From the Square to Octahedra

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Abstract Colwyn Williamson (Notre Dame J. Formal Log. 13:497–500, 1972) develops a comparison between propositional and syllogistic logic. He outlines an interpretation of the traditional square of opposition in terms of propositional logic, that is, the statements corresponding to the corners of the traditional square can be represented with propositional logic operators. His goal is to present a twofold square that preserves the truth conditions of the relationships between the formulas, and define other set of formulas that complete the traditional square to outline an octagon of opposition. We present two octahedra inspired in these squares. The octahedra hold the relations of the traditional square of opposition and also keep (and with some restrictions, extend) the equipollence and immediate inference rules.

Keywords Hexagon • Octagon • Propositional logic • Square of opposition • Syllogistic

Mathematics Subject Classification (2000) Primary 03B05; Secondary 03B22, 03B35, 03B10

> In geometry and logic alike a place is a possibility: something can exist in it.
>
> Ludwig Wittgenstein [6, 3.411]

1 Introduction

In [5] Colwyn Williamson develops a comparison between propositional and syllogistic logic. He outlines an interpretation of the traditional square of opposition in terms of propositional logic, that is, the statements corresponding to the corners of the traditional square can be represented with propositional logic operators. His goal is to present a twofold square that preserves the truth conditions of the relationships between the formulas, and he defines other set of formulas that complete the traditional square to outline an octagon of opposition. The aim of this paper is to lead to the end this reconstruction taking seriously the task stated by Williamson.

J.-Y. Béziau, G. Basti (eds.), *The Square of Opposition: A Cornerstone of Thought*, Studies in Universal Logic, DOI 10.1007/978-3-319-45062-9_15

We present two octahedra inspired in these squares. The octahedra hold the relations of the traditional square of opposition and also keep (and with some restrictions, extend) the equipollence and immediate inference rules. Our goal is threefold: first, to analyze the Williamson's squares and state the basic consequences of his analysis, second, to present an extension of the Williamson's squares, i.e. the octahedra of opposition, and third, we bring to the end Williamson's thesis to get some results concerning the relation between propositional and first-order logic.

In the second section we generate an analysis of the reconstruction of syllogistic logic developed by Williamson in terms of propositional logic. In this part we highlight the main results: (1) consider that the combination of the truth values defines a type of quantifier, and (2) to establish the prevalence of the truth or falsity is relevant in reconstruction. Subsequently, in Sect. 3 these ideas are taken to build two structures that satisfy the constraints presented, but with some difficulties, specifically the asymmetry in the number of rules in each polyhedron. In Part 4 we developed a reinterpretation of the ideas presented to solve the problems. That interpretation is to consider further consequences of the above conditions, the commutativity as an ingredient necessary to define a quantifier square opposition. And finally in the last section we apply the results to the traditional theory.

2 Williamson's Squares

Colwyn Williamson in his work *Squares of opposition: Comparisons between Syllogistic and Propositional Logic*, develops an analysis of propositional and syllogistic logic based on a definition of some Boolean operators. He begins with a definition of the operator K representing the conjunction as follows: $K11 = 1$, $K10 = 0$, $K01 = 0$, $K00 = 0$. The operator K represents the conjunction connective in propositional logic and the 1 and 0 represent truth values *True* and *False*, and the combinations of 1 and 0 represents the possible valuations for the propositional variables, therefore the definition of the operator is 1000.

Taking in account this definition for the logical connectives Williamson defines the following operators: $B = 1101$, $C = 1011$, $D = 0111$, $J = 0110$, $L = 0100$, $M = 0010$, $V = 1110$, $X = 0001$. Williamson uses this resource to elaborate an analysis of the traditional opposition square, and in addition to the later definitions he introduce notation to define the four statements of the corners of the square of opposition as follows:

$Aab :=$ all $a's$ are $b's$
$Eab :=$ no $a's$ are $b's$
$Iab :=$ some $a's$ are $b's$
$Oab :=$ some $a's$ are not $b's$

This notation is used by Williamson to generate the following traditional square of opposition ($TS1$) (Fig. 1).

Fig. 1 TS1

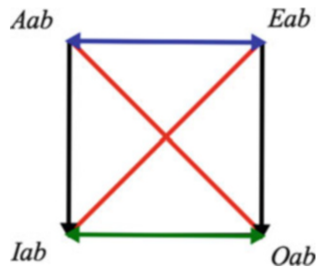
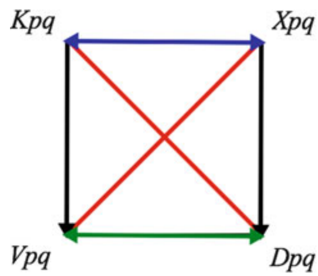


Fig. 2 SP1



We use the standard notation to represent the opposition relations of the square which are represented in Williamson's notation as *D* for contrariety, *J* for contradiction, *C* for subalternation, and *V* for subcontrariety. The first comparison in Williamson's analysis is between the previous square *TS*1 and the following square (which we can call *SP*1) (Fig. 2).

We may assume with Williamson that the *q* in the later square could be consider as predicate of the formulas in the corners, but he finds some problems concerning the *equipollence rule*. The rule consist in define the operator of a formula of some corner in terms of the negation and the operator of the remaining three corners preserving the truth conditions of the initial formula, for example, when we deny[1] the predicate of *Aab* we get a formula with the same truth conditions of *Eab*, namely *Aanb*. Williamson rejects this assumption for the traditional propositional square because the rule of equipollence can't hold in the later square. To verify this take *Kpq* and *Xpq* as analogous of *Aab* ad *Eab*, in according to equipollence rule *KpNq* must be equivalent to *Xpq*, but the equivalent of the later is *KNpNq* and *KpNq* is equivalent to *DNpq*.

There is another reason to reject this square as a faithful propositional representation, in the traditional square only two of the four formulas could be convert, that is *Eab* → *Eba* and *Iab* → *Iba*, but no so with *Aab* → *Aba* and *Oab* → *Oba*. But in the later propositional square all formulas can be converted. These issues make Williamson to generate two squares that correspond exactly with the traditional, that means that the later square don't preserve the restrictions of the traditional square of opposition. The first propositional square is presented in Fig. 3.

[1]The Williamson's notation for negation is: for external negation (de dicto), and for internal negation (de re).

Fig. 3 WP1

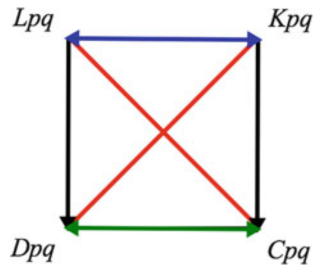
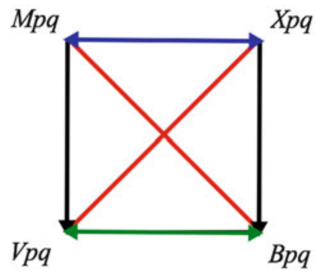


Fig. 4 WP2



Williamson generates a correspondence between the two squares (*TS*1 and *WP*1) associating each Boolean operator formula of *WP*1 with the categorical formulas of *TS*1 in the following sense:

Lpq is analogous to *Aab*
Kpq is analogous to *Eab*
Dpq is analogous to *Iab*
Cpq is analogous to *Oab*

The soundness of this interpretation is confirmed by the preservation of both the rules of equipollence and immediate inference.[2] The second square is shown in Fig. 4.

In this case the link is between *Mpq*, *Xpq*, *Vpq*, and *Bpq* with the categorical formulas *Aab*, *Eab*, *Iab*, and *Oab*, respectively; and the equipollence and immediate inference rules also hold. Williamson remarks two questions concerning the truth conditions of the formulas in the corners of these squares. First "it will be noticed that the operators capable of forming an exact analogue for the traditional square are the ones in which three and only three of the defining values are the same: 1000, 0100, 0010, 0001, 0111, 1011, 1101 and 1110" [5, p. 499]. The second fact is connected with the correspondence by one side between the truth value *True* and the particular quantifier, and by the other side between the truth value *False* and universal quantifier, namely "the operators corresponding to the "universals" of syllogistic are those in which false values predominate, while the operators corresponding to the "particulars" of syllogistic are those in which true values

[2]Simple conversion, conversion *per accidents*, obversion, contraposition, and inversion.

predominate" [5, Idem.]. Williamson emphasize that this correspondence could be some kind of analogue to the medieval distribution theory, but he does not say more.[3]

Williamson note also an absence of symmetry in the comparison, because on the one hand we have one traditional square, and on the other hand we could generate two propositional squares with the above operators. We can assume following Williamson that "there are—or ought to be—two such squares in traditional logic also", and we may call this later sentence the *Williamson's thesis*. In other words, there are eight and not only four, logically independent propositions. Williamson extends the traditional square and add four new quantifiers: *Rab*, *Sab*, *Tab*, and *Uab*; and later he define them as follows:

$Rab \equiv Ananb \equiv Aba$
$Sab \equiv Enanb$
$Tab \equiv Inanb$
$Uab \equiv Onanb \equiv Oba$

These new quantifiers are used by Williamson to present another traditional opposition square analogous to the first ($TS1$) to balance the situation and, evidently, he relates each traditional square of opposition with his counterpart in propositional notation. In this case the relationships are established between the new quantifiers *Rab*, *Sab*, *Tab*, and *Uab* with the later Boolean operators *Mpq*, *Xpq*, *Vpq*, and *Bpq*, respectively. Therefore, the following equivalences also hold in propositional logic:

$Mpq \equiv LNpNq \equiv Lqp$
$Xpq \equiv KNpNq$
$Vpq \equiv DNpNq$
$Bpq \equiv CNpNq \equiv Cqp$

Williamson's interpretation ends with two notes about "certain kind of connection between Syllogistic and propositional logic"[5, p. 500]. First, following Łukasiewicz, Williamson states that "the procedures of traditional logic presuppose laws of propositional calculus"[5, Idem.]; and second, he makes the claim that "syllogistic and propositional logic express, at some level, a common structure of reasoning"[5, Idem.]. We will focus on this assumptions in the final section, and we will give an argument based on some thesis presented in the fourth section to vindicate the words of Łukasiewicz.

3 From Squares to Octahedra

In this section we extend the previous ideas about the propositional interpretation of traditional square of opposition, in specific we will show how to construct two opposition structures based on Williamson's squares. The novelty of this polyhedra is that it satisfy the restrictions concerning the preservation of the equipollence and immediate inference

[3] We will say a few words about that in the final section.

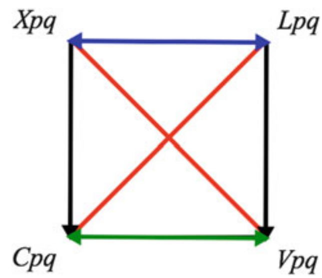


Fig. 5 SP2

rules, but as we will see, this polyhedra has two basic problems related with the rules of the obversion and with the preservation of symmetry of the cited rules; nevertheless, the octahedra has some interesting properties that serve as indication—together with the mentioned difficulties—of the construction of a more complex opposition structure. The main motivation of the extension of the Williamson's squares is to analyze the relation between these squares with the *spurious*[4] squares, namely *SP*1 and a new square *SP*2 with the same problems that the later. Also we think that our extension is relevant because we will see the role payed by *SP*1 and *SP*2 in the representation of the traditional opposition square. Our thesis is twofold, by one side, using *Williamson's thesis* we will show that there is not only one spurious square, but two[5]; and by the other side, we think that if the spurious squares are taken independently they don't satisfy some rules, but if we put all together we may construct a structure that satisfy the restrictions stated by Williamson to make a correct propositional representation of the traditional square, in other words, the spurious squares are intermediaries between the genuine squares.

We begin presenting the spurious squares and consequently we show how join them to the squares presented in the previous section. The first *SP*1 is the one who has presented by Williamson, as we say it has problems with equipollence and immediate inference rules, and for this reason is spurious. For the same reason the square in Fig. 5 is spurious.

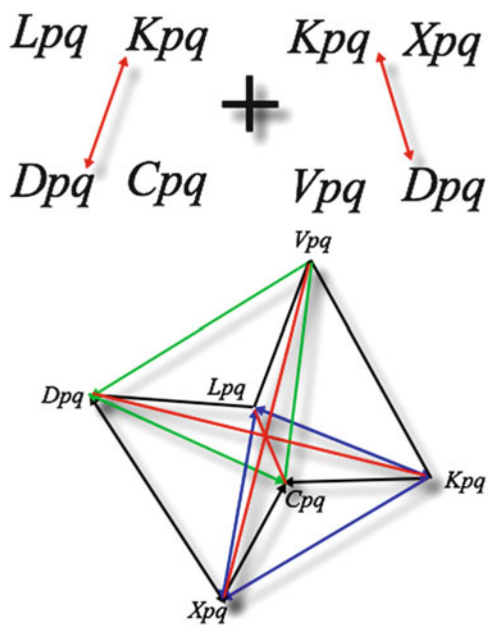
Although this square preserves the main opposition relations it is not a correct representation of the traditional square, to see why take, for example, *Xpq* and *Lpq*, *Xpq* must be equivalent to *LpNq* but it is equivalent to *Kpq* not to *Xpq*. As we say, this two squares are not part of the propositional reconstruction of the theory, to be taken into account in the reconstruction of the propositional representation we must join them to the genuine squares. We begin with the *SP*1 and the *WP*1 squares, in Fig. 6 we can see how we construct the first octahedron from the intersection of the two squares.

The squares intersect perpendicularly taking as point of union the contradictory axis of *Kpq* and *Dpq*. In the picture we have above the two squares with the axis highlighted, but also if we look careful there is another square, the spurious *SP*2. This fact will be analyzed later when we talk about how mix the two octahedra. There are some technical reasons to consider this structure as an suitable reconstruction of the traditional square;

[4]The name was suggested by one of the jurors who reviewed an earlier draft.

[5]I thank one of the jurors for this observation.

Fig. 6 WP1+SP1=D1



in the first place, in the operators V and X three of the defining values are the same, and in the second place in X predominate the false values and in V the true values, because the former is universal and the later particular. Before we move to the presentation of the second octahedron we discuss what properties and rules preserve. The octahedron preserve all the immediate and equipollence rules, but it extends the number of rules in both cases. In the first place we have the equipollence rules:

$Lpq \equiv KpNq \equiv NCpq \equiv NDpNq \equiv NVNpq \equiv XNpq$
$Kpq \equiv LpNq \equiv NDpq \equiv NCpNq \equiv NVNpNq \equiv XNpNq$
$Dpq \equiv CpNq \equiv NKpq \equiv NLpNq \equiv VNpNq \equiv NXNpNq$
$Cpq \equiv DpNq \equiv NLpq \equiv NKpNq \equiv VNpq \equiv NXNpq$
$Vpq \equiv CNpq \equiv DNpNq \equiv NKNpNq \equiv NXpq \equiv NLNpq$
$Xpq \equiv LNpq \equiv KNpNq \equiv NDNpNq \equiv NVpq \equiv NCNpq$

This rules don't have any problem, the only change is in the number. The relevant and interesting modification is in the immediate inference rules, we analyze one by one starting with the simple conversion rule. This rule states that a formula implies another formula with the same operator but subject and predicate exchanged; as we say, this rule is only satisfied by formulas with the E and I quantifier, and for this reason we only have restricted number of them, in specific four. The next rule is conversion *per accidents*. This rule states that an universal formula implies its subaltern with subject and predicate exchanged. In this case we have six formulas that satisfy this rule because we have six subalternation relation. The next one is obversion. In the square $WP1$ we have four obversion rules, this rule states that a formula implies its contrary—in the case of the universal formulas—or its subcontrary—in the case of particular formulas—with the predicated denied. As the $D1$

octahedron have two triangles, one of contraries and other of subcontraries, it is expected that in this polyhedra we have twelve rules of obversion, but the $D1$ only have four rules.[6] The remaining formulas have in common the fact that they preserve some pattern that exhaust the combination of 1 and 0 between p and q as we show below:

$(Kpq \rightarrow XpNq) = 0$ iff $p = q = 1$
$(Xpq \rightarrow LpNq) = 0$ iff $p = q = 0$
$(Vpq \rightarrow DpNq) = 0$ iff $p = 1, q = 0$
$(Dpq \rightarrow VpNq) = 0$ iff $p = 0, q = 1$
$(Lpq \rightarrow XpNq) = 0$ iff $p = 1, q = 0$
$(Xpq \rightarrow KpNq) = 0$ iff $p = q = 0$
$(Vpq \rightarrow CpNq) = 0$ iff $p = 0, q = 1$
$(Cpq \rightarrow VpNq) = 0$ iff $p = q = 1$.

Later we will present a detailed analysis of the question with the help of some additional restrictions to the formulas to make a better propositional reconstruction of the theory with an explanation of this difficulties.

The next rule is contraposition, this rule states that a formula implies another formula whit the same operator, also the subject and predicate are exchanged and negated. The main reason that not all operator satisfies the rule lies in some facts related with the properties of conditional and similar operators, we return on that later. In $WP1$ we only have two rules of contraposition and in the $D1$ we have the same number. The last rule is inversion, this rule states that an universal formula implies its contradictory with the subject denied. The octahedron satisfy three rules of inversion corresponding to the three contradictory axis. Taking in account this facts we may generate the following list of immediate inference rules:

1. $Kpq \rightarrow Kqp$
2. $Dpq \rightarrow Dqp$
3. $Xpq \rightarrow Xqp$
4. $Vpq \rightarrow Vqp$
5. $Lpq \rightarrow Dqp$
6. $Lpq \rightarrow Vqp$
7. $Kpq \rightarrow Cqp$
8. $Kpq \rightarrow Vqp$
9. $Xpq \rightarrow Dqp$
10. $Xpq \rightarrow Cqp$
11. $Lpq \rightarrow KpNq$
12. $KpNq \rightarrow LpNq$
13. $Dpq \rightarrow CpNq$
14. $Cpq \rightarrow DpNq$
15. $Lpq \rightarrow LNqNp$
16. $Cpq \rightarrow CNqNp$

[6]There are many facts that justify this anomaly but that does not discuss now, we will return to the issue in the next section.

17. $Lpq \rightarrow CNpq$
18. $Kpq \rightarrow DNpq$
19. $Xpq \rightarrow VNpq$

The formulas 1–4 are simple conversion, the formulas 5–10 are conversion *per accidents*, the formulas 11–14 are obversion, 15 and 16 are contraposition, and 17–19 are inversion. Now we present the other octahedron together with its list of formulas, but first we explain how to construct the octahedron (Fig. 7).

As in the $D1$ in this octahedron the squares are intersected in a contradictory axis composed by the X and the V operators. The technical restrictions are also satisfied by this polyhedron, i.e. the X is universal and V is particular, and both have three identical values in its definition. Now we will discuss the rules of inference. By one side the octahedron $D2$ preserve the same number of equipollence rules, and there is no anomaly in this kind of rules. By the other side, there are an asymmetry with the later octahedron in the sense that the number of inference rules that preserves are different, the $D2$ only preserves seventeen rules. The following are the equipollence rules:

$Mpq \equiv XpNq \equiv NBpq \equiv NVpNq \equiv NCNpNq \equiv LNpNq$
$Xpq \equiv MpNq \equiv NVpq \equiv NBpNq \equiv NCNpq \equiv LNpq$
$Vpq \equiv BpNq \equiv NXpq \equiv NMpNq \equiv CNpq \equiv NLNpq$
$Bpq \equiv VpNq \equiv NMpq \equiv NXpNq \equiv CNpNq \equiv LNpNq$
$Cpq \equiv NLpq \equiv VNpq \equiv NMNpNq \equiv NXNpq \equiv BNpNq$
$Lpq \equiv NCpq \equiv XNpq \equiv MNpNq \equiv NVNpq \equiv NBNpNq$

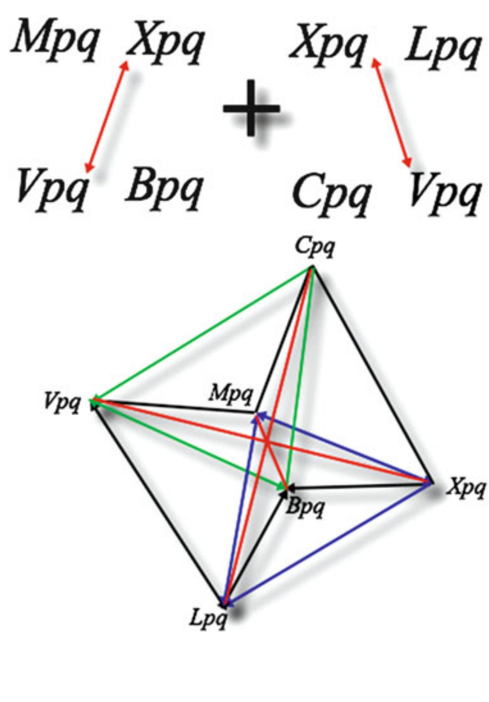


Fig. 7 WP2+SP2=D2

As in the case of $D1$ the main change with respect with the squares lies in the immediate inference rules, now we analyze this issue. The first anomaly is present in the simple conversion rules, in $D1$ we have four rules and here we have only two. In the second place, the $D1$ has six rules of conversion *per accidents* and the octahedron $D2$ hast only four. In the case of the obversion rule we have the same number in the two octahedra but, we have the same situation as in the $D1$, namely, the potential rule schemes that fails in one assignation, as we see below:

$(Bpq \rightarrow CpNq) = 0$ iff $p = q = 1$
$(Cpq \rightarrow BpNq) = 0$ iff $p = q = 0$
$(Lpq \rightarrow MpNq) = 0$ iff $p = 1, q = 0$
$(Mpq \rightarrow LpNq) = 0$ iff $p = 0, q = 1$
$(Vpq \rightarrow CpNq) = 0$ iff $p = q = 1$
$(Xpq \rightarrow LpNq) = 0$ iff $p = q = 0$
$(Lpq \rightarrow XpNq) = 0$ iff $p = 1, q = 0$
$(Cpq \rightarrow VpNq) = 0$ iff $p = 0, q = 1$

We will also give a justification of this facts in the next section. Following with contraposition, the octahedron $D2$ satisfy two more rules that the octahedron $D1$, in this sense we get four rules. And finally the octahedron $D2$ has three rules while the octahedron $D1$ only has two. To end this section we present the list of rules of $D2$ octahedra, and in the next section we try to solve the problems generated by these structures:

1. $Xpq \rightarrow Xqp$
2. $Vpq \rightarrow Vqp$
3. $Mpq \rightarrow Vqp$
4. $Xpq \rightarrow Bqp$
5. $Xpq \rightarrow Cqp$
6. $Lpq \rightarrow Vqp$
7. $Mpq \rightarrow XpNq$
8. $XpNq \rightarrow MpNq$
9. $Vpq \rightarrow BpNq$
10. $Bpq \rightarrow VpNq$
11. $Mpq \rightarrow MNqNp$
12. $Bpq \rightarrow BNqNp$
13. $Lpq \rightarrow LNqNp$
14. $Cpq \rightarrow CNqNp$
15. $Mpq \rightarrow BNpq$
16. $Xpq \rightarrow VNpq$
17. $Lpq \rightarrow CNpq$

4 Solving the Difficulties of the Octahedra: $D1 + D2 =$ Hexagonal Bipyramid of Opposition

In the previous section we have displayed the construction of two octahedra that extend Williamson's squares and preserve the propositional reconstruction of the traditional square of opposition. Despite being a conservative extension the octahedra they have some difficulties relative to the validity of the immediate inference rules and the symmetry of both. In this section we discuss these facts that cause problems and do not allow us to reconstruct faithfully the traditional square in terms of propositional logic; then, based on the analysis we argue in favor of the construction of a more complex structure that connects all the previous polyhedra. This structure is an Hexagonal Bipyramid. The novelty with respect of its construction could be summarize in the following points: (1) with this analysis we can establish some relevant properties needed to understand the restrictions of the rules of traditional square of opposition, (2) we will add a new restriction for a correct reconstruction of the square in terms of propositional logic, namely, the commutativity property; and finally (3) we define essential properties of the four corners of the square of opposition from the point of view of propositional logic. The last point will be emphasized in the final section in which we will apply all the results presented here to the traditional square of opposition.

We begin detailing the steps to form this structure and consequently we analyze the resulting rules. When we analyze these rules we present reasons for the exclusion of the formulas not satisfied in the octahedra and thus solve the problems of the previous section. With this solution we will undermine the asymmetry in the previous interpretation.

The reason for the asymmetry is again that the octahedra are intermediate points in building a more complex figure that more faithfully reconstructs both propositional interpretation of square as the Williamson's thesis. Initially, to show how to pass from the octahedra to the Hexagonal Bipyramid we need to transform the octahedra in hexagons as we see below (Fig. 8).

As we know, the octahedra are only a 3D-representation of a 2D-structure, namely the hexagon of opposition [4, p. 181], [2]. The choice of one representation over the other obeys heuristic questions, in the above case what guided our way to generate the three-

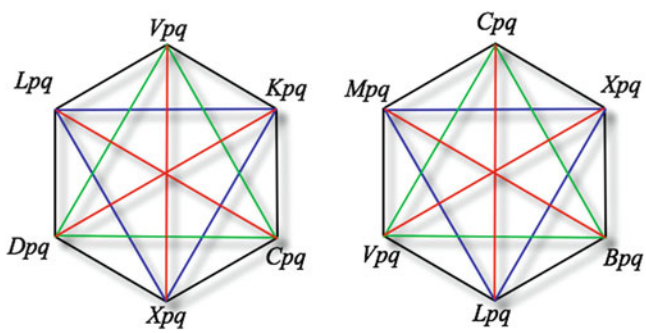


Fig. 8 HD1 and HD2

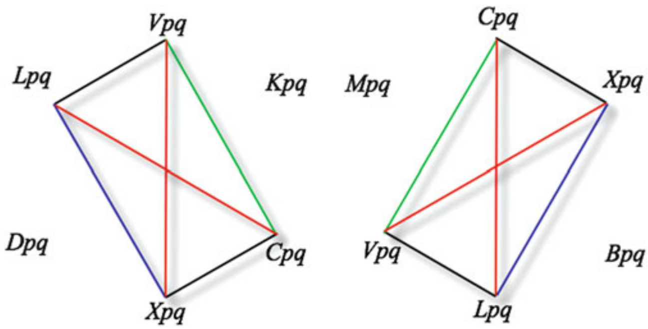


Fig. 9 SP2 in HD1 and HD2

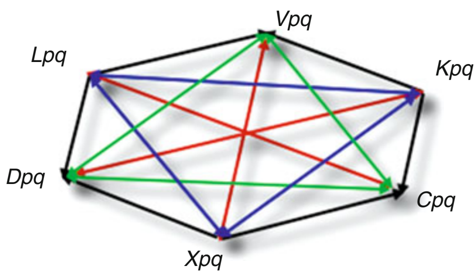


Fig. 10 Base=HD1

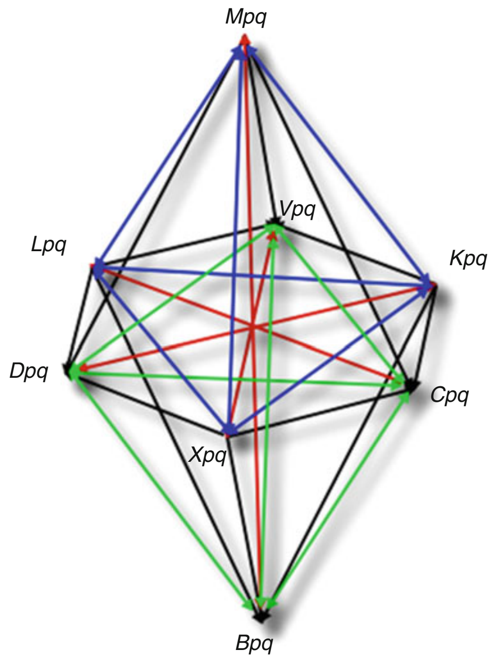
dimensional structure was to highlight two important facts: (1) the function of the spurious square, and (2) to display the asymmetry between the two octahedra, in the sense that the spurious square *SP*2 is in both octahedron and the spurious *SP*1 only the first. These facts are important now because for construction of the Hexagonal Bipyramid we need to consider again the function of the spurious squares. In this case we will take advantage of the visual characteristics of the hexagons to punctuate our thesis.

The feature that we wish to emphasize extracted from the three-dimensional analysis is the presence of the spurious square *SP*2 on both hexagons (octahedra), as we can see in Fig. 9.

We must highlight several facts that support our way of proceeding. As in the previous case, by joining two spurious squares with two genuine squares we take as a point of intersection an axis of contradictories, now what we need is another intersection point between the hexagons, and this is precisely the spurious square. The clue that led us to unite them was precisely the presence of this square on both structures generating asymmetry in the reconstruction.

To generate the Hexagonal Bipyramid (2*PH*) we take as the base the hexagon as shown in Fig. 10. Now, to complete the Hexagonal Bipyramid we need to remove from *HD*2 the vertices that are in the base, i.e., the spurious square *SP*2. This leaves us with a contradictory line going from *Mpq* to *Bpq*. The remainder is to complete the figure by

Fig. 11 2PH



taking the *MB* shaft and cutting the base by the center as shown in Fig. 11. The vertices of this axis works as the tips of the Hexagonal Bipyramid.

The following is to talk about the inference rules generated by this structure, and from this discussion we will study in depth the problems generated in the previous section and the solution that this structure provides. We must highlight several facts: (1) The operators of this figure are sufficient to generate a complete reconstruction of the traditional square from propositional logic; (2) the operators satisfy the constraints identified by Williamson; and (3) the use of these operators in specific vindicates the Williamson's thesis. In addition it should be noted a fact concerning operator properties: In addition to dividing the operators of this structure in "0-predominant" (universal) and "1-predominant" (particular), they can be subdivided into commutative and noncommutative. In this division lies the solution to the above problems and it is the key to understanding the properties of immediate inference rules.

We analyze the square with this distinction. Squares *WP*1 and *WP*2 satisfy a common feature, from the pair of universal operators one is commutative when the other is not, in *WP*1 the *L* is noncommutative and *K* is commutative, and in *WP*2 the *M* is noncommutative and *X* is commutative. The same in the particulars, in the first square *D* commutative and *C* noncommutative and in *WP*2 the *V* is commutative and *B* is not. This makes us suppose that in addition to the restrictions outlined by Williamson to get a correct reconstruction of the traditional square, we can add the following restriction: the commutative/noncommutative combination is distributed symmetrically on the square. In other words, it is not possible to have two adjacent commutative or noncommutative formulas in the square.

We can be more radical and specify this restriction for categorical formulas as follows. The *A* and the *O* corners must contain only a noncommutative operator and the *E* and *I* corner must contain only a commutative operator.[7] This restriction is preserved in all structures generated above, and from it we can now solve the problems encountered. Now we will analyze the rules that satisfies the Hexagonal Bipyramid and why only meets that set, consequently we explain why the other structures left out several potential rules. First, joining the two octahedra an intersection between the rules is generated because there are rules that both structures satisfy, we first present equipollence rules, which only undergo a change in the number. In each Williamson's squares there are sixteen equipollence rules, and in each octahedra there are thirty two rules, and now we have the following list of sixty four rules:

$Lpq \equiv KpNq \equiv NCpq \equiv NDpNq \equiv NVNpq \equiv XNpq \equiv MNpNq \equiv NBNpNq$
$Kpq \equiv LpNq \equiv NDpq \equiv NCpNq \equiv NVNpNq \equiv XNpNq \equiv MNpq \equiv NBNpq$
$Dpq \equiv CpNq \equiv NKpq \equiv NLpNq \equiv VNpNq \equiv NXNpNq \equiv NMNpq \equiv BNpq$
$Cpq \equiv DpNq \equiv NLpq \equiv NKpNq \equiv VNpq \equiv NXNpq \equiv NMNpNq \equiv BNpNq$
$Vpq \equiv CNpq \equiv DNpNq \equiv NKNpNq \equiv NXpq \equiv NLNpq \equiv NMpNq \equiv BpNq$
$Xpq \equiv LNpq \equiv KNpNq \equiv NDNpNq \equiv NVpq \equiv NCNpq \equiv MpNq \equiv NBpNq$
$Mpq \equiv XpNq \equiv NBpq \equiv NVpNq \equiv NCNpNq \equiv LNpNq \equiv KNpq \equiv NDNpq$
$Bpq \equiv VpNq \equiv NMpq \equiv NXpNq \equiv CNpNq \equiv LNpNq \equiv NKNpq \equiv DNpq$

Now we continue with the rules of immediate inference. The first group comprises the simple conversion rules. We have previously said that this rule is generated only between *E* and *I* of the traditional square. From our commutative analysis we can establish that the cause of this is that the formulas that can represent *E* or *I* corners are only formulas with commutative operator. Therefore, the operators susceptible to occupy one of those two corners are commutative, and consequently always preserve simple conversion. For this reason in the first octahedron there are more simple conversion rules that in the second, because the first octahedron has more universal commutative operators. In the second place the conversion rule, as we said states that a universal formula implies his subaltern with subject and predicate interchanged. The reason that there are formulas that do not satisfy this rule is that as the formula involved should be subaltern, one must be commutative if the other is not. In the Hexagonal Bipyramid we have twelve rules, nine present in the octahedra six on the first and four in the second, with a repeated rule present in both, and in addition to these, three new that resulted from the union of the two octahedra.

The next rule is the obversion. In this case there are several facts that highlight. First, we have said that this rule is generated between pairs of contrary or subcontrary formulas. Also, there are twenty four potential rules. Considering contrary and subcontrary relations

[7] This restriction does not exclude the categorical notation or the first order interpretation of the square, and in the last section we will see why.

in the Hexagonal Pyramid we obtain the following list, of which only the first eight are satisfied rules.

1. $Lpq \rightarrow KpNq$
2. $Kpq \rightarrow LpNq$
3. $Dpq \rightarrow CpNq$
4. $Cpq \rightarrow DpNq$
5. $Mpq \rightarrow XpNq$
6. $Xpq \rightarrow MpNq$
7. $Vpq \rightarrow BpNq$
8. $Bpq \rightarrow VpNq$
9. $Kpq \rightarrow XpNq$
10. $Xpq \rightarrow KpNq$
11. $Vpq \rightarrow DpNq$
12. $Dpq \rightarrow VpNq$
13. $Vpq \rightarrow CNqNp$
14. $Xpq \rightarrow LNqNp$
15. $Lpq \rightarrow XNqNp$
16. $Cpq \rightarrow VNqNp$
17. $Bpq \rightarrow CpNq$
18. $Cpq \rightarrow BpNq$
19. $Lpq \rightarrow MpNq$
20. $Mpq \rightarrow LpNq$
21. $Mpq \rightarrow KNqNp$
22. $Kpq \rightarrow MNqNp$
23. $Bpq \rightarrow DNqNp$
24. $Dpq \rightarrow BNqNp$

The remaining are some of those mentioned above that are excluded from the octahedra and generate a pattern on an assignment that makes false (9–20). The second important fact is that there are two connected reasons that cause the last group of formulas are excluded, on the one hand that the operators must satisfy the adjacency of commutativity, so we can not find combinations of rules in which there are two commutative operators or two noncommutative. Although we found relations between commutative and noncommutative in the Hexagonal Bipyramid, this fact is justified because of the spurious squares connect the genuine ones. And this brings us to the second reason, the formulas excluded from this rule belongs to spurious squares. The formulas 9–12 belong to $SP1$ the 13–16 to $SP2$, and the remaining are not in any of the squares presented so far, and that is due to the fact that there are two new spurious squares that result of the union of the two octahedra (Fig. 12).

These squares are only present in the Hexagonal Bipyramid because of its vertices are scattered on both octahedra. They are analogous to the above in the following sense. $SP1$ and $SP3$ are spurious because the former is composed of commutative operators and the second noncommutative operators, these are spurious because they cancel commutativity adjacency. On the other hand, the $SP2$ and $SP4$ are spurious because they do not satisfy equipollence. This is how the problems of the octahedra are cleared and asymmetry is solved. Finally we analyze contraposition and inversion rules.

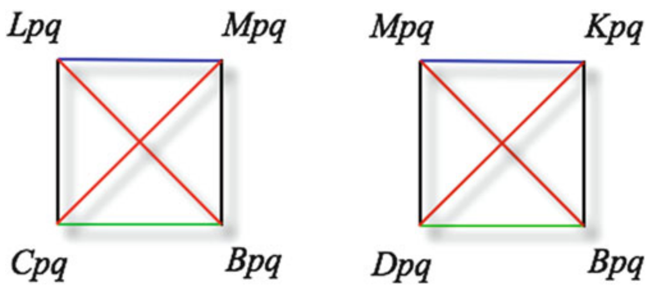


Fig. 12 SP3 and SP4

The contraposition rules are only satisfied by noncommutative operators, therefore there are only four, and that explains why in the octahedra are only two in the first and four in the second; the asymmetry is explained by the predominance of commutative operators in $D1$ and the prevalence of non-commutative $D2$. Finally, inversion rules are satisfied between pairs of contradictory operators, and in this case there is no difficulty, leaving us with the following list of rules. We continue in the las section with the interpretation of this facts in the traditional square.

1. $Kpq \rightarrow Kqp$
2. $Dpq \rightarrow Dqp$
3. $Xpq \rightarrow Xqp$
4. $Vpq \rightarrow Vqp$
5. $Lpq \rightarrow Dqp$
6. $Lpq \rightarrow Vqp$
7. $Kpq \rightarrow Cqp$
8. $Kpq \rightarrow Vqp$
9. $Xpq \rightarrow Dqp$
10. $Xpq \rightarrow Cqp$
11. $Mpq \rightarrow Vqp$
12. $Xpq \rightarrow Bqp$
13. $Lpq \rightarrow Vqp$
14. $Mpq \rightarrow Dqp$
15. $Mpq \rightarrow Cqp$
16. $Kpq \rightarrow Bqp$
17. $Lpq \rightarrow KpNq$
18. $Kpq \rightarrow LpNq$
19. $Dpq \rightarrow CpNq$
20. $Cpq \rightarrow DpNq$
21. $Mpq \rightarrow XpNq$
22. $Xpq \rightarrow MpNq$
23. $Vpq \rightarrow BpNq$
24. $Bpq \rightarrow VpNq$
25. $Lpq \rightarrow LNqNp$

26. $Cpq \to CNqNp$
27. $Mpq \to MNqNp$
28. $Bpq \to BNqNp$
29. $Lpq \to CNpq$
30. $Kpq \to DNpq$
31. $Xpq \to VNpq$
32. $Mpq \to BNpq$

5 From Bipyramid to Octagon of Opposition

In this section we discuss the final part of the analysis with reference to the first square presented: *TS*1. The thesis that we defend to close is related to the bond that—according to Williamson [5, p. 500]—Łukasiewicz established between logic of terms and propositional logic. To do this, we will present two ways to view the Hexagonal Bipyramid in which emphasis is placed on the Williamson's thesis as a unified way to present both squares. Our strategy will be to present the pyramid in traditional notation (A, E, I, O) and consequently order it to form a cube and an octagon, with reference to the two squares; finally we will use notation of first-order logic to show structural similarities and again we use the commutative interpretation to analyze the differences of each vertex. The following figure shows the Hexagonal Bipyramid with traditional notation (Fig. 13).

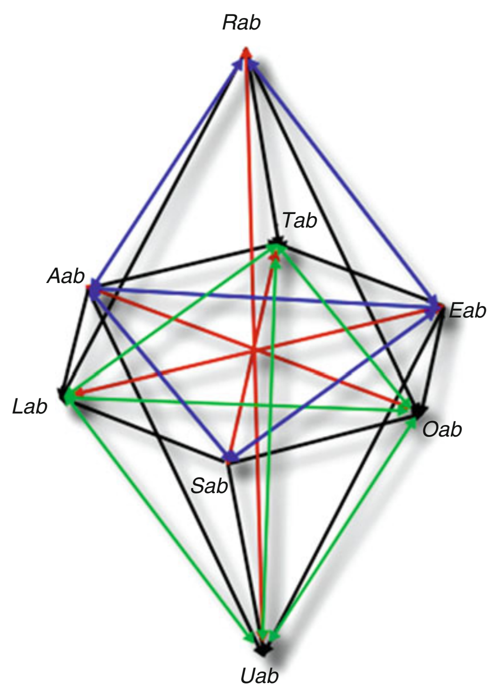


Fig. 13 Traditional 2PH

To better appreciate the link between the two squares we build a cube showing how they connect. This cube in turn can be transformed into a octagon which is simply the interpretation of two-dimensional cube. The cube shows how the two squares are connected from spurious square, in this representation becomes clear its function. The question now is how to interpret the Williamson's thesis from the relationship between these two squares (Fig. 14)?

Our position is that there are two squares, because of an important property of the operators, the inversion. Following to Gottshalk [3, p. 194] '[t]o invert a column of T's and F's is to turn the column upside down". The two squares *WP*1 and *WP*2 are inverse each other, and for that reason both separately satisfy the restrictions indicated for proper reconstruction of traditional logic, but also for that reason together satisfy the constraints. This octagon meets opposition relations in a different order than the other octagons, i.e. medieval octagons [1]; this is also due to inversion. For this reason, we can call this *The Inversion Octagon* (Fig. 15).

This octagon is the ultimate reconstruction of the traditional square, but still we can ask what about the remaining connectives of propositional logic, if we apply Williamson's thesis to get another octagon, this is also one that reconstructs the traditional theory of opposition? We believe that the answer is no because of the following three reasons: (1) the

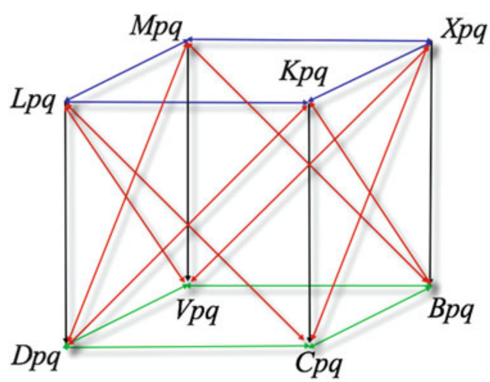


Fig. 14 Cube of opposition

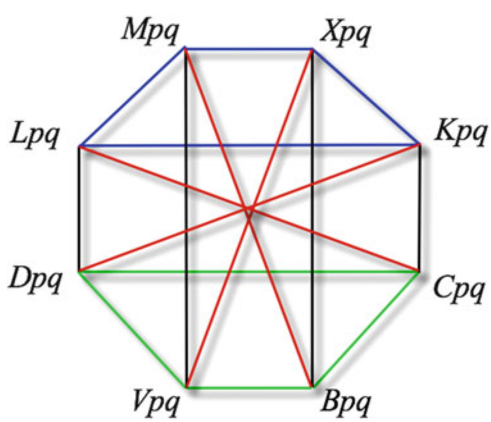


Fig. 15 Inversion octagon

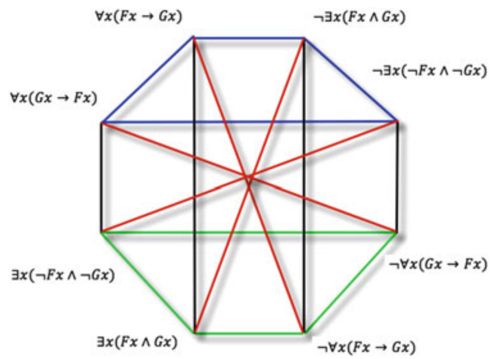


Fig. 16 First-order inversion octagon

remaining operators are not 0-predominant (therefore there is no universal operators), (2) the operators are not 1-predominant(therefore there in no particular operators), and (3) do not meet commutative adjacency. Now to conclude, we analyze this results in the first-order octagon of transposed squares (Fig. 16).

This octagon has the same properties of the previous one and therefore preserves all the equipollence and immediate inference rules presented. Also, is has the same constraints related with commutative adjacency, but in which sense this octagon preserves inversion? We believe that the octagon also satisfies inversion, but in different way depending on whether the formula is commutative or not. For example, take $\forall x(Fx \rightarrow Gx)$ we obtain its inverse only exchanging the F for the G; on the other side take $\exists x(Fx \wedge Gx)$ we obtain its inverse denying Fx and Gx. The first process is applied only to noncommutative formulas and the second to commutative ones. In both cases the inversion is satisfied in the sense that inversion may be defined as the negation of duality, in our octagon if we take again $\exists x(Fx \wedge Gx)$ we obtain its inverse changing the $\wedge$ for its dual $\vee$ and denying them, we obtain $\exists x \neg(\neg Fx \vee \neg Gx)$ which is equivalent to $\exists x(\neg Fx \wedge \neg Gx)$. the same with the remaining corners of the octagon. Finally, we think that this results vindicate the intuition of Łukasiewicz and Williamson [5, p. 500], namely:

> These results cast some light on a certain kind of connection between syllogistic and propositional logic. It has been stressed, especially by Łukasiewicz, that the procedures of traditional logic presuppose laws of propositional calculus. The analogies described above, however, rest on a direct comparison of the logic of terms and the logic of propositions; and they appear to suggest that syllogistic and propositional logic express, at some level, a common structure of reasoning.

6 Conclusion

We may summarize the main results in the following points: (1) Williamson's thesis serve us to generate many opposition structures that hold the constrains imposed in the paper to make a correct reconstruction of the traditional syllogistic logic in terms of proposition logic; (2) we emphasize that the commutativity property play a relevant role in the traditional presentation of the square, and therefore (3) we show the structural connection between these two structures.

Finally, we think that our interpretation of the connectives and quantifiers could be extend to analyze some relevant notions in logic, like the medieval distribution theory, the existential import, and the relation of the spurious square and the *disparate* in medieval octagons of opposition, but it remains open for further work.

Acknowledgements Many thanks to Yessica Ramos for their detailed revisions to previous versions of this paper, and I especially want to thank the two jurors who reviewed earlier versions of this work. Without his detailed comments would have been impossible to reach the conclusions presented here.

References

1. J.M. Campos–Benítez, El octágono medieval de Oposición para oraciones con predicados cuantificados. Tópicos **44**, 177–205 (2013)
2. D. Dubois, H. Prade, From Blanché's hexagonal organization of concepts to formal concept analysis and possibility theory. Log. Univers. **6**, 149–169 (2012)
3. W.H. Gottschalk, The theory of quaternality. J. Symb. Log. **18**(3), 193–196 (1953)
4. H. Smessaert, The classical Aristotelian hexagon versus the modern duality hexagon. Log. Univers. **6**, 171–199 (2012)
5. C. Williamson, Squares of opposition: comparisons between syllogistic and propositional logic. Notre Dame J. Formal Log. **13**, 497–500 (1972)
6. L. Wittgenstein, *Tractatus Logico-Philosophicus* (Routledge & Kegan Paul, London, 1974)

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