A Remark on Measures of Sections of L_p-balls

Alexander Koldobsky and Alain Pajor

Abstract We prove that there exists an absolute constant C so that

$$\mu(K) \leq C\sqrt{p} \max_{\xi \in S^{n-1}} \mu(K \cap \xi^{\perp}) \, |K|^{1/n}$$

for any $p > 2$, any $n \in \mathbb{N}$, any convex body K that is the unit ball of an n-dimensional subspace of L_p, and any measure μ with non-negative even continuous density in $\mathbb{R}^n$. Here $\xi^{\perp}$ is the central hyperplane perpendicular to a unit vector $\xi \in S^{n-1}$, and $|K|$ stands for volume.

1 Introduction

The slicing problem [1, 4, 5, 29], a major open question in convex geometry, asks whether there exists a constant C so that for any $n \in \mathbb{N}$ and any origin-symmetric convex body K in $\mathbb{R}^n$,

$$|K|^{\frac{n-1}{n}} \leq C \max_{\xi \in S^{n-1}} |K \cap \xi^{\perp}|,$$

where $|K|$ stands for volume of proper dimension, and $\xi^{\perp}$ is the central hyperplane in $\mathbb{R}^n$ perpendicular to a unit vector ξ. The best-to-date result $C \leq O(n^{1/4})$ is due to Klartag [15], who improved an earlier estimate of Bourgain [6]. The answer is affirmative for unconditional convex bodies (as initially observed by Bourgain; see also [3, 14, 29]), intersection bodies [10, Theorem 9.4.11], zonoids, duals of bodies with bounded volume ratio [29], the Schatten classes [23], k-intersection bodies [21, 22]; see [7] for more details.

A. Koldobsky (✉)
Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
e-mail: koldobskiya@missouri.edu

A. Pajor
Université Paris-Est, Laboratoire d'Analyse et Mathématiques Appliquées (UMR 8050), UPEM, Marne-la-Vallée F-77454, France
e-mail: alain.pajor@u-pem.fr

B. Klartag, E. Milman (eds.), *Geometric Aspects of Functional Analysis*,
Lecture Notes in Mathematics 2169, DOI 10.1007/978-3-319-45282-1_14

The case of unit balls of finite dimensional subspaces of L_p is of particular interest in this note. It was shown by Ball [2] that the slicing problem has an affirmative answer for the unit balls of finite dimensional subspaces of $L_p,\ 1 \le p \le 2$. Junge [13] extended this result to every $p \in (1, \infty)$, with the constant C depending on p and going to infinity when $p \to \infty$. Milman [27] gave a different proof for subspaces of $L_p,\ 2 < p < \infty$, with the constant $C \le O(\sqrt{p})$. Another proof of this estimate can be found in [22].

A generalization of the slicing problem to arbitrary measures was considered in [18–21]. Does there exist a constant C so that for every $n \in N$, every origin-symmetric convex body K in $\mathbb{R}^n$, and every measure μ with non-negative even continuous density f in $\mathbb{R}^n$,

$$\mu(K) \ \le\ C \max_{\xi\in S^{n-1}} \mu(K \cap \xi^{\perp})\ |K|^{1/n}\ ? \tag{1}$$

For every k-dimensional subspace of $\mathbb{R}^n,\ 1 \le k \le n$ and any Borel set $A \subset E$,

$$\mu(A) = \int_A f(x)dx,$$

where the integration is with respect to the k-dimensional Lebesgue measure on E.

Inequality (1) was proved with an absolute constant C for intersection bodies [18] (see [16], this includes the unit balls of subspaces of L_p with $0 < p \le 2$), unconditional bodies and duals of bodies with bounded volume ratio in [20], for k-intersection bodies in [21]. For arbitrary origin-symmetric convex bodies, (1) was proved in [19] with $C \le O(\sqrt{n})$. A different proof of the latter estimate was recently given in [8], where the symmetry condition was removed.

For the unit balls of subspaces of $L_p,\ p > 2$, (1) was proved in [21] with $C \le O(n^{1/2-1/p})$. In this note we improve the estimate to $C \le O(\sqrt{p})$, extending Milman's result [27] to arbitrary measures in place of volume. In fact, we prove a more general inequality

$$\mu(K) \ \le\ (C\sqrt{p})^k \max_{H\in Gr_{n-k}} \mu(K \cap H)\ |K|^{k/n}, \tag{2}$$

where $1 \le k < n,\ Gr_{n-k}$ is the Grassmanian of $(n-k)$-dimensional subspaces of $\mathbb{R}^n,\ K$ is the unit ball of any n-dimensional subspace of $L_p,\ p > 2, \mu$ is a measure on $\mathbb{R}^n$ with even continuous density, and C is a constant independent of p, n, k, K, μ.

The proof is a combination of two known results. Firstly, we use the reduction of the slicing problem for measures to computing the outer volume ratio distance from a body to the class of intersection bodies established in [20]; see Proposition 1. Note that outer volume ratio estimates have been applied to different cases of the original slicing problem by Ball [2], Junge [13], and Milman [27]. Secondly, we use an estimate for the outer volume ratio distance from the unit ball of a subspace of $L_p,\ p > 2$, to the class of origin-symmetric ellipsoids proved by Milman in [27].

This estimate also follows from results of Davis, Milman and Tomczak-Jaegermann [9]. We include a concentrated version of the proof in Proposition 2.

2 Slicing Inequalities

We need several definitions and facts. A closed bounded set K in $\mathbb{R}^n$ is called a *star body* if every straight line passing through the origin crosses the boundary of K at exactly two points, the origin is an interior point of K, and the *Minkowski functional* of K defined by

$$\|x\|_K = \min\{a \geq 0 : \ x \in aK\}$$

is a continuous function on $\mathbb{R}^n$.

The *radial function* of a star body K is defined by

$$\rho_K(x) = \|x\|_K^{-1}, \qquad x \in \mathbb{R}^n, \ x \neq 0.$$

If $x \in S^{n-1}$ then $\rho_K(x)$ is the radius of K in the direction of x.

We use the polar formula for volume of a star body

$$|K| = \frac{1}{n}\int_{S^{n-1}} \|\theta\|_K^{-n} d\theta. \tag{3}$$

The class of intersection bodies was introduced by Lutwak [25]. Let K, L be origin-symmetric star bodies in $\mathbb{R}^n$. We say that K is the intersection body of L if the radius of K in every direction is equal to the $(n-1)$-dimensional volume of the section of L by the central hyperplane orthogonal to this direction, i.e. for every $\xi \in S^{n-1}$,

$$\begin{aligned}\rho_K(\xi) &= \|\xi\|_K^{-1} = |L \cap \xi^\perp| \\ &= \frac{1}{n-1}\int_{S^{n-1}\cap\xi^\perp} \|\theta\|_L^{-n+1} d\theta = \frac{1}{n-1} R\left(\|\cdot\|_L^{-n+1}\right)(\xi),\end{aligned}$$

where $R : C(S^{n-1}) \to C(S^{n-1})$ is the *spherical Radon transform*

$$Rf(\xi) = \int_{S^{n-1}\cap\xi^\perp} f(x)dx, \qquad \forall f \in C(S^{n-1}).$$

All bodies K that appear as intersection bodies of different star bodies form *the class of intersection bodies of star bodies*. A more general class of *intersection bodies* is defined as follows. If μ is a finite Borel measure on S^{n-1}, then the spherical Radon transform $R\mu$ of μ is defined as a functional on $C(S^{n-1})$ acting by

$$(R\mu, f) = (\mu, Rf) = \int_{S^{n-1}} Rf(x) d\mu(x), \qquad \forall f \in C(S^{n-1}).$$

A star body K in $\mathbb{R}^n$ is called an *intersection body* if $\|\cdot\|_K^{-1} = R\mu$ for some measure μ, as functionals on $C(S^{n-1})$, i.e.

$$\int_{S^{n-1}} \|x\|_K^{-1} f(x) dx = \int_{S^{n-1}} Rf(x) d\mu(x), \qquad \forall f \in C(S^{n-1}).$$

Intersection bodies played a crucial role in the solution of the Busemann-Petty problem and its generalizations; see [17, Chap. 5].

A generalization of the concept of an intersection body was introduced by Zhang [30] in connection with the lower dimensional Busemann-Petty problem. For $1 \le k \le n-1$, the *$(n-k)$-dimensional spherical Radon transform $R_{n-k} : C(S^{n-1}) \to C(Gr_{n-k})$* is a linear operator defined by

$$R_{n-k} g(H) = \int_{S^{n-1} \cap H} g(x)\, dx, \quad \forall H \in Gr_{n-k}$$

for every function $g \in C(S^{n-1})$.

We say that an origin symmetric star body K in $\mathbb{R}^n$ is a *generalized k-intersection body*, and write $K \in \mathcal{BP}_k^n$, if there exists a finite Borel non-negative measure μ on Gr_{n-k} so that for every $g \in C(S^{n-1})$

$$\int_{S^{n-1}} \|x\|_K^{-k} g(x)\, dx = \int_{Gr_{n-k}} R_{n-k} g(H)\, d\mu(H). \tag{4}$$

When $k = 1$ we get the class of intersection bodies. It was proved by Goodey and Weil [11] for $k = 1$ and by Grinberg and Zhang [12, Lemma 6.1] for arbitrary k (see also [28] for a different proof) that the class $\mathcal{BP}_k^n$ is the closure in the radial metric of k-radial sums of origin-symmetric ellipsoids. In particular, the classes $\mathcal{BP}_k^n$ contain all origin-symmetric ellipsoids in $\mathbb{R}^n$ and are invariant with respect to linear transformations. Recall that the k-radial sum $K +_k L$ of star bodies K and L is defined by

$$\rho_{K+_kL}^k = \rho_K^k + \rho_L^k.$$

For a convex body K in $\mathbb{R}^n$ and $1 \le k < n$, denote by

$$\text{o.v.r.}(K, \mathcal{BP}_k^n) = \inf \left\{ \left(\frac{|C|}{|K|} \right)^{1/n} : K \subset C,\ C \in \mathcal{BP}_k^n \right\}$$

the outer volume ratio distance from a body K to the class $\mathcal{BP}_k^n$.

Let B_2^n be the unit Euclidean ball in $\mathbb{R}^n$, let $|\cdot|_2$ be the Euclidean norm in $\mathbb{R}^n$, and let σ be the uniform probability measure on the sphere S^{n-1} in $\mathbb{R}^n$. For every $x \in \mathbb{R}^n$, let x_1 be the first coordinate of x. We use the fact that for every $p > -1$

$$\int_{S^{n-1}} |x_1|^p d\sigma(x) = \frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n+p}{2})}; \tag{5}$$

see for example [17, Lemma 3.12], where one has to divide by $|S^{n-1}| = 2\pi^{(n-1)/2}/\Gamma(\frac{n}{2})$, because the measure σ on the sphere is normalized.

In [20], the slicing problem for arbitrary measures was reduced to estimating the outer volume ratio distance from a convex body to the classes $\mathcal{BP}_k^n$, as follows.

Proposition 1 *For any $n \in \mathbb{N}$, $1 \le k < n$, any origin-symmetric star body K in $\mathbb{R}^n$, and any measure μ with even continuous density on K,*

$$\mu(K) \le \left(o.v.r.(K, \mathcal{BP}_k^n)\right)^k \frac{n}{n-k} c_{n,k} \max_{H \in Gr_{n-k}} \mu(K \cap H)\ |K|^{k/n},$$

where $c_{n,k} = |B_2^n|^{(n-k)/n}/|B_2^{n-k}| \in (e^{-k/2}, 1)$.

It appears that for the unit balls of subspaces of $L_p,\ p > 2$ the outer volume ration distance to the classes of intersection bodies does not depend on the dimension. As mentioned in the introduction, the following estimate was proved in [27] and also follows from results of [9]. We present a short version of the proof.

Proposition 2 *Let $p > 2,\ n \in \mathbb{N},\ 1 \le k < n$, and let K be the unit ball of an n-dimensional subspace of L_p. Then*

$$o.v.r.(K, \mathcal{BP}_k^n) \le C\sqrt{p},$$

where C is an absolute constant.

Proof Since the classes $\mathcal{BP}_k^n$ are invariant under linear transformations, we can assume that K is in the Lewis position. By a result of Lewis in the form of [26, Theorem 8.2], this means that there exists a measure ν on the sphere so that for every $x \in \mathbb{R}^n$

$$\|x\|_K^p = \int_{S^{n-1}} |(x,u)|^p d\nu(u),$$

and

$$|x|_2^2 = \int_{S^{n-1}} |(x,u)|^2 d\nu(u).$$

Also, by the same result of Lewis [24], $K \subset n^{1/2-1/p} B_2^n$.

Let us estimate the volume of K from below. By the Fubini theorem, formula (5) and Stirling's formula, we get

$$\int_{S^{n-1}} \|x\|_K^p d\sigma(x) = \int_{S^{n-1}} \int_{S^{n-1}} |(x,u)|^p d\sigma(x) d\nu(u)$$
$$= \int_{S^{n-1}} |x_1|^p d\sigma(x) \int_{S^{n-1}} d\nu(u) \le \left(\frac{Cp}{n+p}\right)^{p/2} \int_{S^{n-1}} d\nu(u).$$

Now

$$\frac{Cp}{n+p}\left(\int_{S^{n-1}} d\nu(u)\right)^{2/p} \ge \left(\int_{S^{n-1}} \|x\|_K^p d\sigma(x)\right)^{2/p}$$
$$\ge \left(\int_{S^{n-1}} \|x\|_K^{-n} d\sigma(x)\right)^{-2/n} = \left(\frac{|K|}{|B_2^n|}\right)^{-2/n} \sim \frac{1}{n}|K|^{-2/n},$$

because $|B_2^n|^{1/n} \sim n^{-1/2}$. On the other hand,

$$1 = \int_{S^{n-1}} |x|_2^2 d\sigma(x) = \int_{S^{n-1}} \int_{S^{n-1}} (x,u)^2 d\nu(u) d\sigma(x)$$
$$= \int_{S^{n-1}} \int_{S^{n-1}} |x_1|^2 d\sigma(x) d\nu(u) = \frac{1}{n} \int_{S^{n-1}} d\nu(u),$$

so

$$\frac{Cp}{n+p} n^{2/p} \ge \frac{1}{n}|K|^{-2/n},$$

and

$$|K|^{1/n} \ge cn^{-1/p}\sqrt{\frac{n+p}{np}} \ge \frac{cn^{1/2-1/p}}{\sqrt{p}}|B_2^n|^{1/n}.$$

Finally, since $K \subset n^{1/2-1/p}B_2^n$, and $B_2^n \in \mathcal{BP}_k^n$ for every k, we have

$$\text{o.v.r.}(K, \mathcal{BP}_k^n) \le \left(\frac{|n^{1/2-1/p}B_2^n|}{|K|}\right)^{1/n} \le C\sqrt{p},$$

where C is an absolute constant.

We now formulate the main result of this note.

Corollary 1 *There exists a constant C so that for any $p > 2$, $n \in \mathbb{N}$, $1 \le k < n$, any convex body K that is the unit ball of an n-dimensional subspace of L_p, and any* *measure μ with non-negative even continuous density in $\mathbb{R}^n$,*

$$\mu(K) \ \leq \ (C\sqrt{p})^k \max_{H \in Gr_{n-k}} \mu(K \cap H)\ |K|^{k/n}.$$

Proof Combine Proposition 1 with Proposition 2. Note that $\frac{n}{n-k} \in (1, e^k)$, and $c_{n,k} \in (e^{-k/2}, 1)$, so these constants can be incorporated in the constant C. □

Acknowledgements The first named author was partially supported by the US National Science Foundation, grant DMS-1265155.

References

1. K. Ball, Isometric problems in ℓ_p and sections of convex sets, Ph.D. dissertation, Trinity College, Cambridge, 1986
2. K. Ball, *Normed Spaces with a Weak Gordon-Lewis Property*. Lecture Notes in Mathematics, vol. 1470 (Springer, Berlin 1991), pp. 36–47
3. S. Bobkov, F. Nazarov, On convex bodies and log-concave probability measures with unconditional basis, in *Geometric Aspects of Functional Analysis*, ed. by V.D. Milman, G. Schechtman. Lecture Notes in Mathematics, vol. 1807 (Springer, Berlin, 2003), pp. 53–69
4. J. Bourgain, On high-dimensional maximal functions associated to convex bodies. Am. J. Math. **108**, 1467–1476 (1986)
5. J. Bourgain, Geometry of Banach spaces and harmonic analysis, in *Proceedings of the International Congress of Mathematicians (Berkeley, Calif., 1986)* (American Mathematical Society, Providence, RI, 1987), pp. 871–878
6. J. Bourgain, On the distribution of polynomials on high-dimensional convex sets, in *Geometric Aspects of Functional Analysis, Israel Seminar (1989–90)*. Lecture Notes in Mathematics, vol. 1469 (Springer, Berlin, 1991), pp. 127–137
7. S. Brazitikos, A. Giannopoulos, P. Valettas, B. Vritsiou, *Geometry of Isotropic Convex Bodies* (American Mathematical Society, Providence RI, 2014)
8. G. Chasapis, A. Giannopoulos, D.-M. Liakopoulos, Estimates for measures of lower dimensional sections of convex bodies. Adv. Math. **306**, 880–904 (2017)
9. W.J. Davis, V.D. Milman, N. Tomczak-Jaegermann, The distance between certain n-dimensional Banach spaces. Isr. J. Math. **39**, 1–15 (1981)
10. R.J. Gardner, *Geometric Tomography*, 2nd edn. (Cambridge University Press, Cambridge, 2006)
11. P. Goodey, W. Weil, Intersection bodies and ellipsoids. Mathematika **42**, 295–304 (1995)
12. E. Grinberg, G. Zhang, Convolutions, transforms and convex bodies. Proc. Lond. Math. Soc. **78**, 77–115 (1999)
13. M. Junge, On the hyperplane conjecture for quotient spaces of L_p. Forum Math. **6**, 617–635 (1994)
14. M. Junge, Proportional subspaces of spaces with unconditional basis have good volume properties, in *Geometric Aspects of Functional Analysis (Israel Seminar, 1992–1994)*. Operator Theory Advances and Applications, vol. 77 (Birkhauser, Basel, 1995), pp. 121–129
15. B. Klartag, On convex perturbations with a bounded isotropic constant. Geom. Funct. Anal. **16**, 1274–1290 (2006)
16. A. Koldobsky, Intersection bodies, positive definite distributions and the Busemann-Petty problem. Am. J. Math. **120**, 827–840 (1998)

17. A. Koldobsky, *Fourier Analysis in Convex Geometry* (American Mathematical Society, Providence RI, 2005)
18. A. Koldobsky, A hyperplane inequality for measures of convex bodies in $\mathbb{R}^n, n \leq 4$, Discret. Comput. Geom. **47**, 538–547 (2012)
19. A. Koldobsky, A $\sqrt{n}$ estimate for measures of hyperplane sections of convex bodies. Adv. Math. **254**, 33–40 (2014)
20. A. Koldobsky, Slicing inequalities for measures of convex bodies. Adv. Math. **283**, 473–488 (2015)
21. A. Koldobsky, Slicing inequalities for subspaces of L_p. Proc. Am. Math. Soc. **144**, 787–795 (2016)
22. A. Koldobsky, A. Pajor, V. Yaskin, Inequalities of the Kahane-Khinchin type and sections of L_p-balls. Stud. Math. **184**, 217–231 (2008)
23. H. König, M. Meyer, A. Pajor, The isotropy constants of the Schatten classes are bounded. Math. Ann. **312**, 773–783 (1998)
24. D.R. Lewis, Finite dimensional subspaces of L_p. Stud. Math. **63**, 207–212 (1978)
25. E. Lutwak, Intersection bodies and dual mixed volumes. Adv. Math. **71**, 232–261 (1988)
26. E. Lutwak, D. Yang, G. Zhang, L_p John ellipsoids. Proc. Lond. Math. Soc. **90**, 497–520 (2005)
27. E. Milman, Dual mixed volumes and the slicing problem. Adv. Math. **207**, 566–598 (2006)
28. E. Milman, Generalized intersection bodies. J. Funct. Anal. **240**(2), 530–567 (2006)
29. V. Milman, A. Pajor, Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space, in *Geometric Aspects of Functional Analysis*, ed. by J. Lindenstrauss, V. Milman. Lecture Notes in Mathematics, vol. 1376 (Springer, Heidelberg, 1989), pp. 64–104
30. G. Zhang, Section of convex bodies. Am. J. Math. **118**, 319–340 (1996)