Conjunctive Rules in the Theory of Belief Functions and Their Justification Through Decisions Models

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Abstract. In the paper we argue that aggregation rules in the theory of belief functions should be in accordance with underlying decision models, i.e. aggregation produced in conjunctive manner has to produce the order embedded to the union of partial orders constructed in each source of information; and if we take models based on imprecise probabilities, then such aggregation exists if the intersection of underlying credal sets is not empty. In the opposite case there is contradiction in information and the justifiable functional to measure it is the functional giving the smallest contradiction by applying all possible conjunctive rules. We give also the axiomatics of this contradiction measure.

Keywords: Belief functions · Contradiction measures · Aggregation rules · Conjunctive rules

1 Introduction

The theory of belief functions gives us many methods of information fusion. This procedure can be characterized as an aggregation of information sources that allows us to improve its reliability, precision, etc. There are many rules for aggregation of information in the frame of the belief function theory [7]. If each source of information is considered to be reliable then we can use conjunctive rules [2,8] of aggregation that should decrease uncertainty. Because belief functions have various interpretations, the optimal conjunctive rule does not exist. Therefore, in the first part of the paper we propose to justify the application of conjunctive rules based on the underlying decision models. This can be shortly described as follows. Suppose that by using each source of information we can construct the corresponding model of decision making described by a partial preference order on decisions. The union of these orders can be understood as the result of their conjunction. If orders do not contradict each other, then their

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conjunction can be embedded to a partial order. This way also explains when contradiction among sources of information exists or does not. We show that if we choose decisions based on models of imprecise probabilities then sources of information are contradictory if the intersection of the corresponding credal sets is not empty. We show that in this case the contradiction measure giving the smallest contradiction after applying possible conjunctive rules is justifiable, and we propose a number of axioms that leads to its unique choice.

The paper has the following structure. We give first the basic constructions concerning belief functions in Sect. 2. Then we describe some aggregation rules in Sect. 3, in particular, conjunctions rules that generalize the non-normalized Dempster's rule. In Sect. 4 we analyze the relation between decision models based on imprecise probabilities and aggregation rules, and finally, in Sect. 5 we give the axiomatics of the contradiction measure justified in the theory of imprecise probabilities.

2 Some Facts and Notions from the Theory of Belief Functions

Let $X = \{x_1, ..., x_n\}$ be a finite set also called the *frame of discernment* and let 2^X be the powerset of X. Any *belief function* [9] $Bel : 2^X \to [0,1]$ can be defined by a *basic belief assignment* (bba) $m : 2^X \to [0,1]$ with $\sum\limits_{B \in 2^X} m(B) = 1$ as

$$Bel(A) = \sum_{B \in 2^X | B \subseteq A} m(B).$$

A belief function is called *normalized* if $Bel(\emptyset) = 0$. The value $Bel(\emptyset)$ shows the *amount of contradiction* in information described by a belief function. Let Bel be a belief function on 2^X with bba m. Then a set $B \in 2^X$ is called a *focal element* if $m(B) > 0$. The set of all focal elements for a belief function Bel is called the *body of evidence*. If the body of evidence has only one focal element B, then the belief function is called *categorical* and it is denoted by $\eta_{\langle B \rangle}$. Obviously,

$$\eta_{\langle B \rangle}(A) = \begin{cases} 1, & B \subseteq A, \\ 0, & B \not\subseteq A. \end{cases}$$

Any belief function Bel on 2^X can represented as a sum of categorical belief functions as

$$Bel = \sum_{B \in 2^X} m(B) \eta_{\langle B \rangle},$$

where obviously m is the bba of Bel.

In the next we will use the following notations:

- M_{bel} is the set of all normalized belief functions on 2^X and the set of all belief functions including non-normalized ones is denoted by $\bar{M}_{bel}$;
- M_{pr} is the set of all probability measures on 2^X, i.e. normalized belief functions, for which $m(A) = 0$ if $|A| \geqslant 2$.

3 Aggregation Rules in the Theory of Belief Functions

The application of aggregation rules depends on prior information about information sources. We will discuss in detail the conjunctive rules. They are used if each source of information is assumed to be reliable. The following scheme gives us the general approach to construction of conjunctive rules [2,8]. Suppose we have two sources of information described by belief functions $Bel_i = \sum_{A \in 2^X} m_i(A)\eta_{\langle A \rangle}$, $i = 1, 2$. Then the general conjunctive rule can be defined with the help of a joint belief assignment $m : 2^X \times 2^X \to [0, 1]$ that satisfies the following conditions:

$$\begin{cases} \sum_{A \in 2^X} m(A, B) = m_2(B), \\ \sum_{B \in 2^X} m(A, B) = m_1(A). \end{cases} \tag{1}$$

The result of the conjunctive rule is defined as

$$Bel = \sum_{A,B \in 2^X} m(A, B)\eta_{\langle A \cap B \rangle}.$$

Let us notice that if we assume that the sources of information are independent, then the joint belief assignment m is defined as $m(A, B) = m_1(A)m_2(B)$, $A, B \in 2^X$. In the next the last rule of aggregation is referred as the classical conjunctive rule. Dempster's and Yager's rules of aggregation defined as

(1) Dempster's rule [4,9]: $Bel = \frac{1}{1-k} \sum_{A \cap B \neq \emptyset} m_1(A)m_2(B)\eta_{\langle A \cap B \rangle}$, where $k = \sum_{A \cap B = \emptyset} m_1(A)m_2(B)$;
(2) Yager's rule [10]: $Bel = \sum_{A \cap B \neq \emptyset} m_1(A)m_2(B)\eta_{\langle A \cap B \rangle} + k\eta_{\langle X \rangle}$, where k is defined as in (1);

are closely related to the classical conjunctive rule. As one can see they show how the result of the classical conjunctive rule can be transformed to the normalized belief function.

In the theory of belief functions you can find also other rules of aggregation. The *disjunctive rule* [7] is used if at least one source of information is reliable. The result of its application is defined through the joint belief assignment obeying the conditions (1) as

$$Bel = \sum_{A,B \in 2^X} m(A, B)\eta_{\langle A \cup B \rangle}.$$

If the sources of information are independent, then $m(A, B) = m_1(A)m_2(B)$ for all $A, B \in 2^X$, and also $Bel(A) = Bel_1(A)Bel_2(A)$ for all $A \in 2^X$.

The *mixture rule* is used if we can evaluate the reliability of each source of information. Let us assume that we have m sources of information described by belief functions Bel_i, $i = 1, ..., m$, and reliability of i-th source, $i = 1, ..., m$, is evaluated by a non-negative real number r_i and $\sum_{i=1}^{m} r_i = 1$, then the result of the mixture rule is defined as $Bel = \sum_{i=1}^{m} r_i Bel_i$. There are other approaches for accounting reliability of information sources, see for example, Shafer's rule [9]. Let us notice that one can find other rules of aggregation in the theory of belief functions [7] but they can be represented as a combination of the above basic aggregation rules.

4 Aggregation Rules and Decision Models

We will consider decision models in a view of probabilistic interpretation of normalized belief functions. Suppose that each decision is identified with a real valued function (gamble) f on X. The set of all such functions is denoted by K. Let the available information be described by a probability measure $P \in M_{pr}$. Then the preference order $\prec$ on K, based on suspected utility

$$E_P(f) = \sum_{i=1}^{n} f(x_i)P(\{x_i\}),$$

is defined as: decision f_2 is more preferable than decision f_1 ($f_1 \prec f_2$) iff $E_P(f_1) < E_P(f_2)$. If the available information is imprecise then it can be described by a belief function $Bel \in M_{bel}$ or the corresponding credal set $\mathbf{P} = \{P \in M_{pr} | P \geqslant Bel\}$, and we can use several decision rules from the theory of imprecise probabilities [1]:

(a) $f_1 \prec f_2$ iff $E_P(f_1) < E_P(f_2)$ for all $P \in \mathbf{P}$;
(b) $f_1 \prec f_2$ iff $\underline{E}_{\mathbf{P}}(f_1) < \underline{E}_{\mathbf{P}}(f_2)$, where $\underline{E}_{\mathbf{P}}(f) = \inf_{P \in \mathbf{P}} E_P(f)$;
(c) $f_1 \prec f_2$ iff $\bar{E}_{\mathbf{P}}(f_1) < \bar{E}_{\mathbf{P}}(f_2)$, where $\bar{E}_{\mathbf{P}}(f) = \sup_{P \in \mathbf{P}} E_P(f)$;
(d) $f_1 \prec f_2$ iff $\underline{E}_{\mathbf{P}}(f_1) < \underline{E}_{\mathbf{P}}(f_2)$ and $\bar{E}_{\mathbf{P}}(f_1) < \bar{E}_{\mathbf{P}}(f_2)$.

Let us notice that the relation $\prec$ is a strict partial order, and this seems to be natural that we cannot choose an optimal decision if we do not have sufficient information. In the next we will focuse on the rule (a) and analyze its behavior w.r.t. applying aggregation rules.

Assume that we have m information sources described by belief functions $Bel_i \in M_{bel}$, $i = 1, ..., m$. Assume also that each source of information is characterized by the preference order $\rho_i \subseteq K \times K$. Then the result of applying the conjunctive rule to orders ρ_i should be also the preference order $\rho \subseteq K \times K$ and obey the consensus condition $\rho_i \subseteq \rho$, $i = 1, ..., m$, with the meaning that each source of information is reliable. The application of the disjunction rule should give us an order ρ obeying the condition $\rho_i \supseteq \rho$, $i = 1, ..., m$, meaning that $(f_1, f_2) \in \rho$ if this preference is confirmed in each source of information. Observe that the conjunctive rule is not defined if an order $\rho \subseteq K \times K$ with

$\rho_i \subseteq \rho$, $i = 1, ..., m$, does not exist. In this case we say that sources of information are contradictory. The disjunctive rule always exists and it can be defined as $\rho = \bigcap_{i=1}^{m} \rho_i$.

Let us analyze how the above definitions agree with the aggregation rules in the theory of belief functions.

Lemma 1. *Let $Bel \in M_{bel}$ be the result of the conjunctive rule to belief functions $Bel_1, Bel_2 \in M_{bel}$. Let us consider preference orders ρ, ρ_1, ρ_2 that correspond to belief functions Bel, Bel_1, Bel_2 by decision rule (a). Then the preference order ρ for Bel agrees with orders ρ_1 and ρ_2.*

Proof. Clearly, $\mathbf{P}(Bel) \subseteq \mathbf{P}(Bel_i)$, $i = 1, 2$. Thus, applying decision rule (a) implies that $\rho_i \subseteq \rho$, $i = 1, 2$.

Lemma 2. *Let $Bel \in M_{bel}$ be the result of the disjunctive rule to belief functions $Bel_1, Bel_2 \in M_{bel}$. Then $\rho_i \supseteq \rho$, $i = 1, 2$.*

Proof. It is easy to see that $Bel \leqslant Bel_i$, $i = 1, 2$. Thus, $\mathbf{P}(Bel_i) \subseteq \mathbf{P}(Bel)$ and $\rho_i \supseteq \rho$, $i = 1, 2$.

Proposition 1. *Sources of information described by belief functions Bel_1, $Bel_2 \in M_{bel}$ are not contradictory iff $\mathbf{P}(Bel_1) \cap \mathbf{P}(Bel_2) \neq \emptyset$. In this case there is a conjunctive rule with the result $Bel \in M_{bel}$.*

Proof. Sufficiency. Assume that there exists a $P \in \mathbf{P}(Bel_1) \cap \mathbf{P}(Bel_2)$. Consider the preference order ρ, generated by a probability measure P. Obviously, $\rho_i \subseteq \rho$, $i = 1, 2$, where ρ_i is the preference order, generated by Bel_i.

Necessity. Let $\mathbf{P}(Bel_1) \cap \mathbf{P}(Bel_2) = \emptyset$. We will use the well known fact that if we have two disjoint closed convex sets in $\mathbb{R}^n$, then there is a hyperplane separating them. This fact for credal sets $\mathbf{P}(Bel_1)$ and $\mathbf{P}(Bel_2)$ can be formulated as: there is a $f \in K$ such that $E_P(f) > 0$ for all $P \in \mathbf{P}(Bel_1)$ and $E_P(f) < 0$ for all $P \in \mathbf{P}(Bel_1)$. Thus, f is more preferable than $-f$ according to the order ρ_1 and $-f$ is more preferable than f according to the order ρ_2. Clearly, a partial order ρ does not exist, because the above preferences contradict to its asymmetry property.

The existence of the conjunctive rule with properties indicated in the proposition follows from the results in [2].

5 The Axiomatics of Contradiction Measure

Let us consider the measure of contradiction, analyzed in [2,3,5]. Let $R(Bel_1, Bel_2)$ be the set of possible belief functions obtained by the conjunctive rules applied to $Bel_1, Bel_2 \in M_{bel}$. Then the measure of contradiction $Con : M_{bel} \times M_{bel} \to [0, 1]$ is defined as

$$Con(Bel_1, Bel_2) = \inf \{Bel(\emptyset) | Bel \in R(Bel_1, Bel_2)\}.$$

Let us consider its properties indicated in [2]. Further we will use the order $\preccurlyeq$ on $\bar{M}_{bel}$ called *specialization*. Let $Bel_1, Bel_2 \in \bar{M}_{bel}$, then $Bel_1 \preccurlyeq Bel_2$ iff there are representations $Bel_1 = \sum_{i=1}^{N} a_i \eta_{\langle A_i \rangle}$ and $Bel_2 = \sum_{i=1}^{N} a_i \eta_{\langle B_i \rangle}$, such that $\sum_{i=1}^{N} a_i = 1$, $a_i \geqslant 0$, $B_i \subseteq A_i$, $i = 1, ..., n$. It easy to see that $Bel_1 \preccurlyeq Bel_2$ implies $Bel_1 \leqslant Bel_2$ ($Bel_1(A) \leqslant Bel_2(A)$ for all $A \in 2^X$), but the opposite is not true in general [6].

Proposition 2. *The measure of contradiction* $Con : M_{bel} \times M_{bel} \to [0,1]$ *has the following properties:*

A1. $Con(Bel_1, Bel_2) = 0$ *for* $Bel_1, Bel_2 \in M_{bel}$ *iff* $\mathbf{P}(Bel_1) \cap \mathbf{P}(Bel_2) \neq \emptyset$.
A2. Let $Bel_1, Bel_2 \in M_{bel}$, *and let* $\mathcal{A}_i$ *be their corresponding bodies of evidence, then* $Con(Bel_1, Bel_2) = 1$ *iff* $A \cap B = \emptyset$ *for all* $A \in \mathcal{A}_1$ *and* $B \in \mathcal{A}_2$.
A3. $Con(Bel_1, Bel_2) = Con(Bel_2, Bel_1)$ *for all* $Bel_1, Bel_2 \in M_{bel}$*;*
A4. Let $Bel_1 \preccurlyeq Bel'_1$ *and* $Bel_2 \preccurlyeq Bel'_2$, *then* $Con(Bel_1, Bel_2) \leqslant Con(Bel'_1, Bel'_2)$*;*
A5. Let $Bel_1 = (1-a)Bel_1^{(1)} + aBel_1^{(2)}$ *and* $Bel_2 = (1-a)Bel_2^{(1)} + aBel_2^{(2)}$, *where* $a \in [0,1]$ *and* $Bel_i^{(k)} \in M_{bel}$, $i, k = 1, 2$. *Then* $Con(Bel_1, Bel_2) \leqslant (1-a)Con(Bel_1^{(1)}, Bel_2^{(1)}) + aCon(Bel_1^{(2)}, Bel_2^{(2)})$.
A6. Let $Con(Bel_1, Bel_2) = a$, *where* $a \in [0,1]$ *and* $Bel_1, Bel_2 \in M_{bel}$, *then there exist* $Bel_i^{(k)} \in M_{bel}$, $i, k = 1, 2$, *such that* $Bel_i = (1-a)Bel_i^{(1)} + aBel_i^{(2)}$, $i = 1, 2$, $Con(Bel_1^{(1)}, Bel_2^{(1)}) = 0$, *and* $Con(Bel_1^{(2)}, Bel_2^{(2)}) = 1$.

In addition,

(a) $Con(P_1, P_2) = 1 - \sum_{i=1}^{n} \min\{P_1(x_i), P_2(x_i)\}$, $P_i \in M_{pr}$, $i = 1, 2$;
(b) $Con(Bel_1, Bel_2) = \inf\{Con(P_1, P_2) | P_1 \in \mathbf{P}(Bel_1), P_2 \in \mathbf{P}(Bel_2)\}$.

We will consider properties A1–A6 as axioms for a measure of contradiction. We will show later that the measure of contradiction is uniquely defined by this system of axioms. Let us notice that axioms A1–A4 are considered in [5]. Axiom A1 describes the case when sources of information are non-contradictory, and similarly in axiom A2 we describe the case, when information sources are absolutely contradictory. In the last case any evidence (focal element) $A \in \mathcal{A}_1$ taken from the first source of information contradicts to any evidence $B \in \mathcal{A}_2$ from the second source of information. Axiom A3 is the symmetry axiom that follows from the problem statement. Let us observe that axiom A4 reflects the following. If $Bel_i \preccurlyeq Bel'_i$, then Bel'_i describes the same information but with higher precision. Therefore, axiom A4 says that increasing precision can lead to higher contradiction. Axioms A5 and A6 describe how we can evaluate contradiction by dividing information in each source on two parts: axiom A5 says that evaluation produced by dividing on two parts can gives us the result with the higher value of contradiction. Axiom A6 says that it is possible to divide information in each source on two parts such that we can extract parts of information that do not contradict each other and parts that are absolutely contradictory, and this separation defines the value of contradiction.

Lemma 3. *Belief functions $Bel_1, Bel_2 \in M_{bel}$ are absolutely contradictory, i.e. they obey the condition A2, iff there are disjoint sets $A, B \in 2^X$ $(A \cap B = \emptyset)$ such that $Bel_1(A) = Bel_2(B) = 1$.*

Proof. Necessity. Let we use notations from A2 and assume Bel_1, Bel_2 are absolutely contradictory. Let us choose $A = \bigcup\limits_{C \in \mathcal{A}_1} C$ and $B = \bigcup\limits_{C \in \mathcal{A}_2} C$. Then obviously $A \cap B = \emptyset$ and $Bel_1(A) = Bel_2(B) = 1$.

Sufficiency. Assume that there are $A, B \in 2^X$ such that $Bel_1(A) = Bel_2(B) = 1$ and $A \cap B = \emptyset$. Then $\sum\limits_{C \subseteq A} m_1(C) = 1$ and $\sum\limits_{C \subseteq B} m_1(C) = 1$. This means that any focal element for Bel_1 is a subset of A and any focal element for Bel_2 is a subset of B, i.e. belief functions Bel_1, Bel_2 are absolutely contradictory.

Lemma 4. *Let a functional $\Phi : M_{pr} \times M_{pr} \to [0,1]$ obey axioms A1, A2 and A6. Then*

$$\Phi(P_1, P_2) = 1 - \sum_{i=1}^{n} \min\{P_1(x_i), P_2(x_i)\}, P_1, P_2 \in M_{pr}.$$

Proof. In the case of probability measures in possible representations $P_i = (1-a)P_i^{(1)} + aP_i^{(2)}$, where $P_i^{(k)} \in M_{pr}$, $i, k = 1, 2$, $P_1^{(1)} = P_1^{(2)}$, and $P_2^{(1)}, P_2^{(2)}$ are absolutely contradictory, the parameter a is uniquely defined as

$$a = 1 - \sum_{i=1}^{n} \min\{P_1(x_i), P_2(x_i)\}.$$

The probability measures used in these representations can be chosen as

(1) $P_i^{(1)}(\{x_k\}) = \frac{1}{1-a} \min\{P_1(x_k), P_2(x_k)\}$, $i = 1, 2$, for $a \in [0, 1)$;
(2) $P_i^{(2)}(\{x_k\}) = \frac{1}{a}\left(P_i(\{x_k\}) - \min\{P_1(\{x_k\}), P_2(\{x_k\})\}\right)$ for $a \in [0, 1)$;
(3) if $a = 1$, then a probability measure $P_1^{(1)} = P_1^{(2)}$ can be chosen arbitrarily;
(4) if $a = 0$, then absolutely contradictory probability measures $P_2^{(1)}, P_2^{(2)} \in M_{pr}$ can be chosen arbitrary.

Thus, the result from the lemma is proved.

Theorem 1. *Let a functional $\Phi : M_{bel} \times M_{bel} \to [0,1]$ obey axioms A1, A2, A4, and A6. Then it coincides with the contradiction measure Con on $M_{bel} \times M_{bel}$.*

Proof. Let us notice that Lemma 4 implies that functionals Φ and Con coincide on $M_{pr} \times M_{pr}$. Let us show first that $\Phi(Bel_1, Bel_2) \leqslant Con(Bel_1, Bel_2)$ for all $Bel_1, Bel_2 \in M_{bel}$. Property (b) implies that there exist $P_i \in M_{pr}$, $i = 1, 2$, such that $Con(Bel_1, Bel_2) = Con(P_1, P_2)$ and $Bel_i \preccurlyeq P_i$, $i = 1, 2$. Because $\Phi(P_1, P_2) = Con(P_1, P_2)$, axiom A4 implies that $\Phi(Bel_1, Bel_2) \leqslant \Phi(P_1, P_2) = Con(Bel_1, Bel_2)$.

Let us prove that $Con(Bel_1, Bel_2) \leqslant \Phi(Bel_1, Bel_2)$ for all $Bel_1, Bel_2 \in M_{bel}$. Let us assume to the contrary that $Con(Bel_1, Bel_2) > \Phi(Bel_1, Bel_2)$ for some $Bel_1, Bel_2 \in M_{bel}$. Then by axiom A6 there are representations

$$Bel_i = (1-a)Bel_i^{(1)} + aBel_i^{(2)}, i = 1, 2,$$

such that $\Phi(Bel_1, Bel_2) = a$, $\mathbf{P}(Bel_1^{(1)}) \cap \mathbf{P}(Bel_2^{(1)}) \neq \emptyset$, and $\mathbf{P}(Bel_1^{(2)}) \cap \mathbf{P}(Bel_2^{(2)}) = \emptyset$. Consider probability measures

$$P_i = (1-a)P + aP_i^{(2)}, i = 1, 2,$$

where $P \in \mathbf{P}(Bel_1^{(1)}) \cap \mathbf{P}(Bel_2^{(1)})$ and $P_i^{(2)} \in \mathbf{P}(Bel_i^{(2)})$, $i = 1, 2$. Obviously, probability measures $P_1^{(2)}$ and $P_2^{(2)}$ are absolutely contradictory, thus, $\Phi(Bel_1, Bel_2) = Con(P_1, P_2)$. In addition, $P_i \in \mathbf{P}(Bel_i)$, $i = 1, 2$. Thus, by property (b) from Proposition 2, we get $Con(Bel_1, Bel_2) \leqslant Con(P_1, P_2)$, but this contradicts to our assumption.

6 Conclusion

In this paper we show that the choice of aggregation rules has to be in accordance with the underlying decision models, and if we take decision models based on imprecise probabilities then contradiction exists if the intersection of underlying credal sets is not empty. We show that in this case the contradiction measure giving the smallest contradiction by applying possible conjunctive rules is justifiable, and we give the axiomatics of this measure. The important topic of the next research can be the analysis of relations obtained as union of partial preference orders on decisions and how these relations can be used for decision making in case of contradictory information.

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