On a Problem of Farrell and Vershynin in Random Matrix Theory

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Abstract We settle a question of Farrell and Vershynin on the inverse of the perturbation of a given arbitrary symmetric matrix by a GOE element.

1 Introduction

In [1], the authors consider the invertibility of $d \times d$-matrices of the form $D+R$, with D an arbitrary symmetric deterministic matrix and R a symmetric random matrix whose independent entries have continuous distributions with bounded densities. In this setting, a uniform estimate

$$\|(D+R)^{-1}\| = O(d^2) \tag{1}$$

is shown to hold with high probability. The authors conjecture that (1) may be improved to $O(\sqrt{d})$. The purpose of this short Note is to prove this in the case R is Gaussian. Thus we have (stated in the ℓ^2_d-normalized setting).

Proposition *Let T be an arbitrary matrix in $\mathrm{Sym}(d)$. Then, for A (normalized) in GOE, there is a uniform estimate*

$$\|(A+T)^{-1}\| = O(d) \tag{2}$$

with large probability.

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2 Proof of the Proposition

By invariance of GOE under orthogonal transformations, we may assume T diagonal. Let K be a suitable constant and partition

$$\{1, \dots, d\} = \Omega_1 \cup \Omega_2$$

with

$$\Omega_1 = \{j = 1, \dots, d; |T_{jj}| > K\}.$$

Denote $T^{(i)} = \pi_{\Omega_i} T \pi_{\Omega_i} (i = 1, 2)$ and $A^{(i,j)} = \pi_{\Omega_i} A \pi_{\Omega_j} (i, j = 1, 2)$. Since

$$(A^{(1,1)} + T^{(1)})^{-1} = \big(I + (T^{(1)})^{-1} A^{(1,1)}\big)(T^{(1)})^{-1}$$

and

$$\|(T^{(1)})^{-1} A^{(1,1)}\| \le \frac{1}{K} \|A^{(1,1)}\| < \frac{1}{2}$$

with large probability, we ensure that

$$\|(A^{(1,1)} + T^{(1)})^{-1}\| < 1. \tag{3}$$

Next, write by the Schur complement formula

$$(A+T)^{-1} = \begin{pmatrix} (A^{(1,1)} + T^{(1)})^{-1} + (A^{(1,1)} + T^{(1)})^{-1} A^{(1,2)} S^{-1} A^{(2,1)} (A^{(1,1)} + T^{(1)})^{-1} & -(A^{(1,1)} + T^{(1)})^{-1} A^{(1,2)} S^{-1} \\ -S^{-1} A^{(2,1)} (A^{(1,1)} + T^{(1)})^{-1} & S^{-1} \end{pmatrix} \tag{4}$$

defining

$$S = A^{(2,2)} + T^{(2)} - A^{(2,1)} (A^{(1,1)} + T^{(1)})^{-1} A^{(1,2)}. \tag{5}$$

Hence by (4)

$$\begin{aligned} \|(A+T)^{-1}\| &\le C(1 + \|(A^{(1,1)} + T^{(1)})^{-1}\|^2)(1 + \|A\|^2)\|S^{-1}\| \\ &\le C_1 \|S^{-1}\|. \end{aligned} \tag{6}$$

Note that $A^{(2,2)}$ and $A^{(2,1)}(A^{(1,1)}+T^{(1)})^{-1}A^{(1,2)}$ are independent in the A randomness. Thus S may be written in the form

$$S = A^{(2,2)} + S_0 \tag{7}$$

with $S_0 \in \mathrm{Sym}(d)$, $\|S_0\| < O(1)$ (by construction, $\|T^{(2)}\| \leq K$) and $A^{(2,2)}$ and S_0 independent.

Fixing S_0, we may again exploit the invariance to put S_0 in diagonal form, obtaining

$$A^{(2,2)} + S_0' \text{ with } S_0' \text{ diagonal}. \tag{8}$$

Hence, we reduced the original problem to the case T is diagonal and $\|T\| < K + 1$.

Note however that (8) is a $(d_1 \times d_1)$-matrix and since d_1 may be significantly smaller than d, $A^{(2,2)}$ is not necessarily normalized anymore. Thus after renormalization of $A^{(2,2)}$, setting

$$A_1 = \left(\frac{d}{d_1}\right)^{\frac{1}{2}} A^{(2,2)} \tag{9}$$

and denoting

$$T_1 = \left(\frac{d}{d_1}\right)^{\frac{1}{2}} S_0' \tag{10}$$

we have

$$\|T_1\| < \left(\frac{d}{d_1}\right)^{\frac{1}{2}} (K+1) \tag{11}$$

while the condition [cf. (6)]

$$\|(A^{(2,2)} + S_0')^{-1}\| = O(d) \tag{12}$$

becomes

$$\|(A_1 + T_1)^{-1}\| = O(\sqrt{dd_1}). \tag{13}$$

At this point, we invoke Theorem 1.2 from [2]. As Vershynin kindly pointed out to the author, the argument in [2] simplifies considerably in the Gaussian case. Examination of the proof shows that in fact the statement from [2], Theorem 1.2 can be improved in this case as follows.

Claim *Let A be a $d \times d$ normalized GOE matrix and T a deterministic, diagonal $(d \times d)$-matrix. Then*

$$\mathbb{P}[\|(A+T)^{-1}\| > \lambda d] \leq C(1+\|T\|)\lambda^{-\frac{1}{9}}. \tag{14}$$

We distinguish two cases. If $d_1 \geq \frac{1}{C_2}d$, $C_2 > C_1^3$, immediately apply the above claim with d replaced by d_1, A by A_1 and T by T_1. Thus by (11)

$$\mathbb{P}[\|(A_1+T_1)^{-1}\| > \lambda\sqrt{dd_1}] \leq C(1+\|T_1\|)\Big(\frac{d_1}{d}\Big)^{-\frac{1}{18}}\lambda^{-\frac{1}{9}} < C\big(1+\sqrt{C_2}(K+1)\big)\lambda^{-\frac{1}{9}} \tag{15}$$

and (12) follows. If $d_1 < \frac{1}{C_2}d$, repeat the preceding replacing A by A_1, T by T_1. In the definition of Ω_1, replace K by $K_1 = 2K$, so that (3) will hold with probability at least

$$1 - e^{-cK_1^2} = 1 - e^{-4cK^2} \tag{16}$$

the point being of making the measure bounds $e^{-c4^sK^2}$, $s = 0, 1, 2, \dots$ obtained in an iteration, sum up to $e^{-c_1K^2} = o(1)$.

Note that in (13), we only seek for an estimate

$$\|(A_1+T_1)^{-1}\| < O\Big(\frac{\sqrt{C_2}}{C_1}d_1\Big) \tag{17}$$

hence, cf. (12)

$$\|(A_1^{(2,2)} + S'_{1,0})^{-1}\| < O\Big(\frac{\sqrt{d_2}}{C_1}d_1\Big) \tag{18}$$

where $A_1^{(2,2)}$ and $S'_{1,0}$ are defined as before, considering now A_1 and T_1. Hence (13) gets replaced by

$$\|(A_2+T_2)^{-1}\| = O\Big(\frac{\sqrt{C_2}}{C_1}\sqrt{d_1d_2}\Big) \tag{19}$$

where A_2, T_2 are $(d_2 \times d_2)$-matrices,

$$\|T_2\| < \Big(\frac{d_1}{d_2}\Big)^{\frac{1}{2}}(2K+1). \tag{20}$$

Assuming $d_2 \geq \frac{1}{C_2} d_1$, we obtain instead of (15)

$$\mathbb{P}[\|(A_2 + T_2)^{-1}\| > \lambda \frac{\sqrt{C_2}}{C_1} \sqrt{d_1 d_2}] \leq C\big(1 + \sqrt{C_2}(K_1 + 1)\big)\Big(\frac{\sqrt{C_2}}{C_1}\lambda\Big)^{-\frac{1}{9}}$$
$$< C\big(1 + \sqrt{C_2}(K + 1)\big)\ (2C_1^{\frac{1}{9}} C_2^{-\frac{1}{18}})\lambda^{-\frac{1}{9}} \tag{21}$$

and we take C_2 to ensure that $2C_1^{\frac{1}{9}} C_2^{-\frac{1}{18}} < \frac{1}{2}$.

The continuation of the process is now clear and terminates in at most $^2 \log d$ steps. At step s, we obtain if $d_{s+1} \geq \frac{1}{C_2} d_s$

$$\mathbb{P}\Big[\|(A_{s+1} + T_{s+1})^{-1}\| > \lambda\Big(\frac{\sqrt{C_2}}{C_1}\Big)^s \sqrt{d_s d_{s+1}}\Big] < C\big(1 + \sqrt{C_2}(K + 1)\big)2^{-s}\lambda^{-\frac{1}{9}}. \tag{22}$$

Summation over s gives a measure estimate $O(\lambda^{-\frac{1}{9}}) = o(1)$.

This concludes the proof of the Proposition. From quantitative point of view, previous argument shows

Proposition' *Let T and A be as in the Proposition. Then*

$$\mathbb{P}[\|(A + T)^{-1}\| > \lambda d] < O(\lambda^{-\frac{1}{10}}). \tag{23}$$

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Note The author's interest in this issue came up in the study (joint with I. Goldsheid) of quantitative localization of eigenfunctions of random band matrices. The purpose of this Note is to justify some estimates in this forthcoming work.

References

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