frac{q}{n} \pm \frac{p}{nk}$ with $-n < q < n$ and $-k < p < k$, and hence provides a root of $X^{nk} - 1$ which is not a root of $X^n - 1$, i.e. one of the additional roots in $R_{\overline{A}^k}$. In both cases we get a root of $\overline{\mathbf{A}}^k(X)$ and hence Λ' is a spectrum.

For the complete reduction of Fuglede's conjecture to Vuza canons (or to the trivial canon when the deconcatenation process only ever stops with $\{0\} \oplus \{0\}$), one also needs the preservation of the spectral condition under duality (exchanging the

[33] In some cases I was able to predict that any Vuza canon in $\mathbb{Z}_{180}$ with a specific value of R_A could be reduced by demultiplexing to a canon with period 90, implying (T_2), without finding explicitly the canons in question but knowing from the factors in R_A that any complement B of A would be divisible by 2, i.e. that the canon could be demultiplexed.

factors) and prolongation of the other factor in concatenation $B \subset \mathbb{Z}_n$ to $\mathbb{Z}_{kn}$, which are both trivial, as is the zooming operation changing $A \subset \mathbb{Z}_n$ into $kA' \subset k\mathbb{Z}_n$. Gilbert also proved that the condition is invariant under affine transform or even multiplexing under conditions analogous to our computation above, when the new $\mathbb{Z}_{\widetilde{A}}$ is computed from the $\bigcap Z_{A_i}$ (assuming this intersection is spectral in a natural sense).

Example 3.61. Consider $A = \{0,1,4,5\}$, $R_A = \{2,8\}$ and hence A tiles $\mathbb{Z}_n$ for $n = 8$. Triple concatenation of A yields

$$\overline{A}^3 = \{0,1,4,5,8,9,12,13,16,17,20,21\}$$

and $R_{\overline{A}^3}$ is made of all integers below 24 except 6 and 18 (Fig. 3.11). This happens because $\overline{A}^3(X)$ is a pure product of cyclotomic polynomials:[34]

$$\overline{A}^3(X) = \Phi_2\Phi_3\Phi_6\Phi_8\Phi_{12}\Phi_{24}$$

A spectrum for A is [35] $\Lambda = \{0, \frac{1}{2}\} \oplus \{0, \frac{1}{8}\} = \{0, \frac{1}{8}, \frac{1}{2}, \frac{5}{8}\}$.[36] For a spectrum in $\overline{A}^3$ one adds $\frac{0,1,2}{24}$ and finally $\Lambda' = \frac{1}{24}\{0,1,2,3,4,5,12,13,14,15,16,17\}$ with 12 elements as required, whose differences yield all values of $\frac{k}{24}$ barring $\frac{6}{24}$ and $\frac{18}{24}$, as desired.

Detailed algorithms are provided in Section 3.3.

3.3 Algorithms

3.3.1 Computing a DFT

The definition formula is easy to implement in any modern programming language: loop over both the elements of the pc-set and the indexes. In the most general case, for a distribution $f \in \mathbb{C}^{\mathbb{Z}_n}$ one

- Selects (or input) the index k of the coefficient.
- Sets $s = 0$.
- For j from 0 to $n-1$, does $s = s + \mathrm{e}^{-2i\pi kj/n} \times f(j)$.
- Returns the value of s: it is $\widehat{f}(k)$.

[34] Building up rhythmic canons from products of cyclotomic polynomials was tried in [1] and implemented in *OpenMusic*. It is a fairly quick process – list cyclotomic polynomials, select index lists satisfying condition (T_2) and effectuate the corresponding product, discard the result if it is not 0-1, else find the possible outer voices – but omits many canons.

[35] I follow Łaba's recipe in [59].

[36] Search for a spectrum may well require exponential time, unless conjecture 3.54 is true, since the Coven-Meyerowitz conditions can be checked in polynomial time, as pointed out by Kolountzakis.

At worst one can separate real and imaginary parts and compute them separately (the former a sum of cosines, the latter a sum of sines).
Using $\cos(\pi/6) = \sqrt{3}/2, \sin(\pi/6) = 1/2$ and other trigonometric values, one can even compute a DFT by hand (preferably beginning with the kind of geometrical simplifications suggested in Fig. 3.3). Some practical advice: numerical calculations often fail to identify 0, so a routine that tidies the results (turning any $x \in [-10^{-10}, 10^{10}]$ to 0 for instance) is generally a good idea, especially for inverse Fourier transform.

Many high-level environments will provide a ready-made Fourier transform. One has to check which convention is used and perhaps adjust the result. For instance in Mathematica™, the DFT of a pc-set (say $\{0,4,7\}$) as defined in this book could be obtained with the native function **Fourier** by

Fourier $[\{1,0,0,0,1,0,0,1,0,0,0,0\}$, **FourierParameters** $\to \{1,-1\}]$.

Notice that the pc-set is replaced by the associated distribution – this can be automated by something like

Table[If [**MemberQ**[set, k], 1, 0], $\{k, 0, n-1\}$]

unless one prefers to compute one's own DFT with a loop inside a loop, as described above.

Major Scale Similarity

I include in this subsection the computation of **Major Scale Similarity** (MSS) though it is only defined below. One has to input first a temperament (TeT). Say it is given as a table of values in cents – for instance, Werkmeister's fifth TeT is

$$(0, 107.8, 209.8, 305.9, 407.8, 503.9, 611.7, 707.8, 803.9, 911.7, 1007.8, 1109.8).$$

Now define the magnitude of the first Fourier coefficient of a scale[37] (i.e. a table of 7 values in cents) as

$$A(\text{scale}) = \sum_{k=0}^{6} \mathrm{e}^{2\mathrm{i}\pi\,\text{scale}[k]/1200} \mathrm{e}^{-2\mathrm{i}k\pi/7}$$

(beware of your programming language's conventions; here I assume that the first index of a table is 0).

Compute the table of all major scales in the given TeT: starting from the list of indexes $ind = [0,2,4,5,7,9,11]$, run the 12 transpositions, i.e. $ind + k \pmod{12}$, and tabulate

$$\text{scale}[k] = \text{table}(\text{TeT}[(\text{ind}[j]+k) \pmod{12}], j = 0 \ldots 6).$$

With a simple loop, compute the max M and min m of the 12 values $A(\text{scale}[k])$:

- $m = 1000,\ M = 0$.

[37] With Noll's order-dependent definition, see $\mathfrak{F}_{\mathscr{A}}(1)$ inSection 5.2.

- For $k = 0$ to 11, do $x = A(\text{scale}[k])$;
  - If $x > M$ then $M = x$;
  - If $x < m$ then $m = x$;

Now the value of MMS(TeT) is $\dfrac{1}{M-m}$.

3.3.2 Phase retrieval

For convenient reference, I repeat here the algorithm for finding the unknown coefficient in Lewin's problem when one Fourier coefficient is nil:

1. Compute the cardinality of A: it is the sum of the elements of $\text{IFunc}(A,B)$ divided by $\#B$.
2. Compute $\mathscr{F}_A = \dfrac{\mathscr{F}(\text{IFunc}(A,B))}{\overline{\mathscr{F}_B}}$, with two coefficients still indeterminate.
3. Compute the sum of the squared magnitudes of the $n-2$ known coefficients in the last step; subtract the result from $n \times \#A$ to get $2r^2$ and hence r, the magnitude of the missing coefficient.
4. Compute the inverse Fourier transform of $\mathscr{F}_A$ as a function of the missing coefficient $r\,e^{i\varphi}$, where only φ remains unknown.
5. Taking into account that all the values computed in the last step must be 0's or 1's, determine φ; complete the computation of $\mathbf{1}_A$.

To some extent, this algorithm could be used even when A is a multiset.

3.3.3 Linear programming

The matricial formalism mentioned in Section 1.2.3 provides practical solutions to many retrieval problems. In [13], we have used linear programming to good effect for solving equations like $s * \mathbf{1}_A = \mathbf{1}_B$ (which corresponds to finding a linear combination of translates of A equal to B) and the same procedure could be used for solving $\mathbf{1}_A * \mathbf{1}_{-B} = \text{IFunc}(A,B)$ in A, i.e. Lewin's problem, among others like tiling.

Here is the algorithm: given a motif A and a period n for the tiling, consider a vector $x = (x_0, \dots, x_{n-1})$. By linear programming, minimize $x_0 + x_1 + \dots x_n$ under the constraint $\mathscr{A}.x^T = (1,1,\dots 1)$ (this is the tiling condition) and conditions $0 \leqslant x_i \leqslant 1$ for all i (this compels the 'quantity of pc i' to be somewhere between 0 and 1, and hopefully either one or the other).

But though the algorithm seems to work well, it is not formally proved yet that it always provides a solution! For one thing, there may well be multiple solutions (obtained by varying the starting point). For example, for $B = \{0,2,4,6,8,10\} \subset \mathbb{Z}_{12}$, $\text{IFunc}(A, \pm B)$ does not change when A is replaced by $A+2$ and there are at least six different solutions for the same value of $\text{IFunc}(A,B)$. Notice that this method bypasses Fourier transform altogether.

It is advantageous to use an environment wherein linear programming is already implemented (Mathematica, Maple, Fortran, ...).

3.3.4 Searching for Vuza canons

Tilings by translation, i.e. decomposition of cyclic groups in direct sums, gave rise to many conjectures. So far, most of them have proved to be false:

1. Sands and Tidjeman independently believed that any rhythmic canon is deconcatenable, i.e. when $A \oplus B = \mathbb{Z}_n$ then – assuming $0 \in A \cap B$ up to translation – either A or B lies in a strict subgroup of $\mathbb{Z}_n$.
2. Call D the diameter of a finite set of integers A (i.e. up to translation $A \subset \{0, 1 \dots, D\}$), $\mathscr{T}$ the least period of a tiling by A (i.e. A tiles $\mathbb{Z}_{\mathscr{T}}$) and $\mathscr{T}(D)$ the largest $\mathscr{T}$ for all A's with diameter $\leqslant D$. From the case $A = \{0, D\}$, it is clear that $\mathscr{T}(D) \geqslant 2D$; in the other direction, from the pigeonhole principle, it can be shown that $\mathscr{T}(D) \leqslant 2^D$, a rather wide bracket.

The first conjecture was proved false by Szabó ([81]). For the second one, Kolountzakis and others proved that $\gamma D^2 \leqslant \mathscr{T}(D) \leqslant \beta \exp(\alpha \sqrt{D \log D})$ for some constants α, β, γ; the lower bound was since increased to any power of D. The upper bound actually uses Fourier analysis, the factorisation in cyclotomic polynomials, and a sophisticated lower bound for Euler's totient function $\varphi(n) \geqslant \dfrac{Cn}{\log \log n}$ allowing one to construct cyclotomic factors with large degrees. In this section, we will focus on the construction that proves the lower bound and on the similar one by Szabó that disproves Sand's conjecture.

Both constructions start from two basic ideas: first, for composite n, $\mathbb{Z}_n$ can be decomposed as a direct sum (or product) of other cyclic groups (three at least in both cases), enabling one to look at 3D periodic lattices; and second, a very regular tiling (say B is a subgroup of $\mathbb{Z}_n, B = d\,\mathbb{Z}_n$ and A is a complete set of residues modulo d) can be easily perturbed into a very aperiodic tiling. Szabó and Kolountzakis differ in the second part because their aims are different.

Generalised Kolountzakis algorithm

Initially, Kolountzakis starts from an integer $n = 30pq$ and the isomorphism $\mathbb{Z}_n \approx \mathbb{Z}_{3p} \times \mathbb{Z}_{5q} \times \mathbb{Z}_2$ where p, q are large distinct primes with a similar magnitude $\sim D$. He then singles out the two "parallel planes" $P_0 = \mathbb{Z}_{3p} \times \mathbb{Z}_{5q} \times \{0\}$ and $P_1 = \mathbb{Z}_{3p} \times \mathbb{Z}_{5q} \times \{1\}$. He starts from the trivial tiling of $\mathbb{Z}_{3p} \times \mathbb{Z}_{5q}$ by $A = \{0,1,2\} \times \{0,1,2,3,4\}$ and $B = \{0,3,6\dots\} \times \{0,5,10\dots\}$ where B is a subgroup (isomorphic to $\mathbb{Z}_{pq}$) and A, omitted in Fig. 3.15, would appear as a small square. Now for P_0, *a row of* the first factor of B is translated; say $B_0 = \{0,\mathbf{4},6\dots\} \times \{0,5,10\dots\}$ and similarly for P_1 we translate *a column* of the second factor, say $B_1 = \{0,3,6\dots\} \times \{0,\mathbf{8},10\dots\}$. This shatters any periodicity in the tiling. Keeping the same $A \times \{0\}$ as motif and putting $B' = B_0 \times \{0\} \cup B_1 \times \{1\}$ we have an aperiodic tiling of $\mathbb{Z}_{3p} \times \mathbb{Z}_{5q} \times \mathbb{Z}_2$ and by isomorphism a Vuza canon in $\mathbb{Z}_n$. Explicit expression of this isomorphism (given below in 3.3) shows that the diameter of A has the same order of magnitude as $D \sim p \sim q$, whilst $n = 30pq \sim D^2$, proving the worst-case lower bound given *supra*. In Fig. 3.15 we can see at left the regular lattice B, and at right the same perturbed in B'; the first and second planes having respectively a row and a column pushed somewhat out of place (p, q have been reduced to 3 and 4 for the sake of readability).

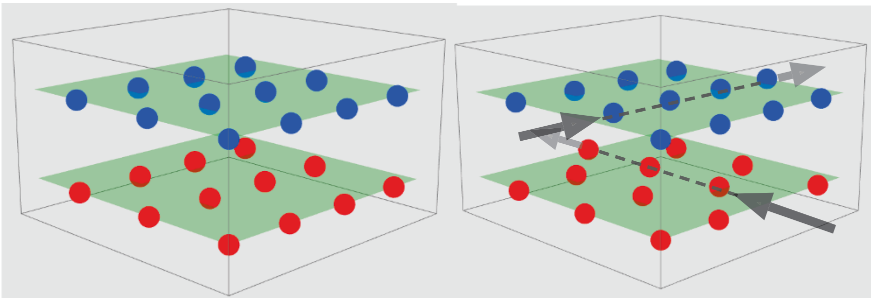

Fig. 3.15. A lattice tiling and its perturbation

I provide without details an example of such a construction, which is useful for building Vuza canons of medium size even though it was devised to prove asymptotic results.

Let $p = 3, q = 5$. Hence $n = 450$. In $\mathbb{Z}_n$ we find that A is $\{0,126,252\} \oplus \{0,100,200,300,400\}$, i.e.

$$A = \{0,2,28,54,100,126,128,154,200,226,252,254,326,352,378\}$$

and $B = \{0,30,60,90\ldots420\} = 30\mathbb{Z}_n$. This corresponds, in 3D, to the triplets with coordinates $(0/3/6, 0/5/10/15/20, 0/1)$ (/ denotes here an arbitrary choice between the values). The perturbation changes $(0,0,0)$ and $(3,0,0)$ to $(2,0,0)$ and $(5,0,0)$ in B_0, and the $(3,5k,1)$ to $(5,5k+2,1)$ in B_1, which yields ultimately the new factor

$$B' = \{15,21,30,45,60,90,100,105,111,120,135,180,195,201,210,225,$$
$$240,250,270,285,291,315,330,360,375,381,390,400,405,420\}.$$

By using five parallel planes instead of two, it is possible to get a tiling of $\mathbb{Z}_{180}$, the minimal value for this construction. One solution is shown in Fig. 3.16.

I will now expound a more general version.

1. Have five numbers a,b,c,p,q such that ap, bq and c are pairwise coprime.
2. Construct the tile
   $A \subset G = \mathbb{Z}_{ap} \times \mathbb{Z}_{bq} \times Z_c$ by $A = \{0,1,2\ldots a-1\} \times \{0,1\ldots b-1\} \times \{0\}$.
3. Construct the lattice complements

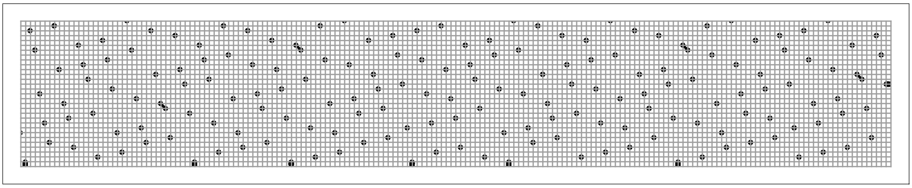

Fig. 3.16. Minimal Vuza canon ($n = 180$) built by Kolountzakis' algorithm

$$B_0 = \{0, a, 2a, \dots (p-1)a\} \times \{0, b \dots (q-1)b\} \times \{0\} \dots$$
$$\vdots$$
$$B_{c-1} = \{0, a, 2a, \dots (p-1)a\} \times \{0, b \dots (q-1)b\} \times \{c-1\}.$$

4. For $k = 0 \dots c-1$ add a perturbation vector ε_k to every element of each B_k, either of the form $\varepsilon_k = (p_k, 0, 0)$ or $\varepsilon_k = (0, p_k, 0)$. The two kinds must be present. Let $B'_k = B_k + \{\varepsilon_k\}$.
5. Compute $B = \bigcup B'_k$. Now $A \oplus B = G$.
6. Turn into a tiling of $\mathbb{Z}_n$ by the canonical linear isomorphism $\Psi : G \to \mathbb{Z}_n$,

$$\Psi(x, y, z) = ux + vy + wz$$

where u is defined modulo by $\Psi(x, y, z) \equiv x \pmod{a} p$ and similar equations, hence $\begin{cases} u \equiv 1 \pmod{a} p \\ u \equiv 0 \pmod{b} q \\ u \equiv 0 \pmod{c} \end{cases}$; so u is a multiple of bcq and we get explicitly $u = bcq \times (bcq)^{-1}$ in $\mathbb{Z}_{ap}$ (similarly for v, w).

This is not guaranteed to yield a Vuza canon, though it usually does. In practice, generate all possible canons by this method and sort out the aperiodic ones.

On the other hand, it is possible to compute R_A quite easily; since in $\mathbb{Z}_n$ one gets

$$A = \{0, u, \dots (a-1)u\} \oplus \{0, v, \dots (b-1)u\}$$

and in polynomials

$$\mathbf{A}(X) = (1 + X^u + X^{2u} + \dots X^{(a-1)u})(1 + X^v + X^{2v} + \dots X^{(b-1)v}) = \frac{X^{au} - 1}{X^u - 1} \frac{X^{bv} - 1}{X^v - 1}.$$

Hence R_A is made of the divisors of au which do not divide u, together with the divisors of bv which do not divide v:

Proposition 3.62. $R_A = \big(\mathrm{Div}(au) \cup \mathrm{Div}(bv)\big) \setminus \big(\mathrm{Div}(u) \cup \mathrm{Div}(v)\big)$.

This easily entails the non-periodicity of A. It is also a clear case of verifying conditions (T_1) and (T_2). It is possible to tell something about R_B (notably proving that it always satisfies condition (T_2)), but since the computation is analogous in the next algorithm, I will only do the latter.

Szabó's algorithm

In [81] the 3D-decomposition is not explicitly made. I will endeavour here to make it so.

Consider three pairs of integers $u_i, v_i, i = 1 \dots 3$ such that $u_i v_i$ and $u_j v_j$ are coprime for $i \neq j$. Let $m_i = u_i v_i$ and $n = m_1 m_2 m_3$. It is convenient to introduce $g_i = n/m_i$, e.g. $g_1 = u_2 v_2 u_3 v_3$.

For an example, let $u_1 = v_1 = 2, u_2 = v_2 = 3, u_3 = v_3 = 5, n = 900$.

Now the three groups G_i generated by the m_i satisfy $G_1 \oplus G_2 \oplus G_3 = \mathbb{Z}_n$. Each can be further decomposed in

$$G_i = \{0, g_i, 2g_i, \dots (u_i - 1)g_i\} \oplus \{0, \frac{n}{v_i}, 2\frac{n}{v_i}, \dots (v_i - 1)\frac{n}{v_i}\} = A_i \oplus B_i.$$

In the example, $G_2 = \{0, 100, 200\} \oplus \{0, 300, 600\}$.

Construct $A = \bigoplus A_i, B = \bigoplus B_i$: we have a tiling since

$$\mathbb{Z}_n = \bigoplus_i (A_i \oplus B_i) = \big(\bigoplus_i A_i\big) \oplus \big(\bigoplus_i B_i\big) = A \oplus B.$$

It is helpful to think of A as 'small change' and B as 'banknotes'.[38]

In the example, $A = \{0, 225\} \oplus \{0, 100, 200\} \oplus \{0, 36, 72, 108, 144\}$ *and* $B = \{0, 30, 60 \dots\} = 30\mathbb{Z}_{900}$.

B is always a subgroup, generated by all three $n/v_i = u_i g_i$, i.e. $B = \frac{n}{v_1 v_2 v_3}\mathbb{Z}_n$.

The idea is to perturbate B using the three dimensions. To ensure that the new B' still tiles with A, Szabó chooses a (circular) permutation σ of $\{1, 2, 3\}$. Remembering that the elements of B can be written as $\sum k_i u_i g_i$, select the $x_{k,i} = k u_i g_i + u_{\sigma(i)} g_{\sigma(i)}$ and replace all $x_{k,i}$ by $x'_{k,i} = x_{k,i} + g_i$.

In the example, if we take $\sigma(i) = i + 1 \pmod 3$ *then we replace* $x_{2,3} = 2u_3 g_3 + u_1 g_1 = 360 + 450 = 810$ *by* $x_{2,3} + g_1 = 810 + 225 = 135$. *On this term, the divisibility by 2 is destroyed, this is how this construction shatters Sand's conjecture. In all,* $\sum v_i = 2 + 3 + 5 = 10$ *elements are changed.*

This destroys the regularity of B but preserves the tiling quality, and perhaps a little more:

Theorem 3.63. *This construction yields a non-deconcatenable, non-demultiplexable, Vuza canon for large enough* u_i, v_i. *However, both factors of the tiling always satisfy condition* (T_2).

The first assertion is proved in [81], at least for composite n greater than 60,060 (though the smallest known counterexample, which uses this construction, lies in $\mathbb{Z}_{900}$). The last assertion appears in the literature, but as far as I know no proof of it has been published before.

[38] Appropriately, one of the very first papers on tilings of integers, *On Number Systems* by Nicolas de Bruijn (1956), originated from the consideration of the British money system.

Proof. Consider the characteristic polynomial

$$\mathbf{B}(X) = 1 + X^m + X^{2m} + \dots = \frac{X^n - 1}{X^m - 1} \quad \text{where } m = u_1 u_2 u_3.$$

In order to turn B into B' we multiply, for all i and all $k = 0 \dots v_i - 1$, the term $X^{ku_i g_i + u_{\sigma(i)} g_{\sigma(i)}}$ by X^{g_i}. In effect we had to $\mathbf{B}(X)$ the polynomials $P_i(X), i = 1 \dots 3$ defined by

$$(X^{g_i} - 1) X^{u_{\sigma(i)} g_{\sigma(i)}} \sum_{k=0}^{v_i - 1} X^{ku_i g_i} = (X^{g_i} - 1) X^{u_{\sigma(i)} g_{\sigma(i)}} \frac{X^n - 1}{X^{u_i g_i} - 1}.$$

Adding these polynomials and multiplying by the explicit form of $\mathbf{A}(X)$ *would prove that the new outer voice* B' *still tiles with A. I will not do it here, since it is already done in [81].*

The cyclotomic factors of this perturbation factor are the Φ_d with $R_{P_i} = d \in (\mathrm{Div}(n) \setminus \mathrm{Div}(u_i g_i)) \cup \mathrm{Div}(g_i)$. Remember that $R_B = \mathrm{Div}(n) \setminus \mathrm{Div}(m)$. Let us elucidate S_B: a prime factor p of u_i can only appear again in v_i by assumption; if it does not then it is cancelled out in the divisors of m, i.e. the prime powers in S_B are those common to u_i and v_i. Any such prime factor being confined to one index i can be labelled p_i, and $p_i^k \in S_B$ only if k is greater than the p_i-valuation of u_i, i.e. p_i^k is *not* a divisor of u_i.

In the example above, $\mathbf{B}(X) = \dfrac{X^{900} - 1}{X^{30} - 1}$ *and* $S_B = \{2 \times 2, 3 \times 3, 5 \times 5\}$.

Such powers still belong to R_{P_i}. So do products of these powers for different indexes i: consider without loss of generality $r = p_1^2 p_2^2$ where p_i is a prime factor of u_i and $v_i, i = 1, 2$ (with valuation 1 to ease the notation). Then r is a divisor of n, of course, but not a divisor of $u_1 g_1 = n / v_1 = p_1^1 \times Q$ where Q is coprime with p_1. A similar verification can be done for P_3. This means that condition (T_2) still holds.

In the example above, $S_{B'} = S_B = 4, 9, 25$ *and we preserve at least* $36, 100, 225$ *and* 900 *in* $S_{B'}$. *Some factors have disappeared but are not required by condition* $(T_2) : 12, 18, 20, 45, 50, 60, 75, 90, 150, 180, 300, 450.$

Matolcsi's algorithm

In [57], Matos Matolcsi devised a neat procedure for an exhaustive search for Vuza canons in a given $\mathbb{Z}_n$. Though this sometimes fails because of computational complexity, it is still worthwhile to study it in the context of this book.

The key to his procedure is a useful lemma in [35]:

Lemma 3.64. *If A satisfies* (T_1) *and* (T_2), *then a complement of A in* $\mathbb{Z}_n$, *i.e. B satisfying* $A \oplus B = \mathbb{Z}_n$, *can be produced by its characteristic polynomial:* $B(X)$ *is the product of the* $\Phi_{p^\alpha}(X^{n/p^{v(p)}})$, *where* $p^\alpha \mid n$ *is* not *in* S_A, *and* $n = \prod_i p_i^{v(p_i)}$ *is the decomposition of n into prime powers (so that* $n/p^{v(p)}$ *is the largest divisor of n coprime with p).*

Example 3.65. Consider $S_A = \{2,8\}$ and $n = 24$.[39] Since $24 = 2^3\,3^1$, the missing prime powers in S_A – which must indeed be in S_B – are $4 = 2^2$ and 3, which are respectively complemented to 24 by coprime prime powers 3 and 8. We compute

$$B(X) = \Phi_4(X^3) \times \Phi_3(X^8) = (1+(X^3)^2)(1+(X^8)+(X^8)^2),$$

hence $B = \{0,6,8,14,16,22\}$ which does tile, for instance with $A = \{0,3,12,15\}$.

Now the idea is to check all possible sets S_A. Begin by choosing n.

- Compute all partitions in two subsets of the set of prime power divisors of n. Keep (usually) the smallest part, which will be S_A (the other being of course S_B).
- Compute the Coven-Meyerowitz complement B for S_A.[40]
- Compute all possible A completing B, using one of the general completion algorithms described in [11].[41] Sort by the different values of R_A, keeping one representative A_i for each possibility.
- Discard all sets R_A that either
  1. ensure that A is periodic, or
  2. ensure that B must be periodic (recalling that R_B must contain at least all divisors of n not in R_A), making use of Theorem 3.28.
- For each remaining representative of possible A's, compute complements B, discarding eventual periodic ones.
- Whatever remains is a Vuza canon.

Details and tables of results are given in [8].

One algorithm that I will not discuss here, though it sounds closely related to harmonic analysis, is the search for a spectrum (cf. Section 3.2.6). Actually it is mostly (as of today) a computational problem; Kolountzakis has studied its complexity and provides strong heuristic reasons for it to be NP-complete, unless the (T_2) conjecture is true. Actually he views this as a strong argument against the latter conjecture!

Exercises

Exercise 3.66. Compute the cyclotomic polynomial Φ_d when d runs over all divisors of 12 (use Eq. 3.1).

Exercise 3.67. X^8+1 is a cyclotomic polynomial. Which one?

Exercise 3.68. Choose some singular pc-set in Table 8.2 and check which of Lewin's conditions is satisfied. Compare with the appropriate Fourier coefficient (e.g. if the augmented triad property is satisfied, check that $a_3 = 0$).

[39] If we start from an actual motif A and n is unknown, n can be taken equal to the lcm of R_A – or any multiple thereof.

[40] This is a simple motif, product of 'metronomes', cf. exercises.

[41] This is the weak point of the algorithm because when B is very regular, both the number of solutions for A and the searching time get considerable.

Exercise 3.69. Is $\{0,2,3,5,7,8\}$ singular or invertible in $\mathbb{Z}_{12}$?

Exercise 3.70. Express a fifth (e.g. $\{0,7\}$) as a linear combination of the 11 other ones.

Exercise 3.71. Compute by hand the DFT of $\{0,1,6,7,11\}$, Berg's sonata's initial pc-set.

Exercise 3.72. Decompose the even elements of $\mathbb{Z}_{32}$ in classes of associated elements, i.e. according to their order.

Exercise 3.73. $A = \{0,1,7,11,17,18,24\} \subset \mathbb{Z}_{30}$. Check that $a_1 = 0$ and that A cannot be decomposed as a reunion of regular polygons.

Exercise 3.74. Prove Proposition 3.20 and/or the next one.

Exercise 3.75. Check that $A = \{0,1,6,10,12,13,15,19\}$, $A' = \{0,2,5,6,11,12,15,17\}$ both tile $\mathbb{Z}_{24}$.

Exercise 3.76. Compute R_A for $A = \{0,5,8,13\}$.

Exercise 3.77. Use Jedrzejewski's recipe and build a Vuza canon.

Exercise 3.78. Prove Theorem 3.49 (discuss on each possible residue i, or read [6]).

Exercise 3.79. Check that $\{0, \frac{1}{8}, \frac{1}{2}, \frac{5}{8}\}$ is a spectrum for $A = \{0,1,4,5\}$ in $\mathbb{Z}_8$.

Exercise 3.80. Finish the computation of the example in $\mathbb{Z}_{900}$ of Szabó's algorithm.

Exercise 3.81. A motif A is such that $S_A = \{2,8,9\}$ and satisfies condition (T_2). Build B that tiles with A using the construction in Lemma 3.64. Use

$$\Phi_{p^\alpha}(X) = 1 + X^{p^{\alpha-1}} + X^{2p^{\alpha-1}} + \dots X^{(p-1)p^{\alpha-1}} = \frac{X^{p^\alpha} - 1}{X^{p^{\alpha-1}} - 1}.$$