(s,p)-Valent Functions

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Abstract We introduce the notion of $(\mathcal{F},p)$-valent functions. We concentrate in our investigation on the case, where $\mathcal{F}$ is the class of polynomials of degree at most s. These functions, which we call (s,p)-valent functions, provide a natural generalization of p-valent functions (see Hayman, Multivalent Functions, 2nd ed, Cambridge Tracts in Mathematics, vol 110, 1994). We provide a rather accurate characterizing of (s,p)-valent functions in terms of their Taylor coefficients, through "Taylor domination", and through linear non-stationary recurrences with uniformly bounded coefficients. We prove a "distortion theorem" for such functions, comparing them with polynomials sharing their zeroes, and obtain an essentially sharp Remez-type inequality in the spirit of Yomdin (Isr J Math 186:45–60, 2011) for complex polynomials of one variable. Finally, based on these results, we present a Remez-type inequality for (s,p)-valent functions.

1 Introduction

Let us introduce the notion of "$(\mathcal{F},p)$-valent functions". Let $\mathcal{F}$ be a class of functions to be specified later. A function f regular in a domain $\Omega \subset \mathbb{C}$ is called $(\mathcal{F},p)$-valent in Ω if for any $g \in \mathcal{F}$ the number of solutions of the equation $f(z) = g(z)$ in Ω does not exceed p.

For example, the classic p-valent functions are obtained for $\mathcal{F}$ being the class of constants, these are functions f for which the equation $f = c$ has at most p solutions in Ω for any c. There are many other natural classes $\mathcal{F}$ of interest, like rational functions, exponential polynomials, quasi-polynomials, etc. In particular, for the class $\mathcal{R}_s$ consisting of rational functions $R(z)$ of a fixed degree s, the number of zeroes of $f(z) - R(z)$ can be explicitly bounded for f solving linear ODEs with

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polynomial coefficients (see, e.g. [4]). Presumably, the collection of $(\mathcal{R}_s, p)$-valent functions with explicit bounds on p (as a function of s) is much wider, including, in particular, "monogenic" functions (or "Wolff-Denjoy series") of the form $f(z) = \sum_{j=1}^{\infty} \frac{\gamma_j}{z-z_j}$ (see, e.g. [13, 16] and references therein).

However, in this note we shall concentrate on another class of functions, for which $\mathcal{F}$ is the class of polynomials of degree at most s. We denote it in short as (s,p)-valent functions. For an (s,p)-valent function f the equation $f = P$ has at most p solutions in Ω for any polynomial P of degree s. We shall always assume that $p \geq s+1$, as subtracting from f its Taylor polynomial of degree s we get zero of order at least $s+1$. Note that this is indeed a generalization of p-valent functions, simply take $s = 0$, and every $(0,p)$-valent function is p-valent.

As we shall see this class of (s,p)-valent functions is indeed rich and appears naturally in many examples: algebraic functions, solutions of algebraic differential equations, monogenic functions, etc. In fact, it is fairly wide (see Sect. 2). It possesses many important properties: Distortion theorem, Bernstein-Markov-Remez type inequalities, etc. Moreover, this notion is applicable to any analytic function, under an appropriate choice of the domain Ω and the parameters s and p. In addition, it may provide a useful information in very general situations.

The following example shows that an (s,p)-valent function may not be $(s+1,p)$-valent:

Example 1.1 Let $f(z) = z^p + z^N$ for $N \geq 10p+1$. Then, for $s = 0,\dots,p-1$, the function f is (s,p)-valent in the disk $D_{1/3}$, but only (p,N)-valent there.

Indeed, taking $P(z) = z^p + c$ we see that the equation $f(z) = P(z)$ takes the form $z^N = c$. So for c small enough, it has exactly N solutions in the $D_{1/3}$. Now, for $s = 0,\dots,p-1$, take a polynomial $P(z)$ of degree $s \leq p-1$. Then, the equation $f(z) = P(z)$ takes the form $z^p - P(z) + z^N = 0$. Applying Chebyshev theorem (for more details see for example [17, Lemma 3.3]) to the polynomial $Q(z) = z^p - P(z)$ of degree p (with leading coefficient 1) we find a circle $S_\rho = \{|z| = \rho\}$ with $1/3 \leq \rho \leq 1/2$ such that $|Q(z)| \geq (1/2)^{10p}$ on S_ρ. On the other hand $z^N \leq (1/2)^{10p+1} < (1/2)^{10p}$ on S_ρ. Therefore, by the Rouché principle the number of zeroes of $Q(z)+z^N$ in the disk D_ρ is the same as for $Q(z)$, which is at most p. Thus, f is (s,p)-valent in the disk $D_{1/3}$, for $s = 0,\dots,p-1$.

This paper is organized as follows: in Sect. 2 we characterize (s,p)-valent functions in terms of their Taylor domination and linear recurrences for their coefficients. In Sect. 3 we prove a Distortion theorem for (s,p)-valent functions. In Sect. 4 we make a detour and investigate Remez-type inequalities for complex polynomials, which is interesting in its own right. Finally, in Sect. 5, we extend the Remez-type inequality to (s,p)-valent functions, via the Distortion theorem.

2 Taylor Domination, Bounded Recurrences

In this section we provide a rather accurate characterization of (s,p)-valent functions in a disk D_R in terms of their Taylor coefficients. "Taylor domination" for an analytic function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is an explicit bound of all its Taylor coefficients a_k through the first few of them. This property was classically studied, in particular, in relation with the Bieberbach conjecture: for univalent f we always have $|a_k| \le k|a_1|$ (see [2, 3, 12] and references therein). To give an accurate definition, let us assume that the radius of convergence of the Taylor series for f is $\hat{R}$, for $0 < \hat{R} \le +\infty$.

Definition 2.1 (Taylor Domination) Let $0 < R < \hat{R}$, $N \in \mathbb{N}$, and $S(k)$ be a positive sequence of a subexponential growth. The function f is said to possess an $(N, R, S(k))$-Taylor domination property if

$$|a_k|R^k \le S(k) \max_{i=0,\dots,N} |a_i|R^i , \quad k \ge N + 1.$$

The following theorem shows that f is an (s,p)-valent function in D_R, essentially, if and only if its lower s-truncated Taylor series possesses a $(p - s, R, S(k))$-Taylor domination.

Theorem 2.2 *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an (s,p)-valent function in D_R, and let $\hat{f}(z) = \sum_{k=1}^{\infty} a_{s+k} z^k$ be the lower s-truncation of f. Put $m = p-s$. Then, $\hat{f}$ possesses an $(m, R, S(k))$-Taylor domination, with $S(k) = \left(\frac{A_m k}{m}\right)^{2m}$, and A_m being a constant depending only on m.*

Conversely, if $\hat{f}$ possesses an $(m, R, S(k))$-Taylor domination, for a certain sequence $S(k)$ of a subexponential growth, then for $R' < R$ the function f is (s,p)-valent in $D_{R'}$, where $p = p(s+m, S(k), R'/R)$ depends only on $m + s$, the sequence $S(k)$, and the ratio R'/R. Moreover, p tends to ∞ for $R'/R \to 1$, and it is equal to $m + s$ for R'/R sufficiently small.

Proof First observe that if f is (s,p)-valent in D_R, then $\hat{f}$ is m-valent there, with $m = p-s$. Indeed, put $P(z) = \sum_{k=0}^{s} a_k z^k + cz^s$, with any $c \in \mathbb{C}$. Then, $f(z)-P(z) = z^s(\hat{f}(z) - c)$ may have at most p zeroes. Consequently, $\hat{f}(z) - c$ may have at most m zeroes in D_R, and thus $\hat{f}$ is m-valent there. Now we apply the following classic theorem:

Theorem 2.3 (Biernacki [3]) *If f is m-valent in the disk D_R of radius R centered at $0 \in \mathbb{C}$ then*

$$|a_k|R^k \le \left(\frac{A_m k}{m}\right)^{2m} \max_{i=1,\dots,m} |a_i|R^i , \quad k \ge m + 1,$$

where A_m is a constant depending only on m.

In our situation, Theorem 2.3 claims that the function $\hat{f}$ which is m-valent in D_R, possesses an $(m, R, \left(\frac{A_m k}{m}\right)^{2m})$-Taylor domination property. This completes the proof in one direction.

In the opposite direction, for polynomial $P(z)$ of degree s the function $f - P$ has the same Taylor coefficients as $\hat{f}$, starting with the index $k = s+1$. Consequently, if $\hat{f}$ possesses an $(m, R, S(k))$-Taylor domination, then $f - P$ possesses an $(s+m, R, S(k))$-Taylor domination. An explicit bound for the number of zeroes of a function possessing Taylor domination can be obtained by using the following result [15, Proposition 2.2.2] (which is announced here as appears in [1]):

Theorem 2.4 ([1, Theorem 2.3]) *Let the function f possess an $(N, R, S(k))$-Taylor domination property. Then for each $R' < R$, f has at most $M = M(N, \frac{R'}{R}, S(k))$ zeros in $D_{R'}$, where M depends only on N, $\frac{R'}{R}$ and on the sequence $S(k)$, satisfying $\lim_{\frac{R'}{R} \to 1} M = \infty$ and $M = N$ for $\frac{R'}{R}$ sufficiently small.*

Now a straightforward application of the above theorem provides the required bound on the number of zeroes of $f - P$ in the disk D_R. □

A typical situation for natural classes of (s, p)-valent functions is that they are (s, p)-valent for any s with a certain $p = p(s)$ which depends on s. However, it is important to notice that essentially *any* analytic function possesses this property, with some $p(s)$.

Proposition 2.5 *Let $f(z)$ be an analytic function in an open neighbourhood U of the closed disk D_R. Assume that f is not a polynomial. Then, the function f is $(s, p(s))$-valent for any s with a certain sequence $p(s)$.*

Proof Let f be given by its Taylor series $f(z) = \sum_{k=0}^{\infty} a_k z^k$. By assumptions, the radius of convergence $\hat{R}$ of this series satisfies $\hat{R} > R$. Since f is not a polynomial, for any given s there is the index $k(s) > s$ such that $a_{k(s)} \neq 0$. Now, we need the following result of [1]:

Proposition 2.6 ([1, Proposition 1.1]) *If $0 < \hat{R} \leqslant +\infty$ is the radius of convergence of $f(z) = \sum_{k=0}^{\infty} a_k z^k$, with $f \not\equiv 0$, then for each finite and positive $0 < R \leqslant \hat{R}$, f satisfies the $(N, R, S(k))$-Taylor domination property with N being the index of its first nonzero Taylor coefficient, and $S(k) = R^k |a_k| (|a_N| R^N)^{-1}$, for $k > N$.*

Applying the above proposition to the lower truncated series $\hat{f}(z) = \sum_{k=1}^{\infty} a_{s+k} z^k$. Thus, we obtain, an $(m, \hat{R}, S(k))$-Taylor domination for $\hat{f}$, for certain m and $S(k)$. Now, the second part of Theorem 2.2 provides the required $(s, p(s))$-valency for f in the smaller disk D_R, with $p(s) = p(s + m, S(k), R/\hat{R})$. □

More accurate estimates of $p(s)$ can be provided via the lacunary structure of the Taylor coefficients of f. Consequently, (s, p)-valency becomes really interesting only for those *classes* of analytic functions f where we can specify the parameters in an explicit and uniform way. The following theorem provides still very general, but important such class. We remark that the second part is known, see [15, Lemma 2.2.3] and [1, Theorem 4.1].

Theorem 2.7 *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be $(s, s+m)$-valent in D_R for any s. Then, the Taylor coefficients a_k of f satisfy a linear homogeneous non-stationary recurrence relation*

$$a_k = \sum_{j=1}^{m} c_j(k) a_{k-j} \tag{1}$$

with uniformly bounded (in k) coefficients $c_j(k)$ satisfying $|c_j(k)| \le C\rho^j$, with $C = e^2 A_m^{2m}$, $\rho = R^{-1}$, where A_m is the constant in the Biernacki's Theorem 2.3.

Conversely, if the Taylor coefficients a_k of f satisfy recurrence relation (1), with the coefficients $c_j(k)$, bounded for certain $K, \rho > 0$ and for any k as $|c_j(k)| \le K\rho^j$, $j = 1, \ldots, m$, then for any s, f is $(s, s+m)$-valent in a disk D_R, with $R = \frac{1}{2^{3m+1}(2K+2)\rho}$.

Proof We need to prove only the first part. Let us fix $s \ge 0$. As in the proof of Theorem 2.2, we notice that if f is $(s, s+m)$-valent in D_R, then its lower s-truncated series $\hat{f}$ is m-valent there. By Biernacki's Theorem 2.3 we conclude that

$$|a_{s+m+1}|R^{m+1} \le \left(\frac{A_m(m+1)}{m}\right)^{2m} \max_{i=1,\ldots,m} |a_{s+i}|R^i \le C \max_{i=1,\ldots,m} |a_{s+i}|R^i,$$

with $C = e^2 A_m^{2m}$. Putting $k = s+m+1$, and $\rho = R^{-1}$ we can rewrite this as

$$|a_k| \le C \max_{j=1,\ldots,m} |a_{k-j}|\rho^j.$$

Hence we can chose the coefficients $c_j(k)$, $k = s+m+1$, in such a way that $a_k = \sum_{j=1}^{m} c_j(k) a_{k-j}$, and $|c_j(k)| \le C\rho^j$, which completes the proof. □

Notice that the bound on the recursion coefficients is sharp, e.g. take $f(z) = [1 - (\frac{z}{R})^m]^{-1}$, in this case, as well as for other lacunary series with the gap m, the coefficients $c_j(k)$ are defined uniquely.

3 Distortion Theorem

In this section we prove a distortion-type theorem for (s, p)-valent functions which shows that the behavior of these functions is controlled by the behavior of a polynomial with the same zeroes.

First, let us recall the following theorem for p-valent functions, which is our main tool in proof.

Theorem 3.1 ([12, Theorem 5.1]) *Let $g(z) = a_0 + a_1 z + \dots$ be a regular non-vanishing p-valent function in D_1. Then, for any $z \in D_1$*

$$\left(\frac{1-|z|}{1+|z|}\right)^{2p} \le |g(z)/a_0| \le \left(\frac{1+|z|}{1-|z|}\right)^{2p}.$$

Now, we are ready to formulate a distortion-type theorem for (s,p)-valent functions.

Theorem 3.2 (Distortion Theorem) *Let f be an (s,p)-valent function in D_1 having there exactly s zeroes $z_1, \dots, z_s$ (always assumed to be counted according to multiplicity). Define a polynomial*

$$P(z) = A \prod_{j=1}^{s} (z - z_j),$$

where the coefficient A is chosen such that the constant term in the Taylor series for $f(z)/P(z)$ is equal to 1. Then, for any $x \in D_1$

$$\left(\frac{1-|z|}{1+|z|}\right)^{2p} \le |f(z)/P(z)| \le \left(\frac{1+|z|}{1-|z|}\right)^{2p}.$$

Proof The function $g(z) = f(z)/P(z)$ is regular in D_1 and does not vanish there. Moreover, g is p-valent in D_1. Indeed, the equation $g(z) = c$ is equivalent to $f(z) = cP(z)$ so it has at most p solutions by the definition of (s,p)-valent functions. Now, apply Theorem 3.1 to the function g. □

It is not clear whether the requirement for f to be (s,p)-valent is really necessary in this theorem. The ratio $g(z) = \frac{f(z)}{P(z)}$ certainly may not be p-valent for f being just p-valent, but not (s,p)-valent. Indeed, take $f(z) = z^p + z^N$ as in Example 1.1. By this example f is p-valent in $D_{1/3}$ and it has a root of multiplicity p at zero. So $g(z) = f(z)/z^p = 1 + z^{N-p}$ and the equation $g(z) = c$ has $N - p$ solutions in $D_{1/3}$ for c sufficiently close to 1. So g is not p-valent there.

4 Complex Polynomials

The distortion Theorem 3.2, proved in the previous section, allows us easily to extend deep properties from polynomials to (s,p)-valent functions, just by comparing them with polynomials having the same zeros. In this section we make a detour and investigate one specific problem for complex polynomials, which is interesting in its own right: a Remez-type inequality for complex polynomial (compare [14, 18]). Denote by

$$V_\rho(g) = \{z : |g(z)| \le \rho\}$$

the ρ sub-level set of a function g. For polynomials in one complex variable a result similar to the Remez inequality is provided by the classic Cartan (or Cartan-Boutroux) lemma (see, for example, [11] and references therein):

Lemma 4.1 (Cartan's Lemma [7], as Appears in [11]) *Let $\alpha, \varepsilon > 0$, and let $P(z)$ be a monic polynomial of degree d. Then*

$$V_{\varepsilon^d}(P) \subset \cup_{j=1}^{p} D_{r_j},$$

where $p \leq d$, and $D_{r_1}, \ldots, D_{r_p}$ are balls with radii $r_j > 0$ satisfying $\sum_{j=1}^{p} r_j^\alpha \leq e(2\varepsilon)^\alpha$.

In [5, 6, 19, 20] some generalizations of the Cartan-Boutroux lemma to plurisubharmonic functions have been obtained, which lead, in particular, to the bounds on the size of sub-level sets. In [5] some bounds for the covering number of sublevel sets of complex analytic functions have been obtained, similar to the results of [18] in the real case. Now, we shall derive from the Cartan lemma both the definition of the invariant $c_{d,\alpha}$ and the corresponding Remez inequality.

Definition 4.2 Let $Z \subset D_1$. The (d, α)-Cartan measure of Z is defined as

$$c_{d,\alpha}(Z) = \min \left(\sum_{j=1}^{p} r_j^\alpha \right)^{1/\alpha}$$

where the minimum is taken over all covers of Z by $p \leq d$ balls with radii $r_j > 0$.

Clearly, the invariant $c_{d,\alpha}(Z)$ satisfies the following basic properties. It is monotone in Z, that is, for $Z_1 \subset Z_2$ we have $c_{d,\alpha}(Z_1) \leq c_{d,\alpha}(Z_2)$. And, also monotone in d, that is, for $d_1 \leq d$ we have $c_{d,\alpha}(Z) \leq c_{d_1,\alpha}(Z)$. Finally, for any $Z \subset D_1$ we have $c_{d,\alpha}(Z) \leq 1$. Note also that the α-dimensional Hausdorff content of Z is defined in a similar way

$$H_\alpha(Z) = \inf \left\{ \sum_j r_j^\alpha \text{ : there is a cover of } Z \text{ by balls with radii } r_j > 0 \right\}.$$

Thus, by the above definitions, we have $H_\alpha^{\frac{1}{\alpha}}(Z) \leq c_{d,\alpha}(Z)$.

For $\alpha = 1$ the $(d, 1)$-Cartan measure $c_{d,\alpha}(Z)$ was introduced and used, under the name "d-th diameter", in [8, 9]. In particular, Lemma 3.3 of [8] is, essentially, equivalent to the case $\alpha = 1$ of our Theorem 4.3. In Sect. 4.1 below we provide some initial geometric properties of $c_{d,\alpha}(Z)$ and show that a proper choice of α may improve the geometric sensitivity of this invariant.

Now we can state and proof our generalized Remez inequality for complex polynomials:

Theorem 4.3 *Let $P(z)$ be a polynomial of degree d. Let $Z \subset D_1$. Then, for any $\alpha > 0$*

$$\max_{D_1} |P(z)| \le \left(\frac{6e^{1/\alpha}}{c_{d,\alpha}(Z)}\right)^d \max_Z |P(z)| \le \left(\frac{6e}{H_\alpha(Z)}\right)^{\frac{d}{\alpha}} \max_Z |P(z)|.$$

Proof Assume that $|P(z)| \le 1$ on Z. First, we prove that the absolute value A of the leading coefficient of P satisfies

$$A \le \left(\frac{2e^{1/\alpha}}{c_{d,\alpha}(Z)}\right)^d.$$

Indeed, we have $Z \subset V_1(P)$. By the definition of $c_{d,\alpha}(Z)$ for every covering of $V_1(P)$ by p disks $D_{r_1}, \dots, D_{r_p}$ of the radii $r_1, \dots, r_d$ (which is also a covering of Z) we have $\sum_{i=1}^d r_i^\alpha \ge c_{d,\alpha}(Z)^\alpha$. Denoting, as above, the absolute value of the leading coefficient of $P(z)$ by A we have by the Cartan lemma that for a certain covering as above

$$c_{d,\alpha}(Z)^\alpha \le \sum_{i=1}^d r_i^\alpha \le e\left(\frac{2}{A^{1/d}}\right)^\alpha.$$

Now, we write $P(z) = A\prod_{j=1}^d (z - z_j)$, and consider separately two cases:

(1) All $|z_j| \le 2$. Thus, $\max_{D_1} |P(z)| \le A3^d \le \left(\frac{2e^{1/\alpha}}{c_{d,\alpha}(Z)}\right)^d 3^d$, as required.
(2) For $j = 1, \dots, d_1 < d$, $|z_j| \le 2$, while $|z_j| > 2$ for $j = d_1 + 1, \dots, d$. Denote

$$P_1(z) = A\prod_{j=1}^{d_1}(z - z_j), \quad P_2(z) = \prod_{j=d_1+1}^{d}(z - z_j),$$

and notice that for any two points $v_1, v_2 \in D_1$ we have $|P_2(v_1)/P_2(v_2)| < 3^{d-d_1}$. Consequently we get

$$\frac{\max_{D_1}|P(z)|}{\max_Z |P(z)|} < 3^{d-d_1}\frac{\max_{D_1}|P_1(z)|}{\max_Z |P_1(z)|}.$$

All the roots of P_1 are bounded in absolute value by 2, so by first part we have

$$\frac{\max_{D_1}|P_1(z)|}{\max_Z |P_1(z)|} \le \left(\frac{2e^{1/\alpha}}{c_{d_1,\alpha}(Z)}\right)^{d_1} 3^{d_1} \le \left(\frac{2e^{1/\alpha}}{c_{d,\alpha}(Z)}\right)^{d} 3^{d_1}$$

where the last inequality follows from the basic properties of the invariant $c_{d,\alpha}(Z)$ described after Definition 4.2. Finally, application of the inequality $H_\alpha(Z) \le c_{d,\alpha}(Z)^\alpha$ completes the proof. □

Let us stress a possibility to chose an optimal α in the bound of Theorem 4.3. Let

$$K_d(Z) = \inf_{\alpha>0} \left(\frac{6e^{1/\alpha}}{c_{d,\alpha}(Z)} \right)^d , \quad K_d^H(Z) = \inf_{\alpha>0} \left(\frac{6e}{H_\alpha(Z)} \right)^{\frac{d}{\alpha}} .$$

Corollary 4.4 *Let $P(z)$ be a polynomial of degree d. Let $Z \subset D_1$. Then,*

$$\max_{D_1} |P(z)| \leq K_d(Z) \max_Z |P(z)| \leq K_d^H(Z) \max_Z |P(z)|.$$

4.1 Geometric and Analytic Properties of the Invariant $c_{d,\alpha}$

In addition to the basic properties of $c_{d,\alpha}$ we also have

Proposition 4.5 *Let $\alpha > 0$. Then, $c_{d,\alpha}(Z) > 0$ if and only if Z contains more than d points. In the latter case, $c_{d,\alpha}(Z)$ is greater than or equal to one half of the minimal distance between the points of Z.*

Proof Any d points can be covered by d disks with arbitrarily small radii. But, the radius of at least one disk among d disks covering more than $d+1$ different points is greater than or equal to the one half of a minimal distance between these points. □

The lower bound of Proposition 4.5 does not depend on α. However, in general, this dependence is quite prominent.

Example 4.6 Let $Z = [a, b]$. Then, for $\alpha \geq 1$ we have $c_{d,\alpha}(Z) = (b-a)/2$, while for $\alpha \leq 1$ we have $c_{d,\alpha}(Z) = d^{\frac{1}{\alpha}-1}(b-a)/2$.

Indeed, in the first case the minimum is achieved for $r_1 = (b-a)/2, r_2 = \cdots = r_d = 0$, while in the second case for $r_1 = r_2 = \cdots = r_d = (b-a)/2d$.

Proposition 4.7 *Let $\alpha > \beta > 0$. Then, for any Z*

$$c_{d,\alpha}(Z) \leq c_{d,\beta}(Z) \leq d^{(\frac{1}{\beta}-\frac{1}{\alpha})} c_{d,\alpha}(Z). \qquad (2)$$

Proof Let $r = (r_1, \ldots, r_d)$ and $\gamma > 0$. Consider $||r||_\gamma = (\sum_{j=1}^d r_j^\gamma)^{\frac{1}{\gamma}}$. Then, by the definition, $c_{d,\gamma}(Z)$ is the minimum of $||r||_\gamma$ over all $r = (r_1, \ldots, r_d)$ being the radii of d balls covering Z. Now we use the standard comparison of the norms $||r||_\gamma$, that is, for any $x = (z_1, \ldots, z_d)$ and for $\alpha > \beta > 0$,

$$||z||_\alpha \leq ||z||_\beta \leq d^{(\frac{1}{\beta}-\frac{1}{\alpha})} ||z||_\alpha.$$

Take $r = (r_1, \ldots, r_d)$ for which the minimum of $||r||_\beta$ is achieved, and we get

$$c_{d,\alpha}(Z) \leq ||r||_\alpha \leq ||r||_\beta = c_{d,\beta}(Z).$$

Now taking r for which the minimum of $||r||_\alpha$ is achieved, exactly in the same way we get the second inequality. □

Now, we compare $c_{d,\alpha}(Z)$ with some other metric invariants which may be sometimes easier to compute. In each case we do it for the most convenient value of α. Then, using the comparison inequalities of Proposition 4.7, we get corresponding bounds on $c_{d,\alpha}(Z)$ for any $\alpha > 0$. In particular, we can easily produce a simple lower bound for $c_{d,2}(Z)$ through the measure of Z:

Proposition 4.8 *For any measurable $Z \subset D_1$ we have*

$$c_{d,2}(Z) \geq (\mu_2(Z)/\pi)^{1/2}.$$

Proof For any covering of Z by d disks $D_1, \dots, D_d$ of the radii $r_1, \dots, r_d$ we have $\pi(\sum_{i=1}^d r_i^2) \geq \mu_2(Z)$. □

However, in order to deal with discrete or finite subsets $Z \subset D_1$ we have to compare $c_{d,\alpha}(Z)$ with the covering number $M(\varepsilon, Z)$ (which is, by definition, the minimal number of ε-disks covering Z).

Definition 4.9 Let $Z \subset D_1$. Define

$$\omega_{cd}(Z) = \sup_\varepsilon \varepsilon (M(\varepsilon, Z) - d)^{1/2},$$

if $|Z| \geq d$, and $\omega_{cd}(Z) = 0$ otherwise. Put $\rho_d(Z) = d\varepsilon_0$, where ε_0 is the minimal ε for which there is a covering of Z with d ε-disks. Note that, writing $y = M(\varepsilon, Z) = \Psi(\varepsilon)$, and taking the inverse $\varepsilon = \Psi^{-1}(y)$, we have $\varepsilon_0 = \Psi^{-1}(d)$.

As it was mentioned above, a very similar invariant

$$\omega_d(Z) = \sup_\varepsilon \varepsilon (M(\varepsilon, Z) - d),$$

if $|Z| \geq d$, and $\omega_{cd}(Z) = 0$ otherwise, was introduced and used in [18] in the real case. We compare ω_{cd} and ω_d below.

Proposition 4.10 *Let $Z \subset D_1$. Then, $\omega_{cd}(Z)/2 \leq c_{d,2}(Z) \leq c_{d,1}(Z) \leq \rho_d(Z)$.*

Proof To prove the upper bound for $c_{d,1}(Z)$ we notice that it is the infimum of the sum of the radii in all the coverings of Z with d disks, while $\rho_d(Z)$ is such a sum for one specific covering.

To prove the lower bound, let us fix a covering of Z by d disks D_i of the radii r_i with $c_{d,2}(Z) = (\sum_{i=1}^d r_i^2)^{1/2}$. Let $\varepsilon > 0$. Now, for any disk D_j with $r_j \geq \varepsilon$ we need at most $4r_j^2/\varepsilon^2$ ε-disks to cover it. For any disk D_j with $r_j \leq \varepsilon$ we need exactly one ε-disk to cover it, and the number of such D_j does not exceed d. So, we conclude that $M(\varepsilon, Z)$ is at most $d + (4/\varepsilon^2)\sum_{i=1}^d r_i^2$. Thus, we get $c_{d,2}(Z) = (\sum_{i=1}^d r_i^2)^{1/2} \geq \varepsilon/2(M(\varepsilon, Z) - d)^{1/2}$. Taking supremum with respect to $\varepsilon > 0$ we get $c_{d,2}(Z) \geq \omega_{cd}(Z)/2$. □

Since $M(\varepsilon, Z)$ is always an integer, we have

$$\omega_d(Z) \geq \omega_{cd}(Z).$$

For $Z \subset D_1$ of positive plane measure, $\omega_d(Z) = \infty$ while $\omega_{cd}(Z)$ remains bounded (in particular, by $\rho_d(Z)$).

Some examples of computing (or bounding) $\omega_d(Z)$ for "fractal" sets Z can be found in [18]. Computations for $\omega_{cd}(Z)$ are essentially the same. In particular, in an example given in [18] in connection to [10] we have that for $Z = Z_r = \{1, 1/2^r, 1/3^r, \ldots, 1/k^r, \ldots\}$

$$\omega_d(Z_r) \asymp \frac{r^r}{(r+1)^{r+1} d^r}, \quad \omega_{cd}(Z_r) \asymp \frac{(2r+1)^r}{(2r+2)^{r+1} d^{r+1/2}}.$$

The asymptotic behavior here is for $d \to \infty$, as in [10].

4.2 An Example

We conclude this section with one very specific example. Let

$$Z = Z(d, h) = \{z_1, z_2, \ldots, z_{2d-1}, z_{2d}\}, \quad z_i \in \mathbb{C}, \ d \geq 2.$$

We assume that Z consists of d, 2η-separated couples of points, with points in each couple being in a distance $2h$. Let $2D(Z)$ be the diameter of the smallest disk containing Z. Assume $h \ll 1$, $2\eta \gg h$.

Proposition 4.11 *Let Z be as above. Then,*

(1) $\omega_d(Z) = dh$.
(2) $\omega_{cd}(Z) = \sqrt{d}h$.
(3) *For* $\alpha > 0$, *we have* $c_{d,\alpha}(Z) \leq d^{\frac{1}{\alpha}} h$.
(4) *For* $\alpha \gg 1$, *we have* $c_{d,\alpha}(Z) = d^{\frac{1}{\alpha}} h$.
(5) *For* $\kappa = [\log_d(\frac{D(Z)}{h})]^{-1}$, *we have* $c_{d,\kappa}(Z) \geq \eta$.

Proof For $\varepsilon > h$, we have $M(\varepsilon, Z) \leq d$, and hence $M(\varepsilon, Z) - d$ is non-positive. For $\varepsilon < h$, we have $M(\varepsilon, Z) = 2d$, and $M(\varepsilon, Z) - d = d$. Thus the supremum of $\varepsilon(M(\varepsilon, Z) - d)$, or the supremum of $\varepsilon(M(\varepsilon, Z) - d)^{\frac{1}{2}}$, is achieved as $\varepsilon < h$ tends to h. Therefore, $\omega_d(Z) = dh$, and $\omega_{cd}(Z) = \sqrt{d}h$.

Covering each couple with a separate ball of radius h, we get for any $\alpha > 0$ that $c_{d,\alpha}(Z) \leq d^{\frac{1}{\alpha}} h$. For $\alpha \gg 1$ it is easy to see that this uniform covering is minimal. Thus, for such α we have the equality $c_{d,\alpha}(Z) = d^{\frac{1}{\alpha}} h$.

Now let us consider the case of a "small" $\alpha = \kappa$. Take a covering of Z with certain disks $D_j, j \leq d$. If there is at least one disk D_j containing three points of Z or more, the radius of this disk is at least η. Thus, for this covering $(\sum_{j=1}^{d} r_j^{\kappa})^{\frac{1}{\kappa}} \geq \eta$. If each disk in the covering contains at most two points, it must contain exactly two, otherwise these disks could not cover all the $2d$ points of Z. Hence, the radius of each disk D_j in such covering is at least h, and their number is exactly d. We have, by the choice of κ, that $(\sum_{j=1}^{d} r_j^{\kappa})^{\frac{1}{\kappa}} \geq d^{\frac{1}{\kappa}} h = D(Z) \geq \eta$. □

Let us use two choices of α in the Remez-type inequality of Theorem 4.3: $\alpha = 1$ and $\alpha = \kappa$. We get two bounds for the constant $K_d(Z)$:

$$K_d(Z) \leq \left(\frac{6e}{c_{d,1}(Z)}\right)^d \quad \text{or} \quad K_d(Z) \leq \left(\frac{6e^{1/\kappa}}{c_{d,\kappa}(Z)}\right)^d.$$

By Proposition 4.11 we have $c_{d,1}(Z) \leq dh$, while $c_{d,\kappa}(Z) \geq \eta$. Therefore we get

$$\left(\frac{6e}{c_{d,1}(Z)}\right)^d \geq \left(\frac{6e}{dh}\right)^d, \quad \text{while} \quad \left(\frac{6e^{1/\kappa}}{c_{d,\kappa}(Z)}\right)^d \leq \left(\frac{6e^{1/\kappa}}{\eta}\right)^d. \tag{3}$$

But $e^{1/\kappa} = e^{\log_d(\frac{D(Z)}{h})} = (\frac{D(Z)}{h})^{\frac{1}{\ln d}}$. So the second bound of (3) takes a form

$$K_d(Z) \leq \left(\frac{6D(Z)}{\eta^{\ln d} h}\right)^{\frac{d}{\ln d}}.$$

We see that for $d \geq 3$ and for $h \to 0$ the asymptotic behavior of this last bound, corresponding to $\alpha = \kappa$, is much better than of the first bound in (3), corresponding to $\alpha = 1$. Notice, that κ depends on h and $D(Z)$, i.e. on the specific geometry of the set Z.

5 Remez Inequality

Now, we present a Remez-type inequality for (s, p)-valent functions. We recall that by Proposition 2.5 above, any analytic function in an open neighborhood U of the closed disk D_R is $(s, p(s))$-valent in D_R for any s with a certain sequence $p(s)$. Consequently, the following theorem provides a non-trivial information for any analytic function in an open neighborhood of the unit disk D_1. Of course, this results becomes really interesting only in cases where we can estimate $p(s)$ explicitly.

Theorem 5.1 *Let f be an analytic function in an open neighborhood U of the closed disk D_1. Assume that f has in D_1 exactly s zeroes, and that it is (s, p)-valent in D_1. Let Z be a subset in the interior of D_1, and put $\rho = \rho(Z) = \min\{\eta : Z \subset D_\eta\}$. Then, for any $R < 1$ function f satisfies*

$$\max_{D_R} |f(z)| \leq \sigma_p(R, \rho) K_s(Z) \max_{Z} |f(z)|,$$

where $\sigma_p(R, \rho) = \left(\frac{1+R}{1-R} \cdot \frac{1+\rho}{1-\rho}\right)^{2p}$.

Proof Assume that $|f(z)|$ is bounded by 1 on Z. Let $z_1, \ldots, z_s$ be zeroes of f in D_1. Consider, as in Theorem 3.2, the polynomial

$$P(z) = A \prod_{j=1}^{l} (z - z_j),$$

where the coefficient A is chosen in such a way that the constant term in the Taylor series for $g(z) = f(z)/P(z)$ is equal to 1. Then by Theorem 3.2 for g we have

$$\left(\frac{1-|z|}{1+|z|}\right)^{2p} \le |g(z)| \le \left(\frac{1+|z|}{1-|z|}\right)^{2p}.$$

We conclude that $P(z) \le (\frac{1+\rho}{1-\rho})^{2p}$ on Z. Hence by the polynomial Remez inequality provided by Theorem 4.3 we obtain

$$|P(z)| \le K_s(Z) \left(\frac{1+\rho}{1-\rho}\right)^{2p}$$

on D_1. Finally, we apply once more the bound of Theorem 3.2 to conclude that

$$|f(z)| \le K_s(Z) \left(\frac{1+R}{1-R}\right)^{2p} \left(\frac{1+\rho}{1-\rho}\right)^{2p}$$

on D_R. □

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