

Valuations on the Space of Quasi-Concave Functions

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Abstract We characterize the valuations on the space of quasi-concave functions on \mathbb{R}^N , that are rigid motion invariant and continuous with respect to a suitable topology. Among them we also provide a specific description of those which are additionally monotone.

1 Introduction

A *valuation* on a space of functions X is an application $\mu : X \rightarrow \mathbb{R}$ such that

$$\mu(f \vee g) + \mu(f \wedge g) = \mu(f) + \mu(g) \quad (1)$$

for every $f, g \in X$ s.t. $f \vee g, f \wedge g \in X$; here “ \vee ” and “ \wedge ” denote the point-wise maximum and minimum, respectively. The condition (1) can be interpreted as a finite additivity property (typically verified by integrals).

The study of valuations on spaces of functions stems principally from the theory of valuations on classes of sets, in which the main current concerns *convex bodies*. We recall that a convex body is simply a compact convex subset of \mathbb{R}^N , and the family of convex bodies is usually denoted by \mathcal{K}^N . An application $\sigma : \mathcal{K}^N \rightarrow \mathbb{R}$ is called a valuation if

$$\sigma(K \cup L) + \sigma(K \cap L) = \sigma(K) + \sigma(L) \quad (2)$$

for every $K, L \in \mathcal{K}^N$ such that $K \cup L \in \mathcal{K}^N$ (note that the intersection of convex bodies is a convex body). Hence, in passing from (2) to (1) union and intersection are replaced by maximum and minimum respectively. A motivation is that the characteristic function of the union (resp. the intersection) of two sets is the maximum (resp. the minimum) of their characteristic functions.

The theory of valuations is an important branch of modern convex geometry (the theory of convex bodies). The reader is referred to the monograph [16]

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for an exhaustive description of the state of the art in this area, and for the corresponding bibliography. The valuations on \mathcal{K}^N , continuous with respect to the Hausdorff metric and rigid motion invariant, i.e. invariant with respect to composition with translations and proper rotations (elements of $O(N)$), have been completely classified in a celebrated result by Hadwiger (see [5–7]). Hadwiger’s theorem asserts that any valuation σ with these properties can be written in the form

$$\sigma(K) = \sum_{i=0}^N c_i V_i(K) \quad \forall K \in \mathcal{K}^N, \quad (3)$$

where c_1, \dots, c_N are constants and V_1, \dots, V_N denote the *intrinsic volumes* (see Sect. 2, for the definition). This fact will be of great importance for the results presented here.

Let us give a brief account of the main known results in the area of valuations on function spaces. Wright, in his PhD thesis [21] and subsequently in collaboration with Baryshnikov and Ghrist [2], characterized rigid motion invariant and continuous valuations on the class of *definable functions* (we refer to the quoted papers for the definition). Their result is very similar to Hadwiger’s theorem; roughly speaking it asserts that every valuation is the linear combination of integrals of intrinsic volumes of level sets. This type of valuations will be crucial in our results as well.

Rigid motion invariant and continuous valuations on $L^p(\mathbb{R}^N)$ and on $L^p(\mathbb{S}^{n-1})$ ($1 \leq p < \infty$) have been studied and classified by Tsang in [17]. Basically, Tsang proved that every valuation μ with these properties is of the type

$$\mu(f) = \int \phi(f) dx \quad (4)$$

(here the integral is performed on \mathbb{R}^N or \mathbb{S}^{n-1}) for some function ϕ defined on \mathbb{R} verifying suitable growth conditions. Subsequently, the results of Tsang have been extended to Orlicz spaces by Kone in [8]. Also, the special case $p = \infty$ was studied by Cavallina in [3].

Valuations on the space of functions of bounded variations and on Sobolev spaces have been recently studied by Wang and Ma respectively, in [14, 19, 20] and [13].

In [4] the authors consider rigid motion invariant and continuous valuations (with respect to a certain topology that will be recalled later on) on the space of convex functions, and found some partial characterization results under the assumption of monotonicity and homogeneity.

Note that the results that we have mentioned so far concern *real-valued* valuations, but there are also studies regarding other types of valuations (e.g. matrix-valued valuations, or Minkowski and Blaschke valuations, etc.) that are interlaced with the results mentioned previously. A strong impulse to these studies have been given by Ludwig in the works [9–12]; the reader is referred also to [18] and [15].

Here we consider the space \mathcal{C}^N of *quasi-concave* functions of N real variables. A function $f : \mathbb{R}^N \rightarrow \mathbb{R}$ is quasi-concave if it is non-negative and for every $t > 0$ the

level set

$$L_t(f) = \{x \in \mathbb{R}^N : f(x) \geq t\}$$

is (either empty or) a compact convex set. \mathcal{C}^N includes log-concave functions and characteristic functions of convex bodies as significant examples.

We consider valuations $\mu : \mathcal{C}^N \rightarrow \mathbb{R}$ which are rigid motion invariant (with the same notion as before for rigid motion transformations), i.e.

$$\mu(f) = \mu(f \circ T)$$

for every $f \in \mathcal{C}^N$ and for every rigid motion T of \mathbb{R}^N . We also impose a continuity condition on μ : if $f_i, i \in \mathbb{N}$, is a *monotone* (either increasing or decreasing) sequence in \mathcal{C}^N , converging to $f \in \mathcal{C}^N$ point-wise in \mathbb{R}^N , then we must have

$$\lim_{i \rightarrow \infty} \mu(f_i) = \mu(f).$$

In Sect. 4.1 we provide some motivation for this definition, comparing this notion of continuity with other possible choices.

There is a simple way to construct valuations on \mathcal{C}^N . To start with, note that if $f, g \in \mathcal{C}^N$ and $t > 0$

$$L_t(f \vee g) = L_t(f) \cup L_t(g), \quad L_t(f \wedge g) = L_t(f) \cap L_t(g). \tag{5}$$

Let ψ be a function defined on $(0, \infty)$ and fix $t_0 > 0$. Define, for every $f \in \mathcal{C}^N$,

$$\mu_0(f) = V_N(L_{t_0}(f))\psi(t_0).$$

Using (5) and the additivity of volume we easily deduce that μ_0 is a rigid motion invariant valuation. More generally, we can overlap valuations of this type at various levels t , and we can further replace V_N by any intrinsic volume V_k :

$$\mu(f) = \int_{(0, \infty)} V_k(L_t(f))\psi(t) dt = \int_{(0, \infty)} V_k(L_t(f)) dv(t), \quad f \in \mathcal{C}^N, \tag{6}$$

where ν is the measure with density ψ . This is now a rather ample class of valuations; as we will see, basically every *monotone* valuation on \mathcal{C}^N can be written in this form. To proceed, we observe that the function

$$t \rightarrow V_k(L_t(f))$$

is decreasing. In particular it admits a distributional derivative which is a non-positive measure. For ease of notation we write this measure in the form $-S_k(f; \cdot)$ where now $S_k(f; \cdot)$ is a (non-negative) Radon measure on $(0, \infty)$. Then, integrating

by parts in (6) (boundary terms can be neglected, as it will be clear in the sequel) we obtain:

$$\mu(f) = \int_{(0,\infty)} \phi(t) dS_k(f; t) \quad (7)$$

where ϕ is a primitive of ψ . Our first result is the fact that functionals of this type exhaust, by linear combinations, all possible rigid motion invariant and continuous valuations on \mathcal{C}^N .

Theorem 1.1 *A map $\mu : \mathcal{C}^N \rightarrow \mathbb{R}$ is an invariant and continuous valuation on \mathcal{C}^N if and only if there exist $(N + 1)$ continuous functions ϕ_k , $k = 0, \dots, N$ defined on $[0, \infty)$, and $\delta > 0$ such that: $\phi_k \equiv 0$ in $[0, \delta]$ for every $k = 1, \dots, N$, and*

$$\mu(f) = \sum_{k=0}^N \int_{[0,\infty)} \phi_k(t) dS_k(f; t) \quad \forall f \in \mathcal{C}^N.$$

The condition that each ϕ_k , except for ϕ_0 , vanishes in a right neighborhood of the origin guarantees that the integral in (7) is finite for every $f \in \mathcal{C}^N$ (in fact, it is equivalent to this fact). As in the case of Hadwiger theorem, the proof of this result is based on a preliminary step in which valuations that are additionally *simple* are classified. A valuation μ on \mathcal{C}^N is called simple if

$$f = 0 \text{ a.e. in } \mathbb{R}^N \quad \Rightarrow \quad \mu(f) = 0.$$

Note that for $f \in \mathcal{C}^N$, being zero a.e. is equivalent to say that the dimension of the support of f (which is a convex set) is strictly smaller than N . The following result is in a sense analogous to the so-called *volume theorem* for convex bodies.

Theorem 1.2 *A map $\mu : \mathcal{C}^N \rightarrow \mathbb{R}$ is an invariant, continuous and simple valuation on \mathcal{C}^N if and only if there exists a continuous function ϕ defined on $[0, \infty)$, with $\phi \equiv 0$ in $[0, \delta]$ for some $\delta > 0$, such that*

$$\mu(f) = \int_{\mathbb{R}^n} \phi(f(x)) dx \quad \forall f \in \mathcal{C}^N,$$

or, equivalently,

$$\mu(f) = \int_{[0,\infty)} \phi(t) dS_N(f; t).$$

Here the equivalence of the two formulas follows from the layer cake principle. The representation formula of Theorem 1.1 becomes more legible in the case of monotone valuations. Here, each term of the sum is clearly a weighted mean of the intrinsic volumes of the level sets of f .

Theorem 1.3 *A map μ is an invariant, continuous and monotone increasing valuation on \mathcal{C}^N if and only if there exists $(N + 1)$ Radon measures on $[0, \infty)$, ν_k , $k = 0, \dots, N$, such that each ν_k is non-negative, non-atomic and, for $k \geq 1$, the support of ν_k is contained in $[\delta, \infty)$ for a suitable $\delta > 0$, and*

$$\mu(f) = \sum_{k=0}^N \int_{[0, \infty)} V_k(L_t(f)) d\nu_k(t), \quad \forall f \in \mathcal{C}^N.$$

We remark that the non-negativity of ν_k depends on the monotone increasing property of μ , as we will see in Sect. 8.

As we already mentioned, and it will be explained in details in Sect. 5.3, the passage

$$\int_{[0, \infty)} \phi_k(t) dS_k(f; t) \longrightarrow \int_{[0, \infty)} V_k(L_t(f)) d\nu_k(t)$$

is provided merely by an integration by parts, when this is permitted by the regularity of the function ϕ_k .

The paper is organized as follows. In the next section we provide some notions from convex geometry. Section 3 is devoted to the basic properties quasi-convex functions, while in Sect. 4 we define various types of valuations on the space \mathcal{C}^N . In Sect. 5 we introduce the integral valuations, which occur in Theorems 1.1 and 1.3. Theorem 1.2 is proved in Sect. 6, while Sects. 6 and 7 contain the proof of Theorems 1.1 and 1.3, respectively.

2 Notations and Preliminaries

We work in the N -dimensional Euclidean space \mathbb{R}^N , $N \geq 1$, endowed with the usual scalar product (\cdot, \cdot) and norm $\|\cdot\|$. Given a subset A of \mathbb{R}^N , $\text{int}(A)$, $\text{cl}(A)$ and ∂A denote the interior, the closure and the topological boundary of A , respectively. For every $x \in \mathbb{R}^N$ and $r \geq 0$, $B_r(x)$ is the closed ball of radius r centered at x ; in particular, for simplicity we will write B_r instead of $B_r(0)$. We recall that a *rigid motion* of \mathbb{R}^N will be the composition of a translation and a rotation of \mathbb{R}^N (i.e. an isometry). The Lebesgue measure in \mathbb{R}^N will be denoted by V_N .

2.1 Convex Bodies

We recall some notions and results from convex geometry that will be used in the sequel. Our main reference on this subject is the monograph by Schneider [16]. As stated in the introduction the class of convex bodies is denoted by \mathcal{K}^N . For $K, L \in$

\mathcal{K}^N , we define the *Hausdorff distance* of K and L as

$$\delta(K, H) = \max\{\sup_{x \in K} \text{dist}(x, H), \sup_{y \in H} \text{dist}(K, y)\}.$$

Accordingly, a sequence of convex bodies $\{K_n\}_{n \in \mathbb{N}} \subseteq \mathcal{K}^N$ is said to converge to $K \in \mathcal{K}^N$ if

$$\delta(K_n, K) \rightarrow 0, \text{ as } n \rightarrow +\infty.$$

Remark 2.1 \mathcal{K}^N with respect to Hausdorff distance is a complete metric space.

Remark 2.2 For every convex subset C of \mathbb{R}^N , and consequently for convex bodies, its dimension $\dim(C)$ can be defined as follows: $\dim(C)$ is the smallest integer such that there exists an affine sub-space of \mathbb{R}^N containing C .

We are ready, now, to introduce some functionals operating on \mathcal{K}^N , the intrinsic volumes, which will be of fundamental importance in this paper. Among the various ways to define intrinsic volumes, we choose the one based on the Steiner formula. Given a convex body K and $\epsilon > 0$, the *parallel set* of K is

$$K_\epsilon = \{x \in \mathbb{R}^N \mid \text{dist}(x, K) \leq \epsilon\}.$$

The following result asserts that the volume of the parallel body is a polynomial in ϵ , and contains the definition of intrinsic volumes.

Theorem 2.3 (Steiner Formula) *There exist N functions $V_0, \dots, V_{N-1} : \mathcal{K}^N \rightarrow \mathbb{R}_+$ such that, for all $K \in \mathcal{K}^N$ and for all $\epsilon \geq 0$, we have*

$$V_N(K_\epsilon) = \sum_{i=0}^N V_i(K) \omega_{N-i} \epsilon^{N-i},$$

where ω_j denotes the volume of the unit ball in the space \mathbb{R}^j . $V_0(K), \dots, V_N(K)$ are called the *intrinsic volumes* of K .

Hence one of the intrinsic volumes is the Lebesgue measure. Moreover V_0 is the Euler characteristic, so that for every K we have $V_0(K) = 1$. The name intrinsic volumes comes from the following fact: assume that K has dimension $j \in \{0, \dots, N\}$, i.e. there exists a j -dimensional affine subspace of \mathbb{R}^N containing K , and j is the lowest number with this property (we will write $\dim(K) = j$). Then K can be seen as a subset of \mathbb{R}^j and $V_j(K)$ is the Lebesgue measure of K as a subset of \mathbb{R}^j . Intrinsic volumes have many other properties, listed in the following proposition.

Proposition 2.4 (Properties of Intrinsic Volumes) *For every $k \in \{0, \dots, N\}$ the function V_k is:*

- *rigid motion invariant;*

- *continuous with respect to the Hausdorff metric;*
- *monotone increasing: $K \subset L$ implies $V_k(K) \leq V_k(L)$;*
- *a valuation:*

$$V_k(K \cup L) + V_k(K \cap L) = V_k(K) + V_k(L) \quad \forall K, L \in \mathcal{K}^N \text{ s.t. } K \cup L \in \mathcal{K}^N.$$

We also set conventionally

$$V_k(\emptyset) = 0, \quad \forall k = 0, \dots, N.$$

The previous properties essentially characterizes intrinsic volumes as stated by the following result proved by Hadwiger, already mentioned in the introduction.

Theorem 2.5 (Hadwiger) *If σ is a continuous and rigid motion invariant valuation, then there exist $(N + 1)$ real coefficients c_0, \dots, c_N such that*

$$\sigma(K) = \sum_{i=0}^N c_i V_i(K),$$

for all $K \in \mathcal{K}^N \cup \{\emptyset\}$.

The previous theorem claims that $\{V_0, \dots, V_N\}$ spans the vector space of all continuous and invariant valuations on $\mathcal{K}^N \cup \{\emptyset\}$. It can be also proved that V_0, \dots, V_N are linearly independent, so they form a basis of this vector space. In Hadwiger's Theorem continuity can be replaced by monotonicity hypothesis, obtaining the following result.

Theorem 2.6 *If σ is a monotone increasing (resp. decreasing) rigid motion invariant valuation, then there exist $(N + 1)$ coefficients c_0, \dots, c_N such that $c_i \geq 0$ (resp. $c_i \leq 0$) for every i and*

$$\sigma(K) = \sum_{i=0}^N c_i V_i(K),$$

for all $K \in \mathcal{K}^N \cup \{\emptyset\}$.

A special case of the preceding results concerns *simple* valuations. A valuation μ is said to be simple if

$$\mu(K) = 0 \quad \forall K \in \mathcal{K}^N \text{ s.t. } \dim(K) < N.$$

Corollary 2.7 (Volume Theorem) *Let $\sigma : \mathcal{K}^N \cup \{\emptyset\} \rightarrow \mathbb{R}$ be a rigid motion invariant, simple and continuous valuation. Then there exists a constant c such that*

$$\mu = cV_N.$$

Remark 2.8 In the previous theorem continuity can be replaced by the following weaker assumption: for every *decreasing* sequence $K_i, i \in \mathbb{N}$, in \mathcal{K}^N , converging to $K \in \mathcal{K}^N$,

$$\lim_{i \rightarrow \infty} \sigma(K_i) = \sigma(K).$$

This follows, for instance, from the proof of the volume theorem given in [6].

3 Quasi-Concave Functions

3.1 The Space \mathcal{C}^N

Definition 3.1 A function $f : \mathbb{R}^N \rightarrow \mathbb{R}$ is said to be *quasi-concave* if

- $f(x) \geq 0$ for every $x \in \mathbb{R}^N$,
- for every $t > 0$, the set

$$L_t(f) = \{x \in \mathbb{R}^N : f(x) \geq t\}$$

is either a convex body or is empty.

We will denote with \mathcal{C}^N the set of all quasi-concave functions.

Typical examples of quasi-convex functions are (positive multiples of) characteristic functions of convex bodies. For $A \subseteq \mathbb{R}^N$ we denote by I_A its characteristic function

$$I_A : \mathbb{R}^N \rightarrow \mathbb{R}, \quad I_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Then we have that $sI_K \in \mathcal{C}^N$ for every $s > 0$ and $K \in \mathcal{K}^N$. We can also describe the sets $L_t(sI_K)$, indeed

$$L_t(sI_K) = \begin{cases} \emptyset & \text{if } t > s, \\ K & \text{if } 0 < t \leq s. \end{cases}$$

The following proposition gathers some of the basic properties of quasi-concave functions.

Proposition 3.2 *If $f \in \mathcal{C}^N$ then*

- $\lim_{\|x\| \rightarrow +\infty} f(x) = 0$,
- f is upper semi-continuous,

- f admits a maximum in \mathbb{R}^n , in particular

$$\sup_{\mathbb{R}^N} f < +\infty.$$

Proof To prove the first property, let $\epsilon > 0$; as $L_\epsilon(f)$ is compact, there exists $R > 0$ such that $L_\epsilon(f) \subset B_R$. This is equivalent to say that

$$f(x) \leq \epsilon \quad \forall x \quad \text{s.t.} \quad \|x\| \geq R.$$

Upper semi-continuity follows immediately from compactness of super-level sets. Let $M = \sup_{\mathbb{R}^N} f$ and assume that $M > 0$. Let $x_n, n \in \mathbb{N}$, be a maximizing sequence:

$$\lim_{n \rightarrow \infty} f(x_n) = M.$$

As f decays to zero at infinity, the sequence x_n is compact; then we may assume that it converges to $\bar{x} \in \mathbb{R}^N$. Then, by upper semi-continuity

$$f(\bar{x}) \geq \lim_{n \rightarrow \infty} f(x_n) = M.$$

□

For simplicity, given $f \in \mathcal{C}^N$, we will denote by $M(f)$ the maximum of f in \mathbb{R}^N .

Remark 3.3 Let $f \in \mathcal{C}^N$, we denote with $\text{supp}(f)$ the support of f , that is

$$\text{supp}(f) = \text{cl}(\{x \in \mathbb{R}^N : f(x) > 0\}).$$

This is a convex set; indeed

$$\text{supp}(f) = \text{cl}\left(\bigcup_{k=1}^{\infty} \{x \in \mathbb{R}^N : f(x) \geq 1/k\}\right).$$

The sets

$$\{x \in \mathbb{R}^N : f(x) \geq 1/k\} \quad k \in \mathbb{N},$$

forms an increasing sequence of convex bodies and their union is convex.

Remark 3.4 A special sub-class of quasi-concave functions is that formed by *log-concave* functions. Let u be a function defined on all \mathbb{R}^N , with values in $\mathbb{R} \cup \{+\infty\}$, convex and such that $\lim_{\|x\| \rightarrow +\infty} f(x) = +\infty$. Then the function $f = e^{-u}$ is quasi-concave (here we adopt the convention $e^{-\infty} = 0$). If f is of this form is said to be a *log-concave* function.

3.2 Operations with Quasi-Concave Functions

Let $f, g : \mathbb{R}^N \rightarrow \mathbb{R}$; we define the point-wise maximum and minimum function between f and g as

$$f \vee g(x) = \max\{f(x), g(x)\}, \quad f \wedge g(x) = \min\{f(x), g(x)\},$$

for all $x \in \mathbb{R}^N$. These operations, applied on \mathcal{C}^N , will replace the union and intersection in the definition of valuations on $\mathcal{K}^N \cup \{\emptyset\}$. The proof of the following equalities is straightforward.

Lemma 3.5 *If f and g belong to \mathcal{C}^N and $t > 0$:*

$$L_t(f \wedge g) = L_t(f) \cap L_t(g), \quad L_t(f \vee g) = L_t(f) \cup L_t(g).$$

As the intersection of two convex bodies is still a convex body, we have the following consequence.

Corollary 3.6 *For all $f, g \in \mathcal{C}^N$, $f \wedge g \in \mathcal{C}^N$.*

On the other hand, in general $f, g \in \mathcal{C}^N$ does not imply that $f \vee g$ does, as it is shown by the example in which f and g are characteristic functions of two convex bodies with empty intersection.

The following lemma follows from the definition of quasi-concave function and the fact that if T is a rigid motion of \mathbb{R}^N and $K \in \mathcal{K}^N$, then $T(K) \in \mathcal{K}^N$.

Lemma 3.7 *Let $f \in \mathcal{C}^N$ be a quasi concave function and $T : \mathbb{R}^N \rightarrow \mathbb{R}^N$ a rigid motion, then $f \circ T \in \mathcal{C}^N$.*

3.3 Three Technical Lemmas

We are going to prove some lemmas which will be useful for the study of continuity of valuations.

Lemma 3.8 *Let $f \in \mathcal{C}^N$. For all $t > 0$, except for at most countably many values, we have*

$$L_t(f) = \text{cl}(\{x \in \mathbb{R}^N : f(x) > t\}).$$

Proof We fix $t > 0$ and we define

$$\Omega_t(f) = \{x \in \mathbb{R}^N : f(x) > t\}, \quad H_t(f) = \text{cl}(\Omega_t(f)).$$

$\Omega_t(f)$ is a convex set for all $t > 0$, indeed

$$\Omega_t = \bigcup_{k \in \mathbb{N}} L_{t+1/k}(f).$$

Consequently H_t is a convex body and $H_t \subseteq L_t(f)$. We define $D_t = L_t(f) \setminus H_t$; our aim is now to prove that the set of all $t > 0$ such that $D_t \neq \emptyset$ is at most countable. We first note that if K and L are convex bodies with $K \subset L$, $\text{int}(L) \neq \emptyset$ and $L \setminus K \neq \emptyset$ then $\text{int}(L \setminus K) \neq \emptyset$, therefore

$$D_t \neq \emptyset \Leftrightarrow V_N(D_t) > 0. \quad (8)$$

It follows from

$$D_t = L_t(f) \setminus H_t \subseteq L_t(f) \setminus \Omega_t(f) = \{x \in \mathbb{R}^N : f(x) = t\},$$

that

$$t_1 \neq t_2 \Rightarrow D_{t_1}(f) \cap D_{t_2}(f) = \emptyset. \quad (9)$$

For the rest of the proof we proceed by induction on N . For $N = 1$, we observe that if f is identically zero, then the lemma is trivially true. If $\text{supp}(f) = \{x_0\}$ and $f(x_0) = t_0 > 0$, then we have

$$L_t(f) = \{x_0\} = \text{cl}(\Omega_t(f)) \quad \forall t > 0, t \neq t_0,$$

and in particular the lemma is true. We suppose next that $\text{int}(\text{supp}(f)) \neq \emptyset$; let $t_0 > 0$ be a number such that $\dim(L_t(f)) = 1$, for all $t \in (0, t_0)$ and $\dim(L_t(f)) = 0$, for all $t > t_0$. Moreover, let $t_1 = \max_{\mathbb{R}} f \geq t_0$. We observe that

$$L_t(f) = \text{cl}(\Omega_t(f)) = \emptyset \quad \forall t > t_1 \quad \text{and} \quad L_t(f) = \text{cl}(\Omega_t(f)) \quad \forall t \in (t_0, t_1).$$

Next we deal with values of $t \in (0, t_0)$. Let us fix $\epsilon > 0$ and let K be a compact set in \mathbb{R} such that $K \supseteq L_t(f)$ for every $t \geq \epsilon$. We define, for $i \in \mathbb{N}$,

$$T_i^\epsilon = \left\{ t \in [\epsilon, t_0) : V_1(D_t) \geq \frac{1}{i} \right\}.$$

As $D_t \subseteq K$ for all $t \geq \epsilon$ and taking (9) into account we obtain that T_i^ϵ is finite. So

$$T^\epsilon = \bigcup_{i \in \mathbb{N}} T_i^\epsilon$$

is countable for every $\epsilon > 0$. By (8)

$$\{t \geq \epsilon : D_t \neq \emptyset\} \text{ is countable}$$

for every $\epsilon > 0$, so that

$$\{t > 0 : D_t \neq \emptyset\}$$

is also countable. The proof for $N = 1$ is complete.

Assume now that the claim of the lemma is true up to dimension $(N - 1)$, and let us prove in dimension N . If the dimension of $\text{supp}(f)$ is strictly smaller than N , then (as $\text{supp}(f)$ is convex) there exists an affine subspace H of \mathbb{R}^N , of dimension $(N - 1)$, containing $\text{supp}(f)$. In this case the assert of the lemma follows applying the induction assumption to the restriction of f to H . Next, we suppose that there exists $t_0 > 0$ such that

$$\dim(L_t(f)) = N, \quad \forall t \in (0, t_0)$$

and

$$\dim(L_t(f)) < N, \quad \forall t > t_0.$$

By the same argument used in the one-dimensional case we can prove that

$$\{t \in (0, t_0) : D_t \neq \emptyset\}$$

is countable. For $t > t_0$, there exists a $(N - 1)$ -dimensional affine sub-space of \mathbb{R}^N containing $L_t(f)$ for every $t > t_0$. To conclude the proof we apply the inductive hypothesis to the restriction of f to this hyperplane. \square

Lemma 3.9 *Let $\{f_i\}_{i \in \mathbb{N}} \subseteq \mathcal{C}^N$ and $f \in \mathcal{C}^N$. Assume that $f_i \nearrow f$ point-wise in \mathbb{R}^N as $i \rightarrow +\infty$. Then, for all $t > 0$, except at most for countably many values,*

$$\lim_{i \rightarrow \infty} L_t(f_i) = L_t(f).$$

Proof For every $t > 0$, the sequence of convex bodies $L_t(f_i)$, $i \in \mathbb{N}$, is increasing and $L_t(f_i) \subset L_t(f)$ for every i . In particular this sequence admits a limit $L_t \subset L_t(f)$. We choose $t > 0$ such that

$$L_t(f) = \text{cl}(\{x \in \mathbb{R}^N : f(x) > t\}).$$

By the previous lemma we know that this condition holds for every t except at most countably many values. It is clear that for every x s.t. $f(x) > t$ we have $x \in L_t$, hence $L_t \supset \{x \in \mathbb{R}^N : f(x) > t\}$; on the other hand, as L_t is closed, we have that $L_t \supset L_t(f)$. Hence $L_t = L_t(f)$ and the proof is complete. \square

Lemma 3.10 *Let $\{f_i\}_{i \in \mathbb{N}} \subseteq \mathcal{C}^N$ and $f \in \mathcal{C}^N$. Assume that $f_i \searrow f$ point-wise in \mathbb{R}^N as $i \rightarrow +\infty$. Then for all $t > 0$*

$$\lim_{i \rightarrow \infty} L_t(f_i) = L_t(f).$$

Proof The sequence $L_t(f_i)$ is decreasing and its limit, denoted by L_t , contains $L_t(f)$. On the other hand, as now

$$L_t = \bigcap_{k \in \mathbb{N}} L_t(f_k)$$

(see Lemma 1.8.1 of [16]), if $x \in L_t$ then $f_i(x) \geq t$ for every i , so that $f(x) \geq t$ i.e. $x \in L_t(f)$. \square

4 Valuations

Definition 4.1 A functional $\mu : \mathcal{C}^N \rightarrow \mathbb{R}$ is said to be a valuation if

- $\mu(\underline{0}) = 0$, where $\underline{0} \in \mathcal{C}^N$ is the function identically equal to zero;
- for all f and $g \in \mathcal{C}^N$ such that $f \vee g \in \mathcal{C}^N$, we have

$$\mu(f) + \mu(g) = \mu(f \vee g) + \mu(f \wedge g).$$

A valuation μ is said to be rigid motion invariant, or simply invariant, if for every rigid motion $T : \mathbb{R}^N \rightarrow \mathbb{R}^N$ and for every $f \in \mathcal{C}^N$, we have

$$\mu(f) = \mu(f \circ T).$$

In this paper we will always consider invariant valuations. We will also need a notion of continuity which is expressed by the following definition.

Definition 4.2 A valuation μ is said to be continuous if for every sequence $\{f_i\}_{i \in \mathbb{N}} \subseteq \mathcal{C}^N$ and $f \in \mathcal{C}^N$ such that f_i converges point-wise to f in \mathbb{R}^N , and f_i is either monotone increasing or decreasing w.r.t. i , we have

$$\mu(f_i) \rightarrow \mu(f), \text{ for } i \rightarrow +\infty.$$

To conclude the list of properties that a valuation may have and that are relevant to our scope, we say that a valuation μ is monotone increasing (resp. decreasing) if, given $f, g \in \mathcal{C}^N$,

$$f \leq g \text{ point-wise in } \mathbb{R}^N \text{ implies } \mu(f) \leq \mu(g) \text{ (resp. } \mu(f) \geq \mu(g)).$$

4.1 A Brief Discussion on the Choice of the Topology in \mathcal{C}^N

A natural choice of a topology in \mathcal{C}^N would be the one induced by point-wise convergence. Let us see that this choice would too restrictive, with respect to the

theory of continuous and rigid motion invariant (but translations would be enough) valuations. Indeed, any translation invariant valuation μ on \mathcal{C}^N such that

$$\lim_{i \rightarrow \infty} \mu(f_i) = \mu(f)$$

for every sequence $f_i, i \in \mathbb{N}$, in \mathcal{C}^N , converging to some $f \in \mathcal{C}^N$ point-wise, must be the valuation constantly equal to 0. To prove this claim, let $f \in \mathcal{C}^N$ have compact support, let e_1 be the first vector of the canonical basis of \mathbb{R}^N and set

$$f_i(x) = f(x - i e_1) \quad \forall x \in \mathbb{R}^N, \quad \forall i \in \mathbb{N}.$$

The sequence f_i converges point-wise to the function $f_0 \equiv 0$ in \mathbb{R}^N , so that, by translation invariance, and as $\mu(f_0) = 0$, we have $\mu(f) = 0$. Hence μ vanishes on each function f with compact support. On the other hand every element of \mathcal{C}^N is the point-wise limit of a sequence of functions in \mathcal{C}^N with compact support. Hence $\mu \equiv 0$.

A different choice could be based on the following consideration: we have seen that $\mathcal{C}^N \subset L^\infty(\mathbb{R}^N)$, hence it inherits the topology of this space. In [3], Cavallina studied translation invariant and continuous valuations on $L^\infty(\mathbb{R}^N)$. In particular he proved that there exists non-trivial translation invariant and continuous valuations on this space, which vanishes on functions with compact support. In particular they cannot be written in integral form as those found in the present paper. Noting that in dimension $N = 1$ translation and rigid motion invariance provide basically the same condition, this suggest that the choice of the topology on $L^\infty(\mathbb{R}^N)$ on \mathcal{C}^N would lead us to a completely different type of valuations.

5 Integral Valuations

A class of examples of invariant valuations which will be crucial for our characterization results is that of integral valuations.

5.1 Continuous Integral Valuations

Let $k \in \{0, \dots, N\}$. For $f \in \mathcal{C}^N$, consider the function

$$t \rightarrow u(t) = V_k(L_t(f)) \quad t > 0.$$

This is a decreasing function, which vanishes for $t > M(f) = \max_{\mathbb{R}^N} f$. In particular u has bounded variation in $[\delta, M(f)]$ for every $\delta > 0$, hence there exists a Radon

measure defined in $(0, \infty)$, that we will denote by $S_k(f; \cdot)$, such that

$-S_k(f; \cdot)$ is the distributional derivative of u

(see, for instance, [1]). Note that, as u is decreasing, we have put a minus sign in this definition to have a non-negative measure. The support of $S_k(f; \cdot)$ is contained in $[0, M(f)]$.

Let ϕ be a continuous function defined on $[0, \infty)$, such that $\phi(0) = 0$. We consider the functional on \mathcal{C}^N defined by

$$\mu(f) = \int_{(0, \infty)} \phi(t) dS_k(f; t) \quad f \in \mathcal{C}^N. \tag{10}$$

The aim of this section is to prove that this is a continuous and invariant valuation on \mathcal{C}^N . As a first step, we need to find some condition on the function ϕ which guarantee that the above integral is well defined for every f .

Assume that

$$\exists \delta > 0 \text{ s.t. } \phi(t) = 0 \text{ for every } t \in [0, \delta]. \tag{11}$$

Then

$$\begin{aligned} \int_{(0, \infty)} \phi_+(t) dS_k(f; t) &= \int_{[\delta, M(f)]} \phi_+(t) dS_k(f, t) \\ &\leq M (V_k(L_\delta(f)) - V_k(M(f))) < \infty, \end{aligned}$$

where $M(f) = \max_{\mathbb{R}^N} f$, $M = \max_{[\delta, \max_{\mathbb{R}^N} f]} \phi_+$ and ϕ_+ is the positive part of ϕ . Analogously we can prove that the integral of the negative part of ϕ , denoted by ϕ_- , is finite, so that μ is well defined.

We will prove that, for $k \geq 1$, condition (11) is necessary as well. Clearly, if $\mu(f)$ is well defined (i.e. is a real number) for every $f \in \mathcal{C}^N$, then

$$\int_{(0, \infty)} \phi_+(t) dS_k(f; t) < \infty \quad \text{and} \quad \int_{(0, \infty)} \phi_-(t) dS_k(f; t) < \infty \quad \forall f \in \mathcal{C}^N.$$

Assume that ϕ_+ does not vanish identically in any right neighborhood of the origin. Then we have

$$\psi(t) := \int_0^t \phi_+(\tau) d\tau > 0 \quad \forall t > 0.$$

The function

$$t \rightarrow h(t) = \int_t^1 \frac{1}{\psi(s)} ds, \quad t \in (0, 1],$$

is strictly decreasing. As $k \geq 1$, we can construct a function $f \in \mathcal{C}^N$ such that

$$V_k(L_t(f)) = h(t) \quad \text{for every } t > 0. \quad (12)$$

Indeed, consider a function of the form

$$f(x) = w(\|x\|), \quad x \in \mathbb{R}^N,$$

where $w \in C^1([0, +\infty))$ is positive and strictly decreasing. Then $f \in \mathcal{C}^N$ and $L_t(f) = B_{r(t)}$, where

$$r(t) = w^{-1}(t)$$

for every $t \in (0, f(0)]$ (note that $f(0) = M(f)$). Hence

$$V_k(L_t(f)) = c (w^{-1}(t))^k$$

where c is a positive constant depending on k and N . Hence if we choose

$$w = \left[\left(\frac{1}{c} h \right)^{1/k} \right]^{-1},$$

(12) is verified. Hence

$$dS_k(f; t) = \frac{1}{\psi(t)} dt,$$

and

$$\int_{(0, \infty)} \phi_+(t) dS_k(f; t) = \int_{(0, M(f))} \frac{\psi'(t)}{\psi(t)} dt = \infty.$$

In the same way we can prove that ϕ_- must vanish in a right neighborhood of the origin. We have proved the following result.

Lemma 5.1 *Let $\phi \in C([0, \infty))$ and $k \in \{1, \dots, N\}$. Then ϕ has finite integral with respect to the measure $S_k(f; \cdot)$ for every $f \in \mathcal{C}^N$ if and only if ϕ verifies (11).*

In the special case $k = 0$, as the intrinsic volume V_0 is the Euler characteristic,

$$u(t) = \begin{cases} 1 & \text{if } 0 < t \leq M(f), \\ 0 & \text{if } t > M(f). \end{cases}$$

That is, S_0 is the Dirac point mass measure concentrated at $M(f)$ and μ can be written as

$$\mu(f) = \phi(M(f)) \quad \forall f \in \mathcal{C}^N.$$

Next we show that (10) defines a continuous and invariant valuation.

Proposition 5.2 *Let $k \in \{0, \dots, N\}$ and $\phi \in C([0, \infty))$ be such that $\phi(0) = 0$. If $k \geq 1$ assume that (11) is verified. Then (10) defines an invariant and continuous valuation on \mathcal{C}^N .*

Proof For every $f \in \mathcal{C}^N$ we define the function $u_f : [0, M(f)] \rightarrow \mathbb{R}$ as

$$u_f(t) = V_k(L_t(f)).$$

As already remarked, this is a decreasing function. In particular it has bounded variation in $[\delta, M(f)]$. Let $\phi_i, i \in \mathbb{N}$, be a sequence of functions in $C^\infty([0, \infty))$, with compact support, converging uniformly to ϕ on compact sets. As $\phi \equiv 0$ in $[0, \delta]$, we may assume that the same holds for every ϕ_i . Then we have

$$\mu(f) = \lim_{i \rightarrow \infty} \mu_i(f),$$

where

$$\mu_i(f) = \int_{[0, \infty)} \phi_i(t) dS_k(f; t) \quad \forall f \in \mathcal{C}^N.$$

By the definition of distributional derivative of a measure, we have, for every f and for every i :

$$\int_{[0, \infty)} \phi_i(t) dS_k(f; t) = \int_{[0, \infty)} u_f(t) \phi_i'(t) dt = \int_{[0, M(f)]} V_k(L_t(f)) \phi_i'(t) dt.$$

On the other hand, if $f, g \in \mathcal{C}^N$ are such that $f \vee g \in \mathcal{C}^N$, for every $t > 0$

$$L_t(f \vee g) = L_t(f) \cup L_t(g), \quad L_t(f \wedge g) = L_t(f) \cap L_t(g). \tag{13}$$

As intrinsic volumes are valuations

$$V_k(L_t(f \vee g)) + V_k(L_t(f \wedge g)) = V_k(L_t(f)) + V_k(L_t(g)).$$

Multiplying both sides times $\phi_i'(t)$ and integrating on $[0, \infty)$ we obtain

$$\mu_i(f \vee g) + \mu_i(f \wedge g) = \mu_i(f) + \mu_i(g).$$

Letting $i \rightarrow \infty$ we deduce the valuation property for μ .

In order to prove the continuity of μ , we first consider the case $k \geq 1$. Let $f_i, f \in \mathcal{C}^N, i \in \mathbb{N}$, and assume that the sequence f_i is either increasing or decreasing with respect to i , and it converges point-wise to f in \mathbb{R}^N . Note that in each case there exists a constant $M > 0$ such that $M(f_i), M(f) \leq M$ for every i . Consider now the sequence of functions u_{f_i} . By the monotonicity of the sequence f_i , and that

of intrinsic volumes, this is a monotone sequence of decreasing functions, and it converges a.e. to u_f in $(0, \infty)$, by Lemmas 3.9 and 3.10. In particular the sequence u_{f_i} has uniformly bounded total variation in $[\delta, M]$. Consequently, the sequence of measures $S_k(f_i; \cdot)$, $i \in \mathbb{N}$, converges weakly to the measure $S_k(f; \cdot)$ as $i \rightarrow \infty$. Hence, as ϕ is continuous

$$\lim_{i \rightarrow \infty} \mu(f_i) = \lim_{i \rightarrow \infty} \int_{[\delta, M]} \phi(t) dS_k(f_i; t) = \int_{[0, M]} \phi(t) dS_k(f; t) = \mu(f).$$

If $k = 0$ then we have seen that

$$\mu(f) = \phi(M(f)) \quad \forall f \in \mathcal{C}^N.$$

Hence in this case continuity follows from the following fact: if f_i , $i \in \mathbb{N}$, is a monotone sequence in \mathcal{C}^N converging point-wise to f , then

$$\lim_{i \rightarrow \infty} M(f_i) = M(f).$$

This is a simple exercise that we leave to the reader.

Finally, the invariance of μ follows directly from the invariance of intrinsic volumes with respect to rigid motions. \square

5.2 Monotone (and Continuous) Integral Valuations

In this section we introduce a slightly different type of integral valuations, which will be needed to characterize all possible continuous and monotone valuations on \mathcal{C}^N . Note that, as it will be clear in the sequel, when the involved functions are smooth enough, the two types can be reduced one to another by an integration by parts.

Let $k \in \{0, \dots, N\}$ and let ν be a Radon measure on $(0, +\infty)$; assume that

$$\int_0^{+\infty} V_k(L_t(f)) d\nu(t) < +\infty, \quad \forall f \in \mathcal{C}^N. \quad (14)$$

We will return later on explicit condition on ν such that (14) holds. Then define the functional $\mu : \mathcal{C}^N \rightarrow \mathbb{R}$ by

$$\mu(f) = \int_0^{+\infty} V_k(L_t(f)) d\nu(t) \quad \forall f \in \mathcal{C}^N. \quad (15)$$

Proposition 5.3 *Let ν be a Radon measure on $(0, \infty)$ which verifies (14); then the functional defined by (15) is a rigid motion invariant and monotone increasing valuation.*

Proof The proof that μ is a valuation follows from (13) and the valuation property for intrinsic volumes, as in the proof of Proposition 5.2. The same can be done for invariance. As for monotonicity, note that if $f, g \in \mathcal{C}^N$ and $f \leq g$, then

$$L_t(f) \subset L_t(g) \quad \forall t > 0.$$

Therefore, as intrinsic volumes are monotone, $V_k(L_t(f)) \leq V_k(L_t(g))$ for every $t > 0$. \square

If we do not impose any further assumption the valuation μ needs not to be continuous. Indeed, for example, if we fix $t = t_0 > 0$ and let $\nu = \delta_{t_0}$ be the delta Dirac measure at t_0 ; then the valuation

$$\mu(f) = V_N(L_{t_0}(f)), \quad \forall f \in \mathcal{C}^N,$$

is not continuous. To see it, let $f = t_0 I_{B_1}$ (recall that B_1 is the unit ball of \mathbb{R}^N) and let

$$f_i = t_0 \left(1 - \frac{1}{i} \right) I_{B_1} \quad \forall i \in \mathbb{N}.$$

Then f_i is a monotone sequence of elements of \mathcal{C}^N converging point wise to f in \mathbb{R}^N . On the other hand

$$\mu(f_i) = 0 \quad \forall i \in \mathbb{N},$$

while $\mu(f) = V_N(B_1) > 0$. The next results asserts that the presence of atoms is the only possible cause of discontinuity for μ . We recall that a measure ν defined on $[0, \infty)$ is said non-atomic if $\nu(\{t\}) = 0$ for every $t \geq 0$.

Proposition 5.4 *Let ν be a Radon measure on $(0, +\infty)$ such that (14) holds and let μ be the valuation defined by (14). Then the two following conditions are equivalent:*

- i) ν is non-atomic,
- ii) μ is continuous.

Proof Suppose that i) does not hold, than there exists t_0 such that $\nu(\{t_0\}) = \alpha > 0$. Define $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$ by

$$\varphi(t) = \int_{(0,t]} d\nu(s).$$

φ is an increasing function with a jump discontinuity at t_0 of amplitude α . Now let $f = t_0 I_{B_1}$ and $f_i = t_0(1 - \frac{1}{i})I_{B_1}$, for $i \in \mathbb{N}$. Then f_i is an increasing sequence in \mathcal{C}^N , converging point-wise to f in \mathbb{R}^N . On the other hand

$$\mu(f) = \int_0^{t_0} V_k(B) d\nu(s) = V_k(B)\nu((0, t_0]) = V_k(B_1) \varphi(t_0)$$

and similarly

$$\mu(f_i) = V_k(B_1) \varphi \left(t_0 - \frac{1}{i} \right).$$

Consequently

$$\lim_{i \rightarrow +\infty} \mu(f_i) < \mu(f).$$

Vice versa, suppose that *i)* holds. We observe that, as ν is non-atomic, every countable subset has measure zero with respect to ν . Let $f_i \in \mathcal{C}^N$, $i \in \mathbb{N}$, be a sequence such that either $f_i \nearrow f$ or $f_i \searrow f$ as $i \rightarrow +\infty$, point-wise in \mathbb{R}^N , for some $f \in \mathcal{C}^N$. Set

$$u_i(t) = V_k(L_t(f_i)), \quad u(t) = V_k(L_t(f)) \quad \forall t \geq 0, \quad \forall k \in \mathbb{N}.$$

The sequence u_i is monotone and, by Lemmas 3.9 and 3.10, converges to u ν -a.e. Hence, by the continuity of intrinsic volumes and the monotone convergence theorem, we obtain

$$\lim_{i \rightarrow \infty} \mu(f_i) = \lim_{i \rightarrow \infty} \int_{(0, \infty)} u_i(t) d\nu = \int_{(0, \infty)} u(t) d\nu(t) = \mu(f).$$

□

Now we are going to find a more explicit form of condition (14). We need the following lemma.

Lemma 5.5 *Let $\phi : [0, +\infty) \rightarrow \mathbb{R}$ be an increasing, non negative and continuous function with $\phi(0) = 0$ and $\phi(t) > 0$, for all $t > 0$. Let ν be a Radon measure such that $\phi(t) = \nu([0, t])$, for all $t \geq 0$. Then*

$$\int_0^1 \frac{1}{\phi^k(t)} d\nu(t) = +\infty, \quad \forall k \geq 1.$$

Proof Fix $\alpha \in [0, 1]$. The function $\psi : [\alpha, 1] \rightarrow \mathbb{R}$ defined by

$$\psi(t) = \begin{cases} \frac{1}{k-1} \phi^{1-k}(t) & \text{if } k > 1, \\ \ln(\phi(t)) & \text{if } k = 1, \end{cases}$$

is continuous and with bounded variation in $[\alpha, 1]$. Its distributional derivative is

$$\frac{1}{\phi^k(t)} \nu.$$

Hence, for $k > 1$,

$$\frac{1}{k-1}[\phi^{1-k}(\alpha) - \phi^{1-k}(1)] = \psi(1) - \psi(\alpha) = \int_{[\alpha,1]} \frac{dv}{\phi^k(t)}.$$

The claim of the lemma follows letting $\alpha \rightarrow 0^+$. A similar argument can be applied to the case $k = 1$. \square

Proposition 5.6 *Let ν be a non-atomic Radon measure on $[0, +\infty)$ and let $k \in \{1, \dots, N\}$. Then (14) holds if and only if:*

$$\exists \delta > 0 \text{ such that } \nu([0, \delta]) = 0. \quad (16)$$

Proof We suppose that there exists $\delta > 0$ such that $[0, \delta] \cap \text{supp}(\nu) = \emptyset$. Then we have, for every $f \in \mathcal{C}^N$,

$$\mu(f) = \int_{\delta}^{M(f)} V_i(L_t(f)) dv(t) \leq V_i(L_{\delta}(f)) \int_{\delta}^{M(f)} dv(t) \quad (17)$$

$$= V_i(L_{\delta}(f))(\nu([0, M(f)]) - \nu([0, \delta])) < +\infty. \quad (18)$$

with $M(f) = \max_{\mathbb{R}^N} f$.

Vice versa, assume that (14) holds. By contradiction, we suppose that for all $\delta > 0$, we have $\nu([0, \delta]) > 0$. We define

$$\phi(t) = \nu([0, t]), \quad t \in [0, 1]$$

then ϕ is continuous (as ν is non-atomic) and increasing; moreover $\phi(0) = 0$ and $\phi(t) > 0$, for all $t > 0$. The function

$$\psi(t) = \frac{1}{t\phi(t)}, \quad t \in (0, 1],$$

is continuous and strictly decreasing. Its inverse ψ^{-1} is defined in $[\psi(1), \infty)$; we extend it to $[0, \psi(1))$ setting

$$\psi^{-1}(r) = 1 \quad \forall r \in [0, \psi(1)).$$

Then

$$V_1(\{r \in [0, +\infty) : \psi^{-1}(r) \geq t\}) = \begin{cases} \psi(t), & \forall t \in (0, 1] \\ 0 & \forall t > 1. \end{cases}$$

We define now the function $f : \mathbb{R}^N \rightarrow \mathbb{R}$ as

$$f(x) = \psi^{-1}(\|x\|), \quad \forall x \in \mathbb{R}^N.$$

Then

$$L_t(f) = \{x \in \mathbb{R}^N : \psi(|x|) \geq t\} = B_{\frac{t}{\psi(t)}}(0),$$

and

$$V_k(L_t(f)) = c \frac{1}{t^k \phi^k(t)} \quad \forall t \in (0, 1],$$

where $c > 0$ depends on N and k . Hence, by Lemma 5.5

$$\int_0^{+\infty} V_k(L_t(f)) dv(t) = \int_0^1 V_k(L_t(f)) dv(t) \geq c \int_0^{+\infty} \frac{dv(t)}{\phi^k(t)} = +\infty.$$

□

The following proposition summarizes some of the results we have found so far.

Proposition 5.7 *Let $k \in \{0, \dots, N\}$ and let ν be a Radon measure on $[0, \infty)$ which is non-atomic and, if $k \geq 1$, verifies condition (16). Then the map $\mu : \mathcal{C}^N \rightarrow \mathbb{R}$ defined by (15) is an invariant, continuous and increasing valuations.*

5.3 The Connection Between the Two Types of Integral Valuations

When the regularity of the involved functions permits, the two types of integral valuations that we have seen can be obtained one from each other by a simple integration by parts.

Let $k \in \{0, \dots, N\}$ and $\phi \in C^1([0, \infty))$ be such that $\phi(0) = 0$. For simplicity, we may assume also that ϕ has compact support. Let $f \in \mathcal{C}^N$. By the definition of distributional derivative of an increasing function we have:

$$\int_{[0, \infty)} \phi(t) dS_k(f; t) = \int_{[0, \infty)} \phi'(t) V_k(L_t(f)) dt.$$

If we further decompose $-\phi'$ as the difference of two non-negative functions, and we denote by ν_1 and ν_2 the Radon measures having those functions as densities, we get

$$\int_{[0, \infty)} \phi(t) dS_k(f; t) = \int_{[0, \infty)} V_k(L_t(f)) d\nu_1(t) - \int_{[0, \infty)} V_k(L_t(f)) d\nu_2(t).$$

The assumption that ϕ has compact support can be removed by a standard approximation argument. In his way we have seen that each valuation of the

form (10), if ϕ is regular, is the difference of two monotone integral valuations of type (15).

Vice versa, let ν be a Radon measure (with support contained in $[\delta, \infty)$, for some $\delta > 0$), and assume that it has a smooth density with respect to the Lebesgue measure:

$$d\nu(t) = \phi'(t)dt$$

where $\phi \in C^1([0, \infty))$, and it has compact support. Then

$$\int_{[0, \infty)} V_k(L_t(f)) d\nu(t) = \int_{[0, \infty)} \phi(t) dS_k(f; t).$$

Also in this case the assumption that the support of ν is compact can be removed. In other words each integral monotone valuation, with sufficiently smooth density, can be written in the form (10).

5.4 The Case $k = N$

If μ is a valuation of the form (10) and $k = N$, the Layer Cake principle provides an alternative simple representation.

Proposition 5.8 *Let ϕ be a continuous function on $[0, \infty)$ verifying (16). Then for every $f \in \mathcal{C}^N$ we have*

$$\int_{[0, \infty)} \phi(t) dS_N(f; t) = \int_{\mathbb{R}^N} \phi(f(x))dx. \tag{19}$$

Proof As ϕ can be written as the difference of two non-negative continuous functions, and (19) is linear with respect to ϕ , there is no restriction if we assume that $\phi \geq 0$. In addition we suppose initially that $\phi \in C^1([0, \infty))$ and it has compact support. Fix $f \in \mathcal{C}^N$; by the definition of distributional derivative, we have

$$\int_{[0, \infty)} \phi(t) dS_N(f; t) = \int_{[0, \infty)} V_N(L_t(f))\phi'(t)dt.$$

There exists $\phi_1, \phi_2 \in C^1([0, \infty))$, strictly increasing, such that $\phi = \phi_1 - \phi_2$. Now:

$$\int_{[0, \infty)} V_N(L_t(f))\phi_1'(t)dt = \int_{[0, \infty)} V_N(\{x \in \mathbb{R}^N : \phi_1(f(x)) \geq s\})ds = \int_{\mathbb{R}^N} \phi_1(f(x))dx,$$

where in the last equality we have used the Layer Cake principle. Applying the same argument to ϕ_2 we obtain (19) when ϕ is smooth and compactly supported.

For the general case, we apply the result obtained in the previous part of the proof to a sequence ϕ_i , $i \in \mathbb{N}$, of functions in $C^1([0, \infty))$, with compact support, which converges uniformly to ϕ on compact subsets of $(0, \infty)$. The conclusion follows from a direct application of the dominated convergence theorem. \square

6 Simple Valuations

Throughout this section μ will be an invariant and continuous valuation on \mathcal{C}^N . We will also assume that μ is *simple*.

Definition 6.1 A valuation μ on \mathcal{C}^N is said to be simple if, for every $f \in \mathcal{C}^N$ with $\dim(\text{supp}(f)) < N$, we have $\mu(f) = 0$.

Note that $\dim(\text{supp}(f)) < N$ implies that $f = 0$ a.e. in \mathbb{R}^N , hence each valuation of the form (19) is simple. We are going to prove that in fact the converse of this statement is true.

Fix $t \geq 0$ and define a real-valued function σ_t on $\mathcal{K}^N \cup \{\emptyset\}$ as

$$\sigma_t(K) = \mu(tI_K) \quad \forall K \in \mathcal{K}^N, \quad \sigma_t(\emptyset) = 0.$$

Let $K, L \in \mathcal{K}^N$ be such that $K \cup L \in \mathcal{K}^N$. As, trivially,

$$tI_K \vee tI_L = tI_{K \cup L} \quad \text{and} \quad tI_K \wedge tI_L = tI_{K \cap L},$$

using the valuation property of μ we infer

$$\sigma_t(K \cup L) + \sigma_t(K \cap L) = \sigma_t(K) + \sigma_t(L),$$

i.e. σ_t is a valuation on \mathcal{K}^N . It also inherits directly two properties of μ : it is invariant and simple. Then, by the continuity of μ , Corollary 2.7 and the subsequent remark, there exists a constant c such that

$$\sigma_t(K) = cV_N(K) \tag{20}$$

for every $K \in \mathcal{K}^N$. The constant c will in general depend on t , i.e. it is a real-valued function defined in $[0, \infty)$. We denote this function by ϕ_N . Note that, as $\mu(f) = 0$ for $f \equiv 0$, $\phi_N(0) = 0$. Moreover, the continuity of μ implies that for every $t_0 \geq 0$ and for every monotone sequence t_i , $i \in \mathbb{N}$, converging to t_0 , we have

$$\phi_N(t_0) = \lim_{i \rightarrow \infty} \phi_N(t_i).$$

From this it follows that ϕ_N is continuous in $[0, \infty)$.

Proposition 6.2 *Let μ be an invariant, continuous and simple valuation on \mathcal{C}^N . Then there exists a continuous function ϕ_N on $[0, \infty)$, such that*

$$\mu(tI_K) = \phi_N(t) V_N(K)$$

for every $t \geq 0$ and for every $K \in \mathcal{K}^N$.

6.1 Simple Functions

Definition 6.3 A function $f : \mathbb{R}^N \rightarrow \mathbb{R}$ is called simple if it can be written in the form

$$f = t_1 I_{K_1} \vee \cdots \vee t_m I_{K_m} \quad (21)$$

where $0 < t_1 < \cdots < t_m$ and K_1, \dots, K_m are convex bodies such that

$$K_1 \supset K_2 \supset \cdots \supset K_m.$$

The proof of the following fact is straightforward.

Proposition 6.4 *Let f be a simple function of the form (21) and let $t > 0$. Then*

$$L_t(f) = \{x \in \mathbb{R}^N : f(x) \geq t\} = \begin{cases} K_i & \text{if } t \in (t_{i-1}, t_i] \text{ for some } i = 1, \dots, m, \\ \emptyset & \text{if } t > t_m, \end{cases} \quad (22)$$

where we have set $t_0 = 0$.

In particular simple functions are quasi-concave. Let $k \in \{0, \dots, N\}$, and let f be of the form (21). Consider the function

$$t \rightarrow u(t) := V_k(L_t(f)), \quad t > 0.$$

By Proposition 6.4, this is a decreasing function that is constant on each interval of the form $(t_{i-1}, t_i]$, on which it has the value $V_k(K_i)$. Hence its distributional derivative is $-S_k(f; \cdot)$, where

$$S_k(f; \cdot) = \sum_{i=1}^{m-1} (V_k(K_i) - V_k(K_{i+1})) \delta_{t_i}(\cdot) + V_k(K_m) \delta_{t_m}(\cdot). \quad (23)$$

6.2 Characterization of Simple Valuations

In this section we are going to prove Theorem 1.2. We will first prove it for simple functions and then pass to the general case by approximation.

Lemma 6.5 *Let μ be an invariant, continuous and simple valuation on \mathcal{C}^N , and let $\phi = \phi_N$ be the function whose existence is established in Proposition 6.2. Then, for every simple function $f \in \mathcal{C}^N$ we have*

$$\mu(f) = \int_{[0, \infty)} \phi(t) dS_N(f; t).$$

Proof Let f be of the form (21). We prove the following formula

$$\mu(f) = \sum_{i=1}^{m-1} \phi(t_i)(V_N(K_i) - V_N(K_{i+1})) + \phi(t_m)V_N(K_m); \quad (24)$$

by (23), this is equivalent to the statement of the lemma. Equality (24) will be proved by induction on m . For $m = 1$ its validity follows from Proposition 6.2. Assume that it has been proved up to $(m - 1)$. Set

$$g = t_1 I_{K_1} \vee \cdots \vee t_{m-1} I_{K_{m-1}}, \quad h = t_m I_{K_m}.$$

We have that $g, h \in \mathcal{C}^N$ and

$$g \vee h = f \in \mathcal{C}^N, \quad g \wedge h = t_{m-1} I_{K_{m-1}}.$$

Using the valuation property of μ and Proposition 6.2 we get

$$\begin{aligned} \mu(f) &= \mu(g \vee h) = \mu(g) + \mu(h) - \mu(g \wedge h) \\ &= \mu(g) + \phi(t_m)V_N(K_m) - \phi(t_{m-1})V_N(K_{m-1}). \end{aligned}$$

On the other hand, by induction

$$\mu(g) = \sum_{i=1}^{m-2} \phi(t_i)(V_N(K_i) - V_N(K_{i+1})) + \phi(t_{m-1})V_N(K_{m-1}).$$

The last two equalities complete the proof. □

Proof of Theorem 1.2 As before, $\phi = \phi_N$ is the function coming from Proposition 6.2. We want to prove that

$$\mu(f) = \int_{[0, \infty)} \phi(t) dS_N(f; t) \quad (25)$$

for every $f \in \mathcal{C}^N$. This, together with Proposition 5.8, provides the proof.

Step 1. Our first step is to establish the validity of this formula when the support of f is bounded, i.e. there exists some convex body K such that

$$L_t(f) \subset K \quad \forall t > 0. \quad (26)$$

Given $f \in \mathcal{C}^N$ with this property, we build a monotone sequence of simple functions, f_i , $i \in \mathbb{N}$, converging point-wise to f in \mathbb{R}^N . Let $M = M(f)$ be the maximum of f on \mathbb{R}^N . Fix $i \in \mathbb{N}$. We consider the dyadic partition \mathcal{P}_i of $[0, M]$:

$$\mathcal{P}_i = \left\{ t_j = j \frac{M}{2^i} : j = 0, \dots, 2^i \right\}.$$

Set

$$K_j = L_{t_j}(f), \quad f_i = \bigvee_{j=1}^{2^i} t_j I_{K_j}.$$

f_i is a simple function; as $t_j I_{K_j} \leq f$ for every j we have that $f_i \leq f$ in \mathbb{R}^N . The sequence of function f_i is increasing, since $\mathcal{P}_i \subset \mathcal{P}_{i+1}$. The inequality $f_i \leq f$ implies that

$$\lim_{i \rightarrow \infty} f_i(x) \leq f(x) \quad \forall x \in \mathbb{R}^N$$

(in particular the support of f_i is contained in K , for every $i \in \mathbb{N}$). We want to establish the reverse inequality. Let $x \in \mathbb{R}^N$; if $f(x) = 0$ then trivially

$$f_i(x) = 0 \quad \forall i \quad \text{hence} \quad \lim_{i \rightarrow \infty} f_i(x) = f(x).$$

Assume that $f(x) > 0$ and fix $\epsilon > 0$. Let $i_0 \in \mathbb{N}$ be such that $2^{-i_0} M < \epsilon$. Let $j \in \{1, \dots, 2^{i_0} - 1\}$ be such that

$$f(x) \in \left(j \frac{M}{2^{i_0}}, (j+1) \frac{M}{2^{i_0}} \right].$$

Then

$$f(x) \leq j \frac{M}{2^{i_0}} + \frac{M}{2^{i_0}} \leq f_{i_0}(x) + \epsilon \leq \lim_{i \rightarrow \infty} f_i(x) + \epsilon.$$

Hence the sequence f_i converges point-wise to f in \mathbb{R}^N . In particular, by the continuity of μ we have that

$$\mu(f) = \lim_{i \rightarrow \infty} \mu(f_i) = \lim_{i \rightarrow \infty} \int_{[0, \infty)} \phi(t) dS_N(f_i; t).$$

By Lemma 3.9, a further consequence is that

$$\lim_{i \rightarrow \infty} u_i(t) = u(t) \quad \text{for a.e. } t \in (0, \infty),$$

where

$$u_i(t) = V_N(L_t(f_i)), \quad i \in \mathbb{N}, \quad u(t) = V_N(L_t(f))$$

for $t > 0$. We consider now the sequence of measures $S_N(f_i; \cdot)$, $i \in \mathbb{N}$; the total variation of these measures in $(0, \infty)$ is uniformly bounded by $V_N(K)$, moreover they are all supported in $(0, M)$. As they are the distributional derivatives of the functions u_i , which converges a.e. to u , we have that (see for instance [1, Proposition 3.13]) the sequence $S_N(f_i; \cdot)$ converges weakly in the sense of measures to $S_N(f; \cdot)$. This implies that

$$\lim_{i \rightarrow \infty} \int_{(0, \infty)} \bar{\phi}(t) dS_N(f_i; t) = \int_{(0, \infty)} \bar{\phi}(t) dS_N(f; t) \tag{27}$$

for every function $\bar{\phi}$ continuous in $(0, \infty)$, such that $\bar{\phi}(0) = 0$ and $\bar{\phi}(t)$ is identically zero for t sufficiently large. In particular (recalling that $\phi(0) = 0$), we can take $\bar{\phi}$ such that it equals ϕ in $[0, M]$. Hence, as the support of the measures $S_N(f_i; \cdot)$ is contained in this interval, we have that (27) holds for ϕ as well. This proves the validity of (25) for functions with bounded support.

Step 2. This is the most technical part of the proof. The main scope here is to prove that ϕ is identically zero in some right neighborhood of the origin. Let $f \in \mathcal{C}^N$. For $i \in \mathbb{N}$, let

$$f_i = f \wedge (M(f)I_{B_i})$$

where B_i is the closed ball centered at the origin, with radius i . The function f_i coincides with f in B_i and vanishes in $\mathbb{R}^N \setminus B_i$; in particular it has bounded support. Moreover, the sequence f_i , $i \in \mathbb{N}$, is increasing and converges point-wise to f in \mathbb{R}^N . Hence

$$\mu(f) = \lim_{i \rightarrow \infty} \mu(f_i) = \lim_{i \rightarrow \infty} \int_{(0, \infty)} \phi(t) dS_N(f_i; t).$$

Let ϕ_+ and ϕ_- be the positive and negative parts of ϕ , respectively. We have that

$$\lim_{i \rightarrow \infty} \left[\int_{(0, \infty)} \phi_+(t) dS_N(f_i; t) + \int_{(0, \infty)} \phi_-(t) dS_N(f_i; t) \right]$$

exists and it is finite. We want to prove that this implies that ϕ_+ and ϕ_- vanishes identically in $[0, \delta]$ for some $\delta > 0$.

By contradiction, assume that this is not true for ϕ_+ . Then there exists three sequences t_i, r_i and $\epsilon_i, i \in \mathbb{N}$, with the following properties: t_i tends decreasing to zero; $r_i > 0$ is such that the intervals $C_i = [t_i - r_i, t_i + r_i]$ are contained in $(0, 1]$ and pairwise disjoint; $\phi_+(t) \geq \epsilon_i > 0$ for $t \in C_i$. Let

$$C = \bigcup_{i \in \mathbb{N}} C_i, \quad \Omega = (0, 1] \setminus C.$$

Next we define a function $\gamma : (0, 1] \rightarrow [0, \infty)$ as follows. $\gamma(t) = 0$ for every $t \in \Omega$ while, for every $i \in \mathbb{N}$, γ is continuous in C_i and

$$\gamma(t_i \pm r_i) = 0, \quad \int_{C_i} \gamma(t) dt = \frac{1}{\epsilon_i}.$$

Note in particular that γ vanishes on the support of ϕ_- intersected with $(0, 1]$. We also set

$$g(t) = \gamma(t) + 1 \quad \forall t > 0.$$

Observe that

$$\int_0^1 \phi_-(t)g(t)dt = \int_0^1 \phi_-(t)dt < \infty.$$

On the other hand

$$\begin{aligned} \int_0^1 \phi_+(t)g(t)dt &\geq \int_0^1 \phi(t)\gamma(t)dt = \sum_{i=1}^{\infty} \int_{C_i} \phi_+(t)\gamma(t)dt \\ &\geq \sum_{i=1}^{\infty} \epsilon_i \int_{C_i} \gamma(t)dt = +\infty. \end{aligned}$$

Let

$$G(t) = \int_t^1 g(s)ds \quad \text{and} \quad \rho(t) = [G(t)]^{1/N}, \quad 0 < t \leq 1.$$

As γ is non-negative, g is strictly positive, and continuous in $(0, 1)$. Hence G is strictly decreasing and continuous, and the same holds for ρ . Let

$$S = \sup_{(0,1]} \rho = \lim_{t \rightarrow 0^+} \rho(t),$$

and let $\rho^{-1} : [0, S) \rightarrow \mathbb{R}$ be the inverse function of ρ . If $S < \infty$, we extend ρ^{-1} to be zero in $[S, \infty)$. In this way, ρ^{-1} is continuous in $[0, \infty)$, and $C^1([0, S))$. Let

$$f(x) = \rho^{-1}(\|x\|), \quad \forall x \in \mathbb{R}^N.$$

For $t > 0$ we have

$$L_t(f) = \begin{cases} \{x \in \mathbb{R}^N : \|x\| \leq \rho(t)\} & \text{if } t \leq 1, \\ \emptyset & \text{if } t > 1. \end{cases}$$

In particular $f \in \mathcal{C}^N$. Consequently,

$$V_N(L_t(f)) = c \rho^N(t) = c G(t) \quad \forall t \in (0, 1],$$

where $c > 0$ is a dimensional constant, and then

$$dS_N(f; t) = c g(t) dt.$$

By the previous considerations

$$\int_{[0, \infty)} \phi_+(t) dS_N(f, t) = c \int_{[0, \infty)} \phi_+(t) g(t) dt = \infty, \quad \int_{[0, \infty)} \phi_+(t) dS_N(f, t) < \infty.$$

Clearly we also have that

$$\int_{[0, \infty)} \phi_+(t) dS_N(f, t) = \lim_{i \rightarrow \infty} \int_{[0, \infty)} \phi_+(t) dS_N(f_i, t),$$

and the same holds for ϕ_- ; here f_i is the sequence approximating f defined before. We reached a contradiction.

Step 3. The conclusion of the proof proceeds as follows. Let $\bar{\mu} : \mathcal{C}^N \rightarrow \mathbb{R}$ be defined by

$$\bar{\mu}(f) = \int_{(0, \infty)} \phi(t) dS_N(f; t).$$

By the previous step, and by the results of Sect. 5.1, this is well defined, and is an invariant and continuous valuation. Hence the same properties are shared by $\mu - \bar{\mu}$; on the other hand, by Step 1 and the definition of $\bar{\mu}$, this vanishes on functions with bounded support. As for any element f of \mathcal{C}^N there is a monotone sequence of functions in \mathcal{C}^N , with bounded support and converging point-wise to f in \mathbb{R}^N , and as $\mu - \bar{\mu}$ is continuous, it must be identically zero on \mathcal{C}^N . □

7 Proof of Theorem 1.1

We proceed by induction on N . For the first step of induction, let μ be an invariant and continuous valuation on \mathcal{C}^1 . For $t > 0$ let

$$\phi_0(t) = \mu(tI_{\{0\}}).$$

This is a continuous function in \mathbb{R} , with $\phi_0(0) = 0$. We consider the application $\mu_0 : \mathcal{C}^1 \rightarrow \mathbb{R}$:

$$\mu_0(f) = \phi_0(M(f))$$

where as usual $M(f) = \max_{\mathbb{R}} f$. By what we have seen in Sect. 5.1, this is an invariant and continuous valuation. Note that it can be written in the form

$$\mu_0(f) = \int_{(0,\infty)} \phi_0(t) dS_0(f; t).$$

Next we set $\bar{\mu} = \mu - \mu_0$; this is still an invariant and continuous valuation, and it is also simple. Indeed, if $f \in \mathcal{C}^1$ is such that $\dim(\text{supp}(f)) = 0$, this is equivalent to say that

$$f = tI_{\{x_0\}}$$

for some $t \geq 0$ and $x_0 \in \mathbb{R}$. Hence

$$\mu(f) = \mu(tI_{\{0\}}) = \phi_0(t) = \mu_0(f).$$

Therefore we may apply Theorem 1.2 to μ_1 and deduce that there exists a function $\phi_1 \in C([0, \infty))$, which vanishes identically in $[0, \delta]$ for some $\delta > 0$, and such that

$$\bar{\mu}(f) = \int_{(0,\infty)} \phi_1(t) dS_1(f; t) \quad \forall f \in \mathcal{C}^1.$$

The proof in the one-dimensional case is complete.

We suppose that the Theorem holds up to dimension $(N - 1)$. Let H be an hyperplane of \mathbb{R}^N and define $\mathcal{C}_H^N = \{f \in \mathcal{C}^N : \text{supp}(f) \subseteq H\}$. \mathcal{C}_H^N can be identified with \mathcal{C}^{N-1} ; moreover μ restricted to \mathcal{C}_H^N is trivially still an invariant and continuous valuation. By the induction assumption, there exists $\phi_k \in C([0, \infty))$, $k = 0, \dots, N - 1$, such that

$$\mu(f) = \sum_{k=0}^{N-1} \int_{(0,\infty)} \phi_k(t) dS_k(f; t) \quad \forall f \in \mathcal{C}_H^N.$$

In addition, there exists $\delta > 0$ such that $\phi_1, \dots, \phi_{N-1}$ vanish in $[0, \delta]$. Let $\bar{\mu} : \mathcal{C}^N \rightarrow \mathbb{R}$ as

$$\bar{\mu}(f) = \sum_{k=0}^{N-1} \int_{(0,\infty)} \phi_k(t) dS_k(f; t).$$

This is well defined for $f \in \mathcal{C}^N$ and it is an invariant and continuous valuation. The difference $\mu - \bar{\mu}$ is simple; applying Theorem 1.2 to it, as in the one-dimensional case, we complete the proof. \square

8 Monotone Valuations

In this section we will prove Theorem 1.3; in particular we will assume that μ is an invariant, continuous and increasing valuation on \mathcal{C}^N throughout. Note that, as $\mu(f_0) = 0$, where f_0 is the function identically zero in \mathbb{R}^N , we have that $\mu(f) \geq 0$ for every $f \in \mathcal{C}^N$.

The proof is divided into three parts.

8.1 Identification of the Measures ν_k , $k = 0, \dots, N$

We proceed as in the proof of Proposition 6.2. Fix $t > 0$ and consider the application $\sigma_t : \mathcal{K}^N \rightarrow \mathbb{R}$:

$$\sigma_t(K) = \mu(tI_K), \quad K \in \mathcal{K}^N.$$

This is a rigid motion invariant valuation on \mathcal{K}^N and, as μ is increasing, σ_t has the same property. Hence there exists $(N + 1)$ coefficients, depending on t , that we denote by $\psi_k(t)$, $k = 0, \dots, N$, such that

$$\sigma_t(K) = \sum_{k=0}^N \psi_k(t) V_k(K) \quad \forall K \in \mathcal{K}^N. \quad (28)$$

We prove that each ψ_k is continuous and monotone in $(0, \infty)$. Let us fix the index $k \in \{0, \dots, N\}$, and let Δ_k be a closed k -dimensional ball in \mathbb{R}^N , of radius 1. We have

$$V_j(\Delta_k) = 0 \quad \forall j = k + 1, \dots, N,$$

and

$$V_k(\Delta_k) =: c(k) > 0.$$

Fix $r \geq 0$; for every j , V_j is positively homogeneous of order j , hence, for $t > 0$,

$$\mu(tI_{r\Delta_k}) = \sum_{j=0}^k r^j V_j(\Delta_k) \psi_j(t).$$

Consequently

$$\psi_k(t) = V_k(\Delta_k) \cdot \lim_{r \rightarrow \infty} \frac{\mu(tI_{r\Delta_k})}{r^k}.$$

By the properties of μ , the function $t \rightarrow \mu(tI_{r\Delta_k})$ is non-negative, increasing and vanishes for $t = 0$, for every $r \geq 0$; these properties are inherited by ψ_k .

As for continuity, we proceed in a similar way. To prove that ψ_0 is continuous we observe that the function

$$t \rightarrow \mu(t\Delta_0) = \psi_0(t)$$

is continuous, by the continuity of μ . Assume that we have proved that $\psi_0, \dots, \psi_{k-1}$ are continuous. Then by the equality

$$\mu(tI_{\Delta_k}) = \sum_{j=1}^k V_j(\Delta_k)\psi_j(t),$$

it follows that ψ_k is continuous.

Proposition 8.1 *Let μ be an invariant, continuous and increasing valuation on \mathcal{C}^N . Then there exists $(N + 1)$ functions ψ_0, \dots, ψ_N defined in $[0, \infty)$, such that (28) holds for every $t \geq 0$ and for every K . In particular each ψ_k is continuous, increasing, and vanishes at $t = 0$.*

For every $k \in \{0, \dots, N\}$ we denote by ν_k the distributional derivative of ψ_k . In particular as ψ_k is continuous, ν_k is non-atomic and

$$\psi_k(t) = \nu_k([0, t)), \quad \forall t \geq 0.$$

Since ψ_k are non-negative functions, by Theorem 2.6, then ν_k are non-negative measures.

8.2 The Case of Simple Functions

Let f be a simple function:

$$f = t_1 I_{K_1} \vee \dots \vee t_m I_{K_m}$$

with $0 < t_1 < \dots < t_m$, $K_1 \supset \dots \supset K_m$ and $K_i \in \mathcal{K}^N$ for every i . The following formula can be proved with the same method used for (24)

$$\mu(f) = \sum_{k=0}^N \sum_{i=1}^m (\psi_k(t_i) - \psi_k(t_{i-1})) V_k(L_{t_i}(f)), \quad (29)$$

where we have set $t_0 = 0$. As

$$\psi_k(t_i) - \psi_k(t_{i-1}) = \nu_k((t_{i-1}, t_i])$$

and $L_t(f) = K_i$ for every $t \in (t_{i-1}, t_i]$, we have

$$\mu(f) = \sum_{k=0}^N \int_{[0, \infty)} V_k(L_t(f)) d\nu_k(t). \quad (30)$$

In other words, we have proved the theorem for simple functions.

8.3 Proof of Theorem 1.3

Let $f \in \mathcal{C}^N$ and let f_i , $i \in \mathbb{N}$, be the sequence of functions built in the proof of Theorem 1.2, Step 2. We have seen that f_i is increasing and converges point-wise to f in \mathbb{R}^N . In particular, for every $k = 0, \dots, N$, the sequence of functions $V_k(L_t(f_i))$, $t \geq 0$, $i \in \mathbb{N}$, is monotone increasing and it converges a.e. to $V_k(L_t(f))$ in $[0, \infty)$. By the B. Levi theorem, we have that

$$\lim_{i \rightarrow \infty} \int_{[0, \infty)} V_k(L_t(f_i)) d\nu_k(t) = \int_{[0, \infty)} V_k(L_t(f)) d\nu_k(t)$$

for every k . Using (30) and the continuity of μ we have that the representation formula (30) can be extended to every $f \in \mathcal{C}^N$.

Note that in (21) each term of the sum in the right hand-side is non-negative, hence we have that

$$\int_{[0, \infty)} V_k(L_t(f)) d\nu_k(t) < \infty \quad \forall f \in \mathcal{C}^N.$$

Applying Proposition 5.6 we obtain that, if $k \geq 1$, there exists $\delta > 0$ such that the support of ν_k is contained in $[\delta, \infty)$. The proof is complete. \square

References

1. L. Ambrosio, N. Fusco, D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems* (Oxford University Press, Oxford, 2000)
2. Y. Baryshnikov, R. Ghrist, M. Wright, Hadwiger's theorem for definable functions. *Adv. Math.* **245**, 573–586 (2013)
3. L. Cavallina, Non-trivial translation-invariant valuations on L^∞ . Preprint, 2015 (arXiv:1505.00089)

4. L. Cavallina, A. Colesanti, Monotone valuations on the space of convex functions. *Anal. Geom. Metr. Spaces* **1**(3) (2015) (electronic version)
5. H. Hadwiger, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie* (Springer, Berlin-Göttingen-Heidelberg, 1957)
6. D. Klain, A short proof of Hadwiger's characterization theorem. *Mathematika* **42**, 329–339 (1995)
7. D. Klain, G. Rota, *Introduction to Geometric Probability* (Cambridge University Press, New York, 1997)
8. H. Kone, Valuations on Orlicz spaces and L^ϕ -star sets. *Adv. Appl. Math.* **52**, 82–98 (2014)
9. M. Ludwig, Fisher information and matrix-valued valuations. *Adv. Math.* **226**, 2700–2711 (2011)
10. M. Ludwig, Valuations on function spaces. *Adv. Geom.* **11**, 745–756 (2011)
11. M. Ludwig, Valuations on Sobolev spaces. *Am. J. Math.* **134**, 824–842 (2012)
12. M. Ludwig, Covariance matrices and valuations. *Adv. Appl. Math.* **51**, 359–366 (2013)
13. D. Ma, Analysis of Sobolev Spaces in the Context of Convex Geometry and Investigations of the Busemann-Petty Problem. PhD thesis, Technische Universität, Vienna, 2015
14. D. Ma, Real-valued valuations on Sobolev spaces. Preprint, 2015 (arXiv: 1505.02004)
15. M. Ober, L^p -Minkowski valuations on L^q -spaces. *J. Math. Anal. Appl.* **414**, 68–87 (2014)
16. R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, 2nd expanded edn. (Cambridge University Press, Cambridge, 2014)
17. A. Tsang, Valuations on L^p -spaces. *Int. Math. Res. Not.* **20**, 3993–4023 (2010)
18. A. Tsang, Minkowski valuations on L^p -spaces. *Trans. Am. Math. Soc.* **364**(12), 6159–6186 (2012)
19. T. Wang, Affine Sobolev inequalities. PhD thesis, Technische Universität, Vienna, 2013
20. T. Wang, Semi-valuations on $BV(\mathbb{R}^n)$. *Indiana Univ. Math. J.* **63**, 1447–1465 (2014)
21. M. Wright, Hadwiger integration on definable functions, PhD thesis, University of Pennsylvania, 2011