A Relationship of Conflicting Belief Masses to Open World Assumption

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Abstract. When combining belief functions by conjunctive rules of combination, conflicting belief masses often appear, which are assigned to empty set by the non-normalized conjunctive rule or normalized by Dempster's rule of combination in Dempster-Shafer theory.

This theoretical study analyses processing of conflicting belief masses under open world assumption. It is observed that sum of conflicting masses covers not only a possibility of a non-expected hypothesis out of considered frame of discernment. It also covers, analogously to the case of close world assumption, internal conflicts of individual belief functions and conflict between/among two or several combined belief functions.

Thus, for correct and complete interpretation of open world assumption it is recommended to include extra element(s) into used frame of discernment.

Keywords: Belief functions · Dempster-shafer theory · Uncertainty · Conflicting belief masses · Internal conflict · Conflict between belief functions · Open world assumption · Transferable Belief Model (TBM)

1 Introduction

When combining belief functions by conjunctive rules of combination, conflicting belief masses often appear. This happens whenever combined belief functions (BFs) are not mutually completely consistent. Conflicting masses are originally considered to be caused by a conflict between belief functions [16] and later, alternatively, by a possibility of having a new hypothesis outside of a considered frame of discernment [17]. The later approach is called open world assumption (OWA).

The original Shafer's interpretation of the sum of all conflicting belief masses does not correctly correspond to the real nature of conflicts between belief functions [1,13], this has motivated a theoretical research and a series of papers on the topic of conflicts of BFs, e.g., [3,6,9–15].

Smets' idea of open world assumption is usually accepted by papers on the Transferable Belief Model (TBM) and on TBM based approaches. Nevertheless Smets' OWA approach hides the real nature of conflicting masses and conflicts of BFs; it mixes conflicts with a possibility of existence of a hypothesis outside of a considered frame of discernment.

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Motivated by a discussion after presentation of author's recent approach to conflicts of BFs [9], we discuss a relationship of the sum of conflicting belief masses and OWA approach in this study, in order to uncover a real nature of the sum of all conflicting masses and to present and analyse interpretations of conflicting belief masses under OWA.

Important basic notions are briefly recalled in Sect. 2. Section 3 presents normalized belief functions under OWA, whereas non-normalized belief functions under OWA are analysed in Sect. 4. Section 5 summarizes the analysed interpretations of OWA approach. Utilizing the presented results, Smets' TBM based on OWA is compared with the classic Shafer's approach to belief functions.

2 Preliminaries

We assume classic definitions of basic notions from the theory of *belief functions* (BFs) [16] on a finite frame of discernment $\Omega_n = \{\omega_1, \omega_2, ..., \omega_n\}$. An exhaustive frame of discernment is considered in the classic Shafer's approach; this is called *closed world assumption*. Alternatively Smets [17] admits a possibility of appearance of a new hypothesis outside of the considered frame of discernment, thus the frame is not exhaustive there; this is called *open world assumption (OWA)*. The sum of conflicting belief masses is interpreted as a mass of a hypothesis(-es) outside of the original frame, BFs are not assumed to be normalized there.

A *basic belief assignment (bba)* is a mapping $m : \mathcal{P}(\Omega) \longrightarrow [0,1]$ such that $\sum_{A\subseteq\Omega} m(A) = 1$; the values of the bba are called *basic belief masses (bbm)*. $m(\emptyset) = 0$ is assumed in the classic approach; $m(\emptyset) \geq 0$ in Smets' OWA approach. A *belief function (BF)* is a mapping $Bel : \mathcal{P}(\Omega) \longrightarrow [0,1]$, $Bel(A) = \sum_{\emptyset\neq X\subseteq A} m(X)$. There is a unique correspondence between m and corresponding Bel thus we often speak about m as of belief function.

A BF is *normalized* if $m(\emptyset) = 0$, thus if $\sum_{\emptyset\neq X\subseteq\Omega} m(X) = Bel(\Omega) = 1$. A BF is *non-normalized* if $m(\emptyset) > 0$, thus if $\sum_{\emptyset\neq X\subseteq\Omega} m(X) = Bel(\Omega) < 1$.

A *focal element* is a subset X of the frame of discernment, such that $m(X) > 0$. If all focal elements are nested, we speak about *a consonant belief function*; if all focal elements have a non-empty intersection, we speak about *a consistent belief function*.

Dempster's (conjunctive) rule of combination $\oplus$ is given as $(m_1 \oplus m_2)(A) = \sum_{X\cap Y=A} K m_1(X) m_2(Y)$ for $A \neq \emptyset$, where $K = \frac{1}{1-\kappa}$, $\kappa = \sum_{X\cap Y=\emptyset} m_1(X) m_2(Y)$, and $(m_1 \oplus m_2)(\emptyset) = 0$, see [16]; putting $K = 1$ and $(m_1 \circledcirc m_2)(\emptyset) = \kappa$ we obtain the *non-normalized conjunctive rule of combination* $\circledcirc$, which is used in OWA approach, see e.g., original Smets' Transferable Belief Model (TBM) [18].

3 Normalized Examples Against a Simple Interpretation of OWA Approach

Let us present several examples in this section. We will start with an extremely illustrative Almond's example [1,6], assuming OWA here.

Example 1. Let us suppose six-element frame of discernment, results of a six-sided die and two independent believers with the same beliefs[1] expressing that the six-sided die is fair: $\Omega_6 = \{\omega_1, ..., \omega_6\} = \{1,2,3,4,5,6\}$, $m_j(\{\omega_i\}) = 1/6$ for $i = 1, ..., 6$, $j = 1, 2$, $m_j(X) = 0$ otherwise. Let $m = m_1 \circledcirc m_2$. We obtain $m(\{\omega_i\}) = 1/36$ for $i = 1, ..., 6$, $m(\emptyset) = 5/6$, $m(X) = 0$ otherwise. Supposing the usual simple OWA interpretation we obtain big belief mass $m(\emptyset) = 5/6$ for a non-expected hypothesis outside of our frame Ω_6, e.g. the die stands on one of its edges or vertices. It seems obvious that such an interpretation is not correct.

An analogous example is presented by W. Liu in [13] on a five-element frame of discernment. We can modify these examples, where both believers have same positive arguments for all hypotheses, by decreasing belief masses of singletons by the same value and putting the removed belief masses to the frame of discernment, or by taking any classic (i.e., normalized with $m(\emptyset) = 0$) non-vacuous[2] symmetric BFs, i.e., by some kind of discounting. Nevertheless, we always obtain positive $m(\emptyset) = 0$, which is hardly interpretable as a belief mass of an unexpected hypothesis, when zero belief mass is assigned to a hypothesis outside of the frame by both of the believers, which are in full accord.

More generally, we can take any couple of numerically same classic non-consistent BFs under OWA, e.g., Example 2 from [6], tossing a coin. We again obtain $m(\emptyset) = 0$, from two believers in full accord with $m_j(\emptyset) = 0$. This is again hardly interpretable as a belief mass of unexpected hypotheses, e.g., coin stands on its edge.

Example 2. Let us suppose for simplicity $\Omega_2 = \{\omega_1, \omega_2\}$ now. Let $m_j(\{\omega_1\}) = 0.5$, $m_j(\{\omega_2\}) = 0.4$, $m_j(\{\omega_1, \omega_2\}) = 0.1$ for $j = 3, 4$, $m_j(X) = 0$ otherwise. Let $m = m_3 \circledcirc m_4$ now. We obtain $m(\{\omega_1\}) = 0.35$, $m(\{\omega_2\}) = 0.24$, $m(\{\omega_1, \omega_2\}) = 0.01$, $m(\emptyset) = 0.4$, $m(X) = 0$ otherwise.

Both believers have same beliefs, they are in full agreement, there is no conflict between them. Assuming OWA the believers have a possibility to assign some belief mass to a new hypothesis unexpected in the frame of discernment using non-normalized BF(s). But they did not use this option, they assigned all the belief masses to non-empty subsets of the considered frame. Thus the positive resulting $m(\emptyset)$ expresses, in accordance with [6], rather *internal conflict* of input BFs than a belief mass assigned to a new hypothesis unexpected in the frame of discernment.

Let us suppose classic internally non-conflicting BFs, thus consonant or more generally consistent BFs now. There is no issue when the BFs are mutually

[1] Do not forget that the equality of BFs is not equivalent to their dependence: dependent BFs, BFs from dependent believers should be same or somehow similar, dependence implies similarity, but same (or very similar) BFs do not imply their dependence.

[2] Combining two vacuous BFs gives $m(\Omega) = 1$, thus $m(\emptyset) = 0$, but vacuous BF does not express the same positive arguments for all hypotheses, it expresses the full ignorance.

consistent, i.e., if common intersection of all their focal elements is non-empty, there is $m(\emptyset) = 0$ in such a case. On the other hand, if our consistent BFs are not mutually consistent, we can obtain the following example:

Example 3. Let us suppose $\Omega_6 = \{\omega_1, ..., \omega_6\}$ again and two simple internally non-conflicting BFs: $m_5(\{\omega_2, \omega_4, \omega_6\}) = 1$, $m_6(\{\omega_1\}) = 1/3$, $m_6(\{\omega_1, \omega_3\}) = 2/3$, $m_i(X) = 0$, otherwise. Combining $m = m_5 \circledcirc m_6$ we obtain $m(\emptyset) = 1$.

Example 3 (Modified). Let us suppose Ω_6 again and two modified BFs Bel_5', Bel_6':

X :	$\{\omega_1\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_4, \omega_6\}$	Ω_6	$\emptyset$
$m_5(X)$:			2/3	1/3	
$m_6(X)$:	2/9	4/9		3/9	
$(m_5 \circledcirc m_6)(X)$:	2/27	4/27	6/27	3/27	12/27

A situation is much more complicated here. Both the input BFs are consistent, thus internally non-conflicting. On the other hand the BFs are not mutually consistent, there is high conflict between them, they are even completely conflicting in the case of the original Example 3. Some part of $m(\emptyset)$ represents conflict between BFs here. Of course another part of $m(\emptyset)$ may be caused by OWA. Because both of the believers assign all their belief masses to non-empty subsets of the frame, even if OWA is considered, we can hardly interpret entire $m(\emptyset)$ as a belief mass assigned to a new hypothesis outside of the frame.

Thus we rather have to consider entire $m(\emptyset)$ or its part to be a *conflict between* BFs (external conflict [11]) than to consider a belief mass of an unexpected hypothesis (or unexpected hypotheses) only.

4 Non-normalized Belief Functions Under OWA

Let us turn our attention to non-normalized BFs in this section. Input BFs explicitly assume or at least admit existence of a new hypothesis unexpected in the considered frame of discernment.

Let us start with an analogy of Example 1, but the believers want to admit the existence of a new hypothesis, thus they assign belief mass $\frac{1}{10}$ outside of the considered frame Ω_6 frame thus to the $\emptyset$:

Example 4. Let us again suppose a six-sided fair die, thus Ω_6, but modified bbms m_1', m_2' this time:

X :	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_4\}$	$\{\omega_5\}$	$\{\omega_6\}$	$\emptyset$
$m_1'(X)$:	0.15	0.15	0.15	0.15	0.15	0.15	0.10
$m_2'(X)$:	0.15	0.15	0.15	0.15	0.15	0.15	0.10
$(m_1' \circledcirc m_2')(X)$:	0.0225	0.0225	0.0225	0.0225	0.0225	0.0225	0.8650

The result is better this time, positive belief masses same for all singletons are obtained. Nevertheless, both believers assign greater masses to any element from the frame than to a new hypothesis. But the resulting belief mass of the empty set is significantly greater than masses assigned to the singletons, even significantly greater than the sum of belief masses assigned to all the singletons from the frame of discernment ($0.865 = m'(\emptyset) > \sum_{i=1}^{6} m'(\{\omega_i\}) = 0.135$). When believers decrease their belief masses assigned to the empty set (see the following modification of the example), the resulting belief mass assigned to the empty set remains almost the same. Thus (a possibility of appearing of) a new hypothesis outside of the frame of discernment is significantly preferred to any hypothesis from the frame, even to the entire frame (as $m(\emptyset) > Bel(\Omega)$).

Example 4 (Modified). Let us suppose Ω_6 again, with different modification of bbms m''_1, m''_2 this time:

X :	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_4\}$	$\{\omega_5\}$	$\{\omega_6\}$	$\emptyset$
$m''_j(X)$:	0.16	0.16	0.16	0.16	0.16	0.16	0.04
$(m''_1 \circledcap m''_2)(X)$:	0.0256	0.0256	0.0256	0.0256	0.0256	0.0256	0.8464

There is a simple mathematical explanation: $m(\emptyset)$ is absorbing element with respect to conjunctive combination, i.e., $m_i(X)m_j(\emptyset)$ goes to $(m_i \circledcap m_j)(\emptyset)$ for any $X \subseteq \Omega$, as there always holds that $X \cap \emptyset = \emptyset$.

There is also an interpretational explanation: in accord with the classic cases studied in [6] the sum of conflicting belief masses $m_{\circledcap}(\emptyset)$ contains also internal conflicts of input belief masses (and conflict between BFs if they are mutually conflicting).

When we want to admit a possibility of an unexpected hypothesis and we do not like to assign positive belief masses directly to the empty set we can either explicitly add a new element(s) representing some unexpected hypothesis(es) into the considered frame of discernment or we can add empty set to the frame.

Let us start with the later option, i.e., addition of the empty set to the frame of discernment. Thus we obtain $\Omega_n^{\emptyset} = \Omega_n \cup \{\emptyset\} = \{\omega_1, \omega_2, ..., \omega_n, \emptyset\}$, especially $\Omega_6^{\emptyset} = \{\omega_1, \omega_2, ..., \omega_6, \emptyset\}$. We can express a possibility of unexpected hypothesis by positive $m(\Omega_n^{\emptyset})$ now. Let us look at Example 5 and its modification applied to $\Omega_6^{\emptyset}$:

Example 5. A six-sided fair die again; and modified bbms on $\Omega_6^{\emptyset}$ this time:

X :	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_4\}$	$\{\omega_5\}$	$\{\omega_6\}$	Ω_6	$\Omega_6^{\emptyset}$	$\emptyset$
$m'''_1(X)$:	0.15	0.15	0.15	0.15	0.15	0.15		0.10	
$m'''_2(X)$:	0.15	0.15	0.15	0.15	0.15	0.15		0.10	
$(m'''_1 \circledcap m'''_2)(X)$:	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525		0.01	0.6750
$m''''_j(X)$:	0.16	0.16	0.16	0.16	0.16	0.16		0.04	
$(m''''_1 \circledcap m''''_2)(X)$:	0.0384	0.0384	0.0384	0.0384	0.0384	0.0384		0.0016	0.7680

We can observe a high belief mass assigned to the empty set at $m'''_1 \circledcap m'''_2$ and $m''''_1 \circledcap m''''_2$, especially in the later case where less masses are assigned to $\Omega_6^{\emptyset}$ in input BFs m''''_1, m''''_2. Thus, the problem of preference of an unexpected hypothesis seems to be solved here, but interpretation of high $(m_1 \circledcap m_2)(\emptyset)$

remains an open issue. Moreover, when we interpret $m(\emptyset)$ or its part as belief mass of an unexpected hypothesis, the unexpected hypothesis is preferred again, as its belief mass comes from two parts: $m(\emptyset)$ and $m(\{\emptyset\})$ (the later is zero in Example 5).

We are going to investigate addition of a new classic element $N(ew)$ representing unexpected hypotheses now. We obtain $\Omega_n^+ = \Omega_n \cup \{N\} = \{\omega_1, \omega_2, ..., \omega_n, N\}$, especially $\Omega_6^+ = \{\omega_1, \omega_2, ..., \omega_6, N\}$. Let us look at Example 4 and its modification 6 applied to Ω_6^+. We can directly assign a belief mass to the additional element N, see Example 6, or analogously to the previous case to entire Ω_6^+ (we obtain numerically same results as in the previous case), see Example 6 (modified). The combination of these two options is of course also a possibility.

Example 6. A six-sided fair die again; and modified bbms on Ω_6^+ this time:

X :	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_4\}$	$\{\omega_5\}$	$\{\omega_6\}$	$N(ew)$	Ω_6^+	$\emptyset$
$m_1^v(X)$:	0.15	0.15	0.15	0.15	0.15	0.15	0.10		
$m_2^v(X)$:	0.15	0.15	0.15	0.15	0.15	0.15	0.10		
$(m_1^v \circledcirc m_2^v)(X)$:	0.0225	0.0225	0.0225	0.0225	0.0225	0.0225	0.01		0.8550
$m_j^{vi}(X)$:	0.16	0.16	0.16	0.16	0.16	0.16	0.04		
$(m_1^{vi} \circledcirc m_2^{vi})(X)$:	0.0256	0.0256	0.0256	0.0256	0.0256	0.0256	0.0016		0.8448

Example 6 (Modified). A six-sided fair die; and modified bbms on Ω_6^+ this time:

X :	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_4\}$	$\{\omega_5\}$	$\{\omega_6\}$	$N(ew)$	Ω_6^+	$\emptyset$
$m_1^{vii}(X)$:	0.15	0.15	0.15	0.15	0.15	0.15		0.10	
$m_2^{vii}(X)$:	0.15	0.15	0.15	0.15	0.15	0.15		0.10	
$(m_1^{vii} \circledcirc m_2^{vii})(X)$:	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525		0.01	0.6750
$m_j^{viii}(X)$:	0.16	0.16	0.16	0.16	0.16	0.16		0.04	
$(m_1^{viii} \circledcirc m_2^{viii})(X)$:	0.0384	0.0384	0.0384	0.0384	0.0384	0.0384		0.0016	0.7680

We can see a high belief mass assigned to the empty set also in the case of Ω_6^+ but this time it does not represent a belief mass of unexpected hypotheses. Belief mass directly assigned to unexpected hypotheses is represented by $m(\{N\})$ and plausibly also by any $m(X)$ for $N \in X$, while $m_i \circledcirc m_j(\emptyset)$ represents conflicts inside and between m_i and m_j as under the close world assumption. If it is useful for a given domain, we can use several additional elements for several unexpected hypotheses, or just one as coarsening of all unexpected hypotheses together. We can see that belief mass of an unexpected hypothesis (of element N) which is less than belief mass of any element of Ω_6 in individual BFs remains less also after combination. Thus an unexpected hypothesis in not preferred now.

Let us suppose an observer which knows European animals only and a frame of discernment Ω_{EA} containing the European animals. Let us move our observer (without any previous knowledge of African animals) to Africa now. When observing a zebra, an assignment of positive belief mass explicitly to a new element N is probably not necessary, as belief masses may be assigned to focal elements $H = \{horse, N\}$ and $H \cup X$ for $X \subseteq \Omega_{EA}$. Observing a crocodile or an elephant some positive belief mass should be probably assigned to N (where its size would be related to a quality of the observation). For some applications one element N for all unknown animals is sufficient, for other applications several new elements, e.g., NM (new mammal), NB (new bird), NR (new reptile), etc., would fit better.

Using additional element N for unexpected hypothesis, we can either make normalization as in classic Shafer's approach; or we can use non-normalized BFs as in Smets' approach, considering that $m(\emptyset)$ represents a size of conflict (both internal and external) of BFs. From the decisional point of view both the options are the same as an element with the highest value of some probabilistic transformation[3] of BFs is usually selected. Usually Smets' pignistic probability [19] or normalized plausibility of singletons [4] (i.e., normalized contour function) is used. For an analysis[4] of probabilistic transformations see, e.g., [4,5].

5 A Comparison of the Approaches

5.1 A Summary of the Presented OWA Approaches

Using just a non-negative $m(\emptyset)$: This is a simple idea and performance. But simple interpretation of $m(\emptyset)$ hides internal conflict(s) of BF(s) and conflict between BFs in results of their combination. Interpreting $m(\emptyset)$ only as belief mass assigned to new unexpected hypotheses significantly prefers possibility of unexpected hypotheses to those which are included in the considered frame of discernment.

Extension of Ω by $\{\emptyset\}$, where belief masses are assigned only to classic focal elements and entire extended $\Omega_n^{\emptyset}$: This simple extension, unfortunately does not cover the issue of interpretation of $m(\emptyset)$, see Example 5.

Extension of Ω by new element(s): Ω^+. This approach increases the size of the frame, thus it also a little bit increases complexity of computation (especially when several new elements are added in a small frame). On the other hand, this approach distinguishes belief masses of unexpected hypothesis(es) from both internal and external conflicts caused by conflicting masses of disjoint focal elements from the frame and also from conflicts caused by conflicting masses of original focal elements and unexpected hypotheses. Moreover, both original and additional hypotheses are managed analogously in this approach, none of them is preferred.

[3] Note, that the pignistic probability gives numerically same results under close and open world assumptions, as normalization is part of pignistic transformation; and that TBM with non-normalized $\circledcirc$ under OWA gives same decisional results as classic Shafer's approach with $\oplus$ and pignistic transformation does. The only difference is that TBM explicitly keeps in $m(\emptyset)$ value of conflict (internal and external conflict together with masses of unexpected hypotheses) until the moment of decision.

[4] Note, that normalized plausibility is consistent with conjunctive combination (they mutually commute), while pignistic transformation is not. Pignistic transformation commutes instead of conjunctive combination with linear combination of BFs.

5.2 When Do the OWA Approaches Coincide?

When there is no necessity or reason to assign belief mass directly to sets of considered hypotheses, we obtain an unexpected hypothesis N in all focal elements in the extended approach. Thus all intersections of focal elements contain N again. There is no reason to assign a positive belief mass to the empty set in the extended approach. Intersections are non-empty under our assumption, thus $\emptyset$ is not a focal element in the extended approach under our assumption. Hence we obtain the following equivalence of focal elements, thus also of the approaches: $X^+ \equiv X$, where $X \subseteq \Omega,\ X^+ \subseteq \Omega^+,\ X^+ = X \cup \{N\}$, and $\{N\} \equiv \emptyset$.

E.g., for two-element frame $\{H(ead), T(ail)\}$ we obtain under our assumption the following equivalence with extended version of the frame $\{H, T, N\}$:

$$\begin{aligned}&\{N\} \equiv \emptyset,\\ &\{H, N\} \equiv \{H\},\\ &\{T, N\} \equiv \{T\},\\ &\{H, T, N\} \equiv \{H, T\}.\end{aligned}$$

We can see that $Bel(\Omega)$ is not only ≤ 1, but it is just $Bel(\Omega) = 0$ under the assumption. We can see that N is preferred in this case as $Bel(\{N\}) \geq 0$. It may be zero in initial BFs, but it may obtain a positive belief mass within combination of two BFs which are not mutually consistent.

We have to notice that we cannot assign any positive belief mass to any focal element from the considered frame Ω in this case which is equivalent to simple interpretation of OWA. Even if we have a fully reliable believer (observer, sensor) and 100 % clear argument (observation, measurement) in favour of an element or a subset of the frame ($\omega_X \in \Omega$ or $X \subseteq \Omega$), focal elements should always contain N, thus they are $\{\omega_X, N\}$ or $X \cup \{N\}$ and $m(\{\omega_X\}) = m(X) = 0$ hence always also $Bel(\{\omega_X\}) = Bel(X) = 0$.

Assuming a criminal example analogous to Smets' Peter, Paul and Mary case, any (partially of fully) contradictive testimonies give (multiples of) their contradictive masses to a person which is out of the frame, thus to unknown person, which is still not suspicious to be an assassin. On the other hand, belief of the entire frame is zero, thus $Bel(\{Peter, Paul, Mary\}) = 0$.

This always holds true under the above assumption and due to the equivalence it also holds true in the simple interpretation of OWA in general.

5.3 A Comparison of Smets' OWA and Classic Shafer's Approaches

A Decisional Point of View. Based on commutativity of normalization with conjunctive combination, i.e., on the fact that $n(n(m_1) \circledcirc n(m_2)) = n(m_1 \circledcirc m_2) = m_1 \oplus m_2$, where $(n(m))(X) = \frac{m(X)}{\sum_{\emptyset \neq Y \subseteq \Omega} m(Y)}$, $\circledcirc$ non-normalized, and $\oplus$ normalized conjunctive combinations, we can see that TBM gives the same results as classic Shafer's approach produces, when pignistic probability is used. This holds true because the first step of the pignistic transformation (generating of $BetP$) is just a normalization. Possibly different results may arise when different probabilistic approaches are used in the approaches: e.g., pignistic transformation in TBM, and plausibility transformation (generation of normalized plausibility of singletons) in the classic approach.

Belief Mass of the Empty Set: $m(\emptyset)$. A positive belief mass of the empty set is a feature which really distinguishes TBM form the classic approach. It is hidden on the normalization step when two or more BFs are combined in the same time in Shafer's approach. It is not even computed there, when input BFs are combined gradually one by one. We have to recall that $m(\emptyset)$ includes not only belief mass of a possible unexpected hypothesis, but also internal conflicts of input BFs, conflict between two or among several input BFs.

If we want correctly use the simple interpretation of $m(\emptyset)$ only as a belief mass of an unexpected hypothesis, then we have to assume that all focal elements include the unexpected hypothesis, hence that $m(X) = Bel(X) = 0$ for any subset X of Ω ($X \subseteq \Omega$), which does not contain the unexpected hypothesis.

Definition Domains. Non-normalized conjunctive rule is defined for any couple (n-tuple) of BFs. Classic Dempster rule is not defined $\sum_{X\cap Y\neq\emptyset} m_1(X)m_2(Y) = 0$, i.e., if $\sum_{X\cap Y=\emptyset} m_1(X)m_2(Y) = 1$. On the other hand in this case we know $m(\emptyset) = 1$ even in the classic approach.

6 Conclusion

In this study, we have studied the nature of conflicting belief masses under open world assumption. Simple interpretation of sum of all conflicting masses from the Transferable Belief Model was analysed. Several variants of extension of frame of discernment with additional element(s) representing unexpected hypothesis(es) was suggested here. Finally, simple interpretation and extension approaches were mutually compared, and condition of their coincidence described.

We have to always keep in mind that the sum of all conflicting belief masses ($m(\emptyset)$) contains not only belief mass which should be assigned to new unexpected hypothesis(es), but also internal conflicts of single belief functions and conflict between belief functions (external conflict) whenever m is a basic belief assignment corresponding to a result of combination of two or more belief functions.

For a correct and complete interpretation of open world assumption it is recommended to include extra element(s) into used frame of discernment.

The presented theoretical results improve general understanding of both the sum of all conflicting masses and conflicts of belief functions under open world assumption. This, consequently, may improve combination of conflicting belief functions and interpretation of results of combination in practical applications under open world assumption.

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