Orbit Point of View on Some Results of Asymptotic Theory; Orbit Type and Cotype

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Abstract We develop an orbit point of view on the notations of type and cotype and extend Kwapien's theorem to this setting. We show that such approach provides an exact equality in the latter theorem. In addition, we discuss several well known theorems and reformulate them using the orbit point of view.

1 Introduction

Let $X = (\mathbb{R}^n, ||\cdot||)$ be an n-dimensional normed space. For a given integer k define by $\alpha(k), \beta(k)$ the smallest possible constants, satisfying

$$\left(E\left\|\sum_{i=1}^{k}\gamma_i x_i\right\|^2\right)^{1/2} \leq \alpha(k)\left(\sum_{i=1}^{k}||x_i||^2\right)^{1/2}$$

and

$$\left(E\left\|\sum_{i=1}^{k}\gamma_i x_i\right\|^2\right)^{1/2} \geq \beta^{-1}(k)\left(\sum_{i=1}^{k}||x_i||^2\right)^{1/2}$$

for any $\{x_i\}_1^k \subset X$ and γ_i independent normalized Gaussian random variables. We say that X has type 2 α where $\alpha = \sup_k \alpha(k)$. Similarly we say that X has cotype 2 constant β where $\beta = \sup_k \beta(k)$. By a result of Tomczak-Jaegermann (see [11]), it is known that $\alpha \leq 2\alpha(n)$ and $\beta \leq 2\beta(n)$. Thus, up to a universal constant we may

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always deal with n-tuples in the definition of type and cotype for n-dimensional spaces. Both notions play an important role in the study of Banach spaces and local theory.

Remark 1.1 In this note we consider only Gaussian type and cotype constants, and we do not deal with Rademacher type and cotype (see [6, 7, 11]).

Before we discuss a few examples, recall that given two n dimensional normed spaces X, Y, the Banach Mazur distance between X, Y is

$$d(X, Y) = \sup\{||T||||T^{-1}|| : T : X \to Y \text{ is an isomorphism}\}.$$

Whenever Y is a Euclidean space, we will denote $d(X, Y)$ by d_X. The next theorem, due to Kwapien, provides an upper bound for d_X through type 2 and cotype 2 constants.

Theorem 1.2 (Kwapien [4]) *Let X be a (finite or infinite) Banach space. Then, X is isomorphic to a Hilbert space if and only if it has a finite type 2 and a finite cotype 2 constants. Moreover, in this case we have $d_X \leq \alpha\beta$, where α is the type 2 constant and β is the cotype 2 constant of X.*

It can be shown that the bound in Theorem 1.2 is not optimal. That is, we can find a space X such that $\alpha\beta$ is of order n, which is clearly not optimal since d_X is always bounded by $\sqrt{n}$ (John's Theorem). In this note we present a new point of view on the above result, which provides us an equality instead of an upper bound in Theorem 1.2. To this end, we present the notion of orbits in normed spaces.

Definition 1.3 Let $x = (x_1, \ldots x_k) \subset X$. We say that a k-tuple $y = (y_1, \ldots y_k)$ belongs to the orbit set of x if there exists $U = (u_{ij}) \in O(k)$ such that

$$y_i = \sum_{j=1}^{k} u_{ij} x_j.$$

The set of all such k-tuples will be denoted by $O(x) = \{Ux : U \in O(k)\}$ and called the orbit of x.

Using this notion, we may define the Gaussian type 2 and cotype 2 of an orbit x as the smallest constants $\alpha(x), \beta(x)$ such that

$$\left(E\left\|\sum_{i=1}^{k} \gamma_i y_i\right\|^2\right)^{1/2} \leq \alpha(x)\left(\sum_{i=1}^{k} ||y_i||^2\right)^{1/2}$$

and

$$\left(E\left\|\sum_{i=1}^{k} \gamma_i y_i\right\|^2\right)^{1/2} \geq \beta^{-1}(x)\left(\sum_{i=1}^{k} ||y_i||^2\right)^{1/2}$$

for all $y \in O(x)$. Clearly, $\alpha(x) = \alpha(y)$ and $\beta(x) = \beta(y)$ for all $y \in O(x)$, so the constants are well defined. Denote

$$g(x,\gamma) = E\left\|\sum_{i=1}^{k} \gamma_i x_i\right\|^2,$$

where $\gamma = \{\gamma_i\}_1^k$. Due to the rotation invariance of the standard Gaussian measure we have that if $y \in O(x)$ then $g(x,\gamma) = g(y,\gamma')$, where $\gamma' = \{\gamma_i'\}_1^k$ are independent Gaussian variables, which are also independent of γ (see e.g. [9, Chap. 2, p. 13]). Hence,

$$\alpha(x)\beta(x) = \inf\left\{\left(\frac{\sum_{i=1}^k ||y_i||^2}{\sum_{i=1}^k ||z_i||^2}\right)^{1/2} : y, z \in O(x)\right\} \tag{1}$$

Using the notion of orbits it is possible to write the exact formula for d_X in Theorem 1.2:

Theorem 1.4 *For any n dimensional normed space X we have*

$$d_X = \sup\{\alpha(x)\beta(x) | x = (x_1, \dots x_k), k = 1, 2, \dots\}.$$

Moreover,

$$d_X \leq 4 \sup\{\alpha(x)\beta(x) : x = (x_1, \dots x_n)\}.$$

Of course, the first formula is correct for infinite dimensional spaces as well.

Remark 1.5 The question of the exact formula for d_X was also considered in the Master Thesis of Limor Ben-Efraim, under the supervision of V. Milman (not published).

Remark 1.6 It was noted by Pivovarov (private communication, 2016) that Theorem 1.4 easily implies that $d_X \leq 4\sqrt{n}$.

In the spirit of Theorem 1.4, it is possible to reformulate several well known theorems regarding embeddings of l_1^k and l_∞^k in X, such as Alon-Milman's theorem (see [1]) and Elton's theorem (see [2]). However, since those theorems involve Rademacher averaging instead of Gaussian averaging, the results will not be precise, as those averages are not equivalent in the general case.

However, the following two theorems may be reformulated in an exact way:

Theorem 1.7 (Figiel-Lindenstrauss-Milman [3]) *Let X be an n dimensional normed space with the unit ball K. Let $x = (x_1, x_2, \dots x_n)$ be an orbit with cotype 2 constant $\beta(x)$. If x is the orthogonal basis of the maximal volume ellipsoid of K then X contains a subspace of dimension $k = cn\beta(x)^{-2}$ that is 2-isomorphic to l_2^k, for some universal constant $c > 0$.*

Theorem 1.8 ([6, Theorem 9.7]) *Let X be an n-dimensional normed space and let $x = (x_1, \dots, x_k) \subset X$ be a k-tuple for some $k \le n$. If $O(x)$ has a 2-type constant α, then the space $E = span\{x_i\}_1^k$ contains a space of dimension $m = \left[c\alpha^2\right]$ which is 2-isomorphic to l_2^m, for some absolute constant $c > 0$.*

It may be an interesting question to analyze the Maurey-Pisier lemma for equivalence of Rademacher and Gaussian averages (see [8, Proposition 3.2]) in this context. However, one should consider a general orbit of cotype q which is not done in this note.

2 Proof of the Extended Kwapien Theorem

Proof Before we proceed with the proof of Theorem 1.4, let us recall a few definitions and facts.

Definition 2.1 An operator $u : X \to Y$ factors through a Hilbert space if there is a Hilbert space H and operators $B : X \to H$ and $A : H \to Y$ such that $u = AB$. Denote by $\Gamma_2(X, Y)$ the space of all such operators, equipped with the norm

$$\gamma_2(u) = \inf\{\|A\|\,\|B\|\}$$

where the infimum is taken over all factorizations of u.

A well known theorem by Lindenstrauss and Pelczynski (see [5], [7, Theorem 2.4], [11, Proposition 13.11]) provides a necessary and sufficient condition when an operator u belongs to $\Gamma_2(X, Y)$:

Theorem 2.2 *$u : X \to Y$ belongs to $\Gamma_2(X, Y)$ if and only if there exists a constant C such that for all n and all $n \times n$ orthogonal matrices (a_{ij}) we have,*

$$\left(\sum_{i=1}^{n}\left\|\sum_{j=1}^{n} a_{ij}ux_j\right\|^2\right)^{1/2} \le C\left(\sum_{i=1}^{n}\|x_i\|^2\right)^{1/2}$$

for all $x_{,1} \dots x_n \in X$. Moreover, $\gamma_2(u)$ coincides with the smallest possible constant C satisfying the above inequality.

Let $x = (x_j)$ be a k-tuple of elements of X and let $(a_{ij}) \in O(k)$. By the definition of Gaussian orbit cotype of x we have

$$\beta(x)^{-1}\left(\sum_{i=1}^{k}\left\|\sum_{j=1}^{k} a_{ij}x_j\right\|^2\right)^{1/2} \le \left(E\left\|\sum_{i=1}^{k}\gamma_i\sum_{j=1}^{k} a_{ij}x_j\right\|^2\right)^{1/2}. \qquad (2)$$

By the definition of Gaussian orbit type we have

$$g(x,\gamma)^{1/2} \le \alpha(x)\left(\sum_{i=1}^{k}\|x_i\|^2\right)^{1/2}. \tag{3}$$

However, since $g(x,\gamma) = g(y,\gamma)$ where

$$y_i = \sum_{j=1}^{k} a_{ij}x_j,$$

we get that

$$\left(\sum_{i=1}^{k}\left\|\sum_{j=1}^{k} a_{ij}x_j\right\|^2\right)^{1/2} \le \alpha(x)\beta(x)\left(\sum_{i=1}^{k}\|x_i\|^2\right)^{1/2}. \tag{4}$$

Thus, the condition of Theorem 2.2 is satisfied with the constant

$$C = \sup_x\{\alpha(x)\beta(x)\}.$$

Clearly, $\beta(x)$ and $\alpha(x)$ are the smallest possible numbers satisfying (2) and (3). Therefore, $\sup_x\{\alpha(x)\beta(x)\}$ is the smallest possible number satisfying (4) for each positive k and each k-frame x. Thus,

$$\gamma_2(Id) = \sup\{\alpha(x)\beta(x)\},$$

However, $\gamma_2(Id) = d_X$ (by definition), so the first part of the proof of Theorem 1.4 is finished. □

Remark 2.3 In the case where $\dim X = \dim Y = n$, one may consider only $n \times n$ orthogonal matrices and the best constant C in Theorem 2.2 is equivalent to $\gamma_2(u)$ up to a factor of 4. This was noted independently by Tomczak-Jaegermann and Pisier (private communication, 2000). Since the result was not published we will provide a different argument which is due to Tomczak-Jaegermann.

To this end, we recall several facts regarding absolutely summing operators (see [7, 11]).

Definition 2.4 Let X and Y be Banach spaces. An operator $u : X \to Y$ is called 2-summing operator if there exists a constant C such that for all finite sequences $\{x_i\} \subset X$:

$$\left(\sum_{i=1}^{k} \|ux_i\|^2 \right)^{1/2} \leq C \sup_{\xi \in X^*, \|\xi\| \leq 1} \left(\sum_{i=1}^{k} |\xi(x_i)|^2 \right)^{1/2}.$$

The smallest possible C satisfying the above is denoted by $\pi_2(u)$ and is called the 2-summing norm of u.

Now we will define a similar concept for an orbit and see how it relates to the definition above. From now on, unless stated otherwise, it is assumed that X is an n-dimensional normed space.

Definition 2.5 Given an operator $u : l_2^k \to X$, denote

$$\pi_2^{(k)}(u) = \sup \left(\sum_{i=1}^{k} \|uf_i\|^2 \right)^{1/2},$$

$$\delta_2^{(k)}(u) = \inf \left(\sum_{i=1}^{k} \|uf_i\|^2 \right)^{1/2},$$

where $\{f_i\}_1^k$ runs over all orthonormal bases of l_2^k.

Given an orbit $x = \{x_1, \dots x_k\} \subset X$ we will denote $\pi_2^{(k)}(x) = \pi_2^{(k)}(u)$, $\delta_2^{(k)}(x) = \delta^{(k)}(u)$ where u is defined by

$$ue_i = x_i, \qquad 1 \leq i \leq k.$$

Remark 2.6 The standard definition of $\pi_2^{(k)}(u)$ slightly differs from definition above. It is defined as the smallest possible constant satisfying

$$\left(\sum_{i=1}^{k} \|ux_i\|^2 \right)^{1/2} \leq C \sup_{\xi \in X^*, \|\xi\| \leq 1} \left(\sum_{i=1}^{k} |\xi(x_i)|^2 \right)^{1/2},$$

for all $x_1, \dots x_k \in X$.

By a theorem of Tomczak-Jaegermann [10] we have that for any operator $u : l_2^k \to X$ of rank n:

$$\pi_2^{(n)}(u) \leq \pi_2(u) \leq 2\pi_2^{(n)}(u). \tag{5}$$

Since the proof of (5) constructs an orthonormal basis (e_j) of l_2^n that satisfies

$$\left(\sum_{i=1}^{n} \|ue_i\|^2\right)^{1/2} \geq \frac{1}{2}\pi_2(u),$$

we get that inequality (5) holds for our definition of $\pi_2^{(n)}(u)$ as well.

An easy consequence of the above is the following lemma:

Lemma 2.7 *For each $k \geq n$ and $x = (x_1, \ldots x_k) \subset X$ there exists $y \in O(x)$ and a subset $y' \subset y$ of cardinality n such that*

$$\pi_2^{(k)}(y) \leq 2\pi_2^{(n)}(y').$$

Proof Let $u : l_2^k \to X$ be the operator defined by $ue_i = x_i$, and denote $E = ker(u)^{\perp}$. Denote by $P : l_2^k \to E$ the orthogonal projection such that $u = u|_E P$. Let $f_1 \ldots f_n \in E$ and $f_{n+1} \ldots f_k \in E^{\perp}$ be another orthonormal basis of l_2^k and denote by $y_i = uf_i$. Clearly,

$$\pi_2^{(k)}(x) \leq \pi_2(u) = \pi_2(u|_E) \leq 2\pi_2^{(n)}(y')$$

where $y' = (y_1, \ldots y_n)$. □

Since $\delta_2^{(k)}$ is not necessarily convex, denote by $\hat{\delta}_2^{(k)}$ the largest convex function that is smaller than $\delta_2^{(k)}$. The norms $\pi_2^{(k)}$ and $\hat{\delta}_2^{(k)}$ are dual norms on $L(l_2^k, X)$ and $L(l_2^k, X^*)$. That is

$$\pi_2^{(k)}(u) = \sup\{|trace(uv)| : v^* \in L(l_2^k, X^*), \hat{\delta}_2^{(k)}(v^*) \leq 1\}.$$

The proof of this fact is similar to the proof presented in [11, Proposition 9.9], for the norms π_2 and δ_2.

By a standard duality argument we get the following corollary.

Corollary 2.8 *Let $u : l_2^k \to X$ be an operator, where $k \geq n$. Let $E = (\ker u)^{\perp}$ with $\dim E = n$, and let P be the orthogonal projection $P : l_2^k \to E$. Define $\tilde{u} : E \to X$ such that $u = \tilde{u}P$. Then we have*

$$\hat{\delta}_2^{(n)}(\tilde{u}) \leq 2\delta_2^{(k)}(u).$$

Now, we may prove the key lemma required for our goal.

Lemma 2.9 *If C satisfies*

$$\forall x = (x_1, \ldots x_n) \subset X, \quad \pi_2^{(n)}(x) \leq C\nu_2^{(n)}(x), \tag{6}$$

then, for all $k > n$,

$$\forall x = (x_1, \dots x_k) \subset X, \quad \pi_2^{(k)}(x) \le 4C\nu_2^{(k)}(x), \tag{7}$$

Proof Denote by X^m the space of all m-tuples of X. Take $x \in X^n$ and consider $u : l_2^n \to X$ an operator defined by $ue_i = x_i$. Clearly, by (6) and the convexity of $\hat{\delta}_2^n$ and $\pi_2^{(n)}$

$$\pi_2^{(n)}(x) \le C\hat{\delta}_2^{(n)}(u).$$

Given $k \ge n$ take $x = (x_1, \dots x_k)$, $y \in O(x)$ and define operator u as above. As before, denote $E = (\ker u)^{\perp}$ and by $P : l_2^k \to E$ the orthogonal projection. Define $\tilde{u} : E \to X$ such that $u = \tilde{u}P$. Let $f_1 \dots f_n \in E$ and $f_{n+1} \dots f_k \in E^{\perp}$ be some orthonormal basis of l_2^k. Denote $y_i = uf_i$ and $y = (y_1, \dots y_k)$, $y' = (y_1, \dots y_n)$. Then,

$$\pi_2^{(k)}(x) = \pi_2^{(k)}(y) \le 2\pi_2^{(n)}(y')$$

and

$$\hat{\delta}_2^{(n)}(\tilde{u}) \le 2\delta_2^{(k)}(u) = \nu_2^{(k)}(y) = 2\nu_2^{(k)}(x).$$

Thus,

$$\pi_2^{(k)}(x) \le 2\pi_2^{(n)}(y') \le 2C\hat{\delta}_2^{(n)}(\tilde{u}) \le 4C\nu_2^{(k)}(x).$$

□

Now we may finish the second part of main theorem. Let $x = (x_1, \dots x_k)$. Notice that by (1), for each k

$$\alpha(x)\beta(x) = \frac{\pi_2^{(k)}(x)}{\nu_2^{(k)}(x)}.$$

Applying Lemma 2.9 we get

$$\sup_{x \in X^k} \alpha(x)\beta(x) = \sup_{x \in X^k} \frac{\pi_2^{(k)}(x)}{\nu_2^{(k)}(x)} \le 4 \sup_{x \in X^n} \frac{\pi_2^{(n)}(x)}{\nu_2^{(n)}(x)}$$

$$= 4 \sup_{x \in X^n} \{\alpha(x)\beta(x)\}$$

and the proof is complete.

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