Maximum and Minimum Principles for the Generalized Fractional Diffusion Problem with a Scale Function-Dependent Derivative

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Abstract In the paper, we prove the necessary condition for the extremum existence in terms of the generalized function-dependent fractional derivatives. By using these results we extend the maximum and minimum principles, known from the theory of differential equations and from diffusion problems with the Caputo derivative of constant or distributed order. We study the fractional diffusion problem, where time evolution is determined by the scale function-dependent Caputo derivative and show that the maximum or respectively minimum principle is valid, provided the source function is a non-positive or a non-negative one in the domain. As an application, we demonstrate how the sign of the classical solution is controlled by the initial and boundary conditions.

Keywords Maximum principle · Minimum principle · Fractional necessary condition for extremum existence · Generalized diffusion equation · Scale function-dependent fractional derivatives

1 Introduction

The paper is devoted to the discussion of properties of classical solutions to generalized multidimensional time-fractional diffusion problems. We shall derive the corresponding maximum/minimum principles and apply them in control of the sign of the solutions.

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We consider the time-fractional diffusion problem with the equation including scale function-dependent fractional derivative:

$$ {}^{c}D^{\alpha}_{0+;[z],t}u(x,t) = L(u) + F(x,t) \quad (x,t) \in \Omega_T := G \times (0,T], \tag{1} $$

where operator L looks as follows

$$ L(u) = \sum_{k=1}^{n} \left(p(x)\frac{\partial^2 u}{\partial x_k^2} + \frac{\partial p}{\partial x_k}\frac{\partial u}{\partial x_k} \right) - q(x)u \tag{2} $$

$$ = p(x)\Delta u + (grad(p), grad(u)) - q(x)u, $$

functions $p \in C^1(\bar{G})$, $q \in C(\bar{G})$ fulfill conditions $p(x) > 0$, $q(x) \geq 0$, $\forall x \in \bar{G}$, G is an open and bounded region in R^n, $\bar{G}$ denotes its closure.

Diffusion equation (1) is subjected to the following initial condition

$$ u|_{t=0} = u_0(x), \quad x \in \bar{G} \tag{3} $$

and boundary condition

$$ u|_S = v(x,t), \quad (x,t) \in S \times [0,T], \tag{4} $$

where S is the boundary of region $G \in R^n$.

The diffusion and advection-diffusion including the generalized fractional derivative were introduced in [1, 2]. Their solutions and properties were studied by means of numerical methods [1–3]. Our aim is to derive the analytical results describing the solutions via maximum/minimum principles. These theorems are developed in the classical differential equations theory as well as in the time-fractional diffusion problems [4–7] and are an important tool in proving the uniqueness results and theorems on continuous dependence of solutions on the problem data. We shall obtain analogous maximum/minimum theorems for models with a fractional time-derivative dependent on the scale function.

The paper is organized as follows. Section 2 contains definitions of the classical solution and of the generalized fractional derivative of the Caputo type, its properties and the preliminary results on the existence condition for maximum and minimum. The version, known in calculus for the first-order derivative, is extended to the case of a two function-dependent fractional derivative. The next part, Sect. 3, includes our main results - maximum and minimum principles for the diffusion problem with the generalized Caputo derivative with respect to the time variable and their application in controlling the sign of the classical solution. The brief conclusion section closes the paper.

2 Preliminaries

In this section, we introduce the basic definitions and properties of the generalized fractional derivative and prove the necessary conditions of the extremum existence expressed in terms of this operator.

First, we recall the notion of the classical solution and define the generalized time-fractional derivative appearing in Eq. (1). In the definition of the classical solution, we restrict the source function and the functions determining the initial and boundary conditions to the continuous ones.

Definition 1 Function u, determined in region $\bar{\Omega}_T := \bar{G} \times [0, T]$ will be called a classical solution to problem (1)–(4) with $F \in C(\Omega_T), u_0 \in C(\bar{G}), v \in C(S \times [0, T])$ iff

$$u \in CW_T(G) := C(\bar{\Omega}_T) \cap W_t^1((0, T)) \cap C_x^2(G) \tag{5}$$

and u fulfills Eq. (1), initial condition (3) and boundary condition (4). $W_t^1((0, T)) \subset C^1((0, T])$ is a function space such that $f \in W_t^1((0, T)) \Longleftrightarrow f' \in L(0, T)$ i.e. derivatives are determined on $(0, T)$ and absolutely integrable in the Lebesgue sense.

Now, we introduce the scale and weight function-dependent fractional derivative which was defined in [8]. We restrict this brief review to the case of order $\alpha \in (0, 1)$ and the left-sided differential operator of the Caputo type. Let us point out that in fractional calculus analogous derivatives are defined and studied for higher orders, in Caputo and Riemann–Liouville versions and in both cases: as the left- and right-sided operators [8, 9].

Definition 2 Let $\alpha \in (0, 1)$. The generalized (two function-dependent) left derivative of the Caputo type is defined as follows

$$^{c}D^{\alpha}_{0+;[z,w]} f(t) = I^{1-\alpha}_{0+;[z,w]} (D_{[z,w,L]} f)(t), \tag{6}$$

where $I^{1-\alpha}_{0+;[z,w]}$ denotes the generalized function-dependent integral operator

$$I^{1-\alpha}_{0+;[z,w]} f(t) = \frac{1}{w(t)\Gamma(1-\alpha)} \int_0^t \frac{w(s)z'(s)f(s)}{[z(t)-z(s)]^{\alpha}} ds \tag{7}$$

and the $D_{[z,w,L]}$-operator is given below

$$D_{[z,w,L]} f(t) = \frac{[w(t)f(t)]'}{w(t)z'(t)}, \tag{8}$$

with weight function $w \in C[0, b]$, scale function $z \in C^1[0, b]$ and $w > 0,\ z' > 0$ in interval $[0, b]$.

The above definition extends the notion of the standard Caputo derivative which is recovered in the case:

$$z(t) = t \quad w(t) = 1 \quad t \in [0, b].$$

We refer the reader to the discussion on generalized two-function dependent fractional derivatives (left and right) enclosed in monograph [9] and to further results in [8], where Caputo type derivatives are constructed and studied as well. Let us recall the differentiation formulas for analogs of power functions when $\beta > 0$:

$$^{c}D^{\alpha}_{0+;[z,w]} \frac{(z(t) - z(0))^{\beta-1}}{w(t)} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \frac{(z(t) - z(0))^{\beta-\alpha-1}}{w(t)}. \tag{9}$$

In particular, for $w = 1$ we have the following simpler definition and differentiation formula:

$$^{c}D^{\alpha}_{0+;[z]} f(t) = I^{1-\alpha}_{0+;[z]} (D_{[z,L]} f)(t), \tag{10}$$

where $I^{1-\alpha}_{0+;[z]}$ denotes the generalized scale function-dependent integral operator

$$I^{1-\alpha}_{0+;[z]} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{z'(s) f(s)}{[z(t) - z(s)]^{\alpha}} ds \tag{11}$$

and the $D_{[z,L]}$-operator is given below

$$D_{[z,L]} f(t) = \frac{f'(t)}{z'(t)}, \tag{12}$$

with scale function $z \in C^1[0, b]$ and $z' > 0$ in interval $[0, b]$. The differentiation formula (9) is of the form

$$^{c}D^{\alpha}_{0+;[z]} (z(t) - z(0))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (z(t) - z(0))^{\beta-\alpha-1}. \tag{13}$$

Now, we shall extend the necessary condition for the maximum existence which was proved in [4–6] in the case of the Caputo derivative and in [7] for the Caputo derivative of distributed order. In the theorem below, we formulate the analogous result for the two-function dependent fractional derivative defined by formula (6). It is a known fact from calculus that if function $f \in W^1_t((0, T)) \cap C[0, T]$ attains a maximum at point $t_0 \in (0, T]$, then $f'(t_0) = 0$. It can be expressed as follows in terms of generalized fractional derivatives.

Theorem 1 *Let function* $f \in W^1_t((0, T)) \cap C([0, T])$ *attain its maximum on interval* $[0, T]$ *at point* $s = t_0$, $t_0 \in (0, T]$. *Then, the generalized two function-dependent Caputo derivative of function* f *fulfills the following inequality for any order* $\alpha \in (0, 1)$

$$^{c}D^{\alpha}_{0+;[z,w]} \frac{f}{w}(t_0) \geq 0. \tag{14}$$

Proof We define an auxiliary function:

$$g(s) := \frac{f(t_0) - f(s)}{w(s)} \quad s \in [0, T]. \tag{15}$$

It is easy to check that function g has the following properties:

$$g(s) \geq 0 \quad s \in [0, T] \tag{16}$$

$${}^c D^{\alpha}_{0+;[z,w]} g(t) = -{}^c D^{\alpha}_{0+;[z,w]} \frac{f}{w}(t) \quad t \in [0, T] \tag{17}$$

$$|g(s)| \leq C_\epsilon |z(t_0) - z(s)| \quad s \in [\epsilon, T], \epsilon \in (0, T), \tag{18}$$

which follow from the fact that: $f \in W^1_t\left((0, t)\right), z \in C^1[0, T], w \in C[0, T], z' > 0, w > 0$. Now, we rewrite the derivative and obtain for any $\epsilon \in (0, t_0)$

$${}^c D^{\alpha}_{0+;[z,w]} g(t_0) = I^{1-\alpha}_{0+;[z,w]} (D_{[z,w,L]} g)(t_0) \tag{19}$$

$$= \frac{1}{w(t_0)\Gamma(1-\alpha)} \int_0^{t_0} \frac{[w(s)g(s)]'}{[z(t_0) - z(s)]^\alpha} ds$$

$$= \frac{1}{w(t_0)\Gamma(1-\alpha)} \int_0^{\epsilon} \frac{[w(s)g(s)]'}{[z(t_0) - z(s)]^\alpha} ds + \frac{1}{w(t_0)\Gamma(1-\alpha)} \int_\epsilon^{t_0} \frac{[w(s)g(s)]'}{[z(t_0) - z(s)]^\alpha} ds$$

$$= I_1 + I_2.$$

Let us note that $f \in W^1_t\left((0, t)\right)$ yields $wg \in W^1_t\left((0, t)\right)$, therefore $(wg)' \in L\left((0, T)\right)$, which means that

$$\forall \delta > 0 \quad \exists \epsilon > 0 \quad |I_1| < \delta. \tag{20}$$

For the I_2 - part we have

$$I_2 = \lim_{s \to t_0} \frac{(z(t_0) - z(s))^{-\alpha} w(s) g(s)}{w(t_0)\Gamma(1-\alpha)} - \frac{(z(t_0) - z(\epsilon))^{-\alpha} w(\epsilon) g(\epsilon)}{w(t_0)\Gamma(1-\alpha)}$$

$$+ \frac{1}{w(t_0)\Gamma(-\alpha)} \int_\epsilon^{t_0} \frac{w(s)g(s)z'(s)}{[z(t_0) - z(s)]^{\alpha+1}} ds.$$

The limit in the above equality vanishes:

$$\lim_{s \to t_0} \frac{|(z(t_0) - z(s))^{-\alpha} w(s) g(s)|}{w(t_0)\Gamma(1-\alpha)}$$

$$\leq ||w|| \lim_{s \to t_0} \frac{|(z(t_0) - z(s))^{-\alpha}| \cdot C_\epsilon |(z(t_0) - z(s))|}{w(t_0)\Gamma(1-\alpha)} = 0,$$

where we applied property (18) and $||\cdot||$ denotes the supremum norm in the $C[0,T]$-space. From $\Gamma(-\alpha)<0$ for $\alpha\in(0,1)$ and property (16) we obtain $I_2\leq 0$ for any $\epsilon\in(0,T)$. Next, from properties (17), (20) we obtain (14). □

The above theorem holds in the case of the scale function-dependent derivative defined by formula (10) when $w=1$.

Corollary 1 *Let function* $f\in W_t^1((0,T))\cap C([0,T])$ *attain its maximum on interval* $[0,T]$ *at point* $s=t_0$, $t_0\in(0,T]$. *Then, the generalized scale function-dependent Caputo derivative of function* f *fulfills the following inequality for any order* $\alpha\in(0,1)$

$$ {}^{c}D^{\alpha}_{0+;[z]}f(t_0)\geq 0. \tag{21}$$

The necessary condition for the minimum existence can also be expressed in terms of a two function-dependent derivative.

Theorem 2 *Let function* $f\in W_t^1((0,T))\cap C([0,T])$ *attain its minimum on interval* $[0,T]$ *at point* $s=t_0$, $t_0\in(0,T]$. *Then, the generalized two function-dependent Caputo derivative of function* f *fulfills the following inequality for any order* $\alpha\in(0,1)$

$$ {}^{c}D^{\alpha}_{0+;[z,w]}\frac{f}{w}(t_0)\leq 0. \tag{22}$$

Proof In the proof, we apply the auxiliary function given in (15). Function g now obeys the inequality:

$$g(s)\leq 0\quad s\in[0,T] \tag{23}$$

and it also fulfills (17), (18). Similar to the previous proof, we split the derivative and obtain for any $\epsilon\in(0,t_0)$

$$ {}^{c}D^{\alpha}_{0+;[z,w]}g(t_0)= \tag{24}$$

$$=\frac{1}{w(t_0)\Gamma(1-\alpha)}\int_0^{\epsilon}\frac{[w(s)g(s)]'}{[z(t_0)-z(s)]^{\alpha}}ds+\frac{1}{w(t_0)\Gamma(1-\alpha)}\int_{\epsilon}^{t_0}\frac{[w(s)g(s)]'}{[z(t_0)-z(s)]^{\alpha}}ds$$

$$=I_1+I_2.$$

Let us note that again for the first term implication (20) holds. For the I_2 - term we have the equality

$$I_2=\lim_{s\to t_0}\frac{(z(t_0)-z(s))^{-\alpha}w(s)g(s)}{w(t_0)\Gamma(1-\alpha)}-\frac{(z(t_0)-z(\epsilon))^{-\alpha}w(\epsilon)g(\epsilon)}{w(t_0)\Gamma(1-\alpha)}$$

$$+\frac{1}{w(t_0)\Gamma(-\alpha)}\int_{\epsilon}^{t_0}\frac{w(s)g(s)z'(s)}{[z(t_0)-z(s)]^{\alpha+1}}ds.$$

The limit in the above equality vanishes as was shown in the previous proof. From $\Gamma(-\alpha) < 0$ for $\alpha \in (0, 1)$ and property (23) we obtain $I_2 \geq 0$ for any $\epsilon \in (0, T)$. Next, from properties (17), (20) we obtain (22). □

From the above necessary condition for the minimum, formulated for the two function-dependent derivative, we obtain the following corollary for the case $w = 1$.

Corollary 2 *Let function* $f \in W_t^1((0, T)) \cap C([0, T])$ *attain its minimum on interval* $[0, T]$ *at point* $s = t_0$, $t_0 \in (0, T]$. *Then, the generalized scale function-dependent Caputo derivative of function* f *fulfills the following inequality for any order* $\alpha \in (0, 1)$

$$ {}^c D^{\alpha}_{0+;[z]} f(t_0) \leq 0. \tag{25}$$

3 Main Results

We shall study the generalized fractional diffusion problem with the diffusion equation (1), the initial condition given in (3) and the boundary conditions determined in (4). Our aim is to derive the maximum and minimum principles for the multidimensional case and to apply these results in a preliminary study of the properties of classical solutions to the problem.

First, applying Corollary 1, we prove the theorem which generalizes the classical maximum principle as well as the result proved in [4–6] for fractional diffusion problems. We extend the fractional maximum principle to the case, where in the diffusion equation the Caputo derivative with respect to the time-variable is replaced with the scale function-dependent derivative of the Caputo type given in (10).

Theorem 3 *Let function* $u \in CW_T(G) := C(\bar{\Omega}_T) \cap W_t^1((0, T)) \cap C_x^2(G)$ *be the classical solution of the generalized time-fractional diffusion equation (1) in region* $\Omega_T := G \times (0, T]$, $G \subset R^n$ *and let* $F(x, t) \leq 0, \quad (x, t) \in \Omega_T$. *Then, either solution* u *is non-positive in* $\bar{\Omega}_T$ *or it attains the positive maximum on set* S_G^T *which means*

$$u(x, t) \leq \max\{0, \max_{(x,t)\in S_G^T} u(x, t)\} \quad (x, t) \in \bar{\Omega}_T, \tag{26}$$

where $S_G^T := (\bar{G} \times \{0\}) \cup (S \times [0, T])$.

Proof Let us assume that thesis (26) is not valid. Then point (x_0, t_0), $x_0 \in G$, $0 \leq t_0 \leq T$ exists such that

$$u(x_0, t_0) > \max_{(x,t)\in S_G^T} \{0, u(x, t)\} = M > 0. \tag{27}$$

We define number $\epsilon := u(x_0, t_0) - M > 0$ and the following auxiliary function:

$$f(x,t) := u(x,t) + \frac{\epsilon}{2}\frac{z(T)-z(t)}{z(T)} \quad (x,t) \in \bar{\Omega}_T. \tag{28}$$

From this definition and assumptions of the theorem we have:

$$f(x,t) \le u(x,t) + \frac{\epsilon}{2} \quad (x,t) \in \bar{\Omega}_T$$

$$f(x_0,t_0) \ge u(x_0,t_0) = \epsilon + M \ge \epsilon + u(x,t) \ge \epsilon + f(x,t) - \frac{\epsilon}{2} \quad (x,t) \in S_G^T.$$

From the above inequality we infer that function f cannot attain its maximum on the S_G^T-part of the boundary of region Ω_T. Therefore point $(x_1,t_1) \in \bar{\Omega}_T$ exists such that $x_1 \in G$ and $0 < t_1 \le T$ and function f attains its maximum at (x_1,t_1). At this point the following inequality is fulfilled

$$f(x_1,t_1) \ge f(x_0,t_0) \ge \epsilon + M > \epsilon.$$

From Corollary 1 and the necessary and sufficient conditions of the existence of the maximum in region Ω_T we obtain the following set of conditions

$${}^c D^{\alpha}_{0+;[z],t} f(x_1,t_1) \ge 0 \quad \alpha \in (0,1)$$

$$grad(f)|_{(x_1,t_1)} = 0 \quad \Delta f|_{(x_1,t_1)} \le 0$$

and the relations for derivatives

$${}^c D^{\alpha}_{0+;[z],t} u(x,t) = {}^c D^{\alpha}_{0+;[z],t} f(x,t) + \tfrac{\epsilon}{2z(T)}\tfrac{(z(t)-z(0))^{1-\alpha}}{\Gamma(2-\alpha)}, \tag{29}$$

$$grad(f) = grad(u), \qquad \Delta u(x,t) = \Delta f(x,t). \tag{30}$$

Now, we are ready to test the behavior of the generalized diffusion operator at point (x_1,t_1)

$$\left({}^c D^{\alpha}_{0+;[z],t} u(x,t) - L(u)\right)|_{(x_1,t_1)}$$

$$= {}^c D^{\alpha}_{0+;[z],t} f(x_1,t_1) + \frac{\epsilon}{2z(T)}\frac{(z(t_1)-z(0))^{1-\alpha}}{\Gamma(2-\alpha)}$$

$$-p(x_1)\Delta f(x_1,t_1) - \left(grad(p)|_{x_1}, grad(f)|_{(x_1,t_1)}\right)$$

$$+q(x_1)\left(f(x_1,t_1) - \frac{\epsilon}{2}\frac{z(T)-z(t_1)}{z(T)}\right) - F(x_1,t_1)$$

$$\ge \frac{\epsilon}{2z(T)}\frac{(z(t_1)-z(0))^{1-\alpha}}{\Gamma(2-\alpha)} > 0.$$

We note that at point (x_1, t_1) the following inequality holds

$$\left({}^{c}D^{\alpha}_{0+;[z],t}u(x,t) - L(u)\right)|_{(x_1,t_1)} > 0,$$

which means that function u is not a solution to Eq. (1). Therefore the assumption (27) is incorrect and the thesis (26) is valid. □

The above theorem is called the maximum principle. We note that in the case $F(x,t) \geq 0$ an analogous result can be formulated. We prove the minimum principle below. The proof is analogous to the proof of the maximum principle, but we now use the condition from Corollary 2.

Theorem 4 *Let function $u \in CW_T(G) := C(\bar{\Omega}_T) \cap W^1_t((0,T)) \cap C^2_x(G)$ be the classical solution of the generalized time-fractional diffusion equation (1) in the region $\Omega_T := G \times (0,T]$, $G \subset R^n$ and let $F(x,t) \geq 0$, $(x,t) \in \Omega_T$. Then, either solution u is non-negative in $\bar{\Omega}_T$ or it attains the negative minimum on set S^T_G which means*

$$u(x,t) \geq \min\{0, \min_{(x,t)\in S^T_G} u(x,t)\} \quad (x,t) \in \bar{\Omega}_T, \tag{31}$$

where $S^T_G := (\bar{G} \times \{0\}) \cup (S \times [0,T])$.

Proof Let us assume that thesis (31) is not valid. Then point (x_0, t_0), $x_0 \in G$, $0 \leq t_0 \leq T$ exists such that

$$u(x_0,t_0) < \min_{(x,t)\in S^T_G}\{0, u(x,t)\} = M_1 < 0. \tag{32}$$

We define number $\epsilon := u(x_0,t_0) - M_1 < 0$ and the following auxiliary function:

$$f_1(x,t) := u(x,t) + \frac{\epsilon}{2}\frac{z(T) - z(t)}{z(T)} \quad (x,t) \in \bar{\Omega}_T. \tag{33}$$

From this definition and by assumptions we obtain:

$$f_1(x,t) \geq u(x,t) + \frac{\epsilon}{2} \quad (x,t) \in \bar{\Omega}_T$$

$$f_1(x_0,t_0) \leq u(x_0,t_0) = \epsilon + M_1 \leq \epsilon + u(x,t) \leq \epsilon + f_1(x,t) - \frac{\epsilon}{2} \quad (x,t) \in S^T_G.$$

From the above inequality we infer that function f_1 cannot attain its minimum on the S^T_G-part of the boundary of region Ω_T. Therefore point $(x_1,t_1) \in \bar{\Omega}_T$ exists such that $x_1 \in G$ and $0 < t_1 \leq T$ and function f_1 attains its minimum at (x_1,t_1). At this point the following inequality is fulfilled

$$f_1(x_1,t_1) \leq f_1(x_0,t_0) \leq \epsilon + M_1 < \epsilon.$$

From Corollary 2 and the necessary and sufficient conditions of the existence of the minimum in region Ω_T, we obtain the following set of conditions

$$ {}^{c}D^{\alpha}_{0+;[z],t} f_1(x_1,t_1) \leq 0 \quad \alpha \in (0,1) $$

$$ grad(f_1)|_{(x_1,t_1)} = 0 \quad \Delta f_1|_{(x_1,t_1)} \geq 0 $$

and the relations for derivatives (29), (30), where we have replaced function f by f_1, hold. Now, we analyze the behavior of the generalized diffusion operator at point (x_1, t_1)

$$ \left({}^{c}D^{\alpha}_{0+;[z],t} u(x,t) - L(u)\right)|_{(x_1,t_1)} $$

$$ = {}^{c}D^{\alpha}_{0+;[z],t} f_1(x_1,t_1) + \frac{\epsilon}{2z(T)} \frac{(z(t_1)-z(0))^{1-\alpha}}{\Gamma(2-\alpha)} $$

$$ -p(x_1)\Delta f_1(x_1,t_1) - \left(grad(p)|_{x_1}, grad(f_1)|_{(x_1,t_1)}\right) $$

$$ +q(x_1)\left(f_1(x_1,t_1) - \frac{\epsilon}{2}\frac{z(T)-z(t_1)}{z(T)}\right) - F(x_1,t_1) $$

$$ \leq \frac{\epsilon}{2z(T)} \frac{(z(t_1)-z(0))^{1-\alpha}}{\Gamma(2-\alpha)} < 0. $$

We note that at point (x_1, t_1) the following inequality holds

$$ \left({}^{c}D^{\alpha}_{0+;[z],t} u(x,t) - L(u)\right)|_{(x_1,t_1)} < 0, $$

which means that function u is not a solution to Eq. (1). Therefore the assumption (32) is incorrect and the thesis (31) is valid. □

The derived minimum and maximum principles can be applied in generalized fractional diffusion problems to prove the uniqueness results and properties of classical solutions. One of the applications are the following corollaries on controlling the sign of the classical solution.

Corollary 3 *Let assumptions of Theorem 4 be fulfilled and*

$$ \min_{(x,t)\in S_G^T} u(x,t) \geq 0. $$

Then, the classical solution u is non-negative.

Proof From Theorem 4 we immediately obtain the thesis

$$ u(x,t) \geq \min\{0, \min_{(x,t)\in S_G^T} u(x,t)\} \geq 0 \quad (x,t) \in \bar{\Omega}_T $$

which means that in the case $F(x, t) \geq 0$ we control the value of the classical solution u via the initial and boundary conditions on the S_G^T-part of the boundary. □

Corollary 4 *Let assumptions of Theorem 3 be fulfilled and*

$$\max_{(x,t)\in S_G^T} u(x, t) \leq 0.$$

Then, the classical solution u is non-positive.

4 Conclusion

In the paper, we extended the necessary conditions for the extremum existence to the version expressed in terms of the generalized scale and weight function-dependent fractional derivative. From these conditions, the corollaries follow, where the existence of minimum or maximum at the given point is connected with the corresponding inequality for the left scale and weight function-dependent derivative of the Caputo type.

The obtained necessary conditions were applied in the proof of maximum and minimum principles for time-fractional diffusion problem (1)–(4). These theorems generalize the known classical results as well as the maximum/minimum principle for diffusion problems with a time-fractional Caputo derivative. In the partial differential equations theory, both for the problems of integer and non-integer order, the maximum/minimum principles are applied to prove uniqueness results for classical solutions and to control the sign of the solution. We demonstrated for the generalized diffusion problem of type (1)–(4) that similar results are valid and follow from the maximum/minimum principles. Further applications are still under investigation.

References

1. Xu, Y., He, Zh, Agrawal, O.P.: Numerical and analytical solutions of new generalized fractional diffusion equation. Comput. Math. Appl. **66**(10), 2019–2029 (2013)
2. Xu, Y., Agrawal, O.P.: Numerical solutions and analysis of diffusion for new generalized fractional advection-diffusion equations. Cent. Eur. J. Phys. **11**(10), 1178–1193 (2013)
3. Xu, Y., He, Zh, Xu, Q.: Numerical solutions of fractional advection-diffusion equations with a kind of new generalized fractional derivative. Int. J. Comput. Math. **91**(3), 588–600 (2014)
4. Luchko, Y.: Boundary value problems for the generalized time-fractional diffusion equation of distributed order. Fract. Calc. Appl. Anal. **12**(4), 409–422 (2009)
5. Luchko, Y.: Maximum principle and its application for the time-fractional diffusion equation. Fract. Calc. Appl. Anal. **14**(1), 110–124 (2011)
6. Luchko, Y.: Maximum principle for the generalized time-fractional diffusion equation. J. Math. Anal. Appl. **351**(1), 218–223 (2009)

7. Al-Refai, M., Luchko, Y.: Analysis of fractional diffusion equations of distributed order: maximum principles and their applications. Analysis **36**(2), 123–133 (2015)
8. Agrawal, O.P.: Some generalized fractional calculus operators and their applications in integral equations. Fract. Calc. Appl. Anal. **15**(4), 700–711 (2012)
9. Kilbas, A.A., Srivastawa, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)