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Jean-Yves Béziau
Gianfranco Basti
Editors

The Square of Opposition: A Cornerstone of Thought

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The Square of Opposition: A Cornerstone of Thought

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ISSN 2297-0282 ISSN 2297-0290 (electronic)
Studies in Universal Logic
ISBN 978-3-319-45061-2 ISBN 978-3-319-45062-9 (eBook)
DOI 10.1007/978-3-319-45062-9

Library of Congress Control Number: 2016961939

Mathematics Subject Classification (2010): 03Axx

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Printed on acid-free paper

This book is published under the trade name Birkhäuser, www.birkhauser-science.com
The registered company is Springer International Publishing AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

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Part I
Introduction

The Square of Opposition: A Cornerstone of Thought

Jean-Yves Béziau and Gianfranco Basti

Abstract We first describe how after having started in Montreux, Switzerland in 2007, the congress on the square of opposition moved to the American University of Beirut in Lebanon in 2012 after a stop at the University Pasquale Paoli in Corsica in 2010. We then describe the square congress at the Pontifical Lateran University in the Vatican in 2014 and the resulting publications.

Keywords Fuzzy logic • Interdisciplinarity • Intuitionistic logic • Modal logic • Paraconsistent logic • Square of opposition • Syllogistic

Mathematics Subject Classification (2000) Primary 00B25; Secondary 00A66, 03A05, 03B22, 03B45; 03B53

1 From Montreux to Beirut, via Corsica

The *World Congress on the Square of Opposition*—SQUARE—is an interdisciplinary event. The idea of the SQUARE congress is to promote interdisciplinarity around a very simple theory that everybody can understand, develop and apply.

The first congress on the square of opposition was organized in Montreux, Switzerland, June 1–3, 2007, by the first author of this paper (hereafter JYB) at the time he was working at the University of Neuchâtel. Neuchâtel is the town of Jean Piaget (1896–1980). He was born there and his father, Arthur Piaget (1865–1952), was the first Rector of the University of Neuchâtel. Jean Piaget had the spirit of research since he was quite young. He started by studying the mollusks of Lake Neuchâtel. His studies then evolved up to vertebrates, in particular rational animals. Interested in intelligence he naturally worked in logic including the theory of opposition. Piaget was a strong promoter of interdisciplinarity. He coined the word "transdisciplinarity", that he thought was better, during a congress in 1970:

> Finally, we hope to see succeeding to the stage of interdisciplinary relations of a superior stage, which should be "transdisciplinary", i.e. which will not be limited to recognize the interactions and or reciprocities between the specialized researches, but which will locate these links inside a total system without stable boundaries between the disciplines [29].

Inspired by Piaget, JYB organized two interdisciplinary events at the University of Neuchâtel where he was working. One in 2005 on *symbolic thinking* and one in 2008 on *imagination*. The 1st SQUARE was however not organized in Neuchâtel but by the

J.-Y. Béziau, G. Basti (eds.), *The Square of Opposition: A Cornerstone of Thought*, Studies in Universal Logic, DOI 10.1007/978-3-319-45062-9_1

Fig. 1 Terence Parsons—1st SQUARE in Montreux in 2007

banks of another lake, Lake Geneva, at the Hotel Helvétie in Montreux, a nice Hotel where was previously organized the first *World Congress and School on Universal Logic*—UNILOG—in 2005. Participants of the 1st UNILOG liked very much this location so it was decided to organize again an event there. Moreover the SQUARE was developed in the same spirit as UNILOG. In both cases the idea is not to construct a big totalitarian system, but to promote exchange of ideas and openness of minds.

The 1st SQUARE gathered people from all over the world from many different fields. Among speakers were Pascal Engel, Jan Woleński, Laurence Horn, Peter Schröder-Heister, Terence Parsons (Fig. 1), Sieghard Beller, Dag Westerståhl. There was a square jazz show and the projection of the movie *Salomé*, a remake of the biblical story based on the square. For the resulting publications see [12] and [13].

The 2nd SQUARE was organized in Corte, at the *University of Corsica Pasquale Paoli*, June 17–20, 2010. The relation between Corsica and Switzerland is not necessarily obvious. Let us however remember that Jean-Jacques Rousseau was asked to write a constitution for Corsica and that JYB had lived in Corsica in his youth (cf. [4]). The proposal to organize the event in Corsica was made at the final round square table in Montreux by Pierre Simonnet, a computer scientist working at the University Pasquale Paoli. He subsequently succeeded to convince his colleagues of this university to organize the event there. They gave their full support, in particular Jean-François Santucci.

Fig. 2 Jean-Louis Hudry, Pierre Simonnet, Pierre Cartier, JYB—2nd SQUARE in Corsica in 2010

For the opening of the event there was a lecture by the Rector of the Corsican Academy, Michel Barat, a philosopher who had also been two times Great Master of the Grande Loge de France (GLDF). Among speakers were Pierre Cartier (Fig. 2), from IHES, Bures-sur-Yvette, with Alexander Grothendieck, the most famous members of Bourbaki of the second generation; Damian Niwinski from the Institute of Informatics, Warsaw University, Poland, editor-in-chief of *Fundamenta Informaticae*; Pieter Seuren from the Max Planck Institute for Psycholinguistics, The Netherlands; Stephen Read from the School of Philosophical and Anthropological Studies, University of Saint-Andrews, Scotland; Hartley Slater, from the University of Western Australia, Perth, Australia; Dale Jacquette from the Department of Philosophy, University of Bern, Switzerland. For the resulting publications of this event see [3] and [11].

The 3rd SQUARE happened at the American University of Beirut (AUB), Lebanon, June 26–30, 2012. Why organizing such square event in this location, at a time where the political situation was quite tense in the region (civil war in Syria, conflict between Lebanon and Israel)? JYB knew about AUB through the famous logician David Makinson, with whom he was in touch since many years, and who had been professor and chair of the department of philosophy of AUB. Later on when JYB was as a Fulbright scholar at UCLA (1994), the director of the Fulbright program was Ann Kerr the widow of Malcom Kerr, who had been president of AUB (assassinated when in office in 1984). Then JYB met Ray Brassier in September 2010 in Maastricht in Netherlands, who was at this time the director of the department of philosophy of AUB. This was at the occasion of the workshop *Cutting the "Not": Workshop on Negativity and Reflexivity* at the Van Eyck Academie organized by Tzuchien Tho. JYB gave there the talk "From classical negation to paranormal negation" where he presented the square of opposition and then started to talk with Brassier about the possibility to organize the SQUARE event at AUB. For the preparation of the event JYB went a first time in Beirut in early 2012 where he met Wafic Sabra, the director of the Center for Advanced Mathematical Sciences of AUB, a former

Fig. 3 Ray Brassier, JYB, Wafic Sabra—3rd SQUARE in Beirut in 2012

student of the physicist David Bohm, with whom JYB had worked. CAMS decided also to support the organization of the event (Fig. 3).

Besides the sponsorship of AUB, the 3rd SQUARE was sponsored by the French, Swiss, Brazilian and Italian embassies in Lebanon. The Swiss ambassador, Ruth Flint, offered a welcome cocktail with Swiss wine and cheese.The ambassador of Brazil, Paulo Roberto Campos Tarrisse da Fontoura, organized a party at the Brazilian cultural center in Beirut for the participants of the event. The program was very rich and varied with speakers such as Musa Akrami from Islamic Azad University, Science and Research Branch of Tehran, Iran; Oliver Kutz from the Department of Informatics, University of Bremen, Germany; Mihir Chakraborty from the Department of Pure Mathematics of University of Kolkata, India; François Nicolas from Ecole Normale Supérieure d'Ulm, Paris; Claudio Pizzi from the Department of Philosophy of the University of Siena, Italy; Saloua Chatti, from the Department of Philosophy of the University of Tunis, Tunisia; Jean Sallantin from LIRMM—CNRS Montpellier, France; Robert L. Gallagher from the Civilization Sequence Program of AUB. For the resulting publications see [14] and [9].

2 The Square at the Pontifical Lateran University, Vatican

At the final round square table in Beirut, Raffaela Giovagnoli from the Pontifical Lateran University (PUL), Vatican, proposed to organize the next edition of the event there. JYB visited the PUL in February 2013 to meet the second author of this paper, Gianfranco

Fig. 4 The entrance of the Pontifical Lateran University

Fig. 5 Bishop Enrico dal Covolo—Rector of PUL

Basti, who was at this time the Dean of the Faculty of Philosophy of PUL and the 4th SQUARE happened at PUL, May 5–9, 2014 (Figs. 4, 5, 6 and 7).

The event was very successful with about 150 participants. The welcome address was given in French by Bishop Enrico dal Covolo, Rector of the Pontifical Lateran University (an English version can be found in the present book, cf. [19]).

Fig. 6 On the *left* Gianfranco Basti, Dean of PUL Faculty of Philosophy discussing with the French Ambassador Bruno Joubert at Villa Bonaparte, Embassy of France in the Holy See

Fig. 7 One of the numerous lecturers from all over the world Juan Campos Benítez, Benemérita Autonomous University, Puebla, Mexico

There was a cocktail at the Polish Embassy in Vatican with the Ambassador Piotr Nowina-Konopka and another one at the beautiful Villa Bonaparte (former property of Pauline Bonaparte, the sister of Napoleon Bonaparte) offered by the Embassy of France in the Holy See, with his Excellency the Ambassador Bruno Joubert. The Polish cockatil was organized by Katarzyna Gan-Krzywoszyńska who had been since the 1st SQUARE one of the executive organizers of the SQUARE event. The French cocktail was organized by Juliette Lemaire and Anne Hénault (University Paris Sorbonne).

There were many speakers including Wolfgang Lenzen from the Department of Philosophy of the University of Osnabrueck, Germany; Rusty Jones from the Department of Philosophy of Harvard University, USA; Henri Prade from IRIT—CNRS, Toulouse, France; John Woods from Department of Philosophy of University of British Columbia, Canada; Bora Kumova from Izmir Institute of Technology, Turkey; François Lepage, from the Department of Philosophy of the University of Montréal, Canada; John N. Martin from the Department of Philosophy of the University of Cincinnati, USA; Patrick Eklund from the Department of Computing Science of Umeá University, Sweden; Lorenzo Magnani from the University of Pavia, Italy; Marcin Schroeder from Akita International University, Japan; Manuel Correia Machuca from the Department of Philosophy, Pontifical Catholic University, Chile with whom it was decided to organize the 5th SQUARE in Easter Island, November, 11–15, 2016.

3 New Investigations and Discoveries About the Square of Opposition

In this book there is a large spectrum of papers showing pretty well the fruitful result of cross-fertilization between various areas generated by the square. The papers published here are part of the lectures present at the 4th SQUARE, others were published in a special issue of *Logica Universalis* (cf. [10]).

As usual we have papers related with the history of the square such as the one by Manuel Correia, "The proto-exposition of Aristotelian categorical logic" [18], about a common source of the Apuleian and the Boethian square. The paper of Spencer Johnstone, "The modal octagon and John Buridan's modal ontology" [24], and Antonino Drago, "From Aristotle's square of opposition to the *tri-unity's concordance*: Cusanus' non-classical arguing" [20] about God, are analyses related to the medieval period.

Then we have three papers which are at the same time reinterpretations of the square from the point of view of modern logic and inputs for new developments, in the spirit of the work of Łukasiewicz. The paper by Bora Kumova, "Symmetric properties of the syllogistic system inherited from the square of opposition" [25], deals with the relations between syllogistic and the square (compare with [28]), the one by Paul Weingartner, "The square of opposition interpreted with a decidable modal logic" [35], is connected with modal logic as is "Two standard and two modal squares of opposition" [30] by Jiří Raclavský.

We have more philosophical papers such as the one by Andrés Bobenrieth, "The many faces of inconsistency" [15], investigating in particular the relation between the square and paraconsistent logic (a topic that has recently been developed in [1, 5, 6]). Raffaela Giovagnoli and Philip Larrey's paper, "Aristotle, Frege and *Second Nature* [23], deals with some fundamental questions regarding rationality and conceptualization. In "There is no cube of opposition" [7] JYB presents a critical analysis of the generalization of a square of opposition to a cube of opposition discussing the question of generalization.

In the follow up we have some theoretical papers. The papers "The unreasonable effectiveness of bitstrings in logical geometry" [33] by Hans Smessaert and Lorenz Demey and "An arithmetization of logical oppositions" [32] by Fabien Schang are both dealing with the binary systematization of the square. "Groups, not squares: exorcizing a fetish" [16] by Walter Carnielli is focusing on the general structure beyond the theory of opposition arguing that the square if just an artificial appearance of it (for more about Klein group and the square, see [31]).

José David García-Cruz in his paper "From the square to octahedra" [22] studies various geometrical figures other than the square, in particular two octahedra improving Colwyn Williamson comparative study of propositional logic and the traditional square. In "Iconic and dynamic models to represent *distinctive* predicates: the octagonal prism and the complex tetrahedron of opposition" [17] Ferdinando Cavaliere is opening the octagonal perspective to the third dimension. Joseph Vidal-Rosset's paper "The exact intuitionistic meaning of the square of opposition" [34] is not about a variation of the square but about the application of the square to one of the most famous variations of classical logic, i.e. intuitionistic logic (similar study was recently developed by François Lepage [26]).

Finally we have three papers dealing with applications of the theory of opposition. This first one, by Christoph Benzmüller and Bruno Woltzenlogel Paleo, "The ontological modal collapse as a collapse of the square of opposition" [2], is about Gödel' proof of the existence of God. The second one by Gert-Jan C. Lokhorst is about prudence, in Greek, *euboulos*, using fuzzy logic: "Fuzzy Eubouliatic logic: A fuzzy version of Anderson's logic of prudence" [27]. The third one by Sascha Benjamin Fink is about mental states and introspection using an octagon: "Why care beyond the square? Classical and extended shapes of oppositions in their application to *introspective disputes*" [21].

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Welcome Address to the Participants of the IV International Congress on: *The Square of Opposition* Vatican City, PUL, May 5–9, 2014

Monsignor Enrico Dal Covolo

Distinguished Professors, esteemed researchers, dear Students,

It is a joy for me and for our University and for its Faculty of Philosophy, an honor, host the fourth edition of the "World Congress of the square of the Opposition" that gathered here over a hundred researchers and scholars, from all over the world, for 5 days of intense work and fruitful debate.

First of all, I want to express my thanks to the Organizing Committee for this event. To Prof. Jean-Yves Béziau of the Federal University of Rio de Janeiro, tireless promoter and organizer of all the congresses on the Square. Their success is evident from the fact that these Congresses, over the years, have seen increase their international fame and their qualifying participation, producing every time, valuable specialized publications, by Publishers and Scientific Journals of international relevance. A special thanks also to Professor Katarzyna Gan-Krzywoszyńska Katarzyna from the Adam Mickiewicz University, in Poznan, Poland, who has closely collaborated with Prof. Beziau in the organization of this event. To her I offer a special greeting, given that only a few weeks ago, together with the Rector of the Mickiewicz University, Professor Professor Bronisław Marciniak, we signed an agreement for Academic Cooperation between our two Universities. We can say that this Congress is also a first fruit of this collaboration. My thanks go also to Prof. Gianfranco Basti, and to Prof. Raffaela Giovagnoli from our Faculty of Philosophy, and of IRAFS (*The International Research Area on Foundations of the Sciences,* www.irafs.org), who cooperated in the organization of this Congress, together with the Events Office of our University.

For me, that I am, as well as Rector of this University, a professor and a researcher in History of Latin Literature is particularly comforting to see how a theme that belongs to the tradition of the Ancient and Medieval age, as the square of the logical opposition, is able to pass through the centuries and the millennia, so to show its incorruptible significance for logic, capable of coagulating around it scholars of various disciplines, not only philosophers, logicians, and mathematicians, but also theologians, historians, psychologists, and even scholars of arts and literature.

This is due to the particular nature of the discipline of logic, "the science that makes sciences all sciences", because it captures in the form of symbols and of relations among

J.-Y. Béziau, G. Basti (eds.), *The Square of Opposition: A Cornerstone of Thought*,
Studies in Universal Logic, DOI 10.1007/978-3-319-45062-9_2

symbols the processes of thought and language in all their expressions, theoretical, practical, aesthetic. In this way, it makes comparable and transparent to the inquiry all languages, verbal and non-verbal, mathematical and humanistic, beyond cultural differences, historical distances, convictions of faith, ideological preconceptions, and language barriers.

Even though I am not an expert in this field, only by reading the abstracts of the many interventions of this Congress, I was able to see once again the power that the instrument of formalization of the philosophical and humanist thought has for the interdisciplinary dialog and for the intercultural exchange. The Interdisciplinary dialog, for the relationship between sciences and humanities; the intercultural dialog for the relationship between the different religions and cultures in a global society. Our society so far, unfortunately, knew the globalization only on the basis of the scientific, economic and the technological exchange. Often, people do not realize that such a type of globalization was made possible in the twentieth century, thanks to the formalization (symbolization and axiomatization) of the logic of mathematical sciences, theoretical and applied, starting from the publication of the *Principia Mathematica* of Alfred N. Whitehead and Bertrand Russell, at the beginning of the twentieth century. They had applied to the entire corpus of mathematical sciences, developed until then, the principles of the logistics of Gottlob Frege, thus inaugurating the new discipline of the *mathematical logic*.

Thanks to it, together with the development, in the second half of the twentieth century of science and technology of communications, which have worked as an enzyme for the entire process, peoples belonging to different cultures and languages, which are very different from the western ones, may acquire a scientific and technological competence—basic and/or advanced—in the various scientific disciplines, in the times of a common course of university degree. On the contrary, only until to the end of nineteenth century, for obtaining the same result, much more years were necessary for learning, with another language, another culture, another way of thinking, without the essential support of the formalization that only can make fast and certain such a process. It was necessary, indeed, firstly learning one or more western languages, secondly understanding some essential aspects of western culture and civilization sometimes very far from the original culture of the student, and, finally!, studying the discipline in question. A sort of frustrating and tedious path to obstacles, generally lasting several decades, which then only a few privileged people could afford. This frustrating learning path, unfortunately, still remains today for those who want to study humanities of other cultures, with enormous waste of resources, and consequent exclusion of traditions sometimes millenarians from the actual debate.

In this way, it is likely to send lost authentic treasures of wisdom in all fields of human knowledge, instilling into the common mentality the idea of an insurmountable cultural relativism, philosophical, ethical and religious, that in fact marginalizes increasingly these forms of knowledge, reducing them to folklore or curiosity for a museum. A loss that is the more serious today, when the overwhelming development of sciences and technologies would require that these contents of millennial wisdom be made easily available to scientifically educated people—the young people before all—from the inside of this type of education.

It seems strange that in the face of such disaster for the reflection that is the more enlightened, and less prone to succumb to the "politically correct" of the nihilism and of the relativism, had not produced its antibodies. They, instead, and fortunately, have been developed along two directions. At the beginning they were separated, but today are largely convergent, for the good of all.

The first direction, better known to philosophers and to humanists, was, throughout the twentieth century up to today, the birth and the development of the phenomenological reflection, with the connected rebirth of the ontological reflection. Two strands of thought largely overlapping, which oppose the false reductionism of scientism, and of his mentor, the bad scientific divulgation that today became an educational problem on a global scale.

The second direction is perhaps less apparently "alternative" to the tyranny of the technological-scientific one-way thought. On the contrary, it is the only one that can be validly oppose it from the inside of the scientific formal languages, and that is virtually underlying all the relations of this Congress.

In fact, simultaneously to the development and the application of the axiomatic method in mathematical logic, for the groundbreaking work of a philosopher rightly entered in the history of the thought of the twentieth century, even though but very little studied and developed in Italy, Clarence I. Lewis, a similar process of axiomatization of its own logic of the various philosophical disciplines, the so-called "intensional logic" appeared. Until that time the subject of study prevailing was the school of phenomenology. A process that started by the axiomatization structures of modal logic common to most of them, inaugurated by Lewis himself. He, in fact, with great foresight had guessed that, since the publication of the first edition of the Principia, the mathematical logic, the formalized "extensional logic", could not apply if not partially, to the analysis of language and philosophical humanist in general. Lewis, who was then a young doctor in philosophy, opposed his "intentional logic" to the application of the extensional logic of the Principia to the analysis of philosophical thought. The project started, on the other hand, immediately after, with the publication and dissemination on a large scale due to Bertrand Russell, of the *Tractatus Logico-Philosophicus* of the young Ludwig Wittengstein. A project that will lead to the birth of the logical neo-positivism movement. A movement that, precisely because it used a formal inadequate tool, which is not that of intensional logic and logical philosophy in general, has often achieved disappointing results, characterized by a substantial scientistic reductionism. This very limit was recognized by Wittgenstein himself after years of isolation and spiritual retreat during the long period spent as missionary of the Carinthian Alps, as a consequence of the trauma of his participation in the World War I. Initially, he remedied by its magisterium in Cambridge and by the theory of multiple "language games", and more than once as in the Tractatus. The language games, with their rules and logical methods necessarily different, but all comparable, through the instrument of the analysis and the formalization of logic, favor the development of different languages and of different disciplines, sciences and humanities.

On the other hand, Lewis, as the pragmatist philosopher he was, just realized the immense potential that the axiomatic method of the Principia would have for the diffusion of scientific knowledge. That is what has widely occurred along the twentieth century, and still continues. "Philosophical logic" can reach important results, pace the "mathematical logic", by acquiring the axiomatic method. A work that Lewis himself begun with the first axiomatization of modal logic, and so, he is the pioneer of the so-called *philosophical logic*. A discipline now developed as the elder sister of the *mathematical logic*, but with a paradoxically sharpest decline, despite the name and not yet sufficiently developed in the humanities, especially in Continental Europe, but rather in the field of computer science.

It is in fact the basis of the so-called “semantic revolution” in the computer science and robotics. From this revolution revolution also arrived applications to consumer: think to the so-called “semantic database” or to the capacity of the artifacts to interact in common language with humans, by interpreting and sometimes anticipating the intentions of the biological party. A process that is at the base of the now incipient “third information revolution” of the *Web3 or* “semantic web”, that will overlap with the current pervasive *Web2* or “web interactive”, that is now a global phenomenon. A phenomenon that, allowing the increasing automation of semantic tasks, is already leading to more and more automation in the field of services, and not only of repetitive tasks in the manufacturing industry it has been hitherto. This revolution, as it is now widely attested even on newspapers and magazines of wide spread, is intended to disrupt deeply not only our culture but also our economy and society as a whole in the coming decades. A revolution which urgently requires critical contribution and address, which only an updated philosophical reflection, ontological and ethical first, may offer.

The discipline of logic and philosophical logic give rise to the so-called “formal philosophy”—in its various meanings, of “formal ontology”, “formal epistemology”, “ethics formal”, “formal aesthetic”, and not only “formal logic” as it was in the past. Unlike the “analytic philosophy”, the formal philosophy does not only use the philosophical logic to analyze the logic of philosophical disciplines and to assess their consistency. It does much more. It formalizes the structures by giving rationality and expressivity to the different philosophical conceptions, no matter how distant in space and time they are from our own, and by making them all comparable, transparent to each other, but especially available to enter in the contemporary debate for the common solution of problems. In this way, each of us can appreciate the genius of a lot of solutions, especially of the past and the most glorious humanistic traditions of all cultures, by creating an ideal *agora,* an actual *global aeropagus*, in which we can invite past and present authors to discuss our problems, in the absence of linguistic and cultural barriers, otherwise hardly to overcome, given the current pluralism—*Ars longa vita brevis*, (to quote the ancients). I think I am correct in saying that your International Congresses on the Square, including the present that we are going to start, are an exemplification of this image and can be true and fruitful for our scientific and humanistic culture.

In summary, the “globalization of humanism” was a great achievement which we can reach today, using the formal languages of logic—better, the “logical”—to play the role that in antiquity and medieval ages of our European culture, the Latin played, whose linguistic logic seized for excellence we are modestly fond. Not for nothing, in the programs of the institutional program of our Faculty of Philosophy we have included well three Latin courses and three courses of logic, which, as far as necessarily the introducers, can form all of our students to have access to the treasures of the universal thought.

For this, at the conclusion of my speech, I wish all of you a good job, with the hope, that is a certainty, that this Congress will bring those fruits, for which not only its organizers, but all the rapporteurs, spent their ingenuity.

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Part II
Historical Perspectives on the Square

The Proto-exposition of Aristotelian Categorical Logic

Manuel Correia

Abstract The aim of this paper is to state that the oldest Western treatises on categorical logic, the one attributed to Apuleius of Madaurus (I–II AD) and the other written by Boethius (VI AD), follow a common plan rather than a common written source. I call this common plan the proto-exposition of categorical logic as much as it can be reconstructed in its formality. Its limits exceed those of Aristotle's written works on logic, and both Apuleius and Boethius suggest that Aristotle's first disciples, Theophrastus and Eudemus, play an important role in its arrangement. After remarking its main characteristics, I describe its limits, by including indefinite terms in syllogistic premises, in connection with three general rules or Axioms making decidable every categorical syllogism. Accordingly, I analyze some study cases in order to distinguish strict conclusions from non-strict ones, which represents a new discernment on existential-import discussion. Then, I remark the importance of the concept of quantitative symmetry by showing how the three general rules can assess the validity of classical hypothetical syllogistic. Thus, the paper presents an extension of Aristotelian logic by taking its proto-exposition as starting point.

Keywords Aristotelian logic • Ancient commentators • Syllogistic • Existential import • Indefinite terms

Mathematics Subject Classification Primary 03-02, Secondary 03B5

1 Introduction

According to Aristotle, categorical logic studies simple propositions. Simple propositions are those in which something of something is affirmed or denied. He defines 'something' as a single thought, which is represented by one univocal name [7, 19b5], and [8, 24a16].[1] He distinguishes between two syntactic forms: two-term propositions and three-term propositions. For instance, "Socrates walks" is a two-term proposition, and "Socrates is just" is a three-term proposition.[2] As such categorical logic opposes to hypothetical

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[1] In the following pages I use the traditional abbreviation for Aristotle's works [7, 8].

[2] The idea of substituting a name for a phrase is as ancient as the first written reports of Aristotelian categorical logic. For instance, Boethius [14, 1176A] says: It may be that the parts of the proposition, which we call terms, occur not only as names but as phrases. Often a phrase is predicated of a phrase, thus

J.-Y. Béziau, G. Basti (eds.), *The Square of Opposition: A Cornerstone of Thought*,
Studies in Universal Logic, DOI 10.1007/978-3-319-45062-9_3

logic, which is the logic of compound propositions. Compound propositions are those in which two or more simple propositions are related by a logical connective. Aristotle also dedicated time to constitute this theory, even if he does not write on it.[3]

Even if Plato's *Sophist* is the most likely source behind this early doctrine,[4] Aristotle made an important theoretical advance by adding (i) a correction on Platonic denial,[5] by introducing (ii) some semantic distinctions between privative, indefinite and negative propositions,[6] and by systematizing (iii) a theory of syllogistic demonstration that has an underlying theory of logical deduction, something that has been very much emphasized in our days by John Corcoran in [17, pp. 73–118].

Aristotle's theoretical effort in categorical logic was promptly arranged, systematized and enlarged by his followers, the old Peripatetics, Theophrastus and Eudemus, giving birth to a long tradition in logic. Their material is lost in a great extent,[7] but it survives partially in many sources: Cicero, Marius Victorinus, Alcinous, Apuleius, Alexander of Aphrodisias, Boethius, Ammonius Hermeias, Philoponus, Stephanus, etc.[8]

In the following figure, I attempt to reconstruct this textual tradition, where some doctrines of Theophrastus in his lost *Prior Analytics* are taken into account to testify adoption or textual dependence.

2 The Proto-exposition in Ancient Logic

The short abridgement on categorical logic titled *Peri Hermeneias*, which is attributed to Apuleius,[9] and Boethius' *De syllogismo categorico,*[10] are the oldest textual references where categorical logic takes the form of a self-contained theory. Obviously, the aim of these treatises is not to comment on Aristotle's logical texts, but to deal with a topic on which they go further the limits of his logical writings. In a brief exam, they distinguish between hypothetical logic and categorical logic, and they add some technical complements to the theory: subalternation, conversion by contraposition, Theophrastus' five indirect syllogistic moods of the first figure, the hypothetical and disjunctive syllogistic moods, some general explanations as why variables are useful in logic, the significance of

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"Socrates and Plato and other students investigate the essence of philosophy". Here the phrase "Socrates and Plato and other students" is the subject and "investigate the essence of Philosophy" is the predicate.

[3] Cf. Aristotle [7, 17a15 and 17a20-21], [8, 40b27, 45b19-20 and 50a39–50b1].

[4] Cf. [20, pp. 42–45].

[5] Cf. [20, p. 44].

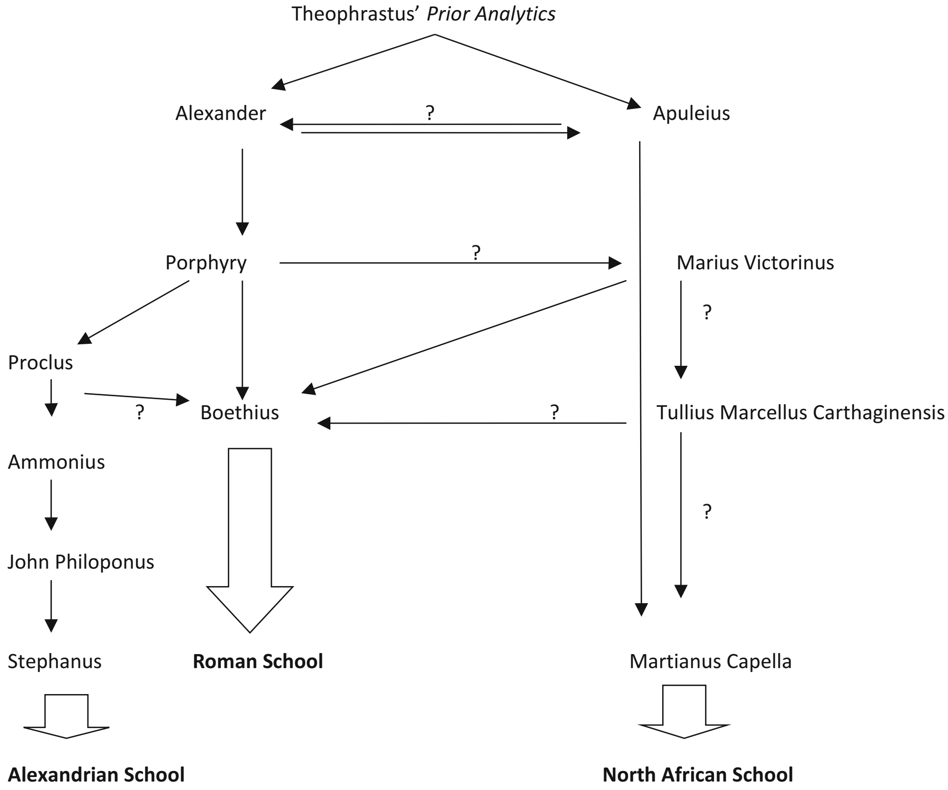
[6] Cf. [7, 19b19-30]. To explain this difficult passage, where Aristotle distinguishes between privative and indefinite propositions, cf. [18, pp. 41–56].

[7] Cf. [10, 31].

[8] The most complete and significant work on the ancient commentators on Aristotle is contained in the Project Ancient Commentators on Aristotle, cf. [29, pp. 1–30].

[9] Bochenski [10].

[10] Boethius [15].



dictum de omni et de nullo, the matters of proposition, the number of categorical and hypothetical propositions, and so on. Apuleius also adds some Stoic elements.

On the other hand, it is a fact that these ancient treatises are very similar to one another.[11] However, it has been difficult to interpret this similitude. Some authors have explained it by attributing textual dependence between Apuleius (if he is the author) and Boethius, as Sullivan in [30, pp. 210–228]. But two problems remain: one is the objection of a common written source from which both Apuleius and Boethius benefit. The other is to response to the fact that both ancient treatises contain many differences too.[12]

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[11] Not only [6] is very similar to [15], but also Boethius' two treatises on categorical logic [15, 16] are shaped alike. To discuss the similitudes between Boethius' two treatises, see [19, pp. 729–745].

[12] Eight differences can easily be grasped: (1) Unlike Apuleius, Boethius seems to be following a source in which Peripatetic distinction between matter and form has been used. This is confirmed when he adds that even though a house is built with splendid materials, if the form is not the correct, there will be not a house at all. The priority of form over matter is a clue for [15] and Greek Aristotelian logic in general. Unlike Boethius, Apuleius in [6] adopts the Stoic triple division of philosophy: moral, rational and physical (cf. [6, I, 265, 1–4]), by saying that he now goes to dissert on the rational part and relates the syllogism with the art of reasoning (*ars disserendi*) and not with the form of human reasoning, as Boethius does. (2) The emblematic teaching concerning the square of opposition is also formally identical

This is why Isaac's hypothesis in [23, p. 27], is more convincing when he thinks of a common plan rather than of a common written source. However, Isaac never described this common plan. This common plan is what I have called the *proto-exposition* of categorical logic, and I argue that it is what makes all the later explanatory accounts on categorical logic similar to one another, for it gives them their formal structure. It is remarkable the fact that Alexander of Aphrodisias in [4] supports the existence of this proto-exposition and its influential role in ancient logic, by giving evidence that he is aware of it.[13] Boethius in [15] gives its most complete account:

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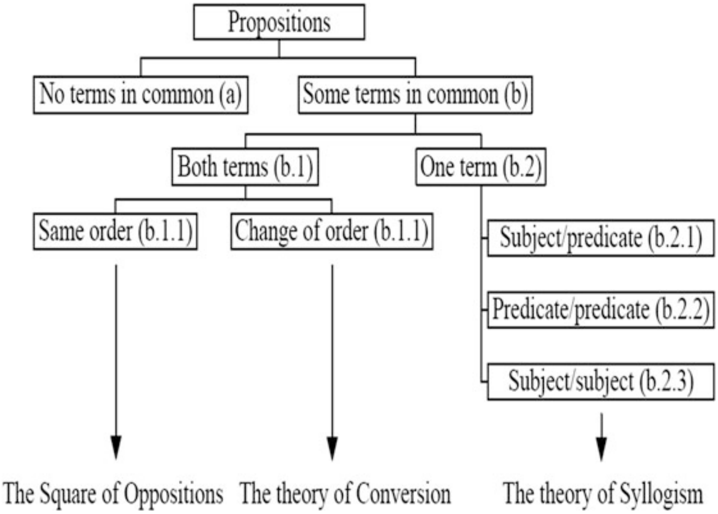
in both authors, but Apuleius recognizes a square ("it is not out of place to form a square", [6, V, 17–19]), but not Boethius, as in his [5] and in his [12] he does not mention the word 'square', but the general expression *descriptio* [12, p. 86], [12, 2, p. 152], [15, p. 21]. (3) There is an enormous difference in style: [15] is serious, slow and sometimes protracted. On the contrary, [6] is agile, dynamic and sometimes superficial. (4) The intentions of the authors: [15] works for the glory of Aristotle, Theophrastus, Eudemus and above all Porphyry, but [6] praises Plato and Apuleius himself. (5) The doctrine of the matters of proposition is an ancient doctrine intending to explain the way in which the predicate of a categorical proposition relates to the subject. It was common to Alexander of Aphrodisias in [3, p. 192, 8], Syrianus (cf. [12, 2, pp. 323, 5–6]), and it is a commonplace in Boethius [16, 29, 11ff], [12, p. 137, 15–16]. Ammonius even gives a report of it in his commentary on *De Interpretatione* (in [5, p. 88, 12–23]). It is recalled by Boethius through the name of *materiebus* in [15, p. 32, 21], by translating the Greek sentence *hulai tes protaseos*. However, Apuleius calls it the doctrine of significations (*significationes*) and relates it to Aristotle's *Topics*. In [6, VI, p. 182, 4–6], Apuleius lists the *significationes*, for he says they are not innumerable (*innumerae*): the property, the genus, the difference, the essence and the accident. These five *significationes* are transformed into necessary, impossible and contingent matters of propositions already in Ammonius. 6. Unlike Boethius' logical certainties, [6] contains some inaccuracies in dealing with conversion by contraposition of particular negative propositions: it says 'Some animal is not rational'—'Some non-rational (thing) is a man', for this is equipollent to the former but it veils the rule: 'Some non-rational is not a non-man': [6, VI, 5–9, p. 183]. It is true that here the critical apparatus is complex, but later in [6, XII, p. 278, 25], Apuleius is again weak. 7. Latin technical terminology is very different in both treatises: 'to divide entirely truth and falsity' (Boethius) stands for *perfecta pugna et integra* (Apuleius [6, V]); negative proposition (Boethius' *negativa propositio*) stands for abdicative proposition (Apuleius' *abdicativa propositio*), etc. 8. The way in which Apuleius and Boethius construe the first mood of categorical syllogism (our traditional *Barbara*) is not one and the same. Apuleius follows the fourth indirect syllogistic figure, while Boethius the first direct one (our traditional *Barbara*). In other words, Apuleius poses the middle term in a *prae-sub* position, while Boethius in a *sub-prae* position. For instance: Every man is a mortal animal. Every Greek is a man. Then, every Greek is a mortal animal (this is Boethius' position); on the other hand, we have: every man is a mortal animal. Every mortal animal is a living creature. Then, every man is a living creature (this is Apuleius' position).

[13] I owe this remark to Christina Thomsen Thörnqvist [16, p. xxii]. She has said that this division procedure is "closely paralleled" in [4, 45, 10 and ff.]. Surely, Alexander is the primal source to Ammonius' commentary on [4, 35, 36 and ff.]) and Philoponus' in [26, 40, 31 and ff.] The remark by Thomsen Thörnqvist in [16, p. xxii], was also noted by Lee [24, pp. 65–74].

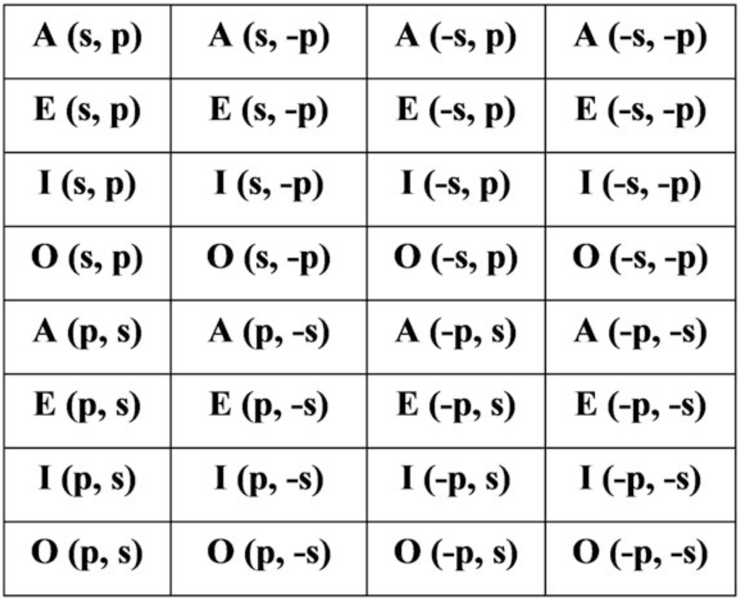


3 The Extension of Categorical Logic

The proto-exposition supposes 8 properties for categorical propositions: two- or three-terms propositions, modal or non-modal, singular/universal subject, definite/indefinite terms, quantity, quality, time, and matter. But it focuses mainly upon quantity and quality, in the following way:

Quantity: Universal/Particular
Quality: Affirmative/Negative.
UA, UN, PA, PN ($2 \times 2 = 4$), which later will be known as A, E, I and O propositions.

The main feature of the proto-exposition of categorical logic is to leaving indefinite terms outside from syllogistic premises. 'Non-man', 'non-just' are indefinite terms. They were defined by Aristotle in [7, 16b9-11], and introduced to propositions by him in [7, 19b19-31], but in [8] indefinite propositions (e.g. "Every man is not-just") are never entering categorical syllogisms (cf. [8, I, 46]). Accordingly, this lack became traditional in all expositions of categorical logic. However, this has changed, as much as we can today introduce indefinite terms in premises and categorical syllogistic consistently. First, we can extend the number of categorical propositions: from the 4 traditional types (A, E, I, and O), we obtain 16 species (without permuting its subject and predicates) and 32 (with permutation).



A (s, p)	A (s, -p)	A (-s, p)	A (-s, -p)
E (s, p)	E (s, -p)	E (-s, p)	E (-s, -p)
I (s, p)	I (s, -p)	I (-s, p)	I (-s, -p)
O (s, p)	O (s, -p)	O (-s, p)	O (-s, -p)
A (p, s)	A (p, -s)	A (-p, s)	A (-p, -s)
E (p, s)	E (p, -s)	E (-p, s)	E (-p, -s)
I (p, s)	I (p, -s)	I (-p, s)	I (-p, -s)
O (p, s)	O (p, -s)	O (-p, s)	O (-p, -s)

Indeed, the traditional 4 propositional forms: A, E, I, O × 2 (double predicative order) × 4 (definite/indefinite subject and predicate) = 32 categorical forms.

Even if the logical possibility of categorical propositions to receive indefinite terms either in their subject or in their predicate or in both is accepted in Aristotelian logic, the consistent incorporation of indefinite terms in syllogistic had not been possible.[14] But with the help of the 3 Axioms described herein and with the understanding that each term of a logical deduction (syllogistic or not) can be definite or indefinite, 256 syllogistic different syllogistic figures come to existence. Indeed, there will be 4 possibilities in each premise, and 4 in the conclusion. Thus, 4 × 4 × 4 = 64. Now, since there are 4 figures, 64 × 4 = 256. Since the number of indirect figures is the same as the direct one, when the indirect figures are added to these direct figures, the total amount is 512 syllogistic figures. Hence, the number of possible syllogistic moods (i.e., valid and invalid) shall be calculated if the number of syllogistic figures is multiplied by the number of variations in both the quantity and the quality that every proposition of a syllogism can have. These are four variances (A, E, I, O), since every proposition of the syllogism (the two premises and the conclusion) has the same possibility of variation, we have 4 × 4 × 4 = 64. From here, 512 × 64 = 32.768:

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[14] Especially relevant here are [11, pp. 35–37], and [32, pp. 145–160]. However, the idea of extending classical syllogistic was already in ancient schools of Logic: see for example [22, XI, 2] and [6, XIII]. Leibniz in his [25] takes the challenge of calculating the number of valid syllogistic moods by integrating the singular and undetermined propositions (something that Aristotle does not consider but the tradition). And in doing so Leibniz is following a long tradition of modern studies of Aristotelian syllogistic. In fact, he cites a previous work by John Hospinianus (cf. [21, pp. 79–92]). Also the nineteenth and twentieth centuries witnessed these studies. For example: J.C. Smith in [28], and O. Bird in [9].

this is the number of all categorical syllogistic moods, either direct or indirect, with definite or indefinite terms.[15]

And when categorical conclusions are observed, as we have explained elsewhere,[16] the formal process to introduce indefinite terms in syllogistic depends on the following three general rules or Axioms:

Axiom of Quantity: the predicate of a negative premise is universally taken and the predicate of an affirmative premise is particularly taken. Hence, to take universally a term T in a proposition (i.e. even if this term is the subject term) is equivalent to take particularly its correspondent conjugate term non-T, and to take particularly a term T is equivalent to take universally its correspondent conjugate term non-T.

Axiom of Particularity: from only particular premises no conclusion follows, and the conclusion of a syllogism is particular if and only if this characteristic is present in one of the premises.

Axiom of Linkage: the quantity of both terms in the conclusion should be the same as that they offer in the premises. The premises common term must be universally taken in one premise and particularly taken in the other premise.

If these 3 Axioms are applied to syllogisms where any of the 32 with-indefinite-terms categorical forms enters, not only the limits of the proto-exposition are exceeded, but also the categorical syllogistic reaches its proper end. Indeed, the axiom of Quantity allows us to define and calculate the quantity of every term in any premise-conclusion argument. Moreover, the axiom of Linkage allows us to calculate with certainty whether the terms in the conclusion have the same quantity as they have in the premises, and (ii) they allow to detect whether the middle term is alternatively universal and particular. This is why the three Axioms allow defining a logical consequence in general: an argument of the premise-conclusion form is conclusive, if the quantity of the terms in the conclusion is the same as the quantity of the very terms in the premises, and the middle term(s) is/are alternatively universal and particular in the premises.

The three Axioms also are useful to distinguish between a *strictly logic argument* from any *restrictively valid argument*. Indeed, if the middle term is alternatively universal and particular in the premises, the argument will be strictly logic (i.e., it will not contain problems with existential import). But, if the middle term is each time taken universally in the premises, the argument will be restrictively valid, i.e., able to be valid if the problem of existential import is resolved.

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[15] Otherwise, every proposition of a categorical syllogism can vary 16 times (4 per quantity and quality—namely: A, E, I and O, which are multiplied by the 4 variations formed by definite or indefinite terms: namely, definite subject and predicate, indefinite subject and predicate, defined subject and indefinite predicate, indefinite subject and definite predicate indefinite, i.e., $16 \times 16 \times 16 = 4.096$. And given that there are 8 figures (4 direct and 4 indirect), the total amount rises to: 32.768 syllogistic moods.

[16] Alvarez and Correia [1, pp. 297–306].

4 Examples

4.1 Strictly Valid Syllogisms

Let start with an AA-A syllogism of the fourth indirect figure[17]:

Every S is P
Every P is Q
Then: Every S is Q

- In the conclusion: S is universally taken. Q is particularly taken (axiom of Quantity). In the premises, S is universally taken (for it is in the domain of a universal quantifier). Q is particularly taken (axiom of Quantity).
- And the middle term is universal in the minor premise and particular in the major premise, fulfilling the function of being alternatively universal and particular in the premises.

4.2 Non-strictly Valid Syllogisms

A syllogistic mood in *Darapti* is as follows:

Every S is P
Every S is Q
Then, some Q is P

- In the conclusion, both Q and P, are particular (Q because of the particular quantifier and P because of the axiom of Quantity).
- In the premises, P and Q are particular (axiom of Quantity).
- However, S is taken universally in both premises, which defines the first of the two cases of syllogistic conclusions affected by existential import, which also affects to *Felapton*, *Fapesmo*, *Fesapo*, and the AA-I and AE-O moods of the third indirect syllogistic figure.

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[17] In this paper, categorical syllogisms are often referred to in the traditional arrangement made by Boethius in yy. Here, at the beginning of Book 2, Boethius says that the expression 'S is in every P'—which is the way in which Aristotle arranges the premises in his *Analytics*- should be taken as 'Every P is a S', without difference. This remark by Boethius is essential to understand why traditionally Barbara has been arranged in a *sub-prae* order: ("And then, let us show in few words what 'to be in every' and 'not to be in every' mean. Indeed, if something belongs to some genus, its entire species will be contained in it, and it would be said that that species *is in every,* i.e. the genus. For example, let the genus animal be, and the species man. Then, given that 'man' is less extensive than 'animal', it is said that < man > is in every animal. Certainly, every man is animal. But if someone says in the reverse way, namely, that something is predicated of something, there will be no difference. Because, it is in the same way as man is in every animal that 'animal' is predicated of every man.") Boethius does not mention that Apuleius arranges the syllogism in the way he avoids, namely, Omne iustum honestum. Omne honestum bonum. Omne igitur iustum bonum est.

Indeed, the three Axioms allow detecting that syllogistic conclusions affected by existential import problem come from two cases of quantitative asymmetry: (a) the first case (in green) is the one we have already described: the middle term is always universal in the premises and it fails to be alternatively universal and particular. And (b) the second case (in yellow) is that in which the subject or the predicate in the conclusion is less extensive than their occurrence in the premise. The case in (b) affects all the subaltern moods and also *Bamalip* and *Baralipton*. All of which is seen in the following figure:

DIRECT MOODS							*Subaltern Moods*		
1	barbara	celarent	darii	ferio			AAI	EAO	6
2	cesare	camestres	festino	baroco			EAO	AEO	6
3	darapti	felapton	disamis	datisi	bocardo	ferison			6
4	bamalip	calemes	dimatis	fesapo	fresison		AEO		6
INDIRECT MOODS							*Subaltern Moods*		
1	baralipton	celantes	dabitis	fapesmo	frisesomorum		EAO		6
2	AEE	EAE	IEO	OAO			AEO	EAO	6
3	AAI	AEO	AII	AOO	IAI	IEO			6
4	AAA	AEE	IAI	IEO			AAI	AEO	6
TOTAL NUMBER									48

4.3 Syllogisms with Indefinite Terms

Every S is non-P
No L is non-P
Then, no S is L

- This is a valid AE-E mood in a *prae-prae* indirect figure, with indefinite middle term.
- L is universal in the conclusion (axiom of Quantity) and universal in the minor premise (because it is under the domain of a universal quantifier). And S is universal in the conclusion and universal in the major premise (being both under the domain of a universal quantifier).
- The middle term, non-P, is particular in the major premise (hence, P is universal here) and universal in the minor premise, and so P here is particular (axiom of Quantity).

The decision method is also able to detect valid syllogistic conclusions in those syllogistic combinations where the traditional rules refuse validity. For example, the following combination OI-O makes a syllogism with a valid conclusion:

No non-S is a non-P
Some T is a non-S
Therefore, some T is not a non-P.

- T is particular in the conclusion (it is under the domain of a particular quantifier) as T it is in the minor premise. And the term non-P is universal in the conclusion (axiom of Quantity) and universal in the major premise (axiom of Quantity).
- The term non-S in the minor premise is particular (hence S is universal here) and universal in the major premise (hence it is particular).
- Accordingly, the syllogism is conclusive.

It must be reminded that the three Axioms are not trivialities, for not every indefinite categorical proposition can be transformed into a categorical proposition without indefinite terms. Indeed, propositions like 'Some non-P is a non-S, "No non S is a non P", etc., are not logical transformations of a simple and without-indefinite-term proposition. Indeed, if every indefinite proposition could be converted into a definite proposition, the Axioms would be useless and there would not be a problem with extending syllogistic with indefinite terms. In that case, the questions by [11, 32, 27], on how to extend classical syllogistic when introducing indefinite terms, would be vacuous demands, but they are not. Certainly, these Axioms extend classical syllogistic, but also enrich logical analysis by identifying a quantity in every syllogistic term.

The implicit law of syllogistic, which was neither in the proto-exposition nor in the tradition, is the symmetry between the quantity of terms in the premises and their quantity in the conclusion. However, the perfect syllogism is that in which the middle term is besides alternatively universal and particular in the premises. For if the middle term is, as explained, in both premises universally taken, or the symmetry between the terms in the conclusion and their occurrence in the premises is lost, the syllogism will suffer the problem of existential import, and it will be only restrictively valid. On the other hand, it is obvious that if the middle term is particularly taken in both premises, there will be no syllogism at all.

The three Axioms make a device for the analysis of syllogistic conclusions rather than mechanical rules of syntactic manipulations of terms. It is supported by the fact that every term in a syllogism has particular or universal quantity. This quantity term is not to be confused with the quantity of the proposition, which is given by quantifiers. Terms reveal quantitative and the Axioms reveal symmetry between the conclusion and the premises. Any asymmetry produces a conclusion that must be qualified. Conclusions can be strict or non-strict.

The Axioms open a new angle in categorical syllogistic, but also in conditional syllogistic conclusions. Indeed, the so called inferential rules *Modus ponendo ponens* and *Modus tollendo ponens*, as well as the two classical disjunctive syllogisms are conditional or hypothetical syllogisms and, as such, they all follow the three Axioms. This is clear when we apply the Axioms to disjunctive syllogisms, and then we accept the formula of equivalence between disjunctive syllogisms and *Modus ponendo ponens* and *Modus* *ponendo Tollens* in the way Boethius taught in his *De hypotheticis syllogismis* [13]. Indeed, Boethius in DHS [13] presents the following disjunctive syllogisms:

It is a or it is b, but it is not a. Therefore, it is b.
It is not a or it is b, but it is a. Therefore, it is b.
It is not a or it is not b, but it is a. Therefore, it is not b.
It is a or it is not b, but it is not a. Therefore, it is not b.

In hypothetical logic no proposition has quantifier. Hence, If we leave aside the Axiom of Particularity, the disjunctive syllogisms are the very expression of the two other Axioms, namely the Axiom of Quantity and the Axiom of Linkage. Thus, in every valid disjunctive syllogism there is quantitative symmetry: the middle term is always alternatively universal and particular, and the term in the conclusion has the same quantity as the very term in the premise. To calculate the quantity of the terms in any disjunctive syllogism, it must be taken into account only two provisos:

1. The logical connective ‘or’ maintains the quantity of the terms in a propositional formula.
2. The single indefinite term is always universal.

And since Boethius (DHS III, 10, 3) agrees with the following equivalence:

It is a or it is b = if it is not a, then it is b.

It will follow that the Axioms are also applicable to the *Modus ponendo ponens* and the *Modus ponendo Tollens*, which is seen in the following:

It is a or it is b, but it is not a. Therefore, it is b. *If and only if:*
If it is not a, then it is b. But it is not a. Therefore, it is b.

It is not a or it is b, but it is a. Therefore, it is b. *If and only if:*
It is a, then it is b. But it is a. Therefore, it is b.[18]

It is not a or it is not b. But it is a. Therefore, it is not b. *If and only if:*
If it is a, then it is not b. But it is a. Therefore, it is not b.

It is a or it is not b, but it is not a. Therefore, it is not b. *If and only if:*
If it is not a, then it is not b. But it is not a. Therefore, it is not b.

The equivalences show that if the Axioms apply to categorical syllogistic they also apply to disjunctive syllogistic and hypothetical syllogistic. This result gives unity to Aristotelian logic, since both branches now, the hypothetical and the categorical, will be governed by a unique set of Axioms.

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[18] This equivalence shows that our modern formula “($p \rightarrow q$)” is equivalent to “ ($\neg p \vee q$)” was consigned first by Boethius’ DHS [13]. It is relevant to notice that Boethius says that the Greek material he used to produce his treatise on hypothetical syllogistic was in Theophrastus and Eudemus.

5 Conclusion

The so called proto-exposition of categorical logic includes a theory of opposition, a theory of conversion and a theory of syllogistic reasoning. The two first parts have already been treated in [1, 2]. Here I have emphasized some advantages that the set of three Axioms (Quantity, Linkage and Particularity) offers to two-premise classical syllogistic as a method of decision.

The first is its simplicity, for if the conclusion does not follow from only particular premises (axiom of Particularity) the axioms of Quantity and Linkage are sufficient to detect whether the syllogism is or is not conclusive: the only characteristics one has to detect is (i) whether the middle term is alternatively universal and particular, and (ii) there is a symmetry between the terms in the conclusion and the very terms in the premises.

The second advantage is the ability to detect problems with existential import and to define them as cases of quantitative asymmetry. There are two cases to consider: since the middle term cannot be particular in the premises, the first case is (i) when there is asymmetry between the terms in the conclusion and the terms in the premises. And the second case is (ii) when the middle term is always universal and not once universal and once particularly taken. This asymmetry in (i) or this asymmetry in (ii) configures a restrictively syllogistic conclusion.

The third advantage is the fact that indefinite terms can enter syllogistic consistently. This is the first time a positive result is presented on this long overdue difficulty and it by itself should call attention to modern logicians interested in logical conclusions.

The fourth advantage is the fact that hypothetical syllogistic and disjunctive syllogistic are also governed by the set of three Axioms, a fact that gives unity to Aristotelian logic and makes us reconsidering the old problem of whether categorical logic should be subordinated to hypothetical logic or rather hypothetical logic should be subordinated to categorical logic.

Acknowledgements This article was made possible thanks to Fondecyt N° 1141135. I am grateful to Prof. Dr. Jean-Yves Beziau for his invitation to the 4th World Congress on the Square of Opposition, May 5-9, 2014, Vatican City, where a first version of this paper was given. I am also thankful to the editor and the referees for their useful comments.

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The Modal Octagon and John Buridan's Modal Ontology

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Abstract In this paper we will argue that the ontology implicit in John Buridan's modal octagon commits him to a form of contingentism. In particular, we will argue that Buridan is committed to denying the validity of the Barcan and converse Barcan formulae.

Keywords John Buridan • Medieval logic • Modal logic • Modal ontology • Necessitism and contingentism

Mathematics Subject Classification (2000) Primary 03A05; Secondary 03B45

1 Introduction

There is a well known interpretive question raised by Aristotle's discussion of the assertoric syllogism. The question is: does Aristotle's assertoric syllogism allow for empty terms? While modern interpreters, often tracing their arguments back to Łukasiewicz's [6], have generally argued that Aristotle's assertoric syllogism does not admit such terms, the medieval interpreters of Aristotle believed Aristotle did and chose to allow them in their own logical theorising. On the standard medieval view, for the proposition 'Every A is B' to be true, in addition to every A being a B, there must also be some A's in existence. Likewise, the proposition 'Some A is not B' is true if either nothing is A *or* there is something which is A and is not be B. The medievals observed that with these truth conditions, one obtained a consistent interpretation of the square of opposition, that is to say, an interpretation that validated all of the usual relationships between contradictory, contrary, sub contrary and subalternate propositions, while allowing for the presence of empty terms.

When the medievals took up the study of the syllogism some expanded the square of opposition to cover modal propositions. In three manuscripts of John Buridan's *Summulae de Dialectica*, we find that Buridan extended the traditional assertoric square of opposition to an octagonal structure designed to illustrate the relationships that exist between propositions of necessity (*de necessario*) and propositions of possibility (*de possibili*). These extensions raise similar questions, now in a modal context, to those raised by the interpretation of assertoric terms in the square of opposition. How were existential commitments to be understood in Buridan's Octagon? Does Buridan's modal logic allow for *possible* objects? i.e. objects that currently do not exist, but will exist or might exist.

J.-Y. Béziau, G. Basti (eds.), *The Square of Opposition: A Cornerstone of Thought*,
Studies in Universal Logic, DOI 10.1007/978-3-319-45062-9_4

As we shall argue, the answer to this is yes. But then, what is the ontological status of these objects that can be A or can fail to be A etc, even if they do not exist? Our goal in this paper is to use the modern debate between contingentism and necessitism to help gain some clarity on the status of possible objects in Buridan's ontology and answer this question. We will argue that Buridan is committed to a kind of contingentism that can be thought of in terms of modern possible worlds semantics with a suppressed existence predicate.

This paper will proceed in three stages. First, we will introduce the requisite historical information about Buridan's modal logic to understand it and highlight some of the philosophically and logically interesting parts of Buridan's theory. Second, we will sketch a semantic reconstruction of Buridan's modal logic which is able to account for all of the inferences in the Octagon of Opposition and the *Treatise on Consequence* using possible worlds semantics. The full details of this system and the textual justification required to show that it accurately reflects Buridan's own remarks about modal logic are more completely developed in [4]. We will use this formal system to help us articulate a clear picture of Buridan's modal logic. Finally, we will address how Buridan's modal logic conceives of possible objects. This will be done by situating Buridan's modal logic within the modern metaphysical debate about contingentism and necessitism. We will argue that Buridan's possible objects are contingentist in nature, being based on a logical system where an 'existence' operation is implicitly assumed though not formally stated. We will conclude with some future directions of research into Buridan's logical theory.

2 Buridan's Modal Logic

Buridan addresses modal logic in a number of his logical works. Our main focus in this paper will be on Buridan's *Treatise on Consequences*, as it contains a compact and complete treatment of Buridan's logic.[1] In the second book of the *Treatise* Buridan develops the inferences that exist between single premise modal inferences, some of which are captured in the modal octagon. In the fourth book Buridan combines the material in the second book with the material in the third book to deal with modal syllogisms. Our goal in this section is to provide a brief summary of Buridan's theory of modality as presented in Book Two of the *Treatise on Consequences*. Our aim in this section is to recall the basics of Buridan's modal theory and to sketch a modal system that is able to recapture Buridan's modal logic. For those who would like to see further details about the how the modal system presented here relates to Buridan's own remarks, they should consult [4].

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[1] All English translations provided in this paper are due to Stephen Read and can be found in [8].

For Buridan, a modal proposition is one that contains a modal term or adjective within the proposition itself:

> It should be noted that propositions are not said to be of necessity or of possibility in that they are possible or necessary, rather, from the fact that the modes 'possible' or 'necessary' occur in them [8, p. 95].[2]

Buridan's point is a syntactic one. For a proposition to be modal, a modal term must occur in the proposition. For example, the proposition 'Every human is an animal' is (according to Aristotle) necessarily true, but the proposition is assertoric, because no modal occurs in the proposition. Likewise, the proposition 'Every human is necessarily running' is a proposition of necessity and it is false.

The next distinction that Buridan draws is between compounded and divided modals. Buridan writes:

> They are called 'composite' when a mode is the subject and a dictum is the predicate, or vice versa . . . They are called 'divided' when part of the dictum is the subject and the other part the predicate. The mode attaches to the copula as a determination of it [8, pp. 96–97].[3]

For Buridan, the standard form of a proposition is: quantifier, subject, verb, predicate.[4] A composite modal proposition is one where the verb is not modalized,[5] but either the subject or the predicate (but not both) is a modal term. The non-modal term is called the dictum of the proposition. In Latin the dictum is designated by using an accusative-infinitive construction. In such a construction the main verb is placed in the infinitive and the terms relating to that verb are placed in the accusative case. There are some challenges with literally translating this into English and it is standard to use dependent clauses to translate the dictum. For example, 'That every B is A is necessary' and 'It is possible that some B is A' are examples of how such propositions are usually translated into English.

In contrast, a divided modal proposition does not contain a modal term, but occurs when the mode is attached to the copula [8, p. 96]. This is best illustrated in English either by the use of verbs like 'can' where the modality is a feature of the verb or by using modal as an adverb. For example, the proposition 'A human can run' or 'A person is of necessity running' are both divided modal propositions for Buridan. In what follows we will often write e.g. 'Every A is necessarily B' for a divided modal proposition.[6]

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[2] Sed notandum est quod propositiones non dicuntur "de necessario" aut "de possibili" ex eo quod sunt possibiles aut necessariae, immo ex eo quod in eis ponuntur isti modi "possibile" aut "necessarium" [1, p. 56].

[3] "Compositae" uocantur in quibus modus subicitur et dictum praedicatur uel econuerso. . . Sed "diuisae" uocantur in quibus pars dicti subicitur et alia pars praedicatur. Modus autem se tenet ex parte copulae, tamquam eius quaedam determinatio [1, p. 57].

[4] This order can be changed in various ways to create non-normal propositions, which will not be treated in this work.

[5] A verb is modalized if either the modality is a 'feature' of the verb e.g. 'can' or if the main verb is modified by a modal adverb, e.g. 'of necessity'.

[6] This is ambiguous in English between 'A is of necessity B' and 'A is necessarily-B' where the hyphen indicates that the modality goes with the term, not with the verb. Unless we say otherwise, 'A is necessarily B' should be read as a divided proposition.

It should be observed that for Buridan, a divided modal proposition is negative if the negation operation occurs in front of the modalized copula or between the modal and the verb as in 'Some B is necessarily not A'. According to Buridan:

> Others are negative, and they are of two sorts. In some the negation occurs in the mode, in that it precedes it, for example, "A human is not possibly an ass" and "No human is possibly an ass." In others the negation does not occur in the mode but follows it, for example, "A human is possibly not white" and "God [p. 58] is necessarily not wicked." Some are in doubt whether these last should properly speaking be called affirmative or negative. But whatever they say, I believe they should be called negative, both because the proposition "B is possibly not A" is equivalent to "B is not necessarily A," which is clearly negative, and because an affirmative proposition is not true if any term supposits for nothing, but "A chimera is necessarily not an ass" is true, and consequently so is "A chimera is possibly not an ass" [8, p. 96].[7]

For Buridan, a proposition is counted as negative when the negation occurs in *front* of either the modal or the verb [9, pp. 38–39]. Otherwise it is positive. This construction is much more natural in Latin where it is perfectly grammatical and intelligible to write something like 'A non est B'. To capture this distinction we will use 'B is non-A' for when the negation occurs after the verb and modifies the term. We will use 'B is not A' when the negation modifies the copula.

This distinction is important, because the truth conditions for negative and positive propositions differ in a number of ways. In assertoric propositions, only positive propositions have 'existential import'. As we have already remarked, if the proposition 'Every A is B' is true then the term A must supposit for something, and everything that A supposits for, B must also supposit for, that is to say everything that is A is also a B.

In contrast, a negative proposition is true if there is nothing which is true of the subject, or to borrow Buridan's terminology, the subject does not supposit for anything. To see the difference, observe that for the proposition 'A person is non-running' to be true, there must be some person who is not running. In contrast, the proposition 'A person is not running' is true even if there are no people in existence.

Buridan also points out that if two negations occur in the modality or the verb, or one occurs in each, then the proposition is equivalent to a positive one. This is because, for Buridan, necessity and possibility are duals in the usual way. He writes:

> From any proposition of possibility, there follows as an equivalent another of necessity and from any of necessity another of possibility, such that if a negation was attached either to the mode or to the dictum or to both in the one it is not attached to it in the other and if it was not attached in the one it is attached in the other, other things remaining the same [8, p. 99].[8]

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[7] Aliae sunt negatiuae, et illae sunt duplices. Quaedam sunt in quibus negatio fertur in modum, quia sic praecedit ipsum ut: Hominem non possibile est esse asinum et: Nullum hominem possibile est esse asinum. Aliae sunt in quibus negatio non fertur in modum sed sequitur ipsum, ut: Hominem possibile est non esse album et: Deum necesse est non esse malum. Et aliqui dubitant utrum istae ultimae debeant simpliciter loquendo dici affirmatiuae aut negatiuae. Ad quod, quidquid dicant aliqui, credo esse dicendum quod ipsae sunt negatiuae, tum quia haec propositio"B potest non esse A" aequipollet isti "B non necesse est esse A", quae manifeste est negatiua, tum quia propositio affirmatiua non esset uera si aliquis terminus pro nullo supponeret, et tamen haec ponitur uera: Chimaeram necesse est non esse asinum et, per consequens, ista: Chimaeram possibile est non esse asinum [1, pp. 57–58].

[8] Ad omnem propositionem de possibili sequi per aequipollentiam aliam de necessario et ad omnem de necessario aliam de possibili, sic se habentes quod si fuerit apposite negatio uel ad modum uel ad dictum

Buridan illustrates this with the following example ‘B is not possibly not A.’ Buridan observes that this is clearly equivalent to an affirmative proposition (‘B is necessarily A’), and so Buridan says he will treat those propositions as affirmative.

Buridan’s analysis of divided modal propositions is one of the unique features of his modal theory. Buridan tells us that:

> It should be realised that a divided proposition of possibility has a subject ampliated by the mode following it to supposit not only for things that exist but also for what can exist even if they do not [8, p. 97].[9]

According to Buridan, in divided modal propositions of possibility, the subject is ampliated to supposit for that which is or can be. When this is taken together with a number of assumptions about equivalences between modal propositions, Buridan is able to provide a uniform account of truth conditions for modal propositions. In particular, Buridan assumes that:

> Now, in the fifth chapter, I take it as Aristotle did and others do too, namely, that “necessarily” and “impossibly not” are equivalent, and “necessarily not” and “impossibly” are also equivalent. For in itself it seems clear that of everything that necessarily is, it is impossible that it not be, and conversely, of everything that necessarily is not, it is impossible that it be. I also take “impossibly” and “not possibly” to be equivalent, because a negation is implicit in the term “impossibly.” So “B is not possibly A” and “B is impossibly A” are equivalent, and similarly, “B cannot be A” and “B is not possibly A,” because “can be” and “is possibly” mean the same. Similarly, “Every B is impossibly A” and “Every B is not possibly A” are equivalent, and “No B is possibly A” and “No B can be A.”
>
> Too, I take it that a universal affirmative contradicts a particular negative, and a universal negative a particular affirmative in the same way, so that in the negative the negation governs the mode. For example, “Every B is possibly A” contradicts “Some B is not possibly A”; similarly, “No B can be A” [contradicts] “Some B can be A.” And similarly for other modes [8, pp. 98–99].[10]

In Conclusion 2 of Book 2 he goes on to use these equivalences together with the ampliation of the subject to provide a uniform treatment of modal propositions of necessity. Buridan writes:

> In every divided proposition of necessity the subject is ampliated to supposit for those that can be. This conclusion seems clear. For otherwise those of necessity would not be equivalent to those of possibility having a negated mode, since in those of possibility the subject is clearly granted to be

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uel ad utrumque in una non apponatur ad illud in alia et si non fuerit apposite in una apponatur in alia, aliis manentibus eisdem [1, p. 61].

[9] Supponendum est quod propositio diuisa de possibili habet subiectum ampliatum per modum sequentem ipsum ad supponendum non solum pro his quae sunt sed etiam pro his quae possunt esse quamuis non sint [1, p. 58].

[10] Deinde, in quinto capitulo, supponam illud quod Aristoteles supponit et communiter alii, scilicet quod aequipollent “necesse esse” et “impossibile non esse”, et etiam aequipollent “necesse non esse” et “impossibile esse”. Quoniam per se uidetur esse manifestum quod omne illud quod necesse est esse ipsum impossibile est non esse, et econuerso, et omne illud quod necesse est non esse ipsum impossibile est esse. Suppono etiam quod aequipollent “impossibile” et “non possibile”, quondam in hoc nomine “impossibile” implicatur negatio. Et ideo istae aequipollent: ‘B non possibile est esse A’ et: ‘B impossibile est esse A’ et similiter: ‘B non potest esse A’ et: ‘B non possibile est esse A’ quia idem significat “potest esse” et “possibile est esse”. Similiter istae aequipollent: ‘Omne B impossibile est esse A’ et: ‘Omne B non possibile est esse A’ et: ‘Nullum B possibile est esse A’ et: ‘Nullum B potest esse A’ [1, pp. 60–61].

> so ampliated. So the proposition "B is necessarily A" is analyzed as "That which is or can be B is necessarily A" and "Every B is necessarily A" is analyzed as "Everything that is or can be B is necessarily A," and similarly for negatives. This is clearly shown if this kind of ampliation is granted for those of possibility [8, p. 100].[11]

The rationale for this conclusion is straight forward. Given that Buridan accepts the usual equivalences between propositions of possibility and necessity, one can argue that e.g. 'Every B is necessarily A' is equivalent to 'Every B is not possibly not A', which is equivalent to 'No B is possibly not A'. A similar argument provides the equivalences between 'No B is A' and 'Every B is not A'. Similar arguments hold in the particular cases. When we turn to a formal presentation of Buridan's system, it is these conditions that will form the basis for the truth conditions for the various modal propositions that Buridan treats.

Before we turn to such a presentation, we should pause to reflect on Buridan's treatment of ampliation. On Buridan's theory, 'Some B is possibly A' is equivalent to 'That which is or can be B is possibly A'. Likewise, 'Some B is necessarily A' is equivalent to 'that which is or can be B is necessarily A'. Why might we think this? According to Buridan this is a general feature about the way these kinds of verbs ampliate their subjects. As an example, consider the proposition, 'Someone labouring was healthy'. Buridan tells us that this can be true in different ways.[12] This proposition is true if there is currently someone labouring who was healthy at some point in the past. The proposition would also be true if there was some person in the past who was labouring and was healthy at that or some other point in the past. The case for future tensed propositions is analogous. This seems clear enough, but what is interesting is that Buridan goes on to remark that:

> Thus, because possibility is about the future and all that is possible, the verb 'can be' similarly ampliates the supposition of the subject to everything which can be [8, p. 71].[13]

This could be one of the reasons that Buridan thinks the divided proposition of possibility and necessity ampliate their subject.[14] Buridan's observation is based on this connection between temporal and modal propositions. Roughly, we could think of this as arguing that,

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[11] In omni propositione de necessario diuisa subiectum ampliatur ad supponendum pro his quae possum esse. Haec conclusio manifeste apparet. Quia aliter illae de necessario non aequipollerent illis de possibili habentibus modum negatum, cum in illis de possibili subiectum manifeste concedatur sic ampliari. Ideo ista propositio: 'B necesse est esse A' exponitur per: 'Quod est uel potest esse B necesse est esse A' et ista: 'Omne B necesse est esse A' exponitur per: 'Omne quod est uel potest esse B necesse est esse A' et simili modo de negatiuis. Et hoc clare patet si sit concessa huiusmodi ampliatio in illis de possibili [1, pp. 56–57].

[12] Buridan uses the term 'causes of truth.' It should be remarked that 'causes of truth' are not the medieval analogue of truth conditions, although they are used in similar ways. See [2, p. 56, fn. 85].

[13] Deinde, quia possibilitas est ad future et omnino ad possibilia, ideo similiter hoc uerbum "potest" ampliat suppositionem subiecti ad omnia quae possunt esse [1, p. 27].

[14] In modern terms we would express this as the following modal 'bridge' principle: $F\phi \rightarrow \Diamond\phi$, where F is a future tense operation.

1. Temporal terms ampliate their subject.
2. If something will be the case, then it can be the case.
3. Therefore: modal terms also ampliate their subjects.

Now, when we turn to the truth conditions for ampliated modal propositions, Buridan offers the following gloss on these propositions:

> It should be realised that a divided proposition of possibility has a subject ampliated by the mode following it to supposit not only for things that exist but also for what can exist even if they do not. Accordingly, it is true that air can be made from water, although this may not be true of any air which exists. So the proposition 'B can be A' is equivalent to 'That which is or can be B can be A' [8, p. 97].[15]

What this means is that there are four ways for the proposition 'Some A is possibly B' to be true:

1. There is something which is A and is B.
2. There is something that is A and can be B (even though it is not B now).
3. There is something that can be A and is B (even though it is not A now).
4. There is something that can be A and can be B (even though it is not A or B now).

It is the final cause of truth that is most interesting here. What exactly are these sorts of objects that Buridan is working with here? During his discussion of Buridan's modal logic, Hughes takes the following digression:

> For a long time I was puzzled about what Buridan could mean by talking about possible but non-actual things of a certain kind. Did he mean by a 'possibly A', I wondered, an actual object which is not in fact A, but might have been or might become, A?... But this interpretation won't do; for Buridan wants to talk, e.g., about possible horses; and it seems quite clear that he does not believe that there are, or even could be, things which are not in fact horses but which might become horses. What I want to suggest here, very briefly, is that we might understand what he says in terms of modern 'possible world semantics'. Possible world theorists are quite accustomed to talking about possible worlds in which there are more horses than there are in the actual world. And then, if Buridan assures us that by 'Every horse can sleep' he means 'Everything that is or can be a horse can sleep' we could understand this to mean that for everything that is a horse in any possible world, there is a (perhaps other) possible world in which it is asleep. It seems to me, in fact, that in his modal logic he is implicitly working with a kind of possible worlds semantics throughout [3, p. 9].

This suggestion by Hughes, that Buridan's modal logic may be thought of as using a kind of 'possible-worlds' semantics has been more fully explored in [4]. In this paper it is shown that these semantics correctly track the single premise and syllogistic validities and invalidities that Buridan claims are valid in the *Treatise on Consequences*. See [4, pp. 14–17] In this paper it is shown how to formalise Buridan's divided modal logic using variable domain KT and we will use the semantics below:

A *Buridan Modal Model* is a tuple:[16]

$\mathfrak{M} = \langle D, W, R, O, c, v \rangle$ such that:

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[15] supponendum est quod propositio diuisa de possibili habet subiectum ampliatum per modum sequentem ipsum ad supponendum non solum pro his quae sunt sed etiam pro his quae possum esse quamuis non sint. Unde sic est uerum quod aer potest fieri ex aqua, licet hoc non sit uerum de aliquo aere qui est. Et ideo haec propositio: B potest esse A aequiualet isti: Quod est uel potest esse B potest esse A [1].

[16] See [4, p. 10].

D and W are non-empty sets. D is the domain of objects and W is a set of worlds.
$R \subseteq W^2$ which is reflexive.
$O : W \to \mathcal{P}(D)$.[17]
$v : W \times PRED \to \mathcal{P}(D)$.
$c : CONS \to D$.

Semantic Abbreviations *Let P be a term, and Q either a term or the negation of a term. Using the semantics we can define the following operations:*

$V'(w, P) = O(w) \cap v(w, P)$
$V'(w, \neg P) = D \setminus (O(w) \cap v(w, P))$
$M(w, Q) = \{d \in D : \textit{ there is some } z \textit{ s.t } wRz \textit{ and } d \in V'(z, Q)\}$
$L(w, Q) = \{d \in D : \textit{ for all } z \textit{ if } wRz \textit{ then } d \in V'(z, Q)\}$

Here the idea is that the operations V', M and L give the extension of a particular term at a particular world. For example, $V'(w, P)$ returns the extension of the predicate for the objects that exist at w while $M(w, P)$ and $L(w, P)$ give the set of objects that are possibly (respectively, necessarily) P at w.

It is then possible to define the modal categorical propositions that Buridan considers in the following way:

Modal Categorical Propositions

$\mathfrak{M}, w \vDash A \overset{L}{a} B$ *if and only if* $M(w, A) \subseteq L(w, B)$ *and* $M(w, A) \neq \emptyset$
$\mathfrak{M}, w \vDash A \overset{L}{e} B$ *if and only if* $M(w, A) \cap M(w, B) = \emptyset$
$\mathfrak{M}, w \vDash A \overset{L}{i} B$ *if and only if* $M(w, A) \cap L(w, B) \neq \emptyset$
$\mathfrak{M}, w \vDash A \overset{L}{o} B$ *if and only if* $M(w, A) \not\subseteq M(w, B)$ *or* $M(w, A) = \emptyset$
$\mathfrak{M}, w \vDash A \overset{M}{a} B$ *if and only if* $M(w, A) \subseteq M(w, B)$ *and* $M(w, A) \neq \emptyset$
$\mathfrak{M}, w \vDash A \overset{M}{e} B$ *if and only if* $M(w, A) \cap L(w, B) = \emptyset$
$\mathfrak{M}, w \vDash A \overset{M}{i} B$ *if and only if* $M(w, A) \cap M(w, B) \neq \emptyset$
$\mathfrak{M}, w \vDash A \overset{M}{o} B$ *if and only if* $M(w, A) \not\subseteq L(w, B)$ *or* $M(w, A) = \emptyset$
$\mathfrak{M}, w \vDash A \overset{Q}{a} B$ *if and only if* $M(w, A) \subseteq M(w, B) \cap M(w, \neg B)$ *and* $M(w, A) \neq \emptyset$
$\mathfrak{M}, w \vDash A \overset{Q}{e} B$ *if and only if* $\mathfrak{M}, w \vDash A \overset{Q}{a} B$
$\mathfrak{M}, w \vDash A \overset{Q}{i} B$ *if and only if* $M(w, A) \cap M(w, B) \cap M(w, \neg B) \neq \emptyset$
$\mathfrak{M}, w \vDash A \overset{Q}{o} B$ *if and only if* $\mathfrak{M}, w \vDash A \overset{Q}{i} B$
$\mathfrak{M}, w \vDash A \overset{\bar{Q}}{a} B$ *if and only if* $M(w, A) \cap M(w, B) \cap M(w, \neg B) = \emptyset$
$\mathfrak{M}, w \vDash A \overset{\bar{Q}}{e} B$ *if and only if* $\mathfrak{M}, w \vDash A \overset{\bar{Q}}{a} B$
$\mathfrak{M}, w \vDash A \overset{\bar{Q}}{i} B$ *if and only if* $M(w, A) \not\subseteq (M(w, B) \cap M(w, \neg B))$ *or* $M(w, A) \neq \emptyset$
$\mathfrak{M}, w \vDash A \overset{\bar{Q}}{o} B$ *if and only if* $\mathfrak{M}, w \vDash A \overset{\bar{Q}}{i} B$

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[17] This is corrected. The original reads $O : W \to D$ [4, p. 10].

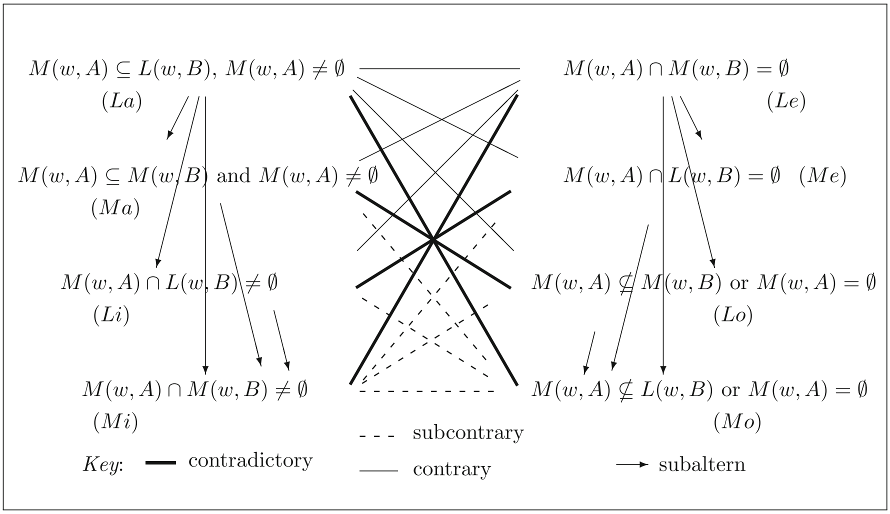

Fig. 1 Buridan's modal octagon of opposition

As we have already seen, Buridan claims that such propositions give rise to an octagon of opposition which is shown below in Fig. 1.[18] It is shown in [4] that they do in fact do this, and that the semantics given above, combined with a natural reconstruction of Buridan's assertoric syllogism, can be used to formalise all of Buridan's divided modal logic.

From this, we can see that a natural way to understand 'Everything that is possibly A' or 'Something that is necessarily not B' is to view these as quantifying over all of the objects that exist at various possible worlds and are A or are not B.

3 The Ontological Implications of Buridan's Modal Logic

With this possible worlds based analysis of Buridan's modal logic in place, we now want to answer the question, 'What sorts of objects are Buridan's possible objects?' by looking at where Buridan would stand on the debate between necessitism and contingentism. Surprisingly, a number of Buridan's remarks have bearing on this debate. Very briefly, we can sum up the two positions with the following quote:

> Call the proposition that it is necessary what there is *necessitism* and its negation *contingentism*. In slightly less compressed form, *necessitism* says that necessarily everything is necessarily something; still more long-windedly: it is necessary that everything is such that it is necessary that something is identical with it [10, p. 3].

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[18] The Octagon pictured here can be found in [4, p. 11] and is a modified version of a diagram made by Stephen Read.

Our methodology for answering these questions is as follows. In [10] Williamson sets out a number of arguments for necessitism and reflects on a number of consequences that follow from rejecting necessitism. What we will do is see in which places Buridan goes along with Williamson (or it seems like he would) and in which places he differs from Williamson. The idea is that we will use Williamson's *Modal Logic as Metaphysics* as giving us a collection of criteria for identifying someone who holds to some flavour of necessitism. We use *Modal Logic as Metaphysics* as this is one of the most recent and most through defences of necessitism.

Our goal here is twofold. First, historically, by comparing Buridan's views to this modern question we will hopefully gain a somewhat better understanding of exactly what Buridan was doing with his modal theory. Second, we will see how Buridan's logic and his theory connect to this interesting metaphysical debate and see how Buridan's modal logic relates to modern modal logic and modal metaphysics. In doing this we will also have the opportunity to focus on a few features of Buridan's modal logic that will help us better understand Buridan's position. In order to accomplish this, we will first sketch some of the key features of necessitism. After doing this, we will look at the inferences and principles that Buridan accepts and see if they commit Buridan to either necessitism or contingentism or are consistent with both. We will argue that Buridan's position on a number of features of his modal language are not compatible with necessitism. In particular, Buridan seems to be committed to denying the Barcan and Converse-Barcan formulae. As we shall see, this places his modal logic in tension with necessitism but leaves it consistent with contingentism.

3.1 *Necessitism and Contingentism: The Case of Modal Logic as Metaphysics*

In his recent book, [10] Williamson offers a spirited, vigorous and insightful defence of necessitism. As we have already remarked, this is the view that "it is necessary that everything is such that it is necessary that something is identical with it" [10, p. 3]. Williamson offers a number of arguments for this position within his book and he identifies a number of key principles that either follow from necessitism, imply it, or are required for us to formulate the relevant distinctions between the two positions. The first distinction we will need is the distinction between the predicative reading of a modal attribution and the attributive reading of a modal attribution. According to Williamson:

> Someone might object that it is absurd to postulate a non-concrete possible stick, because being concrete is necessary for being a stick. But that is to mistake the intended sense of 'possible stick'. The objector reads 'x is a possible stick' as equivalent to something like 'x is a stick and x could have existed'. Call that the *predicative* reading. On this reading, it is trivially necessary that all sticks are concrete.

> On the relevant alternative reading 'x is a possible stick' is simply equivalent to 'x could have been a stick'. Call that the *attributive reading* of 'possible stick'... it is not necessary that all possible sticks are sticks on the attributive reading [10, p. 10].[19]

As Williamson later points out the necessitist will (unless context or other features of proposition require a different reading) want to conceive of modal attributions following the attributive reading. Not only does this help the necessitist avoid being confused with other, less plausible theories (e.g. Meinongianism) [10, pp. 18–21] but it also avoids some trivialising issues with the theory.

The next feature of this debate that is worth highlighting here is that the quantifiers used in the formulation of necessitism and contingentism need to be understood as unrestricted quantifiers that range over absolutely everything.

> Both necessitists and contingentists can also use quantifiers with various restriction, and may even regard such uses as typical of everyday discourse. In particular, necessitists can simulate contingentist discourse by tacitly restricting their quantifiers to the concrete. Then they sound like contingentists, saying 'Concrete things are only contingently something'. But they just mean that concrete things are only contingently something *concrete*. The restriction makes the words express different claims from those they express when used unrestrictedly. The disagreement is made explicit only when both sides use their quantifiers unrestrictedly. In what follows, our interest is in the unrestricted uses [10, p. 15].

In what follows in our treatment of Buridan it will be important to establish that he views the quantifiers in his modal theory as sufficiently non-restricted to not run afoul of this issue. To see why this is an interpretive problem, say that we argue for the conclusion that Buridan is a contingentist. One natural response would go, 'you cite evidence X,Y and Z showing that Buridan needs to reject necessitism, but it is consistent with what Buridan says, that these quantifiers be read in a restricted way.' Interpretively, there is a helpful warning here. I am not aware of any medieval discussions concerning anything quite like the modern debates about unrestricted generality. Hence, it will not be clear what Buridan thinks on the matter and as such, we will need to present some evidence about how Buridan understands his modal propositions.

From a formal perspective, perhaps the most important feature of necessitism is its commitments to the Barcan and Converse Barcan Formulae. Williamson writes:

> The metaphysical disputes discussed in Chapter I between contingentism and necessitism turns out to be intimately connected with some technical issues in quantified modal logic, over two principles usually known as the Barcan formula and its converse. When those principles are interpreted in the relevant way, they are typically accepted by necessitists, and rejected by contingentists. Indeed, in some natural logical settings, each of them is equivalent to the central necessitist claim that necessarily everything is necessarily something [10, p. 31].

What is important to observe is that, working in classical first order logic from the validity of the Barcan Formula (BF), the Converse Barcan Formula (CBF), and necessitation, it is possible to derive the necessitist claim that "necessarily everything is necessarily something" [10, p. 38].

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[19] Throughout this book, Williamson holds that 'concrete' and 'abstract' are not best thought of as contradictory pairs, i.e. that something is non-concrete if and only if it is abstract, but are better thought of as contraries. See [10, p. 7].

While there are other important features of necessitism that Williamson highlights, the collection of quotes and thumbnail sketches of the view are sufficient for what will follow. Williamson also points out a number of consequences that, he argues, the contingentist is under pressure to adopt. The main one which interests us is the following:

> The challenge to contingentists is to identify a fallacy in Barcan Marcus's proof…They have a natural line. Her proof involves the claim that $\neg\exists y(x = y)$ strictly implies $\exists x\neg\exists y(x = y)$, in other words:
> (8) $\Box(\neg\exists y(x = y) \rightarrow \exists x\neg\exists y(x = y))$…
> As we have seen, contingentists cannot accept (8) as a theorem, where (8) is the necessitation of (11)$[(\neg\exists y(x = y) \rightarrow \exists x\neg\exists y(x = y))]$…thus a contingentist must either reject (11) as a theorem or reject the rule of necessitation…First, suppose that the contingentist rejects (11) as a theorem. But (11) is a theorem of standard non-modal first-order logic. It is simply an instance of 'existential generalization', $A \rightarrow \exists xA$. Thus the contingentist is under pressure to adopt some form of 'free logic' in which that principle is not unrestrictedly valid [10, p. 39].

As we have already seen, Buridan does appear to be working in something that is somewhat like a 'free logic'. For Buridan, the truth of any particular affirmative proposition requires that there be some object which is truly predicated of both the subject and the predicate. As such, it is natural to think that Buridan would reject the move from $\neg\exists yx = y$ to $\exists x\neg\exists yx = y$ as the first is a negative proposition while the second is affirmative. Williamson goes on to point out that by duality, the contingentist is also required to deny the principle $\forall vA \rightarrow A$, which Buridan also would deny.

As a brief foreshadowing of Buridan, it is worth observing that, in the eyes of at least one metaphysician, there are analogues of the Barcan and Converse Barcan principles identified by Buridan. In his book *The Nature of Necessity*, Plantinga observes that:

> Jean Buridan once remarked that
> (31) Possibly everything is F
> does not in general entail:
> (32) Everything is possibly F.
> That is, he rejected
> (33) necessarily, if possibly everything is F, then everything is possibly F.
> His counter-example is as follows. God need not have created anything; hence it is possible that
> (34) Everything is identical with God.
> It does not follow from this, he says, that everything is possibly identical with God. You and I, for example are not
> [7, p. 58].

Sadly, Plantinga does not include references for where Buridan makes this remark and it is not actually clear what passage in Buridan he has in mind. What seems likely here is that he is extrapolating from a number of Buridan's counter-examples in Book 2 where Buridan starts from the assumption that God is the only one creating. We will have quite a bit more to say about this counter-example of Buridan's in what follows.

3.2 *Quantification in Buridan*

Before we turn explicitly to see how Buridan's modal logic relates to necessitism and contingentism, we should pause and think about how quantification works in Buridan's modal logic. As we already saw, there is a natural way that necessitists can express contingentist questions speaking within their logical framework, namely by restricting the quantification of their quantifiers.

Buridan's logic also has very general resources that allow him to express various sorts of restricted quantifiers. In particular, Buridan has a consistent way to cancel the ampliation of various predicates. Up until this point, one may have thought that if an ampliative term is present, or if the copula has ampliative force, then the there is no way to present a narrower reading of the modal proposition. E.g. there may be no way to express the idea that only those things that currently exist can be B. Throughout his writing in the *Treatise on Consequence* Buridan uses the phrase 'quod est X' (that which is X) as a way of making explicit the ampliation of a particular subject term when it may be ampliated by its predicate. Elsewhere Buridan observes that ampliation is blocked in cases where 'quod est' is used. He remarks that:

> This conclusion has six parts. The first is clear because this construction, "B is A," permits the ampliation of the subject if the predicate is ampliative, for example, "A human is dead." But the [other] construction, "That which is B is A," does not permit the ampliation of the subject, namely, of "B"; for [B] is contracted and restricted to the present by the verb "is" in the present tense, which precedes it. Thus if the predicate is ampliative, "B is A" has more causes of truth than "That which is B is A," and from many to fewer is not a good consequence [8, p. 83]. . . Note that a proposition with the subject ampliated by the predicate should be analysed by a disjunctive subject combining the present tense with the tense or tenses appropriate to the ampliation, for example, "B will be A" as "That which is or will be B will be A," and "A human is dead" as "The one who is or was a human is dead," and "The Antichrist can be a man" as "He who is or can be the Antichrist can be a man," [8, p. 84].[20]

There are a few important things to observe here. First, observe that Buridan is using ampliation in its most general sense. He illustrates this by using examples of both tense and modality. Second, the examples he discusses include cases where a predicate such as 'dead' causes the subject to ampliate, but Buridan also considers temporal and modal cases where the predicate *term* does not have ampliative force, as in the case of 'A' or in the case of 'man'. Indeed, the expression 'ampliated by the predicate' seems to be used in a slightly loser way here than we might expect. In the cases of 'B will be A', and 'The Antichrist can

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[20] Ista conclusio habet sex particulas. Prima patet quia iste modus loquendi "B est A" permittit ampliationem subiecti si praedicatum sit ampliatiuum, ut: Homo est mortuus. Sed iste modus loquendi "quod est B est A" non permittit ampliationem subiecti, scilicet ipsius B; <contrahitur> enim et restringitur ad praesentia per hoc uerbum "est", praesentis temporis, quod praecedit ipsum. Ideo, si praedicatum sit ampliatiuum, ista "B est A" habet plures causas ueritatis quam ista "quod est B est A", et a pluribus ad pauciores non erat bona consequentia. . . Notandum est quod propositio de subiecto ampliato per praedicatum exponenda est per disiunctionem in subiecto temporis praesentis ad tempus uel tempora ad quod uel ad quae fit ampliatio, ut: 'B erit A' 'Quod est uel erit B erit A' et: 'Homo est mortuus' 'Qui est uel fuit homo est mortuus' et: 'Antichristus potest esse homo' 'Qui est uel potest esse antichristus potest esse homo' et: 'Rosa intelligitur' 'Quod est uel fuit uel erit uel potest esse rosa intelligitur' [1, pp. 42–43].

be a man' the cause of the ampliation is not due to the predicate 'A' or 'man' respectively, but is due to the presence of the modals 'will' and 'can' respectively.

How does this relate to quantification? In the following way. We have already seen that, for Buridan, 'Some A is B' is true if there is an object of which we can say, 'This thing is A' and 'This same thing is B'. Thus, if the supposition of the subject term ranges over everything that can fall under the subject, the quantification inherits the range given by the ampliation of the subject and the predicate. As we have just seen, in the case of temporal propositions, it quantifies over all things that were, if the ampliation is to the past. Likewise, if the ampliation is modal, then it ampliates to the possible, as we have earlier seen Buridan claim. What this suggests is that Buridan is intending his ampliation to range over everything that falls under a particular class: Everything that was, everything that can be etc. If one wishes to restrict such quantifiers, then on Buridan's account one needs to make use of the relevant restrictive clauses. Given this, it seems a fair extrapolation of Buridan's views that he intended the proposition to range over all of the relevant objects in question.

Because of how Buridan uses his quantifiers, there is a way that Buridan could mimic both necessitist and contingentist readings of various modal propositions. On the contingentist reading (according to Williamson), 'Every A is necessarily B' states that 'Everything that is concretely A is necessarily B' while the necessitists would hold that 'Every A is necessarily B' states that 'Everything that is concretely A or is non-concretely A is necessarily B'. Notice that all we have done here is made the range of the quantification explicit in both cases. It is also instructive to notice the parallel with how Buridan sets up his modal framework. These sorts of quantifiers give Buridan a way to talk about either sort of quantification, regardless of which reading he would regard as the correct reading of the proposition.

As such, it seems to me that it would be an unmotivated view of Buridan's modal logic to argue that he is implicitly restricting his quantification to only range over concrete objects. For such a reading to be plausible, a gloss would need to be offered "although this may not be true of any A which exists" which either restricts the range of the supposita of air in this passage or argues that here Buridan means to only speak of concrete objects. The second disjunct seems to go directly against what is said in the passage[21] while the first disjunct goes against the spirit of Buridan's unrestricted ampliation of the subject. As such it seem that a fair extrapolation of Buridan's logic is to see him quantifying over absolutely everything.[22]

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[21] This assumes that 'if something does not exist then it is not concrete', a principle which seems to not be ruled out by anything Williamson has said, and is in keeping with the spirit of non-concrete objects.

[22] Again, it should be stressed that this is an extrapolation from Buridan's views as presented in the *Treatise on Consequence*.

3.3 Predicative and Attributive Readings

As we have already seen the distinction between predicate and attributive readings of the modal operations are rather important for understanding and formulating necessitism. What is interesting to observe here is that, broadly speaking, Buridan's ways of reading the various terms within his modal logic are either attributive readings or do not fall under either. As we have already seen, Buridan reads divided modal propositions as ranging over the things that 'can be A' or 'are necessarily B'. Formally, we treated these as ranging over classes of object in the domain. For example recall that $A \overset{L}{a} B$ is true if and only if $M(w, A) \subseteq L(w, B)$ and $M(w, A) \neq \emptyset$. Recall that $M(w, A)$ is defined as $M(w, A) = \{d \in D : \text{ there is some z s.t wRz and } d \in V'(z, A)\}$ and analogously for $L(w, B)$. What is important to see here is that the formal readings offered match Williamson's gloss on the attributive reading, assuming that by 'x could have been a stick' he intends the modality 'could' to be read as a possibility and not as a counter-factual. $M(w, A)$ picks out the class of all objects that could have been A. Likewise $L(w, A)$ picks out the class of all objects that are necessarily A. The point here is that Buridan seems to situate his discussion of modal logic within an attributive framework. Given Buridan's ampliative reading of the subject and his views about the expository syllogism, this is not surprising. On Buridan's account, the predicative reading of 'x is a possible stick', namely 'x is a stick and x could have existed' is too narrow in its ampliative force. First, such a reading does not cover the cases where x could have been a stick and x could have existed. Second, this seems to be analysing the modal operations as a composition of two disjunctions, which Buridan rejects in the *Treatise on Consequence* [1, pp. 56–57]. Buridan's point was that we should not analyse 'some stick can exist' as 'either there is something that is a stick and it can exist or there is something that can be a stick and can exist'.

3.4 Barcan & Converse Barcan

So we have already seen one major point where Buridan breaks with Williamson and another where he is in step. How does Buridan's logic fair on the Barcan and Converse Barcan formulae? Near the end of Book Two of the *Treatise on Consequences* Buridan propounds the following conclusion:

> From no affirmative composite of possibility does there follow a divided one of possibility with the mode affirmed, or conversely, except that from an affirmative composite with an affirmed dictum there follows a divided particular affirmative [8, p. 110].[23]

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[23] Ad nullam compositam affirmatiuam de possibili sequi aliquam diuisam de possibili de modo affirmato nec econuerso, praeterquam ad compositam affirmatiuam de dicto affirmato sequitur particularis affirmatiua diuisa [1, p. 65].

What this tells us is that ‘It is possible that ‘Some A is B’’ entails ‘Some A is possibly B’, but that in the other cases there is no valid inference.[24] From what we have already seen, this makes sense. Reading, ‘It is possible that ‘Some A is B’’ as telling us that there is some world where ‘Some A is B’ is true, we know from what Buridan has already said, that this is only true if there is some object, say D, such that ‘This D is A’ and ‘The same D is B’. But then, it is possible that ‘This D is A’ and it is possible that ‘This D is B’. As we have already seen, it follows by an expository syllogism that ‘Some A is possibly B.’ At first glance, this might seem to look like Buridan endorsing the inference: $\Diamond\exists x(Ax \wedge Bx)$ implies $\exists x \Diamond (Ax \wedge Bx)$.[25] However, this is actually not the case, and hinges on a unique feature of Buridan’s modal logic. The operations M and L are rather difficult to express in first-order modal logic. More to the point, $\exists x \Diamond (Ax \wedge Bx)$ is not equivalent to $A \overset{M}{i} B$, as the quantifier unduly restricts the admissible objects to the world at which the quantifier is evaluated, and so gets the ampliation of the terms wrong. We can construct the relevant counter-models as follows:

$$\begin{array}{ll} D = \{a, b\} & W = \{w, v, u\} \\ O(w) = \{a\} & O(v) = \{a, b\} \\ V'(w, A) = \{\emptyset\} & V'(w, B) = \{\emptyset\} \\ V'(u, A) = \{b\} & V'(u, B) = \{a, b\} \end{array}$$

It is an easy exercise in first order modal logic to verify that $A \overset{M}{i} B$ is true here while $(\Diamond A \wedge \Diamond B) \rightarrow \Diamond(A \wedge B)$ is not. The problem is that the quantification used here does not range over the specific world at which the formula is evaluated, but should range over all of the objects at all of the worlds. So, at least here, it seems Buridan is not committed to either of the Barcan Formulae.

In fact, Buridan’s account provides us with a few possible counter-examples to the Barcan and Converse Barcan Formulae. A counter-example to the Barcan Formula is easily seen to follow from Buridan’s consideration of the definition of possibility modals. Recall that Buridan said: “Accordingly, it is true that air can be made from water, although this may not be true of any air which exists [8, p. 97]”.[26] Let us assume that this situation does indeed obtain, there is some air that can be made from water. Let us assume further that there is currently no water but that there will be. Then, we have $A \overset{M}{i} W$ (reading A for air and W for water) is true, but $\exists x \Diamond (Ax \wedge Wx)$ is not true because no water currently exists.

For the Converse Barcan Formula, things are a little bit more tricky but Buridan will reject it, given the following sorts of remarks:

> As to whether the proposition ‘A horse is an animal’ is necessary, I believe it is not, speaking simply of a necessary proposition, since God can annihilate all horses all at once, and then there would be no horse; so no horse would be an animal, and so ‘A horse is an animal’ would be false, and so it would not be necessary. But such [propositions] can be allowed to be necessary, taking conditional

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[24] This is equally clear since, even if something can be A and can be B, it does not entail that something can be A and B at the same world. i.e. $(\Diamond A \wedge \Diamond B) \rightarrow \Diamond(A \wedge B)$ is not valid.

[25] There are some in the modern literature who have suggested Buridan may have implicitly endorsed this principle, however for different reasons. See [5, pp. 158, 160, fn. 56].

[26] Unde sic est uerum quod aer potest fieri ex aqua, licet hoc non sit uerum de aliquo aere qui est [1, p. 58].

> or temporal necessity, analysing them as saying that every human is of necessity an animal if he or she exists, and that every human is of necessity an animal when he or she exists [8, p. 141].[27]

Informally, what Buridan is pointing out here is that it is entirely possible for all objects to cease existing. It is within the power of God to bring it about that no horses exist, or in fact ever existed. More to the point, such objects also lose all of the properties that they might have, upon ceasing to exist concretely. As such, it seems that Buridan allows for objects to pass out of existence.

As is well known, we can use this to construct a counter-example to the Converse Barcan Formula along the usual lines. Let us assume that some horse exists. Then, clearly given what Buridan has said above, it is clearly possible that this horse does not exist and hence, $\exists x \Diamond \neg Ex$, where Ex stands for 'x exists'. However, since Buridan maintains that horses (and objects more generally) lose their properties once they cease to exist, $\Diamond \exists x \neg Ex$, will turn out to be impossible on Buridan's view, as it would require the existence of a non-existent object.

Hence, from what we have seen, while Buridan is does make use of an attributive reading of the modal operations, he seems committed to rejecting both the Barcan and the Converse Barcan formulae. With this in place, we have good reason to think that Buridan would not accept the core tenets of necessitism, but would instead opt for some form of contingentism.

4 Conclusion

Our aim in this paper was to argue that the ontology implicit in the octagon of opposition and Buridan's *Treatise on Consequences* is contingentist in nature. To advance this conclusion we have argued that Buridan is committed to working in a sort of 'free logic', where only affirmative categorical propositions have existential import, that he appears committed to the rejection of the Barcan formula, and that from some of the remarks he makes, he is also committed to the rejection of the Converse Barcan formula.

At this point there are two major questions that this work leaves open. First, the analysis presented here crucially hangs on the point that Buridan's modal logic does not implicitly make use of the Converse Barcan formula. While there is a formal reconstruction of the divided fragment of Buridan's modal syllogism, [4, pp. 14–17] this work does not treat composite modal propositions, and does not treat the most likely places where Buridan could make implicit use of either formulae. While the remarks we have sketched above would make it dubious that Buridan should accept their soundness, a complete formal

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[27] Utrum autem haec propositio sit necessaria 'Equuus est animal', crederem quod non, loquendo simpliciter de propositione necessaria quia deus potest simul adnihilare omnes equos, et tunc nullus equus esset; ideo nullus equus esset animal, et sic ista esset falsa 'Equus est animal', ergo ipsa non esset necessaria, quamuis tamen tales possint concedi necessariae necessitate conditionali uel temporali, secundum tales expositiones quod omnis homo de necessitate est animal si ipse est et quod omnis homo de necessitate est animal quando ipse est [1, p. 112].

treatment of Buridan's modal logic should be provided, both for its own interest, and to remove all doubts on this point.

Second, we have not attempted to show what sort of contingentist Buridan is, or how his approach to modal logic relates to other, modern contingentists. Given Buridan's nominalism, which includes a rejection of the necessary existence of propositions, this suggests that Buridan's logic would be unique and perhaps offer a different perspective for philosophers who are not comfortable with the idea that propositions necessarily exist. While this has not been undertaken in any detail here, I believe this would be interesting and could prove fruitful.

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From Aristotle's Square of Opposition to the "Tri-unity's Concordance": Cusanus' Non-classical Reasoning

Antonino Drago

Abstract It is well-known that Cusanus (1401–1464) introduced the surprising notion of the coincidence of opposites, which in fact shrinks Aristotle's square of opposition into a segment. Almost a century ago Cassirer suggested that Cusanus had looked for a new kind of logic. Indeed, an accurate inspection of Cusanus' texts shows that in order to discover new names of God by means of coincidences of opposites, Cusanus invented several names belonging to different kinds of non-classical logic—i.e. positive, paraconsistent, modal and intuitionist—, which were formalised in the last century. When, in his more important book, he invented an intuitionist name he implicitly reasoned about it according to the intuitionist square of opposition so precisely that he was able to organize his theories in a new way; it was based not on axioms- principles, but on the search for a new method for solving a general problem. Moreover, he wanted to refer his reasoning not to the square of opposition but to a new logical scheme, a "tri-unity of concordance", for which he suggested an original definition. Here this tri-unity is represented by means of a geometrical figure.

Keywords Cusanus • Square of opposition • Non-classical logics

Mathematics Subject Classification Primary 03A05, Secondary 03B53, 03B22, 03B45, 03B20

1 Cusanus' Philosophical Theology as a Logical Problem

Modern scholars gave various evaluations of the philosophico-theological works by Cusanus (1401–1464)—referred to also as Nicholas of Kues, Nicolaus Cusanus and Nicholas of Cusa.

Some scholars have evaluated negatively both the language and the contents of these works. By mixing together dogmatic notions of Christian faith, pedagogical exhortations, imaginative illustrations of new ideas, analogies and arguments, Cusanus' writings seem so obscure as to be considered inconsistent (Duhem [42, pp. 262–263]: *tours de passe-passe*; results of an *audace déraisonnable*; Vansteenberg [70, p. 287]: *broderies*; Hopkins [52, pp. 3–28]). In particular, some scholars have disqualified his celebrated notion of a *coincidentia oppositorum* as a *jonglerie de mots* [42, p. 262], or as a *jeu verbal,* or *jeu mental* [49, pp. 290 and 291].

J.-Y. Béziau, G. Basti (eds.), *The Square of Opposition: A Cornerstone of Thought*, Studies in Universal Logic, DOI 10.1007/978-3-319-45062-9_5

Other scholars have admired Cusanus' writings. They recognized two merits in him: not only that he surprisingly anticipated some subsequent achievements of modern science [14, 66]; but that he also introduced, more or less effectively, the notion of the infinity into Western thinking, also formulating lucidly the problem of human knowledge so that he was the first scholar to investigate the method with which the mind thinks; for these reasons he is often presented as the first modern Western philosopher [14, p. 10], [12, vol. I, Chap. 1].

In order to elucidate his thought some scholars studied the applications of his method to mathematical problems; but no new result for a general interpretation was obtained [60]. However, about the relationship between theology and mathematics in Cusanus, a new perspective was suggested two years ago by Albertson [1]. Here, however, it is the logical aspects of his writings that will be examined.

According to a common opinion, the logical outlook of Humanism was a "purely negative attitude, a mere rejection of Scholasticism" [8, vol. I, pt. IV, 36 A], [10, pp. 78–85]. Cusanus also opposed the representatives of Aristotelian logic, accusing them of constituting an *Aristotelis secta.*[1] He never appealed to Aristotle's square of opposition and nor did he make use of the syllogism. However, he much appreciated logic: "... logic is, as Aristotle said, a most exact instrument for pursuit both of the truth and the truthlike. Hence when the intellect finds [what is true], it recognizes [it] and eagerly embraces [it]." [32, Chap. I, p. 1282, no. 4]. Moreover, he appreciated Aristotle's logical work, but only as a particular way of reasoning, because it relies on the law of non contradiction; which however is inadequate for thinking about God. About this point he lucidly wrote:

> The Philosopher [Aristotle] certainly seems... to have come upon nothing which is sufficient [to name God]. For not even reason [*ratio*] attains to what precedes reason; and even less can any of the arts produced by reason furnish a way to what is unknown to all reasons. The Philosopher held it to be most certain that an affirmation contradicts a negation and that both cannot at the same time be said of the same thing, since they are contradictories. He said this on the basis of the reason's concluding it to be true... [Instead he did not see] that that to which he gives the name 'first principle' (*primum principium*) does not suffice for showing the way to the truth which the mind contemplates beyond [the *ratio*].... in [this] manner Aristotle closed off himself a way for viewing the truth ([31, Chap. 19, pp. 1149–1150, nos. 88–89]; I put the word *ratio* for Hopkins's "reasoning")

Was Cusanus successful in suggesting a new way of logical thinking? About this point I took seriously what Cassirer suggested almost a century ago:

> When Cusanus' theology abandons the scholastic logic, i.e. the logic... which undergoes the principle of contradiction and of the excluded middle, yet it requires a new kind of mathematical logic, which does not exclude the coincidence of the opposites; rather it needs of this opposition, just of the coincidence of the absolutely great and the absolutely little, as a stable principle and necessary vehicle of the proceeding knowledge. [14, p. 15]

Indeed the rejection of Aristotelian logic did not lead Cusanus to renounce rational *reasoning*. About this point in his first important book he asked his friend to "...

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[1] Cusanus [24, p. 463, no. 6]. I quote here and in the following (except for few cases) from the translation in English laguage by Jasper Hopkins. His site, http://jasper-hopkins.info/, includes almost all Cusanus' books.

receive ... *a mode of reasoning* such as the following—a mode which great labor has rendered very pleasure to me." ([20], final proposition of the "Prologue", p. 4, no. 1; emphasis added); and moreover at the beginning of his most important book he asked his conversant: "unless you are compelled by reason, you will reject as unimportant everything you will hear from me." [31, Chap. I, p. 1108, no. 2].

In fact, he introduced surprising novelties. First, he maintained that there exist two faculties in our mind. The first one is *ratio*, which is regulated by the law of non-contradiction and hence by the Aristotelian logic; in particular, it manages the building of mathematics, a science praised by Cusanus because "... we have no certain knowledge except mathematical knowledge". (Cusanus 1440, p. 936, no. 44), precisely because the above logical law perfectly applies to this field of knowledge.

> But when [the critic of my writings] alleges that both the fundamental principle-of-knowledge (which is enfolded in the principle "everything either is or not is [the case]") and all inferences are destroyed, he is misconceiving. For.... [Aristotle's] logic and any [past] philosophical investigation do not attain unto seeing [with the mind's eye and with apprehension-by-the-intellect]. [24, p. 469, no. 14]

Indeed, Cusanus wanted to think about God by means of one more faculty, the *intellectus,* which advances through intuitive and creative steps, called by him *coniecturae,* a word *grosso modo* corresponding to conjectures, or surmises. As a whole, this is a transcending process [*transcessus*].

He then parallels the capabilities of the two faculties of the human mind to the two following kinds of vision; parallel to the *ratio*, the vision of what "is being sought (from various inferences and in the manner of a tracker) by one who is wandering on the terrain"; and parallel to the *intellectus* the vision of learned ignorance "which elevates someone, in the way that a higher tower does" (Ibidem, p. 470, no. 16)

He claimed that this new way of reasoning according to the *intellectus* was the basis of a new kind of theology, i.e. a new method of thinking God. Since he did not define his way of reasoning in modern logical terms, I scrutinized his main philosophical writings in order to extract those parts that are more meaningful in the light of logic, by exploiting also the several kinds of non-classical logic which were formalized only some decades ago.[2] I will prove that, although he did not present a rigorous logical method (Sects. 2 and 3), he employed several kinds of non-classical logic—i.e. positive, paraconsistent, modal and intuitionist—, which will be distinguished from classical logic by means of plain features (Sects. 4, 5 and 6). His first result was that the series of his names for God prove to be progressive in logical terms. Moreover it will be shown that his reasoning anticipated most of the features of intuitionist logic: (*i*) the systematic use of innumerable doubly negated propositions for which the double negated law fails (Sect. 6); (*ii*) the characteristic reasoning through numerous *ad absurdum* arguments (Sect. 7); (*iii*) the application of

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[2] An analysis in the light of non-classical logic was attempted in 1982. The author wanted to state the new logical law followed by Cusanus; but he failed to formalize it correctly [74, p. 120]. An interpretation of Cusanus's coincidence of opposites through paraconsistent logic was given by Ursic [69]; it is different from the one in the following Sect. 4. An attempt to define Cusanus's logic philosophically was made by Caramella [11]. I leave aside the several philosophical attempts to assimilate Cusanus' logic to Hegel's dialectical logic because [46] decisively refuted them.

the principle regulating this new way of reasoning, i.e. the principle of sufficient reason; which was explicitly enounced by Cusanus; e.g.: "It is not the case that anything is created unreasonably"[3]; (*iv*) as an alternative to the deductive organization of a theory a new kind of organization (Sect. 8); and (*v*) the use of the square of opposition in non-classical logic, obtained in a similar way to the modern 'negative translation' (Sect. 9).

However he wanted to refer his way of reasoning, not to the square of opposition, but to a new logical structure, a "tri-unity of concordance". In Sect. 10 I will show its main logical features. A geometrical representation of it is offered.

No previous knowledge of specific notions of non-classical logic is required from the reader. Rather, the reader has to take into account that new conclusions about Cusanus' thinking will be obtained by overcoming the following five prejudices: (*i*) all non-classical kinds of logic are deviant kinds of logic; (*ii*) doubly negated propositions belong to primitive languages; (*iii*) *ad absurdum* proofs are always invertible into direct proofs; (*iv*) the only systematic organization of a theory is the deductive one; (*v*) the principle of sufficient reason is either a useless or a misleading principle.

2 Cusanus' Logical Effort to Name the Infinite God

Before Western civilization generated mass rationalism and atheism, the existence of God was an indisputable certainty. A basic problem was rather to find the appropriate way to name Him, who is *per se* an unknowable Being. This was a logical problem inasmuch as the ancient logic was a logic of terms.[4]

About the above problem nothing was suggested by ancient Greek logic beyond making use of Aristotle's square of opposition. Usually people attribute positive names to God, hence they exploit the *AffIrmo* side of the square. In the history of mankind innumerable names of God have been accumulated under the thesis A of the square of opposition ("All S is P", where P stands for 'God'). But such positive names lead directly to pantheism, i.e. a primitive conception of religion; whereas the names according to thesis I ("Some S is P"), by looking for specific objects or ideas that may represent God in some way (Truth, Good, Love, etc.), lead to both idolatry and fetishism.

On the other hand, a minority theological school, called 'negative theology', named God through negative words; for instance Infinite, Ineffable, Inaccessible, etc. However, the nEgO side of the square of opposition is not suitable either. Indeed, thesis E ("No S is P") separates the faithful from God; it opposes man's hopes (and rejects the incarnation dogma of Christian theology); whereas thesis O ("Some S is not P"), is a trivial proposition lacking any information about God.

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[3] Cusanus [31, p. 1123, Sect. 9, no. 32]. Here and in the following the relevant negations will be emphasised in order to make easier the recognition of a doubly negated proposition.

[4] I recall that a term in ancient logic is a word which functions as subject or predicate in a proposition. Instead modern logic joins together propositions.

Aware of these difficulties, others have conceived the names of God by means of analogies. However, through them our mind can only produce vague allusions to the divine beings.

The main problem of Cusanus' philosophical theology was exactly that of solving the problem of approaching knowledge of God by means of a suitable name. In an early period of his life, Cusanus named God through positive names, then negative names (indeed he is considered by most scholars as a prominent representative of negative theology) and also analogies.

> In the earlier works Cusanus remained within the general scholastic position that God is the superior being or *primum ens*, but we can only know *that* He is. As to *what* He is, that remains incomprehensible; and we are reduced to inadequate metaphors and analogies. [16, pp. 271–272]

Later, he lucidly illustrated in a booklet that previous names stress our insurmountable distance from God [22, 23,[5] nos. 6ff.]. Among the traditional names of God even the Hebrew tetragrammaton was considered as inadequate by Cusanus [20, book I, Chap. xxiv, p. 40, no. 75]. Indeed, any name cannot be the true name of God, since He, being the first principle, cannot be defined otherwise than by Himself.

"Nevertheless, the mind's acute gaze sees the Beginning more precisely through one mode of signifying than another." [31], Chap. 2, p. 1110, no. 6]. Hence, Cusanus stressed that one can however discover a verbal expression which is less inadequate than all others; he looked for more appropriate ways of conceiving the divine beings.[6]

Moreover, in the same booklet of the year 1445 he appears fully aware of the need to find a new way of thinking God, because His name overcomes the *ratio*:

> He "is not nothing" and "is not something", "Because God is beyond nothing and something" [22, 23, p. 303, no. 9] "He is not ineffable, though He is beyond all things effable... [, He] is not [even] the foundation of the contradiction ... [because He] is prior of any foundation [of contradiction too]...[; and] whatever can be said disjunctively or conjunctively, whether consistently or contradictorily, does not befit Him." (Ibidem, p. 303, no. 10)

Because, as Cusanus puts it, the *ratio* cannot perceive Him just as the sight cannot see the light. (Ibidem, p. 304, no. 14)

Notice that in the above quotations reasoning through *ratio* is qualified by means of all the operations of modern propositional and predicate logic, i.e. conjunctions, disjunctions and of course negation; also the quantifiers are implicitly included inasmuch as he deals with the existence of divine beings and the totality of beings in the world. Hence his operative logical basis is complete. His logical tools well qualified his search for a new method of reasoning about an essentially transcendent being.

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[5] The date of this work can be located in one of the years between 1440 and 1445; hence the book may be contemporary to Cusanus [20].

[6] He was moved to this aim by his belief that man is an *imago Dei*; hence, when a man approaches God, he cannot be deceived in his expectations [20, book I, Chap. 1, I]; rather the problem is to discover the most suitable intellectual method to name God.

3 Cusanus' Surprising Notion of the *Coincidentia Oppositorum*

In the first important book he illustrated a new idea he received when travelling back from the East,[7] the *coincidentia oppositorum,* i.e. the coincidence of the opposites.[8] He considered at the same time a term and its opposite, or even its negation [20, book I]. Here Cusanus was aware of starting a totally new intellectual adventure in reasoning outside Aristotle's logic—in particular, his square of opposition and, in modern terms, outside classical logic—when he claimed:

> ... the endorsement of this [method of a coincidence of opposites] is the beginning of the ascent to the [new] theology ... [which lead to] leap higher [than Aristotle]. [24, Chap. 6, p. 463, no. 6]

It passed unrecognised that an authoritative scholar of mathematical logic, Beth, suggested a similar logical process in his major book, although seen in the light of the construction of "a principle". [6, book I, Chap. 1, Sect 4. "Aristotle's Principle of the Absolute", pp. 9–12].

> A considerable number of arguments in speculative philosophy are based on a certain principle, which is in most cases tacitly assumed. This principle has been applied with remarkably virtuosity by Aristotle. [6, p. 9]

It follows from a plain consideration. Let x and y be in a relation (e.g. x presupposes y). An absolute being X participates only partially in a relation with a finite being (in the above example, because X is not presupposed by anything else); so that X *has and has not* this relation (several instances of this principle, belonging to the theories of Aristotle, Newton, Kant and Marx are listed). The previously emphasised proposition is then formalised in mathematical logic; it is apparent that the formula includes a formal contradiction.

Beth adds the remark:

> It will be clear that the unrestricted applications of the Principle of the Absolute must sooner or later lead to incorrect conclusions. So Kant was undoubtedly right in observing that it cannot be considered as constituting in itself a reliable instrument of proof. [However] Of course in special cases conclusions drawn from the Principle of the Absolute may be correct. [6, pp. 11–12]

As a consequence, Cusanus' *coincidentiae oppositorum* far from being an absurdity, or a merely mystical way of speaking, is the best logical way to approach an absolute being,

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[7] In this intellectual adventure he was supported by faith in the double nature of Christ, who being a true man and a true God, represents the living reconciliation of the world of the natural beings and the world of the divine beings; according to ancient thinking these two worlds are incommensurable, but according to the Christian dogma of the incarnation they can be reconciled.

[8] According to Aristotle [2, Sect. 3, 9–10], there exist four kinds of oppositions: (i) between "relative terms" (for ex., between the double and the half), (*ii*) between contraries (for ex., good and evil), (*iii*) between terms expressing privation and possession (for ex., to be blind and sight), (*iv*) between an affirmative term and its negation (for ex., sitting and not-sitting). The last opposition is a radical one, since in this case the principle of the non-contradiction holds true. This principle was stated by Aristotle in the following terms: "The same attribute cannot at the same time belong and not belong to the same object under the same point of view". [3, βοοκ Ις, πτ. 3]. In fact Aristotle declared that "the first principle" was the following: "It is truly impossible that contrary determinations belong to the same object at the same time." [4].

although this way leads out of classical logic. We see that his challenge to the *Aristotelis secta*, which knew only than classical logic, was surely well-founded and important.

In order to apply this notion in the best possible way Cusanus made use of the mathematical technique of ancient times, proportions,[9] where magnitudes are compared as greater or lesser than a given one. He first stressed that infinite beings do not obey proportions; then, he suggested a new knowledge of God by joining the two endpoints located at infinity; i.e. the coincidence of the two opposites, the absolute *Maximum* and the absolute *Minimum* [20, book I, Chap. 4, p. 10].[10]

In his next important book [21] he illustrated this new method. A *coincidentia oppositorum* is not a mechanical process, but an inventive process, it is a *coniectura* performed by the *intellectus*, the mind's faculty distinct from the *ratio*.[11] Cusanus' aim was to achieve through it as far as possible insights into the realm which stands beyond the "walls of Paradise", constituted by the law of non-contradiction of A and $\neg A$ [26, Chap. 10, p. 700, no. 44]. As Cusanus puts it: "For the limit of every mode of signification that belongs to names is the wall [constituted by the coincidence of the contradictories] beyond which I see You" [26, Chap. 13, pp. 703–704, no. 52].

In order to better appreciate Cusanus' thinking, let us compare it with the basic notions of the ancient logic, i.e. Aristotle's great logical achievements, both the syllogism and the square of opposition.

He considered the syllogism in a new light.

> ... reason makes inferences—logically or reasonably—from an enfolding to an unfolding, doing so by investigating one and the same thing in terms of differences. For example there is present in the conclusion of a syllogism the same thing that is present in the premises; but it is in the major premises in an enfolded way, in the conclusion as an unfolded way ... Therefore the rational domain encompasses the coincidence of the enfolding and unfolding. (Cusanus 1442, book II, Chap. 1, p. 201, no. 78)

In a transcendent view a syllogism also proves to be a coincidence of opposites!

His *coincidentia oppositorum*, when understood as a *concidentia contradictorum,* coalesces the opposite theses of the square of opposition, respectively A and E, I and O; so that the square is now a mere 'segment' joining two theses only. They prove to be separate because each of them represents a different quantifier; either 'there exists', $\exists$, or 'for all', $\forall$ (Fig. 1).

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[9] Fearing that the comparison of two magnitudes produces an incommensurability, Eudox introduced into geometry the mathematical technique of proportions, because it certainly avoids infinity.

[10] Here one notices that Cusanus' notion of *minimum* ambiguously means the least but also the zero, which was never used by him.

[11] In order to acquire mathematical notions from which he obtained *intellectus*' conjectures about God, Cusanus intensively explored the extrapolation at infinity—obtained by the *ratio*—of some finite mathematical notions (e.g. a closed polygon whose number of sides grows to infinity is extrapolated to a circle; then the *intellectus* conjectures these mathematical notions—polygon and circle—to conceive metaphysical beings—God is at the same time polygon and circle—which he often invites the reader to "see"). After claiming that his method was successful also in the mathematical problem of squaring a circle, this application received disqualifying evaluations; for instance Regiomontanus wrote *ridicula*. [45, p. 179, fn. no. 312], [60, pp. 50, 233]. However, after 1459 Cusanus' thought no longer relied on mathematical notions.

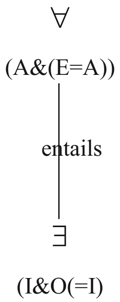

Fig. 1 The square of opposition stretched to a coincidence of contradictories

Hence, the coincidence of opposites leads his mind beyond the walls of Paradise, where the entire world is basically composed of either global entities or singular entities; no space is left for a single negation operation.

In a chapter of one of his books [26, Chap. 10] he described the world he saw through the coincidence of opposites, i.e. beyond the wall of the contradiction law. Cusanus stated that in this world nothing is non-existent:

> You [God] speak to *nothing* as if it were something, and You summon nothing to [become] something, and nothing hears you, because that which was nothing becomes something." [26, Chap. 10, p. 699, no. 42]

This description agrees with a kind of logic rejecting any negation.

Mathematical logic was born in the second half of the nineteenth century. Around the year 1900 new kinds mathematical logic arose; but it was not before the '30s that they were recognised as kinds of logic that were independent of classical logic. At present there exists a mathematical logic representing the lack of negation. It is the positive logic, which is less than minimal logic (which adds one more axiom), which in its turn is less than intuitionist logic (obtained by adding one more axiom), which eventually is less than classical logic (obtained by adding the law of the double negation) (Grize 1970, pp. 208–210). Hence, Cusanus definitely abandoned positive and negative theologies, which both make use of classical logic, in order to investigate a new kind of theology belonging to a non-classical logic.

The above implicit way Cusanus appealed to positive logic represents a first way of experiencing new intellectual situations according to non-classical logic. In the following sections his experiences of three more ways will be illustrated.

4 Cusanus' Second Way of Thinking the New Logical Situation: Paraconsistent Logic

Since in classical logic all propositions follow from a contradiction and thus the entire discourse is useless, logicians have maintained that a contradiction invalidates the entire system to which it belongs. However, Cusanus opposed to his critics that his introduction of the coincidence of opposites is not invalidating his discourse. (see for instance Cusanus 1442, book II, Chap. 1, p. 201, no. 78; [30, Sect. 13, p. 705, no. 55]).

As a fact, in recent times some mathematical logicians have presented logical systems which although including contradictions do not explode in the totality of propositions. This kind of mathematical logic is called paraconsistent logic (see G. Priest, K. Tanaka, Z. Weber, 2013).

The founder of paraconsistent logic, N.A. Vasiliev has obtained such a system from a criticism to Aristotle's square of opposition; instead of four theses, he stated the following three propositions: "S is A", "S is not A", "S is and is not A"; he called the last one the "indifferent judgement" and he considered it as the characteristic proposition of his logic [5]. Indeed, the most simple way to deal with the paraconsistent logic is to consider three values, i.e. true, false and true and false [63, Sect 3.5]. We recognise that the last value precisely represents a coincidence of contradictories in the sense intended by Cusanus; in fact Vasiliev recalled him [72, p. 332].

An inspection of Cusanus' texts shows some instances of Vasiliev's indifferent proposition; each of them is marked by two asterisks *, one before and one after it.

> For to the question whether God exists there can be no more unrestricting response than that... it is not the case that * either He exists or that He does not exist and... He both exists and does not exist* [21, Sect. 5, p. 172, no. 21].

> It is neither the case that * He is named or is not named nor the case He both is named and is not named.* [22, p. 303, no. 10].

> It not the case that *He is nothing or that He is not nothing; nor He both nothing and not nothing *. [22, p. 303, no. 11]

> Next you see that the contradictories are negated of the unnameable Beginning, so that * it is not the case that *is not* and is not the case that is *both is and is not*,* and not is not the case that it *either is or is not*. [28, no. 19]

Several scholars saw in Cusanus' coincidence of opposites no more than a mystical contemplation which is typical of an extreme Platonism. Instead, the above propositions show that Cusanus' way of thinking, although including coincidences of opposites, may be formalised at least in paraconsistent logic.

Since I interpreted the paraconsistent logic by means of a simple technique of non-classical logic, [37], I will come back to this kind of non-classical logic in Sect. 6, where I will deal with the intuitionist logic.

5 A Third Way of Cusanus' Dealing with the New Logical Situation: The Modalities

Cusanus invention of God's names through coincidences of opposites achieved a decisive result when he invented the new name *poss-est* (= *posse* + *esse*, or *posse* and *esse*), i.e. a word directly expressing a coincidence of opposites, here the two opposite aspects pervading the complete world, i.e. possibility and actuality.[12]

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[12] These two words also recall the two opposite philosophies of the ancient times, Plato's and Aristotle's, that Cusanus tried to reconcile by writing each of his several books according to either one [65, pp. 15ff.].

> … *there* [in the *Possest*] the being and the not being are not contradictory of each other.- nor do any other opposites which either affirm or deny a distinct state of affairs.” [30, p. 928, no. 26]

In the last two years of his life, he worked on the previous name, *Possest*, so as to make it independent of a coincidence of opposites.[13] He obtained the two new names *Posse* and *Posse ipsum* [33].

By including the word *posse*, the above names all belong to the well-known modal logic. It is surely different from classical logic, although it was to some extent employed also by Aristotle. Notice that by making use of negations negative theology does not manifest any change in the kind of logic; instead, to call God by means of modal names overcomes negative theology and introduces Cusanus into a new logical world, which is, as we at present well know, non-classical.

Modal logic was formalized only a century ago. It is remarkable that four centuries before this event, through the above names of God, Cusanus places modal logic above classical logic, which is capable of naming God positive or the negative names alone.

Here we meet a deeply-rooted prejudice, according to which kinds of logic other than classical logic should be considered “deviant kinds of logic” [51]. But in last decades there have been great advances in these kinds of logic; they have proliferated, knowledge of each of them has greatly improved and they have been applied to several fields, first of all to computer science; so that at the present time classical logic—with its absolute distinction between true and false—appears to be an extreme logic.

However, it is a hard task to formalize the above name in present modal logic.[14] After five centuries of logical progress, Cusanus’ modal names still constitute remarkable formal problems. However the name *Possest* is rather an operation that translates from the modal logic of the *posse* to the classical logic of the *est*; or equally, from the modal square of opposition to the classical square of opposition. Recently a general translation from modal logic to classical logic has been introduced [7, pp. xii, Sect. 2.4 and p. 120]. It is accomplished by determining the value of a variable added in order to represent the modality. *Possest* is the best linguistic expression for representing this translation, since it determines the values (*est*) of the variable-modality (*posse*).

However, notice that the modal words “possible”, “necessary”, “must”, etc. are all in an intuitive sense equivalent to double negations. For instance, the first word is equivalent to: “It is <u>not</u> true that it does <u>not</u> exist …” ($\neg\neg\ \exists x$, which is not equivalent to “Being”, or “Exists”). Hence, the above modal names may be all translated into doubly negated

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[13] Cusanus did not suggest by which specific logical steps he achieved this name. One may suppose that the public charge of pantheism made by Wenk [73] led Cusanus to stress as far as possible that God is not the World, but is *in* the World; where the word “in” implies a modality to be discovered.

[14] Each of Cusanus’ new names of God implicitly, but also in an essential way, includes a universal quantifier. Hence, in modern logic each of the above names properly belongs to predicate logic. But the formalisation of the predicate calculus of modal logic is disputable [47]. In addition, the suitable quantifier for God is dubious; it may be either total or existential. Moreover according to Cusanus’ philosophy the main feature of God is, before existence or omnipresence, Oneness, to which no logical modern operator corresponds.

propositions.[15] This fact allows us to consider modal names together with the doubly negated names which will be considered in the next section.

6 The Fourth Way of Cusanus' Dealing with the New Logical Situation: The Double Negations of Intuitionist Logic

Two years after the book *De Possest*, Cusanus progressed to a new original name of God. Again he devoted an entire book in which he both proposed and illustrated this new name: *De li non aliud* (*On the Not-Other*). Most scholars have considered it to be Cusanus' major work on the subject of how to know God, although he excluded it from the collection of his books (maybe in order to add further improvements to it or to revise all previous books in the light of this one). It was printed in the year 1888 and eventually it received a first critical edition in 1944 [17, pp. 203–204].

Here we have to overcome a second deeply rooted prejudice according to which only primitive languages make use of double negations.[16]

When presenting the above name, Cusanus declared that he had come to it after long intellectual work:

> It is that which for many years I sought by way of the coincidence of opposites—as the many books which I have written about this speculative matter bear witness. [31, Chap. 4, pp. 1113–1114, no. 12; further positive qualifications are added in Chap. 19, p. 1149, no. 87]

Furthermore, he defined it as "the most precise" name [31, Chap. 2, p. 1110, no. 5] (although subsequently he wrote books insisting on new names of modal kinds: *posse, posse ipsum*).

However the name is not presented as a coincidence of opposites, but as anterior to the same "other" [31, Chaps. 1–4];

> It is seen prior to all positing or removing." [31, Chap. 4, p. 1114, no. 12]; in other terms, the negation does not exists before the not-other; thus, it is not obtained by negating the latter word. . . . which I understand Not-Other . . . cannot be expressed in different ways by different [words]. [31, Chap. 4, p. 113, no. 11] . . . because God is not *other* than [any] other, He is Not-other, although Not-other and other seem to be opposed. [31, Chap. 6, p. 1118, no. 21]

Indeed Cusanus emphasised that it is inappropriate to attribute a contradictory nature to negation, other, with respect to its corresponding double negation, not-other:

> . . . if someone had asked Aristotle, "What is other?" he surely could have answered truly, "It is not other than other." And, if the questioner had thereupon added, "Why is other other?" Aristotle could rightly have answered as at first, "Because it is not other than other." And thus, he would

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[15] At present these intuitive equivalences are formalised by translating modal logic *via* the S4 model into intuitionist logic [15, pp. 76ff.].

[16] Horn [53, pp. 82ff] and [54, pp. 111–112]. For a long time linguists ostracised double negations; this explains why the importance of DNPs was rarely noticed. In the following I will disregard an analysis of the various kinds of doubly negated propositions, because I assume that the ancient philosophers used this linguistic figure by intuition, i.e. by referring more to the intended semantic than to formal rules.

> have seen that not-other and other do not contradict each other as contradictories. [31, Chap. 19, p. 1150, no. 88]

He was so convinced that he had entered a non-Aristotelian kind of logic that he persuaded even a revived Aristotle to agree with him.

The double negation in Not-Other makes manifest that Cusanus’ thinking did not belong to Aristotle’s square of opposition, which includes at most single negations. Hence Cusanus’ new name went beyond the scope of Aristotle’s square.

In addition, notice that this new name of God is presented by Cusanus as a double negation which is not equivalent to the corresponding affirmative word *Idem,* i.e. “the same”: “But notice that “Not-other” does not signify as much as does “same”.” [32, Chap. 14, p. 1304, no. 41] In fact, the nature of this name is not conclusive but explorative-inductive, it well represents a *coniectura* produced by the *intellectus*.

> PETER: I cannot mentally conceive what It [= the Not-other] is. / NICHOLAS: If you were able to conceive it, then by no means would it be the *Beginning-of-all-things*, which signifies all in all. For every human concept is a concept of some one thing. But Not-other is prior to [all] concept since a concept is not other than a concept. Therefore, Not-Other may be called the Absolute Concept, which is indeed seen mentally but which, notwithstanding, is not conceived. [31, Chap. 20, p. 1152, no. 94]

So to Cusanus it was clear that he had left both ancient Greek thought and the dominant Western thought; it is only in Eastern culture that one can find similar conceptions. What one scholar wrote is fully justified: “It is possible to support the view that this name is the most original one by Cusanus”. [58, p. 181].[17]

Among the names suggested by Cusanus, *Non Aliud* appears to be very important also for reasons pertaining to modern logic.

First of all, notice that if the proposition P holds true, then it is equivalent to $\neg\neg P$ and classical logic holds true; for instance, “I have five euro”, implies that “It is not the case that I do not have five euro” since it expresses a well-verifiable fact. It is the reverse implication which is here called into question.[18] The name not-other is a double negation lacking, as Cusanus stressed, a corresponding affirmative word, supported by verifiable evidence; indeed, the content of either “the same” or “equal to everything” proves to be idealistic for a man, because its verification requires an infinite number of tests (“... precise equality befits only God .... [whereas in our world] equality between different things is *actually* impossible.” [20, book II, Chap. 1, p. 58, nos. 91, 92].

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[17] The great importance of this invention for the history of philosophy was emphasised by [59, p. 171]: ““Cusanus’ entire thought may be considered as a renewed and continual effort to link together the first and the second hypotheses of Plato’s dialogue [*Parmenides*] through the very difficult task of thinking that dizziness of the thought which is constituted by the One: from this viewpoint the Not-Other is nothing other than one of the most original re-formulations that the history of the Western thought never knew of the tremendous question put by the father of philosophy in his most enigmatic and troubling work, the *Parmenides*.”

[18] For clarity’s sake, let us consider one more instance of such a proposition. A Court judges a defendant “acquitted owing to insufficient evidence of guilt”; i.e. the Court did not collect sufficient evidence for deciding either to send the man in prison or to give him the freedom. Since the above proposition is not equivalent to the corresponding affirmative proposition (“innocent”), in this case the law of the double negation fails.

On the other hand, in order to verify the truth of the doubly negated name one has to verify that, once a being is given as “other” than God, the difference between “”other” and God can be expressed with a “not”.

In sum, the most important feature of the name *non aliud* is to be a double negation lacking an equivalent affirmative verbal expression, simply because no one can obtain objective evidence of God’s positive existence. Without this evidence Cusanus maintains that the most correct way to name Him is to have recourse to a double negation. Surely his result belongs to a non-Aristotelian logic.

Let us again recall that ancient logic was a logic of terms. Since in classical logic the terms all refer to reality, more or less idealized, the use of double negations without a corresponding affirmative term was unjustified. (This explains the ancient tradition of the dictum in all languages: “Two negations affirm”, which correctly represents the double negation law of classical logic). Here, Cusanus’ innovation of the Not-other anticipated the relevance of the double negations in the modern logical calculus of propositions.

A very remarkable fact is, that the above mentioned failure of the double negation law has to be considered the most characteristic feature of intuitionist with respect to the classical logic.[19] Hence, the name Not-Other for God, owing to the failure of this law, surely belongs to intuitionist logic.

Cusanus introduced this name Not-other not through a coincidence of opposites, but by pondering on the definition. Let us recall that according to Aristotle

> The definition is a discourse which expresses the essence. In such a case it is provided either a discourse in place of a name, or a discourse in place of a discourse [; in this case the form of the definition is the following one:] ... is.... [52, book vi]

On the other hand, Cusanus did not mean the definition in the sense of an identification of the *definiendum* with the *definiens*; of course, the references of the definition are only when one is dealing with beings of reality. Cusanus, however, meant a Socratic definition, i.e. as an *ad excludendum* process: “we remove from the excellences of the cause the defects we find in what is caused“. [19, p. 330, no. 5]. Remarkably, this proposition is a doubly negated proposition which is not equivalent to the corresponding affirmative one (which in Aristotle’s words is: “...to express the essence...”) for lack of operatively based evidence of the latter proposition. I will call this kind of doubly negated proposition a DNP.

Cusanus’ thought relied heavily on DNPs; an inspection of Cusanus’ texts shows that each of them includes a lot of DNPs.[20] For reasons of space I quote some relevant

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[19] Prawitz and Malmnaes [62], Grize [50, pp. 206–210], Dummett [43, pp. 17–26], Troelstra and Van Dalen [68, pp. 56ff]. Notice also that according to modern logic the failure of the principle of the excluded middle, which Cusanus sometimes referred to, is equivalent to the failure of the double negation law.

[20] Drago [38, 39]. Notice that sometimes a single word includes two negations; e.g. “in-variant” (which does not mean “constant”); moreover, the word “only” represents a double negation, because it is equivalent to “nothing other than...”. In addition, sometimes one has to discover a covert negation within a negated proposition. For example, there is only one way to resolve the paradoxical aspect of the celebrated title of his first book, *De docta ignorantia*; it has to be completed by a second negative word: *Ignorantia docta [ex infinitis rebus]* (An ignorance learnt by infinite beings) [20], Title; the same for the title of the first chapter: “How it is that knowing is not knowing [the infinite beings]”). These two words

instances only. The above mentioned proposition: "… we have no certain knowledge except mathematical knowledge" ($\neq$ mathematics is our certainty) is a DNP. Remarkably, Cusanus defined both notions transcending the *ratio*, i.e. *intellectus* and *coniectura*, through DNPs. The *intellectus* is "the otherness of the infinite Oneness" ($\neq$ the identity of finite Oneness) [21, book II, Chap. 16, p. 249, no. 167]. The same holds true for the definition of its product: ”A surmise is a positive assertion that in the otherness shares not in its totality the truth as it is [in itself] ($\neq$ is a partial truth)."[21] Hence, both definitions belong to intuitionist logic. Also remarkable is the proposition reiterated in several books: "For the intellect apprehends nothing which it has not found in itself." ($\neq$ what is found in itself) [32, Chap. 29, p. 1332, no. 86].

In the previous section we remarked that modal logic may be considered equivalent to intuitionist logic. Moreover, according to a previous work [37] one of Vasiliev's indifferent propositions may be translated into a DNP.[22] Hence Cusanus' names may be compared within the intuitionist logic only. Since even at present time modal logic is not sharply defined, [47] modal names are surely less precise than an intuitionist name (Also Cusanus [31, Chap. 7, p. 1120, no. 23] stated: "If Not-other ceased…, [then] the actuality and the possibility of the beings which Not-other precedes cease.") Even less precise are the indifferent propositions of paraconsistent logic. Hence, also according to the all above-mentioned kinds of non-classical logic, the word "Non-Aliud" proves to be, as Cusanus claimed, more accurate than the others. By means of this name Cusanus' thinking was decisively introduced into the realm of non-classical logic.

If the reader suspects that thinking through DNPs is cumbersome and therefore very rare, he should consider that an analogy is also DNP, because it is equivalent to "It is not the case that the two things at issue are not equal….".[23] Hence, by including analogies, names and adjectives of the modal kind and DNPs, natural language makes a great use—contrary to the above mentioned prejudice—of propositions belonging to non-classical logic.[24]

(*docta ignorantia*) being a characteristic sentence of the scholars of so-called "negative" theology, one may suspect that; when these scholars wrote merely negative propositions they often unwarily meant DNPs. It is significant that also in modern mathematical logic it is called "negative translation" the translation of formulas through double negations [68, p. 57].

[21] I preferred Vescovini's translation [18, p. 234] of the words "*Coniectura est positiva assertio in alteritate veritatem uti est participans"* to Hopkins' one (Cusanus 1442, book I, Chap. 11, p. 190, no. 57).

[22] [37] I interpreted Vasiliev's three propositions respectively as follows: $\neg\neg P \rightarrow P$; $\neg\neg P$ *does not* $\rightarrow P$; and $\neg\neg P \rightarrow P$ *and* $\neg\neg P$ *does not* $\rightarrow P$.

[23] Incidentally, this fact proves that Augustine's suggestion that God can be named through analogies actually represents a unconscious way to escape from classical logic in order to introduce theology into non-classical logic.

[24] In fact, some interpreters of his books, although ignorant of non-classical logic, were aware of this way of Cusanus' of conceiving logic. E.g. [67, pp. 147–151] illustrated the concept of "not-other" through some typical characteristic features of reasoning in intuitionist logic. A similar penetrating insight is in [9], who assumed infinity and conjecture as interpretative categories; the latter is the best representative of DNPs in Cusanus' thought. He rightly stressed that in the book *Non Aliud* ”theology is derived from logic." [9, p. 147].

7 Cusanus’ Searching for a New Way of Reasoning: His *Ad Absurdum* Arguments

In spite of his departure from Aristotle’s logic, Cusanus claimed to be *reasoning* in a rational way when he was transcending the real world to obtain some features of the metaphysical One. He claimed this point also in the title of Chap. 3 of one of his last books, where he remembers how he developed his ideas: “The line-of-reasoning by which reason pursues [wisdom]”. [32, Chaps. 2–5].

He never made use of the common tool for reasoning in ancient times, i.e. the syllogism. He rather reasoned according to intuitive implications, such as “to be before to . . . ”.[25]

Let us rather recall that the definitions of both *coniectura* and *transcessus* are two DNPs. Since from a single DPN no affirmative proposition follows, they have to govern a number of DNPs. Hence, Cusanus’ reasoning has to start from a DNP, $\neg\neg P$, and then continue through further DNPs. This chain of DNPs has to conclude in the only way intuitionist reasoning can, i.e. an *ad absurdum* argument (AAA).

In point of fact each of Cusanus’ books’ without assuming general axioms, but only common knowledge, compose DNPs into units of reasoning, each one being a chain of DNPs ending with an AAA (often revealed by the words “otherwise” in the argument, and after by the word “therefore”).

For instance, in his first important book [20, book I] the 53 lines of the first section include 20 DNPs. The mere sequence of these DNPs is enough to preserve the logical thread of Cusanus’s text, provided that one implements them by adding a few connecting propositions. In other words, the sequence of all DNPs circumscribes the logical content of the text [38].

These 20 DNPs compose three AAAs. The first concerns the ability of our mind to attain the truth (in the following two propositions I depart from Hopkins translation):

> “. . . the inborn judgement faculty, satisfying the aim to know, ensures that this attraction is not vain . . . ”
>
> “. . . if the judgement was otherwise, it would be successful perchance [= not always], as when sickness misleads the taste or an [false] opinion misleads reason”.

Two more relevant instances of AAA are the following ones:

> Suppose someone sees how if Not-Other were removed, it is not the case that either the other or nothing would remain, since Not-other is the Nothing[ness] of nothing. Then he sees that in all things Not-other is in all things and nothing is nothing. [31, prop. vii, p. 1114][26]

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[25] Remarkably, he is able to symbolize the notions and concepts; in Cusanus [31, Chaps. 15–16] he calls A “what is signified by Not-other”. But this is not sufficient to prove he was reasoning. Nor is a *coniectura*, being only one proposition, enough to prove that he is reasoning in any kind of logic.

[26] A more subtle instance of an AAA is the following one (where I inserted some words to make it more apparent): “NICHOLAS: Tell me, then, what is Not-other? [Do you accept the following *absurdum*:] Is it other than Not-other? FERDINAND: [Absurd! Since it is] Not at all other. NICHOLAS: So [it is the same than] Not-other.” [31, book I, p. 1109, no. 4].

> The loftiest level of the contemplative reflection is Possibility itself, the Possibility of all possibility . . . For how would [intellect's activity of contemplation] be possible <u>without</u> possibility<u>?</u> ([33, p. 1431, no. 17], Introductory statement; the insertion specifies Hopkins' insertion "contemplation". Notice that the question mark alludes to a negation: "<u>No</u> way.")

Here we have to overcome a third logical prejudice. Although recently natural deduction proved that one can reason in parallel in either classical or intuitionist logic, [61] several logicians maintain that true reasoning belongs to classical logic only (Haack 1970, pp. 37–38). A more subtle version of this prejudice concerns the typical argument pertaining to intuitionist logic, i.e. the AAA. It is currently maintained that such a proof can be inverted into a direct proof, provided that the thesis is exchanged with the conclusion [64, p. 15], [48]. But this exchange presupposes that the DNP which concludes an AAA is translated according to the double negation law of classical logic into an affirmative proposition. Hence, without this application of the classical law of the double negation the AAA is not invertible.

In sum, in his books Cusanus several times reasoned rationally with ingenuity through both DNPs and AAAs which are all governed by intuitionist logic. Cusanus dealt exhaustively with each subject; an occurrence of an AAA is the most accurate mark of this rational reasoning. I conclude that Cusanus' new kind of theology includes rational reasoning.

8 Cusanus' Search for a New Way of Reasoning: His Use of a New Model of Organisation of a Theory

In previous works I have shown that several important scientific theories—from S. Carnot's thermodynamics to Lobachevsky's non-Euclidean geometry, to Einstein's theory of quanta, to Kolmogorov's formalization of intuitionist logic—were organised by their respective authors in a way that was alternative to the well-known deductive model [40].

Here we have to overcome a fourth prejudice which has been perpetuated by almost all mathematicians admiring Euclid's system and eventually elevated to an a priori premise by Hilbert, i.e. there exists only one systematic organization of a theory, the deductive one. This prejudice was contradicted in formal terms by Goedel's theorems, stating that this kind of organization is not able to represent even the simplest mathematical theory, arithmetic [71, p. 356].

A comparative analysis of the original texts of the above mentioned scientific theories suggests the following characteristic development [40]. A theory of such kind starts not from axiom-principles, but from a universal problem which is unsolvable by common tools (I call this alternative organization problem-based: PO).

The theory then looks for a new scientific method capable of solving the problem. It then reasons through DNPs which are grouped into units of argument; each of these units starts from a sub-problem which is then solved by means of an AAA; whose conclusion—again a DNP works as a methodological principle for the next unit of argument. A final AAA concerning the main problem concludes a doubly negated predicate $\neg\neg T$, which in the intuitionist square of opposition—obtained by the ‘negative translation’ of the classical one—represents thesis A.

At this point, the author, in the belief that he has collected enough argumentative evidence, translates the above conclusion into the corresponding affirmative predicate T representing the thesis A of the square of opposition of classical logic. The logical formula of this translation of the universal predicate, i.e. the thesis A, formally implies the translation of all the theses of the intuitionist square of opposition into the corresponding theses of the classical square; and hence the translation of the entire intuitionist logic—which governed the reasoning in the previous part of the theory—into the classical one.

This author’s move appears to be justified by an application of the principle of sufficient reason (PSR), which is formally represented by the same logical formula as the previous translation from the intuitionist thesis A to the classical A; hence, it represents in the most general terms, i.e. for all possible theories, the same previous translation.[27]

From the new predicate the author then derives in classical logic all the relevant consequences. Hence, this translation of both the final DNP and the logigcas a whole amounts to a leap from a problem-based organization theory to a deductive organization.

Although Cusanus was a cardinal who wrote on Christian dogmas, no theory of his philosophical books is of the deductive kind, i.e. derived from some assured truths. The common theoretical model of all these books is easily recognised. Rather than being based on a few axioms-principles, each of his books looks for a new method able to solve a basic problem—often the problem of which is the best name of God—through AAAs and the application of the PSR.

E.g. in the above considered first instance of an AAA the first proposition is the universal thesis $\neg\neg T$; the subsequent rejection of the absurdum (“successful perchance”) allows the author to state the corresponding T: “Wherefore, we say that a sound, free intellect knows to be true that which is apprehended by its affectionate embrace.” [20, book I, Sect. 1, p. 5, no. 2]

Let us consider one more instance of this translation- It is constituted by the 20 “Propositions” which at the end of the book *De Non Aliud* summarise the entire content of the theory. Also here, in order to solve the problem of naming God, Cusanus reasoned through DNPs, which are linked together in three “units of reasoning”, each including an AAA (in respectively the propositions VII, VIII and IX). In prop. IX these units eventually conclude a universal intuitionist predicate. “Whatever the mind sees, it does not see without Not-other.” The achieved universality leads the author to translate the final predicate into the corresponding affirmative one. “Therefore, [the mind] sees . . . that all things have from Not-other their names and quiddities and whatever else they have.” Afterwards, from the new predicate he derives, according to classical logic, all possible consequences, i.e. the subsequent eleven propositions.

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[27] On the role played by this principle see my paper [40].

When he translated from the universal predicate to the affirmative proposition, Cusanus appealed only implicitly to PSR. But in the book *De non Aliud* he stated it twice: "… for it is not the case that anything is created unreasonably." [31, Chap. 9, p. 1123, no. 32]; " "Nothing is in vain" ([31], last of the 20 "Propositions"); but also in (Cusanus 1463, Chap. 11, p. 1404, no. 35) we have: "… nothing is done without a reason…" These clear-cut enunciations are not surprising since, according to a metaphysical view, the previous name for God, *Posse = est*, appears to be the most general expression of this principle.

Here we meet a fifth prejudice which concerns the PSR. This principle is commonly disqualified as a vague philosophical principle. Certainly, an unrestricted application of it leads to fanciful consequences. But in some cases one obtains a plausible hypothesis which can be tested with experimental data. Indeed, in history its validity was supported by e.g. Enriques [36, 44] and it was exploited by several scientific theories; e.g. it was applied by Markov when he founded the theory of the constructive mathematics.[28]

In conclusion, in his writings Cusanus introduced non-classical reasoning which presents all the foundational features of an alternative organization of a scientific theory. Hence he rightly claimed to have introduced an entirely new theoretical system, i.e. a third kind of theology beyond positive and negative theology [20, book I, Chap. 24–26].

9 Cusanus' Implicit Introduction of a Non-classical Square of Opposition

In *De Non Aliud* Cusanus stressed that the new name, Not-other, characterises any being, because one can define e.g. sky as "…the sky [is] not other than the sky" [31, Chap. 1, p. 1109]. Hence, Cusanus changed each thesis of the square of opposition by inserting the two words, "not other" before "P"; in this way he obtained a translation of Aristotle's square of opposition into a new one. In modern logic we have Kolmogorov-Goedel-Kuroda's 'negative translation' of the classical logic into intuitionist logic. This translation adds (according to some rules) a pair of negations to each predicate. In Kolmogorov's version one is "simultaneously inserting double negations before all subformulas of the predicate (including the predicate itself)".[29] In comparison with it, Cusanus's translation lacked the addition of a double negation before the quantifiers (which were not distinguished as such before the nineteenth century) (Table 1).

In sum, Cusanus' intuitionist-like square of opposition is produced according to a clever, although inadequate, translation. In any case, in the history of logic Cusanus was the first to introduce a substantial part of the intuitionist square of opposition.

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[28] Markov [57, p. 5]. In this case the application of the PSR was recognised by [43, p. 22]. It is remarkable that Markov suggested restricting the application to a decidable predicate resulting from an AAA.

[29] Troelstra van Dalen [68, p. 59]. A second translation suggested by Goedel-Gentzen is performed by adding two negations before each prime and in addition by substituting $\neg\forall\neg$ for the existential quantifier and by doubly negating the LEM (Ibidem, p. 57). A third translation, Kuroda's one, "is obtained as follows: insert $\neg\neg$ *after* each occurrence of $\forall$ and in front at the whole formula." (Ibidem, p. 59).

Table 1 Three squares of opposition

Aristotle's square of opposition		KGGK's translation of it in Intuitionist logic		Implicit Cusanus' translation of it	
$\forall$	$\forall\neg$	$\forall\neg\neg$	$\forall\ \neg$	$\forall\neg\neg$	$\forall\ \neg$
$\exists$	$\exists\ \neg$	$\neg\neg\ \exists$	$\neg\neg\ \exists\neg$	$\exists\neg\neg$	$\exists\ \neg$

However, in Cusanus's square the theses including the total quantifier are the same as the intuitionist ones; whereas Cusanus' thesis O is classical and also Cusanus' thesis I is stronger than the intuitionist one, as is shown by the table in [43, p. 29].

In conclusion, did Cusanus correctly reason according to intuitionist logic? The answer is "yes" for propositional calculus, but it is "no" for predicate logic, which is the specific logic for dealing with infinite beings, owing to the differences in the two kinds of square of opposition. However, since Cusanus' reasoning referred more to the existential predicate applied to the divine beings, to the total predicate for their global features, the differences of his square from the intuitionist one do not matter. Therefore, according to predicate intuitionist logic the inadequacy of Cusanus' logical translation does not invalidate his way of reasoning.[30]

10 The Search for Oneness in Pluralism: Defining the Tri-unity of Concordance

Although he stated that "... when oneness proceeds into otherness, it stops at number four." [21, book II, Chap. 6, p. 212, no. 99] and saw the number four in a multitude of activities of the mind [25, book II], he did not explicitly refer to the square of opposition. Rather he, as a neo-Platonist and hence a henological metaphysician, saw Oneness as the highest principle, capable of composing any multiplicity.[31] According to Cusanus oneness should be seen at the same time in God, in the *intellectus*, in the *ratio* and in senses (Cusanus 1442, book I, Chaps. 5–9). Hence he wanted rather to characterize a logical process culminating in a oneness.

> By partaking of the One all things are that they are... Therefore, you have need of no other consideration than that you seek out the identity that is present in the diversity of the things which you are to investigate, i.e. that you seek out the oneness that is present in the otherness. For then you will see, in the otherness of the contracted beings, the "modes", as it were, of Absolute Oneness. [21, book II, Chap. 1, p. 198, no. 71]

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[30] Moreover, Cusanus also reasoned according to the modal square of opposition, because the last quoted AAA is modal (see also [30, p. 916, no. 4, p. 928, no. 27]). However, the modal syllogism is not present in his writings; actually, it was rejected by most Humanists [10, p. 81].

[31] Indeed he claimed e.g. to have obtained the concordance of both philosophers [30, p. 925, no. 21] and theologians [30, p. 925, no. 21].

But to what multiplicity does oneness refer? He was interested in a multiplicity which can represent a totality, although in reductive terms. Surely, not a two-fold multiplicity which represents a contraposition, but a threefold multiplicity, as in geometry a triangle represents the minimal figure including space, the minimal polygon.[32]

In the book *De Li Non Aliud* he achieved this logical aim at a linguistic level. Having defined God as Not-other, he obtained a surprising name for the Tri-unity: "The not-other is no other than the not-other".[33] ([31, Chap. 5, p. 1116, no. 18]; also in [32, Chap. 14, p. 1303, no. 40]) Through the threefold reiteration of the same words it expresses at the same time three beings and oneness.

In *De Non Aliud* the Cardinal is asked why "the trine and one God is signified by "Not-other", when the not-other precedes all numbers [and hence, the number three and the number one]" ([31, Chap. 5, p. 1116, no. 18]; I changed Hopkins' "since" in "when"). He believed that he had shown through an AAA that in this case the three is not different from the one:

> All things are seen from what has been said—seen on the basis of a single rational consideration ... the Beginning, which is signified by "Not-other", defines itself. Therefore, let us behold its unfolded definition: viz. that Not-other is Not-other than Not-other. If the same thing, repeated three times is the definition of the First, as you recognize [it to be], then assuredly the First is triune—and of no other reason than that it defines itself. If it did not define itself, it would not be the First; yet, since it defines itself, it shows itself to be trine. Therefore, you see that out of the perfection there results a trinity which, nevertheless, (since you view it prior to other) you can neither number nor assert to be a number. For this trinity is not other than onenesss, and [this] oneness is not other than trinity. For the trinity and the oneness are not other than the simple Beginning which is signified by "Not-other". [31, Chap. 5, pp. 1116–1117, no. 18]

The idea of this structure is expressed by Cusanus through the differences between the three cases of a proposition, i.e. affirmative, negative and doubly negated:

> Now it is evident that those who do not attain unto the fact that *not-other* is not the *same* and that *not-same* is *other* cannot grasp the fact that [the three qualifications of both the trinity of God and the mind, i.e.] Oneness, Equality and Union are the same in essence but are not the same one another. ([29, book II, Chap. 8, p. 1023, no. 107]; see also [31, Chap. 5])

If we apply the same sentence structure to different affirmative words we obtain at most a tautology and when we do the same to different negative words we obtain merely nonsensical propositions. Hence, there exists no other verbal expression that includes both features of trinity and unity. The double negations play an essential role in the human mind in achieving close approximations to Tri-unitiy. Among them the Not-other is the only suitable one.

One more reason for searching for tri-unity is that also the mind, which defines itself, is tri-unitarian [25, Chap. 11, p. 574, no. 133]

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[32] In his opinion, for this same reason God cannot be quaternary [20, book I, Chap. 20].

[33] Since in intuitionist logic three negations are equivalent to one, the complex of six negations is equivalent to two negations; hence, we have remained within intuitionist logic.

> Just as the First Beginning of all things, including our mind, is shown to be triune, ... so our mind ... makes itself to be the triune beginning of its own rational products. For only reason is the measure of multitude, magnitude and of composition ... Therefore our mind is a distinguishing, a proportioning and a combining beginning." (Cusanus 1442, book I, Chap. 1, pp. 165–166, no. 6)[34]

Hence, Cusanus was interested in a one-three-fold logical structure expressing a definition defining itself. This result coincides with the trinity of the three Persons of the Christian God. But Flasch stressed that:

> The structure [illustrated by Cusanus] matter-form-connection is a constraint for every mind; it expresses the nature of the thought. On this solid ground Cusanus *constructs* his Trinitarian philosophy, or more exactly *this is his philosophy of the Trinity*. This philosophy appears not as if it wants to make comprehensible or plausible the Christian faith in the Trinity. It intends itself as a proof which is immanent to each kind of logic and science, and even to whatsoever spiritual activity. ([45, pp. 316–317]; emphasis added)

In sum, he wanted to refer to a *concordance* of the Tri-unity. Notice that in classical logic this verbal expression is also a twofold oxymoron, because the three cannot be the one and this contradiction cannot be considered a concordance.

Only in this light was Cusanus interested in the basic tool of ancient reasoning, the syllogism. In a work before the *De non Aliud* he considered a particular syllogism where the three propositions are all universal, each without alterity. Their roles are compared to both the three Persons of the Trinity and the three faculties of the intellective soul (memory, intellect and will; they are illustrated five pages later, in no.s 26ff.).

> Therefore, in the oneness-of-essence of this syllogism of three propositions that are equal in all respects there shines forth the essential oneness of the intellective soul—shines forth as in the intellective soul's logical or rational work. [27, 28, p. 846, no. 12]

Hence, Aristotle's basic tool of the *ratio*, a syllogism, represents at best an analogy to the triunity.

I close my long study by representing through a geometrical figure Cusanus' thinking about this logical structure—where I name the third person of the Trinity with a double negation, Im-material (a word which seems to me to be more appropriate than the inane affirmative Spirit, holy or not).

The figure represents the relationships among the three Persons by means of only the basic logical operations employed by Cusanus in [31], i.e. either negations ($\neq$) or double negations ($\neg \neq$). Notice that they are less numerous and very different from the relationships—alternate/subalternate, contrary/subcontrary, contradictory—between a pair of theses in Aristotle's square of opposition. They, however, enjoy the self-duality property for addition of double negations (Fig. 2).[35]

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[34] By means of an independent analysis of the birth of modern science I put forward evidence that this rational product is trinitarian in nature [41].

[35] The three Persons are characterized by Cusanus in the way started by Raymond Lull; for instance: the lover, the loved and love; generating, generated and generation, etc.

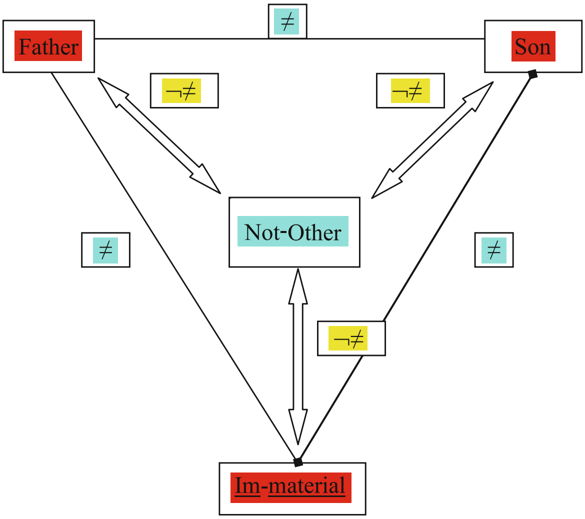

Fig. 2 A geometrical interpretation of Cusanus' Tri-unity. (⇔) and (¬≠) not-other, (≠) different

11 Conclusions

The following appraisal of Cusanus' thinking written by Lanza del Vasto fifty years ago is remarkable:

> The conciliation of the opposites at the infinity is the subject of the philosophy of Nicholas of Cues (*De Docta Ignorantia*). It is, in our opinion, the most important contribution to the Western thinking after Aristotle and St. Thomas, since he lays the basis for a new logic, for a *Novissimum Organon*, the Logic of the Infinity, according to which the principle of non-contradiction is changed into a law of fusion and transformation. (By ignoring that the opposite extremes achieve a reconciliation only at infinity and in God, and by having located their "synthesis" at the [superficial] level of Becoming, Phenomenon, History, Hegel's Dialectics remains a clever juggler's trick exulting in merely approximate abstractions). [55, p. 78]

It is well-known that a century ago Cassirer recognised Cusanus' merit of being the first to lead the human mind to conquer infinity [14, pp. 11ff.]. Now we can attribute to him also the merit that Cassirer (and later Lanza del Vasto and Bonetti) merely hinted at, i.e. the merit of having introduced a new logical rationality. Indeed, he was capable of anticipating some kinds of non-classical logic. His consistency in such ways of reasoning and his adherence to the alternative organization of a theory allowed him to anticipate much of modern epistemology.

In retrospect, we see that more than three centuries elapsed before some scientists—Avogadro, S. Carnot and Lobachevsky—reasoned through DNPs in a more precise way than he did.[36] However, they were favoured by their respective subjects, which, being

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[36] Drago [40]. These theories are all of great relevance to the history of science: Avogadro's was the first accurate atomic theory; S. Carnot's founded the first non-mechanistic theory of physics, i.e. thermodynamics; Lobachevsky introduced the first non-Euclidean geometry. Their texts have been analysed through their DNPs by some papers cited in [40].

scientific in nature, offered much more experimental evidence than that at the disposal of Cusanus.

One more interesting question is when in the history of science the idea of an alternative organization of a theory, already employed by Cusanus, was eventually suggested. The answer is D'Alembert [34]. After him, in theoretical physics it occurred and was illustrated by L. Carnot's book of mechanics (Carnot in 1783, pp. 101–103); in mathematics it occurred in the book on non-Euclidean geometry [35, 56]. Nonetheless, in the following this alternative organization was for a long time ignored.

Why was such an important result not recognized for so long? Unfortunately, (1) in all his books Cusanus did not circumscribe his results, so that to most scholars these results seemed ill-founded. (2) He excluded from the list of his books the most advanced one, *De non Aliud*. (3) Since at first sight Cusanus's language appears to be that of a neo-Platonist, the interpreters of his thinking tried to determine the continuity or the discontinuity of Cusuanus' thinking with Plato's philosophy, although the latter philosophy in turn presents significant difficulties. Such studies were unsuccessful because, as we saw, neither *Possest* nor *Non-Aliud* are concepts; these names introduced Cusanus to an essentially new philosophy with respect to Plato's. (4) He misled the interpreters in some ways. Initially he relied on mathematics (numbers and geometry). Several scholars exhausted their energies in interpreting this aspect of his works, although they did not achieve useful or clear mathematical results. (5) On the other hand, his real achievement was in logic, as we saw in the above. Yet he presents a logical tool which is disputable, i.e. the coincidence of opposites. At present we recognize in it no more than an introductory and inaccurate tool for obtaining DNPs. (6) It may have been due to the inadequacy of his logical tool that in the last three years of his life Cusanus abandoned the *De Non Aliud* and tried to develop a more manageable notion of *posse*. (7) Each of his last books, the *Compendium* and the *De Apice Theoriae*, was apparently planned to illustrate the framework of his research; yet, the reader cannot recognize in these books a single thread linking together all the previous books.

If one adds the above five common prejudices illustrated in the above, it is easy to understand why Cusanus' writings were so difficult to understand for almost five centuries and even after the different kinds of non-classical logic had been formalized. Even at the present time they possess a logical novelty (the triunity) that remains unexplored.

Acknowledgement I am grateful to Prof. David Braithwaite for having revised my poor English and to an anonymous referee for an important suggestion.

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Part III

Reinterpretations of the Square

Symmetric Properties of the Syllogistic System Inherited from the Square of Opposition

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Abstract The logical square Ω has a simple symmetric structure that visualises the bivalent relationships of the classical quantifiers A, I, E, O. In philosophy it is perceived as a self-complete possibilistic logic. In linguistics however its modelling capability is insufficient, since intermediate quantifiers like few, half, most, etc cannot be distinguished, which makes the existential quantifier I too generic and the universal quantifier A too specific. Furthermore, the latter is a special case of the former, i.e. A⊂I, making the square a logic with inclusive quantifiers. The inclusive quantifiers I and O can produce redundancies in linguistic systems and are too generic to differentiate any intermediate quantifiers. The redundancy can be resolved by excluding A from I, i.e. ^{2}I=I-A, analogously E from O, i.e. ^{2}O=O-E. Although the philosophical possibility of A⊂I is thus lost in ^{2}I, the symmetric structure of the exclusive square ${}^2\Omega$ remains preserved. The impact of the exclusion on the traditional syllogistic system $\mathbb{S}$ with inclusive existential quantifiers is that most of its symmetric structures are obviously lost in the syllogistic system ${}^2\mathbb{S}$ with exclusive existential quantifiers too. Symmetry properties of $\mathbb{S}$ are found in the distribution of the syllogistic cases that are matched by the moods and their intersections. A syllogistic case is a distinct combination of the seven possible spaces of the Venn diagram for three sets, of which there exist 96 possible cases. Every quantifier can be represented with a fixed set of syllogistic cases and so the moods too. Therefore, the 96 cases open a universe of validity for all moods of the syllogistic system $\mathbb{S}$, as well as all fuzzy-syllogistic systems ${}^n\mathbb{S}$, with n-1 intermediate quantifiers. As a by-product of the fuzzy syllogistic system and its properties, we suggest in return that the logical square of opposition can be generalised to a fuzzy-logical graph of opposition, for $2<n$.

Keywords Fuzzy logic • Reasoning • Set theory • Syllogisms

Mathematics Subject Classification 03B22 Abstract deductive systems, 03B35 Mechanization of proofs and logical operations, 03B52 Fuzzy logic, 03C55 Set-theoretic model theory, 03C80 Logic with extra quantifiers and operators

J.-Y. Béziau, G. Basti (eds.), *The Square of Opposition: A Cornerstone of Thought*,
Studies in Universal Logic, DOI 10.1007/978-3-319-45062-9_6

1 Introduction

The logical square of opposition, in short the square Ω, is an ancient construct of Aristotle [1] that depicts all possible relationships among the four classical quantifiers, universal, existential and their negations. It visualises the consistency of the relationships in terms of philosophical possibilities. An immediate application of the square are the well known categorical syllogisms, in all 256 possible combinations within the four syllogistic figures. We will refer to the 256 moods as the syllogistic system $\mathbb{S}$.

The square and the syllogistic system have been extensively analysed in the history of logic, however mostly separately from each other. Especially the square has become increasingly controversial in pragmatical discussions and has therefore been extended to various forms of n-polytopes [24]. However, such extensions were mostly not reflected on the syllogistic system, not until modern logic emerged in the century of Frege [12]. For instance, reduction of a syllogism, by changing an imperfect mood into a perfect one [30]. Conversion of a mood, by transposing the terms, and thus drawing another proposition from it of the same quality [22, 23]. Unfortunately, such extensions on the syllogistic system were in turn not reflected back on the square.

Initial generalisations of quantifiers were introduced in linguistics [25], at a time, where computing became popular in science, along with discussions about the possibility of artificial intelligence [37]. Cardinalities of quantifiers have forced logicians to rethink [2] about related logics, such as intermediate quantifiers, like several, few, many, most in syllogisms [31]. Fuzzifications of quantifiers [8, 40] and cardinality-based fuzzy quantifications [7, 11], have enabled approximate reasoning [40], fuzzy-logical generalisations of syllogisms [27, 41] and eventually their reflections on the square [26, 31].

In order to be able to algorithmically calculate precise truth values of syllogistic moods [18], for any fuzzy-logical generalisation of the syllogistic system, first properties and dynamics of fuzzy-moods need to be well understood, such as varying validities, symmetries and equalities. Some of them have already been discussed partially in the literature, for instance, validity of moods with classical quantifiers using diagrammatic proves [29, 36]. Such approaches are the closest to our algorithmic calculations of truth ratios for moods [18]. Further, symmetry and equality of moods analysed based on Aristotle's heuristics and geometric properties [34], validity of moods with intermediate quantifiers using axiomatic [27] or algebraic approaches [38]. Eventually, such findings about a fuzzy syllogistic system should help in verifying the logical consistencies of the used quantifiers by using their reflections on extended versions of the square.

Promising is that most of the empirically obtained truth values for the 256 moods are close to our algorithmically calculated truth ratios [18]. For instance philosophical studies confirm that syllogistic reasoning does model human reasoning with quantified object relationships [14]. For instance in psychology, studies have compared five experimental studies that used the full set of 256 syllogisms [6, 28] about different subjects. Two settings about choosing from a list of possible conclusions for given two premisses [9, 10], two settings about specifying possible conclusions for given premisses [15], and one setting about deciding whether a given argument was valid or not [16]. It has been found that the results of these experiments were very similar and that differences in design appear to have had little effect on how human evaluate syllogisms [6].

Inference logics like modus ponens or modus tollens, are some simplified derivations from syllogisms [35]. Since they have no quantities any more, they cannot capture any fuzzy-quantified propositions. Whereas fuzzy-quantified syllogisms can formalise the whole range of linguistic quantities and thus can provide more powerful inferences. Ones the capabilities of inferencing with fuzzy-syllogistic systems $^{n}\mathbb{S}$ are fully revealed, they may become a preferred tool for approximate reasoning in artificial intelligence.

After formalising the square of opposition, we provide formalisations for the syllogistic system, its properties and a fuzzy syllogistic system. Finally, we introduce a fuzzy-logical square of opposition and its generalisation, the fuzzy-logical graph of opposition.

2 Logical Square of Opposition

The square reflects symmetric relationships between quantifiers that seam to be consistent in terms of philosophical possibilities, but prove to be impractical in engineering, as some of the possibilities develop redundancies, with which distinctive decision making is not possible.

The square Ω consists of four quantifiers $\psi \in \{A, E, I, O\}$, two affirmative A:All and I:Some, their negations, E:All Not and O:Some Not respectively, and all possible six relationships amongst them (Fig. 1):

$$\Omega = \{(A, E, I, O) | R_{sa}(A, I), R_{cr}(A, E), R_{cd}(A, O), R_{cd}(E, I), R_{sa}(E, O), R_{sc}(I, O)\}$$

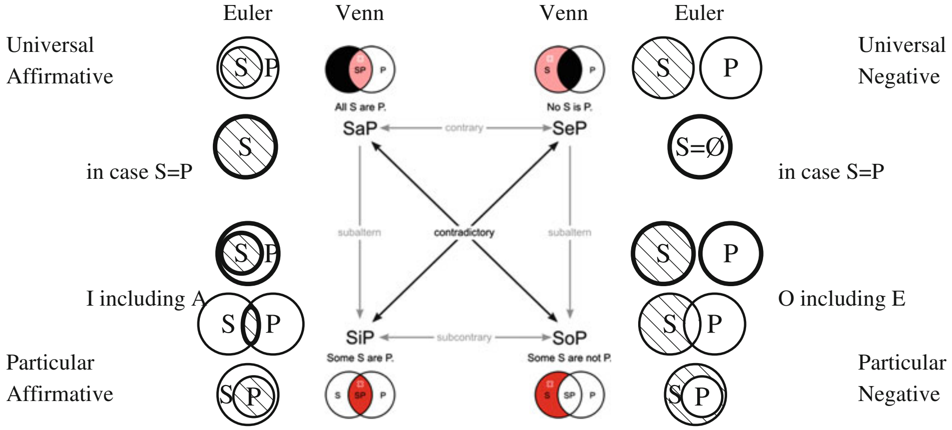


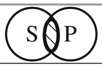
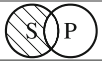
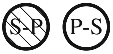
Fig. 1 The square of opposition Ω with Euler and Venn diagram representations of the quantifiers with all Gergonne relations [13]

where R_{sa} is subaltern, R_{cr} is contrary, R_{cd} is contradictory, R_{sc} is subcontrary and only R_{sa} is unidirectional, R_{cr}, R_{cd}, R_{sc} are bidirectional (Fig. 1).

Set-theoretic visualisations of the quantifiers [33] help understanding the logical cases every quantifier encapsulates and help identifying overlapping partial equalities among them (Table 1). These logical cases, to which we will refer later in the text as syllogistic cases, form the essential data for our algorithmic calculations of truth ratios for the syllogistic moods. Although Venn diagrams are more popular in the literature, because they provide a more compact representation, we prefer Euler diagram, as we can visualise every logical cases of a quantifier in a distinct diagram. Logical cases of quantifiers are sometimes referred to as states [3].

Depending on different pragmatical considerations, the cases (c) of I and O are further separated in the literature (Table 1). Some consider them as invalid [5] and some include them as valid [39] for a given domain. Since case (c) of I is equivalent to proposition A, A becomes a special case of I. Similarly, since case (c) of O is equivalent to proposition E, E becomes a special case of O. We will refer to existential quantifiers that include the universal cases as inclusive and to those that exclude the universal cases as exclusive quantifiers.

Table 1 Logical case of inclusive and exclusive quantifiers represented in Euler diagrams and space diagrams

Quantifier Ψ	Proposition Φ	Logical case/disjoint space[a]		
		(a)	(b)	(c)
A	ALL S are P		Ø	Ø
				
E	ALL S are NOT P[b]		Ø	Ø
				
I[c]	SOME S are P			
				
O	SOME S are NOT P			
				

[a] Logical cases are in the first row of every quantifier, equivalent disjoint spaces are in the second row

[b] We will use ALL NOT interchangeably with NO. Whereas the quantifier "NOT ALL" is not interchangeable with No [4]!

[c] For the quantifier I and O, we exclude the case of equality S=P. Otherwise the syllogistic system of two sets would reduce down to a system of one set; in general from n to (n−k), for all k equal sets

3 Categorical Syllogisms

A categorical syllogism can be defined as a logical argument that is composed of two logical propositions for deducing a logical conclusion, where the propositions as well as the conclusion consist each of a quantified object-property relationship.

3.1 Syllogistic Propositions

In general, a proposition is a statement that can specify multiple objects and properties. Since a property itself may recursively become an object with properties, we will denote a property as well as an object. Additionally, we will use further terms interchangeably, object, propositional variable and set.

A syllogistic proposition has a fixed structure, consisting of one object and one quantifying property:

$$\text{Syllogistic proposition} : \Phi = \mathrm{S}\psi\mathrm{P}$$

where S and P denote sets, such that S is categorised on P with $\psi = \{A, E, I, O\}$.

3.2 Syllogistic Figures

A syllogism consists of two premising propositions and one concluding proposition. The first proposition specifies a quantified relationship between the objects M and P, the second proposition between S and M, the conclusion between S and P (Table 2).

Below triple is a more general definition of a categorical syllogism, without distinguishing figures:

$$\begin{aligned}\text{Syllogistic figures} : (\psi_1\psi_2\psi_3\mathrm{F}) &= (\Phi_1, \Phi_2, \Phi_3) \\ &= (\{\mathrm{M}\psi_1\mathrm{P}, \mathrm{P}\psi_1\mathrm{M}\}, \{\mathrm{S}\psi_2\mathrm{M}, \mathrm{M}\psi_2\mathrm{S}\}, \mathrm{S}\psi_3\mathrm{P})\end{aligned}$$

where Φ_1 and Φ_2 denote the first and second premising propositions and Φ_3 denotes the concluding proposition.

Table 2 Syllogistic figures F

	Figure $(\Psi_1\Psi_2\Psi_3\mathrm{F})^a$			
Syllogism	1	2	3	4
Φ_1 = First Premise	MΨP	PΨM	MΨP	PΨM
Φ_2 = Second Premise	SΨM	SΨM	MΨS	MΨS
Φ_3 = Conclusion	SΨP	SΨP	SΨP	SΨP

[a] $\Psi = \{A, E, I, O, U\}$; $F = \{1, 2, 3, 4\}$

Since the propositional operator ψ may have 4 values, 64 syllogistic moods are possible for every figure and 256 moods for all 4 figures in total. For instance, AAA1 constitutes the mood MAP, SAM-SAP in Fig. 1.

3.3 *Syllogistic Moods and Cases*

Syllogistic moods are well known as categorical syllogism, whereas syllogistic case and truth ratio are relative new concepts for syllogisms [18].

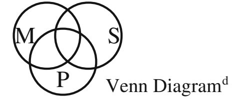
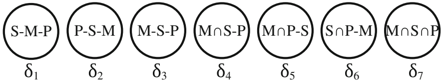
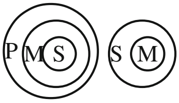
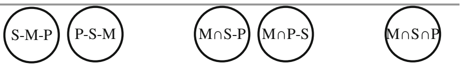
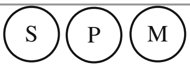
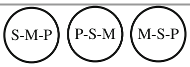
Syllogistic moods ($\psi_1\psi_2\psi_3$F) can be defined with the following tuple constructor:

$$\text{Syllogistic mood of propositions} : (\psi_1\psi_2\psi_3F) = (\Phi_1\Phi_2\Phi_3F, \tau)$$

where $\tau = [0, 1]$ denotes the truth ratio of the mood in figure F={1, 2, 3, 4}.

For three sets, there are 7 possible distinct spaces, which can be easily identified in the Venn diagram (Table 3). From these 7 spaces, in total 128 combinations can be generated, out of which, only 96 are valid for the above quantifier restrictions (Table 1) and only these allow us to uniquely distinguish the space combinations that are matched by every mood

Table 3 Sample syllogistic cases Δ_j

Syllogistic case		
Binary code $\Delta_j = \delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7$[a]	Euler diagram	Space diagram[b]
$\Delta_{96} = 1\,1\,1\,1\,1\,1\,1$[c]		
$\Delta_{78} = 1\,1\,0\,1\,1\,0\,1$		
$\Delta_{81} = 1\,1\,1\,0\,0\,0\,0$		

Binary coding and alternative diagrams of sample combinations for the 7 possible distinct spaces, generated from set relationships between M, S, P

[a] Binary coding of all possible distinct space combinations Δ_j, $j = [1, 96]$ that can be generated for three sets

[b] Every circle of a space diagram represents exactly one distinct sub-set of $M \cup P \cup S$

[c] $\delta_i = 0$: space i is empty; $\delta_i = 1$: space i is not empty; $i = [1, 7]$

[d] A Venn diagram depicts all possible intersections for any given number of sets, while every set is drawn within a single closed area, where some spaces may be empty. Whereas Euler diagrams never show-empty spaces

Table 4 Sample syllogistic moods, their truth cases, truth ratios and sample interpretations

Mood $\psi_1\psi_2\psi_3$F	AAAI, AAI1	EEI1, 2, 3, 4	AAI2
Cases Δ_i	t:0100101	t: 0110010 t: 1010010 t: 1110010	t: 0001101 t: 0010101 t: 0011001 t: 0011101
		f: 1110000	f: 0001100 f: 0011100
Truth ratio τ	lt/(lt+0f)=1 .0[a]	3t/(3t+l f)=075	4t/(4t+2f)=0.67
Interpretation of false cases[b]	Ø	At least $P \cap S \neq \emptyset$ is missing	At least $P \cap S \neq \emptyset$ is missing
Example	ALL primates are mammals	ALL NOT are {Turks. Christian}	ALL birds can fly
	ALL humans are primates	ALL NOT are {Orientals, Turks}	ALL raptors can fly
	{ALL, SOME} humans are mammals	SOME Orientals are Muslim	SOME: raptors are birds
Interpretation of Example	Concluding with ALL is true, probably without exception; concluding with SOME is true only for the possible ALL case in SOME	All four examples that can be loaded into the four moods are possibly more true than false, however possibly not fully true	Since at least bats are raptors, but no birds, concluding with MOST is possibly more true

[a] t=true case; f=false case

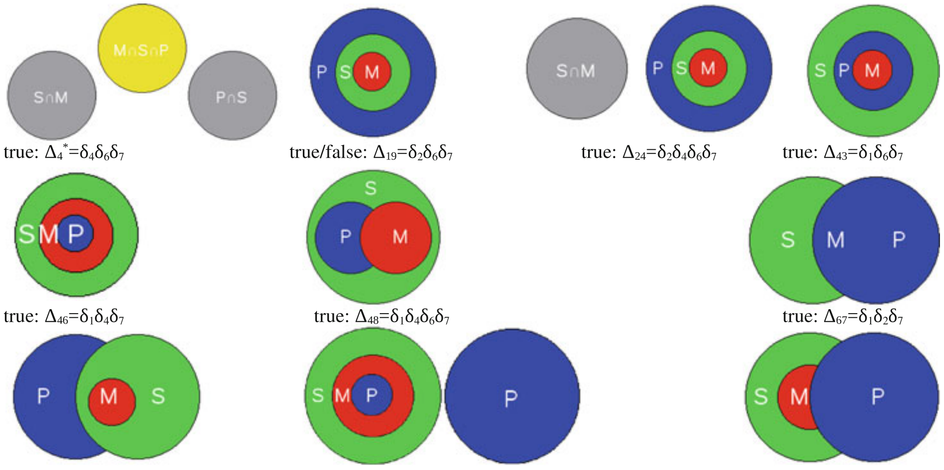
[b] The conclusions of the examples assume that $P \cap S \neq \emptyset$ is given with a value of the truth ratios equal to τ of the mood

(Table 4):

$$\text{Distinct space combinations}: \Delta_j = \{\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7 | \exists_{m,p,s} m \in M \wedge p \in P \wedge s \in S \\ \rightarrow m, p, s \in \delta_1 \cup \delta_2 \cup \delta_3 \cup \delta_4 \cup \delta_5 \cup \delta_6 \cup \delta_7\}$$

where Δ_j with j=[1, 96] are all possible combinations Δ_j of δ_i with i=[1, 7], whereby every Δ_j is the union of distinct spaces, such that at least one element from every set M, P, S must be in the union [43]. The distinct spaces δ_i are named in (Table 3). These combinations Δ_j are exactly all those matched by propositions and conclusions of the 256 moods. The union of all Δ_j, with j=[1, 96], is the universe of all possible truth cases of all 256 moods. Therefore we refer to these 96 combinations as syllogistic distinct cases. Every mood matches some of the cases according following rules:

$$\text{Syllogistic mood of cases}: \psi_1\psi_2\psi_3{}^{\Delta} = \{_{k=1}\cap^2{}_{j=1} \cup^{96} \Delta_j \in \Phi_k \rightarrow \Delta_j \in \Phi^{\Delta}{}_k\}$$



true: $\Delta_4^*=\delta_4\delta_6\delta_7$ true/false: $\Delta_{19}=\delta_2\delta_6\delta_7$ true: $\Delta_{24}=\delta_2\delta_4\delta_6\delta_7$ true: $\Delta_{43}=\delta_1\delta_6\delta_7$

true: $\Delta_{46}=\delta_1\delta_4\delta_7$ true: $\Delta_{48}=\delta_1\delta_4\delta_6\delta_7$ true: $\Delta_{67}=\delta_1\delta_2\delta_7$

true: $\Delta_{68}=\delta_1\delta_2\delta_6\delta_7$ true: $\Delta_{74}=\delta_1\delta_2\delta_4\delta_7$ true: $\Delta_{76}=\delta_1\delta_2\delta_4\delta_6\delta_7$

Fig. 2 10 syllogistic cases Δ^j of the mood IAI4 in $\mathbb{S}$ and of ${}^{2/1}\mathrm{IA}^1\mathrm{I}4$ in ${}^2\mathbb{S}$

where $\Phi^{\Delta}{}_k$ is the set of cases, out of the universal set of all cases Δ_j, j=[1,96], that satisfy the proposition Φ_k on all spaces of every case $\Delta_j = \delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7$. The cases that represent the premiss of the mood, are then calculated by intersecting the cases of the propositions $\Phi^{\Delta}{}_1 \cap \Phi^{\Delta}{}_2$. Out of this set of premising cases $\Phi^{\Delta}{}_1 \cap \Phi^{\Delta}{}_2$, the concluding proposition Φ_3 determines now the true Λ^t and false Λ^f cases of the mood:

$$\text{True syllogistic cases}: \Lambda^t = \Delta \in (\Phi^{\Delta}{}_1 \cap \Phi^{\Delta}{}_2) \wedge \Delta \in \Phi_3 \to \Delta_j \in \Phi^{\Delta}{}_3$$

$$\text{False syllogistic cases}: \Lambda^f = \Delta \in (\Phi^{\Delta}{}_1 \cap \Phi^{\Delta}{}_2) \wedge \Delta \notin \Phi_3 \to \Delta_j \notin \Phi^{\Delta}{}_3$$

where Λ^t and Λ^f is the set of all true and false matching cases of a particular mood, respectively. Since every quantifier ψ always matches a fixed number of syllogistic cases and any particular combination thereof in a mood $\psi_1\psi_2\psi_3{}^{\Delta}$ results in the equal set of cases, this set of cases remains fixed for every particular mood.

For instance, the two premisses Φ_1 and Φ_2 of the mood IAI4 of the syllogistic system $\mathbb{S}$, match the 10 syllogistic cases $\Phi^{\Delta}{}_3 = \Lambda^t = \{\Delta_4, \Delta_{19}, \Delta_{67}, \Delta_{24}, \Delta_{43}, \Delta_{46}, \Delta_{68}, \Delta_{74}, \Delta_{48}, \Delta_{76}\}$, which are all true for the conclusion Φ_3 as well. Thus the mood has no false cases $\Lambda^f = \varnothing$ (Fig. 2).

3.4 Truth Ratios of a Mood

The truth ratio of a mood is calculated by relating the amounts of the two sets Λ^t and Λ^f with each other. Consequently the truth ratio τ becomes either more true or more false:

$$\text{Truth ratio} : \tau \in \{\tau^t, \tau^f\}$$

$$\text{More true truth ratio} : \tau^t \in \{|\Delta^f| < |\Delta^t| \rightarrow 1 - |\Delta^f|/(|\Delta^t| + |\Delta^f|) = [0.545, 1]\}$$

$$\text{More false false ratio} : \tau^f \in \{|\Delta^t| < |\Delta^f| \rightarrow |\Delta^t|/(|\Delta^t| + |\Delta^f|) = [0, 0.454]\}$$

where $|\Lambda^t|$ and $|\Lambda^f|$ are the numbers of true and false syllogistic cases, respectively. A fuzzy-syllogistic mood is then defined by assigning an Aristotelian mood $\psi_1\psi_2\psi_3F$ the structurally fixed truth ratio τ:

$$\text{Fuzzy-syllogistic mood} : (\psi_1\psi_2\psi_3F, \tau)$$

The truth ratio identifies the degree of truth of a particular mood, which we will associate further below in fuzzy-syllogistic reasoning with generic vagueness of inferencing with that mood.

For instance, the two premisses Φ_1 and Φ_2 of the mood IAO3, match 10 syllogistic cases, of which nine are true for the conclusion $\Phi_{3,}\Lambda^t = \{\Delta_4, \Delta_{24}, \Delta_{43}, \Delta_{46}, \Delta_{48}, \Delta_{67}, \Delta_{68}, \Delta_{74}, \Delta_{76}\}$ and one is false $\Lambda^f = \{\Delta_{19}\}$.

4 Structural Analysis

Our objective is to analyse the whole syllogistic system $\mathbb{S}$ of 256 moods, in order to reveal pure structural properties of the system and the moods. For that purpose, we will not consider any semantic interpretations on the moods and we will not apply the elimination rules of Aristotle.

4.1 Assumptions

Following assumptions allow us to perform a pure structural analysis of the system $\mathbb{S}$:

- Classical existential quantifiers: Universal cases included in I and O (Table 1a)
- Inclusive moods: All 256 moods considered, no mood elimination rules or heuristics applied
- Horizontal propositions: Major-minor proposition hierarchy not interpreted
- Set-theoretic: No distinction between the propositional variables subject and predicate
- Syllogistic cases: 96 distinct space combinations assumed to be the universal set of all possible set-theoretic truth cases of the 256 moods
- Normalised truth values: Truth ratios of moods in $\tau = [0, 1]$

4.2 True Syllogistic Moods

24 moods are discussed in the literature since ancient times, to be the only true ones out of the 256 moods. Based on different restrictions that can be made for the value rages of the quantifiers, different numbers of valid moods can be obtained. Accordingly the mood AAO4 is considered to be conditionally true. However, our algorithmic approach calculates the very same 24 true moods, plus AAO4, namely anasoy [18], without any additional conditions for AAO4 [19], but the above assumptions (Table 1) for all moods.

Everyone of these 25 moods matches only true cases, but no false cases (Appendix 2):

$$\text{Syllogistic subsystem of true moods}: \mathbb{S}_1 = \{(\Phi_1\Phi_2\Phi_3, \tau)|\tau = 1.0\}; |\mathbb{S}_1| = 25$$

The number of total cases matched by any mood in $\mathbb{S}_1$ varies from 1 to 11.

4.3 Properties of the Syllogistic System

The algorithmic approach [18] enables revealing various structural properties of the syllogistic system. Some of them are presented here.

4.3.1 Equality

Out of the 256 moods there are 136 distinct moods, in terms of identical true and false cases matched per mood and equal truth ratios. In that sense $256 - 136 = 120$ moods are redundant. For instance, the 25 true moods can be reduced to 11 distinct moods (Fig. 3). For instance, AAA1=AAI1, AAO4=AAI4 or EIO1=EIO2=EIO3=EIO4.

4.3.2 Point-Symmetry

All moods are pairwise point-symmetric in terms of the syllogistic cases they match and in terms of their truth ratios.

Pairs have equal propositional quantifiers, but shifting concluding quantifiers. Almost all moods, i.e. 250, shift from O to A, in total 63 pairs, or from I to E, in total 62 pairs. Thus, the observed point-symmetry of moods is as follows:

$$\text{Point-symmetric mood}: (\psi_1\psi_2\text{OF}^{\Delta}, \tau_t) = (\psi_1\psi_2\text{AF}^{\Delta}, \tau_f = 1 - \tau_t); (\psi_1\psi_2\text{IF}^{\Delta}, \tau_t)$$
$$= (\psi_1\psi_2\text{EF}^{\Delta}, \tau_f = 1 - \tau_t)$$

where Δ denotes that the moods match mutually equal cases. However, only for the following eight moods the quantifiers shift reverse, from A to O in $\text{AAA1}^{\Delta}=\text{AAO1}^{\Delta}$ and from E to I in $\text{EAE1}^{\Delta}=\text{EAI1}^{\Delta}$, $\text{EAE2}^{\Delta}=\text{EAI2}^{\Delta}$, $\text{AEE2}^{\Delta}=\text{AEI2}^{\Delta}$ and $\text{AEE4}^{\Delta}=\text{AEE4}^{\Delta}$.

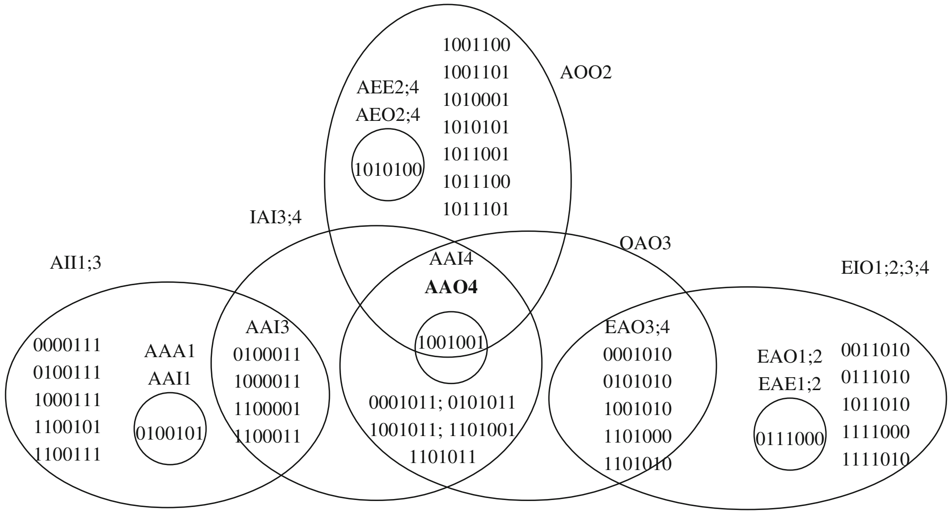


Fig. 3 Set-theoretical relationships between syllogistic moods that are true in case of inclusive existential quantifiers. The inclusive syllogistic system $\mathbb{S}_1$ of true moods

Interesting is that these exceptional moods occur only amongst the fully true $\tau = 1.0$ moods.

Because of the above mood equalities, half of the 136 distinct moods, 68 moods, have 68 such point-symmetric counterparts (Appendix 2). For the 25 fully true and 25 fully false moods one can define a point-symmetric syllogistic subsystem:

$$\text{Point-symmetric syllogistic subsystems} : \mathbb{S}_1 = {\mathbb{S}^{-1}}_0$$

$$\text{Syllogistic subsystem of false moods} : \mathbb{S}_0 = \{(\Phi_1\Phi_2\Phi_3, \tau)|\tau = 0.0\}; |\mathbb{S}_0| = 25$$

where -1 in the exponent denotes point-symmetry, in terms of point-symmetric moods. Equal moods in $\mathbb{S}_1$ have their point-symmetric counterparts in $\mathbb{S}_0$. Thus distinct moods in $\mathbb{S}_0$ are also 11.

The same symmetry exists for the remaining 206 moods in the interval (0,1), this time however without any exceptional quantifier shift (Appendix 1):

$$\begin{aligned}\text{Point-symmetric syllogistic subsystems} : \mathbb{S}_{(1,0.545]} &= {\mathbb{S}^{-1}}_{[0.454,0)}; |\mathbb{S}_{(1,0.545]}| \\ &= |\mathbb{S}_{[0.454,0)}| = 103\end{aligned}$$

Out of the 206 moods in the range (0,1), 114 are distinct. Half of them 57 are in $\mathbb{S}_{\tau t}$ and half in $\mathbb{S}_{\tau f}$.

Interesting is that from the above subsystems, only moods in $\mathbb{S}_1$ are partially point-symmetric amongst each other (Fig. 3), respectively for $\mathbb{S}_0$. However, this partial symmetry is week, as it is observed only on the number of syllogistic cases of the moods and their relationships, but not on the distinct space combinations of the cases.

Since the truth ratio τ assigns every mood a vagueness, even before introducing fuzzy-quantifiers to the Aristotelian syllogistic system, we refer to $\mathbb{S}$ as the fuzzy-syllogistic system. Note that the truth ratio is a structural property that is constant, as long as the above assumptions hold.

4.3.3 Case Distribution

The 96 syllogistic distinct cases span the universal set, in which every mood matches a fixed number of cases. The distribution of these matches over the whole 256 moods shows interesting symmetric properties, which seam to be reflections of the above discussed symmetries.

Every mood has 0 to 65 true and 0 to 65 false distinct cases. The sum of all true and false cases matched per mood varies from 1 to 73 cases, out of the total possible 96 cases. For instance, mood AAA1 has only 1 true and 0 false case, in total 1 case, whereas mood OIA1 has 6 true and 65 false cases, in total 71 cases. Hence the truth ratio of AAA1 is $\tau = 1.0$, fully true, and that of OIA1 is $\tau = 0.084$, which is almost false.

For instance, mood OOO2 with 61 true and 11 false cases has truth ratio $\tau = 0.847$, which is mostly true, and its point-symmetric counterpart OOA2 with 11 true and 61 false cases has truth ratio $\tau = 0.153$, which is mostly false. With 72 cases in total, they match exactly 75 % of the universe.

Further details about case distributions and properties of the subsystems $S_{(1,0.545]}$ and $S_{[0.454,0)}$ will be provided elsewhere, since that discussion requires considerably more space.

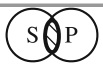
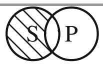
5 Fuzzy Syllogistic System

The basic fuzzy syllogistic system consists of 256 moods that has constant truth ratios in [0, 1]. It can be further fuzzified, by introducing fuzzy-logical propositions, which can be model with fuzzy sets or fuzzy quantifications. By using fuzzy quantifiers we construct a fuzzy-quantified syllogistic system, in which some symmetric properties of the classical syllogistic system degrade, already with crisp sets. Here we discuss initial steps of an approach for gradually fuzzifying quantifiers towards a fuzzy-quantified syllogistic system and discuss the resulting fuzzy-logical square of opposition.

5.1 Fuzzy Quantification

Some of the symmetric properties of the syllogistic system are due to the inclusive existential quantifiers I and O (Table 1 logical cases a). Also, it is these cases that introduce the logical system redundancy, enable abduction of A as well as I from A and abduction of E as well as O from E, thus make the logical system undecidable on these cases. Most

Table 5 Logical cases of exclusive existential quantifiers represented with Euler diagrams and disjoint spaces

Quantifier ψ	Proposition Φ	Logical case/disjoint space[a]		
		(a)	(b)	(c)
^{2}I	ONLYSOME S are P	Ø		
				
^{2}O	ONLYSOME S are NOT P	Ø		
				

[a] Logical cases are in the first row of every quantifier, equivalent disjoint spaces are in the second row

engineering systems cannot decide with such properties. Especially linguistic systems can decide the more effectively, the finer the quantifier granularities are adapted to semantics and pragmatics [17].

We start by fuzzifying the existential quantifiers I into ^{2}I and O into ^{2}O (Table 5):

$$^2\mathrm{I} = \mathrm{I} - \mathrm{A} = \text{“SOME are, but not ALL”} = \text{“ONLYSOME are”}; |^2\mathrm{I}| = [1, |\mathrm{A}| - 1]$$

$$\begin{aligned} ^2\mathrm{O} &= \mathrm{O} - \mathrm{E} = \text{“SOME are NOT, but not ALL”} \\ &= \text{“ONLYSOME are NOT”}; |^2\mathrm{O}| = [1, |\mathrm{A}| - 1] \end{aligned}$$

The value range of exclusive existential quantifiers exclude |A|, whereas inclusive quantifiers include |A|. Based on the exclusive quantifiers ^{2}I and ^{2}O, we elaborate now the smallest possible fuzzy-syllogistic system $^n\mathbb{S}$, n=2. The exponent n determines the granularity of distinct quantifiers, i.e. n=2 affirmative and 2 negative. With increasing number of quantifiers 2<n, the granularity of the total quantifier value range increases, which may be associated with further linguistic quantifiers, like, few, several, most, many (Table 6). Sometimes these are referred to as intermediate quantifiers. Since I encapsulates A, the two are not distinct. Analogously, E and O are not distinct.

Because the universal quantifiers A and E are equal in all systems $\mathbb{S}$ and $^n\mathbb{S}$, 1<n, we do not need to distinguish them with an exponent.

5.2 *Fuzzy Syllogistic Moods*

Moods $^2\psi_1\psi_2\psi_3$F of the fuzzy-syllogistic system $^2\mathbb{S}$ are constructed analogously and with the same propositions ($\Phi_1\Phi_2\Phi_3$F), but they match less truth cases and get different truth ratios τ:

$$\text{Fuzzy syllogistic mood of propositions} : {}^2(\psi_1\psi_2\psi_3\mathrm{F}) = {}^2(\Phi_1\Phi_2\Phi_3\mathrm{F}, \tau) \in {}^2\mathbb{S}$$

Table 6 Value ranges of affirmative fuzzy quantifiers[a] of n fuzzy-syllogistic systems $^{n}\mathbb{S}$

<table>
<tr><th colspan="2">Syllogistic System</th><th colspan="11">Fuzzy quantifier ψ[b]</th></tr>
<tr><td>Aristotelian</td><td>$\mathbb{S}$</td><td>A = All</td><td colspan="10">I=Some(including A)</td></tr>
<tr><td rowspan="6">Fuzzy</td><td>$^{2}\mathbb{S}$</td><td>A = All</td><td colspan="10">^{2}I=Somk=OnlySome (excluding A)</td></tr>
<tr><td>$^{3}\mathbb{S}$</td><td>A = All</td><td colspan="5">$^{3/2}$I=Most</td><td colspan="5">$^{3/1}$I=Several</td></tr>
<tr><td>$^{4}\mathbb{S}$</td><td>A = All</td><td colspan="4">$^{4/3}$I=Most</td><td colspan="2">$^{4/2}$I=Half</td><td colspan="4">$^{4/1}$I=Several</td></tr>
<tr><td>$^{5}\mathbb{S}$</td><td>A = All</td><td colspan="3">$^{5/4}$I=Many[c]</td><td colspan="2">$^{5/3}$I=Most</td><td colspan="2">$^{5/2}$I=Several</td><td colspan="3">$^{5/1}$I=Few</td></tr>
<tr><td>$^{6}\mathbb{S}$</td><td>A = All</td><td colspan="2">$^{6/5}$I=Many</td><td colspan="2">$^{6/4}$I=Most</td><td colspan="2">$^{6/3}$I=Half</td><td colspan="2">$^{6/2}$I=Several</td><td colspan="2">$^{6/1}$I=Few</td></tr>
<tr><td>$^{n}\mathbb{S}$</td><td>A = All</td><td>$^{n/n-1}$I</td><td colspan="8">...</td><td>$^{n/1}$I</td></tr>
</table>

[a] Negative quantifiers are arranged analogously

[b] Column breadths are not drawn proportional to the overall value range or to oilier quantifiers or systems

[c] Discussions of relationships between linguistic quantifiers, for instance whether Many>Most or Many<Most, does not effect the system syntax, but its semantics and therefore is left to linguistics

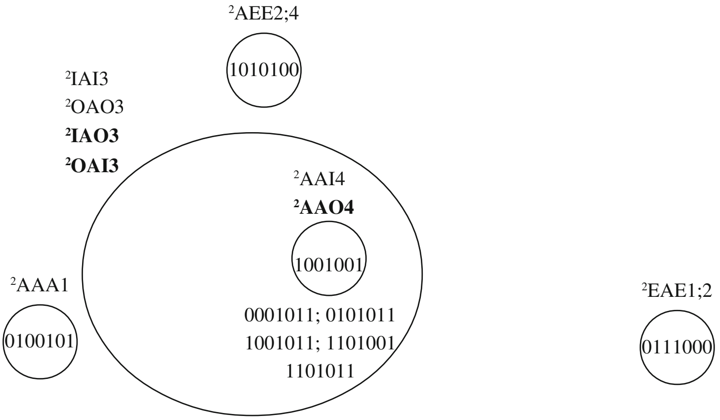


Fig. 4 Set-theoretical relationships between syllogistic moods that are true in case of exclusive existential quantifiers. The exclusive syllogistic system $^{2}\mathbb{S}_1$ of true moods

where $^{2}\psi = \{A, E, {}^{2}I, {}^{2}O\}$. For instance, the mood IAI4 in $\mathbb{S}$ with inclusive existential quantifier I, becomes $^{2/1}$IA^{1}I4 in $^{2}\mathbb{S}$ with the exclusive existential quantifier $^{2/1}$I. The conclusion Φ_3 of the mood, does not match the case Δ_{46} any more. Thus the mood has one false case $\Lambda^{f} = \{\Delta_{46}\}$ and 9 true cases, $\Lambda^{t} = \{\Delta_4, \Delta_{19}, \Delta_{24}, \Delta_{43}, \Delta_{48}, \Delta_{67}, \Delta_{68}, \Delta_{74}, \Delta_{76}\}$, $\Phi^{\Delta}{}_3 = \Lambda^{t} \cup \Lambda^{f}$ (Fig. 2).

The fuzzy syllogistic system $^{2}\mathbb{S}$ has 11 true fuzzy syllogistic moods, of which some are equal. Thus they produce 5 distinct groups of moods (Fig. 4, Appendix 3 $^{2}\mathbb{S}_1$):

$$\text{True} : {}^{2}\mathbb{S}_1; |{}^{2}\mathbb{S}_1| = 11$$

The remaining 245 moods of $^{2}\mathbb{S}$ can be categorised in terms of truth ratio ranges into further four subsystems:

- MoreTrue: $^{2}\mathbb{S}_{(1,0.5)}; |{}^{2}\mathbb{S}_{(1,0.5)}| = 70$

- HALFTRUEHALFFALSE: ${}^2\mathbb{S}_{0.5}$; $|{}^2\mathbb{S}_{0.5}| = 16$
- MOREFALSE: ${}^2\mathbb{S}_{(0.5,0)}$; $|{}^2\mathbb{S}_{(0.5,0)}| = 119$
- FALSE: ${}^2\mathbb{S}_0$; $|{}^2\mathbb{S}_0| = 40$

The linguistic terms that we use to express the vagueness of the subsystems may be used analogously for the subsystems of $\mathbb{S}$ [19].

5.2.1 Truth Ratio Distribution

It is interesting to observe that 16 moods that are true in $\mathbb{S}$, become false in ${}^2\mathbb{S}$ and that two moods that are false in $\mathbb{S}$ become true in ${}^2\mathbb{S}$.

OAI3 limano and IAO3 nomali are two moods that are false in $\mathbb{S}$, i.e. OAI3, IAO3 $\notin \mathbb{S}_1$, OAI3, IAO3 $\in \mathbb{S}_{[0,1)}$, but turn true in ${}^2\mathbb{S}$, i.e. ^{2}OAI3, ^{2}IAO3 $\in {}^2\mathbb{S}_1$ (Fig. 4, Appendix 3 ${}^2\mathbb{S}_1$).

Out of the 16 moods that become false in ${}^2\mathbb{S}$ (Appendix 3 ${}^2\mathbb{S}_{[0,0.89]}$), five moods, ^{2}EAO1, ^{2}EAO2, ^{2}AAI1, ^{2}AEO2, ^{2}AEO4 all turned to zero. These moods were true in $\mathbb{S}$, but turned to 100 % false, only by excluding the universal cases from the existential quantifiers, i.e. they would become true only with universal cases. In fact, if we replace in these moods ^{2}I with A and ^{2}O with E, we get EAE1, EAE2, AAA1, AEE2, AEE4, which are all true moods, found both, in ${}^2\mathbb{S}$ as well as in $\mathbb{S}$ and all have a single syllogistic case. Thus this scenario exemplifies clearly that inclusive existential quantification can turn some moods to true, whereas without universal cases the moods would remain fully false.

This observation can be generalised, such that the truth ratios of many moods with existential quantifiers decrease, whereas some increase, amongst which limano and nomali even increase to 100 % true.

5.3 *Properties of the Fuzz-Quantified Syllogistic System*

In general, the number of equal moods per truth ratio increases from $\mathbb{S}$ to ${}^2\mathbb{S}$, point-symmetry vanishes (Appendix 1), more moods hit a lower truth ratio and the total number of matched syllogistic cases decreases, which includes more false cases than true cases.

Every mood has 0 to 40 true and 0 to 48 false distinct cases. The sum of all true and false cases matched per mood varies from 1 to 54 cases, out of the total possible 96 cases.

For instance, the moods OOO2=OOA2^{-1} now become ^{2}OOO2 with 40 true and 8 false cases gets truth ratio $\tau = 0.833$, which is close to OOO2, and ^{2}OOA2 with 6 true and 42 false cases gets truth ratio $\tau = 0.125$, which is close to OOA2. With 48 cases in total, both match exactly 50 % of the universe. Most point-symmetric counterparts in $\mathbb{S}$ do not even preserve the same number of total cases in ${}^2\mathbb{S}$, like these two moods do.

5.4 Generic Fuzzy-Syllogistic Systems

We have defined the Aristotelian syllogistic system $\mathbb{S}$ as fuzzy-syllogistic, as moods have truth ratios that can be interpreted as degree of vagueness in inferencing with them. Further we have defined the fuzzy-quantified syllogistic system $^2\mathbb{S}$, in which the philosophically possible universal cases are excluded from the existential quantifiers. In further steps towards generic fuzzy-syllogistic systems $^n\mathbb{S}$, $2<n$, the value range of the existential quantifiers of $^2\mathbb{S}$ are further partitioned, in general into n-1 partitions, each representing a fuzzy-existential quantifier (Table 6).

The systems $\mathbb{S}$ and $^2\mathbb{S}$ constitute the basic generic syllogistic systems, in terms of truth rations. Truth ratios are calculated from syllogistic cases and those are based on the set-theoretical logical cases (Table 1, case b and c). All fuzzy-existential quantifiers $[^{n/n-1}I, {}^{n/1}I]$ of $^n\mathbb{S}$ are valid on exactly these same logical cases. Therefore, the truth ratio τ of any particular mood $^2(\Phi_1\Phi_2\Phi_3F, \tau) \in {}^2\mathbb{S}$ is equal in the same mood with all further partitioned n-1 existential quantifiers $^n(\Phi_1\Phi_2\Phi_3F, \tau) \in {}^n\mathbb{S}$.

For instance, the truth ratio $\tau = 0.888$ of the mood $^2IAI4$ is equal for all moods with any further partitioned I, like $^{3/2}IA^2I4$, $^{3/1}IA^2I4$, $^{3/2}IA^1I4$, $^{3/1}IA^1I4$ or $^{6/5}IA^5I4$, $^{6/4}IA^5I4$, $^{6/3}IA^5I4$ etc.

For instance, the truth ratio $\tau = 1$ of the mood $^{2/1}OA^1I3$ is equal for all moods with any further partitioned O or I, like $^{3/2}OA^2I4$, $^{3/1}OA^2I4$, $^{3/2}OA^1I4$, $^{3/1}OA^1I4$ or $^{6/5}OA^5I4$, $^{6/4}OA^5I4$, $^{6/3}OA^5I4$ etc.

6 Extensions to the Square of Opposition

In order to verify the consistency of the quantifier relationships of the various fuzzy-syllogistic systems $^n\mathbb{S}$, $1<n$, we now present extensions to the Aristotelian square of opposition Ω.

6.1 Fuzzy-Logical Square of Opposition

The quantifier relationships of the fuzzy syllogistic system $^2\mathbb{S}$ imply the same visual structure like the original square of opposition (Fig. 1), however without universal cases in the existential quantifiers.

We will denote the fuzzy-logical square of opposition with $^2\Omega$ and refer to it in short as the exclusive square:

$$^2\Omega = \{(A, E, {}^2I, {}^2O) | R_{sa}(A, {}^2I), R_{cr}(A, E), R_{cd}(A, {}^2O), R_{cd}(E, {}^2I), R_{sa}(E, {}^2O), R_{sc}({}^2I, {}^2O)\}$$

where $^2\Omega$ has two affirmative quantifiers. In the same manner we have identified $^2\mathbb{S}$ as the smallest possible fuzzy syllogistic system, we identify the exclusive square $^2\Omega$ as the smallest possible fuzzy-logical square.

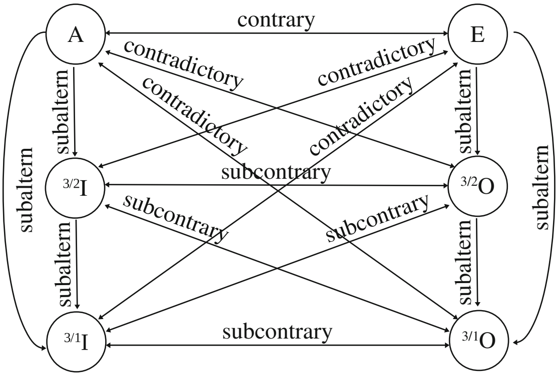
6.2 Fuzzy-Logical Graph of Opposition

For every further partition of the existential quantifiers (Table 6), we will extend the classical square analogously step-wise and eventually generalise the exclusive square ${}^2\Omega$ to a fuzzy-logical graph of opposition ${}^{\mathrm{n}}\Omega$.

Our first extension of ${}^2\Omega$ is ${}^3\Omega$ (Fig. 5), which verifies the logical quantifier relationships of ${}^2\mathbb{S}$. Following new relationships emerge in ${}^3\Omega$:

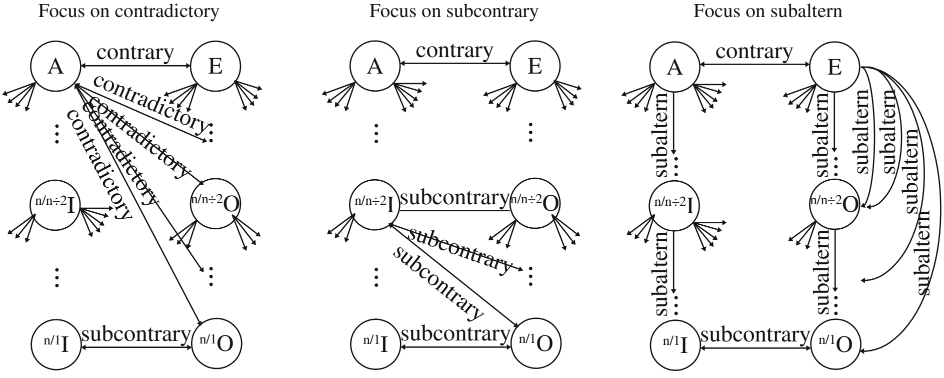
- Subaltern: Any existential quantifier is subaltern to the universal quantifier, so is any smaller existential quantifier to any greater one.
- Subcontrary: Any existential quantifier is subcontrary to any negative existential quantifier.

The structure of ${}^{\mathrm{n}}\Omega$ (Fig. 6) is obtained, by simply replicating the new relationships of ${}^3\Omega$, for every further partitioning existential quantifier. The relationships are analogous to those of Buridan or Celaya [24].



3/2I: Most 3/2O: MostNot
3/1I: Several 3/1O: SeveralNot

Fig. 5 3-quantified fuzzy-logical graph of opposition ${}^3\Omega$ with three fuzzy existential quantifiers and traditional relationships



n/n÷2: index n÷2 of existential quantifier, in ${}^{\mathrm{n}}\Omega$ with n quantifiers

Fig. 6 n-quantified fuzzy-logical graph of opposition ${}^{\mathrm{n}}\Omega$ with $n - 1$ fuzzy existential quantifiers and traditional relationships

7 Discussion

We have used the fuzzy-logical graph of opposition $^{n}\Omega$ for verifying possible logical relationships between the quantifiers of the fuzzy-syllogistic systems $^{n}\mathbb{S}$. Generalisations to the classical square of opposition, are not new in the literature. We shall discuss one similar approach that appears to be related to ours.

In some recent work, the validity of fuzzy syllogism have been analysed based on the concept of intermediate quantifiers and 105 moods have been heuristically identified as valid [32], structurally [27] and algebraically [38] validated and verified on a generalisation of the square of opposition [26]. For instance, fuzzy-quantified derivations of the mood AAI1, like AAT1, AAK1, AAP1 (T=most; K=many; P=almost all) are reported to be valid. However, according our truth ratio calculations that are based on the above quantifier definitions (Table 1), the mood AAI in $\mathbb{S}$ has $\tau = 1$, but turns false in $^{2}\mathbb{S}$, i.e. ^{2}AAI1 has $\tau = 0$. The mood turns false in $^{2}\mathbb{S}$, because the only syllogistic case of the mood is 0100101 and that is true only for the A case of the inclusive quantifier I, the very one that is excluded in ^{2}I (Table 1 logical cases a for I). As we have discussed above, this mood has $\tau = 0$ in all systems $^{n}\mathbb{S}$, 1<n. Since the cardinalities of the fuzzy-quantifiers T, K, P are all smaller or equal than ^{2}I, i.e. $T < K < P \leq {}^{2}I$, none of those moods can be true according to our calculations.

In general, according to our calculations, any mood of any system $^{n}\mathbb{S}$ is true, only if it has at least one premising universal quantifier. Otherwise moods have truth ratios in $\tau < 1$.

The same authors verify their intermediate quantifiers visually on different shapes of generalised squares of oppositions, which are all very similar to each other and partially similar with our fuzzy-logical graph of opposition $^{n}\Omega$. Only few differences are worth mentioning:

- Number of quantifiers are constant at five; whereas $^{n}\Omega$ has a finite number n of quantifiers.
- Contradictory and subcontrary are defined only between some specific quantifiers; whereas in $^{n}\Omega$, every quantifier has either contradictory or subcontrary relations to all smaller contrapositive quantifiers, which is a derivation from the basic fuzzy-logical negation [40], e.g. $\neg^{n/n-1}O = {}^{n/n-2}I \cup {}^{n/n-3}I \cup \ldots \cup {}^{n/1}I$.
- The quantifier Some is used; whereas Some is explicitly not used in any graph $^{n}\Omega$, as Some has a historically rooted pre-defined value-range in the Aristotelian square that covers all philosophically possible values (Fig. 1).

8 Conclusion

We have analysed the classical syllogistic system $\mathbb{S}$ in terms of 96 syllogistic cases, which span the universal value range of all moods of all systems ${}^n\mathbb{S}$ and in which moods match some of them either true or false. We have identified equal moods in terms of cases and truth ratios, point symmetry in terms of cases and truth ratios and the symmetric case distributions. We have presented the point symmetry of the subsystems $\mathbb{S}_1 = \mathbb{S}^{-1}{}_0$ and $\mathbb{S}_{(1,0.545]} = \mathbb{S}^{-1}{}_{[0.454,0)}$. The symmetric structures are obviously not only due to the square Ω, but also caused by the combinatorial ordering of the premising propositional variables.

We have discussed the properties of the smallest possible fuzzy syllogistic system ${}^2\mathbb{S}$ and revealed why the symmetric structures of $\mathbb{S}$ almost vanish in ${}^2\mathbb{S}$. We have introduced the smallest possible fuzzy syllogistic square of opposition ${}^2\Omega$ and suggested an approach for generalising it to a fuzzy-logical graph of opposition ${}^n\Omega$ with 2n fuzzy quantifiers.

Currently we are testing the feasibility of the generic system ${}^n\mathbb{S}$ on fuzzy-syllogistic ontologies [20] and fuzzy-syllogistic reasoning with such ontologies [21, 42].

Appendix 1: Distinct Groups of Moods in $\mathbb{S}$ and ${}^2\mathbb{S}$

The Aristotelian syllogistic system $\mathbb{S}$ consists of 136 distinct groups of moods, in terms of equal truth ratios (Fig. 7). One can observe the fully point-symmetric distribution of the values around $\tau = 0.5$. Truth ratios as well as the number of moods in the groups are symmetric.

The fuzzy-syllogistic system ${}^2\mathbb{S}$ consists of 70 distinct groups of moods, in terms of equal truth ratios (Fig. 8). However neither truth ratios nor the number of moods in the groups are symmetrically distributed around $\tau = 0.5$ any more.

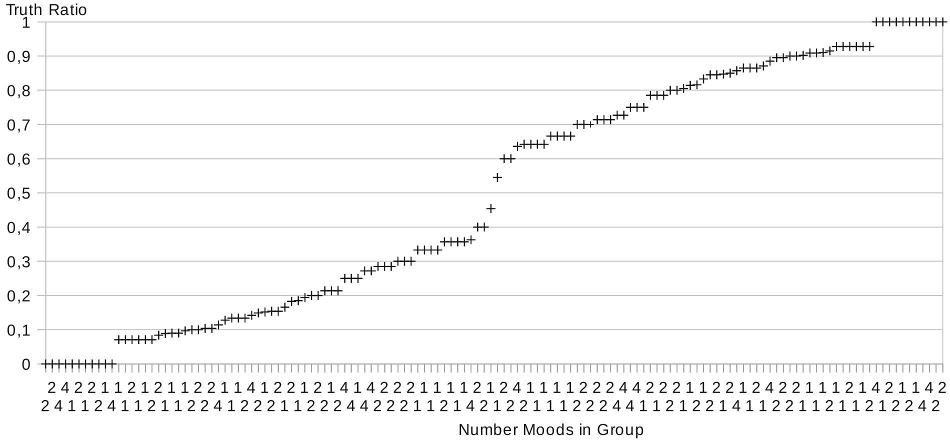


Fig. 7 136 distinct groups of moods of $\mathbb{S}$, sorted in ascending order of truth ratio τ (inclusive logic)

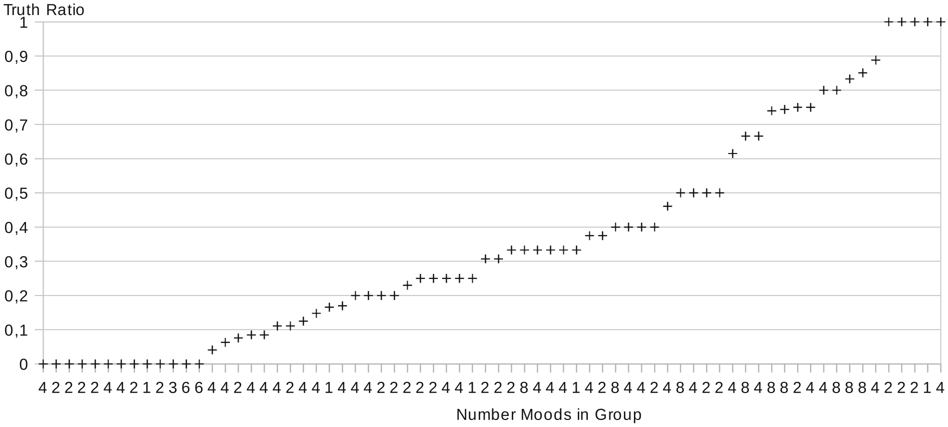


Fig. 8 70 distinct groups of moods of $^2\mathbb{S}$, sorted in ascending order of truth ratio τ (exclusive logic)

Table 7 In case of inclusive existential quantifiers, true moods of $\mathbb{S}_1$, and their point-symmetric counterparts in $\mathbb{S}_0$, showing numbers of true cases t and false cases f of their truth ratios τ

$\mathbb{S}_1$: True 1.0					$\mathbb{S}_0$: False 0.0				
Moods in group	$\psi_1\psi_2\psi_3$F	τ	t	f	Moods in group	$\psi_1\psi_2\psi_3$F	τ	t	f
2	AAA1; AAI1	1.000	1	0	2	AAO1; AAE1	0.000	0	1
2	AAO4; AAI4	1.000	1	0	2	AAA4; AAE4	0.000	0	1
4	AEO2;4; AEE2;4	1.000	1	0	4	AEA2;4; AEI2;4	0.000	0	1
4	EAE1;2; EAO1;2	1.000	1	0	4	EAI1;2; EAA1;2	0.000	0	1
1	AAI3	1.000	4	0	1	AAE3	0.000	0	4
2	EAO3;4	1.000	5	0	2	EAA3;4	0.000	0	5
1	AOO2	1.000	9	0	1	AOA2	0.000	0	9
2	AII1;3	1.000	10	0	2	AIE1;3	0.000	0	10
2	IAI3;4	1.000	10	0	2	IAE3;4	0.000	0	10
1	OAO3	1.000	11	0	1	OAA3	0.000	0	11
4	EIO1;2;3;4	1.000	11	0	4	EIA1;2;3;4	0.000	0	11

Appendix 2: Moods with Inclusive Existential Quantifiers

In case of inclusive existential quantifiers 25 moods are 100 % true, i.e. have truth ratio $\tau = 1.0$, because they have only true cases t. 25 moods are 100 % false, i.e. have truth ratio $\tau = 0.0$, because they have only false cases f (Table 7). Some moods are equal in terms of their syllogistic cases, as they match exactly the same cases out of the possible 96 cases. For instance, AII1 has 10 cases and AII3 has the very same cases.

Table 8 In case of exclusive existential quantifiers, true moods of $^{2}\mathbb{S}_{1}$ and false turned moods in $^{2}\mathbb{S}_{[0,0.89]}$, showing numbers of true cases t and false cases f of their truth ratios τ

$^{2}\mathbb{S}_{1}$: Remained/turned true 1.0			
$^{2}\psi_1\psi_2\psi_3$F	τ	t	f
^{2}AAA1	1.000	1	0
$^{2/1}$EA^{1}E1;2	1.000	1	0
^{2}A^{1}E^{1}E2;4	1.000	1	0
^{2}AA^{1}I4;^{2}AA^{1}O4	1.000	1	0
$^{2/1}$IA^{1}I3;$^{2/1}$OA^{1}O3;$^{2/1}$IA^{1}O3;$^{2/1}$OA^{1}I3	1.000	6	0

False in $\mathbb{S}_{(0,1)}$			
$\psi_1\psi_2\psi_3$F	τ	t	f
OAI3	0.909	10	1
IAO3	0.900	9	1

$^{2}\mathbb{S}_{[0,0.89]}$: Turned false in [0,0.89]			
$^{2}\psi_1\psi_2\psi_3$F	τ	t	f
$^{2/1}$IA^{1}I4	0.890	8	1
$^{2/1}$E^{1}I^{1}O1;2	0.800	8	2
$^{2/1}$E^{1}I^{1}O3;4	0.667	4	2
$^{2/1}$EA^{1}O3;4	0.800	4	1
$^{2/1}$EA^{1}O1;2	0.000	0	1
^{2}A^{1}O^{1}O2	0.750	6	2
^{2}AA^{1}I3	0.750	3	1
^{2}AA^{1}I1	0.000	0	1
^{2}A^{1}I^{1}I1	0.700	6	3
^{2}A^{1}I^{1}I3	0.500	3	3
^{2}A^{1}E^{1}O2;4	0.000	0	1

Everyone of the 25 true moods has a point-symmetric counterpart, in terms of the particular cases they match. For instance, AOO2 has 9 cases and AOA2 has the very same cases, but for AOO2 all cases are true, whereas for AOA2 all those cases are false.

Appendix 3: Moods with Exclusive Existential Quantifiers

In case of exclusive existential quantifiers 9 moods remain 100 % true and two $^{2/1}$IA^{1}O3 and $^{2/1}$OA^{1}I3 turn 100 % true. 16 moods turn false with truth ratios τ ranging in $[0, 0.89]$ (Table 8). Some moods become equal in terms of their syllogistic cases. For instance, $^{2/1}$IA^{1}I3, $^{2/1}$OA^{1}O3, $^{2/1}$IA^{1}O3 and $^{2/1}$OA^{1}I3 reduce all to the very same 6 cases.

The syllogistic system with exclusive existential quantifiers shows considerably less symmetric properties in terms of syllogistic cases and truth ratios.

Acknowledgements Thanks are due to Mikhail Zarechnev for developing applications that generate various data sets of the fuzzy syllogistic systems $\mathbb{S}$, $^{2}\mathbb{S}$ and $^{6}\mathbb{S}$, for analysis purposes.

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The Square of Opposition Interpreted with a Decidable Modal Logic

Paul Weingartner

Abstract In connection with Aquinas modal interpretation of the square of opposition the paper interprets the 24 syllogistic modes by a decidable modal logic. Those 15 modes which are not making existential presuppositions are theorems of it right away whereas the other 9 modes are theorems when adding the possibility of the antecedent.

Keywords Decidable many-valued logic • Decidable modal logic • Relevant logic • Syllogistics

Mathematics Subject Classification (2010) Primary 03B45; Secondary 03B50

In his short article *De Propositionibus Modalibus* Thomas Aquinas interpreted the square of opposition with the help of modalities. He observed that all the essential relations of the square, contradictions in the diagonal, contraries, subcontraries and subalternities, are preserved if one puts the following modalities into the four corners (see Fig. 1):

In this paper it will be shown that the 24 syllogistic modes can be interpreted in this way by a decidable modal logic in such a way that those 15 which do not make existential presuppositions are valid in this modal logic and the remaining ones are valid if one adds the premise that the antecedence is possible. This modal logic is based on the 6-valued propositional logic *RMQ* which has relevance properties and was constructed in order to avoid paradoxes which come up if two-valued classical propositional logic is applied outside logic and mathematics, i.e. to empirical sciences.[1]

1 The Underlying System *RMQ*

1.1 RMQ Has Properties of Relevance

The underlying system *RMQ* was developed with the intention to construct a propositional logic which contains its own semantics and obeys some important criterion of relevance.

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[1] Weingartner [7–9]. Thomas Aquinas [4].

J.-Y. Béziau, G. Basti (eds.), *The Square of Opposition: A Cornerstone of Thought*,
Studies in Universal Logic, DOI 10.1007/978-3-319-45062-9_7

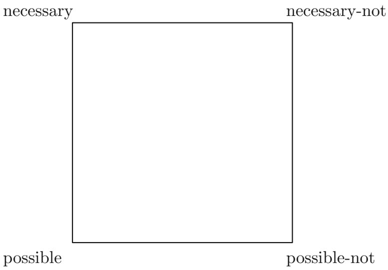

Fig. 1 Modalities in four corners

This relevance criterion was developed together with Gerhard Schurz[2] in order to avoid different types of paradoxes in the domains of scientific explanation, disposition predicates, scientific confirmation, verisimilitude, Quantum Physics and Deontic Logic.

The common cause of these paradoxes are principles of CPC (Classical Propositional Logic) that contain irrelevant elements (such elements that can be replaced by any others or can be reduced to others) in the conclusion or in the consequence class. The relevance criterion has two parts, a replacement part RC and a reduction part RD. The main idea of the first part of the relevance criterion called replacement criterion (RC)—is to forbid those parts of a consequence (conclusion) of a valid inference which can be replaced (on one or more occurrences) by any arbitrary part (wff) salva validitate of the inference.

For example the classically valid principle of addition $p \rightarrow (p \vee q)$ allows to introduce a new sentence q which has nothing to do with the premises and can be replaced by any other sentence (wff) salva validitate of the inference. It is important to realize that this principle is the chief cause for the following paradoxes in different domains: Hesse's paradox of confirmation, Goodman's paradox, paradox in the definition of verisimilitude, Ross paradox.

Similarly, the classical valid ex falso quodlibet principle $\neg p \rightarrow (p \rightarrow q)$ is the chief cause for the disposition paradox and also for the paradoxes of Derived Obligation and Commitment.[3] We do not think that the mentioned paradoxes are a special type of paradoxes just in this domain. On the contrary, the underlying cause is much more general. It consists of some very tolerant properties of Classical Logic concerning valid inferences which are properties of irrelevance in the sense that something which can be replaced (in the consequence-class) by anything arbitrary cannot be relevant. The second part of the relevance criterion is a reduction criterion (RD) which reduces redundant repetitions, double negations, splits complex wffs into smallest conjuncts.

The system *RMQ* has two concepts of validity, a weaker one (material or classical validity) and a stronger one (strict validity). All the theorems of classical 2-valued propositional logic (CPC) are materially (classically) valid in *RMQ*. Those of CPC which obey a certain relevance restriction are strictly valid in *RMQ*. The so restricted system is called *RMQ** and avoids most of the well-known paradoxes in different areas where CPC is applied.

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[2] Schurz and Weingartner [3].

[3] This has been shown in Weingartner [6].

1.2 The Modal Logic Contained in *RMQ*

The modal logic contained in *RMQ* has two kinds of necessity and two kinds of possibility, 14 different modalities altogether (7 positive and 7 negative ones). It derives from a similar 6-valued matrix-calculus.[4] Although both systems do not have strong necessitation (i. e. if $\vdash p$ then $\vdash Lp$) the system *SS*1*M* contains all theorems of CPC as strictly (necessarily) valid, whereas *RMQ* allows as strictly valid only those theorems of CPC which (approximately) satisfy the criteria RC and RD (for avoiding the well-known difficulties), yet including all the important traditional principles of CPC. The modal system contained in *RMQ* is similar to the modal system *T* (of Feys or von Wright) concerning many theorems, except necessitation for all CPC-valid formulas. It also includes Brouwer's system *B*, though without necessitation w. r. t. all CPC-valid formulas.

1.2.1 Modal Theorems of *RMQ*

(1) $Lp \Leftrightarrow \neg M \neg p$
(2) $Mp \Leftrightarrow \neg L \neg p$
(3) $L \neg p \Leftrightarrow \neg Mp$
(4) $\neg Lp \Leftrightarrow M \neg p$
(5) $LLp \Leftrightarrow \neg MM \neg p$
(6) $MMp \Leftrightarrow \neg LL \neg p$
(7) $LL \neg p \Leftrightarrow \neg MMp$
(8) $\neg LLp \Leftrightarrow MM \neg p$
(9) $Lp \Rightarrow p$
(10) $p \Rightarrow Mp$
(11) $LLp \Rightarrow Lp \Rightarrow MLp \Rightarrow p \Rightarrow LMp \Rightarrow Mp \Rightarrow MMp$
(12) $L(p \to q) \Rightarrow (Lp \to Lq)$
(13) $L(p \wedge q) \Leftrightarrow (Lp \wedge Lq)$
(14) $M(p \wedge q) \Rightarrow (Mp \wedge Mq)$
(15) $(Lp \vee Lq) \Rightarrow L(p \vee q)$
(16) $L(p \to q) \Rightarrow (Mp \to Mq)$
(17) $L(p \vee q) \Rightarrow (Lp \vee Mq)$
(18) $[L(p \to q) \wedge p] \Rightarrow q$
(19) $[L(p \to q) \wedge \neg q] \Rightarrow \neg p$
(20) $[L(p \to q) \wedge L(q \to r)] \Rightarrow L(p \to r)$
(21) $L(p \to q) \Rightarrow L(\neg q \to \neg p)$

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[4] Weingartner [5]. This system was called there *SS*1*M*.

1.2.2 Comparison With Other Modal Systems

RMQ is well comparable to the Modal System T. Although T has a strong rule of necessitation which is not valid in *RMQ*, many of the main theorems of T are strictly valid in *RMQ*, too. Thus all theorems (1)–(10) of 1.2.1 are theorems of T. Furthermore, theorems (12)–(14), (16), (17) and (20) are theorems of T. T and *RMQ* behave also similar concerning important invalid wffs. For example in both systems $(p \rightarrow q) \vee (q \rightarrow p)$ is valid, but $(p \Rightarrow q) \vee (q \Rightarrow p)$ is invalid. Moreover, $(p \wedge q) \rightarrow (p \rightarrow q)$ is valid in T and materially valid in *RMQ*, but $(p \wedge q) \Rightarrow (p \rightarrow q)$ is invalid in both.

Concerning the modal system $S4$, *RM* does not have the reduction theorem $LP \rightarrow LLp$ of $S4$, which is also valid in $S5$, because it is a task of *RMQ* to distinguish two kinds of necessity, a stronger one and a weaker one; analogously *RMQ* distinguishes between two kinds of possibility, a stronger and a weaker one. Nevertheless, *RMQ* has the same number of different modalities as $S4$ (7 positive and 7 negative ones). But in *RMQ* they can be ordered into one line of strict implication (cf. theorem (11) above), whereas in $S4$ this is not possible and the non-modal proposition p is deductively connected only with Lp and Mp, but not with any other modality. Concerning modal system $S5$, the axiom which leads form $S4$ to $S5$, $Mp \rightarrow LMp$, is materially, but not strictly, valid in *RMQ*.

1.2.3 The Modal Logic of RMQ Contains the Syllogistic

This Modal Logic with the theorems of 1.2.1 and which is exactly defined in Sect. 1.2.1 below contains the whole Syllogistic with all 24 modes by a simple straightforward interpretation: The four sentence-types A, E, I, O of the Square of Opposition are represented by the modal square of opposition; i. e. the A and E sentence by *necessary* and the I and O sentence by *possible*.

1.3 Definition of the System RMQ

The system RMQ can be defined as the set of all formulas, which satisfy the matrix $M =< T, F, \neg, \vee, \wedge, \rightarrow, L >$ where $T = \{1, 2, 3\}$, $F = \{4, 5, 6\}$ and the operations $\neg$, $\vee$, $\wedge$, $\rightarrow$, L are defined as follows:

p	$\neg p$	$p \vee q$	1 2 3 4 5 6
1	6	1	1 1 1 1 1 1
2	5	2	1 2 2 2 1 2
3	4	3	1 2 3 1 3 3
4	3	4	1 2 1 4 4 5
5	2	5	1 1 3 4 5 5
6	1	6	1 2 3 5 5 6

$p \wedge q$	1 2 3 4 5 6	$p \to q$	1 2 3 4 5 6	Lp
1	1 2 3 4 5 6	1	1 2 3 5 5 6	1
2	2 2 3 4 6 6	2	1 1 3 5 5 5	3
3	3 3 3 6 5 6	3	1 2 1 4 5 5	6
4	4 4 6 4 5 6	4	1 2 3 1 3 3	6
5	5 6 5 5 5 6	5	1 2 2 2 1 2	6
6	6 6 6 6 6 6	6	1 1 1 1 1 1	6

$p \leftrightarrow q = df[(p \to q) \wedge (q \to p)]$

RMQ obeys the usual interdefinability between necessity (L) and possibility (M):$Mp = df\neg L\neg p$. Hence, the matrices for all the 7 positive modalities in RMQ are as follows:

p	$\neg p$	LLp	Lp	MLp	p	LMp	Mp	MMp
1	6	1	1	1	1	1	1	1
2	5	6	3	1	2	1	1	1
3	4	6	6	6	3	1	1	1
4	3	6	6	6	4	1	1	1
5	2	6	6	6	5	6	4	1
6	1	6	6	6	6	6	6	6

As it follows from the definition, every well-formed formula (wff) of *RMQ* is unambiguously determined by a particular matrix, according to the definition in 1.3, possessing either 6 or 36 or 216 ..., etc (in general $6^n, n = 1, 2, \ldots$) values. And any such particular matrix represents some well formed formula (wff) of the system *RMQ*.

Observe further that in a representation of CPC by matrices (truth tables) $A \vdash B$ coincides with (valid) $A \to B$; and for the representation of CPC by *RMQ*, $A \vdash B$ coincides with (valid) $A \to B$ (material implication) since all valid formulas of CPC are materially valid in *RMQ* (see 1.4 (6)).

1.4 Properties of the Underlying System RMQ

(1) *RMQ* is a 6-valued matrix system (3 values for truth, 3 for falsity) and so it contains its own semantics. Every well-formed formula of *RMQ* is unambiguously determined by a particular matrix which contains 6^n values for $n (n = 1, 2, \ldots)$ different propositional values.

(2) *RMQ* is motivated by two relevance criteria called replacement (RC) and reduction (RD), which avoid difficulties in the application of logic (see (6) and (7) below).[5]
(3) *RMQ* is consistent and decidable.
(4) *RMQ* has the finite model property.
(5) *RMQ* has two concepts of validity: a weaker one (classically valid which is identical with materially valid) and a stronger one (strictly valid). All theorems of two-valued Classical Logic (Classical Propositional Calculus CPC) are at least classically valid, that is materially valid, in *RMQ*. Only a restricted class of them are strictly valid in *RMQ*.
(6) The validity of a proposition is decided by calculating the highest value (cv) in its matrix. If $cv = 3$ the proposition (formula) is classically valid, that is materially valid. If $cv = 2$ the proposition (formula) is strictly valid.
(7) The strictly valid theorems of *RMQ* avoid a great number of well-known paradoxes in the domain of scientific explanation, law statements, disposition predicates, verisimilitude, ... etc.[6]
(8) The strictly valid theorems of *RMQ* avoid the well-known difficulties when logic is applied to physics; especially those with commensurability, distributivity and with Bell's inequalities.[7]
(9) *RMQ* is closed under transitivity of implication, and under modus ponens.
(10) *RMQ* also contains a modal system with 14 modalities, where Lp (necessary p) has the matrix: 1 3 6 6 6 6. $MP = \neg L \neg p$ (possible p).

1.5 *Theorems of RMQ*

1.5.1 Conventions

If → is the main connective, then → means that the formula (wff) is only materially valid ($cv = 3$); ⇒ means that the formula is at least strictly valid (i. e. valid with L in front; $cv \leq 2$); $(p \Rightarrow q) = df\, L(p \rightarrow q)$. Also wffs of which the main connective is ∨ (disjunction) may be only materially valid, i. e. if their $cv = 3$.

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[5] For an exact formulation of these criteria see Weingartner [7] Sect. 2 and [9] Sect. 2.

[6] See Weingartner [7] Sect. 4.3 and [8] Sect. 2.4.

[7] See Weingartner [7] Sects. 2.1, 2.2 and 4.2, 4.4. and [8] Sect. 2.

1.5.2 Theorems of CPC Which Are only Materially Valid (But Strictly Invalid) in *RMQ*

(1)	$\neg p \to (p \to q)$	Ex falso quod libet
(2)	$\neg p \to [p \to (q \wedge \neg q)]$	Ex falso quod libet
(3)	$(p \to \neg p) \to [p \to (q \wedge \neg q)]$	Ex falso quod libet
(4)	$p \to p \vee q)$	Redundant Element(s)
(5)	$p \to [p \vee (q \wedge \neg q)]$	Redundant Element(s)
(6)	$p \to [(p \wedge q) \vee (p \wedge \neg q)]$	Redundant Element(s)
(7)	$p \to (q \to p)$	Adding premise
(8)	$(p \wedge q) \to (p \leftrightarrow q)$	Conjunction and implication
(9)	$(p \wedge q) \to (p \to q)$	Conjunction and implication
(10)	$(p \wedge q) \to [(p \wedge r) \vee (q \wedge \neg r)]$	Conjunction and disjunction
(11)	$[p \wedge (q \vee r)] \leftrightarrow [(p \wedge q) \vee (p \wedge r)]$	Distribution
(12)	$[(p \vee q) \wedge (p \vee r)] \leftrightarrow [p \vee (q \wedge r)]$	Distribution
(13)	$p \vee (p \to q)$	intuitionistically invalid $cv = 3$
(14)	$(p \to q) \vee (p \to \neg q)$	intuitionistically invalid $cv = 3$
(15)	$(p \to q) \vee (\neg p \to q)$	intuitionistically invalid $cv = 3$

With the exception of the last three (13)–(15), the principles (CPC theorems) (1)–(10) are separated as classically valid bunt irrelevant by the Replacement Criterion RC: it is easily seen that at least one occurrence of the variable q (or p or r) in the consequent can be replaced by any variable salva validitate (veritate) of the CPC-theorem. The equivalence $p \leftrightarrow q$ in CPC-theorem (8) has to be split into two implications $(p \to q \wedge p \to q)$ in order to apply RC.

The direction $\to$ of (11) and (12) is forbidden by the Reduction Criterion RD. The last three (13)–(15) cannot be designated that way because the main connective is not an implication, but a disjunction.

However, for the system *RMQ* the form of the wff is not essential because the decision whether the wff is materially valid or strictly valid is determined by the cv of the matrix (cf. 1.4(6) above) which represents the wff. Thus, it can easily be checked that all the above listed principles (1)–(15) of CPC are (only) materially valid, but not strictly valid in *RMQ*.

1.5.3 Basic Theorems of CPC Which are Strictly (or Necessarily) Valid (Valid with L in Front) in *RMQ*

The main theorems which are strictly valid, are recognisable by their main connective $\Rightarrow$. We shall denote that subsystem of *RMQ* which contains only theorems which are strictly valid in *RMQ* by *RMQ**.

(1)	$(p \wedge q) \Leftrightarrow (q \wedge p)$	Commutation
(2)	$(p \vee q) \Leftrightarrow (q \vee p)$	Commutation
(3)	$[p \wedge (q \wedge r)] \Leftrightarrow [(p \wedge q) \wedge r)]$	Association
(4)	$p \Rightarrow p$	
(5)	$p \Leftrightarrow \neg\neg p$	Double negation
(6)	$p \wedge q \Rightarrow p$	Simplification
(7)	$p \wedge q \Rightarrow q$	Simplification
(8)	$p \vee p \Rightarrow p$	Simplification
(9)	$[(p \rightarrow q) \wedge p] \Rightarrow q$	Modus ponens
(10)	$[(p \rightarrow q) \wedge \neg q] \Rightarrow \neg p$	Modus tollens
(11)	$(p \rightarrow q) \Rightarrow (\neg q \rightarrow \neg p)$	Contraposition
(12)	$[(p \rightarrow q) \wedge (q \rightarrow r)] \Rightarrow (p \rightarrow r)$	Hypothetic syllogism (transitivity of →)
(13)	$[(p \vee q) \wedge \neg p] \Rightarrow q$	Disjunctive syllogism
(14)	$(p \wedge q) \Rightarrow \neg(\neg p \vee \neg q)$	De Morgan's law
(15)	$(p \vee q) \Rightarrow \neg(\neg p \wedge \neg q)$	De Morgan's law
(16)	$(\neg p \wedge \neg q) \Rightarrow \neg(p \vee q)$	De Morgan's law
(17)	$(\neg p \vee \neg q) \Rightarrow \neg(p \wedge q)$	De Morgan's law
(18)	$[(p \wedge q) \vee (p \wedge r)] \Rightarrow [p \wedge (q \vee r)]$	Distribution
(19)	$[p \vee (q \wedge r)] \Rightarrow [(p \vee q) \wedge (p \vee r)]$	Distribution
(20)	$[(p \wedge q) \vee (p \wedge \neg q)] \Rightarrow p$	
(21)	$[p \rightarrow (q \wedge r)] \Rightarrow [(p \rightarrow q) \wedge (p \rightarrow r)]$	
(22)	$[(p \rightarrow r) \vee (q \rightarrow r)] \Rightarrow [(p \wedge q) \rightarrow r]$	
(23)	$[r \rightarrow (p \rightarrow q)] \Rightarrow [(r \wedge p) \rightarrow q]$	
(24)	$\neg(p \wedge \neg p)$	Principle of non-contradiction strictly valid
(25)	$p \vee \neg p$	Principle of excluded middle or tertium non datur strictly valid

Those distribution laws which are strictly (or necessarily) valid (i. e. valid with L) are just the ones which must hold in a logic applicable to empirical sciences including Quantum Physics.

2 Syllogistics and the Square of Opposition Interpreted with the Modal Logic of *RMQ*

2.1 Historical Remarks: Aristotle

In his book on interpretation Chap. 12 Aristotle describes the modalities by saying which are contradictory and which are contrary opposites. He does not order them into a modal square of opposition but lists all modes (ibid. 22a ff.): "possible to be and not possible to be cannot be said truly of one and the same thing because these statements are opposites. And possible not to be and not possible not to be cannot be said truly at the same time of one and the same thing…"

He continues in Chap. 12 first by stating equivalences, for example: not possible to be is necessary not to be. Then by stating implications, for example: what is necessary not (the case) is (implies) what is not necessary (ibid. 22b). Many laws of Modal Logic are described in Chap. 13.[8]

2.2 Historical Remarks: Thomas Aquinas

Before Thomas Aquinas states the Modal Square of Opposition he begins to explain what a model proposition is and what a modus is. We quote the beginning part of this text since it is interesting in his formal and linguistic character and contains the essential part of Tarski's truth condition (Tarski's biconditional):

> "Since the modal proposition gets its name from 'modus', to know what a modal proposition is we must know what a *modus* is. Now a *modus* is a determination of something effected by a nominal adjective determining a substantive, e. g. 'white man', or by an adverb determining a verb. But it is to be known that modes are threefold, some determining the subject of a proposition, as a white man runs, some determining the predicate, as 'Socrates is a white man', or 'Socrates runs quickly', some determining the composition of the predicate with the subject, as that Socrates is running is impossible, and it is from this alone that a proposition is said to be modal. Other propositions, which are not modal, are said to be assertoric (*de inesse*).
>
> The modes which determine the composition are six: 'true', 'false', 'necessary', 'possible', 'impossible' and 'contingent'. But 'true' and 'false' add nothing to the signification of assertoric propositions; for there is the same significance in 'Socrates runs' and it is true that Socrates runs (on the one hand), and in "Socrates is not running" and "it is false that Socrates is running" (on the other). This does not happen with the other four modes, because there is not the same significance in 'Socrates runs' and 'that Socrates runs is impossible (or necessary)'. So we leave 'true' and 'false' out of consideration and attend to the other four. Now because the predicate determines the subject and not conversely, for a proposition to be modal the four modes aforesaid must be predicated and the verb indicating composition must be put as subject. This is done if an infinitive is taken in place of the indicative verb in the proposition, and an accusative in place of the nominative. And it (the accusative and infinitive clause) is called 'dictum', e. g. of the proposition 'Socrates runs' the dictum is 'that Socrates runs' (*Socratem currere*). When then the dictum is posited as subject and a mode as predicate, the proposition is modal, e. g. 'that Socrates runs is possible'. But if it be converted it will be assertoric, e. g. 'the possible is that Socrates runs'.
>
> Of modal propositions one kind concerns the dictum, another concerns things. A modal (proposition) concerning the dictum is one in which the whole dictum is subjected and the mode predicated, e. g. 'that Socrates runs is possible'. A modal (proposition) concerning things is one in which the mode interrupts the dictum, e. g. 'for Socrates running is possible' (*Socratem possibile est currere*). But it is to be known that all modals concerning the dictum are singular, the mode being posited as inherent in this or that as in some singular thing. But … modals concerning things are judged to be universal or singular or indefinite according to the subject of the dictum, as is the case with assertoric propositions. So that 'for all men, running is possible' is universal, and so with

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[8] For studies of Aristotle's Modal Logic cf. Bochenski [1, 2].

> the rest. It should further be known that modal propositions are said to be affirmative or negative according to the affirmation or negation of the mode, not according to the affirmation or negation of the dictum. So that . . . this modal ‘that Socrates runs is possible’ is affirmative, while ‘that Socrates runs is not possible’ is negative.”[9]

Thomas Aquinas then states the difference between modality *de dicto* and modality *de re* and the respective forms of such modal propositions. Finally he distinguishes four orders: The first, he says, is *possibile est esse* and its equivalent forms; the second *possibile est non esse* (and equivalences) the third *impossibile est esse* (and equivalences) the fourth *necesse est esse* (and equivalences).

Then he gives the description of the modal square of opposition: The fourth order is contrary to the third, the first order is subcontrary to the second, the third is contradictory to the first and the fourth contradictory to the second, finally the first is subaltern to the fourth and the second to the third. Before he gives the picture for the modal square of opposition (see Fig. 2) he cites a verse used for teaching logic in medieval times that explains the Latin words in the four squares: “Primus amabimus, edentulique secundus. Tertius illiace, purpurea reliquus.”

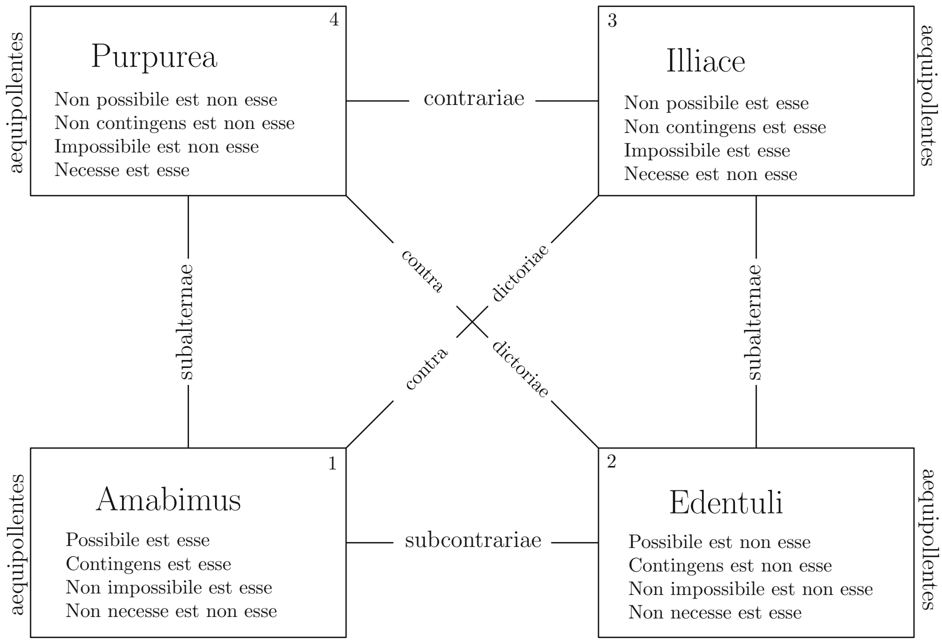

Fig. 2 Square of opposition

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[9] Thomas Aquinas, De Propositionibus Modalibus 719–721 [4]. Translation by Bochenski [2], 29.09.

2.3 Syllogistic A, E, I and O: Propositions Interpreted in the Modal Logic of RMQ

The interpretation is very simple. *A* and *E* propositions are interpreted with necessity, *I* and *O* propositions with possibility. We use only propositional logic. The subject-predicate relation in the *A* and *E* proposition we interpret with an implication; the subject-predicate relation in the *I* and *O* proposition we interpret with a conjunction. The propositions containing subject term middle term and predicate term are represented by the propositions p, q, r as follows:

Proposition containing the subject term	…	p
Proposition containing the middle term	…	q
Proposition containing the predicate term	…	r

The translation of the *A*, *E*, *I*, *O* propositions into modal propositions of *RMQ* are as follows:

$$\begin{array}{lll} SaP & = df & L(p \rightarrow r) \\ SeP & = df & L(p \rightarrow \neg r) \\ SiP & = df & M(p \wedge r) \\ SoP & = df & M(p \wedge \neg r) \end{array}$$

2.4 The Syllogistic Modes Are Theorems of the Modal Logic of RMQ

Since *RMQ* including its modal logic is decidable every syllogistic mode interpreted in the way explained above is decidable too or follows from *RMQ*. "Follows" or "logically follows" means in *RMQ* just that the representative implicational proposition gets the value $cv = 3$, or if it strictly follows the value $cv = 2$. The syllogistic mode is decidable by showing that its matrix in *RMQ* gets a $cv = 3$ or ≤ 3 cf. Sect. 1.4, (6).

In fact all 24 syllogistic modes get a $cv = 1$, i. e. are strongly true in *RMQ*. This holds for the 15 syllogisms which do not make existential presuppositions (when interpreted in First Order Predicate Logic) straightforwardly, and for the remaining ones if one adds the premise that the antecedence is possible.

1st Figure:

$L(q \to r) \land L(p \to q) \xrightarrow{1} L(p \to r)$ Barbara

$L(q \to \neg r) \land L(p \to q) \xrightarrow{1} L(p \to \neg r)$ Celarent

$L(q \to r) \land M(p \land q) \xrightarrow{1} M(p \land r)$ Darii

$L(q \to \neg r) \land M(p \land q) \xrightarrow{1} M(p \land \neg r)$ Ferio

$L(q \to r) \land L(p \to q) \land Mp \xrightarrow{1} M(p \land r)$ Barbari

$L(q \to \neg r) \land L(p \to q) \land Mp \xrightarrow{1} M(p \land \neg r)$ Celaront

2nd Figure:

$L(r \to \neg q) \land L(p \to q) \xrightarrow{1} L(p \to \neg r)$ Cesare

$L(r \to q) \land L(p \to \neg q) \xrightarrow{1} L(p \to \neg r)$ Camestres

$L(r \to \neg q) \land M(p \land q) \xrightarrow{1} M(p \land \neg r)$ Festino

$L(r \to q) \land M(p \land \neg q) \xrightarrow{1} M(p \land \neg r)$ Baroco

$L(r \to \neg q) \land L(p \to q) \land Mp \xrightarrow{1} M(p \land \neg r)$ Cesaro

$L(r \to q) \land L(p \to \neg q) \land Mp \xrightarrow{1} M(p \land \neg r)$ Camestrop

3rd Figure:

$L(q \to r) \land L(q \to p) \land Mq \xrightarrow{1} M(p \land r)$ Darapti

$L(q \to \neg r) \land L(q \to p) \land Mq \xrightarrow{1} M(p \land \neg r)$ Felapton

$M(q \land r) \land L(q \to p) \xrightarrow{1} M(p \land r)$ Disamis

$L(q \to r) \land M(q \land p) \xrightarrow{1} M(p \land r)$ Datisi

$M(q \land \neg r) \land L(q \to p) \xrightarrow{1} M(p \land \neg r)$ Bocardo

$L(q \to \neg r) \land M(q \land p) \xrightarrow{1} M(p \land \neg r)$ Ferison

4th Figure:

$M(r \land q) \land L(q \to p) \xrightarrow{1} M(p \land r)$ Dimaris

$L(r \to \neg q) \land M(q \land p) \xrightarrow{1} M(p \land \neg r)$ Fresison

$L(r \to q) \land L(q \to p) \xrightarrow{1} L(p \neg r)$ Camenes

$L(r \to q) \land L(q \to p) \land Mr \xrightarrow{1} M(p \land r)$ Bamalip

$L(r \to q) \land L(q \to \neg p) \land Mp \xrightarrow{1} M(p \land \neg r)$ Camenop

$L(r \to \neg q) \land L(q \to p) \land Mq \xrightarrow{1} (p \land \neg r)$ Fesapo

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Two Standard and Two Modal Squares of Opposition

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Abstract In this study, we examine modern reading of the Square of Opposition by means of intensional logic. Explicit use of possible world semantics helps us to sharply discriminate between the standard and modal ('alethic') readings of categorical statements. We get thus two basic versions of the Square. The Modal Square has not been introduced in the contemporary debate yet and so it is in the centre of interest. Some properties ascribed by medieval logicians to the Square require a shift from its Standard to Modal version. Not inevitably, because for each of the two squares there exists its mate which can be easily confused with it. The discrimination between the initial and modified versions of the Standard and Modal Square enable us to separate findings about properties of the Square into four groups, which makes their proper comparison possible. The disambiguation so achieved leads to the solution of various puzzles often mentioned in recent literature.

Keywords Modal logic • Modal Square of opposition • Possible world semantics • Square of opposition

Mathematics Subject Classification Primary 03B45 · Secondary 03B60 · 03B65 · 03C80

1 Introduction

Does "All chimeras are creatures" entail "Some chimeras are creatures"? Which of the two statements has existential import? These are examples of several persistent and controversial questions related to the Square of Opposition which are addressed in this paper. Rather than the classical (traditional) reading of the Square, the starting point of the present investigation is the *Standard Square* of modern logic textbooks. Only a few enrichments to it are made when utilizing intensional logic which is capable to fix e.g. partiality failures related to existence issues.

The main contribution of our investigations consists in a disambiguation of the prevailing discourse about the Square.[1] We show a rival of the *modern reading* of the

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[1] Needles to remind the present reader of the recent wave of scholars, papers, books and events organized by J.-Y. Béziau which are focused on Squares, Hexagons and other figures displaying oppositions (cf. [3]).

J.-Y. Béziau, G. Basti (eds.), *The Square of Opposition: A Cornerstone of Thought*,
Studies in Universal Logic, DOI 10.1007/978-3-319-45062-9_8

Standard Square, we call it *modified* modern reading. Not in the former, but in the latter Square the entailment of "Some non-self-identical objects are non-self-identical objects" from "All non-self-identical objects are non-self-identical objects" holds, the particular statement is thus not false because of existential import. The modified modern reading of the Standard Square simply treats another quadruple of sentences than the unmodified modern reading. Gottschalk's pioneer modern work on the Square [13] seems to contain this second kind of Square.

But another progress must be made because there remain questions unanswered by the first attempt just mentioned. We will thus expose two readings of the *Modal Square* of Opposition, i.e. the Square deploying modal versions of categorical statements. Altogether, we thus treat a tetrad of related Squares. This move is required because some categorical sentences and puzzles involving them implicitly presuppose modality. (Of course, our results are also applicable for the case of explicit modality.) For example, if "All chimeras are creatures" is meant as a necessary (*de dicto*) statement, then in every world a chimera exists, it is a creature; this holds also in at least one world and the entailment of "Some chimeras are creatures" thus holds.

The Modal Square of Opposition comes in two versions, the *modern reading* and the *modified* modern reading. The latter one has been met already by medieval logicians. We will find e.g. that contrariety and subcontrariety do not hold in it. This Square is thus not a simple projection of the modified modern reading of the Standard Square in which the two relations hold.

The Modal Square of the modern reading is the most novel one. Not only because of this, most findings stated in this paper focus, directly or less directly, on this Square. We find, for example, that subalternation is nearly valid in it, the exception only being made by void properties which are ignored even by some contemporary metaphysicians.

The present paper is organized as follows. We begin, Sect. 2, with a brief introduction to the logic convenient for our purposes, viz. Transparent Intensional Logic. In Sect. 3, we expose modern reading of the Standard Square, stating familiar and also some less familiar facts. The modified modern reading of the Standard Square is exposed in Sect. 4, where we compare it with Gottschalk's Square. This Section is followed by a short Sect. 5 which treats the modified reading of the Modal Square. Section 6 concerns modal reading of categorical statements and is preparatory for Sect. 7 in which we expose the modern reading of the Modal Square. The conclusions close the paper in Sect. 8.

2 A Brief Introduction to Transparent Intensional Logic

2.1 Semantic Scheme, Constructions, Type Theory, Deduction

Pavel Tichý's *Transparent Intensional Logic* (*TIL*) which we use here for its convenience is a substantial modification of Church's typed λ-calculus [8]. Rivalling the well-known system by Montague, the most important applications of TIL are in semantics of natural language (propositional attitudes, subjunctive conditionals, modalities, verb tenses, etc.).[2] There are many reasons why to adopt TIL: as a framework, it is rather huge and so is capable to treat many phenomena not only in logical analysis of natural language as suggested in this paragraph or Sect. 2.2.

The *semantic scheme* employed in TIL involves the level of *hyperintensions* the need for which was repeatedly argued in recent literature[3]:

- expression E *expresses:*
- construction C, i.e. *meaning* explicated as hyperintension; E *denotes*, C *constructs*:
- intension/extension, i.e. explication of *denotation*.[4]

Constructions are structured abstract entities of algorithmic nature (for their careful description and defence see esp. [34]). They have an 'intensional principle of individuation': every object is constructed by infinitely many equivalent, but not identical constructions. For instance, the number eight is constructed e.g. by multiplying four by two or the square root of sixty four, which are two distinct, yet congruent constructions. Every construction C can be specified by i. the object O constructed by C, ii. the way how (by means of which subconstructions) C constructs O.

Constructions are usually written by familiar λ-terms such as:

"$\boldsymbol{X}$" | "x" | "$[\boldsymbol{F}\boldsymbol{X}]$" | "$\lambda x[\boldsymbol{F}x]$".

Note thus that constructions—not usual set-theoretic entities—are direct semantic values of TIL λ-terms. The behaviour of four basic kinds of constructions can be described in a simplified way as follows. Dependently on valuation v,

i. the trivialization $\boldsymbol{X}$ v-constructs directly the entity (a non-construction or construction) X;
ii. the variable x_k v-constructs the kth object in the sequence (a part of v) of objects of the type the variable ranges over;
iii. the composition $[\boldsymbol{C}\,\boldsymbol{C_1}\ldots\boldsymbol{C_n}]$ (where $\boldsymbol{C}, \boldsymbol{C_1}, \ldots, \boldsymbol{C_n}$ are any constructions) v-constructs the value (if any) of the n-ary function (if any) v-constructed by $\boldsymbol{C}$ at the argument (if any) v-constructed by $\boldsymbol{C_1}, \ldots, \boldsymbol{C_n}$;
iv. the closure $\lambda x_1 \ldots x_n\boldsymbol{C}$ v-constructs the n-ary function from (strings of) values of $x_1, \ldots, x_n$ to the corresponding results of $\boldsymbol{C}$.

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[2] For applications of TIL see esp. [12, 25, 34, 35]. For simplicity reasons, I will entirely suppress temporal parameter below.

[3] One of several reasons for adoption of hyperintensions is this. According to possible world semantics (PWS), the meaning of all true mathematical sentences is one and the same, viz. the proposition true in all possible worlds. Consequently, the argument "Alice believes that $8 = 8$. Therefore, Alice believes that $2 \times 4 = \surd 64$." is evaluated as valid, which is intuitively not, Alice cannot be omniscient. Obviously, there is a structuredness issue which is relevant to the invalidity of such arguments. Semantic theory of TIL solves the paradox of omniscience by discriminating between the proposition true in all possible worlds, which is not the meaning but denotation of the sentences, and its infinitely many constructions differing in their structure.

[4] Of course, it may happen that a construction is denoted by an expression which expresses a higher-order construction of the denoted construction. Moreover, expression can lack denotation or even meaning.

Constructions of well-known binary logical or mathematical operations will be written in infix manner. Constructions of form [$\boldsymbol{C}\ w$] will be abbreviated to $\boldsymbol{C}_w$. Brackets will often be omitted; sometimes, they will be eliminated with help of so-called dot convention, whereas the dot indicates the left-hand bracket and the corresponding right-hand bracket should be imagined as far right as it is consistent.

TIL utilizes an instance of Tichý's *type theory*, which is a substantial modification of Church's simple theory of types [8]. Let base B be a non-empty class of pairwise disjoint collections (sets) of primitive objects, e.g.

$$\mathrm{B}_{\mathrm{TIL}} = \{\iota, o, \omega\},$$

where ι is the type of individuals, o is the type of (two) truth-values and ω is the type of possible worlds. The hierarchy of (first-order) types is defined inductively as follows (where $\xi_{(i)}$ is any type):

i. Any member of B is a *type over B*.
ii. If $\xi_1, \ldots, \xi_n, \xi$ are types over B, then $(\xi\xi_1 \ldots \xi_n)$—i.e. the collection of all total and partial n-ary functions from $\xi_1, \ldots, \xi_n$ to ξ—is a *type over B*.

Tichý [34] ramified his simple type theory to enable non-circular quantification over constructions. But this feature of TIL will not be, with some exceptions, used here for simplicity reasons. Nevertheless, the typing of constructions used below is in full conformity with Tichý's late type theory (cf. [34], Definition 16.1).

For *logical analysis* of natural language expressions we utilize both extensions and intensions over $\mathrm{B}_{\mathrm{TIL}}$. *Intensions* are (total or partial) functions from possible worlds; they are of type $(\xi\omega)$, which will be abbreviated to "ξ_ω". They comprise *propositions*, *properties*, etc. Intensions are chosen for denotation of expressions whose reference varies across the logical space, e.g. "(be) dog", "the U.S. president", "It rains in Paris".

Now let us list types of some main objects we discuss below ("/" abbreviates "v-constructs an object of type"; "$X, Y/\xi$" abbreviates "$X/\xi, Y/\xi$"):

- x/ξ (an object belonging to the type ξ, briefly: a ξ-object; below, ξ is usually ι)
- $\mathbf{U}/(o\xi)$ (the universal ξ-class); $\boldsymbol{\emptyset}/(o\xi)$ (the total empty ξ-class)
- $p,q/o_\omega$ (a proposition); let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be any constructions of the propositions P and Q
- $f, g/(o\xi)_\omega$ (a property of ξ-objects; its *extension* in W is of type $(o\xi)$; let $\boldsymbol{F}$ and $\boldsymbol{G}$ be any constructions of ξ-properties F and G
- w/ω (a possible world)
- $\boldsymbol{\forall}^\xi/(o(o\xi))$ (the class containing the universal ξ-class); $\boldsymbol{\exists}^\xi/(o(o\xi))$ (the class containing all nonempty ξ-classes)
- $c^k, d^k/*_k$ (a k-order construction)
- $\mathbf{1}, \mathbf{0}/o$ (the truth values True, False); o/o (a truth value)
- $\boldsymbol{\neg}/(oo)$ (the classical negation); $\boldsymbol{\wedge}, \boldsymbol{\vee}, \boldsymbol{\rightarrow}/(ooo)$ (the classical conjunction, disjunction, material conditional); boldface will be suppressed
- $\boldsymbol{=}^\xi, \boldsymbol{\neq}^\xi\ /(o\xi\xi)$ (the familiar relations between ξ-objects); "$^\xi$" will be usually suppressed even in the case of other functions/relations
- $\boldsymbol{\subseteq}/(o(o\xi)(o\xi))$ (the familiar relation between ξ-classes)
- $\boldsymbol{\cap}/((o\xi)(o\xi)(o\xi))$ (the familiar operation on ξ-classes).

The expressions “Fido is a dog” and “There is a dog” are analysed in TIL as expressing the propositional constructions (i.e. constructions of o_ω-objects) $\lambda w[\mathbf{Dog}_w\ \mathbf{Fido}]$ and $\lambda w.\exists\lambda x[\mathbf{Dog}_w\ x]$, where $\mathbf{Fido}/\iota$ (the individual); $\mathbf{Dog}/(o\iota)_\omega$ (the ι-property).

Tichý [32, 33] proposed also a system of *deduction* for his simple type theory. Sequents of the system are made from matches. The match $x{:}\boldsymbol{C}$ means that the variable x v-constructs the same ξ-object as the (compound) construction $\boldsymbol{C}$. Deduction rules are made from sequents. *Definitions* can be viewed as certain rules of form $\vdash x{:}\boldsymbol{C} \Leftrightarrow x{:}\boldsymbol{D}$, where “$\Leftrightarrow$” indicates interderivability of the matches written on its sides. Let “$\boldsymbol{C} \Leftrightarrow \boldsymbol{D}$” abbreviate “$\vdash x{:}\boldsymbol{C} \Leftrightarrow x{:}\boldsymbol{D}$”. A simple example of definition: $\boldsymbol{\emptyset} \ \ \Leftrightarrow_{df} \lambda x\mathbf{0}$ (note that classes are construed as characteristic functions).

2.2 Fixing Partiality; Properties of Propositions and Their Constructions

Partial functions adopted in TIL enable us to aptly model partiality phenomena such as empty descriptions or gappy propositions but also results of existential import.

Partiality of functions usually causes abortiveness of constructions. A composition $[\boldsymbol{C}\ x]$ is *v-improper*, i.e. v-constructing nothing at all, if the function constructed by $\boldsymbol{C}$ is not defined for the value of x.

Adoption of partial functions and improper constructions has a strong impact on the logical theory, since most classical laws do not hold (cf. [26]). They have to be amended. *De Morgan Law for exchange of quantifiers*, for instance, must be protected against the case when the extension of the property F or G is a partial class (i.e. a characteristic function undefined for some arguments), which would cause v-improperness of $[[f_w x] \rightarrow [g_w x]]$ and then invalidity of the law[5]:

$$\neg\forall\lambda x.[f_w x]! \rightarrow [g_w x]! \Leftrightarrow \exists\lambda x.\neg[[f_w x]! \rightarrow [g_w x]!].$$

“$[\ldots w \ldots]!$” abbreviates “$[\mathbf{True}^{\mathbf{T}\pi}{}_w\ \ \lambda w'[\ldots w' \ldots]]$” which serves as a ‘*definiteness operator*’. The total notion of truth of propositions involved in it is definable by

$$[\mathbf{True}^{\mathbf{T}\pi}{}_w\ p] \Leftrightarrow_{df} \exists\lambda o.[p_w = o] \wedge [o = \mathbf{1}],$$

where $\mathbf{True}^{\mathbf{T}\pi}/(oo_\omega)_\omega$ (the property of propositions).[6] We will return to improperness and the role of ! in existential import in Sect. 3.4.

The second important feature of our logical framework is the fact that properties of constructions of certain objects correspond to (‘supervene on’) properties of those objects. We may therefore simply speak about properties of objects without a strict need to speak also about properties of constructions of those objects (in consequence of this, the reader is assumed to compile appropriate definitions of notions applicable to constructions herself).

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[5] On the right side of $\Leftrightarrow$, ! can be omitted. Below, we will also use ! even if it is not inevitable.

[6] Compare it with $[\mathbf{True}^{\mathbf{P}\pi}{}_w\ p] \Leftrightarrow_{df} [p_w = \mathbf{1}]$. To the two notions of truth there correspond two notions of falsity. Cf. [27] for analysis of truth in TIL.

To illustrate, the truth$^{\pi}$ of a proposition P makes all constructions of P true*; truth of constructions is thus easily definable with help of truth of propositions:

$$[\mathbf{True}^{\mathbf{PT}*\boldsymbol{k}}{}_{w}\ c^{k}] \Leftrightarrow_{df} [\mathbf{True}^{\mathbf{T}\boldsymbol{\pi}}{}_{w}\ {}^{2}c^{k}],$$

where $\mathbf{True}^{\mathbf{PT}*\boldsymbol{k}}/(o*_{k})_{\omega}$ (the property of k-order constructions). The construction of form ${}^{2}\boldsymbol{C}$ (called "double execution") v-constructs the object, if any, v-constructed by what is v-constructed by $\boldsymbol{C}$.

For another important example:

$$[p \models^{\pi} q] \quad \Leftrightarrow_{df} \forall\lambda w[p_{w}! \rightarrow q_{w}!]$$

$$[c^{k} \models d^{k}] \Leftrightarrow_{df} [{}^{2}c^{k} \models^{\pi} {}^{2}d^{k}]$$

where $\models^{\pi}\ /(oo_{\omega}o_{\omega})$ (in fact, $\models^{\pi}$ is the relation $\subseteq$ between classes of worlds); $\models\ /(o*_{k}*_{k})$ (the relation between constructions), which is preferred of the two notions. Below, we will steadily omit brackets of "$[c^{k} \models d^{k}]$" as well as proper indication that the constructions flanking $\models$ are introduced, using their trivializations, as constructions per se.

3 Modern Reading of the Standard Square of Opposition

3.1 Categorical Statements

Within TIL, the four familiar *categorical statements*, each being a propositional construction, are captured with help of "!":

Abbreviated form	Full form	Usual verbal expression
$\boldsymbol{A}$	$\lambda w.\forall\lambda x.[\boldsymbol{F}_{w}x]! \rightarrow [\boldsymbol{G}_{w}x]!$	"Every *F* is *G*."
$\boldsymbol{E}$	$\lambda w.\forall\lambda x.[\boldsymbol{F}_{w}x]! \rightarrow \neg[\boldsymbol{G}_{w}x]!$	"No *F* is *G*."
$\boldsymbol{I}$	$\lambda w.\exists\lambda x.[\boldsymbol{F}_{w}x]! \wedge [\boldsymbol{G}_{w}x]!$	"Some *F* is *G*."
$\boldsymbol{O}$	$\lambda w.\exists\lambda x.[\boldsymbol{F}_{w}x]! \wedge \neg[\boldsymbol{G}_{w}x]!$	"Some *F* is not *G*."

Each abbreviated form is definable by its full form, e.g.

$$\boldsymbol{A}_{w} \Leftrightarrow_{df} \forall\lambda x.[\boldsymbol{F}_{w}x]! \rightarrow [\boldsymbol{G}_{w}x]!$$

Each full form construction is the logical analysis of the respective verbal expression. Below, we will sometimes loosely speak also about those verbal expressions as categorical statements. Moreover, we will often speak about categorical statements with help of the expressions "***A***"—"***O***".

For the reason discussed in the end of Sect. 3.4, some authors formalize *O*-statements with help of ¬ and ∀. In our construal, which takes an advantage of !, the 'negation first' forms of *O*- (etc.) statements are equivalent (because of the law of quantifier exchange) to the non-'negation first' ones:

$$\lambda w.\exists\lambda x.[\boldsymbol{F}_{w}x]! \wedge \neg[\boldsymbol{G}_{w}x]! \Leftrightarrow \lambda w.\neg\forall\lambda x.[\boldsymbol{F}_{w}x]! \rightarrow [\boldsymbol{G}_{w}x]!.$$

The four categorical statements can equivalently be rephrased using generalized quantifiers All, Some and No, adding here also the nameless [2] quantifier NotAll[7]:

$[[\mathbf{All}\, f_w]\, g_w]$	$\Leftrightarrow_{df}$	$\forall\lambda x.[f_w x]! \rightarrow [g_w x]!$	(btw. $\Leftrightarrow [f_w \subseteq g_w]$)
$[[\mathbf{No}\, f_w]\, g_w]$	$\Leftrightarrow_{df}$	$\forall\lambda x.[f_w x]! \rightarrow \neg\, [g_w x]!$	(btw. $\Leftrightarrow$ $[f_w \cap g_w] = \emptyset$)
$[[\mathbf{Some}\, f_w]\, g_w]$	$\Leftrightarrow_{df}$	$\exists\lambda x.[f_w x]! \wedge [g_w x]!$	(btw. $\Leftrightarrow$ $[f_w \cap g_w] \neq \emptyset$)
$[[\mathbf{NotAll}\, f_w]\, g_w]$	$\Leftrightarrow_{df}$	$\neg\forall\lambda x.[f_w x]! \rightarrow [g_w x]!$	(btw. $\Leftrightarrow$ $\neg[f_w \subseteq g_w]$)

where **All**, **Some**, **No**, **NotAll**/$((o(o\xi))(o\xi))$. We thus have e.g. “All Fs are Gs” with the meaning $\lambda w[[\mathbf{All}\ \boldsymbol{F}_w]\ \boldsymbol{G}_w]$.

3.2 The Modern Reading of the Standard Square

As is well known, noteworthy logical relations between couples of categorical statements are often couched in the square while its edges and diagonals represent relations obtaining between statements written in its vertices. The *modern reading* of the Square does not fully preserve the classical construal, thus the only important relation is contradictoriness; the square is then rather a big “X”[8], see Fig. 1 below.

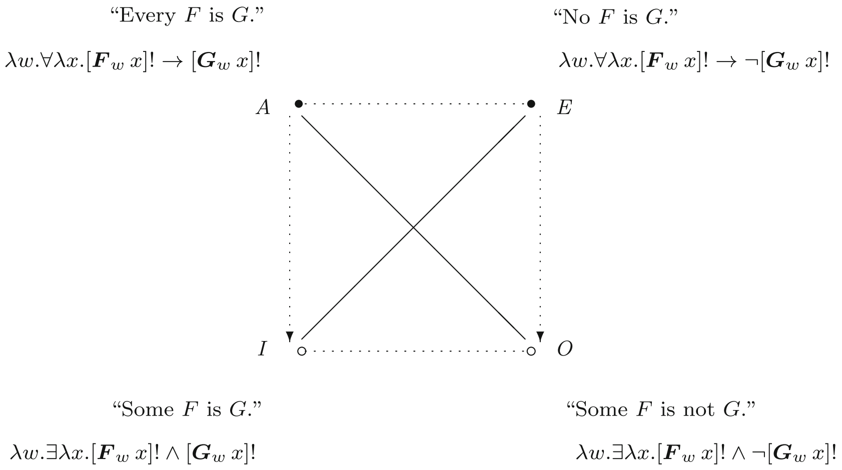

Fig. 1 The Standard Square in its modern reading

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[7] Adopted from [23, 24]. The definitions of **All** and **Some** are borrowed from [32] where Tichý showed also concise proofs of various relations between constructions involving them. Further remarks: definienda suggested in brackets are usual in the topic of generalized quantifiers; common construal of generalized quantifiers as (binary) relations between classes presupposes Schönfinkel’s reduction (‘currying’) which is not generally valid in the logic utilizing partial functions.

[8] Cf. [16].

— —	Contradictoriness	———	The relation holds
— →	Subalternation	- - - - - -	The relation holds with an exception
•— —•	Contrariety	· · · · · · · · ·	The relation does not hold
○— —○	Subcontrariety		

Legend.

3.3 *Contradictoriness and Equivalences*

The notion of *contradictoriness* is definable by

$$[\mathbf{Contradictory}\ \boldsymbol{P}\,\boldsymbol{Q}] \Leftrightarrow_{df} \boldsymbol{\forall}\lambda w.\neg[\boldsymbol{P}_w! \leftrightarrow \boldsymbol{Q}_w!],$$

where **Contradictory**/$(oo_\omega o_\omega)$ (the relation between propositions).[9] ∀ (or ∃) is here of type (oo_ω) and can be written also as "□" (or "◇").

Usual formalization of relations in the Square—according to which P and Q are contradictories iff $\neg(P \leftrightarrow Q)$, contraries iff $P \uparrow Q$, subcontraries iff $P \vee Q$, Q is subalternate to P iff $P \to Q$—ignores modality, which obfuscates the non-trivial reasons why the relations do not generally hold in the modal interpretation of the Square (cf. Sects. 7.3 and 7.4).

Let us add that two well-known equivalence relations not diagrammed in the above Square, viz. *contraposition* (e.g. "Every F is G" ⇔ "Every non-G is non-F") and *obversion* (e.g. "No F is G" ⇔ "Every F is non-G"), are fully confirmed on this reading. For that sake, one utilizes the function Non- of type $((o\xi)_\omega(o\xi)_\omega)$, while $[[\mathbf{Non\text{-}}\ f]_w\ x] \Leftrightarrow_{df} \neg[f_w\ x]$.[10]

3.4 *Problems of Existential Import*

As is well known, the Standard Square is constructed in the modern reading only to preserve contradictories (and contrapositions and obversions), whereas subalternation and (sub)contrariety[11] are omitted because of existential import. The 'term' $\boldsymbol{F}$ has *existential import* in a statement $\boldsymbol{P}$ of which it is a subconstruction iff $\boldsymbol{P}$ is not true for there is no F in W; we then say that $\boldsymbol{P}$ lacks existential import iff $\boldsymbol{F}$ can have no existential import in it.

Suppose that there is no F in W. The (intuitive) I- and O-statements are then in natural sense false in W, $\boldsymbol{F}$ has existential import in them (however, cf. the modal reading below). But A- and E-statements are on the modern reading true in W, not false, because the

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[9] Recall that this concept can be utilized for an apt definition of contradictoriness between propositional constructions.

[10] [22, 24]. This definition does not blur the (Aristotle's) difference between 'infinite' ("non-") and 'finite' (¬) negation because 'infinite' negation is definable rather by $[[\mathbf{Non}^{\mathbf{T}}\mathbf{\text{-}}\ f]_w\ x] \Leftrightarrow_{df} \neg[f_w x]!$.

[11] Abbreviating thus "contrariety and subcontrariety".

modern logic explicates the (intuitive) A- and E-statements as lacking existential import. In a consequence of this,

$\boldsymbol{A} \not\models \boldsymbol{I}$ and $\boldsymbol{E} \not\models \boldsymbol{O}$.

From this, invalidity of subalternation and (sub)contrariety follows.

To preserve subalternation and (sub)contrariety would require strange existential assumptions such as that only affirmative statements have existential import, which was recently embraced by Parsons [19, 20]. Not only that such proposal gives up the logical basis of the modern reading, it also contradicts our intuition. For instance, "All chimeras are creatures" or "All ogres are ogres" are usually understood as true, thus lacking existential import, i.e. not entailing existence of chimeras or ogres. For another intuitive fact, "Some women are not mothers" naturally entails existence of women, having thus existential import, which is likewise abandoned on Parsons' construal.

Now let us explain how partiality relates to existential import. Suppose that $[\boldsymbol{F}_w x]$ v-constructs nothing (is v-improper). Then, e.g. the construction $\lambda w.\forall\lambda x.[\boldsymbol{F}_w x]\rightarrow[\boldsymbol{G}_w x]$ is false because $\rightarrow$ does not receive an argument, $\lambda x.[\boldsymbol{F}_w x] \rightarrow[\boldsymbol{G}_w x]$ thus v-constructs a partial, not the universal, class and so $\forall$ returns 0 to it.

There are two possible causes why $[\boldsymbol{F}_w x]$ v-constructs nothing: i. the property F is not defined for the given world W; ii. $\boldsymbol{F}_w$ v-constructs a partial class which is not defined at W for the value of x—the property F is inapplicable to it. If F is not defined for other values of x, it is an example of existential import as well. Both causes of failure are fixed by employing ! because $[\boldsymbol{F}_w x]!$ v-constructs 0 on such v.

The v-improper construction $[\boldsymbol{F}_w x]$ is a subconstruction of all constructions $\boldsymbol{A}$–$\boldsymbol{O}$. If not containing !, each of them would be false on such v. Thus, e.g. $\boldsymbol{O}$ would not be contradictory to $\boldsymbol{A}$. Contradictoriness would then only be preserved if $\boldsymbol{O}$ and $\boldsymbol{E}$ were in 'negation first' forms, i.e. $\boldsymbol{O}$ being $\lambda w.\neg\forall\lambda x.[\boldsymbol{F}_w x]\rightarrow[\boldsymbol{G}_w x]$, not $\lambda w.\exists\lambda x.[\boldsymbol{F}_w x]\wedge\neg[\boldsymbol{G}_w x]$. This 'negation first' form of O-statements was already proposed by Aristotle in *De Interpretatione* ([1], see e.g. [20]), but it is doubtful whether it was for the same reason we suggest here.

3.5 *Subalternation, Contrariety and Subcontrariety*

In the classical construal, the proposition Q is *subaltern* of P iff Q must be true if P is true and P must be false if Q is false. Using TIL,

$[\mathbf{Subaltern}\ \boldsymbol{Q}\,\boldsymbol{P}] \Leftrightarrow_{df} \forall\lambda w[\boldsymbol{P}_w! \rightarrow \boldsymbol{Q}_w!]$.[12]

The definiens clearly shows the relationship of subalternation to entailment from single statements: these relations are identical because $[\mathbf{Subaltern}\ \boldsymbol{Q}\,\boldsymbol{P}] \Leftrightarrow \boldsymbol{P} \models^{\pi} \boldsymbol{Q}$.

Since on the modern reading $\boldsymbol{A} \not\models \boldsymbol{I}$ and $\boldsymbol{E} \not\models \boldsymbol{O}$, subalternation does not generally hold. The lack of subalternation invalidates contrariety and subcontrariety because the left conjuncts of their definiens assume $\boldsymbol{A} \models \boldsymbol{I}$ and $\boldsymbol{E} \models \boldsymbol{O}$.

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[12] **Subaltern**, **Contrary**, **Subcontrary**/$(oo_{\omega}o_{\omega})$ (the relations between propositions).

In the classical construal, two propositions are *contraries* iff they cannot both be true but can both be false. As Sanford [29, p. 96] noticed, the second condition (i.e. the second conjunct) in the definiens cannot be omitted, as many recent authors do, because contradictions would be contrary as well. Thus,

$$[\mathbf{Contrary}\ \boldsymbol{P}\,\boldsymbol{Q}] \Leftrightarrow_{df} \boldsymbol{\forall}\lambda w[\boldsymbol{P}_w! \rightarrow \neg\boldsymbol{Q}_w!] \wedge \boldsymbol{\exists}\lambda w[\neg\boldsymbol{P}_w! \wedge \neg\boldsymbol{Q}_w!].$$

Classical example: ***A*** and ***E***. On the modern reading, there is no example because $\boldsymbol{A} \not\models \neg\boldsymbol{E}$.[13]

In the classical construal, two propositions are *subcontraries* iff they cannot both be false but can both be true. Again, the second condition cannot be omitted [29, p. 96] because contradictions would be subcontrary. Thus,

$$[\mathbf{Subcontrary}\ \boldsymbol{P}\,\boldsymbol{Q}] \Leftrightarrow_{df} \boldsymbol{\forall}\lambda w[\neg\boldsymbol{P}_w! \rightarrow \boldsymbol{Q}_w!] \wedge \boldsymbol{\exists}\lambda w[\boldsymbol{P}_w! \wedge \boldsymbol{Q}_w!].$$

Classical example: $\neg\boldsymbol{I}$ and ***O***. On the modern reading, there is no example because $\boldsymbol{E} \not\models \neg\boldsymbol{A}$.

3.6 *The Modern Reading of the Standard Square and Truth-Conditions*

To understand semantic behaviour of the categorical statements of the Square, it is useful to rethink their truth-conditions. Firstly realize that each categorical statement of the modern reading attributes something to the class C_i which is v-constructed, on a particular valuation v, by the construction $\boldsymbol{C_i}$ which is the body of that kind of statement. Every categorical statement is of form $\lambda w.\boldsymbol{Q_i}\boldsymbol{C_i}$, where Q_i is the corresponding quantifier, i.e. ∀, ¬∃, ∃, or ¬∀.[14] The truth-condition of a categorical statement consists in a certain quality attributed to C_i. For instance, an A-statement is true iff the particular class C_A v-constructed by $\lambda x.[\boldsymbol{F}_w\ x]! \rightarrow [\boldsymbol{G}_w\ x]!$ is identical with U; the truth-condition is thus $C_A = \mathrm{U}$.

Each quantifier can be defined utilizing the corresponding truth-condition. The universal quantifier ∀, for instance, is nothing but the only class which includes U, viz. {U}; and it is thus definable accordingly. Here is a list of such definitions and their more common set-theoretic versions written in ordinary notation:

Statement	Truth-condition	Quantifier definition	Alternative definition
A	$C_A = \mathrm{U}$	$[\boldsymbol{\forall}\ c] \Leftrightarrow_{df} [c = \mathbf{U}]$	$\forall =_{df} \{\mathrm{U}\}$
E	$C_E = \emptyset$	$[\neg\boldsymbol{\exists}\ c] \Leftrightarrow_{df} [c = \boldsymbol{\emptyset}]$	$\neg\exists =_{df} \{\emptyset\}$
I	$C_I \neq \emptyset$	$[\boldsymbol{\exists}\ c] \Leftrightarrow_{df} [c \neq \boldsymbol{\emptyset}]$	$\exists =_{df} \mathcal{P}(\mathrm{U})\text{–}\{\emptyset\}$
O	$C_O \neq \mathrm{U}$	$[\neg\boldsymbol{\forall}\ c] \Leftrightarrow_{df} [c \neq \mathbf{U}]$	$\neg\forall =_{df} \mathcal{P}(\mathrm{U})\text{–}\{\mathrm{U}\}$

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[13] $\neg\boldsymbol{E}$ is a natural abbreviation of $\lambda w.\neg[\mathbf{True}^{\mathbf{T}\pi_w}\ \boldsymbol{E}]$. Analogously for $\neg\boldsymbol{A}$, $\neg\boldsymbol{I}$, $\neg\boldsymbol{O}$.

[14] We will treat the compound symbols of quantifiers as simple. Their obvious definitions see below.

The nature of truth-conditions of categorical statements yields an illuminative explanation why subalternation and (sub)contrariety are invalid on the modern reading. Suppose there is no F in W. Then,

$$C_A = \mathrm{U},\ C_E = \mathrm{U},\ C_I = \emptyset,\ C_O = \emptyset$$

(if there is an F in W, we usually get another quadruple of classes). This quadruple evidently preserves contradictoriness. However, it does not preserve subalternation and (sub)contrariety because these relations are dependent on $\boldsymbol{A} \models \boldsymbol{I}$ and $\boldsymbol{E} \models \boldsymbol{O}$. But $\boldsymbol{A} \models \boldsymbol{I}$ would hold if C_A were U and C_I were also U, which would match the inclusion of {U} to $\mathcal{P}(\mathrm{U})$–{$\emptyset$} (cf. the first and the third row of the above table). Analogously for the case of $\boldsymbol{E} \models \boldsymbol{O}$.

4 Modified Modern Reading of the Standard Square

4.1 The Standard Square of Opposition: Two Readings

We are going to put forward the *modified modern reading* of the Standard Square, according to which not only contradictoriness but also subalternation and (sub)contrariety hold. Yet it does not employ all four standard categorical statements.

Without noticing its difference from the unmodified modern reading, the modified reading seems to be suggested already by Gottschalk [13, p. 195] as the Square of Quaternality (cf. also [7], pp. 315–316). In recent literature, the confusion persists because the modified reading is often introduced without a proper notification.[15]

4.2 The Modified Modern Reading of the Standard Square

We have seen above that, on the modern reading of the Standard Square, the four standard quantifiers apply to heterogeneous collection of four, not necessarily distinct classes C_A–C_O v-constructed by four distinct constructions. On the modified modern reading, however, the vertices are 'decorated' by a more tight class of constructions with one and the same body, namely one construction v-constructing one particular class C.

Below, we write simply the (schematic) constructions $\lambda w.\boldsymbol{\forall C}$, $\lambda w.\neg\boldsymbol{\exists C}$, etc. because the particular form of the construction of C does not matter—provided $\lambda w.\boldsymbol{Q_i C}$ is still a categorical statement (at least in a weaker sense). We may label the schematic statements by "A'"–"O'". Realize that it may happen that e.g. $A' = A$ (the case when $\boldsymbol{C}$ is $\lambda w.[\boldsymbol{F}_w x]! \to [\boldsymbol{G}_w x]!$).

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[15] A rare example of their distinguishing as Apuleian Square and the (Gottschalk's) Logical Quatern can be found in [30, p. 294].

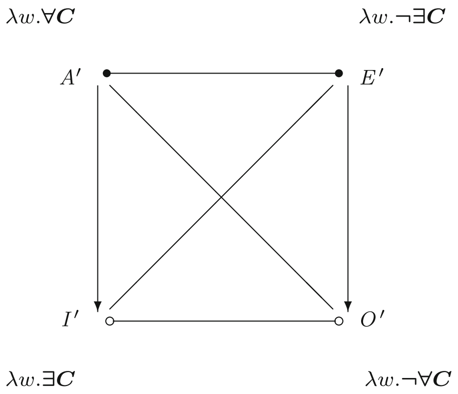

Fig. 2 The Standard Square in its modified modern reading

In comparison with the categorical statements displayed in the above Sect. 3.6, we have a distinct quadruple of statements, but with similar truth-conditions:

Statement schema	Truth-condition
$\lambda w.\boldsymbol{\forall C}$	$C = \mathrm{U}$
$\lambda w.\neg\boldsymbol{\exists C}$	$C = \emptyset$
$\lambda w.\boldsymbol{\exists C}$	$C \neq \emptyset$
$\lambda w.\neg\boldsymbol{\forall C}$	$C \neq \mathrm{U}$

As indicated in our diagram below, all classical rules including subalternation and (sub)contrariety hold in this Square. They are confirmed for obvious reasons such as $\forall \subset \exists$, which justifies $\lambda w.\boldsymbol{\forall C} \models \boldsymbol{\lambda}\ w.\boldsymbol{\exists C}$ and subalternation of $\lambda w.\boldsymbol{\exists C}$ to $\lambda w.\boldsymbol{\forall C}$ thus holds. See Fig. 2.

Though it may contain statements such as $\lambda w.\boldsymbol{\exists}\lambda x.[\boldsymbol{F}_w x]!\rightarrow[\boldsymbol{G}_w x]!$ which are not frequently expressed in ordinary language, the modified reading is natural if we consider possible quantified forms of one basic statement, e.g.:

$\lambda w.\boldsymbol{\forall}\lambda x.[\boldsymbol{F}_w x]!\rightarrow[\boldsymbol{G}_w x]!$	Normal categorical statement, viz. ***A***
$\lambda w.\neg\boldsymbol{\exists}\lambda x.[\boldsymbol{F}_w x]!\rightarrow[\boldsymbol{G}_w x]!$	Unusual categorical statement
$\lambda w.\boldsymbol{\exists}\lambda x.[\boldsymbol{F}_w x]!\rightarrow[\boldsymbol{G}_w x]!$	Unusual categorical statement
$\lambda w.\neg\boldsymbol{\forall}\lambda x.[\boldsymbol{F}_w x]!\rightarrow[\boldsymbol{G}_w x]!$	Equivalent to normal categorical statement ***O***

Here are two examples where such Square is useful. Firstly note that the interpretation of *I*-statements as containing → instead of ∧ is inevitable for explanation why e.g. "All non-self-identical objects are non-self-identical objects" entails the statement "Some non-self-identical objects are non-self-identical objects" while the latter statement is considered true despite the nonexistence of non-self-identical objects. An analogous explanation can be provided for "All chimeras are creatures" entailing "Some chimeras are creatures", though a more convenient explanation reads the two statements as modal ones (cf. Sects. 5.1 and 5.2 below).

4.3 Gottschalk's Square and Duality

In [13], Gottschalk proposed Theory of Quaternality, which is a model of many possible Squares of Opposition. A particular form employing standard quantifiers resembles our modified reading. Gottschalk introduced it as the *Square of Quaternality for Restricted Quantifiers* employing statements

$$(\forall x \in F)(Gx),\ (\forall x \in F)\neg(Gx),\ (\exists x \in F)(Gx),\ (\exists x \in F)\neg(Gx),$$

where F is a nonempty class. He considered it to be the Square in the traditional form [13, p. 195].

However, this claim might be challenged: $x \in F$ is a condition on which x is G or non-G; all formulas thus contain an implicit → with the condition as its antecedent. For example, his I-statement of his Square is in fact $\exists x((Fx) \rightarrow (Gx))$, not $\exists x((Fx) \wedge (Gx))$ of the modern reading. Note also that Gottschalk's Square is not the Square of our modified reading because the very same instance of his Square contains $\exists x((Fx) \rightarrow \neg(Gx))$ which differs from $\exists x((Fx) \rightarrow (Gx))$ of the corresponding modified reading.

Gottschalk [13, p. 193] and lately e.g. Brown [7] and Westerståhl [37] studied the Square with help of the notion of *duality*. From an input categorical statement one derives the other three as indicated in the following table:

ϕ	(the original statement)
Contradual of ϕ	By 'negation' of ϕ's variables
Dual of ϕ	By exchange of ϕ's dual constants ($\forall$ for $\exists$, $\vee$ for $\wedge$, $\neg\leftarrow$ for $\rightarrow$)
Negational of ϕ	By exchange of ϕ's dual constants and 'negation' of ϕ's variables

The theory aptly describes relations between statements $\lambda w.\mathbf{\forall C}$, $\lambda w.\mathbf{\neg\exists C}$, $\lambda w.\mathbf{\exists C}$, $\lambda w.\mathbf{\neg\forall C}$ of our modified reading.

However, Gottschalk's (contra)duality work only for the modified reading, not for the modern reading of the Square. For instance, the contradual of $\forall x((Fx) \rightarrow (Gx))$ is $\forall x(\neg(Fx) \rightarrow \neg(Gx))$, not the familiar $\forall x((Fx) \rightarrow \neg(Gx))$.

Brown [7], Westerståhl [37, 38], D'Alfonso [9] suggested in fact a remedy to this problem by proposing another notion of dual. This way they returned their attention from Gottschalk's modified to non-modified reading of the Square. They explicitly introduced "inner negation" ("post-complement") which places ¬ properly inside a formula. The dual of $Q(F,G)$ is then its outer and inner negation (here Q is a generalized quantifier as a binary relation). Adapting definitions from [7, 9]:

Outer negation	$\neg Q(F,G) \quad =_{df} (\mathcal{P}(\mathrm{U}^2) - \{Q\})(F, G)$
Inner negation	$Q(F,\neg G) \quad =_{df} Q(F, U - G)$
Dual	$(Q(F,G))^{dual} =_{df} \neg Q(F,\neg G)$

5 Modified Reading of the Modal Square of Opposition

5.1 Two Readings of the Modal Square

We are going to introduce two readings of the *Modal Square of Opposition*, i.e. the Square whose vertices are 'decorated' by modal ('alethic') categorical statements. Each *modal operator* M_i—i.e. $\Box$, $\neg\Diamond$, $\Diamond$, or $\neg\Box$—involved in such statements is a 'predicate' for propositions, i.e. quantifier for worlds; each is of type (oo_ω) (a class of propositions). In modal (*de dicto*) categorical statements, the operators are applied to propositions constructed by categorical statements. The two Modal Squares differ analogously as the Standard ones: one of them uses quadruple of statements with the same body, while the other does not.

5.2 The Modified Reading of the Modal Square

We begin with the *modified reading* which deploys statements of form $\lambda w.\boldsymbol{M_i C}$, whereas $\boldsymbol{P}$, a categorical statement, is a construction of P.[16] In each particular quadruple of statements projected on the vertices, the propositional construction $\boldsymbol{P}$ is one and the same. This modified reading of the Modal Square seems to be nothing but a type-theoretic variant of the modified reading of the Standard Square.

Analogously to the modified reading of the Standard Square, this reading is natural when one considers various modal (*de dicto*) versions of one given statement. But it is even

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[16] This reading of the Square as concerning modal and even deontic notions can be found in Leibniz (cf. [14]), however, it was known already in the thirteenth century [15, 36]. In modern era, it was met by Gottschalk [13, p. 195] and Blanché [6], see also [10].

more natural, cf. e.g. the following particular example (accompanied by the traditional medieval terminology):

"Necessarily, all horses are animals"	*necesse est esse*
"Not possibly, all horses are animals"	*impossibile est esse*
"Possibly, all horses are animals"	*possibile est esse*
"Not necessarily, all horses are animals"	*possibile non est esse*

In the following table, the four schematic statements—which can be labelled "A'^M"–"O'^M"—are arranged together with their truth-conditions and related definitions of modal quantifiers. Let L (as "logical space") be the universal class of possible worlds, an object of type o_ω, i.e. the necessarily true proposition; in such context, $\emptyset$ is the empty class of worlds, i.e. the necessarily false proposition.[17] See Fig. 3.

Statement schema	Truth-condition	Quantifier definition	Alternative definition
$\lambda w.\Box \boldsymbol{P}$	$P = \mathrm{L}$	$[\Box\, p] \Leftrightarrow_{df} [p = \mathbf{L}]$	$\Box =_{df} \{\mathrm{L}\}$
$\lambda w.\neg \Diamond \boldsymbol{P}$	$P = \emptyset$	$[\neg\Diamond\, p] \Leftrightarrow_{df} [p = \boldsymbol{\emptyset}]$	$\neg\Diamond =_{df} \{\emptyset\}$
$\lambda w. \Diamond \boldsymbol{P}$	$P \neq \emptyset$	$[\Diamond\, p] \Leftrightarrow_{df} [p \neq \boldsymbol{\emptyset}]$	$\Diamond =_{df} \mathcal{P}(\mathrm{L})-\{\emptyset\}$
$\lambda w.\neg\Box \boldsymbol{P}$	$P \neq \mathrm{L}$	$[\neg\Box\, p] \Leftrightarrow_{df} [p \neq \mathbf{L}]$	$\neg\Box =_{df} \mathcal{P}(\mathrm{L})-\{\mathrm{L}\}$

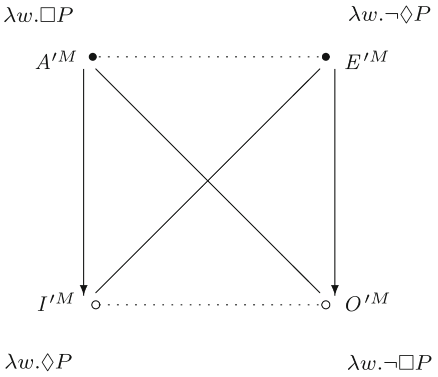

Fig. 3 The Modal Square in its modified modern reading

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[17] (Schematic) statements such as $\lambda w.\Box \boldsymbol{P}$ are analytic—they are constructions constructing constant propositions, which is apparent from their 'bodies' (e.g. $\Box \boldsymbol{P}$) which contain no free possible world variables.

5.3 Subalternation, Contrariety, Subcontrariety

The modified reading of the Modal Square is isomorphic to the modified reading of the Standard Square and it may seem that not only contradictoriness, but also subalternation and (sub)contrariety hold.

Subalternation indeed holds. To demonstrate its validity, we may utilize the following way of reasoning: $\Diamond$ is $\mathcal{P}(\mathrm{L})-\{\emptyset\}$, thus $\Box$ (i.e. $\{\mathrm{L}\}$) is subclass of $\Diamond$; consequently, $\lambda w.\boldsymbol{\Box P} \models \lambda w.\boldsymbol{\Diamond P}$, which justifies the subalternation of $\lambda w.\boldsymbol{\Diamond P}$ to $\lambda w.\boldsymbol{\Box P}$.

Nevertheless, (sub)contrariety does not hold. The reason is that truth or falsity of modal statements is stable across the logical space. For instance, the *A*- and *E*-statements "Necessarily, all horses are animals" and "Not possibly, all horses are animals" are in no world both false; contrariety is thus lost. Quite analogously for subcontrariety: the *I*- and *O*-statements "Possibly, all horses are animals" and "Not necessarily, all horses are animals" are in no world both true.

6 Modal Reading of Categorical Statements

6.1 Requisites

The novel modern reading of the Modal Square proposed below is based on the assumption that in ordinary discourse we often understand categorical sentences such as "Every *F* is *G*." as expressing a certain necessary connection between *F* and *G*. (The statements are thus analytic, i.e. necessarily true or necessarily false.) This connection is sometimes made explicit by inserting "*by definition*" or even "*necessarily*", e.g. "Every horse is, by definition, an animal.". Tichý [32, Sect. 42] suggested reading such sentences as talking about so-called *requisites*. We will adopt and extend his proposal.

Tichý defined the notion of requisite and essence for both individual 'concepts', called *offices*, and for properties; the two notions differ not only in type. Essence is everything that is necessary for an object to become such and such; essence is a certain collection of requisites. A *requisite* is thus one of conditions, i.e. a property, an object must possess to become such and such. For example, (BE) WINGED is a requisite of the individual office PEGASUS, it is thus a property an individual must possess in *W* to be Pegasus in *W*.

In the case of properties, requisites are particular 'subproperties' of a property. For example, (BE AN) ANIMAL is one of many requisites of (BE A) HORSE. Below, we will employ just this notion of requisite. It is definable by

$$[\mathbf{Requisite}\ g\, f] \Leftrightarrow_{df} \Box\lambda w.\forall\lambda x.[f_w\ x]! \rightarrow [g_w\ x]!,$$

where $\mathbf{Requisite}/(o(o\xi)_\omega(o\xi)_\omega)$ (the relation between ξ-properties). Note that entailment of a proposition Q from a proposition P, i.e. in fact $P \subseteq Q$, is a medadic case of entailment between properties (in every world W, the extension of $F \subseteq$ the extension of G). Thus,

$$[\mathbf{Entails}\ f\, g] \Leftrightarrow_{df} [\mathbf{Requisite}\ g\, f].$$

Utilizing the preceding definition, we get two modal categorical statements in their two equivalent forms[18]:

	'Intensional' form	Form with explicit modality	Usual expression
$\boldsymbol{A^M}$	λw[**Requisite** $\mathbf{G\,F}$]	$\lambda w.\Box\ \lambda w.\forall\lambda x.[\mathbf{F}_{w}x]!\rightarrow[\mathbf{G}_w x]!$	"Necessarily, every *F* is *G*."
$\boldsymbol{O^M}$	$\lambda w.\neg$[**Requisite** $\mathbf{G\,F}$]	$\lambda w.\Diamond\ \lambda w.\exists\lambda x.[\mathbf{F}_{w}x]!\wedge\neg[\mathbf{G}_w x]!$	"Possibly, some *F* is not *G*."

6.2 *Potentialities*

The second notion needed for investigation of the modal version of the Square must be comparable with the notion of requisite. Let us call this novel, but not entirely unfamiliar notion "*potentiality*". To explain, an individual who possesses the property (BE A) HORSE has to be an animal; but the property (BE A) HORSE admits the individual being white or fast, etc. The properties (BE) WHITE, (BE) FAST are thus mere potentialities of the property (BE A) HORSE.

On our definition, a property *G* is a potentiality of *F* if there is at least one possible world in which at least one individual *F* possesses *G*:

[**Potentiality** $g\,f$] $\Leftrightarrow_{df}$ $\Diamond\lambda w.\exists\lambda x.[f_w\ x]!\wedge[g_w\ x]!$,

where **Potentiality**/$(o(o\xi)_\omega(o\xi)_\omega)$ (the relation between ξ-properties).

We complete the quadruple of modal (*de dicto*) categorical statements:

	'Intensional' form	Form with explicit modality	Usual expression
$\boldsymbol{I^M}$	λw[**Potentiality** $\mathbf{G\,F}$]	$\lambda w.\Diamond\lambda w.\exists\lambda x.[\mathbf{F}_w x]!\wedge[\mathbf{G}_w x]!$	"Possibly, some *F* is *G*."
$\boldsymbol{E^M}$	$\lambda w.\neg$[**Potentiality** $\mathbf{G\,F}$]	$\lambda w.\Box\lambda w.\forall\lambda x.[\mathbf{F}_w x]!\rightarrow\neg[\mathbf{G}_w x]!$	"Necessarily, no *F* is *G*."

6.3 *On the Relationship of Requisites and Potentialities*

Let us briefly compare the notions of requisite and potentiality. Here are convenient examples of true modal categorical statements:

"Being an animal is a requisite of being a horse."	*An* A^M*-statement*
"Being non-self-identical is not a potentiality of being a horse."	*An* E^M*-statement*
"Being black is a potentiality of being a horse."	*An* I^M*-statement*
"Being winged is not a requisite of being a horse."	*An* O^M*-statement*

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[18] Analogously as above, statements such as λw[**Requisite** $\mathbf{G\,F}$] are analytic—they are constructions constructing constant propositions, which is apparent from their 'bodies' which (normally) contain no free possible world variables.

We can realize that a property which is a non-requisite of the property (say) (BE A) HORSE is either its mere potentiality—it is e.g. the property (BE) BLACK—, or it is a property no horse can ever instantiate—it is e.g. the property (BE) NON-SELF-IDENTICAL. On the other hand, the property (BE) NON-SELF-IDENTICAL is a non-potentiality of the property (BE A) HORSE; it can never be instantiated and it is thus also a non-requisite of properties such as (BE A) HORSE. Note also that many requisites, e.g. (BE AN) ANIMAL, (BE) FOUR LEGGED, of a property such as (BE A) HORSE are its potentialities because it is fulfilled that there is at least one world in which at least one horse possesses the property; potentialities which are not requisites of a given property can be set apart as *pure potentialities*.

There is a remarkable connection of the notions of requisite and potentiality with the notions of essential and accidental property. A property is *essential for/accidental for* an individual iff in every world W/at least one world W, the individual instantiates the property in W. A property is *essential/accidental* iff in every world W/at least one world W, there is an individual who instantiates the property in W.[19] Now a requisite/potentiality G of a property F is definable as an essential/accidental property for every bearer of F.

7 Modern Reading of the Modal Square of Opposition

7.1 Modern Reading of the Modal Square

In the modern reading of the Modal Square, we 'decorate' vertices of the square by the modal versions of standard categorical statements, i.e. by $\boldsymbol{A^M}$–$\boldsymbol{I^M}$.[20] Each statement is of form $\lambda w.\boldsymbol{M_i P_i}$, where $\boldsymbol{P_i}$ is $\boldsymbol{A}$, $\boldsymbol{E}$, $\boldsymbol{I}$, or $\boldsymbol{O}$. See Fig. 4.

Abbreviated form	Statement schema	Truth-condition
$\boldsymbol{A^M}$	$\lambda w.\Box \boldsymbol{P_A}$	$P_A = \mathrm{L}$
$\boldsymbol{E^M}$	$\lambda w.\neg\Diamond \boldsymbol{P_E}$	$P_E = \emptyset$
$\boldsymbol{I^M}$	$\lambda w.\Diamond \boldsymbol{P_I}$	$P_I \neq \emptyset$
$\boldsymbol{O^M}$	$\lambda w.\neg\Box \boldsymbol{P_O}$	$P_O \neq \mathrm{L}$

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[19] For an exhaustive study of such notions see [22] from which we borrow our definitions.

[20] After finishing my paper, M. Duží—who is also using Tichý's logic—reminded me that she proposed the essentials of the modal modern reading of the Square in her presentation [11]. She also proposed to call properties concepts, which leads to the quadruple of statements about their two basic relations; schematically: "The concept F is subsumed by/compatible with the concept G/non-G".

$\lambda w.[\textbf{Requisite}\ \boldsymbol{G}\ \boldsymbol{F}]$

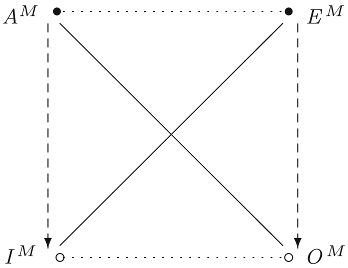
"Necessarily, every F is G."

$\lambda w.\Box\lambda w.\forall\lambda x[\boldsymbol{F}_w\ x]! \rightarrow [\boldsymbol{G}_w\ x]!$

$\lambda w.\neg[\textbf{Potentiality}\ \boldsymbol{G}\ \boldsymbol{F}]$

"Necessarily, none F is G."

$\lambda w.\Box\lambda w.\forall\lambda x[\boldsymbol{F}_w\ x]! \rightarrow \neg[\boldsymbol{G}_w\ x]!$



$\lambda w[\textbf{Potentiality}\ \boldsymbol{G}\ \boldsymbol{F}]$

"Possibly, some F is G."

$\lambda w.\Diamond\lambda w.\exists\lambda x[\boldsymbol{F}_w\ x]! \wedge [\boldsymbol{G}_w\ x]!$

$\lambda w.\neg[\textbf{Requisite}\ \boldsymbol{G}\ \boldsymbol{F}]$

"Possibly, some F is not G."

$\lambda w.\Diamond\lambda w.\exists\lambda x[\boldsymbol{F}_w\ x]! \wedge \neg[\boldsymbol{G}_w\ x]!$

Fig. 4 The Modal Square in its modern reading

7.2 Usual Lack of Existential Import

Using our modal reading of categorical statements, we immediately resolve the well-known puzzle concerning existential import and (especially) *A*- and *O*-statements. On natural reading, "Every griffin is a creature" has no existential import and is true in every *W*, since it amounts to true saying that a certain property is a requisite of another property. The sentence "Some griffin is not a creature" is false in every *W* and has likewise no existential import regardless of the existence of griffins.

It might then seem that modal categorical statements never have existential import. The exception from this rule concerns mainly *I*-statements employing properties which have no instance in any possible world, being thus empty in the whole logical space. For example, the *I*-statement "Possibly, some non-self-identical individual is *G*" is false because of existential import of the term "non-self-identical individual". Properties such as (BE) NON-SELF-IDENTICAL, (BE A) HORSE WHICH IS NOT HORSE, (BE A) BACHELOR WHO IS MARRIED etc. can be called *void properties*.[21] Admittedly, they can be called '*contradictory beings*'.

Because of 'contradictory beings', some modal *O*-statements have existential import as well, consider e.g. "Possibly, some non-self-identical individual is not *G*". This may

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[21] Cf. [22], where void properties are defined and related to accidental, essential and partly essential properties. However, there is only a partial correspondence of the quadruple of those properties with the Square studied below because requisites and potentialities are not essential or accidental properties, but kinds of properties essential for or accidental for (cf. Sect. 5.3).

seem a bit odd: one might object that e.g. the property (BE AN) ANIMAL is not a property of any particular non-self-identical individual and so the *A*-statement "Necessarily, every non-self-identical individual is an animal" is false and the *O*-statement "Possibly, some non-self-identical individual is not an animal" is consequently true, i.e. without existential import. In that case, however, the *A*-statement must be understood as involving $\wedge$, rather than the usual $\rightarrow$. Though it might seem a little bit puzzling, both properties (BE AN) ANIMAL and (BE A) NON-ANIMAL are requisites of the property (BE) NON-SELF-IDENTICAL, yet no non-self-identical individual possesses them.

Because of the lack of existential import, *weakened modes of syllogisms*—e.g. Darapti: "All *H* are *G*", "All *H* are *F*", "Therefore, some *F* are *G*"—are valid on this modal interpretation, only with few exceptions containing 'contradictory beings'. There is a hypothesis, not examined here, that medieval logicians, who accepted the weakened modes, purposely ignored these 'contradictory beings' because the properties simply have no possible instances.

7.3 Subalternation

Similarly as on the modern reading of Standard Square, subalternation does not generally hold because $\boldsymbol{A^M} \not\models \boldsymbol{I^M}$ and $\boldsymbol{E^M} \not\models \boldsymbol{O^M}$. The invalidity of $\boldsymbol{A^M} \models \boldsymbol{I^M}$ and $\boldsymbol{E^M} \models \boldsymbol{O^M}$ is caused by 'contradictory beings'.

For an example consider the *A*-statement "Necessarily, everybody who shaves all and only those who do not shave themselves is a barber". It is true because the '*F*-property' is a requisite—even an essence—of the property (BE A) (Russellian) BARBER. The *A*-statement does not entail its corresponding *I*-statement "Possibly, somebody who shaves all and only those who do not shave themselves is a barber" which is false because such barber cannot exist; (BE A) (Russellian) BARBER is thus not a potentiality of the '*F*-property'.[22] In other words, if '*F*-property' is not a potentiality of the respective '*G*-property', subalternation of *I*-statement to the respective *A*-statement does not hold.

To show invalidity of subalternation in the case of *E*- and *O*-statements, consider the true *E*-statement "Necessarily, no non-self-identical individuals are self-identical individuals" (while $G = \text{non-}F$) and the corresponding *O*-statement "Possibly, some non-self-identical individuals are self-identical individuals" which is false (of course, unless we use the 'if-reading' of the sentence as in the end of Sect. 4.3).

7.4 Contrariety and Subcontrariety

Similarly as in the modified modern reading of the Modal Square, contrariety and subcontrariety do not generally hold. To see it, consider an instance of the above definiens

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[22] Cf. [28].

of contrariety, $\lambda w.\forall\lambda w[\boldsymbol{A^M}{}_w! \rightarrow \neg \boldsymbol{E^M}{}_w!] \wedge \exists\lambda w[\neg \boldsymbol{A^M}{}_w! \wedge \neg \boldsymbol{E^M}{}_w!]$. If $\boldsymbol{A^M}$ is true, there is no W in which it would be false. The right conjunct cannot be satisfied, contrariety of $\boldsymbol{A^M}$ to $\boldsymbol{E^M}$ does not generally hold. Analogously for the case of subcontrariety.

With our modal reading of the Square, we are ready to understand also the puzzling fact noticed by Sanford [29, p. 95]: contrary statements cannot be both false if one of them is true in all circumstances. For instance, "All squares are rectangles" is a necessary *A*-statement thus the respective *E*-statement "No squares are rectangles" is necessarily false. This affects also subcontrariety because the truth of ***A*** transfers to ***I***, analogously for ***E*** and ***O***. On our approach, it is clear that this feature is peculiar to statements which are equivalent (in a synchronically given language) to their modal versions, e.g. "All squares are rectangles" ⇔ "Necessarily, all squares are rectangles", "No squares are round" ⇔ "Necessarily, no squares are round", whereas in the Modal Square (sub)contrariety does not hold.

7.5 On the Modal Hexagons of Opposition

To remind the reader, Gottschalk [13, p. 195] introduced and investigated Modal ('Alethic') Hexagon of Opposition with two new modal quantifiers U and Y.[23] The Hexagon was independently discovered and developed by Blanché [6] who defined the quantifier Y already in [5, p. 370]. Here are definitions of the respective two notions and truth-conditions of statements involving them:

Statement schema	Truth-condition	Quantifier definition	Alternative definition
$\lambda w.\mathbf{U}\boldsymbol{P}$	$P = \mathrm{L} \vee P = \emptyset$	$[\mathbf{U}\ p] \Leftrightarrow_{df} [\Box\ p] \vee [\neg\Diamond\ p]$	$\mathrm{U} =_{df} \{\mathrm{L}, \emptyset\}$
$\lambda w.\mathbf{Y}\boldsymbol{P}$	$P \neq \mathrm{L} \wedge P \neq \emptyset$	$[\mathbf{Y}\ p] \Leftrightarrow_{df} [\Diamond\ p] \wedge [\neg\Box\ p]$	$\mathrm{Y} =_{df} \mathcal{P}(\mathrm{L}) - \{\mathrm{L}, \emptyset\}$

The modal quantifier U is well known in philosophy as *analytic* or *non-contingency* [4, 6, 10, 13]. The modal quantifier Y can aptly be called (purely) *contingent* ([5, 6, 13]; see [17], cf. also [21, 31]). Here are analogous names for familiar modal quantifiers: □—*necessary*; ¬◇—*impossible*; ◇—*possible*; ¬□—*nonnecessary*.

Obviously, there are two readings of the Modal Hexagon: either with statements of form $\lambda w.\boldsymbol{M_i P_i}$, or $\lambda w.\boldsymbol{M_i P}$. Both Hexagons inherit properties of the Squares involved in them. For instance, in the modified reading diagrammed below, subalternation holds but (sub)contrariety does not. This is neglected by the most recent writers (admittedly, they assume the *weakened (sub)contrariety* which is satisfied also by contradictory statements, cf. Sect. 3.5 above, see Fig. 5).

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[23] We keep here the original notation though it clashes with our previous use of "U".

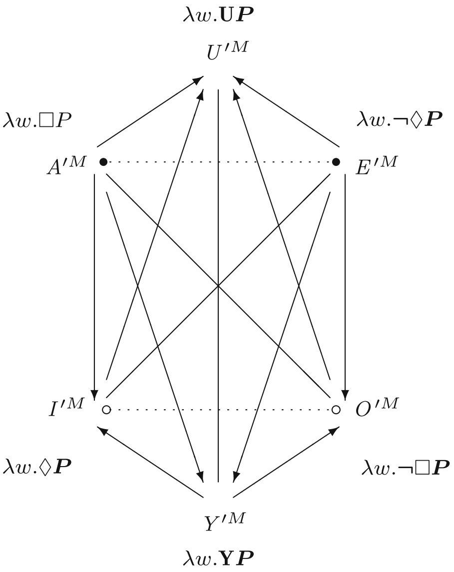

Fig. 5 The Modal Hexagon in its modified modern reading

8 Conclusions

To repeat the main ideas of this paper, there are two modern readings of the Standard Square of Opposition:

i. the well-known Square of modern logic textbooks, for which subalternation and (sub)contrariety do not hold; see Sect. 3;
ii. the less known Square encountered already by Gottschalk; on this modified modern reading of the Square, all classical relations, including subalternation and (sub)contrariety hold; see Sect. 4.

We compared the two Squares, reviewing even some familiar facts. We showed that certain confusions or strange claims occurring in the literature can be explained as results of a shift from i. to ii. For instance, the entailment of "Some ogres are ogres" from "All ogres are ogres" holds in ii. but not in i.

But there are also two modern readings of the Modal Square of Opposition, they employ modal versions of categorical statements:

iii. the Modal Square known already to medieval logicians is nothing but another form of the Square ii.; contradictoriness and subalternation hold in it; but rigidity of truth or falsity of modal categorical statements invalidates contrariety and subcontrariety; see Sect. 5;
iv. the novel Modal Square which is a certain modal version of the well-known non-modal Square i.; see Sect. 7.

Again, some confusions occurring in the literature were easily disentangled by pointing mainly to shifts from some standard to some modal Square.

As regards the Square iv., subalternation does not hold in it only because of existence of rare properties, called ‘contradictory beings’, which cannot be instantiated. Admittedly, one might dismiss these properties as non-properties thus subalternation would generally hold. Common modal (*de dicto*) categorical statements employed in iv. lack existential import. An exclusion of ‘contradictory beings’ would then lead to validity of weakened forms of syllogisms (e.g. Darapti) without an exception. (Sub)contrariety is invalidated for the same reason as in iii., i.e. because of analytic character of the modal (*de dicto*) statements—if true, they cannot be possibly false (and vice versa) as the definition of (sub)contrariety requires.

The Square iv. is interesting as an interpretation also because pre-modern tendencies to adopt some form of essentialism. The shift from i. to iv. can thus nicely explain oppositions when we shift from ordinary discourse about contingent things to discourse in which fixed meaning relations plays a role, e.g. in ‘mythological’ discourse [18, p. 409], cf. the examples with chimeras or ogres. Future work on the topic may focus on modal syllogistic utilizing mainly both Modal Squares, enriching the investigation by combinations with non-modal or modal *de re* statements.[24]

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[24] The work on the paper has been partly supported by the grant of the Czech Science Foundation (GACR) ‘Semantic Notions, Paradoxes and Hyperintensional Logic Based on Modern Theory of Types,’ registration no. GA16-19395S.

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Part IV
Philosophical Perspectives on the Square

The Many Faces of Inconsistency

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Abstract To think about inconsistencies involves reflecting on several basic notions widely used in order to talk about human knowledge and actions, such as negation, opposition, denial, assertion, truth, falsity, contradiction and incompatibility, just to name the more perspicuous ones. All of them are regularly used in natural language and for each one several definitions or conceptions have been proposed throughout the history of Western thought. That being so we tend to think that we have a good enough intuitive understanding of them. Yet a closer examination shows many ways in which "contradiction" and related words can be understood. Thus, a more precise definition would help to clarify their meaning and assist us to use them in a more appropriate manner. In this paper I will try to clarify these notions and thus make a terminological proposal. The general background will be the reflexion on paraconsistency. A main purpose will be to show that the confusion between contraries and contradictories—although they were clearly distinguished in the original square of opposition—is very common and it paves the way to the rejection of all forms of "inconsistencies" without making distinctions, and also to the wrong assumption that regarding all the main aspects the effects of contrary opposition are equivalent to the ones of contradictory opposition.

Keywords Inconsistency • Contradiction • Contrary • Incompatibility • Negation • Paraconsistency

Mathematics Subject Classification Primary 03B65, Secondary 03B53

1 Contradictories and Contraries

In order to study 'contradiction' and 'contradictory' together we could take 'contradictoriness' as a generic term embracing both of them, which also may include 'contradict'. And from this broad perspective we can see two main ways to characterize it: one is based on the notion of negation or denial and the other is based on the interplay between true and false. The first points out the structural antagonism determined by the use of negation or by the act of denial, so it can be called a «syntactic characterization» considering that it emphasizes the role played by one (or some) linguistic element(s). The second deals with the truth-values and uses them in order to specify one form of opposition, so it can be

J.-Y. Béziau, G. Basti (eds.), *The Square of Opposition: A Cornerstone of Thought*, Studies in Universal Logic, DOI 10.1007/978-3-319-45062-9_9

called a «semantic characterization». Moreover, to form a contradiction in the first sense the main requirement would be to know how to use the rule (or rules) of negation within that language, while in the second it would be based on the meaning of the utterances involved.

An interesting question is how we can have the definition of contradictories without an immediate reference to contradiction. The obvious place to look at is the traditional view, beginning with the definition given by Aristotle, that two statements are contradictories when they cannot both be true and cannot be both false, and contraries when they cannot both be true but can both be false. And if we want to enquire about the justification for the difference between the two situations, the usual explanation uses the famous "square of opposition", which—it is important to remember—is "a post-Aristotelian device" [4, p. 75]. Let's see how it works.

If we have two statements that seem to be «opposite», how do we know if they are contraries or contradictories? Well, we check if they are general (universal, if preferred) or particular, and also if they are affirmative or negative. Thus, if we have two general statements, yet being one affirmative and the other negative, we have two good candidates for contraries. But there is something else that is required: they must be talking about the «same thing». Here there is a main problem: The traditional approach would point to the fact that we are talking about a logical structure that is based on terms and their combinations—'All S are P' (A), 'Some S are P' (I), 'No S are P' (E) and 'Some S are not P' (O)—, thus in order to see if the statements are talking about the same thing, we have to start checking if they are using the same terms. But that is not enough because we are dealing here with statements (or propositions, if preferred) and not with sentences, so it is necessary to check if they are actually uttered and how they are uttered, in order to see that the uses of the terms in both statements are exactly the same. Finding out this can be difficult, in the case of contraries we have to check if the terms are used in the same sense when used in the positive and in the negative statements, but it is much more difficult in the case of contradictories, in which we have to check not only that, but also if nothing changes when they are used in universal and particular statements.

The situation here is similar to the one that we encounter with the definition of contradiction based on negation, yet even worse. There we saw that we will be able to say that we have a contradiction only when we have the conjunction of one categorical statement (without any denotational problem) and its negation. Here we will have contradictories only when we have two categorical statements (without any problem with respect to denotation) of which one has to be general and the other particular and one has to be positive and the other negative, and also they have to use the same terms in the same sense and context.

It is important to stress that, in this definition of contradictories, the feature that one statement is the negation of the other is something that is not present in the definition and also is not used as an assumption; rather, it is something that is «inferred» or «proved». This is clearly stated in an important dictionary of logic, which first says that "the sentences *A* and *O* are *contradictory*, *E* and *I* are also *contradictory*, *A* and *E* are *contrary*" [8, p. 189] and later introduces this feature in terms of predicate logic [8, p. 190]. So, it does not start using negation, as was the case in the definition of contradiction, but instead it starts with a definition that does not refer to negation and later on establishes that among two proposition that fit into the definition one has to be the negation of the other. In my view, this expresses as its best the difference between the characterization based on negation and the one based on the opposition of truth-values. If we start with one definition, then we *arrive* at the other notion. In the classical view that is always the case, but I think it is worth examining that transition in detail.

Let me put things in a somehow different way. If we start just considering the four kinds of statements (*A, E, I, O*), it is quite a legitimate question to ask which one is the negation of the other. We are not dealing with negation as a sentential operator, as was before, so we are not going to find 'it is not the case that' attached to any sentence. So taking the most relevant case, if we start with 'All S are P', which is its negation: 'No S are P' or 'Some S are not P'? Intuitively there are good reasons to think that is the first one: they are the same kind of statements, i.e., general statements, and, secondly, one is asserting that there is a property applicable to all members of a kind of objects, and the other is saying that for no member of that kind that property is applicable. For example, what can be «more opposite» to 'all men are mortal' than to say 'no men is mortal'? If we say 'some men are not mortal' it is clearly opposite, but it is an opposition limited to some cases, but in the first option it is an opposition for all cases.

The answer from the traditional conception refers to the definition given by Aristotle for contraries and contradictories. Two statements that we know cannot both be true, then they can be either contraries or contradictories. We need more to be able to distinguish between them: we need to know if they may be both false or if they cannot be both false. In the first case they will be contraries and in the second contradictories. And then the argument would be that in this second respect there is a radical difference between contradictories statements, while the contraries are equal. Thus, contradictories are different not only in one but in two respects, and the conclusion would be that there is a stronger opposition between them.

But here a question arises: how can we know that two statements cannot be both false. Without any doubt that is a very complicated question that involves several philosophical problems, but here I would want to focus on one point. A possible response could be that we can know that two statements cannot be both false in the same or in a similar way as we can know that two statements cannot be both true, whatever that might be. So there is no reason to emphasise the difficulties just in one instance. In fact, my response would be that even accepting, for the sake of the discussion, that knowing that two statements cannot be both false is comparable with knowing that they cannot be both true, it is a quite different thing to accept that knowing that two statements cannot be both false is comparable with knowing that they may both be false. The former requires a very strong justification while the second could be more easily grounded, to start with the default assumption that any statement may be false. So, around the question about what negates a statement—regarding its contrary and its contradictory—, it is worth asking why we should restrict ourselves to consider something as a denying statement only when strong prerequisites are fulfilled if we can have weaker prerequisites available. Once again the classical option ends up restricting its own applicability.

Let us go back to the claim that knowing that two statements cannot be both false is comparable to knowing that they cannot be both true. If we inquire about what kind of knowledge that would be, a common answer is that we can know that based on «logical grounds», and actually in the traditional conception it is often stated that only in virtue of logical grounds can two statements be considered contradictory—in a «technical» sense—.[1] This would generally mean that for some pair of statements we can know that is «logically impossible» for them to be both true, and the same for the case of being both false. This, again, points to an important philosophical problem with a long-standing tradition. But here we can concentrate on what might be the basic form of the argument, which could be based on the «standard» definition of logical possibility, for instance:

> Where ϕ is a proposition, it can be understood as: (1) ϕ is logically possible; its negation entails a contradiction. [. . .] [5, p. 706] [entry written by Ruth Barcan Marcus].

From that the next step would be to change possible for impossible and say that a statement is logically impossible if it entails a contradiction.[2] Then, the argument will be restated saying that we can know that two statements cannot be both true, or—along side—both false, because it would be logically impossible for them to be both true, or both false; which, following closely the proposed definition, would mean: a statement that asserts that they are both true (false) will entail a contradiction. So we are back to the notion of contradiction and the ground for establishing that two statements cannot be both be true (false) is by knowing that a statement that denies that would entail a contradiction. Recapitulating, we start with a definition for contradictories that does not make use of the notion of contradiction but when we want to understand it, we have to appeal to that notion. It does not look as if we have gone very far, but at least we are talking of something "entailing a contradiction" rather than "being a contradiction", and also what entails that contradiction is not the original statements but something that is said about them. We are one level up. This shift of level happened when we started talking about what we need to know about a pair of statement in order to be able to apply to them the given definitions of contradictories or contraries, but still what implies the contradiction is in a different level than the contradictory or contrary statements. So contradictory statements are such not because they together entail a contradiction—which could be the case—but because a contradiction will follow from the denial of what is said about them.

It is noteworthy that this act of appealing to a contradiction holds not only in relation to contradictories, but also to contraries. Furthermore, that is also the case for what are traditionally called 'subcontraries' (I vs. O), whose definition—in similar terms—would be: two statements that cannot both be false but can both be true (cf. [4, p. 75]). There we have again a 'cannot be both . . . ' condition. So the notion of contradiction is involved not only in the understanding of contradictories but also of contraries and subcontraries, the only difference is that for contradictories the 'cannot be both . . . ' clause is used twice while for the other it is only used once. However, if we examine that other side, that is, when it is said that two statements 'can be both false' (for the case of contraries) or 'can be both true' (for subcontraries), facing the question about how we can know that, one plausible option would be to keep the same line of «logical ground» and say that it would be logically possible for them to be both true (false). Following the definition of logical possibility previously quoted, that can be understood as saying that two statements

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[1] For example the glossary in Sainsbury says:

"contradictories: two propositions are contradictories iff it is logically impossible for both to be true and logically impossible for both to be false." [13, p. 394]

[2] "What is logically impossible involves some kind of contradiction, [. . .]" [13, p. 15].

can be both true (false) because the negation of the statement that says it will entail a contradiction. It has to be pointed out that here we are not making claims about the logical possibility of each statement separately, which would be something like: one statement is logically possible because its negation entails a contradiction, a claim that could be later conjoined with a similar claim about the other statement; what interests us here is not whether a statement is logical possible in absolute terms but relative to the other one. That is the point of the three definitions of opposition: to link two statements through an internal correlation between their truth and/or falsity.

So following this appeal to «logical possibility», the notion of contradiction becomes the key aspect of the whole square of opposition, but—as I said previously—the contradiction appears here in a different level; the definition of contradictories has not being made reducible to term of contradiction, but instead its meaning has been explained by appealing to the eventuality of a contradiction being entailed at a different level.

There is something that has to be mentioned here. That is the question of the existential import of categorical propositions[3] or as it is also called existential commitment.[4] It has lead to a modern clarification or correction with respect to the traditional square of opposition, which in a general formulation can be presented like this:

> For these relations to hold [contradictories, contraries, subcontraries, subalterns], an underlying existential assumption must be satisfied: the terms serving as subjects of propositions must be satisfied, not empty (e.g., ‘man’ is satisfied and ‘elf’ empty). Only the contradictory opposition remains without that assumption. Modem interpretations of categorical propositions exclude the existential assumption; thus, only the contradictory opposition remains in the square. [1, p. 875 (entry written by Ivan Boh)])

But it also can be presented in a way in which the relations of contraries and subcontraries hold for non-empty subject terms (for example in Marciszewski [8, p. 190 f.]), but that restriction is not added for contradictories. So, in these versions the relation of contradictories still applies even to statements that talk about objects that do not exist (even though, Strawson [15] qualified this point[5]). Anyhow, the notion of contradictories stands as a grounding relation in the square of oppositions.

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[3] That is the denomination in Church [3].

[4] This is used sometime by Strawson [15, pp. 164 ff].

[5] Strawson [15, pp. 164 ff.], examining the question about the interpretation of the system of traditional logic, especially by means of modern predicate logic, addresses the problem of existential import and says that in the case of the particular statements (*I* and *O*) the only reasonable solution is that they do carry existential commitment, but for the case of the universal (*A* and *E*) he present it as a dilemma: “Either the *A* and E forms have existential import or they do not. If they do, one set of laws has to be sacrificed as invalid; if the do not, another set has to go. Therefore no consistent interpretation of the system as a whole, within the prescribed limits is possible.” (Strawson [15, p. 165]) He develops the dilemma in a more formal way maintaining that one option is to take the A and O statements just in terms of their standard formalization in the predicate logic, respectively ‘$\sim(\exists x)(fx. \sim gx)$’ and ‘$\sim(\exists x)(fx.gx)$’, and the other is to conjoin these formalizations with “an assertion of the existence as far as the first ‘term’ is concerned (‘$(\exists x(fx)$’)” [15, p. 165]. Then he shows that in the first option the rule for contradiction still holds, but the rules for contraries and subcontraries do not hold, while in the second option the rule for contraries is preserved but not for subcontraries and contradictories. Yet, to me what I think is more interesting is his conclusion about the later: “A and O are no longer contradictories, but only contraries; since, while both cannot be true for a given example, both may be false, in the case where the positively existential component of A is false. Similarly, E and I are only contraries.” [15, p. 166]

From a wider perspective, this development brings in the concern about the content of the statements involved in the different types of opposition, since it makes explicit underlying existential assumptions and the need of satisfying them in order to be able to apply the standard definitions of those types. And that links us with a different way of addressing the question about how can we know that two statements cannot be both false (or true) that will not just be based on the notion of logical impossibility. This option would have to take on board concerns about the content of the statements in order to consider them as being in one kind of opposition or another. Roughly speaking, that is tantamount to using other available information about the statements, different from their inner logical structure, in order to justify the claim that they cannot be both false (or true). This may open the door to invoke other kinds of impossibilities: metaphysical, epistemological, rational, scientific, mathematical, psychological, etc., in order to sustain the claim that two statements cannot be both true (or false), and, thus, paving the way to consider them as contraries or contradictories.

An unlimited number of questions can arise about these other types of impossibilities, but here I want to put forward just two. First, what role will the notion of contradiction play within them? If we take again a standard definition of those different types of possibility involved, we frequently find expressions like 'consistent with metaphysical necessities', 'consistent with scientific laws', 'consistent with what is known', and so on (cf. [5, p. 706]). Consequently, the definition of each type of impossibility would have to be articulated in terms of 'inconsistent with . . . ', but then the question about what is meant by *inconsistent* becomes fundamental. There is no doubt that the notion of inconsistency has always been closely associated with the notion of contradiction, so in order to understand how these other types of impossibilities work we will have to examine the link between inconsistency and contradiction. That would be looked at in the next section.

The second question that is relevant here is the following: If by appealing to these other types of impossibilities we establish that two statements cannot be both true and cannot be both false, is that enough reason to consider them contradictories? (*mutatis mutandis* for the case of contraries). At first sight, nothing in the definition for contradictories prevents that, but it could be argued that the definition has its context, and in this case it is the context of the logical relation between terms within categorical statements and the oppositions that can be established based on that; so any relation of opposition established using different grounds cannot be properly considered contradictories, in this precise sense. In short, to be contradictory is a logical relation and has to be based on logical grounds. A different argument could be that the only way we can establish that two statements cannot both be false is by means of logical constraints, then, despite any other information that we might use, if we effectively establish that two statements are contradictories, that has to be based—in one way or other—on logical grounds. So, either there are not contradictories or they are contradictories for logical reasons. The result of both arguments may be that it is only in virtue of logical constraints that two statements can be considered contradictories. I will address this question on its due course.

I would like to close this section by considering whether this appeal to other types of information can also affect the conception based only on logical impossibilities. As we saw there, the key element was the notion of some statements 'entailing a contradiction'. Now we can ask if it is possible for those other kinds of information to be involved in that situation. Again, the «definitional» attitude would say that entailment is a logical notion and contradiction has to be understood as ‘the conjunction of a statement and its negation’. But then we will only be able to say that two statements are contradictories just in the case that the negation of the assertion that states that they cannot be both true and both false will entail a statement compounded of the conjunction of a statement and its negation. And this does look like a set of prerequisites that can be easily or—at least—frequently fulfilled. On the other hand the aim could be to enlarge the applicability of the notion of logical impossibility, and that could be done by replacing that strict understanding of ‘entailing a contradiction’ with something that could be pictured as being closer to ‘an inconsistency will follow’; hence, other relations of consequence between statements will have to be considered, as well as inconsistency as a wider notion. In that context, the non-logical information about the related statement would have an important role to play.

In this section we have encountered the notion of inconsistency twice. It can be seen as a bridge between strict logical notions and more general questions. So it seems that is about time to examine it closely.

2 Inconsistency

In the case of the notions of inconsistency and “inconsistent” there is not one standard definition that can be easily recognized as such, which makes it quite different to what happens with contradiction (which, as we have seen, is normally understood as the conjunction of one statement and its negation) and contradictories (two statements that cannot be both true and cannot be both false). This may have many consequences, but the most apparent is that the terms inconsistency or inconsistent tend to be used more «loosely» than the others, even in contexts where terminological precision is highly appreciated. A common way of dealing with this problem is to try to specify some meanings of the terms and then stick to them. This is usually done by focusing on the application of the adjective ‘inconsistent’ to different nouns, and then the kind of inconsistency would be mainly determined by the notion to which it is applied; yet sometimes this is not enough, so another adjective or adverb is used to make precise the intended meaning of ‘inconsistent’. In this section, concentrating on logical notions, we will see how this may work.

In contemporary logic it is an established tradition to talk about two senses for consistency: a syntactic and a semantic sense. Their definitions can be found in many places. About the first type of consistency there is little variance, since the definitions are usually structured in terms of the non-derivability of a contradiction (understood as what I have presented as the standard definition).

Then, focusing on sentences and/or statements I think it is still possible to give a definition that could express the standard notion of syntactic consistency; something like this:

> Def. CONSY: A set of statements [sentences] is syntactically consistent if for no statement [sentence] is it the case that the conjunction of that statement [sentence] and its negation is derivable from that set of statements [sentences] using the accepted rules of derivability.

In the case of *semantic consistency* there is more variation. To keep things separated and for other reasons that will emerge later, I think we can have two definitions of semantic consistency running in parallel. Following the structure of def. CONSY, they can be formulated thus:

> Def. CONSE1: A set of statements [sentences] is semantically consistent (sense 1) if for no statement [sentence] it is the case that both that statement [sentence] and its negation are logically implied by the set of statements [sentences].
>
> Def. CONSE2: A set of statements [sentences] is semantically consistent (sense 2) if they can all be true under at least one interpretation.

At this point, we can go back to the original problem in this section, i.e., the notion of inconsistency and/or inconsistent. We have three definitions articulated in terms of different ways of being consistent, so we can have the same number of definitions for the opposite situations:

> Def. INCONSY: A set of statements [sentences] is syntactically inconsistent if there is a statement [sentence] for which it is the case that the conjunction of that statement [sentence] and its negation is derivable from that set of statements [sentences] using the accepted rules of derivability.
>
> Def. INCONSE1: A set of statements [sentences] is semantically inconsistent (sense 1) if there is a statement [sentence] for which it is the case that both that statement [sentence] and its negation are logically implied by the set of statements [sentences].
>
> Def. INCONSE2: A set of statements [sentences] is semantically inconsistent (sense 2) if they cannot all be true under any interpretation.

As I have anticipated, here there is no general definition of inconsistency but only definitions of what is to be an ‘inconsistent set of’ statements or sentences; so ‘inconsistent’ is just an adjective applied to those entities. That restriction comes from the way in which the definitions of ‘consistent’ were given.

3 Contradict and Contrary

So far we have been talking about names and adjectives, but what about verbs? There is no definition for them in philosophical or logical dictionaries, so we are only left with language dictionaries. Searching through them we find ‘contradict’ in any English dictionary, we find ‘contrary’ as a verb in very few dictionaries (while it is always found as name, adjective and adverb), and we find no verb that would share the root with inconsistent. The definitions that NSOED gives are these:

> contradict [. . .] [obsolete]1 v.t. Speak against; oppose in speech; forbid. l16-m18. 2 v.t. & i. Deny a statement made by (a person); affirm the contrary of (a statement etc.). l16. 3 Of a statement, action, etc.: be contrary to, go counter to. l16. (NSOED [10, p. 496])
>
> contrary [. . .] 1 vt. Oppose, thwart; contradict; do what is contrary to. obs.[olete] exc.[ept] dial.[ect, dialectal, -ly] ME. [obsolete]2 v.i. Act, speak, or write in opposition. lME-l16. (NSOED [10, p. 498])

Out of the five given meanings, three are presented as obsolete, but—what is more important—both meanings for ‘contrary’ (as a verb) are obsolete, a fact confirmed by any native English speaker. So if we want to talk about the action of stating the contrary of any statement, or to refer to what the contraries do to each other, we are only left with the second definition of ‘contradict’ in its second part: “affirm the contrary of . . . ”. Furthermore, if we want to talk about the action of saying the contradictory, or what the contradictories do to each other, we have to use again the second definition of ‘contradict’ stressing its first part “deny a statement . . . ”, or use the third definition, but then “be contrary to” appears again. Summing up, if we have two statements and we want to say that they cannot be both true nor both false, we will say they ‘contradict each other’, but if we want to say that they cannot be both true but may be both false, we cannot say, without being archaic, that they ‘contrary each other’. The normal option is to use the available word: ‘contradict’, which would fit perfectly in the definition given by dictionaries. Nevertheless, it sounds awkward—particularly after all these pages analysing definitions—to say that ‘contraries contradict each other’, but normally we do not talk about these notions in general, but use them as referred to specific statements, and then it is quite frequent to hear that ‘statement x contradicts statement y’ when the relation of opposition between them is not that of the contradictories but of contraries. Even in expressions like ‘statement x states the opposite to statement y’, the ambiguity will still be there; a less ambiguous option would be ‘statement x states the contrary to statement y’, but then ‘contrary’ (as an adjective) has a wide range of meanings (cf. NSOED [10, p. 497 f.]). A more precise option would be ‘statements x and statement y are contraries’, but there the diversity of meanings of ‘contrary’ could be involved and it may not be clear enough that the intended meaning is the one given in the square of oppositions.

It seems that the situation is the same in all the languages that use the Latin roots to refer to these traditional oppositions.[6] And even in German where there is a duality between the German and the Latin roots, and the Latin terms are used in the context of the square of opposition, somehow as a «technical term», the ambiguity is still present in the usual terms used for what in English would be ‘contradict’.[7]

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[6] For example in Spanish, ‘*contradecir*’—the equivalent to contradict—is defined in the «canonical dictionary», this way:

“contradecir (Del. lat. *contradicere, -onis*) Decir uno lo contrario de lo que el otro afirma, o negar lo que da por cierto.” (Real Academia Española: *Diccionario de la Lengua Española*. Madrid: Epasa-Calpe, 1992, p. 556).

It is almost the same as in NSOED definition 2, (translated it would be: “to say the contrary to what someone else asserts or to deny what he takes as true”). So both cases are clearly stated there.

[7] In the context of the square of opposition ‘*kontradiktorishenGegensatz*’ and ‘*konträrenGegensatz*’ are use for ‘contradictory opposition’ and ‘contrary opposition’, respectively [cf. *Die Philosophie* (Mannheim: DudenVerlag, 1985) p. 151]. Meanwhile, the verb ‘*widersprechen*’ is the term that normally will be used to translate ‘contradict’ and the noun ‘*Widerspruch*’ for ‘contradiction’, but also the term ‘*Gegensatz*’ is commonly used to express the confrontation between two statements, so it can be translated as ‘opposition’ but also as ‘contrariety’ as well as ‘contradiction’ (cf. Waibl, E./Herdina, P.: *German Dictionary of Philosophical Terms*, vol. 1 *German-English*. München: K. G. Saur/London: Routledge, 1997, p. 99). Similarly with related words as the verb ‘*entgegengesetzten*’, the adjective ‘*entgegengesetzt*’, and the noun ‘*Entgegensetzung*’. In fact, Hegel used ‘*Gegensatz*’ for what is known as his ‘antithesis’ (cf. ibid.).

The situation is quite peculiar: we have several names and adjectives—contradiction, self-contradiction, contradictories, contradictory, contrary, contraries, inconsistency, inconsistent—and some of them can be understood in different senses, as we have seen, but we have one verb—contradict—that can express the actions or interactions related with them. I think that this yields to a layout that generates important conceptual confusions. Mainly, when the word 'contradict' is used to describe the interaction between two elements, it can lead to the assumption that these elements are contradictories or together they would constitute a contradiction, which in several cases is not correct because they do not fulfil the two conditions that are required to have a proper contradiction. In sum, this preponderance of the verb 'contradict' encourages several «loose» uses of the notion of contradiction and/or several «analogical» uses of it and the other terms. Acknowledging this can have far-reaching consequences, some of which I will try to point out in what follows.

4 Some References to Previous Works

We have studied different definitions of the main notions related to inconsistency, so we can now turn to study the relations among them. Although it is generally accepted that they are all closely related, it is not so easy to find proposals that explicitly deal with their interaction. In the spirit of examining accounts that can be considered as defending the classical view, I think that Strawson and Sainsbury are two authors that can be quite helpful, and I cannot touch on the subject without making some very short comments about their proposals.

In Strawson's first book, published 50 year ago, there is an entire chapter devoted to the subject, and he uses it as a gateway to the whole of logic; his analysis points out several problems that, in my view, are at the heart of the problematic concerning inconsistencies. Even though it is rare to see references to this text in recent discussions of these issues, with the important exception of Horn 1989, we ought to take it into account.

Sainsbury is a contemporary author that has been especially concerned with paradoxes and with the challenge posed to the classical position by paraconsistent logic, particularly Priest's proposals [11, 12]. He has also written [13] a general introduction to logic where he addresses the relations among the main notions related to inconsistency. Although in that text he does not mention paraconsistent logic or dialetheism, his presentation touches on some of the points that we will be dealing with, so it deserves attention. However, the definition of contradiction given by Sainsbury is the one that Slater [14] uses when criticizing paraconsistent logic.

In the next section, I will present a terminological proposal, which, trying to reap what has been studied so far, will seek to arrange together what, in my view, are the main elements surrounding the issue of inconsistency.

5 Terminological Proposal

As we have seen there are many definitions of the notions related with contradictoriness. So far I have only presented notions that assume classic logical principles as a general framework, but even in that case there are differences in relation to which term is the appropriate one for each distinguishable situation. If we were to examine the uses in natural languages, sciences, legal theory, etc., we will surely find even wider differences. I do not think it is possible to establish a framework that would be able to settle the whole matter, particularly due to the tendency for «analogical uses» of such words. Every new context may yield other uses of the same words, and nothing can prevent that. However, I would like to present a way of using these words that may help to prevent some confusion, which can be instrumental in avoiding the extrapolations from one notion to another or others.

Let me start with the basic case: the relation between two statements. There are two general dualities that seem to me the most relevant: inconsistent/consistent and opposite/non-opposite. They do not coincide. My proposal is to consider these terms as follows: Two statements are **inconsistent** if they cannot both be true, two statements are **consistent** if they can both be true; two statements are **non-opposite** if they can both be true and can both be false, otherwise they are **opposite**. In all these definitions what is considered is only the two statements and their mutual interaction. The next step is to bring in the traditional definitions: two statements are **contrary** if they cannot both be true but may both be false, two statements are **contradictory** if they cannot both be true neither can both be false, and two statements are **subcontrary** if they can they both be true but cannot both be false. In all this, the only new term is 'non-opposite', but it could be replaced by a longer phrase like 'two statements that are not opposite to each other'.

Putting these notions together, we will have that two inconsistent statements can be either contrary or contradictory, while two consistent statements can be either subcontrary or non-opposite, but two opposite statements can be contrary, contradictory or subcontrary. That is why these dualities do not coincide: subcontraries are on the side of opposite, but also on the side of consistent; in other words, the duality inconsistent/consistent has two in each side, while opposite/non-opposite has three in the first and one in the second.

Then, **contraries** would be the generic name for both statements that are contrary, **contradictories** for the ones that are contradictory, **subcontraries** for the ones that are subcontrary, and **opposites** for the ones that are opposite. Instead of 'contraries' one could also use **contrary statements** and, similarly, **contradictory statements** and **subcontrary statements**. Unfortunately this does not work so well for the other two notions; yet searching for a name we may say that two statements that are inconsistent conform to an **inconsistency**, but I do not see a generic name for them, apart from '**inconsistent statements**'; the case of 'consistent' is worse because to say that two statements that are consistent conform to a «consistency» sounds awkward, so we are left just with **consistent statements**.

As you can see the aim is to capture as much as possible of the traditional terminology. Contraries, contradictories and subcontraries, conform to three traditional form of opposition, so here they are seen as 'opposite statements'. The only difference with those traditional denominations may be in the relation of subalternation, which traditionally is explained in the context of the «square of opposition» saying that the true of the 'superaltern' (can be A or E) implies the truth of the 'subaltern' (I and O, respectively), but not conversely (cf. [1, p. 875]). The fact that they are explained in that context may lead to think that they are considered as 'opposite'; however, that is more a relation of consequence than a relation of opposition. Taking a traditional example, to say that 'all men are mortal' and 'some men are mortal' are «opposite sentences», seem to go very much against the normal use of the term. Let us compare this with the other cases: two subcontraries, like 'some men are mortal' and 'some men are not mortal', can be considered as opposite, but not as inconsistent, while two contraries, like 'all men are mortal' and 'no man is mortal', are clearly opposite and inconsistent, and also two contradictories, like 'all men are mortal' and 'some men are not mortal'. On the other hand, the superaltern and subaltern statements can both be true, and the same is the case for two subcontrary statements, so to them applies the given definition of 'consistent statements'.

Here it is important to remind ourselves that I am using the given generic definition of contradictory, contrary and subcontraries, and not the ones with the specifications of the square of opposition, so there is no restriction about being universal and particular and also not in the subject-predicate relation. For example, 'Mr. X went to the right' and 'Mr. X went to the left' would be contraries, despite being both particular statements, and, in parallel, 'object Y is animate' and 'object Y is inanimate' would be contradictories (assuming that every object has to be either animate or inanimate). It is difficult to find an example of subcontraries where both are general statements, but consider: 'all ambidextrous people can use their right hand skilfully' and 'all ambidextrous people can use their left hand skilfully'; then, ruling out the case of a mental or physical disease or any other particular physical limitation, and assuming that all human beings in normal physical conditions can at least use one of their hands skilfully, we can say that those two statements can be both true but cannot be both false.

Now lets compare this with the definitions that we studied in last chapter. Apart from those generic definitions, this proposal is based on the second of the semantic definitions of consistency (def. CONSE2) and also the second of the semantic definitions of inconsistency (def. INCONSE2). So it is convenient to examine how it fit, with the other definitions. In a standard situation, both contradictories and contraries—defined in the traditional way—fulfil the condition stated in the other semantic definition of inconsistency (def. INCONSE1), i.e., they will logically imply a statement and its negation. That also holds for the case of contradictories without negation presented by Sainsbury. The converse (if for any case where def. INCONSE1 applies, then def. INCONSE2 also applies) is a little more complicated; the question is, in other words, whether given that a pair of statements logically imply a statement and also its negation, from that it would follow that they cannot both be true. Using the standard notion of logical implication the first part may be understood as follows: if those two statements are true, then there is a statement that is true and also its negation is true. But classically a statement and its negation cannot both be true, so in principle there is never such a statement, consequently, by *modus tolens*, the two statements cannot be true. This fits with def. INCONSE2 and with the definition of 'inconsistent' that I have used.

Considering def. INCONSY, it all depends on the rules of inference. Nevertheless, in classical logic, up to first order predicates level, in virtue of its completeness, if we can apply def. INCONSE1 we can also apply def. INCONSY, and also vice versa, due to soundness. Then, if we start with two statements that fulfil def. INCONSE2, then—as we have seen—they would fulfil def. INCONSE1, and then they will also fulfil def. INCONSY; and similarly in the opposite direction. Outside the domain of classical elementary logic that assurance is not there. Consequently, in each case it will depend on which rules of inference are maintained and what kind of semantics is used. But that does not affect the heart of my proposal, rather quite the contrary. For in those other situations we will have two or three parallel definitions of 'inconsistent', and different senses in which two statements can be considered as inconsistent, and then the task would be to find out if they coincide for all cases or not. Moreover, considering that for each definition of 'inconsistent' there is a corresponding definition of 'consistent', so the duality inconsistent/consistent would still be there, yet instantiated by each of the different senses, if they differ.

So far I have been explicitly dealing with the case of two statements, but considering that some definitions are in terms of 'set of statements' then the extension is straightforward. A set of statements is **inconsistent** if they cannot all be true (where 'inconsistent triad' is just a particular case), **consistent**, if they can all be true. The other duality can be expressed like this: a set of statements that contains **non-opposite statements**, if they together can all be true together and can all be false, otherwise it contains **opposite statements**. That works well for more than two statements, but also for two statements. As I said that two statements are inconsistent, I can also say here I say that they together constitute an 'inconsistent set'; I said two statements are **non-opposite**, here that together they constitute 'a set of non-opposite statements', and the same for 'consistent' and 'opposite'.

In the case of 'contrary', 'contradictory' and 'subcontrary', although they seem to be more linked to their origin in the case of two statements, they may well be extended to more than two. Then we can say that a plural number of statements are **contrary** if they together cannot all be true but may all be false, **contradictory** if they together cannot all be true neither can they all be false, and **subcontrary** if they together can all be true but cannot all be false. However, to keep with the tradition, I think the names 'contraries', 'contradictories' and 'subcontraries' should be reserved for the case of two statements, but they constitute a specific case within 'contrary statements', 'contradictory statements' and 'subcontrary statements'. We may say that a set of statements is contrary if it contains contrary statements, and similarly for contradictory and subcontrary; yet for the first two the name commonly used is 'inconsistent set', which for me is fine as far it is kept in mind that it can contain either contrary or contradictory statements, or both. In the case of a set containing subcontrary statements this could be also described as a 'set containing opposite statements that are also consistent', but 'subcontrary set' seems better. Finally, for a set that contains contrary, contradictory and subcontrary statements the best denomination seen to be 'set containing opposite statements', because there is nothing shorter that could be similar to 'inconsistent set'.

Now, if for a single statement we were going to use the names 'contrary', 'contradictory' and 'subcontrary', then the obvious question would be: 'contrary to what?' (and similarly for the other two cases). If it is to another statement, then they together would fit in what I have said so far in this section, but if the answer is 'to itself'. One possibility is that it is statement compounded of two contradictories, contraries, or subcontraries. The first case could be designated as ‘two contradictory statements stated together’, which *mutatis mutandis* can be extended to the other two. However, the standard definition of ‘**contradiction**’ is the conjunction of one statement and its negation, which has to be preserved for the aim of capturing as much as possible of the normal uses of the terms. But that does not cover all the cases of ‘two contradictory statements stated together’ (i.e., when one statement is the negation of the other), and then the question is what do we do with the others. The problem is in Sainsbury’s and Strawson’s texts, the first was not explicit about it, and the second accept a ‘wider sense’ of contradiction that covers even contraries. My proposal is to use the denomination ‘**contradiction without negation**’ for the conjunction of two contradictory statements such that neither of them is the negation of the other, which may sound odd but will make explicit the possibility of having two contradictories conjoined without one being the negation of the other. This preserves the difference between the syntactic and semantic characterization (thus overcoming the objections against the tendency of assuming that both characterizations always coincide), without breaking the normal association of ‘contradiction’ with ‘contradictories’ and ‘contradictory statements’. Furthermore, ‘**inconsistent statement**’ would be the generic term that would cover ‘two contrary statements stated together’ and ‘two contradictory statements stated together’.

The latter would have a particular status because although it will cover the case of ‘contradiction without negation’, which is an explicit conjunction, it would remain as the appropriate denomination when the two statements are stated together without a conjunction. Furthermore, an advantage of this scheme is that the name ‘contradiction’ (without qualification) would remain linked with the notion of negation, but then it could refer to different types of negation, depending on how that notion is defined in each system, so we can have contradictions with classical, intuitionistic and paraconsistent negations—just to mention the more relevant for my purposes—, but that would also be the case for any other characterization of negation. Then, a contradiction with classical negation, which can be called ‘classical contradiction’, would be placed side by side with the ‘contradiction without negation’, both belonging to the type of ‘two contradictory statements stated together’. However, when other types of negations were to be used, then it would have to be examined if for all cases a statement and its negation (using the other type of negation) cannot both be true and cannot both be false; if that is the case, then that other contradiction would also be placed in the group of ‘two contradictory statements stated together’, otherwise, we will have to see if they constitute an inconsistent statement (i.e., they cannot be both true) or if something else is the case.

The other possibility is when the statement to which ‘contradictory’, ‘contrary’ and ‘subcontrary’ is applied is not a compound statement. That takes us to the discussion on atomic statements, where several reservations related to the term ‘self-contradiction’ can be raised. There is not much to add here, apart from two comments. First, possible names would be ‘self-contradictory statement’, ‘self-contrary statement’ and ‘self-subcontrary statement’, yet the first two could be called ‘self-inconsistent statement’; which now can be seen as based on the present terminological proposal. Moreover, I said that it may be better to use the denomination ‘internally inconsistent statement’, because it makes more explicit where the inconsistency comes from, or at least where it does not come from. Second, the case of a ‘self-subcontrary statement’, that is, not being a compounded statement, seems to me unfeasible. I presented several reservations about the feasibility of a proper ‘self-contradictory atomic statement’, and showed how in many cases what we may have is at the most a ‘self-contrary statement’, and the point was that it could be seen that the atomic statement was somehow stating together two characterizations that could not be both true (like in ‘that bachelor is married’), but the difficult part is whether they cannot both be false. The case of subcontraries is particular because we will have to have the difficult part, yet excluding the easy one. I cannot see any case in which this may hold. However, this is not the same as saying that there cannot be any ‘subcontrary predicates’, a matter that will be addressed next, because if there is any subcontrary predicates the point would still be if it is possible to apply them together conforming a single atomic statement. If I am right, that yields an interesting consequence for my schema, that is, the duality opposite/non-opposite does not seem to be of much use in the case of atomic statements, since the possibility of a ‘single consistent and opposite statement’ does not seem relevant, apart from the case of a statement compounded by subcontraries, case in which such a description is much better.

All the definitions that I have given are presented in terms of statements, but they can be extended to sentences, propositions and formulae. Such an extension depends on accepting that it is possible to apply to these entities both the syntactic and semantic characterizations, that is, if one accepts that it can be properly said about two sentences, propositions or formulae, that they cannot both be true and/or cannot both be false, and also that one is the negation of the other. Moreover, if the case of two is accepted then one would have to analyse whether there would be any additional problems for the case of more than two and for the case of just one, which would allow us to generalize the definitions in terms of sets of sentences, propositions or formulae. In the background of such extensions would be the whole problematic around truth-bearers, which I have mentioned but somehow bypassed. The main reason for that has been that I think that it would distract the attention from the main issues that I wanted to address here, and I do not think it is necessary to assume a fixed position about that polemic in order to approach them. Nevertheless, let me say (just to be open about it, even though the reader probably has already noticed) that I am in favour of the thesis of the statements as the truth bearers. Yet, that polemic in general does not interest me as much as a more specific one: what can be ‘contradiction bearers’, but now we can use a more generic designation: ‘inconsistency bearers’; that is, the issues around which elements or items can be inconsistent with each other (or even with themselves). Considering that only two of the three given definitions of inconsistency are directly based on the notions of truth and falsity, then, in order to deal with the other definition, i.e., the syntactic one, the discussion about inconsistent bearers would not be restricted to what it is covered by the polemic about truth bearers. Moreover, it would also have to deal with the notions of contrary and contradictory terms. So it seems to me that they are two different discussions, although closely related. In addition, it is not that only one affects the other but it can be both ways. Actually, as we have seen, Strawson uses considerations about ‘what can be inconsistent with what’ as a step in his argument for the thesis that statements are the true bearers.

Going back to the proposed definitions, as I have said, they are articulated in terms of statements, but they can be extended to sentences, propositions and/or formulae as long as one considers that any of these entities can be true or false, but also if one thinks that negation is an operation among them (these options may run independently). There we will have ‘inconsistent sentences’, ‘inconsistent propositions’ and/or ‘inconsistent formulae’, and *mutatis mutandi* for the other definitions. They can also be extended to ‘inconsistent beliefs’ and similar, as long as it is accepted that two beliefs cannot both be true (false) and/or that one belief is the negation of another.

Let us focus now on the more general terms. I have used ‘contradictoriness’ as embracing terms for both ‘contradiction’ and ‘contradictory’, which may also take in ‘contradict’ but without including its use for contrary statements. Above I said that ‘two statements that are inconsistent conform an inconsistency’, and also proposed ‘opposites’ as a denomination for two opposite statements. Considering that I have distinguished three notions of ‘inconsistent’, it has to be explained how that affects these denominations. About the term ‘inconsistency’ I think that the best thing to do is to acknowledge two main uses of it: as the name for the relation between inconsistent elements, and, by extension, as the generic name for elements that are inconsistent. It does not seem to me that these two uses lead to much confusion because if necessary it can be established in which of the two senses the term is used; however, I think that the first use is more precise, while for the second it may be better to say ‘inconsistent elements’ (or whatever one is talking about). Regarding the plural case, ‘inconsistencies’ allow us to talk about more than one instance of the relation of being inconsistent, but it may also be used as a generic name for several inconsistent elements. This duality does not seem harmful because whenever we have one side we will have the other. In all these options the key element is the term ‘inconsistent’, so these variations depend on to which kind of element it is applied; however, there is a more substantial diversity that comes from the different senses in which the term can be understood: one syntactic and two semantic. Even though, in each case it is possible to replace ‘inconsistent’ for ‘syntactically inconsistent’ or ‘semantically inconsistent’ (and, if necessary, differentiating somehow between the two semantic senses), it is important to ask to which sense the term inconsistency is going to be applied; my answer is to all of them: whenever there are elements that are inconsistent, in any of the described senses, one may say that there is an inconsistency. They are three different senses and none of them has prominence over the others. It may be useful in each particular case to elucidate which is the relevant sense, but in order to address them in general the best option is to use ‘inconsistency’. Furthermore, if one wants to talk about the issues related to the situation characterized by one or several elements being inconsistent with another(s), again ‘inconsistency’ seems also to be appropriate, possibly articulated in expressions like ‘the question of inconsistency’ and similar. Thus, at the very abstract level, we have ‘**inconsistency**’ as the generic term, and ‘**contradictoriness**’ and ‘**contrariness**’ as species.

So far I have presented my terminological proposal for names and adjectives, so it is time to address the verbs. As we saw, ‘contradict’ is the term normally used to designate the interaction between contradictory statements (sentences, . . .), but it is also used for contrary statements; even more, it is also used for the relation between one statement and its negation. I have said that these diverse uses of the same term contribute greatly to generate confusion among these uses. Consequently, it would be very helpful to distinguish these meanings by using different terms, but I do not think that is possible by means of verbs without being at odds with a well-established use in natural languages (at least in English and Spanish, and very likely in other major European languages). My proposal is to use not a verb alone, but the following expression: '**to be contradictory to**' and '**to be contrary to**'; but then both would be covered by the generic expression '**to be inconsistent with**'. This latter expression is frequently used to refer to both situations, and it does not yield to confusion as 'contradict' does, a reason for which it is preferable. Moreover, whenever 'contradict' is used, even though it would be possible—in principle—to elucidate in which of the two senses is used, it can be understood as a generic term that is equivalent to 'to be inconsistent with'. In the line of Strawson's approach, if something is contrary to something, that does not exclude the possibility that they can be contradictory to each other, because what is being said—by the first expression—is that they cannot be true together, which does not exclude the possibility that it may be the case that they cannot be false together. This is not the case when the terms 'contradictories' and 'contraries' are used, because they exclude each other. We have seen that the two semantic definitions of 'inconsistent' are applicable to both contradictories and contraries, and with respect to the syntactic definition it may depend on the inference rules adopted but at least within classical logic that also holds. For the case of subcontrary statements (sentences, . . .), the expression 'to be subcontrary to' would be precise but maybe too jargon-like, so it seems better to use '**to be opposite to, but not inconsistent with**'. Then, '**to be opposite to**' covers this case, together with the cases of 'to be contradictory to' and 'to be contrary to', and—consequently—also 'to be inconsistent with'.

In the case of the opposition between statements, if the opposition is determined by classical negation as a statement operator, then the expression 'to be contradictory to' would be appropriate, this matches with the customary use. But if what is used is somehow different, then in each case it would have to be assessed what kind of opposition is being articulated. Let me just mention two examples without going deeper into them for the moment:, using another characterization of negation it may be the case that, concerning a statement and its negation (using that alternative negation), we will be able to say that they cannot both be true, but we are not able to say that they cannot both be false, so they will only be contrary to each other. Furthermore, using some form of paraconsistent negation it can be the case that a statement and its negation will be both true, in which case they would not even fulfil the condition that I have presented as characteristic of 'to be contrary to', but they will fulfil the conditions stated by at least one of the definitions of inconsistent (def. INCONSY and/or def. INCONSE2). A classical reply to that would be to say that they are not proper negations, as issue that requires a full debate, but for the purpose of this terminological proposal it would not be appropriate to presuppose any outcome of it, so it has to leave open the possibility of calling these other operator 'negations'. Then, to express in general the relation of an statement (sentence, . . .) with its negation (whatever kind it is) a useful expression would be '**to be the negation of**'; although, it may be made more specific adding some designator for the kind of operator involved (classical, strong, weak, paraconsistent, paracomplete, etc.).

Let me draw a schema of the main divisions of this terminological proposal:

- Statements
  - opposite
    - inconsistent
      - contrary
      - contradictory
    - subcontrary
  - non-opposite

6 Terms

When I addressed the most general words, like ‘inconsistency’, ‘contradictoriness’ and ‘contrariness’, I talked about ‘elements’ because being the most general definitions they have to cover, at least in principle, statements, sentences, propositions, formulae, beliefs and similar notions. Moreover, there is no reason, at this point, to exclude the possibility of these elements being just terms, so let us consider ‘contradictory terms’ and ‘contrary terms’. Brady [2, p. 61] defines the former as being mutually exclusive and jointly exhaustive of the universe of discourse, while the latter is defined only as mutually exclusive.

If we enquire about their relation with the notions that we have been dealing with, we can find answers like this:

> contradiction is the counterpart, on the level of sentences, of the semantic relation of meaning incompatibility at the level of lower order constituents. [...] Linguistically, the concept of incompatibility appears as antonymy, which is a relation between expressions. (Katz [7, p. 144 f.])

Yet trying to characterize more formally the notion of antonymy, the same author says that considering the notions of maleness and femaleness, we can assume that they are “incompatible and jointly exhaustive of the sexual domain” [7, p. 146] or “that the concepts are not jointly exhaustive (taking the term ‘hermaphrodite’ into consideration, for example)” [7, p. 146]. Then we can have antonymy with and without exhaustion, so it would be the counterpart not of the relation among contradictory statements (sentences) but of contrary ones.

Taking a term as a generic notion that covers both names and predicates, and not restricted to one word but including expressions,[8] my proposal is to use **incompatible terms** as the counterpart of ‘inconsistent statements (sentences, ...)’. Then, we will

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[8] Following definitions like the one in Honderich 1995:

“term. A word or phrase denoting an individual or class, or the propositional component it expresses. Thus ‘John is a man’ contains two terms ‘John’ and ‘man’ (or ‘is a man’), denoting John and the set of have **contrary terms** when they are only incompatible in some domain of discourse, and **contradictory terms** when they are also jointly exhaustive of the domain[9]; thus, the definitions given by Brady [2] apply, as long as 'incompatible' is understood as an equivalent to 'mutually exclusive'. Considering the possibility of having terms that are jointly exhaustive but not incompatible, a denomination that seems to me adequate is **opposite yet compatible terms**; then, **opposite terms** can be used as a denomination that would include these but also incompatible terms.

This last possibility could also be called 'subcontrary terms', but it would be more distant from the habitual use of the words, and the proposed denomination makes the intended sense more explicit. Actually, in Brady [2], the definition of 'subcontrary' is only given for propositions. That could be explained by the fact that it is much more difficult to find terms that are jointly exhaustive but not mutually exclusive in a domain of discourse. I have already presented my reservations about the possibility of an 'atomic subcontrary statement'; nevertheless, the situation is different regarding terms, because we have to consider the possibility of establishing compound predicates. A good example is: 'greater than or equal' and 'less than or equal'—and other similar expressions—because if the two quantities compared are equal, then both terms would be applicable; thus, in that case there is not incompatibility, but for all the other cases in the specific domain (like the natural numbers) either one term or the other would have to be applicable, so they will jointly exhaust that domain. In general, any pair of terms that overlap in some cases but together cover the whole domain of discourse, would be 'opposite yet compatible terms'. I think that the most relevant cases are not vague predicates that may overlap, but terms that their overlapping are not taken as a problem, or as a failure of being more precise. It may be said that it is in principle possible to establish or insolate a term for those overlapping cases, and, then, use two different terms for all the other cases (in my example would be 'equal to', and then 'greater than' and 'less than'). That is right, but then we would have just made a transformation: from 'two opposite yet compatible terms' into 'three incompatible terms'. In this new outcome, the three terms will exhaust the domain of discourse, but not by pairs, so they will fit in the classical law of trichotomy.[10] However, that does not imply that the first situation cannot hold, but rather that it can be transformed into something else.

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men respectively. More generally, any word or phrase that determines the proposition expressed. In this sense, the above sentences contains the syncategorematic term 'is', which does not denote and individual or class." [5, p. 869 (Entry written by Wayne A. Davis)]

[9] Horn [6, p. 268] presents a different proposal: what I have called 'incompatible terms', there are called "contraries (incompatibles)"; my 'contrary terms' are called "mediate (weak) contraries" (which are subdivided into "simple (reductive) contraries" and '"polar (absolute) contraries", subdivision that I do not have; my 'contradictory terms' are called "immediate (strong, logical) contraries". The main reason for such differences is that he thinks, following Aristotle, that terms cannot be contradictories, only propositions (cf. Ibid. p. 39). I will come back to this in the next chapter, but for the moment let me point out that, for the purposes of his chap. 5, he also uses contradictories for terms (cf. [6, p. 269]).

[10] It is interesting to see how it is presented: "Law of Trichotomy. Also called the law of comparability. In general, a division of entities into three sets that are pairwise disjoint (that is, non-overlapping) and exhaustive." [4, p. 61] That is followed by specific definitions for the theory of real numbers and for set theory.

That takes us to a further point, which is the application of these notions to more than two. The definitions given in Brady 1967 only talk about the case of two terms, but their extension to more than two terms does not seem to pose any problems as long as they are all incompatible with each other; then, if they also exhaust the domain of discourse, we can have a 'contradictory set of terms', if not, we will have a 'contrary set of terms'. An interesting feature of the former is that if we take its terms by pairs, or any other proper subset, then they would conform to just a 'contrary sets of terms'. This just emphasizes the fact that both contradictory and contrary are notions that express a correlation among some specific elements, so their applicability strictly depends on which are the elements considered, but it also shows how closely related they are to each other. An even more interesting issue is the converse situation, that is, having a contrary set of terms, how can we conform a contradictory set of terms; then, the key question is what has to be added to get jointly exhaustiveness. I will address this issue in the next chapters, so allow me to leave it, for the moment, as an open question.

Let us consider now the case of one single term. Considering that I am using 'term' in a wide sense, which includes expressions, then, it can be the case that a term is compounded of two terms that are contradictory or contrary. That kind of situation take us to consider the possibility of having a self-contradictory atomic statement, where what was contradictory was either the name or the predicate in themselves. I do not have anything to add apart from stressing that since these terms are compounded, then the correlative character of contradictory and contrary is clearly present. It is quite different in the situation where it is not a compound term, because if there is not at least two notions involved I do not see how it would be possible to establish some kind of contradictory or contrary relation, simply because there will not be something that would be contradictory with something else; reflexivity is not feasible. And that includes the case of 'opposite yet compatible terms'; since to have some form of opposition there must be some difference. In short, the notion of a contradictory or contrary set of terms may include the case of just one term only if it is a compound one; furthermore, that holds for all cases of opposite terms.

7 Incompatibility

I have presented 'incompatible terms' as the counterpart of 'inconsistent statements (sentences, ...)', but considering that the latter covers the three senses of 'inconsistent', we have to consider what happens with 'incompatible'. Apart from the definitions in language dictionaries,[11] I have found only one relevant definition in a reference book of philosophy

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[11] The most relevant definitions in the NSOED are:

"**Incompatible** [...] adj. **1** Incapable of existing together in the same person, opposed in character, discordant. (Foll. by *with*, [obsolete] *to*.) LME. [...] **c** Of an item of equipment: unable to be used in conjunction with some other item. M20. **2** Unable to agree or be in harmony together, at variance. M16. [...]" (NSOED [10, p. 1339]).

or logic, in Mautner [9]; it defines incompatible as “not compatible”, and then:

> **compatible***adj.* Two beliefs, theories, etc. are compatible if and only if they can be true together. Two facts, events, states of affairs, etc. are compatible if and only if the occurrence of one does not rule out the occurrence of the other.
>
> *Compatible* and *consistent* are near synonyms. There are, however, some subtle differences: one point of difference is that compatible is mainly used in respect of exactly *two* items, whilst *consistent* is used in respect of any number of items. Another point of difference is that when two items are both objects, events, states of affairs (rather than thoughts, beliefs, statements, theories), they are said to be *compatible* (rather than consistent). Similarly, we say two colours are incompatible; to say that they are inconsistent would sound odd. [9, p. 101]

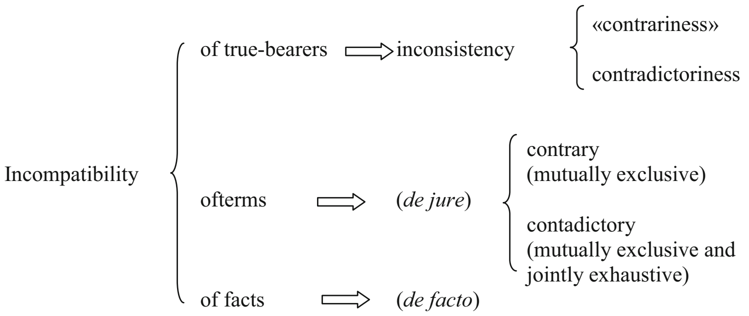
Considering that the three definitions of ‘inconsistent’ that I have presented came from definitions of ‘consistent’, now, the situation would be similar. In the first paragraph we have two definitions of ‘compatible’, but applied to different entities; our very familiar ‘can be true together’ is applied to “belief, theories, etc.”, but then in the next paragraph it is indirectly implied that for those entities (adding explicitly thought and statements) it would be better to say that they are consistent. That is exactly what I have been doing so far, and then this first definition matches with def. CONSE2, and consequently with def. INCONSE2. The second definition is for “fact, events, state of affairs, etc.” and adapting it for ‘incompatible’ it would say ‘they are incompatible if and only if the occurrence of one does rule out the occurrence of the other’; in the next paragraph, after explicitly adding objects, it is said that ‘compatible’ is a better word to use about all of them, instead of consistent. Then, ‘incompatible’ appears as the proper term for the relation between two colours. This second definition does not have a direct equivalence in the definitions of ‘inconsistent’ that I have presented, which is part of the reason why I think the distinction must be made explicit somehow.

From a wide perspective ‘incompatibility’ would be the most generic term, which may describe the relation among facts, objects, events, state of affairs, but also thoughts, beliefs, statements, theories, although ‘inconsistent’ is a more specific designation for the relation among the latter ones. Now let’s enquire about terms. If ‘incompatible’ can be applied to this very wider range of items, there is no doubt that it can be applied to terms, which somehow are in the middle between the first ones and the late ones. But then, none of the given definitions for ‘inconsistent’ are directly applicable to terms: they are not truth-bearers so they cannot be true together (or otherwise), and the relations of syntactic or semantic consequence do not hold among terms but among statements (or sentence, proposition, formulae). Similarly, the definition of incompatibility articulated as “the occurrence of one rules out the occurrence of the other” also does not seem appropriate because terms are, rather, used or applied in some expressions, and the occurrence of one term does not have *de facto* any effect in the occurrence of another. Consequently a parallel definition can be proposed: ‘Two terms are incompatible if the application of one rules out the application of the other’, so the application of one term would have a *de jure* effect in the application of another.

We have ended up with three kinds of incompatibilities: *de facto* incompatibility, incompatibility among terms, and inconsistency. Each has to do with a different kind of entities or elements, but conceptually they are closely related. In the traditional classical view—so to speak—, they must all fit together, so *de facto* incompatibilities should be expressed as incompatibilities among terms, and then these being reflected in the setting-up of possible inconsistencies. But, in my view, a general main problem is that each one can be seen as having its own dynamic, which is a source of important difficulties in relation to the idea of all of them matching together. I have discussed this on Bobenrieth 2003, but for the moment let me stress that as a result of analysis of definitions in the last chapter and the present terminological proposal, the notion of inconsistency has come out as placed in the wider context of incompatibilities. That can be seen as something obvious, but I think that keeping it in mind can be very helpful in order to deepen our understanding of the whole issue of inconsistencies.

Before finishing, I would like to add that I do not fully agree with the quoted definition when it says that compatible is "used mainly in respect of exactly *two* items", which is proposed as one of the differences with *consistent*. It may be that there is a tendency for using 'compatible' when it is related to two elements, because it is the most notorious case, but there is no reason not to use it for more than two. Following the same case mentioned at the end, saying that two colours are incompatible sounds better that saying that they are inconsistent, but the same is the case for any plural number of colours. Furthermore, there are many other distinctions that aim to establish incompatibilities among more than two items (for example: past, present and future; solid, liquid and gaseous, etc.). In my proposed denomination, all of these would be, in principle, cases of incompatible terms.

Let me draw another schema:



With these I conclude this terminological proposal. Its purpose has been not so much to prescribe some kind of correct use of the terms but to differentiate some meanings and assign some denominations to them. The words that I have used are not so important as recognizing that there are different situations that require some specific designations. Although I have tried to be as close as possible to the common use of words, in many cases I have made distinctions that go beyond it. I hope that the result is not too artificial. Anyhow, in case of any important disagreement about the most suitable word or expression for some situation, the reader can «read» his or her preferred option instead of my suggestion. Any terminological usage will have its pros and cons, but my main aim has been to show some differences that are relevant for the reflection about inconsistencies and related issues.

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Aristotle, Frege and "Second Nature"

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Abstract Aristotle proposed a "naturalistic" epistemological perspective that rests on some fundamental notions:

– Perceptual judgment (passivity and activity),
– Simple propositions (subject and predicate),
– Complex propositions (syllogisms).

As is well known, the "Square of Opposition" provides the possibility of a fruitful classification of reality that is made of things, species and genus. Frege introduced a new form of notation that is exemplified in his *Begriffschrift* and changed the Aristotelian square. He introduced a conception of judgment that entails a fundamental relationship with a "second nature". Starting from this background, McDowell and Brandom present two original views of the "second nature" which are subject to some criticisms.

Keywords Second nature • Concepts • Rationality • Normativity

Mathematics Subject Classification 00-02, 03AXX

1 Second Nature as Conceptual

What is "second nature"? We mean (aside from any anthropological alternatives, interesting in themselves) the characterization of human beings as rational; namely, concept-using and discursive creatures. So, there is a nature we share with animals and a peculiar second nature, which reveals our conceptual capacities. We want to sketch these two natures while placing them in a fruitful relationship. This means that concepts must be world-involving.

Frege's recognition of true judgments means recognition of true thoughts, where a thought is the sense of a linguistic expression; its meaning is a concept; namely, the corresponding function. But also a proper name has a sense (the way in which we think of an object) and a meaning (*Bedeutung*) that is the object to which the name refers. The distinction between "function", namely the fixed part of an expression and "argument", namely the variable part of it, plays the fundamental epistemological role to indicate when the argument is "determinate" or "indeterminate". This very distinction is relevant for specifying a new notation of "generality", which differs from the Aristotelian one and rests on a substitutional strategy [1].

J.-Y. Béziau, G. Basti (eds.), *The Square of Opposition: A Cornerstone of Thought*,
Studies in Universal Logic, DOI 10.1007/978-3-319-45062-9_10

Frege's logic developed sophisticated logical relations among concepts that rest on the fundamental notion of "unsaturedness". Here, we can represent the falling of an individual under a concept by F(x), where x is the subject (argument) and F() is the predicate (function), and where the empty place in the parentheses after F indicates non-saturation. The distinction between "judgeable content" and "judgment", namely the way from a thought to its truth-value, entails a distinction between "cognitive content" and "semantic content". We need both in order to have a plausible definition of the nature of judgment. But, Brandom and McDowell seem to privilege only one of them.

Brandom and McDowell follow the main idea of Wilfrid Sellars: «The essential point is that in characterizing an episode or a state as that of *knowing*, we are not giving an empirical description of that episode or state; we are placing it in the logical space of reasons, of justifying and being able to justify what one says» [2]. We can consider perceptual judgments as the product of two types of capacities: the capacity to respond to environmental stimuli and the capacity of taking a position in the game of giving and asking for reasons [3]. Otherwise, we can imagine two different logical spaces: the space of impressions and the "normative" space of knowledge, of the "normative" relations with the world (for example, justification) [4]. The motive of these distinctions is that the natural response to environmental stimuli is a necessary condition of empirical knowledge, but not a sufficient one. A parrot can reliably respond to the presence of a red thing by uttering the sound, «That's red» and we can also suppose that an observer can do the same under the same circumstances. Consequently, we can conclude that the parrot and the observer share the same *reliable differential responsive dispositions*. These capacities are the ones on which empiricism builds its cognitive basis and so its "Myth of the Given". Sellars distinguishes the capacities to respond to stimuli from the observational knowledge (the whole of true beliefs): true beliefs are responses by the application of concepts. The observer responds generally to red things by asserting "that" there is something red. To respond reliably to red things means to make a certain kind of move, i.e., to take a position in the game of giving and asking for reasons, to commit oneself to a certain content playing the role of premise and conclusion of inferences. This account differs from the Fregean one as Frege points on the inferential articulation that is "internal" to a concept. Differently, on Sellars view, the response of the observer possesses a conceptual content because it occupies a knot on the net of inferential relations. The parrot does not treat "red" as implying "colored", implied by "scarlet" and incompatible with "green". In this sense, assertions have a pragmatic sense that corresponds to the undertaking of a specific type of normative attitude: the undertaking of commitment. The cognitive commitment possesses therefore an inferential structure: by performing an assertion the agent commits herself to its use as premise from which certain conclusions can be derived. The noninferential descriptions do not form an autonomous level of language: a game that can be played without contemporarily playing another one. To grasp a concept corresponds to the use of a word: concepts are acquired in the process of learning a language. This process requires two elements: the inferential know-how that allows the speaker the connection of different sentences and the social acknowledgement of that know-how as sufficient for the speech acts of the speaker to have the sense of commitments and entitlements to inferentially articulated claims.

2 The Space of Reasons

Sellars’ inferentialism could cause a certain kind of “deformation” of the space of reasons that makes it difficult to clarify the very nature of knowledge. «The deformation is an interiorization of the space of reasons, a withdrawal of it from the external world. This happens when we suppose we ought to be able to achieve flawless standings in the space of reasons by our own unaided resources, without needing the world to do us any favour» [5]. This leading idea is at the basis of McDowell’s reading of Frege’s account. Frege introduced the distinction between sense and reference, which means that propositions can exist even though the proper name that occurs in the sentence that expresses them do not refer. McDowell presented a Fregean reading that included object dependent propositions and contrasted them to Frege’s senses which he took to be object-independent [6]. He accepted also Evans interpretation to make his view stronger [5, 7]. On Frege’s account, a thought can play the role of cognitive content “and” the role of truth value bearer (nota). This option is refused by the direct reference theory and revisited by Evans and McDowell. “For example, if Ted is watching a cat on earth and Twin-Ted (Ted’s replica on Twin Earth) is watching a cat on Twin Earth (of course a precise replica of the earth cat, this being Twin Earth), then their thoughts would have the same narrow content; i.e., from looking inside their heads their thoughts would be indistinguishable. In spite of that their objects of thought are not the same due to Ted having beliefs about the cat on Twin Earth, and thus two different animals figuring in the propositions that Ted and Twin-Ted believe. The natural conclusion to draw from this type of thought experiment was that Frege was wrong, it was not a single entity that played the role of cognitive content and the role of truth-value bearer. Instead, we had a divergence as the cognitive significance criterion individuated the narrow content, or what is in the head, while the truth-conditional criterion individuated the objects of thought” [8]. This divergence does not belong to Frege’s account; at the same time, it shows the importance of objects individuating our thoughts. But, McDowell and Evans deny the divergence view that rests on the distinction between narrow and wide contents. In this sense, Ted and Twin-Ted do not have the same narrow contents, because they do not admit a mode of presentation (sense) independent of reference. Actually, we can have thoughts without reference. First, as Frege shows, we can make sense of terms like Pegasus, Santa Claus, Zeus etc. Second, mental causation can produce thoughts that can be shared by different persons who act and feel the same accordingly.

Contrary to McDowell, who maintains that immediate certainty of responsibly expressed perceptual judgments exist, Brandom specifies that this is the only way we have to speak about immediate certainty. But justification has to do with a different concept of space of reasons, which does not require experience. McDowell is wrong, as he does not consider the social articulation of the space of reasons. The idea of learning the inferential use of a concept is bound to social attitudes that imply “responsibility’” and “authority”. The game of giving and asking for reasons becomes, therefore, dependent on the social practices by which we recognize commitments and entitlements. The “scorekeeper” takes the place of the Sellarsian knower and becomes a “social role”. The scorekeeper is the one who is able to reliably recognize inferentially articulated commitments that constitute the content of beliefs. He possesses an “expressive” rationality as the capacity to perform inferences in the game of giving and asking for reasons.

According to Hegel, the very nature of negation is incompatibility, which is not only formal but also material, i.e., entails material properties as, for example, “triangular”. In this sense, we can say that *non-p* is the consequence of anything materially incompatible with *p*. From an idealistic point of view we cannot objectively acknowledge relations of material incompatibility unless we take part in processes and practices by which we subjectively acknowledge the incompatibility among commitments. This is the reason why to apply a concept is to occupy a social position, i.e., to undertake a commitment (to take responsibility of justifying it or to be entitled to it). Thus, judgments, as the minimum unit of experience, possess two sides: the subjective side which indicates who is responsible for the validity of his claims, and the objective one, which indicates whatever the speaker considers as responsible for the validity of his/her claims. Through specific attitudes we can specify the social dimension of knowledge. The *de dicto* ascription such as “he believes that…”, determines the content of a commitment from a subjective point of view, i.e., from the point of view of the one who performs a certain claim. The *de re* ascription such as “he believes **of** this thing that…”, determines the content of a commitment from an objective point of view, i.e., the inferential commitments the scorekeeper must acknowledge [9–13]. How does this acknowledgment happen? We can use the above mentioned ascriptions. If, for example, I am a scorekeeper who performs the *de dicto* ascription «Vincenzo says that this golden agaric must be cooked in butter» and contemporarily I acknowledge that the mushroom is totally similar to an *amanita caesarea* (a good golden agaric) yet it is dangerous because it is an *amanita muscaria* (an evil golden agaric), I can isolate the content of Vincenzo’s assertion through the *de re* ascription «Vincenzo says **of** this golden agaric that it must be cooked in butter» and make explicit the commitments I undertake and the ones I refuse from an objective point of view [14].

3 Cognitive Content, Semantic Content and Second Nature

Let us now follow Danielle Macbeth’s analysis to understand the limits of McDowell’s and Brandom’s accounts about empirical judgment. «According to Frege, both object names and concept words at once express senses, which, as Brandom argues in *Making It Explicit*, just are inferential contents, and also, in favored cases, designate objects and concepts. A thought *qua* thinkable is an inferentially articulated Fregean sense. It belongs to the realm of freedom and is not world-involving. But that same thought is judgeable, available to be acknowledged as true, just if the relevant object names and concept words designate objects and concepts respectively. A *judgeable* content must designate a truth-value; for to judge just is to advance from a thought to a truth-value. A judgeable content is thus essentially world-involving. Brandom’s founding insight, we can now say, is an insight into thinkable, or as we might say, *cognitive* content; it is an insight into the nature of a thought insofar as it is available to be grasped by a thinker. McDowell’s founding insight, by contrast, is an insight into the nature of judgeable, or as we might say, *semantic* content; it is an insight into the nature of a thought as it is available to be acknowledged as true by a thinker» [15]. This point of view requires the world be in conceptual shape not only in Frege's sense of including concepts as the *Bedeutung* of concept-words but also as including the *senses*, the *Sinn*, of concept words. We have this result if we do not distinguish cognitive and semantic content. «On Frege's account, the dictates of reason as set in place by the *Bedeutung* of object names and concept words have as such only the *potential* rationally to constrain our thinking. That potential is realized in our developing adequate inferentially articulated conceptions of how things are and thereby the eyes to see. His account is perfectly compatible with a conception of nature as the realm of law» [16].

Brandom, on the contrary, collapses semantic content into cognitive content as the contents of concept words (which belong to the realm of Fregean sense) are exhausted by their inferential relations. He maintains that reality is not conceptual and we simply know about it using our discursive practices. «Brandom is right to think that cognitive content is exhausted by inferential relations and as such belongs wholly to the faculty of spontaneity as it contrasts with the faculty of receptivity; but he is wrong to think that that conception of content can serve as a conception of judgeable content. Experience must function as a tribunal for judgment, and we can understand how it can as soon as we recognize that semantic content is different from cognitive content, that it is world-involving» [17]. The Fregean innovation of the Aristotelian notation introduces epistemological consequences, which rest on the notions of "judgeable content" and "judgment". To conceive ourselves as knowers we need to take a step beyond the idea of true perception. This move means to consider our "second nature" and, as MacBeth points out, Brandom, McDowell and Frege together help us to see this kind of human "actualization". «To acquire the eyes to see things as they are is to be acculturated into a sufficiently advanced scientific tradition where this is at once a matter of our acquiring adequate conceptions of things *and* of the world acquiring a face, a presence for us. Our capacity to know together with the capacity of the world to be known is then fully actualized in successful (i.e., correct) judgments» [18].

4 "First" and "Second" Nature in Classical Thought

The Fregean distinction concerning "Second Nature" is said to have sprung from Aristotle's writings and indicates the emergence of human intelligence as a unique phenomenon in the natural world.[1] Nature (φυσις), in this sense, would refer to all that is independent of human intelligence, the "natural world", as that which surrounds us and which exists separate from culture. The ancient Greeks by and large supported an authentic "ecological" world view: the natural world was something to be respected and human intelligence had the goal of harmonizing with the natural order so that life flourished. Kitto in his classical work, *The Greeks*, has explained perhaps better than anyone the peculiar mindset of a people that forged the basis of Western culture [19].

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[1] Cf. Aristotle, *Peri Hermeneias*, Part 7; also see *Categories*, Part 2, where he writes explicitly: "Of things themselves some are predicable of a subject, and are never present in a subject. Thus 'man' is predicable of the individual man, and is never present in a subject. By being 'present in a subject' I do not mean present as parts are present in a whole, but being incapable of existence apart from the said subject."

An argument can be made to include human intelligence in this natural world (that it is part of "nature"), but for our purposes here we will elaborate on the distinction that divides nature in to two. The Second Nature is that which owes it foundation to human thought. Aristotle was convinced that animals exhibited some sort of intelligence, albeit different in essence from human intelligence. He referred to them as the "brutes", although certainly not in a derogatory fashion. The brutes were able to perceive their environment and act on certain conditions in order to achieve their instinctive goals. Every living thing possessed a "soul", which for Aristotle was another name of the life principle ("form"). Even though Aristotle did not believe in a revealed religion, he did conclude that humans possess souls which are immortal, and that they continue to exist even after separation from bodies. As is well known, Plato also taught the same doctrine yet he differed on the journey of souls after their separation from bodies.

In his *On the Soul*, Aristotle argues that the human intellect is capable of actions which transcend the material substratum of the body/soul union and that therefore is capable of an act of existence which is "higher" than that of the body:

> «Actual knowledge is identical with its object: in the individual, potential knowledge is in time prior to actual knowledge, but in the universe as a whole it is not prior even in time. Mind is not at one time knowing and at another not. When mind is set free from its present conditions it appears as just what it is and nothing more: this alone is immortal and eternal [. . .] and without it nothing thinks» [20].

Here, Aristotle refers to "mind" as opposed to "intellect" because it is separate from individual intellects yet provides each one with the *act* necessary for cognition. Therefore, it is unclear if Aristotle held that actual people's souls are immortal (as in a personal immortality). The medieval scholar, Thomas Aquinas, will incorporate Aristotle's notion of a separate mind and place it in each individual human being, thus arriving at the Christian view of personal immortality.

Although it is true that "there is nothing in the intellect which has not passed through the senses" (a phrase which appears often in Aquinas), the intellect is capable of actions which go beyond the potentialities of the body. The most common of such actions would be conceptual thought, i.e., the universalization of the material element that results from the process of abstraction. Through the apprehension of empirical data from the external world, the passive intellect elaborates the sensorial data through the various internal senses (such as the common sense, the fantasy, the memory and the cogitative sense) and the active intellect then takes that "phantasmata" and renders it knowable in act. The universal concept is thus seen as transcendent with regards to the (sensorial) process needed to create it. For judgment to occur, the rational intellect would apply the verb "is" to the mental content and arrive at an affirmation (or negation), as in the example, "This is a cat".

It is commonly understood that non-human intellects are incapable of such mental processes. In other words, animals do not possess an active intellect, yet only passive. Instead of a "cogitative" sense, the brutes have the "estimative" sense which allows them to calculate the value of the external object before them, the result of their own perceptive process. Such a power allows the rabbit to flee when it perceives the fox, "estimating" the danger of the object perceived. According to Aristotle, the rabbit does not "judge" the value of the object, for that would require a more powerful intellect (which humans possess).

The study of animal cognition expanded greatly after Darwin. Many scholars are convinced that some animals are indeed capable of conceptualization, and not simply instinctive response.[2] A friend of mine, Bill Penn, retired professor of logic and ethics at St. Edward’s University in Austin, Texas, has a ranch in western Colorado where he cares for six horses and 200 acres of land. Musing about the question of animal intelligence, he asked me how his horses realize that I am not a threat to them? He suggests that, in some way, they have a type of “universal notion” which tells them that not only is Bill non-threatening, but all human beings are thus (unless otherwise demonstrated, for example, when someone truly wants to harm them in a concrete way). I could tell that they certainly recognize Bill by his voice and scent as different from me, yet when I approach them they do not flee. How do they “know” that all humans can be trusted (at least generically) unless they have some notion of similar members of a class?

Another example from Colorado conveys a similar idea. Every morning, Bill loads the humming bird feeder on the deck with sugar water, and throughout the day the humming birds arrive and drink the nectar. One morning, he forgot to add the water and went fishing down by the river, about 300 yards from the house. Within a half hour, several dozen humming birds started dive bombing towards him as he stood on the shore of the river. He could not understand why this was happening and was sincerely perplexed (and slightly scared) until he realized that he had not filled the feeder. Once he did so, the dive bombing stopped. In order to explain such a phenomenon, one would seem to need to attribute to the humming birds a notion of causal agency, attracting the attention of the one who replenishes the nectar in the feeder in order to fill it up. Crows and pigeons are said to be quite cunning in this regard.[3]

Although it is difficult to provide a comprehensive evaluation of the differences between human and animal intelligence, it is very apparent that there are differences. Aristotle understood the human mind as capable of “modelling” nature, in the sense that extra-mental reality becomes present to the soul in an intentional way. There are two modes of existence for objects: a real mode of existence and an intentional one. The famous phrase, “The soul becomes, in a way, all things” means that objects exist in the soul through the intellect which gives rise to their representations. By way of the process of abstraction, the intellect creates an image or representation of objects and then performs judgments.

Aquinas, commenting Aristotle, insists that the mental representation of reality is not the object of knowledge, but rather *things* are the objects of knowledge. The mental representation is that *by which* the mind knows [21]. It is clear that representations cannot be the very objects of knowledge, because that would lock the cognitive subject in an immanent loop, without ever being able to achieve knowledge of the external world. The fact that we possess scientific knowledge which is efficacious and progressive implies

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[2] For a recent, well-documented text on this subject, see *How Animals See the World. Comparative Behavior, Biology and Evolution of Vision*, ed. By Olga Lazareva, Toru Shimizu, Edward Wasserman. Oxford University Press, 2012.

[3] Cf. Candace Savage, *Crows: Encounters with the Wise Guys of the Avian World*, Greystone Books, 2005. As a sign of crows’ advanced smarts, Savage cites Kacelnik’s 2002 study in the journal *Science* on a captive New Caledonian crow that bent a straight piece of wire into a hook to fetch a bucket of food in a tube.

that we do in fact achieve an understanding of extra-mental reality through our thought process. Throughout the Medieval period, this notion becomes known as the doctrine of intentionality. Intentionality is simply the way that reality and the mental are related.

It is in this sense that we can understand Frege's term of "Second Nature": First Nature is that which exists in the real world, and Second Nature is the same Nature's existence in the mind, an *intentional* existence, which depends on the intellectual capabilities of human beings. From this understanding, it is logical to foresee the advent of some of the great philosophical questions of modernity: how do the representations arise in the intellect? (problem of perception); are such representations innate in the mind? (John Locke's dilemma); do our mental representations correspond to real objects? (Immanuel Kant's problem of the bridge); can these representations be expressed linguistically? (problem addressed by the "Linguistic Turn"); how do our assertions "hook" onto the world? (Hilary Putnam's quandary); are representations truly universal or are they culturally conditioned? (problem of post-modernity); as well as others.

The possibility of creating models of reality allows human beings to know the empirical world. From the Aristotelian point of view, the modelling of nature captures the essence of objects and holds such essences in the intellect, albeit in a non-physical way. In what way the intentional essence is constructed from sense data arising from the object, as opposed to an immanent construction by the powers of the mind, is a controversy which survives even today. Quine addresses this issue by way of *conceptual schemes* which refutes the Kantian distinction of the difference between analytic propositions and synthetic propositions [22]. Reality, according to Quine, is relative to conceptual schemes, and this gives rise to his notion of *ontological relativity*. Donald Davidson attempts to restore ontological objectivity in his classical essays, *On the Very Idea of Conceptual Scheme* [23]. «In giving up dependence on the concept of an uninterpreted reality, something outside all schemes and science, we do not relinquish the notion of objective truth—quite the contrary. Given the dogma of a dualism of scheme and reality, we get conceptual relativity, and truth relative to a scheme. Without the dogma, this kind of relativity goes by the board» [24].

In a way, Davidson returns to Aristotle, recognizing the *causal* influence that objects have on our mental activity. The relationship between objects and mental activity is certainly complex, and it will continue to perplex us. Some scholars look to the neurosciences to solve these mysteries, and there have certainly been advances in this field. Robert T. Knight at the University of California Berkeley has made effective progress in this sense with his Cognitive Neuroscience Research Laboratory. One spectacular experiment involved epilepsy patients which were wired with deep neural sensors attached to computers. The patients listened to words and were asked to think about them without pronouncing the words. The words were then heard over the speakers of the computer. Some patients later told Bob Knight, "You're reading my mind!"[4]

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[4] For the complete scientific study, see Robert T. Knight, *Reconstructing Speech from Human Auditory Cortex* in http://www.plosbiology.org/article/info%3Adoi%2F10.1371%2Fjournal.pbio.1001251, January 31, 2012.

5 Conclusion

As a concluding provocation, a further hypothesis can be suggested. Third Nature. If Second Nature is what occurs when the human mind models reality, could it not be suggested that Third Nature is what occurs when autonomous robots model reality? Rapid progress in artificial intelligence (AI) leads to such a suggestion. In just several months, the capacity to model nature given to Google's self-driving cars has increased almost exponentially.[5] These cars can now calming drive in urban traffic, addressing hundreds of seemingly unpredictable situations (unforeseen construction work on the road, animals running across the street, stop lights which don't work). Google claims that their cars have driven for more than 700,000 miles with no accidents.[6] Any advanced AI system utilizes complex models of nature which allow that system to interact with the environment. As these systems get better and create more complex models (for example, IBM's *Watson* which competed with the best two *Jeopardy!* champions and defeated them), we can certainly ask if those systems give rise to a Third Nature, understanding that modeling nature from their perspective is quite different than the way the human mind does it.

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There Is No Cube of Opposition

Jean-Yves Béziau

Abstract The theory of opposition has been famously crystallized in a square. One of the most common generalizations of the square is a cube of opposition. We show here that there is no cube such that each of its faces is a square of opposition. We discuss the question of generalization and present two other generalizations of the theory of opposition to the third dimension: one based on Blanché's hexagon of opposition, the other on the square of contrariety.

Keywords Cube of opposition • Generalization • Hexagon of opposition • n-Opposition • Square of opposition

Mathematics Subject Classification (2000) Primary 03A05; Secondary 00A30; 03B45, 03B53, 03B22, 03B50.

1 The Cube of Opposition: An Obvious Geometrical Generalization

An obvious way to generalize the square of opposition is to consider a cube of opposition. Many cubes of opposition have been presented in the literature (see e.g. [19, 27, 29–31, 39, 40, 49, 50]). The cube is a an immediate generalization that one may have for the theory of opposition driven by a geometrical spirit. From a square we can go to other polygons: a pentagon, a hexagon, a heptagon, . . . , a chiliagon. And such generalizations also exist in the literature (see e.g. [26, 36, 37] and in general all the recent publications on the square: [6, 14–18]).

The cube is a nice generalization in the sense that we keep the square shape but at the same time we go to the third dimension. Something is preserved and at the same time there is a change, an expansion. This double contrasting aspect—preservation with transformation—is a key feature of generalization. But this is here only from the geometrical point of view.

Another fundamental aspect should be taken into account, and that is: the relation between the theory and what it is supposed to represent. There is an interaction between the geometrical figure of the square and the theory of opposition and this interaction also has to be preserved. Generalization should not just be on the geometrical side, it should also be on the side of what the geometrical object is supposed to represent. This side is

J.-Y. Béziau, G. Basti (eds.), *The Square of Opposition: A Cornerstone of Thought*,
Studies in Universal Logic, DOI 10.1007/978-3-319-45062-9_11

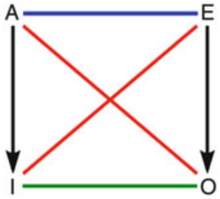

Fig. 1 Abstract coloured square of opposition

not a dark side and only one side. The square has internal and external structures that can be colourfully represented. The square can be seen as built on a red cross (the heart of the square), which is then "circled" by a top blue line of contrariety, a bottom green line of subcontrariety and two black arrows of subalternation (Fig. 1).[1]

The result presented in this paper shows that there is no straightforward generalization of the theory of opposition from a square to a cube, in the sense that there is no cube of opposition such that each of its six faces is a square of opposition as represented in Fig. 1.

The title of our paper is deliberately provocative. It is to stress that if one wants to promote the idea of a cube of opposition, (s)he has to carefully explain and/or justify what (s)he is doing. We will let the proposers of such cubes of opposition do the job. Here we will present two other generalizations of the square into the third dimension which are not cubes, and explain why they are good generalizations of the square of opposition.

2 The Square of Opposition: A Flag for the Theory of Opposition

The theory of opposition has famously been crystallized in a square. This crystallization became very important, exceeding the theory itself. It is not exaggerated to say that the square of opposition became the flag of the theory of opposition. And this is not necessarily a problem, this is quite a nice flag. This is indeed a better flag than most of countries' flags where there is no visible connection between the image and what it is supposed to represent. Let us examine two cases shown in Fig. 2.

The flag of Switzerland is a white cross on a red square. Like the square of opposition, it is a square, but here the square apparently has no special meaning. It is a pure question of regularity/symmetry in harmony with the cross which is inside. At the end everything is square in the Swiss flag. This can be seen as a compass indicating rationality and organization. Generally flags are rectangular. Only two sovereign states have a flag with a shape of a square: Switzerland and Vatican. We have organized congresses on the square in both of these countries (The first SQUARE in Montreux in 2007 and the fourth SQUARE at the Pontifical Lateran University in 2014). The others were organized in countries with rectangular flags. This is the case of the third SQUARE that took place at the American University of Beirut in 2012. The flag of Lebanon is also made of some red and white

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[1] We introduced this colouring of the square in 2003, cf. [2].

Fig. 2 Flags of Switzerland and Lebanon

Fig. 3 A typical representation of a giraffe

geometric shapes but it has moreover at the middle of it a tree, known as the Lebanon cedar. This tree is the symbol of Lebanon, because it is *typical* of Lebanon. The square of opposition is also a *typical exemplification* of the theory of opposition. However such kind of "typicity" is not the same as the one of the Lebanon cedar.

It is also not the same as the picture of a giraffe as represented in Fig. 3. Such a picture is a schematic representation of a giraffe emphasizing its main features corresponding to the standard definition of this animal: a long-necked, spotted quadruped ruminant. The class of giraffes is a class of homogeneous "things", so it is easier to "typify" them. What about the class of all animals? The giraffe in Fig. 3 certainly is not a good typical example of animal. It is not *general* enough. If we exemplify the notion of animal through this picture, this may give the idea that all animals are quadruped. Choosing the square represented as in Fig. 1 this may also give the idea that opposition is necessarily a quadruped ... If we generalize the square of opposition to a cube or a hypercube, maybe the square may still serve as a good example, considering it is the first and simplest form. It would be the same as to consider 1 as a typical example of number.

Can we say that the square is a *symbol* for the theory of opposition like the balance for justice, or the two parallel lines for identity (Fig. 4)? A symbol can be defined as a sign where there is a connection between the sign and what it represents, as opposed to arbitrary signs (cf. Saussure [51] and Beziau [8] for a semiotic hexagon). The two signs of Fig. 4 are doubly symbolic: (1) They are stylized pictures; (2) They represent an idea through

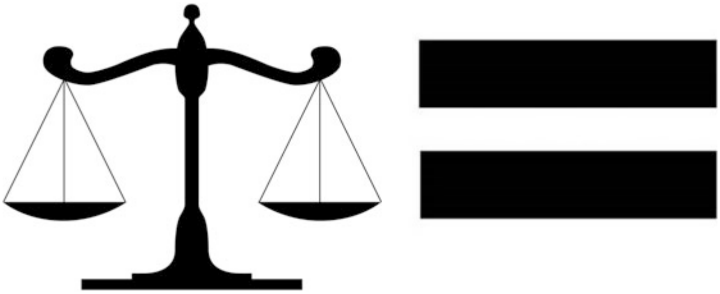

Fig. 4 Symbols of justice and equality

a typical concrete example. We call the first aspect of symbolization *pictogrammatic symbolization* and the second aspect *ideal symbolization*.[2]

The picture of the giraffe presented in Fig. 3 can be considered as doubly symbolic. But there are two slight differences on both sides of the symbolization procedure. On the one hand it is not completely stylized—not so simplified both in form and colour, on the other hand it is not so ideally symbolic since the reality it describes, the species of giraffes, is not so ideal. The square of opposition presented in Fig. 3 is more stylized, despite the fact that there are still colours. But colours are used here in a different way than in the case of the giraffe. The colours themselves are symbolic like in the case of traffic signs; in the case of the giraffe the colours are purely descriptive.[3] Considering the ideal aspect of symbolization, the theory of opposition is much more ideal than the species of the giraffes. The question we have to investigate is if the figure of the square is as good an idealization through particularization as is the balance for justice or the two parallel lines for equality.

Since justice and identity are very heterogeneous, the objects singled out to represent them are necessarily too particular. The art of ideal symbolization is to convey the general idea through a particular concrete instantiation. The problem is that with the theory of opposition we are going from the particular to the general, whereas this is not the case with justice, as the theory of justice did not start with a balance. Going from the particular to the general is very common in science, in particular in mathematics (see e.g. [33]).

Figure 1 can be viewed as an abstract structure having many different instantiations. The letters A, E, I, O can be seen as variables that can be interpreted as different kinds of propositions or different kinds of concepts. Let us just give two examples: the square of modalities and the square of speed (Fig. 5). The square of modalities can be interpreted as a square of concepts (necessary, possible, impossible, not necessary) or of correlated

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[2] We have elaborated this distinction in our paper "La puissance du symbole" [11] published in the book *La pointure du Symbole* [10] which is the result of the interdisciplinary workshop we organized at the University of Neuchâtel in 2005. Saussure gives as an example of symbol the balance but he does not specify the double aspect of symbolization.

[3] Our choice for the colours of the square as in Fig. 1 was more or less intuitive: red for contradiction, because it is the strongest opposition, black for subalternation, because it is not an opposition. The choice of blue and green was more intuitive, we didn't know at this time the RBG theory which was later on formalized by Dany Jaspers using the theory of opposition, see [35].

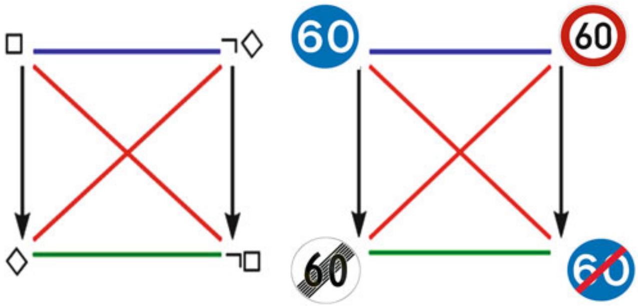

Fig. 5 Square of modality and square of speed

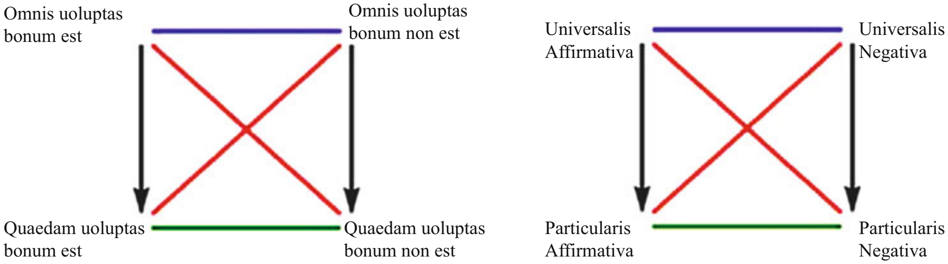

Fig. 6 The voluptuous square of Apuleius and the corresponding categorisation

propositions (It is necessary that it will rain, etc ...). These modalities can also be interpreted in a deontic way (obligatory, prohibited, ...) of which the square of speed is a particular case related to action.

Historically speaking the situation developed the other way round. First a particular square was developed, a square related to Aristotle's theory of proposition, which classifies the propositions in four categories.[4] There are here already two levels: the categories themselves (universal affirmative, universal negative, particular affirmative and particular negative) and specific examples. In Fig. 6 on the left we have the original "typical" example given by Apuleius, the voluptuous square. It is very easy to understand through this particular example the corresponding categorical generalization, which is on the right.

People have generally not stuck to the original exemplification of Apuleius or/and to the Aristotelian categorization, but many have stuck to the square (and two of its avatars: the quantificational and modal squares) as if the theory of opposition was limited and/or reducible to that. Sticking to the original square is the same as to stick to natural numbers, not considering other numbers. But generalization in mathematics is not the product of

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[4] The square of opposition is an interesting way to classify propositions and it can be seen as a tool for classification, which is at once more complex yet more compact than the most famous classificatory structure, the tree—about the theory of classification see [47].

pure fantasy. Irrational numbers are the by-product of rationality, more specifically the reduction to the absurd (Fig. 7).

Aristotle's theory of proposition led to a specific configuration of the theory of opposition. By abstraction a certain structure is manifested and then applied back to many particular cases. This procedure is common in mathematics where structures like algebraic structures were extracted from some specific cases, studied by themselves and applied back to some concretes cases.[5] Two famous cases are groups and lattices. Some people even had the funny idea that everything is (or has the structure of) a group. Other people had a similar idea about lattices. In fact at some point lattices were called "structures", as if they were the quintessence of structures (see [32]). But the idea of structures was indeed the next step in generalization by abstraction in mathematics.

Saying that the theory of opposition is nothing more than the square of opposition would be the same as saying that geometry is nothing more than Euclidean geometry or that numbers are nothing more than natural numbers. Nevertheless we can use the square of opposition as a flag for the theory of opposition because it was the first manifestation of it. This is a phenomenon common in thought and language. "Alpinism" means mountain climbing, not only climbing the Alps. Some people are trying to detach it from its particularism and replace it by "mountaineering". Another possibility would be to talk about "Everestism", considering that Mount Everest is the highest mountain on earth (the name of this mountain is related to George Everest, the uncle of Mary Everest Boole, the wife of George Boole). Using proper names, another option would be "Saussurism", in memory of Horace-Bénédict de Saussure, one of the main promoters of Alpinism. For the square it is also common to attach it to Aristotle, Apuleius or Boethius. When one is talking about the Apuleian square, we know it is about opposition, not just about a geometrical shape or/and Apuleius. This conveys the idea of the theory of opposition.

Saussure was not the first to climb the Mount Blanc, nor George Everest was the first to climb the Mount Everest. And probably Apuleius is not the first to have drawn a square of opposition (see [28]), as it has been claimed by Bocheński [24, 25] and Sullyvan [52] and supported by Londey and Johanson:

> Historians of logic are agreed that, although Aristotle stated the principal logical relations between the four types of categorical proposition, he did not invent the heuristic diagram, traditionally known as the Square of Opposition, which maps those relations. This diagram has been part of the staple fare of students of elementary logic for centuries, but modern writers do not always show any certainty about its origin, or its original form. It is not uncommonly thought to be a medieval invention, or is simply glossed as 'traditional' in a way which implies either a medieval or post-medieval origin. However, Bocheński and Sullyvan correctly locate the first known occurrence of the diagram in the *Peri Hermeneias*. "The Apuleian square of opposition", Appendix B of [41, p. 108].

Let us point out that nobody has seen a square of opposition drawn by Apuleius though Londey and Johanson correctly say that Apuleius gives a "set of instructions on how to draw the figure and how to label the relations to be charted". But Laurence Horn pointed out that Aristotle also had a square in mind (see [34]).

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[5] Let us point out here that there is a difference between generalization reached by induction and generalization reached by abstraction from a single example. There can be some mixed cases. In the case of the square it looks more like pure abstraction than induction.

3 The Proof that There Is No Cube of Opposition

We now present the proof that there is no cube of opposition. Firstly we present an abstract proof and secondly a visual proof.

Theorem *There is no cube of opposition such that each side of it is a square of opposition.*

Abstract Proof (1) Suppose that we have a cube of opposition such that each of the six faces of it is a standard square of opposition. (2) At a vertex v of a cube we have a triple point where three edges A, B, C coterminate and three faces X, Y, Z meet. (3) Any pair of these three faces share one of these three edges, and any pair of these three edges form two adjoining sides of one of these three faces. (4) According to the definition of a square of opposition, when we have two edges meeting at a corner of a square, one should be black (subalternation) and one should not be black (either green or blue). (5) Therefore one of the edges meeting at v must be black, let's say A. (6) If B is black too, then, according to (3), A and B are two edges of a square, say X, meeting at a corner of this square, so X is not a square of opposition, this contradicts (1). (7) So B is not black. (8) Then according to (4) C has to be black. (9) According to (3) B and C meet at a corner of one of the squares, say Y. (10) Since B is not black, according to (4), C must be black. (11) But then A and C are two black edges meeting at a corner of the third square Z, so Z is not a square of opposition, this contradicts (1).

Visual Proof There is a more visual and more direct way to prove this result.[6] Consider the following situation.

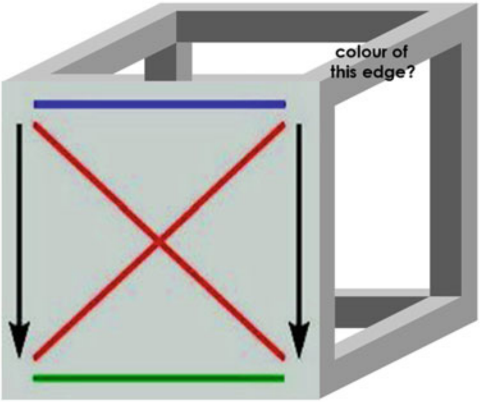

Fig. 7 No cube of opposition

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[6] About recent advances on visual reasoning see e.g. [43].

We have put a square, in the standard position, on the front side of the cube. It is easy to understand that there is no loss of generality putting the square in this position.

Now let us consider the diagonal edge on the top right. It can be blue, green or black.

Blue: then the upper side of the cube is not a square of opposition
Green: then again the upper side of the cube is not a square of opposition
Black: then the right side of the cube is not a square of opposition

In the three cases, due to colouring, we immediately see why the mentioned sides are not squares of opposition even without stating explicitly the above proposition (4).

4 Two Other Three-Dimensional Generalizations of the Square of Opposition

In this section we will discuss two other generalizations of the square of opposition: the hexagon of opposition and n-opposition theory. They are also related to the third dimension but in a different way than the cube. In a way which is at the same time more indirect and more fundamental. These two generalizations have in common the fact that the first motivation of their development is not the third dimension, but they naturally and even imperatively lead to it. They are also tightly related to each other, n-opposition theory being a generalization of the hexagon of opposition.

The hexagon of opposition appeared in the 1950s, and as often in the evolution of science, which can be seen as a general movement of human thought, it is not the idea of one isolated person. Different people had independently the same idea at more or less the same time. Let us note that this is what happened with many-valued logic which was independently developed by Peirce, Post, Bernays, and Łukasiewicz. This does not mean that everybody had exactly the same idea. There is something in common, but it is presented and developed in different ways, and this can lead to a theory which is a blend of ideas, or alternatively one of them develops more than the other ones and dominates.

In the case of the hexagon, this is rather the second case, as Robert Blanché developed the hexagon of opposition in a systematic and continuous way over more than 10 years starting in the mid 1950s (see [20–23]). We have already written a paper on the hexagon entitled "The Power of the Hexagon" (see [6] and edited a special issue of *Logica Universalis* on the hexagon (see [6]) so we will not enter much into details here. We will just discuss the hexagon from the issue of generalization.

Blanché's hexagon of opposition is not a two-dimensional generalization of the square of opposition in the sense that sides are added. Let us point out that the square of opposition is not just a square, it has a structure made of three oppositions and the further notion of subalternation. If we add sides, how to adjust the structure and what is the motivation?

Blanché's construction is based on a true philosophical inquiry about the theory of opposition dealing in particular with the original exemplification of the square, the square of quantification. Blanché solved one of the main problems of the square of quantification. There were two problems with the traditional square, both related with the I-corner, the "existential" corner. The first problem is the question of *existential import* that we will not

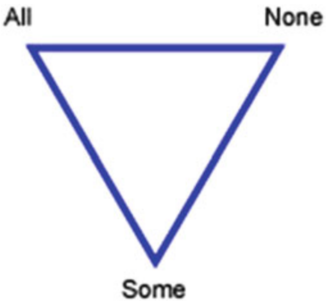

Fig. 8 Quantificational triangle of contrariety

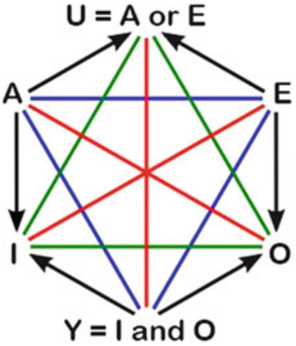

Fig. 9 Blanché's hexagon of opposition

discuss here (a very romantic topic which bears some similarity with the question of *sex of angels*). The second problem is about *some*. It was pointed out by several people that the I-corner does not correspond to the meaning of the quantifier some and for this reason people wanted to replace the square by a triangle (Fig. 8).

Blanché, instead of staying with just the triangle, constructed a hexagon through a star by tying this triangle of contrariety with a triangle of subcontrariety (Fig. 9).

In which sense can we say that this hexagon of opposition is a generalization of the square? Firstly in the sense that the hexagon has a high degree of generality, it can be applied to many different situations (see [8, 9, 12, 13]). But this is also the case of the blue triangle which is the heart of hexagon. Secondly it is a natural *extension* of the square, which is recovered inside the hexagon (and two more squares appear as can be seen by rotating the hexagon). It is natural to consider the conjunction of the I and the O corners and the disjunction of the A and E corners. This makes sense, and it is supported by a nice internal structure. So the hexagon is like a flourishing of the square. It is a natural *complexification* of the square. Such kinds of developments contrast with trivial generalizations.

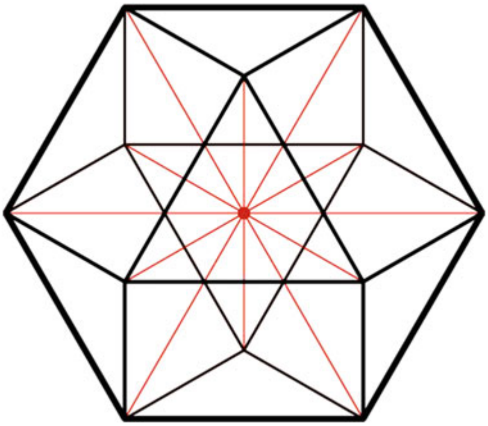

Fig. 10 The cuboctahedron—with a structure of 4 hexagons

The way from the hexagon to the third dimension is also a kind of flourishing. Having discovered that the negation of necessity was a paraconsistent negation (see [1, 4, 5]) I wanted to systematically study the relations between negation and modalities (see [3, 7]). This can lead to an octagon. It is one option, but I wanted to preserve the star/hexagon structure, therefore I built three hexagons of opposition and a natural way of relating them is to construct a three-dimensional object. So I built such a structure. As noted by Alessio Moretti and Hans Smessaert to whom I communicated my idea at this time a fourth hexagon shows up in this three-dimensional construction, that I first saw as a stellar dodecahedron but that they identified as a cuboctahedron considering the surfaces generated by subalternation (Fig. 10). The surface of this cuboctahedron is made of an alternation of six squares with eight triangles. None of these squares is a square of opposition, all the edges are subalternations, that's why they are in black in Fig. 10. In red we have links of contradiction. We have not put the blue and green edges of contrariety and subcontrariety but at the end we have 12 interlaced squares of opposition inside this three-dimensional object because we have 4 hexagons and inside a hexagon of opposition there are 3 squares of opposition.

Let us now have a look at the other generalization which also leads to the third dimension, *n*-opposition theory. As we have seen, the hexagon is constructed by putting two triangles together. The heart of the hexagon is the blue triangle of contrariety. Such a triangle can be seen as breaking/extending the dichotomy promoted by the school of Pythagoras (cf. the table of opposition). This triangle was not designed by Aristotle, but Aristotle promoted the notion of contrariety which is intimately related to it.

On the other hand contrariety is not necessarily limited to trichotomy. We can go to tetrachotomy and draw some squares of contrariety. The similarity of a square of contrariety and the standard square of opposition is only in the structure/shape of the edges, but all edges correspond to contrariety, like with the triangle of contrariety; so using colour, we have a blue square (Fig. 11).

In the same way as in the construction of the hexagon, we can consider a dual green square of subcontrariety and tie the two squares together using red contradictory edges. This gives birth to an octagon of opposition. This is a natural generalization of the hexagon.

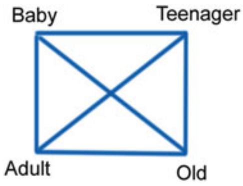

Fig. 11 Square of contrariety of ages

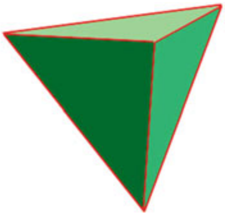

Fig. 12 Tetrahedron

We have gone from a 3-contrariety (a contrariety of three terms) to a 4-contrariety (a contrariety of four terms). Contrariety can work with only two terms but it is not limited to three terms. We can go to four terms and more. In fact it seems that "child" is missing in the square of Fig. 11, so it would be better to have a pentagon.

I promoted this generalization of the theory of the hexagon of opposition and Alessio Moretti baptized this "*n*-opposition theory" (cf. [44]). The expression is quite ambiguous because the number of oppositions is still the same in all the cases, we are not adding more oppositions, we are staying only with the three basic ones: contrariety, subcontrariety and contradiction. 3-opposition theory is Blanché's hexagon, 4-opposition theory is when we consider four contrarieties (and four subcontrarieties), etc. But the main contribution of Moretti is much more interesting than this ambiguous terminology (which he likes to abbreviate as N.O.T.= *n*-opposition theory).

Moretti had the idea that it would be better to have the same distance between the four vertices. This is not the case in a square because the diagonals are longer than the sides. For the standard square this is not necessarily a problem, because the diagonals do not correspond to the same notion of opposition as the sides. One may defend the idea that contradiction is longer because it is stronger. But if we consider a blue square of contrariety then the asymmetry is a defect. So Moretti suggested that it would be better to consider a tetrahedron (Fig. 12). Such a geometrical object is three-dimensional. We go here to the third-dimension by a kind of accidental necessity.

If we want now to generalize the construction of the hexagon for 4-opposition, we construct a dual green tetrahedron of subcontrariety and putting the two together we arrive at the object represented in Fig. 13, which is called a "stellated octahedron".

The idea of tetrahedron can be generalized to any number of vertices, the name for such an object is "simplex" and the name for a composition of two simplexes is a bi-simplex. Moretti used these geometrical objects to generalize Blanché's hexagon of opposition. His theory of *n*-opposition takes a bi-simplicial form and even a poly-simplicial from (see [45, 46] and also [42, 48]).

Fig. 13 Stellated octahedron for 4-opposition theory

Let us summarize the story. We have the square of opposition which is a two-dimensional object. Considering trichotomy is quite natural and does not lead immediately to the third dimension. It leads to a hexagon. On the one hand instead of going to an octagon where the basic figure of the triangle is somewhat lost, we can go to the third dimension constructing a cuboctahedron—all this staying in 3-opposition theory. On the other hand we can generalize this theory going to 4-opposition theory, then we have to go directly to the third dimension right at the start.

In the case of the cuboctahedron, the move to the third dimension is motivated by the preservation of a triangular structure. The triangle leads to the third dimension, this is quite homogeneous and harmonious. In the case of 4-opposition, although the way of going to the third dimension is more subtle than in the case of the cube of opposition (internal structural necessity) this theory can also be criticized. Why go from trichotomy to tetrachotomy? does this make sense from a philosophical point of view? The move from 2 to 3 can already be criticized, as Kant puts it: only dichotomy is a priori (see [38]). But trichotomy can indeed be defended, either from the point of view of reality or from a transcendental viewpoint, or both. The structure of thought can be seen as trichotomic. This is an idea more or less promoted by Blanché, justifying his hexagon (see [23]). We can consider that at the level of signs everything can be reduced to dichotomy but that thought is essentially trichotomic. Tetrachotomy looks much more empirical. There are four seasons (in some regions of the earth), but can we say that there are four kinds of ages? As we have said, five would be a better division.

The four points of the compass are a rather arbitrary squaring of space. Maybe everything is round, in space and time (Fig. 14).

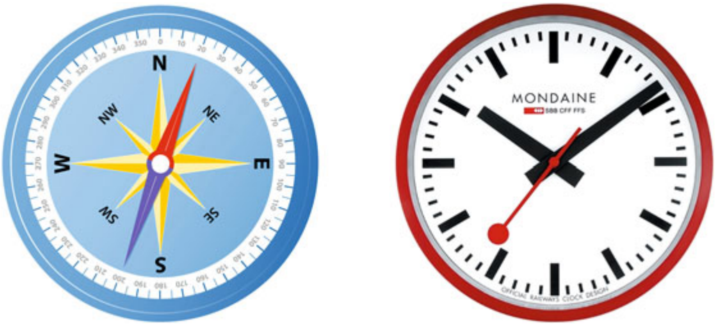

Fig. 14 Circles of space and time

Acknowledgements Thanks to Catherine Chantilly and Robert Purdy for discussion and comments.

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Part V
Theoretical Investigations on the Square

The Unreasonable Effectiveness of Bitstrings in Logical Geometry

Hans Smessaert and Lorenz Demey

Abstract This paper presents a unified account of *bitstrings*—i.e. sequences of bits (0/1) that serve as compact semantic representations—for the analysis of Aristotelian relations and provides an overview of their effectiveness in three key areas of the Logical Geometry research programme. As for *logical* effectiveness, bitstrings allow a precise and positive characterisation of the notion of logical independence or unconnectedness, as well as a straightforward computation—in terms of bitstring length and level—of the number and type of Aristotelian relations that a particular formula may enter into. As for *diagrammatic* effectiveness, bitstrings play a crucial role in studying the subdiagrams of the Aristotelian rhombic dodecahedron, and different types of Aristotelian hexagons turn out to require bitstrings of different lengths. The *linguistic* and *cognitive* effectiveness of bitstring analysis relates to the scalar structure underlying the bitstrings, and to the difference between linear and non-linear bitstrings.

Keywords Aristotelian diagram • Bitstrings • Cognitive effectiveness • Diagrammatic effectiveness • Linguistic effectiveness • Logical effectiveness • Logical geometry • Unconnectedness

Mathematics Subject Classification (2000) Primary 03G05, 68T30; Secondary 03B65, 06A07, 00A66, 97C30

1 Introduction

The central aim of the research programme of *Logical Geometry* (henceforth abbreviated as LG) is to develop an interdisciplinary framework for the study of logical diagrams.[1] LG has focussed on constructing logical diagrams for (1) *logical* systems such as syllogistics with subject negation [10], syllogistics with singular propositions [41], modal logic [40] and public announcement logic [9], (2) *linguistic* systems such as those involving subjective quantifiers [45] and generalised quantifiers [40], and (3) *conceptual* systems, such as those involving the Aristotelian and duality relations themselves [13] and the metalogical concepts of tautology and satisfiability [11].

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[1] For more detailed information, see the website www.logicalgeometry.org.

J.-Y. Béziau, G. Basti (eds.), *The Square of Opposition: A Cornerstone of Thought*, Studies in Universal Logic, DOI 10.1007/978-3-319-45062-9_12

LG studies both the abstract-logical properties and the visual-geometrical properties of logical diagrams. As far as the *abstract-logical* properties of logical diagrams are concerned, LG investigates a range of topics including the information contents of the Aristotelian relations [43], the difference between opposition and implication relations [43], the intricate connection between Aristotelian and duality relations [17, 42], the context-dependence of Aristotelian relations [10], logical complementarities between Aristotelian diagrams [44–46], Boolean subfamilies and Boolean closures of Aristotelian diagrams [41]. These abstract-logical topics are studied from the perspective of logic itself [10], but also from those of formal semantics [40, 42], group theory [8, 17] and lattice theory [14, 40].

As for the *visual-geometrical* properties of logical diagrams, the LG framework studies, among others, the relation between Aristotelian and Hasse diagrams [14, 40], differences between 2D and 3D diagrams [40, 45], subdiagrams embedded inside larger diagrams [8, 41, 44–46], geometrical complementarities between Aristotelian diagrams [44–46], informational and computational equivalence of Aristotelian diagrams [15, 16, 48] and cognitive aspects of Aristotelian and duality diagrams [8, 14]. For the analysis of these visual-geometrical topics LG makes crucial use of insights from disciplines such as cognitive psychology [14, 44], group theory [8, 17], diagrams design [14, 44] and computer graphics [10].[2]

The LG programme also studies the historical development of logical diagrams, focussing on their use in the works of distinguished authors such as John Buridan [19] and J. N. Keynes [10]. Finally, LG has also explored the potential roles of logical diagrams in logic education [11] and the interface between formal and natural languages [13].

In its investigations, LG makes extensive use of *bitstrings*, i.e. sequences of bits (0/1) that serve as compact representations of the formulas' semantics. These bitstrings have turned out to be an extremely powerful tool, yielding both quantitative and qualitative results as well as raising interesting new questions. The main aims of this paper are hence (1) to present a unified account of bitstrings in LG and (2) to provide an overview of their effectiveness in the various areas of LG.[3]

The paper is organised as follows. Section 2 introduces bitstrings and discusses some of their basic properties. The next three sections survey the effectiveness of bitstrings in three key areas of LG. In particular, Sect. 3 goes into the logical effectiveness of bitstrings, while Sect. 4 deals with their diagrammatic effectiveness, and Sect. 5 addresses their linguistic and cognitive effectiveness. Finally, Sect. 6 draws some conclusions and points out some prospects and challenges.

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[2] Some of these abstract-logical and visual-geometrical properties are also studied (for Aristotelian diagrams) in Moretti's oppositional geometry framework [28, 29].

[3] The paper thus stands in a long tradition of work discussing the 'unreasonable effectiveness' of a variety of mathematical tools and techniques for a variety of purposes [4, 21, 22], which was initiated by Wigner's famous [49].

2 Bitstrings in Logical Geometry

Bitstrings are sequences of bits (0/1) which serve as compact combinatorial representations, both of the denotations of formulas in *logical systems* (such as classical propositional logic, first-order logic, modal logic and public announcement logic), and of concepts from *lexical fields* (such as comparative quantification, subjective quantification, color terms and set inclusion relations).[4] As such, there is no limitation on the length of bitstrings: they may consist of any number of bit positions. For example, bitstrings consisting of up to 16 bit positions have already proved useful in LG [10]. However, most of the properties and applications to be discussed in this paper can already be described by means of much shorter bitstrings. For ease of presentation, we will therefore mainly work with bitstrings of length 4, which allow us to encode various interesting logical fragments, such as the $2^4 = 16$ formulas of classical propositional logic with 2 propositional variables p and q, and the 16 formulas from the modal logic $\mathsf{S5}$ with 1 propositional variable p, as illustrated in Table 1. If a formula φ is encoded by the bitstring b, we write $\beta(\varphi) = b$. In other words, β is a function mapping a formula φ onto its bitstring b.[5] Bitstrings can be characterised in terms of their *level*, i.e. the number of positions with value 1. Hence, for bitstrings of length 4, the top half in Table 1 contains the 4 level 1 (L1) bitstrings 1000, 0100, 0010 and 0001 and their 4 contradictory L3 bitstrings 0111, 1011, 1101 and 1110. The bottom half in Table 1 then consists of the 6 L2 bitstrings as well as the L0 and L4 bitstrings 0000 and 1111.

The Aristotelian relations are standardly defined as relations holding between two *formulas*. Relative to a logical system S (which is assumed to be bivalent, and have all the Boolean connectives), two formulas φ, ψ are said to be

$$\begin{array}{llll}
\mathsf{S}\text{-}\textit{contradictory}\ (CD_{\mathsf{S}}) & \text{iff} & \mathsf{S} \models \neg(\varphi \wedge \psi) & \text{and} \quad \mathsf{S} \models \neg(\neg\varphi \wedge \neg\psi), \\
\mathsf{S}\text{-}\textit{contrary}\ (C_{\mathsf{S}}) & \text{iff} & \mathsf{S} \models \neg(\varphi \wedge \psi) & \text{and} \quad \mathsf{S} \not\models \neg(\neg\varphi \wedge \neg\psi), \\
\mathsf{S}\text{-}\textit{subcontrary}\ (SC_{\mathsf{S}}) & \text{iff} & \mathsf{S} \not\models \neg(\varphi \wedge \psi) & \text{and} \quad \mathsf{S} \models \neg(\neg\varphi \wedge \neg\psi), \\
\textit{in}\ \mathsf{S}\text{-}\textit{subalternation}\ (SA_{\mathsf{S}}) & \text{iff} & \mathsf{S} \models \varphi \rightarrow \psi & \text{and} \quad \mathsf{S} \not\models \psi \rightarrow \varphi.
\end{array}$$

This definition shows that the Aristotelian relations are sensitive with respect to the logical system S [10, 18]. If the system is clear from the context, we will usually omit it, and simply talk about 'contrariety' instead of 'S-contrariety', and so on. As will be discussed in more detail in Sect. 3, this definition is fundamentally 'hybrid' in nature: the relations

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[4] The original formulation of bitstring semantics in Smessaert [40] was inspired by considerations from generalised quantifier theory about partitioning the powerset of the quantificational domain. As demonstrated in Chatti [5, 6], however, an informal precursor of this technique was already used by Avicenna in the eleventh century AD. Conceptually very similar techniques are the *setting* approach of Pellissier [30], the *valuation spaces* account of Seuren [37, 39] and the *question-answer semantics* of Schang [33].

[5] Note that Moretti [29] and Schang [34] use a bitstring-like notation to encode the Aristotelian relations themselves (as well as possible generalisations of these relations). Within the LG framework, however, bitstrings do not encode *relations* between formulas, but rather (the denotations of) the *formulas* as such. Finally, note that it is not always immediately clear how to define the bitstring mapping β precisely; however, a systematic way for achieving this is available (also see Sect. 6 and [18]).

Table 1 Bitstrings (BS) for the 16 formulas of classical propositional logic (CPL) and the modal logic S5

S5	CPL	BS	BS	CPL	S5
$\Box p$	$p \wedge q$	1000	0111	$\neg(p \wedge q)$	$\neg\Box p$
$p \wedge \neg\Box p$	$\neg(p \rightarrow q)$	0100	1011	$p \rightarrow q$	$\neg p \vee \Box p$
$\neg p \wedge \Diamond p$	$\neg(p \leftarrow q)$	0010	1101	$p \leftarrow q$	$p \vee \neg \Diamond p$
$\neg \Diamond p$	$\neg(p \vee q)$	0001	1110	$p \vee q$	$\Diamond p$
p	p	1100	0011	$\neg p$	$\neg p$
$\Box p \vee (\neg p \wedge \Diamond p)$	q	1010	0101	$\neg q$	$\neg\Box p \wedge (p \vee \neg \Diamond p)$
$\Box p \vee \neg \Diamond p$	$p \leftrightarrow q$	1001	0110	$\neg(p \leftrightarrow q)$	$\neg\Box p \wedge \Diamond p$
$\Box p \wedge \neg\Box p$	$p \wedge \neg p$	0000	1111	$p \vee \neg p$	$\Box p \vee \neg\Box p$

CD, *C* and *SC* are defined in terms of whether the formulas can be true together and whether they can be false together,[6] whereas *SA* is defined in terms of implication or truth propagation [43].

Completely analogously, the Aristotelian relations can be defined as holding between two *bitstrings*. Two bitstrings b_1 and b_2 of length ℓ are said to be

$$\begin{array}{llllll} \textit{contradictory}\ (CD) & \text{iff} & b_1 \wedge b_2 = 0\cdots0 & \text{and} & b_1 \vee b_2 = 1\cdots1, \\ \textit{contrary}\ (C) & \text{iff} & b_1 \wedge b_2 = 0\cdots0 & \text{and} & b_1 \vee b_2 \neq 1\cdots1, \\ \textit{subcontrary}\ (SC) & \text{iff} & b_1 \wedge b_2 \neq 0\cdots0 & \text{and} & b_1 \vee b_2 = 1\cdots1, \\ \textit{in subalternation}\ (SA) & \text{iff} & b_1 \wedge b_2 = b_1 & \text{and} & b_1 \vee b_2 \neq b_1. \end{array}$$

If two formulas φ and ψ cannot be true together, the meet of the corresponding bitstrings $\beta(\varphi)$ and $\beta(\psi)$ equals the bottom element of the Boolean algebra $\{0, 1\}^\ell$, namely the L0 bitstring $0\cdots0$.[7] Similarly, φ and ψ cannot be false together, whenever the join of the bitstrings $\beta(\varphi)$ and $\beta(\psi)$ equals the top element of the Boolean algebra $\{0, 1\}^\ell$, namely the Lℓ bitstring $1\cdots1$. The Aristotelian relation holding between any two formulas can then easily be determined by computing the meet and join of their bitstring counterparts. In other words, the formulas φ and ψ stand in some Aristotelian relation (as defined for S) if and only if $\beta(\varphi)$ and $\beta(\psi)$ stand in that same relation (as defined for bitstrings). This can be seen as a manifestation of the representation theorem for finite Boolean algebras [20, Chap. 15].

In contrast to the setting approach of Pellissier [30], the mapping β assigns a *semantics* to the formulas. More in particular, each bit provides an answer to a meaningful (binary) question. In the case of S5, for instance, the bit positions encode answers to the following

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[6] The $\neg(\varphi \wedge \psi)$ part in these definitions specifies whether the formulas can be true together, while the $\neg(\neg\varphi \wedge \neg\psi)$ part specifies whether the formulas can be false together. Note that these clauses explicitly use the $\neg$-connective to express that a formula is false, and thus assume the classicality of the underlying logical system S. In non-classical (e.g. many-valued) logics, the informal condition that two formulas cannot be true (resp. false) together can be formalised in many different, non-equivalent ways.

[7] The Boolean operations on bitstrings are defined bitwise, i.e. as operations of negation, conjunction or disjunction computed bit position by bit position. For example, $\neg 1100 = 0011$, $1100 \wedge 1010 = 1000$ and $1100 \vee 1010 = 1110$.

questions about sets of possible worlds (PWs), where φ is a modal formula containing the propositional variable p:

Is φ true if p is true in all PWs?	yes/no
Is φ true if p is true in the actual world but not in all PWs?	yes/no
Is φ true if p is true in some PWs but not in the actual world?	yes/no
Is φ true if p is true in no PWs?	yes/no

The examples below illustrate how the bitstrings of length 4 that the β-function assigns to the formulas of S5 are a compact way to represent a quadruple of yes/no answers to the questions above:

$$\begin{array}{lcccl} \beta(\Diamond p) & = & 1110 & = & \langle \text{ yes, yes, yes, no } \rangle \\ \beta(\Diamond p \wedge \Diamond\neg p) & = & 0110 & = & \langle \text{ no, yes, yes, no } \rangle \\ \beta(\Diamond\neg p) & = & 0111 & = & \langle \text{ no, yes, yes, yes } \rangle \end{array}$$

The fact that the S5-formula in the middle example is the conjunction of the upper and lower formulas nicely corresponds to its bitstring being the meet of the upper and lower bitstrings as well as to its quadruple of answers being the meet of the upper and lower quadruples.

3 Logical Effectiveness

This section discusses two prime examples of the logical effectiveness of bitstring semantics. First of all, bitstrings allow us to provide a precise and positive characterisation of the notion of logical independence or unconnectedness. Secondly, the number and type of Aristotelian relations that a particular formula may enter into can straightforwardly be computed on the basis of the length and the level of its bitstring representation.

3.1 Characterizing Unconnectedness

As was mentioned in the previous section, the original set of Aristotelian relations is hybrid. In [43] two other sets of logical relations are defined in order to account for this hybrid nature, namely the *opposition* relations and the *implication* relations. The set of opposition relations is uniformly defined in terms of whether the formulas can be true together and whether they can be false together, and is obtained by removing subalternation from the original set of Aristotelian relations and replacing it with the relation of non-contradiction:

Opposition relations between bitstrings. Two bitstrings b_1 and b_2 of length ℓ are

contradictory (*CD*)	iff	$b_1 \wedge b_2 = 0 \cdots 0$	and	$b_1 \vee b_2 = 1 \cdots 1$,
contrary (*C*)	iff	$b_1 \wedge b_2 = 0 \cdots 0$	and	$b_1 \vee b_2 \neq 1 \cdots 1$,
subcontrary (*SC*)	iff	$b_1 \wedge b_2 \neq 0 \cdots 0$	and	$b_1 \vee b_2 = 1 \cdots 1$,
non-contradictory (*NCD*)	iff	$b_1 \wedge b_2 \neq 0 \cdots 0$	and	$b_1 \vee b_2 \neq 1 \cdots 1$.

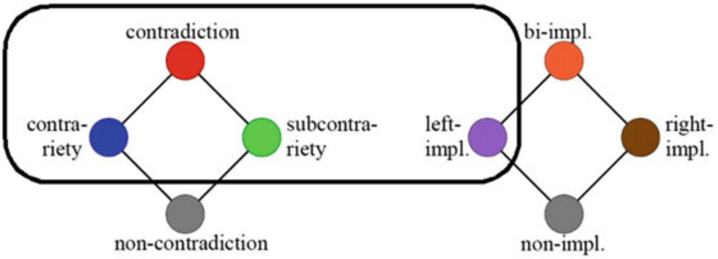

Fig. 1 Aristotelian relations as hybrid between opposition and implication relations

The set of implication relations, by contrast, is uniformly defined in terms of implication or truth propagation. The starting point is the relation of subalternation which was removed from the Aristotelian relations and relabeled as left-implication since the implication holds from the first/left formula to the second/right formula (but not vice versa). The three extra implication relations then correspond to implication from right to left (right-implication), two-way implication (bi-implication), and absence of implication in either direction (non-implication):

Implication relations between bitstrings. Two bitstrings b_1 and b_2 of length ℓ are in

bi-implication (BI)	iff	$b_1 \wedge b_2 = b_1$	and	$b_1 \vee b_2 = b_1$,
left-implication (LI)	iff	$b_1 \wedge b_2 = b_1$	and	$b_1 \vee b_2 \neq b_1$,
right-implication (RI)	iff	$b_1 \wedge b_2 \neq b_1$	and	$b_1 \vee b_2 = b_1$,
non-implication (NI)	iff	$b_1 \wedge b_2 \neq b_1$	and	$b_1 \vee b_2 \neq b_1$.

In Fig. 1 the hybrid nature of the Aristotelian relations is visualised: the relations of contradiction, contrariety, and subcontrariety are taken from the set of opposition relations on the left, whereas subalternation corresponds to left-implication from the set of implication relations on the right.[8]

In [43] the lattices for the two sets of relations in Fig. 1 are argued to be ordered by information level: they reveal parallel hierarchies of informativity, with the least informative relations at the bottom, and the most informative ones at the top. From an informational perspective, the four Aristotelian relations can be considered maximally informative.[9]

This information perspective also sheds new light on the notion of unconnectedness. Classically, two formulas are said to be *unconnected* if and only if they do not stand in any Aristotelian relation whatsoever.[10] As illustrated in Fig. 2, the information perspective provides an alternative, positive characterisation of unconnectedness in terms of the two

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[8] Despite their conceptual independence, there are several close connections between the sets of opposition and implication relations—e.g. $LI(b_1, b_2)$ iff $C(b_1, \neg b_2)$ [43, Lemma 3]. The latter essentially captures Schang's [35] claim that subalterns can be seen as contradictories of contraries; also see [43, Footnote 18].

[9] The absence of the two informative implication relations of bi-implication and right-implication can be accounted for independently, see [43].

[10] Many authors refer to this same notion as *logical independence*, e.g. see [1, 23, 31, 37].

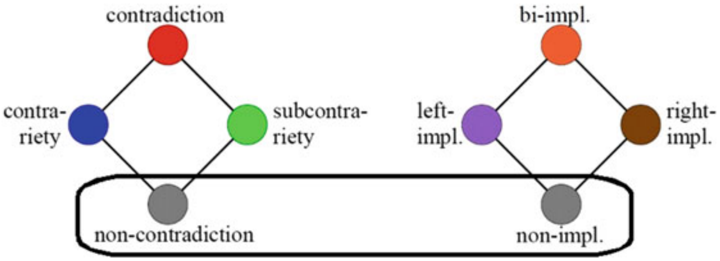

Fig. 2 Unconnectedness as the combination of non-contradiction and non-implication

least informative opposition and implication relations, viz. non-contradiction and non-implication respectively.

It can be shown that unconnectedness requires bitstrings of length at least 4: if two formulas φ and ψ are unconnected, then their bitstring representations $\beta(\varphi)$ and $\beta(\psi)$ need to consist of at least 4 bit positions. Since unconnectedness is defined as the combination of non-contradiction and non-implication, and the latter two themselves are both characterised in terms of two conditions, unconnectedness involves four conditions altogether. By virtue of non-contradiction, two unconnected formulas φ and ψ can be true together and can be false together. In terms of their bitstring representations, this means that there must be at least one bit position in which both $\beta(\varphi)$ and $\beta(\psi)$ have a value 1, and at least one bit position in which both $\beta(\varphi)$ and $\beta(\psi)$ have a value 0 respectively. By virtue of non-implication, there can be no implication relation in either direction between two unconnected formulas φ and ψ. In terms of their bitstring representations, this means that there must be at least one bit position in which $\beta(\varphi)$ has a value 1 and $\beta(\psi)$ has a value 0, and conversely, that there must be at least one bit position in which $\beta(\psi)$ has a value 1 and $\beta(\varphi)$ has a value 0. Since these four conditions on bit positions are logically independent, φ and ψ can only be unconnected if their bitstrings $\beta(\varphi)$ and $\beta(\psi)$ consist of at least 4 bit positions.[11] By contraposition, it also holds that if the formulas in an Aristotelian diagram can be encoded by bitstrings of length 3, then that diagram cannot contain any unconnectedness, i.e. every pair of its formulas stands in some Aristotelian relation.

Consider the three examples of Aristotelian hexagons for $\mathsf{S5}$ in Fig. 3. The best-known hexagon is no doubt the strong Jacoby-Sesmat-Blanché (JSB) hexagon in Fig. 3a: it can be encoded by bitstrings of length 3, and thus does not contain any unconnectedness.[12] It is important to stress that having bitstrings of length 4 is a necessary condition for

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[11] This dual perspective on unconnectedness can already be found in the works of the fourteenth century logician John Buridan. He characterised unconnected formulas negatively as “obeying no law, neither the law of contradictories, nor the law of contraries, nor the law of subcontraries, nor that of subalterns”, whereas according to his positive characterisation, “such propositions can be true at the same time … and they can both be false at the same time … [and] it is impossible that one should follow from the other” [3, p. 81]. Also see [19].

[12] The strong JSB hexagon in Fig. 3a is named after Jacoby [24], Sesmat [36] and Blanché [2].

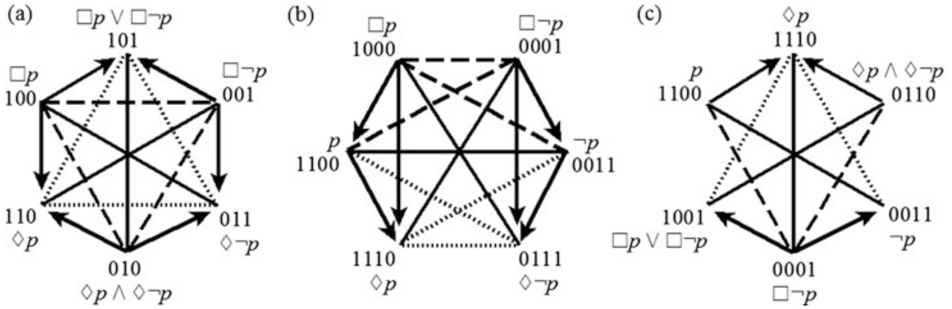

Fig. 3 Three Aristotelian hexagons for S5: (**a**) strong Jacoby-Sesmat-Blanché, (**b**) Sherwood-Czeżowski, (**c**) unconnected-4

unconnectedness, but not a sufficient condition. In other words, it is perfectly possible to have Aristotelian diagrams that require an encoding by means of bitstrings of length at least 4, and that yet do not contain any unconnectedness. A case in point is the Sherwood-Czeżowski (SC) hexagon of Fig. 3b, which requires bitstrings of length 4, but in which every pair of formulas nevertheless does stand in some Aristotelian relation.[13] By contrast, the unconnected-4 (U4) hexagon in Fig. 3c does contain unconnectedness (e.g. the formulas p and $\Diamond p \wedge \Diamond\neg p$ are unconnected), and therefore its formulas can only be encoded by bitstrings of length at least 4.[14]

3.2 *Calculating Logical Relations*

A second illustration of the logical effectiveness of the bitstring approach concerns the way in which, for any bitstring of length ℓ and level i, we can use simple combinatorial arguments on bitstrings[15] to calculate the number of:

contradictories	$\#CD$	$= 1$
contraries	$\#C$	$= 2^{\ell-i} - 1$
subcontraries	$\#SC$	$= 2^{i} - 1$
non-contradictories	$\#NCD$	$= (2^{\ell-i} - 1)(2^{i} - 1)$

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[13] The SC hexagon in Fig. 3b is named after William of Sherwood [25, 26] and Czeżowski [7].

[14] The U4 hexagon in Fig. 3c is called 'unconnected-4' because it contains exactly 4 pairs of unconnected formulas; it has recently been studied in [38] and [44].

[15] The combinatorial arguments for $\#CD$, $\#C$ and $\#SC$ can also be found in [35] (where they are based on Schang's question-answer semantics).

If we take a level 1 bitstring of length 3, for instance, then $\ell = 3$ and $i = 1$, which yields the following distribution over the 4 opposition relations:

$$\begin{array}{lll} \#CD & & = 1 \\ \#C & = 2^{\ell-i} - 1 = 2^{3-1} - 1 = 2^2 - 1 = 4 - 1 & = 3 \\ \#SC & = 2^i - 1 = 2^1 - 1 = 2 - 1 & = 1 \\ \#NCD & = (2^{\ell-i} - 1)(2^i - 1) = 3 \times 1 & = 3 \end{array}$$

Notice that the total number of relations equals $2^3 = 8$, since every bitstring of length 3 stands in an opposition relation to itself and to the 7 other bitstrings of length 3 (i.e. including the bottom element 000 and the top element 111). For example, for the L1 bitstring 100 the distribution looks as follows:[16]

$$\begin{array}{rl} CD[100] & = \{011\} \\ C[100] & = \{010, 001, 000\} \\ SC[100] & = \{111\} \\ NCD[100] & = \{110, 101, 100\} \end{array}$$

Completely analogously, taking a level 2 bitstring of length 4 ($\ell = 4$ and $i = 2$) yields the following distribution over the 4 opposition relations:

$$\begin{array}{lll} \#CD & & = 1 \\ \#C & = 2^{\ell-i} - 1 = 2^{4-2} - 1 = 2^2 - 1 = 4 - 1 & = 3 \\ \#SC & = 2^i - 1 = 2^2 - 1 = 4 - 1 & = 3 \\ \#NCD & = (2^{\ell-i} - 1)(2^i - 1) = 3 \times 3 & = 9 \end{array}$$

For the L2 bitstring 1100, for instance, the $2^4 = 16$ bitstrings are distributed over the opposition relations in the following manner:

$$\begin{array}{rl} CD[1100] & = \{0011\} \\ C[1100] & = \{0010, 0001, 0000\} \\ SC[1100] & = \{1011, 0111, 1111\} \\ NCD[1100] & = \{1000, 0100, 1010, 1001, 0110, 0101, 1100, 1110, 1101\} \end{array}$$

Finally, it can be shown that for bitstrings on non-extreme levels (i.e. which are on level i, for $1 < i < \ell - 1$), we have $\#CD < \#C, \#SC < \#NCD$. There thus exists a perfect inverse correlation between (1) the numbers of opposition relations that those bitstrings enter into, and (2) the informativity ordering of the opposition relations shown in Figs. 1 and 2:

$$\begin{array}{rccccc} \text{Number of relations} & \#CD & < & \#C, \#SC & < & \#NCD \\ \text{Informativity ordering} & CD & > & C, SC & > & NCD \end{array}$$

Notice, furthermore, that if $i \approx \frac{\ell}{2}$, then $\#C \approx \#SC$. In other words, bitstrings in middle levels have similar numbers of contraries and subcontraries, which straightforwardly

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[16] For any binary relation R on a set A, the *R-image* of an element $a \in A$ is defined as $R[a] := \{a' \in A \mid (a, a') \in R\}$.

corresponds to the fact that contrariety and subcontrariety occupy the same intermediate level of informativity in the lattices of Figs. 1 and 2 [43].[17]

4 Diagrammatic Effectiveness

This section presents two key examples of the diagrammatic effectiveness of bitstring analysis. First, bitstrings play a crucial role in studying the subdiagrams of the Aristotelian rhombic dodecahedron. Second, in establishing an exhaustive typology of all possible Aristotelian hexagons, different types of hexagons turn out to require bitstrings of different lengths.

4.1 Subdiagrams of the Aristotelian Rhombic Dodecahedron

The JSB hexagon in Fig. 3a is *Boolean closed*: every contingent Boolean combination of formulas in this hexagon is (logically equivalent to) a formula that already belongs to it. It thus visualises the entire Boolean algebra $\{0, 1\}^3$, except for its $\top$-element 111 and $\bot$-element 000. The SC hexagon in Fig. 3b, by contrast, is *not* Boolean closed: the disjunction of the two top vertices, for instance, is itself not (logically equivalent to) a vertex of the hexagon. The construction of the Boolean closure of bitstrings of length 4 has led to the discovery of the rhombic dodecahedron (RDH)—a 3D polyhedron with 12 rhombic faces and 14 vertices—for the visualisation of the Boolean algebra $\{0, 1\}^4$, represented by bitstrings of length 4 [40]. In order to describe the internal structure of this RDH and to present an exhaustive typology of all Aristotelian diagrams that can be embedded inside RDH, bitstrings again play a crucial role. The $2^4 - 2 = 14$ contingent bitstrings of $\{0, 1\}^4$ constitute 7 pairs of contradictories (PCDs). These 7 PCDs can be subdivided into 4 C-PCDs—which correspond to the 4 diagonals of the *cube* embedded in RDH and connect the L1 and L3 bitstrings—and 3 O-PCDs—which correspond to the 3 diagonals of the *octahedron* embedded in RDH and connect pairs of L2 bitstrings. This so-called CO-perspective then yields an exhaustive typology of the subdiagrams of RDH in terms of how many C-PCDs and how many O-PCDs they consist of. For example, both the strong Jacoby-Sesmat-Blanché hexagon in Fig. 3a and the Sherwood-Czeżowski hexagon in Fig. 3b are C^2O^1 hexagons, whereas the unconnected-4 hexagon in Fig. 3c is a C^1O^2 hexagon [46, 47].

As far as embedding smaller Aristotelian diagrams into bigger ones is concerned, the classical result in the literature is that the RDH contains six strong JSB hexagons [27, 28, 30, 32, 40, 45]. Bitstrings turn out to be a very powerful tool to study such embeddings. If we consider two bit positions, for example the second and third, then

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[17] The application of combinatorial techniques to bitstrings has generated many more results that are relevant for LG than the few simple ones described in this subsection. A more comprehensive and mathematically detailed overview can be found in [12].

Table 2 Bitstring compression from length 4 to length 3

1**01**1	1**10**1	1**00**1	⇝	1**0**1	1**11**0	⇝	1**1**0
1**01**0	1**10**0	1**00**0	⇝	1**0**0	0**11**1	⇝	0**1**1
0**01**1	0**10**1	0**00**1	⇝	0**0**1	0**11**0	⇝	0**1**0
0**01**0	0**10**0	(0**00**0)	⇝	(0**0**0)	(1**11**1)	⇝	(1**1**1)

the 14 contingent bitstrings of length 4 can be partitioned into a group of 8 bitstrings having different values in those positions—the left-hand side of Table 2—and a group of 6 bitstrings having identical values in those positions—the right-hand side of Table 2.[18] The latter group constitutes a strong JSB hexagon, whereas the former group constitutes its complementary Buridan octagon [44–46]. Although we are dealing with bitstrings of length 4, the six contingent bitstrings in the right half of Table 2—with identical values in their second and third bit positions—can thus be 'compressed' into bitstrings of length 3, which constitute the JSB hexagon in Fig. 3a.[19]

There are exactly $\frac{4\times3}{2} = 6$ ways in which bitstrings of length 4 can have identical (resp. different) values in two of their bit positions, and these correspond exactly to the 6 strong JSB hexagons (resp. Buridan octagons) embedded inside RDH:[20]

$[b]_2 = [b]_3$	$[b]_1 = [b]_2$	$[b]_3 = [b]_4$	$[b]_1 = [b]_4$	$[b]_1 = [b]_3$	$[b]_2 = [b]_4$
JSB1	JSB2	JSB3	JSB4	JSB5	JSB6

For the modal logic S5, the first three JSB hexagons are presented in terms of classical, paraconsistent and paracomplete negation in [1]. The fourth JSB hexagon was discovered independently in [28, 40] and the fifth and sixth JSB hexagons are introduced in [30, 40].[21]

We have just seen that the strong JSB hexagons inside RDH can be characterised by means of bitstring constraints of the form $[b]_i = [b]_j$ (for distinct $i, j \in \{1, 2, 3, 4\}$). It can be shown that all other types of Aristotelian diagrams embedded inside RDH can also be characterised by means of other, more complex bitstring constraints. For example, SC hexagons are characterised by bitstring constraints of the form $[b]_i \neq [b]_j \wedge ([b]_i = [b]_k \rightarrow [b]_i = [b]_\ell)$ (for pairwise distinct $i, j, k, \ell \in \{1, 2, 3, 4\}$); the concrete SC hexagon shown in Fig. 4b corresponds to taking $i = 1, j = 4, k = 3$ and $\ell = 2$.

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[18] Of course, the top and bottom elements 1111 and 0000 also have identical values in their second and third bit positions, but as usual, these are ignored in Aristotelian diagrams, which explains the numerical discrepancy between the two groups.

[19] For example, by collapsing the second and third bit positions, the bitstrings 1**00**0 and 0**11**0 for $\Box p$ and $\Diamond p \wedge \Diamond\neg p$ in RDH are compressed into the bitstrings 1**0**0 and 0**1**0 in Fig. 3a, respectively.

[20] We will write $[b]_i = [b]_j$ to express the condition that a bitstring b has the *same* values in bit positions i and j.

[21] Note that the corresponding six hexagons for CPL were already discovered in [32] and that [27] establishes the connection between S5 and CPL.

4.2 An Exhaustive Typology of Aristotelian Hexagons

A second illustration of the diagrammatic effectiveness of bitstrings concerns the typology of Aristotelian hexagons. A first question to be answered is how many hexagons can be constructed with bitstrings of length ℓ. Although strictly speaking there are 2^ℓ bitstrings of length ℓ, the restriction to contingent bitstrings means we generally only consider $2^\ell - 2$ bitstrings of length ℓ. The following combinatorial formula captures the number of hexagons with bitstrings of length ℓ:

$$\frac{(2^\ell - 2)(2^\ell - 4)(2^\ell - 6)}{3! \times 2^3}$$

Bitstrings are chosen in contradictory pairs (PCDs): choosing one bitstring automatically means choosing its contradictory as well. Hence, in order to select a hexagon, only three 'choices' need to be made in the numerator of this fraction, and the number of bitstrings from which we can choose each time decreases by 2 instead of 1. The denominator captures the variety of presentations of a given hexagon: 3! represents the number of permutations of 3 PCDs, while 2^3 reflects the fact that each of these 3 PCDs occurs inside the hexagon with a given 'orientation' (e.g. 1000—0111 versus 0111—1000).[22] Applying the formula above to bitstrings of length 3–7 yields the following numbers of hexagons:

$\ell = 3$	$\ell = 4$	$\ell = 5$	$\ell = 6$	$\ell = 7$
$\frac{6\times4\times2}{48}$	$\frac{14\times12\times10}{48}$	$\frac{30\times28\times26}{48}$	$\frac{62\times60\times58}{48}$	$\frac{126\times124\times122}{48}$
1	35	455	4495	39,711

Secondly, bitstrings have proved their computational importance in generating all possible types of Aristotelian hexagons (and their Boolean subtypes). They thus allow us to answer the question which types of hexagons exist and which lengths of bitstrings each type requires. As discussed before, the strong JSB hexagon in Fig. 3a requires bitstrings of length 3, whereas the Sherwood-Czeżowski and unconnected-4 hexagons in Fig. 3b,c require bitstrings of length 4. Three other types of Aristotelian hexagons can be distinguished: the weak JSB hexagon [30] and the (strongest Boolean subtype of the) unconnected-12 hexagon also require bitstrings of length 4, whereas the (strongest Boolean subtype of the) unconnected-8 hexagon is the only type requiring bitstrings of length 5.[23] A combination of mathematical reasoning and exhaustive computational verification has demonstrated that there exist no types of Aristotelian hexagons that require length 6 or higher (up to Boolean subtype).

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[22] See [16] for a detailed comparison of the relationship between the number of presentations of a hexagon on the one hand, and the number of geometrical symmetries/rotations of a regular hexagon on the other.

[23] From the bitstring characterisations of the strong and weak JSB hexagons, it follows that a JSB hexagon is strong iff it is Boolean closed.

5 Linguistic and Cognitive Effectiveness

This section briefly introduces two topics illustrating the linguistic and cognitive effectiveness of bitstring analysis, namely that of the scalar structure underlying bitstrings, and that of the difference between linear and non-linear bitstrings.

5.1 Scalar Structure in Bitstrings

In addition to its logical and diagrammatical effectiveness, bitstring semantics also generates new questions about the linguistic and cognitive aspects of the encoded expressions. Two related questions are (1) what is the relative weight or strength of individual bit positions inside bitstrings? and (2) what is the scalar or linear structure of the underlying conceptual domain? To illustrate these questions, note that the semantics of the basic operators of modal logic, predicate logic and total orders in Fig. 4 can all straightforwardly be captured in terms of bitstrings of length 3. Nevertheless, there does seem to be a clear intuitive difference in the relative weight of the individual bit positions in these cases, in the sense that some bit positions correspond to *points* on a cognitive scalar structure (or 'logical space'), whereas other bit positions correspond to *intervals* on that structure. In the case of the modal operators and the quantifiers in Fig. 4a,b, for instance, the first and third bit position encode the end points of the scale, whereas the second bit position encodes the intervening interval. With the ordering relations in Fig. 4c, by contrast, the second bit position encodes the central reference point on the scale, whereas the first and third positions encode the intervals extending to the left and to the right of that reference point.

The tripartitions in Fig. 4a,b can then be seen as the result of superimposing two bipartitions that each consist of one point and one interval, e.g. *all* vs. *not all* (with the point on the left and the interval on the right) and *some* vs. *no* (with the interval on the left and the point on the right). By contrast, the scalar structure of total orders in Fig. 4c can either be seen as being primitively tripartite in nature, or alternatively as being the result of superimposing two bipartitions that each consist of two intervals, viz. $\geq$ vs. $<$ on the one hand, and $>$ vs. $\leq$ on the other (so that the central reference point of the tripartite scale ($=$) only arises out of the interaction between these two bipartitions).

It should be emphasised that the distinction between point- and interval-interpretations of bit positions is primarily relevant from a linguistic or cognitive perspective, and does not go beyond the realm of classical Boolean algebra. In particular, the scalar structures in Fig. 4a,c all share the same Boolean structure. For example, for all three scalar structures, the negation of the middle bit position is identical to the join of the leftmost and rightmost bit positions ($\neg 010 = 101 = 100 \vee 001$), regardless of whether that middle bit position corresponds to an interval (as in Fig. 4a,b) or to a point (as in Fig. 4c).

Moving from bitstrings of length 3 to those of length 4, some quadripartite scalar structures can be seen as refinements of an underlying tripartite scalar structure, while others seem to be primitively quadripartite in nature, or to be the result of superimposing

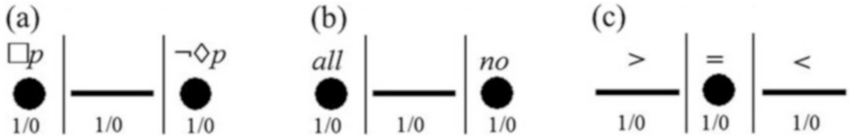

Fig. 4 Points versus intervals in bitstrings of length 3

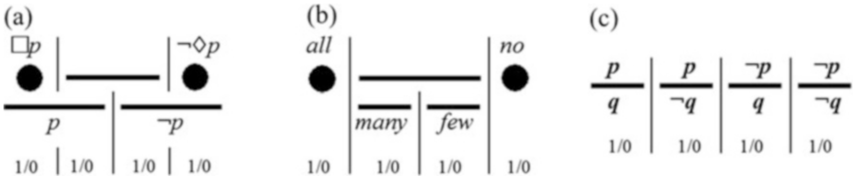

Fig. 5 Bitstrings of length 4 as refinements of bitstrings of length 3

two bipartitions. For example, the quadripartite scale of the modal logic S5 in Fig. 5a can be seen as the result of superimposing a bipartition for the bare modalities (p vs. $\neg p$) onto the original tripartition of Fig. 4a ($\Box p$ vs. $\Diamond p \wedge \Diamond \neg p$ vs. $\neg \Diamond p$). Similarly, the bipartition with the subjective quantifiers *many* and *few* in Fig. 5b can be seen as a further refinement of the original interval of the second bit position in Fig. 4b [45]. With the formulas of CPL in Fig. 5c, by contrast, the scalar structure can either be seen as being primitively quadripartite in nature (with each bit position corresponding to a row in the classical truth tables), or alternatively as being the result of superimposing two independent bipartitions (viz. p vs. $\neg p$ and q vs. $\neg q$).

5.2 Linear Versus Non-linear Bitstrings

From a mathematical or algebraic perspective we cannot distinguish between 'linear' bitstrings—such as 1010, where all four bit positions are linearly ordered with respect to each other—and 'non-linear' bitstrings—such as $1^{0}_{1}0$, where the precise ordering between the second and the third bit position is left unspecified. From a linguistic or cognitive perspective, however, such a difference does become relevant. *Linear* bitstrings imply that all questions—i.e. all bit positions—about a lexical field can be situated on a single dimension. For the realms of comparative and proportional quantification this does indeed seem to be the case. *Non-linear* bitstrings, by contrast, imply that the various questions belong to fundamentally distinct dimensions, as was argued to be the case for the modalities of S5 and the scale with *many* and *few* in Fig. 5a,b.

It should be emphasised that from a mathematical perspective, linear and non-linear bitstrings have the same Boolean structure. For example, a non-linear bitstring such as $1^{0}_{1}0$ consists of *four* bit positions that each have *exactly one* of the values 1 and 0 (just like the linear bitstring 1010). In particular, the non-linear bitstring $1^{0}_{1}0$ should not be seen as consisting of *three* bit positions, with the second position containing *both* the values 1 and

0. The latter perspective might also prove useful (e.g. for assigning bitstring semantics to non-classical logics), but by allowing certain bit positions to be simultaneously 1 and 0, it constitutes a far more radical departure from the realm of classical Boolean algebra than the non-linear bitstrings proposed here.

In future research, empirical hypotheses will be formulated concerning the cognitive complexity of various lexical fields (e.g. in terms of processing times), and possible correlations with the scalar and (non-)linear nature of their underlying bitstring representations will be investigated.

6 Conclusion

In this paper we have presented a unified account of bitstrings and provided an overview of their effectiveness in three key areas of the Logical Geometry research programme. As for *logical* effectiveness, bitstrings first of all allow us to provide a positive characterisation of the notion of unconnectedness as the combination of two conditions for non-contradiction and two conditions for non-implication, thus requiring bitstrings of length at least 4. Secondly, the number and type of Aristotelian relations that a particular formula may enter into can straightforwardly be computed on the basis of the length and the level of its bitstring representation. The number of opposition relations ($\#CD < \#C, \#SC < \#NCD$) turns out to be inversely correlated with the informativity level of these relations.

Furthermore, two key examples have been discussed regarding the *diagrammatic* effectiveness of bitstring semantics. On the one hand, bitstrings play a crucial role in studying the subdiagrams of the Aristotelian rhombic dodecahedron. A case in point is the embedding of 6 strong JSB hexagons in RDH, which can be accounted for in terms of the 6 ways in which a bitstring of length 4 can be compressed into a bitstring of length 3 by collapsing bit positions with identical values. On the other hand, the exhaustive typology of all possible Aristotelian hexagons reveals that different types of hexagons require bitstrings of different lengths. Four types require a bitstring length of 4 (the weak JSB, the Sherwood-Czeżowski, the Unconnected-4 and the Unconnected-12 hexagons), whereas the strong JSB hexagon only requires length 3 and the Unconnected-8 hexagon requires length 5.

Finally, two topics have briefly illustrated the *linguistic* and *cognitive* effectiveness of bitstring analysis. First of all, scalar structures underlying the bitstrings may differ from one another as to which bit positions correspond to points on the scale and which positions to intervals. Some quadripartite scalar structures can be considered as refinements of originally tripartite structures, whereas others are inherently quadripartite. Secondly, bitstrings are called linear or non-linear depending on whether the underlying binary questions relate to one single dimension or to different dimensions.

As illustrated throughout the paper, bitstrings have proved extremely useful in Logical Geometry so far. Nevertheless, bitstring analysis in its original formulation (as presented in this paper) still exhibits a number of limitations. First of all, it is not always clear how 'sensitive' bitstrings are to the specific properties of the underlying logical system: two formulas may enter into different Aristotelian relations with one another depending on the logical system and should therefore be assigned different bitstrings accordingly. Secondly, the complex interplay between Boolean and Aristotelian structure requires further investigation: some fragments which have an isomorphic Aristotelian structure may nevertheless not be isomorphic from a Boolean point of view. Thirdly, the current approach does not provide a systematic strategy for establishing a bitstring semantics for any fragment $\mathcal{F}$ of any logical system S [9]. In ongoing research we are developing a more mathematically mature version of bitstring semantics that is able to overcome these different limitations [18].

Acknowledgements We thank Dany Jaspers, Alessio Moretti, Fabien Schang and Margaux Smets for their comments on earlier versions of this paper. The second author holds a postdoctoral fellowship from the Research Foundation–Flanders (FWO).

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An Arithmetization of Logical Oppositions

Fabien Schang

Abstract An arithmetic theory of oppositions is devised by comparing expressions, Boolean bitstrings, and integers. This leads to a set of correspondences between three domains of investigation, namely: logic, geometry, and arithmetic. The structural properties of each area are investigated in turn, before justifying the procedure as a whole. To finish, I show how this helps to improve the logical calculus of oppositions, through the consideration of corresponding operations between integers.

Keywords *k*-base system • Bitstring • Chasles' relation • Opposite • Opposition • Question-answer semantics • Vectors

Mathematics Subject Classification Primary 03B35, Secondary 03B05, 03B65

1 Introduction

The arithmetization of the logic of oppositions is usually taken to mean a process that consists in translating the logical relations between formulas (in a language L) into arithmetic relations between integers (in the domain of positive integers $\mathbf{N}^*$).

This work can be considered from a structural point of view. That is, just as Descartes made a connection between geometry and arithmetic through analytic geometry, the present paper relies upon the fact that there are blatant analogies between the abstract areas of geometry, logic, and arithmetic. There also exists a form of serendipity in this paper, since the final result is derived from a completely different domain area of discourse—the Chinese *Book of Changes* (or *Yiking*). There is, fairly obviously, no causal or logical connection between the latter and logical oppositions, however, a comparison of both domains leads to a fruitful explanation thanks to an analogy between elements of their common structure.

The content of the paper runs as follows. In the first section, we propose a broad historical background of binary systems. In the second section, the geometry of oppositions is considered from the Aristotelian square to recent research developments. In the third section, a logic of opposition is characterized by an algebraic calculus on non-Fregean valuations. In the fourth and final section, we introduce an arithmetic version of oppositions.

J.-Y. Béziau, G. Basti (eds.), *The Square of Opposition: A Cornerstone of Thought*, Studies in Universal Logic, DOI 10.1007/978-3-319-45062-9_13

2 A Historical Background of Binary Systems

We start with three main works of Leibniz, the aim of which was (to some extent) to build correspondences between arithmetic and metaphysics: “De progressione Dyadica” (1678), his correspondence (from 1697 to 1703) with Father Joachim Bouvet (1697-1703), a French Jesuit and mathematician sent to China by Louis XIV—above all, his article “Explication de l’arithmétique binaire qui se sert des seuls caractères 0 & 1 avec des remarques sur son utilité et sur ce qu’elle donne le sens des anciennes figures chinoises de Fohy” (1703) is a sample of the way in which Leibniz aimed towards his ambitious project of a *calculus ratiocinator*. Fohy, or Fu Xi (3rd millenary B.C.), is the legendary ancestor of the Chinese who is supposed to have created the well-known *Book of Changes*, or Yiking. The reason that Leibniz took this religious book into consideration is because of an analogy between the structuration of hexagrams and his own works on the arithmetic binary number system. Furthermore, the German philosopher tried to show that a metaphysical interpretation of the Yiking would help to corroborate the Christian metaphysics regarding the origins of Being and Nothingness. We will not discuss this aspect of Leibniz’s philosophy, instead, our aim is to show some striking similarities between the Yiking, Leibniz’s binary number system, and a Boolean theory of logical oppositions.

Let us consider the binary number system first. Our common number system is decimal, since it consists in ten basic units every numeral is composed of. Leibniz’s binary number system relies on the two basic units well-known to computer scientists, viz. the Boolean bits 0 and 1. More generally, there is a systematic way of transcribing in our usual decimal (or 10-base) system the number X of an arbitrary number system according to its base and its length. Thus:

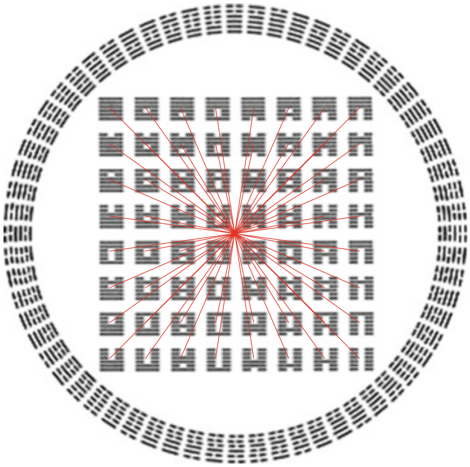
For any k-base number system in which numbers $(\mathrm{X})_{[k]}$ are sequences of n items, its decimal coding is characterized as follows:

$$(\mathbf{a}_1 \ldots \mathbf{a}_n)_{[k]} = \Big(\Big(k^{n-1} \times \mathbf{a}_1\Big) + \cdots + \big(k^0 \times \mathbf{a}_n\big)\Big)_{[10]}$$

Let us take an example. What is the appropriate decimal coding of a 2-base number $\mathrm{X} = (\mathbf{a}_1 \ldots \mathbf{a}_n)_{[2]}$ like, e.g., $(101110)_{[2]}$? Its decimal transcription is the integer 46, or $(46)_{[10]}$, starting from an integer of base $k = 2$ and length $n = 6$. Indeed,

$$\begin{aligned}(101110)_{[2]} &= \big(\big(2^{6-1} \times 1\big) + \big(2^{5-1} \times 0\big) + \big(2^{4-1} \times 1\big) + \big(2^{3-1} \times 1\big) + \big(2^{2-1} \times 1\big) \\ &\quad + \big(2^{1-1} \times 0\big)\big)_{[10]} \\ &= \big(\big(2^5 \times 1\big) + \big(2^4 \times 0\big) + \big(2^3 \times 1\big) + \big(2^2 \times 1\big) + \big(2^1 \times 1\big) + \big(2^0 \times 1\big)\big)_{[10]} \\ &= ((32 \times 1) + (16 \times 0) + (8 \times 1) + (4 \times 1) + (2 \times 1) + (1 \times 0))_{[10]} \\ &= (32 + 0 + 8 + 4 + 2 + 0)_{[10]} \\ &= (46)_{[10]}\end{aligned}$$

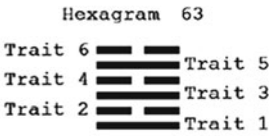
Here, Leibniz saw a connection between his 2-base number system and the Chinese Yiking, especially the graphic representation of it as set out by the philosopher, numerologist and poet Shao Yong (1012–1077). It is a set of $2^6 = 64$ *gua* or hexagrams, that is, 6-tuples of lines organized both in a circular and a quadratic form (see Appendix).

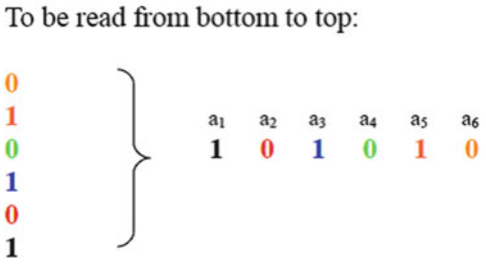


A certain square of contradiction or “earth structure” is included inside the above mandala, or “heaven structure”, in the sense that each pair of contradictory terms is similarly related by central symmetry in the square—and it is the same sort of symmetry which accounts for the distribution of each of the components in the circular mandala. The square and the circle include the same $2^6 = 64$ hexagrams as their components, where 2 is the number *m* of distinctive data for each item and 6 the number *n* of items in a given *n*-gram.

Starting from the Daoist picture of the world, each entity of the world is a combination of two basic elements. Subsequently, each discontinuous line of hexagrams symbolizes the passive element Yin, which Leibniz made correspond to the integer 0; each continuous line symbolizes the active element Yang, symbolized by Leibniz with the integer 1. Furthermore, each “gua” (hexagram) historically results from a sequence of two added trigrams, resulting in $2^{3+3} = 2^6 = 64$.

However, the Chinese numbering of the hexagrams differs from the Leibnizian representation with respect to its direction (bottom-up for the former, left-right for the latter). For example, the 63rd hexagram is 101010 and has the decimal value $(101010)_{[2]} = (42)_{[10]}$.

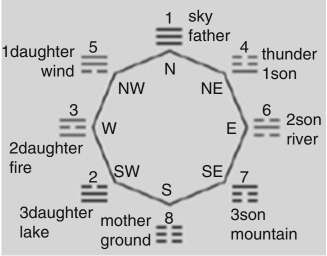




Therefore, the Chinese classification seems to rely on purely contingent rules of interpretation that have nothing to do with arithmetic. So why should we follow Leibniz's binary number system, in this respect?

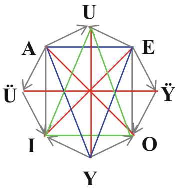
One reply is to point to the blatant resemblance between the following two examples and recent developments in the theory of opposition.

The first case is the so-called *diagram of unity*, occurring in the flag of South Korea; it also resorts to a logical octagon of opposition, which is an expansion of the Aristotelian square.



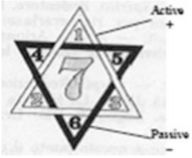
The flag of South Korea stems from four basic elements: Qian (Heaven), Kan (Water), Kun (Earth), and Li (Fire). However, its four components result from a combination of three lines whose exhaustive set is depicted clockwise in the middle figure. By adapting the eight trigrams of the preceding right hand figure to the Leibnizian 2-base number system, we obtain these definitions: Thunder (first son): $(001)_{[2]} = 1$; River (second son): $(010)_{[2]} = 2$; Mountain (third son): $(100)_{[2]} = 4$; Ground (mother): $(000)_{[2]} = 0$; Lake (third daughter): $(011)_{[2]} = 3$; Fire (second daughter): $(101)_{[2]} = 5$; Wind (first daughter): $(110)[2]_{[2]} = 6$; Sky (father): $(111)_{[2]} = 7$. It is important to note that the boldface numbers placed above the trigrams do not correspond to their decimal translation: they represent ordinal numbers (**1** for "the 1st, trigram", **2** for the 2nd trigram, and so on), just as in the various sequences proposed throughout the history of the Yiking (Shao Yong, Jin Fang, Mawangdui).

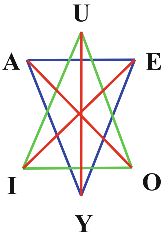
The same distribution of relations occurs in the below logical octagon below, which is a double extension of the initial square of oppositions: two additional pairs of edges supplement the hexagon vertically –**U** and **Y**, and horizontally –**Ü** and **Ÿ**. The resulting octagon is a combination of the diagrams studied in [2–4]: the two hexagons **AUEOYI** and **AEŸOIÜ** are combined to form the octagon **AUEŸOYIÜ**, which was also mentioned in [1] for studies on modal logic.



The isomorphism between the Yiking trigram and logical octagons does not mean that they match with each other, however: each vertex of the logical octagon corresponds to a bitstring of length $n = 4$, whereas trigrams are bitstrings of length $n = 3$. This is the case, because the first and eighth trigrams (Sky and Ground, respectively) have the maximal and minimal values 7 and 0; these extrema correspond to tautology and antilogy in logical polygons, and both can be located at the center of intersection of all the logical contradictories (see page 7).

The second case is the *Seal of Solomon*, also occurring in a flag—that of Israel. Whilst the Korean flag has just been compared to the logical octagon, the present one also corresponds to one diagram of logical oppositions: Blanché's logical hexagon of oppositions **AUEOYI**, once the latter has been deprived from its surrounding relations of subalternation.





This truncated version of the hexagram is a six-point starlike figure of the the *Magen David* (Shield of David), to be also compared with the Hindu Shatkona. Each edge of the Seal is marked with an integer from 1 to 6: 4 for **A**, 1 for **U**, 5 for **E**, 3 for **O**, 6 for **Y**, 2 for **I**. The center of the star is marked with the integer 7, resulting from the sum of each of its diagonals:

$$(1 + 6) = (2 + 5) = (3 + 4) = 7$$

The connection link between this mystic star and the logical hexagon is not obvious at first sight, given that the opposition between numbers and sentences is not supposed to obey one and the same ordering process. Yet such a connection is made possible through arithmetic, especially Leibniz's 2-base system, as will be shown in the section devoted to the logic of opposition.

It should be taken for granted that the above two examples of the Yiking and the Seal of Solomon do not constitute a plea for numerology. The present paper does not purport to show the "power" of numbers in order to explain how things are organized in the world, as is the case with the 64 exhaustive elements of the world in the Yiking. Rather, this paper is an essay on *numerical logic* and deals with the "power" of numbers in order to show how people think logically. It is a matter of hermeneutics, *i.e.*, how to interpret signs in a language, whether signs denote integers in number theory or propositions in logic.

To investigate this, three languages are compared in the next sections: geometrical, logical, and arithmetic languages, assuming that each of the three areas consists of a set of structured elements with specific relations between them.

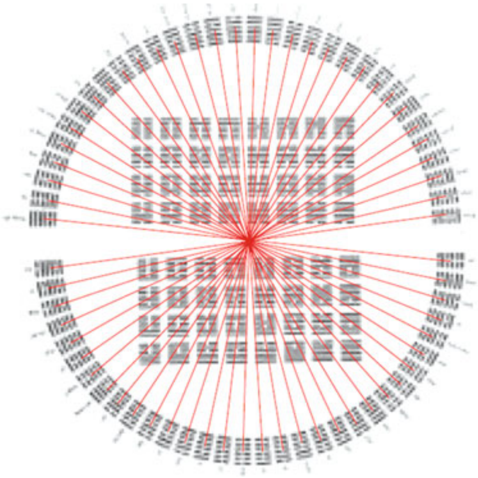
3 Geometry of Oppositions

The most famous geometry of oppositions is the so-called Aristotelian square, originating from the logical corpus of the Stagirite. Although the latter did not himself consider the logical oppositions of contradiction and contrariety in this way, the "Aristotelian" square refers to the set of 6 logical oppositions between the categorical propositions **A**,**E**,**I**,**O** of syllogistics. Despite some isolated developments of the geometry of oppositions in the history of logic—*e.g.*, Buridan's logical octagon of *de re* modalities, a new impetus has been given much later by the French philosopher and logician Robert Blanché. As exemplified above, Blanché's hexagon has enriched the Aristotelian square with two additional vertices **Y**,**U** in the vertical sense and given several interpretations to the six resulting vertices: quantified, modal, but also conceptual in a broader way. Earlier, the Polish logician Czeżowski extended Aristotle's categorical square by introducing the singular propositions **Ÿ**,**Ü**. This suggests that there was no transcendental limitation on the initial square of oppositions, insofar as more or less than four propositions may be opposed to each other. Moreover, propositions are not the only sort of meaningful entities that can be represented in such geometrical figures: concepts, modalities, and so on, may also be defined in terms of opposition.

A more systematic treatment of logical oppositions has been recently proposed by philosophers, linguists, and mathematicians [1–5–6–10]. Some precise explanations have been given there about the minimal and maximal extensions of the historical core pattern of geometrical oppositions, namely: the square. There are not only different extensions of the opposed edges in an arbitrary figure, but also different ways of representing these extensions according to their inner structure.

Even given this precedence, there may remain some reluctance towards such a logical procedure: why and how to afford a geometrical representation for logical oppositions? But, whilst the core logical notion of consequence seems absent from this special discipline, we see at least two advantages of a geometry of oppositions.

On the one hand, these structured geometries offer a harmonious representation of logical contradictions by central symmetry. This has been also seen twice above, both with logical hexagons and octagons as well as in the mystic figures of the Yiking and the Seal of Solomon. The central symmetry is shown there by sets of intersecting lines between red contradictories: two lines in a square, three lines in a hexagon, four lines in a octagon, and so on, until a circle displaying a hypothetically complete set of n relations in a perfect shape (recall Shao Yong's aforementioned mandala).



This suggests that such a way of organizing oppositions goes beyond the scientific domain of modern logic and already occurred a long time ago in prescientific domains of thought.

On the other hand, a geometrical figure may also be used to depict the structural completeness of a finite set of logical relations. To the question regarding how many logical relations there can be in a given set of propositions or concepts, the answer is that this depends on the number of relata to be opposed to each other in a given geometrical structure. Thus, a logical structure (square, hexagon, and the like) is said to be complete if and only if all the logical relations can be displayed in it. Following [8–11], structural completeness can be explained as follows:

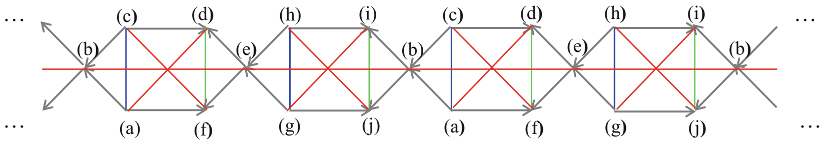
> For any oppositional structure with m relata, the number of n-ary relations between any relata is a combination $C^n_m = m!/\,(n!\,(m-n)!)$.

For example, the Aristotelian square of oppositions is a structure that includes $m = 4$ relata; hence there is a set of $C^2_4 = 4!/\,(2!\,(4\text{–}2)!) = 6$ binary relations between its relata: **AE**,**AO**,**AI**,**EO**,**EI**,**OI**.

A related problem was raised in [2]: how to display every such opposition between the classical (bivalent) binary sentences in one and the same geometric structure?

Binary sentences are of the form $p \circ q$, where $\circ$ is a binary connective. There is a total set of $2^{nm} = 2^{22} = 2^4 = 16$ such connectives in the classical or two-valued logic, where n stands for the number of truth-values (T for truth, F for falsehood) and m for the number of connected propositions (p and q).

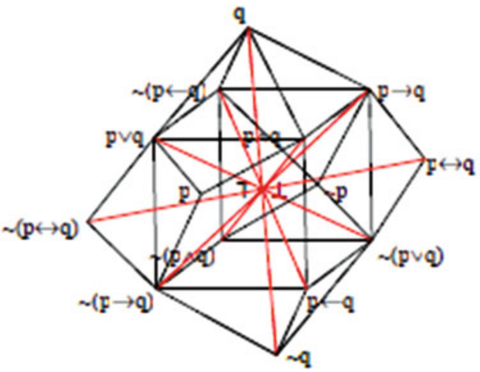
A partial solution has been proposed by [2] in the form of a hexadic structure. It is a repeated two-by-two connection of hexagons, each being related by two common vertices up and down and forming a whole DNA-shaped structure of ten elements (a)–(j).



(a) $p\wedge q$; (b) $p\leftrightarrow q$; (c) $\sim(p\vee q)$; (d) $\sim(p\wedge q)$; (e) $\sim(p\leftrightarrow q)$; (f) $p\vee q$; (g) $\sim(p\leftarrow q)$; (h) $\sim(p\rightarrow q)$; (i) $p\leftarrow q$; (j) $p\rightarrow q$

Albeit an extension of the single hexagon, the above structure is still incomplete since 6 among the 16 binary connectives are still missing there.

A solution has been found by [6] in order to introduce all binary connectives into one and the same structure while following the basic criterion of central symmetry for contradictories. The result is a tetraicosahedron of logical oppositions, central symmetry requiring a transition from 2D to 3D geometry. It is a very complex figure of $C^2_{16} = 16!/\,(2!\,(16–2)!) = 120$ binary relations including 8 contradictories, in which the special connectives of tautology and antilogy are located in the center of the structure. A common feature with the Seal of Solomon is the occurrence of special values in the center of the figure: tautology and antilogy, while the Seal has the sum of any opposed integers as its core value. The link between the latter degenerate connectives $\top,\bot$ and the arithmetic notion of sum will be explained in the final section.



The above geometric structure [5, 6] includes 6 squared faces; however, these are not Aristotelian squares because they fail to have some of the usual logical relations in them in such a 3D structure. The latter also includes 10 hexagons, whether regular or irregular. Some of these figures have been studied at length elsewhere, especially in [5–10], and it is not the purpose of the present paper to scrutinize the geometric features of logical oppositions. Rather, our point is to show how the link can be made between geometry and arithmetic through logic and with respect to the same issue: opposition.

4 Logic of Oppositions

The following logic of oppositions departs from mainstream systems by focusing on the concept of opposition instead of consequence. It is not a logical system endowed with a set of theorems; rather, its purpose is to give an abstract formal definition of logical relations including cases of opposition.

It is well-known that the truth-functional classical or two-valued logic is not able to characterize each of the logical relations of opposition: if any two sentences φ_1, φ_2 are said to be contraries (or subcontraries), what of the truth-value of φ_2 if φ_1 is false (or true)? The impossibility to treat oppositions as functions thus necessitates another formal device to deal with logical oppositions.

4.1 Opposition and Opposites

This has been developed in several respects [7–8–9], inspired by two previous formal devices: P^{Q}-semantics, introduced in [5]; Modal Quantified Algebra, elaborated in [10]. Let us consider the resulting general theory of oppositions. It consists of two main sections, namely, a semantic of the relations of opposition, and a complementary theory of opposites, which are relata formed by opposition-forming operators.

The basic semantics embracing both areas is a constructive Question-Answer Semantics (hereafter, **QAS**). It is an algebraic and *non-Fregean* semantics: in a Fregean semantics, logical values are “truth-values” (true, or false) corresponding to single non-structured objects and are assigned to only one category of objects, viz. propositions. In **QAS**, logical values are structured context-dependent objects and are assigned to any meaningful expression (not only sentences, but also concepts or individuals). In the case of sentential expressions, the meaning of any sentence is afforded by a finite sequence of answers to corresponding questions about the sentence. It results in an alternative coding of the logical values of any sentence φ_i: not T or F, but a bitstring of relative length. The process of valuation can be defined generally as follows:

For any meaningful expression φ, there are m^n logical values in a semantic with:

- n questions $\mathbf{Q}(\varphi) = \langle \mathbf{q}_1(\varphi), \ldots, \mathbf{q}_n(\varphi)\rangle$
- m sorts of answer $\mathbf{A}(\varphi) = \langle \mathbf{a}_1(\varphi), \ldots, \mathbf{a}_n(\varphi)\rangle$, every element $\mathbf{a}(\varphi)$ mapping into $\{m{-}1, \ldots, 0\}$.

The presentation of logical relations in the form of a question-answer game helps to approach an important problem concerning their cardinality, namely: how many logical oppositions can there be? Although it is usually said that the Aristotelian square is a set of four “oppositions”, a more scrutinized investigation will show that this is not so.

In the following logic of oppositions, only $m = 2$ sorts of answer are used as single bits: yes (symbol: 1), and no (symbol: 0). A corresponding semantics of classical oppositions consists in interpreting binary sentences in the form of sequences of 4-bits, each logical value standing for their Disjunctive Normal Form. The sense

$\mathbf{Q}(\varphi) = \langle \mathbf{q}_1(\varphi),\mathbf{q}_2(\varphi),\mathbf{q}_3(\varphi),\mathbf{q}_4(\varphi)\rangle$ of these binary sentences $\varphi = p\circ q$ is specified by an ordered set of 4 questions about their possible valuations: $\mathbf{q}_1(\varphi)$: "$v(p) = v(q) = \mathrm{T}$?"; $\mathbf{q}_2(\varphi)$: "$v(p) = \mathrm{T}$, $v(q) = \mathrm{F}$?"; $\mathbf{q}_3(\varphi)$: "$v(p) = \mathrm{F}$, $v(q) = \mathrm{T}$?"; $\mathbf{q}_4(\varphi)$: "$v(p) = v(q) = \mathrm{F}$?".

This results in a set of $2^4 = 16$ logical values $\mathbf{A}(\varphi) = \langle \mathbf{a}_1(\varphi),\mathbf{a}_2(\varphi),\mathbf{a}_3(\varphi),\mathbf{a}_4(\varphi)\rangle$, each of these characterizing one of the 16 binary sentences of two-valued logic.

$p \circ q = \varphi$	$\mathbf{A}(p \circ q) = \mathbf{A}(\varphi)$	$p \circ q = \varphi$	$\mathbf{A}(p \circ q) = \mathbf{A}(\varphi)$
$p\vee \sim p$	$1111 = \mathrm{T}$	$p \leftrightarrow q$	1001
$p \vee q$	1110	q	1010
$p \leftarrow q$	1101	$\sim q$	0101
$p \rightarrow q$	1011	$\sim(p \vee q)$	0001
$\sim(p \wedge q)$	0111	$\sim(p \leftarrow q)$	0010
p	1100	$\sim(p \rightarrow q)$	0100
$\sim(p \leftrightarrow q)$	0110	$p \wedge q$	1000
$\sim p$	0011	$\perp = p\wedge \sim p$	0000

Once the logical values are set out, it becomes possible to achieve a calculus of oppositions and opposites [7]. All related concepts are characterized by two interconnected logical forms.

On the one hand, oppositions are relations the form $\mathrm{Op}(\varphi_1,\varphi_2)$ and can be read as "φ_1 stands in an opposition of ... to φ_2". The generic relation $\mathrm{R} = \mathrm{Op}$ is depicted in all geometric structures we dealt with previously, Aristotle's square, Blanché's hexagon, and so on, with the relata φ_1,φ_2, and the relation Op, corresponding to vertices and lines.

On the other hand, opposites are opposition-forming operators mapping on expressions and forming non-identical expressions in a given set of logical values. The generic opposition-forming operator is of the form $\mathrm{op}(\varphi_1) = \varphi_2$. This includes its output value and can be read as "a ... of φ_1 is φ_2", so that any application of an operator op yields a corresponding relation from the set $\mathrm{Op} = \{\mathrm{CT},\mathrm{CD},\mathrm{SCT},\mathrm{NCD}\}$.

Let $\sqcap$ and $\sqcup$ be the Boolean operations of meet and join such that, for any single answer $\mathbf{a}_i(\varphi)$:

$$\mathbf{a}_i(\varphi_1) \sqcap \mathbf{a}_i(\varphi_2) = min\,(\mathbf{a}_i(\varphi_1),\ \mathbf{a}_i(\varphi_2))$$

$$\mathbf{a}_i(\varphi_1) \sqcup \mathbf{a}_i(\varphi_2) = max\,(\mathbf{a}_i(\varphi_1),\ \mathbf{a}_i(\varphi_2))$$

Then for every relation $\mathrm{Op}(\mathbf{A}(\varphi_1), \mathbf{A}(\varphi_2)) = \mathrm{Op}(\mathbf{A}(\varphi_1), \mathbf{A}(\mathrm{op}(\varphi_1)))$, we have the following distinctive valuations:

Contrariety: CT

$$\mathrm{Op}(\varphi_1, \varphi_2) = \mathrm{CT}(\varphi_1, \mathrm{ct}(\varphi_1)) \text{ iff}$$

$$\mathbf{A}(\varphi_1) \sqcap \mathbf{A}(\varphi_2) = \perp \text{ and } \mathbf{A}(\varphi_1) \sqcup \mathbf{A}(\varphi_2) \neq \mathrm{T}$$

Contradictoriness: CD

$$\mathrm{Op}\,(\varphi_1, \varphi_2) = \mathrm{CD}\,(\varphi_1, \mathrm{cd}\,(\varphi_1)) \ \text{ iff}$$

$$\mathbf{A}\,(\varphi_1) \sqcap \mathbf{A}\,(\varphi_2) = \perp \text{ and } \mathbf{A}\,(\varphi_1) \sqcup \mathbf{A}\,(\varphi_2) = \top$$

Subcontrariety: SCT

$$\mathrm{Op}\,(\varphi_1, \varphi_2) = \mathrm{SCT}\,(\varphi_1, \mathrm{sct}\,(\varphi_1)) \ \text{ iff}$$

$$\mathbf{A}\,(\varphi_1) \sqcap \mathbf{A}\,(\varphi_2) \neq \perp \text{ and } \mathbf{A}\,(\varphi_1) \sqcup \mathbf{A}\,(\varphi_2) = \top$$

Non-Contradictoriness: NCD

$$\mathrm{Op}\,(\varphi_1, \varphi_2) = \mathrm{NCD}\,(\varphi_1, \mathrm{ncd}\,(\varphi_1)) \ \text{ iff}$$

$$\mathbf{A}\,(\varphi_1) \sqcap \mathbf{A}\,(\varphi_2) \neq \perp \text{ and } \mathbf{A}\,(\varphi_1) \sqcup \mathbf{A}\,(\varphi_2) \neq \top$$

It is worthwhile to note that the well-known relation of subalternation SB does not appear in the above list: its valuation cannot be made distinct from non-contradictoriness, and another way to individuate SB goes by the following definition:

Subalternation: SB

$$\mathrm{Op}\,(\varphi_1, \varphi_2) = \mathrm{SB}\,(\varphi_1, \mathrm{sb}\,(\varphi_1)) \ \text{ iff}$$

$$\mathbf{A}\,(\varphi_1) \sqcap \mathbf{A}\,(\varphi_2) = \mathbf{A}\,(\varphi_1) \ \text{ and } \mathbf{A}\,(\varphi_1) \sqcap \mathbf{A}\,(\varphi_2) \neq \mathbf{A}\,(\varphi_2)$$

Subalternation has two peculiar features. For one thing, it has a converse relation of superalternation $\mathrm{SB}^{-1} = \mathrm{SP}$, by reverting the relation order between its relata φ_1 and φ_2. Also, it is not characterized in the same way as the previous relations: these are defined by a combinatorial game of minimal and maximal valuations, $min(\varphi_i) = \perp$ and $max(\varphi_i) = \top$, depending upon whether their meet and join values result or not in either of these extreme valuations. There is still one such combination that has never been introduced in the usual theory of opposition: *unconnectedness* [10, 11], which imposes absolutely no constraint on the relata. However, this further relation cannot be defined by the sole terms of opposition and requires another sort of questioning: not only about difference, but also about identity.

4.2 Identities and Differences

The above definitions show that the concept of subalternation does not answer to the same set of questions as the other four relations. Following [11], one can see the whole as a set of two distinctive questionings about modes of difference and identity.

On the one side, the sense of logical *oppositions* is given by two meta-questions about *differences* in valuations:

$$\mathbf{Q}\,(\mathrm{Op}\,(\varphi_1,\varphi_2)) = \langle \mathbf{q}_1(\mathrm{Op}\,(\varphi_1,\varphi_2))\,,\ \mathbf{q}_2(\mathrm{Op}\,(\varphi_1,\varphi_2))\rangle$$

$$\mathbf{q}_1(\mathrm{Op}\,(\varphi_1,\varphi_2)):\ \text{“}\mathbf{a}_i(\varphi_1) = 0 \Rightarrow \mathbf{a}_i(\varphi_2)\ \ 1?\text{”}$$

$$\mathbf{q}_2(\mathrm{Op}\,(\varphi_1,\varphi_2)):\ \text{“}\mathbf{a}_i(\varphi_1) = 1 \Rightarrow \mathbf{a}_i(\varphi_2)\ \ 0?\text{”}$$

Contradiction, contrariety, subcontrariety and unconnectedness are rendered by these questions: $\mathbf{A}\,(\mathrm{CD}\,(\varphi_1,\varphi_2)) = \langle 1,1\rangle$, $\mathbf{A}\,(\mathrm{CT}\,(\varphi_1,\varphi_2)) = \langle 0,1\rangle$, $\mathbf{A}\,(\mathrm{SCT}\,(\varphi_1,\varphi_2)) = \langle 1,0\rangle$, and $\mathbf{A}\,(\mathrm{NCD}\,(\varphi_1,\varphi_2)) = \langle 0,0\rangle$.

On the other side, the sense of logical implications is given by two meta-questions about *identities* in valuation:

$$\mathbf{Q}\,(\mathrm{Imp}\,(\varphi_1,\varphi_2)) = \mathbf{q}_1\langle(\mathrm{Imp}\,(\varphi_1,\varphi_2)\,),\ \mathbf{q}_2(\,\mathrm{Imp}\,(\varphi_1,\varphi_2))\rangle$$

$$\mathbf{q}_1(\mathrm{Imp}\,(\varphi_1,\varphi_2)):\ \text{“}\mathbf{a}_i(\varphi_1) = 1 \Rightarrow \mathbf{a}_i(\varphi_2)\ \ 1?\text{”}$$

$$\mathbf{q}_2(\mathrm{Imp}\,(\varphi_1,\varphi_2)):\ \text{“}\mathbf{a}_i(\varphi_1) = 0 \Rightarrow \mathbf{a}_i(\varphi_2)\ \ 0?\text{”}$$

Subalternation is rendered transparent by this set of questionings, highlighting its peculiarity with respect to the preceding relations of opposition: $\mathbf{A}\,(\mathrm{SB}\,(\varphi_1,\varphi_2)) = \langle 1,0\rangle$. Special headings are assigned in [11] to the other three identity relations: bi-implication for $\mathbf{A}\,(\mathrm{Imp}\,(\varphi_1,\varphi_2)) = \langle 1,1\rangle$, right implication for $\mathbf{A}\,(\mathrm{Imp}\,(\varphi_1,\varphi_2)) = \langle 0,1\rangle$, and non-implication for $\mathbf{A}\,(\mathrm{Imp}\,(\varphi_1,\varphi_2)) = \langle 0,0\rangle$.

On the one hand, it quickly appears that the 16 binary connectives result from a combination of the above four questionings, thus accounting for their cardinality. However, the questioning giving rise to the sixteen Disjunctive Normal Forms includes notions of compossibility and asks if any two sentences *can* be true or false together or not. At the same time, the meta-questioning that has just been used for relations of difference and opposition deals with necessary relations, having to do with metalogical possibilities and necessities.

On the other hand, every logical relation results from a combination of identities and differences, viz. between the bits of a given expression and those of its relatum. In this sense, it can be said that *every logical relation is a logical opposition*, once opposition is defined in **QAS** as a Boolean difference of relative degree between bitstrings. It has been argued in [11] that subalternation is not a relation of opposition at all, as depicted by the above distinction between questionings about differences and identities (subalternation proceeds from the latter). Admittedly, no logical relation is to be properly called an "opposition" if its relata are compatible with each other. But since this also holds for subcontrariety and unconnectedness, our reply is that a calculus of opposites helps to show that any different relata can be formed by means of *contradictory* oppositions between some of their single bits. Thus, a constructive definition of relations entails that different relata may be opposed in some way to each other, whilst being possibly compatible with respect to their whole bitstrings.

4.3 Negations

It has been argued in [7] that each of these opposition-forming operators is a special case of *negation*: a difference-forming operator that turns a given value into another one. However, these operators are not functions in the usual sense of the word. That is to say, most of these are not one-to-one mappings turning one input value into another single one. Rather, the opposition-forming operators op are mostly one-many mappings: to one input value corresponds more than one output value.

On the one hand, contradiction is the only extensional opposition-forming operator. In other words, the operator $\text{op}_x = \text{cd}$ is a bijection. This is shown arithmetically by the fact that there is only one integer $\mathbf{A}(\varphi_1) + \mathbf{A}(\varphi_2) = max(\varphi_i)$. By derivation, only classical negation plays the role of a contradiction-forming operator.

On the other hand, contrariety-forming operators can be associated to what is called a *paracomplete* negation, whilst subcontrariety-forming operators behave like *paraconsistent* negation. Such a correspondence relies upon the nature of consequence from the perspective of logical oppositions. As far as one can see, subalternation is the best candidate to render the notion of consequence in a Boolean version of logical oppositions. Indeed, every premise is such that it entails a number of consequences, and this number is dependent upon the ways every yes-answer of the premise can be contained within them. Taking the case of conjunction, $p \wedge q$, Boolean oppositions combined with an informal definition of subalternation helps to show that binary conjunction has as many consequences as logical contraries. According to [1], subaltern expressions correspond to contradictories of contraries.

It has also been shown in [8, 11] that the number of subalterns and contrary is relative to the bitstring $\mathbf{A}(\varphi_i)$ characterizing any expression φ_i. Thus, not every expression has contraries or subcontraries, by virtue of its Boolean value, whereas every expression has its own contradictory. In this way, we get information both about the number of logical opposites and the nature of any opposition between arbitrary relata.

Moreover, a general calculus of oppositions can be achieved through two basic operators, cd and ct, in addition to the method of substitution. Then, for every expression φ_i:

$$\text{sb}(\varphi_i) = \text{cd}(\text{ct}(\varphi_i))$$

$$\text{sct}(\varphi_i) = cd(\text{sp}(\varphi_i))$$

$$\text{sp}(\varphi_\text{i}) = \text{ct}(\text{cd}(\varphi_\text{i}))$$

However, the behavior of op as a one-many operator is troublesome from a computational point of view: how to determine the value of, for example, the contrary of a contradictory expression, if there may be more than one output value? The next section helps to clarify these issues.

5 Arithmetic of Oppositions

In the preceding algebraic *logic* of oppositions, the logical import of relations relies upon a Boolean calculus of expressions. Again, in the general logical form

$$\sum(\varphi)_{[k]} = \sigma_1(\varphi)_{[k]} + \cdots + \sigma_{\mathrm{n}}(\varphi)_{[k]},$$

$$\text{where } \sigma_i(\varphi)_{[k]} = (k_{i-1} = \mathbf{a}_i\,(\varphi))$$

Arithmetic counterparts of the logical relation Op, and its corresponding logical operators op, are the relation of difference $\oplus$ and its corresponding operators of differentiation $\pm$ in $\mathbf{N}^*$. In addition, just as we expressed the view that oppositions are formed by means of opposition-forming operators, we also can say that, for any pair of integers $\Sigma(\varphi_i)_{[k]}$, $\Sigma(\varphi_j)_{[k]}$:

$$\oplus\left(\sum(\varphi_i)_{[k]}, \sum(\varphi_j)_{[k]}\right) \times \oplus\left(\sum(\varphi_i)_{[k]}, \pm\left(\sum(\varphi_i)_{[k]}\right)\right).$$

In other words, the relation $\oplus\,(\Sigma(\varphi_i)_{[k]},\Sigma(\varphi_j)_{[k]})$ stands for an arithmetic difference expressed by an integer of $\mathbf{N}^*$ such that $(\Sigma(\varphi_i)[k]) \neq \Sigma(\varphi_j)_{[k]}$. It corresponds to Leibniz's 2-base number and its decimal coding, such that

$$\oplus\left(\Sigma(\varphi_i)_{[2]}, \Sigma(\varphi_j)_{[2]}\right) = 2^{n-1} \times \left(\mathbf{a}_1\,(\varphi_i) - \mathbf{a}_1\,(\varphi_j)\right) + \cdots + 2^0 \times \left(\mathbf{a}_n\,(\varphi_i) - \mathbf{a}_n\,(\varphi_j)\right)$$

The set of binary sentences and their arithmetic value can be listed as follows.

$p \circ q = \varphi$	$\mathbf{A}\,(p \circ q)$	$\sum(p \circ q)$	$p \circ q = \varphi$	$\mathbf{A}\,(p \circ q)$	$\sum(p \circ q)$
$p\vee \sim p$	1111	15	$p \leftrightarrow q$	1001	9
$p \vee q$	1110	14	q	1010	10
$p \leftarrow q$	1101	13	$\sim q$	0101	5
$p \rightarrow q$	1011	11	$\sim(p \vee q)$	0001	1
$\sim(p \wedge q)$	0111	7	$\sim(p \leftarrow q)$	0010	2
p	1100	12	$\sim(p \rightarrow q)$	0100	4
$\sim(p \leftrightarrow q)$	0110	6	$p \wedge q$	1000	8
$\sim p$	0011	3	$p\wedge \sim p$	0000	0

For example, let $\varphi_1 = p\wedge q$, $\varphi_2 = p\vee q$. On the one hand, binary sentences φ_1 and φ_2 are related by a relation of subalternation Op($\mathbf{A}(\varphi_1)$,$\mathbf{A}(\varphi_2)$) = SB(1000,1110). On the other hand, the operation turning the arithmetic value of the first relatum into the second one is –6:

$$\begin{aligned}\oplus\left(\Sigma(\varphi_1)_{[2]},\Sigma(\varphi_2)_{[2]}\right) &= \left(2^{4-1}\times(1–1)\right)+\left(2^{3-1}\times(0–1)\right)+\left(2^{2-1}\times(0–1)\right)+\left(2^{1-1}\times(0–0)\right)\\ &= (8\times 0)+(4\times(–1))+(2\times(–1))+(1\times 0)\\ &= 0+(–4)+(–2)+0\\ &= –6\end{aligned}$$

The second step of our analogy consists in giving an arithmetic sense to the logical concept of opposition.

5.1 Arithmetical Oppositions

It may seem impossible to talk about oppositions between integers. Although I have already shown that concepts can be opposed to each other without referring to truth-values, it hardly makes sense to say that, e.g., 7 is opposed to 5 by some relation of opposition. For this purpose, an additional arithmetic criterion is required, namely, the introduction of maximal or minimal integers, just as tautologies and antilogies play this role of extreme values in algebraic logic. Thus for every sequence of n items φ_i in a k-base system:

- its maximum or maximal value is $\Sigma(\varphi_i) = k^n–1$
  Example: if $k = 2$ and $n = 4$, then $max(\Sigma(\varphi_i)) = 2^4–1 = 15$
- its minimum or minimal value is $\Sigma(\varphi_i) = 0$

Maximal value helps to understand why every vertex has only one contradictory: there is only one $\Sigma(\varphi_2)$ such that $\Sigma(\varphi_1) + \Sigma(\varphi_2) = \max(\Sigma(\varphi_i))$, and $\mathrm{Op}(\Sigma(\varphi_1), \Sigma(\varphi_2)) = \mathrm{CD}(\Sigma(\varphi_1), \Sigma(\varphi_2))$.

A third notion has to be introduced in order to keep the 2-base system of Boolean bits: *summand*, which denotes every single component or term $\sigma(\varphi_i)$ of the addition $\Sigma(\varphi_i)$. This item helps to define the logical opposition between two arithmetic terms through their characteristic bitstrings, thereby focusing on the essential role of Boolean algebra to connect logic and arithmetic.

Arithmetic oppositions can now be defined with the help of the above three main concepts of maximal value, minimal value, and summand.

For every logical relation $\mathrm{Op}(\Sigma(\varphi_1),\Sigma(\varphi_2))$ between integers, we have:

Contrariety

$$\mathrm{Op}(\Sigma(\varphi_1)\,,\Sigma(\varphi_2)) = \mathrm{CT}(\Sigma(\varphi_1)\,,\mathrm{ct}(\Sigma(\varphi_1)))\ \text{iff}$$

- $\sigma\,(\varphi_1) \neq 0 \Rightarrow\!/\ \sigma\,(\varphi_2) = 0$
- $\sigma\,(\varphi_1) = 0 \Rightarrow\!/\ \sigma\,(\varphi_2) \neq 0$

Example: CT($p \wedge q, p \wedge \sim q$)

$$\Sigma\,(p \wedge q) = 8 = 8 + 0 + 0 + 0;\ \ \Sigma\,(p \wedge \sim q) = 0100 = 0 + 4 + 0 + 0.$$

Contradictoriness

$$\mathrm{Op}\,(\Sigma\,(\varphi_1)\,,\Sigma\,(\varphi_2)) = \mathrm{CD}\,(\Sigma\,(\varphi_1)\,,\mathrm{cd}\,(\Sigma\,(\varphi_1)))\ \text{ iff}$$

- $\sigma\,(\varphi_1) \neq 0 \Rightarrow \sigma\,(\varphi_2) = 0$
- $\sigma\,(\varphi_1) = 0 \Rightarrow \sigma\,(\varphi_2) \neq 0$

Example: CD($p \wedge q$,$\sim(p \wedge q)$)

$$\Sigma\,(p \wedge q) = 8 = 8 + 0 + 0 + 0; \Sigma\,(\sim(p \wedge q)) = 7 = 0 + 4 + 2 + 1.$$

Subcontrariety

$$\mathrm{Op}\,(\Sigma\,(\varphi_1)\,,\Sigma\,(\varphi_2)) = \mathrm{SCT}\,(\Sigma\,(\varphi_1)\,,\mathrm{sct}\,(\Sigma\,(\varphi_1)))\ \text{ iff}$$

- $\sigma\,(\varphi_1) \neq 0 \Rightarrow/\ \sigma\,(\varphi_2) = 0$
- $\sigma\,(\varphi_1) = 0 \Rightarrow \sigma\,(\varphi_2) \neq 0$

Example: SCT($p \vee q$, $\sim(p \wedge q)$)

$$\Sigma\,(p \wedge q) = 14 = 8 + 4 + 2 + 0;\ \ \Sigma\,(\sim(p \wedge q)) = 7 = 0 + 4 + 2 + 1.$$

Non-contradictoriness

$$\mathrm{Op}\,(\Sigma\,(\varphi_1)\,,\Sigma\,(\varphi_2)) = \mathrm{NCD}\,(\Sigma\,(\varphi_1)\,,\mathrm{ncd}\,(\Sigma\,(\varphi_1)))\ \text{ iff}$$

- $\sigma\,(\varphi_1) = 0 \Rightarrow /\sigma\,(\varphi_2) \neq 0$
- $\sigma\,(\varphi_1) \neq 0 \Rightarrow /\sigma\,(\varphi_2) = 0$

Example: NCD(p, q)

$$\Sigma\,(p \wedge q) = 12 = 8 + 4 + 0 + 0;\ \ \Sigma\,(p \vee q) = 6 = 0 + 4 + 2 + 0.$$

Subalternation

$$\mathrm{Op}\left(\Sigma\left(\varphi_1\right), \Sigma\left(\varphi_2\right)\right) = \mathrm{SB}\left(\Sigma\left(\varphi_1\right), \mathrm{sb}\left(\Sigma\left(\varphi_1\right)\right)\right) \text{ iff}$$

- $\sigma\left(\varphi_1\right) \neq 0 \Rightarrow \sigma\left(\varphi_2\right) \neq 0$
- $\sigma\left(\varphi_1\right) = 0 \Rightarrow /\sigma\left(\varphi_2\right) \neq 0$

Example: SB($p \wedge q, p \vee q$)

$$\Sigma\left(p \wedge q\right) = 8 = 8 + 0 + 0 + 0;\ \Sigma\left(p \vee q\right) = 14 = 8 + 4 + 2 + 0.$$

In this way, arithmetic oppositions can make sense thanks to a correspondence with logical values in **QAS** and their arithmetic properties. As such, the third and final stage of our analogy concerns the link between arithmetic and geometry of oppositions.

5.2 Analytic Geometry of Oppositions

The analogy between arithmetic and geometry is made in the light of what Descartes devised under the heading of analytic geometry. Its present version includes three main components, namely: coordinates, identity, and opposition.

Just as in analytic geometry, expressions like points are to be defined by *coordinates* in a space. Let A and B be any two points in a 2-dimensional vector space; then $\mathrm{AB}^{\rightarrow} = (\mathrm{x_B}-\mathrm{x_A};\ \mathrm{y_B}-\mathrm{y_A})$. In our arithmetic of oppositions, coordinates are bitstrings and vectors are turned into constants arithmetic functions $\oplus$ between integers.

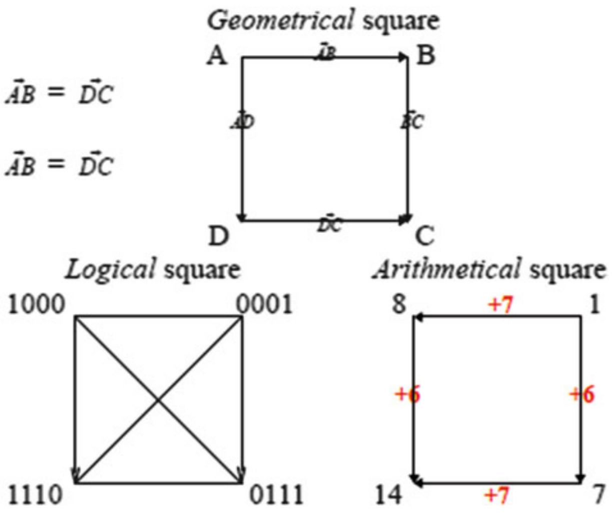
Example: let ± for +2, $\Sigma(\varphi) = 3$; hence $\pm\left(\varphi\right) = 3 + 2 = 5$.

Furthermore, *identity* has a precise definition both in analytic geometry and arithmetic.

Let A,B,C be any three points in a vector space. Then the corresponding vectors $\mathrm{AB}^{\rightarrow} = u^{\rightarrow}$ and $\mathrm{BC}^{\rightarrow} = v^{\rightarrow}$ are *identical* in a given space if and only if they have:

- the same direction
- the same sense
- the same norm

In the structured geometry of oppositions, the first criterion means that any two lines relating vertices are parallel with each other. The second criterion has one logical counterpart: subalternation, where the arrow between vertices indicates the sense of entailment from premise to conclusion. The third criterion cannot be explained in logical terms: a norm is a distance that cannot be rendered in a logical space, and only an arithmetization of oppositions can do this with the help of an operator of differentiation ± .



Finally, the idea of opposite vectors is the most important and proceeds as the converse of identity. However, “opposite” is to be made in a particular sense of *maximal* opposition.

Two vectors $u^{\rightarrow}$ and $v^{\rightarrow}$ are *opposed* to each other if and only if they have opposite coordinates, such as: $u^{\rightarrow} = -v^{\rightarrow}$.

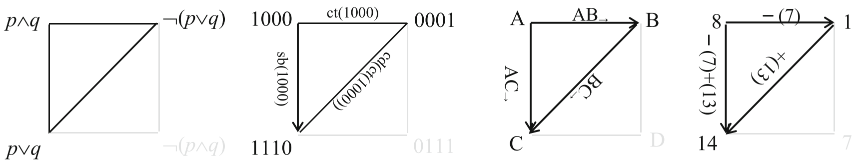
If such geometric relations and properties are verified in a logical space of oppositions, modulo its Boolean transcription of logical values in **QAS**, then the same features of analytic geometry should equally hold in our arithmetic of oppositions. An interesting result can be obtained in this respect, concerning the calculus of oppositions already suggested in [7] but limited by the inner constraints of one-many operators.

Another relevant result for our own purposes is the application of the so-called Chasles' relation in analytic geometry.

For any points A,B,C in an affine space, we have:

$$AC^{\rightarrow} = AB^{\rightarrow} + BC^{\rightarrow}$$

The same properties can be observed in the 2D geometry of oppositions, as illustrated by the following half-squares of opposition.



Given the definitions of coordinates and vectors given above, the correspondence of diagrams shows that Chasles' relation applies to 2D geometries only. That it is, to Aristotle's square, but also Blanché's hexagon, Buridan's octagon, and so on. However, a reference to Pellissier's tetraicosahedron also shows that it does not hold for 3D geometries: the identity criteria of vectors are lost once the third dimension of depth is introduced between vertices. For this reason, we limit our correspondence result to 2D geometries for the same reason Chasles' relation holds for affine spaces only.

Thanks to this arithmetization of logical oppositions, it is also possible to identify any logical relation between expressions. The one-many operators of logical oppositions made this impossible, whereas a calculus of single values helps to define any sort of relation on the basis of its relata. So, the correspondence also confirms the plurality of logical negations as expounded in [7] from a purely Boolean perspective of logical negations.

Echoing the previous picture of oppositions as a set of partial identities and differences, it is easily seen that not every *double negation* amounts to mere affirmation in a logical calculus of oppositions: far from that, given the occurrence of double mixed negations like, e.g., ct(cd(φ_i)). Thus, for any opposition-forming operators op_x and op_y:

$$\mathrm{op}_x\mathrm{op}_y\,(\varphi) = \varphi \text{ iff } x = y = \mathrm{cd}$$

This does not hold for the other logical oppositions such as contrariety. A contrary of a contrary of φ may be (or not) another contrary of φ, and the same holds for every one-many mapping whereas contradictoriness proceeds as a genuine function, i.e. a one-one operator.

More importantly, arithmetic helps to overcome the obstacle of one-many operators by specifying the Boolean calculus of opposites: although there is not only one contrary, for example, an arithmetic calculus helps to operate between single identified opposites from particular relata.

To give an example of such a calculus, let us take any binary sentence of Pellissier's tetraicosahedron as a starting point; then let us check the final value of its opposite through a finite sequence of operations between the 16 integers from 0 to 15.

Let $\mathbf{A}(p \wedge q) = 1000 = \Sigma(p \wedge q) = 8$. Then:

$8–5 = 3 = 0 + 0 + 2 + 1 = \mathbf{A}(0011) = \Sigma(\neg p)$, $0011 \subseteq \mathrm{ct}(1000)$;
therefore, $\mathrm{Op}(8,3) = \mathrm{CT}(p \wedge q, \neg q)$.
$3 + 11 = 14 = 8 + 4 + 2 + 0 = \mathbf{A}(1110) = \Sigma(p \vee q)$, $1110 \subseteq \mathrm{sct}(0011)$;
therefore $\mathrm{Op}(3,14) = \mathrm{SCT}(p \wedge q, \neg q)$.
$14–2 = 12 = 8 + 4 + 0 + 0 = \mathbf{A}(1100) = \Sigma(p)$, $1100 \subseteq \mathrm{sp}(1110)$;
therefore, $\mathrm{Op}(14,12) = \mathrm{SP}(p \vee q, p)$.
$8–5 + 11\text{-}2 = 12 = 8 + 4 + 0 + 0 = \mathbf{A}(1100) = \Sigma(p)$, $1100 \subseteq \mathrm{sb}(1000)$;
therefore, $\mathrm{Op}(8,12) = \mathrm{SB}(p \wedge q, p)$.

To summarize:

- p is a subaltern of $p \wedge q$: $\mathrm{sb}(p \wedge q) \supseteq p$.
- p is a contrary of a subcontrary of a superaltern of $p \wedge q$: $\mathrm{ct}(\mathrm{sct}(\mathrm{sp}(p \wedge q))) \supseteq p$.

It is worthwhile recalling that the above operations are not equations $x = \mathrm{op}(y)$ between single values but, rather, inclusions $x \subseteq \mathrm{op}(x)$ between singletons and sets. This is the case, because there is not only one subaltern to binary conjunction, and the above sequence is just one possible way to move from an expression to one of its subalterns through arithmetic extensions.

6 Conclusion

In this paper, I have proposed an arithmetization of logical oppositions, based on a structural identity between three domains: logic, geometry, and arithmetic. Moving from one domain to another one through the central device of Boolean algebra is possible by assuming that any operation of a given domain finds its counterpart in the other.

This correspondence result can be summarized in the following table:

	Logic	Geometry	Arithmetic
(1)	φ_1	A	$x \in \mathrm{N}^+$
(2)	$\varphi_2 \subseteq \mathrm{op}\,(\varphi_1)$	B	$y \in \mathrm{N}^+$
(3)	$\mathrm{Op}(\varphi_1, \varphi_2)$	[AB]	$\oplus(x,y)$
(4)	op	$u^{\rightarrow} = \mathrm{AB}^{\rightarrow}$	$\pm(x) = y$

According to this table:

1. expressions are like different points in geometry and different integers in arithmetic;
2. opposites are like differentiated points in geometry and differentiated integers in arithmetic;
3. opposition is like a segment in geometry and a numerical difference in arithmetic;
4. opposition-forming operators are like norms in analytic geometry and constant operations of addition or subtraction in arithmetic.

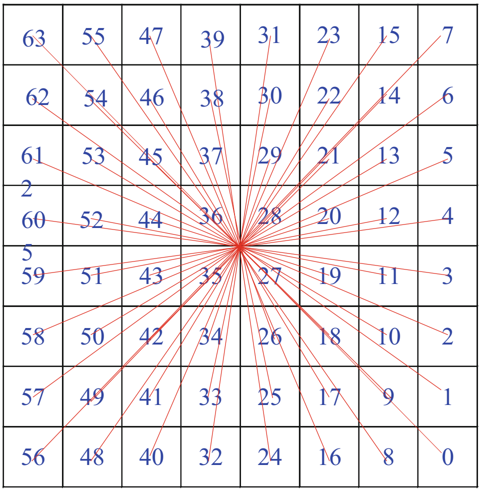
The whole process may be seen as a legacy of Leibniz's achievements in logic. That is, despite its limited results, the algebraic logic of oppositions is an instantiation of what Leibniz took as a *lingua characteristic*. Each relation of opposition is definite in a finite set of expressions, and not every element of a given language can be compared to any other in terms of oppositions. Likewise, the proposed arithmetic may be understood in terms of a *calculus ratiocinator*, by computing the logical relation of expression through their arithmetic values.

A number of further problems have not been addressed in the present paper and should be developed in later works. For instance, how to construct a general algebraic logic for any sorts of sentences beyond the sole binary sentences $p \circ q$ and its 16 elements? This will depend upon the capacity of **QAS** to characterize every meaningful expression in the form of a bitstring. This has been achieved with modal expressions [8–10], but only under the proviso that their sense be given by a different set of questionings.

Furthermore, the above arithmetic calculus relies upon a decimal coding of Boolean values. The underlying game of yes-no answers thereby assumed a bivalent algebra {1,0}. What of the meaning of expressions in non-Boolean algebras? If bivalence is rendered by yes-no answers 1-0 and results in a 2-base number system, then many-valuedness amounts to a higher set of 1, …, k answers in **QAS** (where $k > 2$) and should require a corresponding k-base system.

Appendix: A Constructive Geometry of Logical Relations

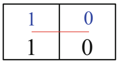
In the first section of the paper, a historical reference has been made to Shao Wong's ordering of the 64 hexagrams. Its striking feature is that it also respects the central symmetry of contradictory oppositions between the Boolean bitstrings—and their corresponding blue integers, here below, as is the case in all contemporary gatherings of logical geometry.



63	55	47	39	31	23	15	7
62	54	46	38	30	22	14	6
61 2	53	45	37	29	21	13	5
60 5	52	44	36	28	20	12	4
59	51	43	35	27	19	11	3
58	50	42	34	26	18	10	2
57	49	41	33	25	17	9	1
56	48	40	32	24	16	8	0

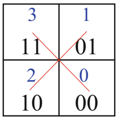
We propose in the following a similar constructive representation of logical oppositions: all are decreasing quadrangles, of length L and width l. Each progression of a given 2^n quadrangle consists in duplicating it either horizontally (from left to right) when n is odd, or vertically (from top to bottom) when n is even. The resulting figure is either a rectangle, such that $L = 2\,l$ whenever n is odd, or a square, such that $L = l$ whenever n is even. Each quadrangle is a complete set of bitstrings from the minimal value 0 to the maximal value $2^n - 1$, and the new ordering also preserves the properties of vectors (except Chasles' relation).

$2^1 = 2$



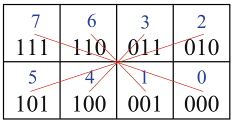
1 1	0 0

$2^2 = 4$



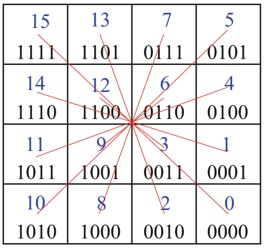
3 11	1 01
2 10	0 00

$2^3 = 8$



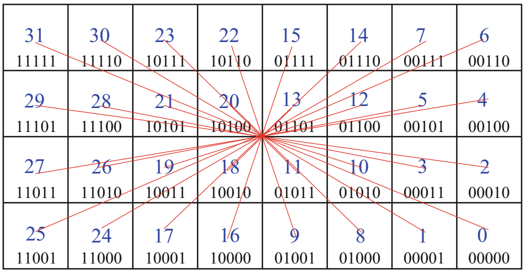
7 111	6 110	3 011	2 010
5 101	4 100	1 001	0 000

$2^4 = 16$



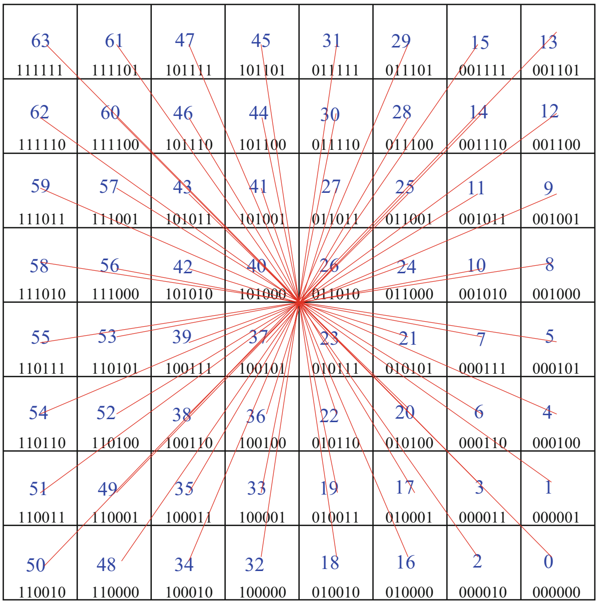
15 1111	13 1101	7 0111	5 0101
14 1110	12 1100	6 0110	4 0100
11 1011	9 1001	3 0011	1 0001
10 1010	8 1000	2 0010	0 0000

$2^5 = 32$



31 11111	30 11110	23 10111	22 10110	15 01111	14 01110	7 00111	6 00110
29 11101	28 11100	21 10101	20 10100	13 01101	12 01100	5 00101	4 00100
27 11011	26 11010	19 10011	18 10010	11 01011	10 01010	3 00011	2 00010
25 11001	24 11000	17 10001	16 10000	9 01001	8 01000	1 00001	0 00000

$2^6 = 64$



63 111111	61 111101	47 101111	45 101101	31 011111	29 011101	15 001111	13 001101
62 111110	60 111100	46 101110	44 101100	30 011110	28 011100	14 001110	12 001100
59 111011	57 111001	43 101011	41 101001	27 011011	25 011001	11 001011	9 001001
58 111010	56 111000	42 101010	40 101000	26 011010	24 011000	10 001010	8 001000
55 110111	53 110101	39 100111	37 100101	23 010111	21 010101	7 000111	5 000101
54 110110	52 110100	38 100110	36 100100	22 010110	20 010100	6 000110	4 000100
51 110011	49 110001	35 100011	33 100001	19 010011	17 010001	3 000011	1 000001
50 110010	48 110000	34 100010	32 100000	18 010010	16 010000	2 000010	0 000000

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Groups, Not Squares: Exorcizing a Fetish

Walter Carnielli

Abstract I argue that the celebrated Square of Opposition is just a shadow of a much deeper relationship on duality, complementarity, opposition and quaternality expressed by algebraic means, and that any serious attempt to make sense of squares and cubes of opposition must take into account the theory of finite groups. By defining a group as *triadic* if all its elements, other than the identity, have order 3, I show that a natural notion of *triality group* acting on three-valued structures emerges, generalizing the intuitions of duality and quaternality.

Keywords Boolean groups • Group theory • Square of opposition • Triadic groups

Mathematics Subject Classification (2000) Primary 03A05; Secondary 05C25

1 Contra Quadratus?

In their textbook [12], when discussing duality and commutativity, Halmos and Givant make explicit something which should be obvious, but which is rather embarrassing for the majority of philosophers, mathematicians and logicians: is the dual of a Boolean polynomial that represents a proposition in classical logic[1] such as $p \vee q$, the polynomial $p \wedge q$, the polynomial $\neg p \vee \neg q$, or the polynomial $\neg p \wedge \neg q$? All they comply a form of duality, complementarity, or opposition. To make things clear for algebraic considerations, given a polynomial $f(p, q)$ in two variables, they define the *complement* of $f(p, q)$ (sometimes also called the *external negation* of $f(p, q)$) as $\neg f(p, q)$, the *contradual* of $f(p, q)$ (sometimes also called the *internal negation* of $f(p, q)$) as $f(\neg p, \neg q)$, and the *dual of* $f(p, q)$ as $\neg f(\neg p, \neg q)$. Thus duality, in a proper sense, is the composite effect of internal and external negations.

These three notions are intimately related, to a point that composing any two of them defines the third: what happens, as they comment, is that there is a group of order four acting in the set of propositions, not a group of order two: this group is precisely the famous *Vierergruppe V* proposed by Felix Klein in 1884 (also called Klein 4-group), as

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[1] A Boolean polynomial is the analogous of an ordinary polynomial, employing a finite number of Boolean operations $\wedge$ and $\vee$ on a finite number of elements in a Boolean algebra.

J.-Y. Béziau, G. Basti (eds.), *The Square of Opposition: A Cornerstone of Thought*,
Studies in Universal Logic, DOI 10.1007/978-3-319-45062-9_14

we shall see. Thus, naive duality has no place in the deep relationship between algebra and logic: what holds is quaternality, as already remarked by Gottschalk in [10]. This is what lies behind De Morgan laws and behind many results of universal Boolean algebra, geometry, topology, and several other areas.

Depicting this from a squared perspective, we obtain the following figure, which we may call the Square of Quaternality:

$$\begin{tikzcd}
f(p,q) \arrow[r, leftrightarrow, "\text{contraduals}"] \arrow[d, leftrightarrow, "\text{duals}" description] \arrow[dr, leftrightarrow, "\text{complements}" description] & f(\neg p,\neg q) \arrow[d, leftrightarrow, "\text{duals}" description] \arrow[dl, leftrightarrow] \\
\neg f(\neg p,\neg q) \arrow[r, leftrightarrow, "\text{contraduals}"'] & \neg f(p,q)
\end{tikzcd}$$

The same quaternality relations hold of course for quantifiers, not only in classical predicate logic but also for most modal logics and for generalized quantifiers (as for instance for the quantifiers ‘most’, ‘many’, ‘rarely’, as treated in [4], as much as their internal logics are endowed with a negation sharing the relevant features of classical negation):

$$\begin{tikzcd}
Q \arrow[r, leftrightarrow, "\text{contraduals}"] \arrow[d, leftrightarrow, "\text{duals}" description] \arrow[dr, leftrightarrow, "\text{complements}" description] & Q- \arrow[d, leftrightarrow, "\text{duals}" description] \arrow[dl, leftrightarrow] \\
-Q- \arrow[r, leftrightarrow, "\text{contraduals}"'] & -Q
\end{tikzcd}$$

where Q is $\forall$, $\exists$, $\Box$, $\Diamond$ or a generalized quantifiers (as the ones in [4], and several other). Thus, for instance, it holds:

$$\begin{tikzcd}
\forall \arrow[r, leftrightarrow, "\text{contraduals}"] \arrow[d, leftrightarrow, "\text{duals}" description] \arrow[dr, leftrightarrow, "\text{complements}" description] & \forall\neg \arrow[d, leftrightarrow, "\text{duals}" description] \arrow[dl, leftrightarrow] \\
\exists \arrow[r, leftrightarrow, "\text{contraduals}"'] & \neg\exists
\end{tikzcd}$$

and

$$\begin{tikzcd}
\Box \arrow[rr, leftrightarrow, "\text{contraduals}"] \arrow[dd, leftrightarrow, "\text{duals}"'] \arrow[ddrr, leftrightarrow] & & \Box\neg \arrow[dd, leftrightarrow, "\text{duals}"] \arrow[ddll, leftrightarrow] \\
& \text{complements} & \\
\Diamond \arrow[rr, leftrightarrow, "\text{contraduals}"'] & & \neg\Diamond
\end{tikzcd}$$

More significant for our purposes, Q can also be a *restricted quantifier*, as for instance, as the ones usually employed in set theory: $(\forall x \in S(x))P(x)$ and $(\exists x \in S(x))P(x)$, defined respectively by $(\forall x)(S(x) \to P(x))$ and $(\exists x)(S(x) \land P(x))$.

Taking into account that the traditional forms of syllogistic quantification are symbolized in contemporary logic as particular cases of restricted quantifiers

1. Universal affirmative **A**: Every S is P, $(\forall x \in S(x))P(x)$
2. Universal negative **E**: No S is P, $(\forall x \in S(x))\neg P(x)$
3. Particular affirmative **I**: Some S is P, $(\exists x \in S(x))P(x)$
4. Particular negative **O**: Some S is not P, $(\exists x \in S(x))\neg P(x)$

the Square of Quaternality holds also for such traditional forms as follows:

$$\begin{tikzcd}
A \arrow[rr, leftrightarrow, "\text{contraduals}"] \arrow[dd, leftrightarrow, "\text{duals}"'] \arrow[ddrr, leftrightarrow] & & E \arrow[dd, leftrightarrow, "\text{duals}"] \arrow[ddll, leftrightarrow] \\
& \text{complements} & \\
I \arrow[rr, leftrightarrow, "\text{contraduals}"'] & & O
\end{tikzcd}$$

However, the illustrious Square of Opposition works in terms of relative validity, replacing the useful algebraic notions of duality (or quaternality) by much less significant semantical relations (such as the notions of contrary and subcontrary) and entailment relation (such as subalternation). Recalling the standard definitions:

- Two propositions are *contraries* iff they cannot both be true but can both be false;
- Two propositions are *contradictory* iff they cannot both be true and they cannot both be false;
- Two propositions are *subcontraries* iff they cannot both be false but can both be true;
- A proposition B is a *subaltern* of a proposition A iff B must be true if A is true, and A must be false if B is false.

$$\begin{tikzcd}
A \arrow[rr, leftrightarrow, "\text{contraries}"] \arrow[dd, "\text{subaltern}"'] \arrow[ddrr, leftrightarrow] & & E \arrow[dd, "\text{subaltern}"] \arrow[ddll, leftrightarrow] \\
& \text{contradictories} & \\
I \arrow[rr, leftrightarrow, "\text{subcontraries}"'] & & O
\end{tikzcd}$$

We also see that the Square of Opposition lacks symmetry, as the semantic relation of subalternation is not symmetrical. The quaternality relations of complementarity, contraduality and duality are related to the semantic relations of contrariety, contradictoriety, subcontrariety and subalternation, though in a complicated (and perhaps uninteresting) way: although the complementary polynomials $f(p,q)$ and $\neg f(p,q)$ are always contradictory, the contraduals $f(p,q)$ and $f(\neg p, \neg q)$ can be contraries or subcontraries. For instance, the contraduals $\neg p \vee q$ and $p \vee \neg q$ cannot be simultaneously false, but can be simultaneously true, hence are subcontraries. Similarly, the contraduals $\neg p \wedge q$ and $p \wedge \neg q$ cannot be simultaneously true, but can be simultaneously false, hence are contraries. Duals are neither contraries nor subcontraries, e.g. $p \wedge q$ and $p \vee q$ can be both true and both false. Subalternation is even more awkward: for instance, taking in particular the Boolean polynomial $f(p,q)$ as $p \wedge q$ gives:

$$\begin{tikzcd}
p\wedge q \arrow[rr, leftrightarrow, "\text{contraries}"] \arrow[dd, "\text{subaltern}"'] \arrow[ddrr, leftrightarrow] & & \neg p\wedge\neg q \arrow[dd, "\text{subaltern}"] \arrow[ddll, leftrightarrow] \\
& \text{contradictories} & \\
p\vee q \arrow[rr, leftrightarrow, "\text{subcontraries}"'] & & \neg(p\wedge q)
\end{tikzcd}$$

while this does not hold by taking $f(p,q)$ as $p \vee q$.

An illuminating exercise is to determine which pair of Boolean functions are contraduals, contraries, or subcontraries, and to compare this with their classification into contradictory, contrary, and subcontrary. This is easily done by, for instance, representing all the classical Boolean polynomials $f(p,q)$ as expressions of the form $a_1pq + a_2p + a_3q + a_4$ with coefficients in the field $\mathbb{Z}_2$ (see [3, 5] or [6]) which readily gives the following tables, separating the 16 binary Boolean polynomials $f(p,q)$ into classes grouping complementary, contradual, and dual functions:

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \wedge \neg q$	$p \vee q$
0	0	0	1	1	0
0	1	0	1	0	1
1	0	0	1	0	1
1	1	1	0	0	1

p	q	$p \wedge \neg q$	$\neg(p \wedge \neg q)$	$\neg p \wedge q$	$p \vee \neg q$
0	0	0	1	0	1
0	1	0	1	1	0
1	0	1	0	0	1
1	1	0	1	0	1

p	q	pXORq	$\neg(p$XOR$q)$
0	0	0	1
0	1	1	0
1	0	1	0
1	1	0	1

p	q	pXNORq	$\neg(p$XNOR$q)$
0	0	1	0
0	1	0	1
1	0	0	1
1	1	1	0

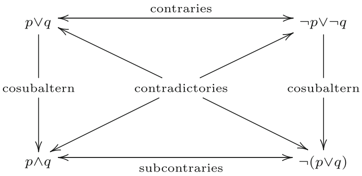
The remaining are 0-ary and unary functions, namely, the constants, projections and their negations:

p	q	0	1	p	$\neg p$	q	$\neg q$
0	0	0	1	0	1	0	1
0	1	0	1	0	1	1	0
1	0	0	1	1	0	0	1
1	1	0	1	1	0	1	0

Clearly, complementary functions are contradictory, contradual functions are sometimes contrary and sometimes subcontrary, and dual functions are can be simultaneously contrary and subcontrary (or, neither contrary nor subcontrary). XOR and XNOR are self-duals and self-contraduals, thus self-contrary and self-subcontrary: this just testifies, if anything, for the weakness of the Square of Opposition. As an attempt to (at least partly) remedy this lack of symmetry, a "new" form of Square of Opposition will hold by defining a relation of *cosubalternation* as follows:

- A proposition B is a *cosubaltern* of a proposition A iff B must be false if A is false, and A must be true if B is true.

The following Square of Co-Opposition now holds by putting, e.g., $f(p, q)$ as $p \vee q$:



The Square of Quaternality, thus, gives rise to two symmetry-lacking squares: the traditional one, preserving truth (subalternation), and a new one, preserving falsity (cosubalternation). Of course, B is a cosubaltern of A iff A is a subaltern of B.

Model-theoretic relations also enjoy a form of algebraic duality: recall that a sentence is said to be *satisfiable* in a given domain if there are assignments to its free variables that make it true, and is *valid* if every assignment to its free variables makes it true. Since a sentence is invalid if and only if its negation is satisfiable, and is unsatisfiable if and only if its negation is valid, satisfiability and validity are dual notions. Moreover, if A is satisfiable and B is valid, A and B are subcontrary, and if A is unsatisfiable and B is invalid, A and B are contrary.

What we see from such examples is that the Square of Quaternality is immensely richer than the shallow Square of Opposition with its lack of symmetry and its mediocre capacity of being generalized. Why does it attract such attention? Let us postpone a tentative reply for a moment.

2 The Triality Group: Generalizing Duality and Quaternality

The German-Jewish mathematician Amalie Emmy Noether was the first to note the intrinsic relationship between symmetry and abstract structures, like rings or groups. Her legacy, in the form of Noetherian rings, Noetherian groups, Noetherian equations and so on made clear that the theory of groups is, as sometimes said, another way to treat symmetries. Noether's ideas explain the connection between symmetry and conservation laws in Physics: conservation of energy comes from time symmetry, and conservation of momentum comes from space symmetry. Conservation laws of quantum mechanics can be derived from properties of symmetry of physical systems (see Chap. 52, Vol. I of [8]).

The combinatorialist Frank Harary, author of the one of the most important books on graph theory [13] and who contributed to give the field a broader relevance, is a co-author of a study (cf. [11]) of Boolean group models for the analysis of sexual symbolism in New Guinea tribes applicable to a wide array of symbolic systems, not only interesting to anthropology but also to logic.

The results of [11] are significant for ethnographic research and for assumptions concerning the logic of mathematical representation of cultural structures. Hage and Harary pay special attention to Boolean groups and cubes, starting from a simple definition of group as a set X endowed with an operation $*$ satisfying the axioms of:

1. (Closure) If $x, y \in X$ then $x * y \in X$.
2. (Associativity) For all $x, y, z \in X$, $(x * y) * z = x * (y * z)$.
3. (Identity) There exists an identity element $i \in X$ such that for all $x \in X$, $i*x = x*i = x$.
4. (Inversion) For each $x \in X$ there exists an element $x^{-1} \in X$ (called the inverse of x) such that $x * x^{-1} = x^{-1} * x = i$

An important class of groups are the *permutation groups*, whose elements are bijective functions from a finite set M (the permutations) onto itself and whose group operation is the composition of permutations. Permutations of a set M with m elements are usually written in cyclic notation $\sigma = (\sigma(1), \sigma(1), \cdots, \sigma(k))$, for $k \leq m$. So, for instance, if $M = \{0, 1, 2, 3\}$, a permutation σ of M with $\sigma(0) = 2$, $\sigma(2) = 3$, $\sigma(3) = 0$, and where $\sigma(1) = 1$ is a fixed point, is written as $(0, 2, 3)$. The importance of permutation groups can hardly be exaggerated: according to a famous theorem of Arthur Cayley (see [9]), every group is isomorphic to a group of permutations.

The permutation group consisting of all $n!$ permutations of n objects is called the *symmetric group*, denoted by S_n. It is obvious that S_2 is isomorphic with Z_2, the group of residues modulo 2 (odd-even), which by its turn coincides with classical negation (i.e., classical negation is just a cycle in the values 0 and 1).

A *Boolean group* B_n is a finite abelian group in which every element different from the identity i has order two (that is, $x * x = i$ for all x).

Boolean groups have closed connections with classic propositional logic: the set of propositional sentences forms a Boolean group under the operation of equivalence $\Leftrightarrow$ with the identity element 1 (*verum*): indeed, it is enough to check that the following laws are valid:

(associativity): $p \Leftrightarrow (q \Leftrightarrow r) = (p \Leftrightarrow q) \Leftrightarrow r.$
(identity: $p \Leftrightarrow 1 = p.$
(inversion): $p \Leftrightarrow p = 1.$

The same holds for the operation of 'exclusive disjunction' $\veebar$ (also known as *XOR*), but now with the identity element 0 (*falsum*):

(associativity): $p \veebar (q \veebar r) = (p \veebar q) \veebar r.$
(identity): $p \veebar 0 = p.$
(inversion): $p \veebar p = 0.$

It is well known (see e.g [13], p.163) that every Boolean group is isomorphic to a sum of n copies of the group S_2 for some positive integer n, where the sum of two permutation groups X and Y is defined by all permutations xy obtained from the juxtaposition of permutations $x \in X$ and $y \in Y$.

As an example, the well-known Klein 4-group V, often referred to in structural studies (such as by Lévi-Strauss in various occasions, see e.g. his canonical formula for the structure of myths in chapter 11 of [15]) is defined by the following group table:

$*$	i	a	b	c
i	i	a	b	c
a	a	i	c	b
b	b	c	i	a
c	c	b	a	i

Clearly, Klein's V is a Boolean group (every element is its own inverse), and it thus follows that the V is isomorphic to the sum $S_2 \oplus S_2$. Thus the four smallest Boolean groups are: $B_1 = S_2, B_2 = S_2 \oplus S_2, B_3 = S_2 \oplus S_2 \oplus S_2$, and $B_4 = S_2 \oplus S_2 \oplus S_2 \oplus S_2$. B_2 can be written as the sequences $\langle 00, 01, 10, 11 \rangle$ and B_3 as $\langle 000, 001, 010, 100, 011, 101, 110, 111 \rangle$.

The group operation (under this representation) is componentwise addition modulo 2, that is, $\langle a_1, \cdots, a_n \rangle \oplus \langle b_1, \cdots, b_n \rangle = \langle a_1 + b_1, \cdots, a_n + b_n \rangle$ where $a_1 + b_1$ is calculated modulo 2, so e.g. $\langle 01 \oplus 11 = 10 \rangle$; B_2 is generated by the (unity) vectors 01 and 10.

Seduced by the Erlangen Program of Felix Klein (for whom topology was the primary form of conceiving space), Jean Piaget proposed a theory of oppositions connected to his view on the development of cognitive behavior (see [7] and [19]) centered on his *INRC* group,[2] a group of 4 elements that combines two cyclical, 2-element groups. *INRC* is $\mathbb{Z}_2 \times \mathbb{Z}_2$, the direct product of two copies of the cyclic group of order 2.

The group *INRC* can be seen as acting on quadruples (x_1, x_2, x_3, x_4) of 0's and 1's, such that its four components $\{I, N, R, C\}$ produce the following actions on quadruples:

- $I(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4)$ Identity
- $N(x_1, x_2, x_3, x_4) = (x_1 + 1, x_2 + 1, x_3 + 1, x_4 + 1)$ Negation
- $R(x_1, x_2, x_3, x_4) = (x_4, x_3, x_2, x_1)$ Reciprocal
- $C(x_1, x_2, x_3, x_4) = (x_4 + 1, x_3 + 1, x_2 + 1, x_1 + 1)$ Correlative

It is clear that $NR = C$, $NC = R$, $RC = N$, and $NN = RR = CC = I$ where the product means composition, and moreover composition is commutative, so the group table is exactly the Klein's group V (up to isomorphism):

$*$	I	N	R	C
I	I	N	R	C
N	N	I	C	R
R	R	C	I	N
C	C	R	N	I

On their turn, both the Klein's 4-group V and the *INRC* are isomorphic to the Gottschalk's group of order four, considering complement, dual, contradual and identity as operations on propositions (see also [17]).

---

[2] I am not here interested in Piaget's theory, nor in the abundant criticism around it: the only thing interesting here is the coincidence involving Klein's group V.

From the geometric viewpoint, V, $INRC$, and Gottschalk's group are also isomorphic to the smallest non-trivial dihedral group. The dihedral groups D_n are the group of symmetries of a regular polygon of n sides, containing exactly $2n$ different symmetries: n rotational symmetries and n reflection symmetries.

The dihedral group D_2 (isomorphic with the group Klein's V) is generated by the reflection σ across the vertical axis and the rotation ρ of $180°$. The elements of D_2 can then be represented as $\{e, \rho, \sigma, \rho\sigma\}$, where e is the identity or null transformation and $\rho\sigma$ is the reflection composed with the rotation, which coincides with the reflection across the horizontal axis. The geometric action of D_2 on the letter L is illustrated below:

L, ⅃, ⅂, Γ

Operations of rotation and reflection in general do not commute: for $n > 2$ the group D_n is not abelian; so D_2 (besides the trivial D_1) is the only commutative dihedral group.

The Boolean groups B_n are depicted by the graphs called n-cubes, having as vertexes the 2^n points constituted by the binary sequences of length n, with two points adjacent whenever their sequences differ in exactly in one places (that is to say, in terms of coding theory they have Hamming distance equal 1). The four smallest cubes are 0–1 (expressing classical negation) the "square of oppositions", the "cube of oppositions" and the "hypercube of oppositions in four-dimensional space", which has not found any logical or anthropological application yet.

By taking a closer look at Boolean polynomials, it becomes clear, looking at them from the point of view of algebraic polynomials as in [3] (see also [1]), that the Boolean polynomials $f(p, q)$ in classical propositional logic reduce to all 16 expressions[3] of the form $a_1pq + a_2p + a_3q + a_4$, for a_i in the two-element field $\mathbb{Z}_2 = \{0, 1\}$.

This gives a precise way to generalize the notions of duality, contraduality and complementarity of [10] to several other, not necessarily bivalued, logics with negation: it is just a matter of understanding which permutation lies behind negation. In more general terms, a more complex group may be acting over negation, and this will define more general duality, contraduality, complementarity, and perhaps yet new notions of symmetry.

For the case of three-valued logics, the six possible permutations of the truth-values $M = \{0, 1, 2\}$ are $(0, 1)$, $(0, 2)$, $(1, 2)$, $(0, 1, 2)$, and $(0, 2, 1)$, plus the identity. Negations of the well-known three-valued logics of Peirce, Bochvar Kleene and Lukasiewicz, among others, are permutations (although several other three-valued negations are not). Recalling that any three-valued logical operator (including negations) can be seen as a polynomial $f(p, q)$ in two variables over the three-element field $\mathbb{Z}_3$ (for details see [1, 3], [5] or [6]), then a generalization of Gottschalk's complement, dual and contradual (as in [10]) of a three-valued operator $f(p, q)$ for $\sim$ a negation which is a permutation of the truth-values

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[3] It should be clear that the operation of multiplication is interpreted as conjunction or meet $\wedge$, and addition is interpreted as exclusive disjunction or symmetric difference $\veebar$. Addition *does not* represent disjunction $\vee$, which would perhaps explain how exclusive disjunction, rather than standard inclusive disjunction, would be a more natural basis for logic.

$M = \{0, 1, 2\}$ are:

$$\sim f(p, q),\ \sim f(\sim p, \sim q) \text{ and} f(\sim p, \sim q), \text{as before, plus} \sim^2 f(p, q), \sim^2 f(\sim p, \sim q),$$
$$f(\sim^2 p, \sim^2),\ \sim f(\sim^2 p, \sim^2 q) \text{ and } \sim^2 f(\sim^2 p, \sim^2 q).$$

We conclude then, that there is a group of order 9 (let us call it the *triality group*) acting on three-valued sentences so as to define a precise notion of triality. Since any group of order 9 is isomorphic to either $\mathbb{Z}_9$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$ (see [9]), as a direct consequence of the well-known Theorem of Finitely Generated Abelian Groups (see [18], Theorem 4.2.10, p. 103), and since the triality group has no element of order 9 (i,e., it is not cyclic), the triality group is then obviously isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$. This represents thus a direct generalization of Klein's V and Gottschalk's group of quaternality and of Boolean groups to three-valued structures (by calling a group *triadic* if all its elements, other than the identity, have order 3).

3 Summing Up

I have argued that the algebraic structure behind Boolean groups, their generalizations, and their isomorphic versions is what makes them relevant for understanding symmetries: as each B_n is a subgroup of B_{n+1}, they form increasingly complex structures, and any symbolic system may be regarded as a subsystem of a larger one, as suggested in [11]. For instance, each face of the cube Q_3 is Q_2 so the "cube of oppositions" contains six "squares of oppositions", and so on, expressing combinatorial manifolds of complex binary oppositions. In 1936 D. König, the graph theorist who proposed the famous König's Lemma, a denumerable form of the Axiom of Choice, conjectured that every finite group is the group of symmetries of a finite (undirected) graph, a result proved by R. Frucht in 1939. Frucht's theorem essentially says that for any finite group, there is a graph G such that the group of automorphisms of G is isomorphic to the given group. In particular, this will hold for any group of symmetries, so there will be infinitely many abstract forms of 'squares of symmetries' to play with, of which our notion of triality is just an example.

A perfect quantifier in many-valued logic is a quantifier that can generate all others by means of negations, as in the classical existential and universal quantifiers. By using the power group enumeration theorem, a combinatorial-algebraic generalization of the classical enumeration techniques of Georg Polya, it was possible in [2] to define the notion of distribution quantifiers, and to characterize all perfect quantifiers in 3-valued logics, showing that there are 360 such quantifiers in 3-valued logics. A characterization of this kind is only possible by applying the action of groups, not by any static diagram.

The Square of Opposition (as well as similar diagrammatic devices) is a rather poor structure in comparison with Gottschalk's group of order four and its isomorphic version of Klein's 4-group V: indeed, V is highly non-trivial, being the smallest non-cyclic group, with importance for the study of symmetry in chemistry and physics, and has played a relevant historical role in Klein's program.

The notion of quaternality is amply clarified from a group-theoretical perspective by taking into account that there are exactly four functions that are one-to-one mappings (automorphisms) of the set of Boolean polynomials onto itself: the identity function, the complement function (defined by the external action of negation), the contradual function (defined by the internal action of negation) and the dual function (defined by the internal and the external actions of negation). The above defined triality group is an immediate generalization of this idea. I am not against the trend of using diagrams in logic, but I am not trying to depict a diagram for the triality group (which would be a kind of "star of triality"). The insistence on diagramming everything may be harmful to the generalizations, as warned in the harsh criticism against the proliferation of diagrams and graphical systems lacking formal semantics in computer programming found in [16].

If we are interested into a generalizations of Gottschalk's notion of duality, as the above suggested notion of triality, groups are essential and the Square of Opposition alone is of no help; perhaps the penchant some people have for the square comes from the Aristotelian elements and qualities (earth: cold and dry, water: cold and wet, air: hot and wet, fire: hot and dry), popularized by Apuleius and Boethius as a pedagogical device, and applied to modal propositions by twelfth-century logicians (see [14]). Recognizing that our understanding in many things has evolved since the Aristotelian doctrines, historically interesting as it can be, and that sophisticated structures are naturally behind linguistics, logic, set theory, category theory, topology, geometry, philosophy, and anthropology would heal and liberate people from a squared fetish.

Acknowledgements I acknowledge support from FAPESP Thematic Project LogCons 2010/51038-0, Brazil and from the National Council for Scientific and Technological Development (CNPq), Brazil. I am thankful to the five anonymous referees who have read, commented and criticized this paper.

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Part VI
Expansions and Variations of the Square

From the Square to Octahedra

José David García-Cruz

Abstract Colwyn Williamson (Notre Dame J. Formal Log. 13:497–500, 1972) develops a comparison between propositional and syllogistic logic. He outlines an interpretation of the traditional square of opposition in terms of propositional logic, that is, the statements corresponding to the corners of the traditional square can be represented with propositional logic operators. His goal is to present a twofold square that preserves the truth conditions of the relationships between the formulas, and define other set of formulas that complete the traditional square to outline an octagon of opposition. We present two octahedra inspired in these squares. The octahedra hold the relations of the traditional square of opposition and also keep (and with some restrictions, extend) the equipollence and immediate inference rules.

Keywords Hexagon • Octagon • Propositional logic • Square of opposition • Syllogistic

Mathematics Subject Classification (2000) Primary 03B05; Secondary 03B22, 03B35, 03B10

> In geometry and logic alike a place is a possibility: something can exist in it.
>
> Ludwig Wittgenstein [6, 3.411]

1 Introduction

In [5] Colwyn Williamson develops a comparison between propositional and syllogistic logic. He outlines an interpretation of the traditional square of opposition in terms of propositional logic, that is, the statements corresponding to the corners of the traditional square can be represented with propositional logic operators. His goal is to present a twofold square that preserves the truth conditions of the relationships between the formulas, and he defines other set of formulas that complete the traditional square to outline an octagon of opposition. The aim of this paper is to lead to the end this reconstruction taking seriously the task stated by Williamson.

J.-Y. Béziau, G. Basti (eds.), *The Square of Opposition: A Cornerstone of Thought*, Studies in Universal Logic, DOI 10.1007/978-3-319-45062-9_15

We present two octahedra inspired in these squares. The octahedra hold the relations of the traditional square of opposition and also keep (and with some restrictions, extend) the equipollence and immediate inference rules. Our goal is threefold: first, to analyze the Williamson's squares and state the basic consequences of his analysis, second, to present an extension of the Williamson's squares, i.e. the octahedra of opposition, and third, we bring to the end Williamson's thesis to get some results concerning the relation between propositional and first-order logic.

In the second section we generate an analysis of the reconstruction of syllogistic logic developed by Williamson in terms of propositional logic. In this part we highlight the main results: (1) consider that the combination of the truth values defines a type of quantifier, and (2) to establish the prevalence of the truth or falsity is relevant in reconstruction. Subsequently, in Sect. 3 these ideas are taken to build two structures that satisfy the constraints presented, but with some difficulties, specifically the asymmetry in the number of rules in each polyhedron. In Part 4 we developed a reinterpretation of the ideas presented to solve the problems. That interpretation is to consider further consequences of the above conditions, the commutativity as an ingredient necessary to define a quantifier square opposition. And finally in the last section we apply the results to the traditional theory.

2 Williamson's Squares

Colwyn Williamson in his work *Squares of opposition: Comparisons between Syllogistic and Propositional Logic*, develops an analysis of propositional and syllogistic logic based on a definition of some Boolean operators. He begins with a definition of the operator K representing the conjunction as follows: $K11 = 1$, $K10 = 0$, $K01 = 0$, $K00 = 0$. The operator K represents the conjunction connective in propositional logic and the 1 and 0 represent truth values *True* and *False*, and the combinations of 1 and 0 represents the possible valuations for the propositional variables, therefore the definition of the operator is 1000.

Taking in account this definition for the logical connectives Williamson defines the following operators: $B = 1101$, $C = 1011$, $D = 0111$, $J = 0110$, $L = 0100$, $M = 0010$, $V = 1110$, $X = 0001$. Williamson uses this resource to elaborate an analysis of the traditional opposition square, and in addition to the later definitions he introduce notation to define the four statements of the corners of the square of opposition as follows:

$Aab :=$ all $a's$ are $b's$
$Eab :=$ no $a's$ are $b's$
$Iab :=$ some $a's$ are $b's$
$Oab :=$ some $a's$ are not $b's$

This notation is used by Williamson to generate the following traditional square of opposition ($TS1$) (Fig. 1).

Fig. 1 TS1

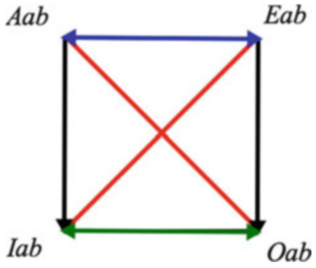
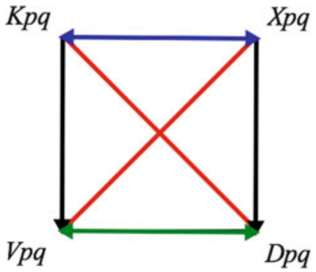


Fig. 2 SP1



We use the standard notation to represent the opposition relations of the square which are represented in Williamson's notation as *D* for contrariety, *J* for contradiction, *C* for subalternation, and *V* for subcontrariety. The first comparison in Williamson's analysis is between the previous square *TS*1 and the following square (which we can call *SP*1) (Fig. 2).

We may assume with Williamson that the *q* in the later square could be consider as predicate of the formulas in the corners, but he finds some problems concerning the *equipollence rule*. The rule consist in define the operator of a formula of some corner in terms of the negation and the operator of the remaining three corners preserving the truth conditions of the initial formula, for example, when we deny[1] the predicate of *Aab* we get a formula with the same truth conditions of *Eab*, namely *Aanb*. Williamson rejects this assumption for the traditional propositional square because the rule of equipollence can't hold in the later square. To verify this take *Kpq* and *Xpq* as analogous of *Aab* ad *Eab*, in according to equipollence rule *KpNq* must be equivalent to *Xpq*, but the equivalent of the later is *KNpNq* and *KpNq* is equivalent to *DNpq*.

There is another reason to reject this square as a faithful propositional representation, in the traditional square only two of the four formulas could be convert, that is *Eab* → *Eba* and *Iab* → *Iba*, but no so with *Aab* → *Aba* and *Oab* → *Oba*. But in the later propositional square all formulas can be converted. These issues make Williamson to generate two squares that correspond exactly with the traditional, that means that the later square don't preserve the restrictions of the traditional square of opposition. The first propositional square is presented in Fig. 3.

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[1] The Williamson's notation for negation is: for external negation (de dicto), and for internal negation (de re).

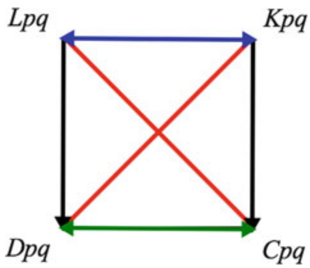

Fig. 3 WP1

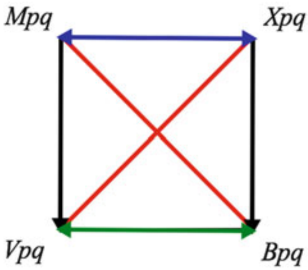

Fig. 4 WP2

Williamson generates a correspondence between the two squares ($TS1$ and $WP1$) associating each Boolean operator formula of $WP1$ with the categorical formulas of $TS1$ in the following sense:

Lpq is analogous to *Aab*
Kpq is analogous to *Eab*
Dpq is analogous to *Iab*
Cpq is analogous to *Oab*

The soundness of this interpretation is confirmed by the preservation of both the rules of equipollence and immediate inference.[2] The second square is shown in Fig. 4.

In this case the link is between *Mpq*, *Xpq*, *Vpq*, and *Bpq* with the categorical formulas *Aab*, *Eab*, *Iab*, and *Oab*, respectively; and the equipollence and immediate inference rules also hold. Williamson remarks two questions concerning the truth conditions of the formulas in the corners of these squares. First "it will be noticed that the operators capable of forming an exact analogue for the traditional square are the ones in which three and only three of the defining values are the same: 1000, 0100, 0010, 0001, 0111, 1011, 1101 and 1110" [5, p. 499]. The second fact is connected with the correspondence by one side between the truth value *True* and the particular quantifier, and by the other side between the truth value *False* and universal quantifier, namely "the operators corresponding to the "universals" of syllogistic are those in which false values predominate, while the operators corresponding to the "particulars" of syllogistic are those in which true values

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[2] Simple conversion, conversion *per accidents*, obversion, contraposition, and inversion.

predominate" [5, Idem.]. Williamson emphasize that this correspondence could be some kind of analogue to the medieval distribution theory, but he does not say more.[3]

Williamson note also an absence of symmetry in the comparison, because on the one hand we have one traditional square, and on the other hand we could generate two propositional squares with the above operators. We can assume following Williamson that "there are—or ought to be—two such squares in traditional logic also", and we may call this later sentence the *Williamson's thesis*. In other words, there are eight and not only four, logically independent propositions. Williamson extends the traditional square and add four new quantifiers: *Rab*, *Sab*, *Tab*, and *Uab*; and later he define them as follows:

$Rab \equiv Ananb \equiv Aba$
$Sab \equiv Enanb$
$Tab \equiv Inanb$
$Uab \equiv Onanb \equiv Oba$

These new quantifiers are used by Williamson to present another traditional opposition square analogous to the first ($TS1$) to balance the situation and, evidently, he relates each traditional square of opposition with his counterpart in propositional notation. In this case the relationships are established between the new quantifiers *Rab*, *Sab*, *Tab*, and *Uab* with the later Boolean operators *Mpq*, *Xpq*, *Vpq*, and *Bpq*, respectively. Therefore, the following equivalences also hold in propositional logic:

$Mpq \equiv LNpNq \equiv Lqp$
$Xpq \equiv KNpNq$
$Vpq \equiv DNpNq$
$Bpq \equiv CNpNq \equiv Cqp$

Williamson's interpretation ends with two notes about "certain kind of connection between Syllogistic and propositional logic"[5, p. 500]. First, following Łukasiewicz, Williamson states that "the procedures of traditional logic presuppose laws of propositional calculus"[5, Idem.]; and second, he makes the claim that "syllogistic and propositional logic express, at some level, a common structure of reasoning"[5, Idem.]. We will focus on this assumptions in the final section, and we will give an argument based on some thesis presented in the fourth section to vindicate the words of Łukasiewicz.

3 From Squares to Octahedra

In this section we extend the previous ideas about the propositional interpretation of traditional square of opposition, in specific we will show how to construct two opposition structures based on Williamson's squares. The novelty of this polyhedra is that it satisfy the restrictions concerning the preservation of the equipollence and immediate inference

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[3] We will say a few words about that in the final section.

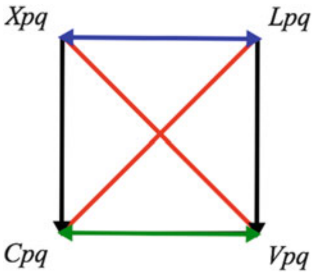

Fig. 5 SP2

rules, but as we will see, this polyhedra has two basic problems related with the rules of the obversion and with the preservation of symmetry of the cited rules; nevertheless, the octahedra has some interesting properties that serve as indication—together with the mentioned difficulties—of the construction of a more complex opposition structure. The main motivation of the extension of the Williamson's squares is to analyze the relation between these squares with the *spurious*[4] squares, namely $SP1$ and a new square $SP2$ with the same problems that the later. Also we think that our extension is relevant because we will see the role payed by $SP1$ and $SP2$ in the representation of the traditional opposition square. Our thesis is twofold, by one side, using *Williamson's thesis* we will show that there is not only one spurious square, but two[5]; and by the other side, we think that if the spurious squares are taken independently they don't satisfy some rules, but if we put all together we may construct a structure that satisfy the restrictions stated by Williamson to make a correct propositional representation of the traditional square, in other words, the spurious squares are intermediaries between the genuine squares.

We begin presenting the spurious squares and consequently we show how join them to the squares presented in the previous section. The first $SP1$ is the one who has presented by Williamson, as we say it has problems with equipollence and immediate inference rules, and for this reason is spurious. For the same reason the square in Fig. 5 is spurious.

Although this square preserves the main opposition relations it is not a correct representation of the traditional square, to see why take, for example, Xpq and Lpq, Xpq must be equivalent to $LpNq$ but it is equivalent to Kpq not to Xpq. As we say, this two squares are not part of the propositional reconstruction of the theory, to be taken into account in the reconstruction of the propositional representation we must join them to the genuine squares. We begin with the $SP1$ and the $WP1$ squares, in Fig. 6 we can see how we construct the first octahedron from the intersection of the two squares.

The squares intersect perpendicularly taking as point of union the contradictory axis of Kpq and Dpq. In the picture we have above the two squares with the axis highlighted, but also if we look careful there is another square, the spurious $SP2$. This fact will be analyzed later when we talk about how mix the two octahedra. There are some technical reasons to consider this structure as an suitable reconstruction of the traditional square;

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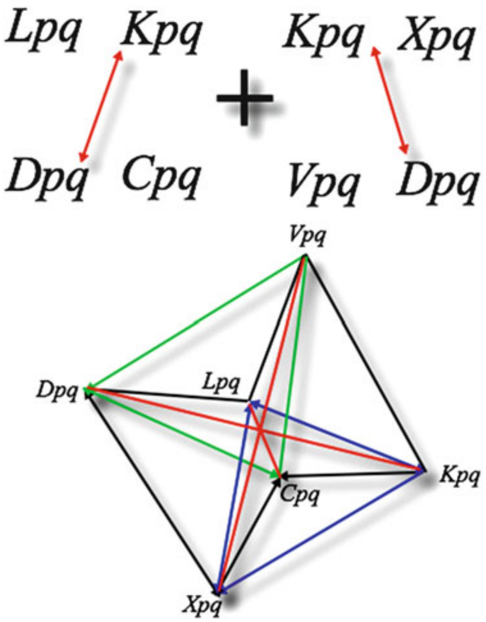
[4] The name was suggested by one of the jurors who reviewed an earlier draft.

[5] I thank one of the jurors for this observation.

Fig. 6 WP1+SP1=D1



in the first place, in the operators V and X three of the defining values are the same, and in the second place in X predominate the false values and in V the true values, because the former is universal and the later particular. Before we move to the presentation of the second octahedron we discuss what properties and rules preserve. The octahedron preserve all the immediate and equipollence rules, but it extends the number of rules in both cases. In the first place we have the equipollence rules:

$Lpq \equiv KpNq \equiv NCpq \equiv NDpNq \equiv NVNpq \equiv XNpq$
$Kpq \equiv LpNq \equiv NDpq \equiv NCpNq \equiv NVNpNq \equiv XNpNq$
$Dpq \equiv CpNq \equiv NKpq \equiv NLpNq \equiv VNpNq \equiv NXNpNq$
$Cpq \equiv DpNq \equiv NLpq \equiv NKpNq \equiv VNpq \equiv NXNpq$
$Vpq \equiv CNpq \equiv DNpNq \equiv NKNpNq \equiv NXpq \equiv NLNpq$
$Xpq \equiv LNpq \equiv KNpNq \equiv NDNpNq \equiv NVpq \equiv NCNpq$

This rules don't have any problem, the only change is in the number. The relevant and interesting modification is in the immediate inference rules, we analyze one by one starting with the simple conversion rule. This rule states that a formula implies another formula with the same operator but subject and predicate exchanged; as we say, this rule is only satisfied by formulas with the E and I quantifier, and for this reason we only have restricted number of them, in specific four. The next rule is conversion *per accidents*. This rule states that an universal formula implies its subaltern with subject and predicate exchanged. In this case we have six formulas that satisfy this rule because we have six subalternation relation. The next one is obversion. In the square $WP1$ we have four obversion rules, this rule states that a formula implies its contrary—in the case of the universal formulas—or its subcontrary—in the case of particular formulas—with the predicated denied. As the $D1$

octahedron have two triangles, one of contraries and other of subcontraries, it is expected that in this polyhedra we have twelve rules of obversion, but the $D1$ only have four rules.[6] The remaining formulas have in common the fact that they preserve some pattern that exhaust the combination of 1 and 0 between p and q as we show below:

$(Kpq \rightarrow XpNq) = 0$ iff $p = q = 1$
$(Xpq \rightarrow LpNq) = 0$ iff $p = q = 0$
$(Vpq \rightarrow DpNq) = 0$ iff $p = 1, q = 0$
$(Dpq \rightarrow VpNq) = 0$ iff $p = 0, q = 1$
$(Lpq \rightarrow XpNq) = 0$ iff $p = 1, q = 0$
$(Xpq \rightarrow KpNq) = 0$ iff $p = q = 0$
$(Vpq \rightarrow CpNq) = 0$ iff $p = 0, q = 1$
$(Cpq \rightarrow VpNq) = 0$ iff $p = q = 1$.

Later we will present a detailed analysis of the question with the help of some additional restrictions to the formulas to make a better propositional reconstruction of the theory with an explanation of this difficulties.

The next rule is contraposition, this rule states that a formula implies another formula whit the same operator, also the subject and predicate are exchanged and negated. The main reason that not all operator satisfies the rule lies in some facts related with the properties of conditional and similar operators, we return on that later. In $WP1$ we only have two rules of contraposition and in the $D1$ we have the same number. The last rule is inversion, this rule states that an universal formula implies its contradictory with the subject denied. The octahedron satisfy three rules of inversion corresponding to the three contradictory axis. Taking in account this facts we may generate the following list of immediate inference rules:

1. $Kpq \rightarrow Kqp$
2. $Dpq \rightarrow Dqp$
3. $Xpq \rightarrow Xqp$
4. $Vpq \rightarrow Vqp$
5. $Lpq \rightarrow Dqp$
6. $Lpq \rightarrow Vqp$
7. $Kpq \rightarrow Cqp$
8. $Kpq \rightarrow Vqp$
9. $Xpq \rightarrow Dqp$
10. $Xpq \rightarrow Cqp$
11. $Lpq \rightarrow KpNq$
12. $KpNq \rightarrow LpNq$
13. $Dpq \rightarrow CpNq$
14. $Cpq \rightarrow DpNq$
15. $Lpq \rightarrow LNqNp$
16. $Cpq \rightarrow CNqNp$

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[6] There are many facts that justify this anomaly but that does not discuss now, we will return to the issue in the next section.

17. $Lpq \rightarrow CNpq$
18. $Kpq \rightarrow DNpq$
19. $Xpq \rightarrow VNpq$

The formulas 1–4 are simple conversion, the formulas 5–10 are conversion *per accidents*, the formulas 11–14 are obversion, 15 and 16 are contraposition, and 17–19 are inversion. Now we present the other octahedron together with its list of formulas, but first we explain how to construct the octahedron (Fig. 7).

As in the $D1$ in this octahedron the squares are intersected in a contradictory axis composed by the X and the V operators. The technical restrictions are also satisfied by this polyhedron, i.e. the X is universal and V is particular, and both have three identical values in its definition. Now we will discuss the rules of inference. By one side the octahedron $D2$ preserve the same number of equipollence rules, and there is no anomaly in this kind of rules. By the other side, there are an asymmetry with the later octahedron in the sense that the number of inference rules that preserves are different, the $D2$ only preserves seventeen rules. The following are the equipollence rules:

$Mpq \equiv XpNq \equiv NBpq \equiv NVpNq \equiv NCNpNq \equiv LNpNq$
$Xpq \equiv MpNq \equiv NVpq \equiv NBpNq \equiv NCNpq \equiv LNpq$
$Vpq \equiv BpNq \equiv NXpq \equiv NMpNq \equiv CNpq \equiv NLNpq$
$Bpq \equiv VpNq \equiv NMpq \equiv NXpNq \equiv CNpNq \equiv LNpNq$
$Cpq \equiv NLpq \equiv VNpq \equiv NMNpNq \equiv NXNpq \equiv BNpNq$
$Lpq \equiv NCpq \equiv XNpq \equiv MNpNq \equiv NVNpq \equiv NBNpNq$

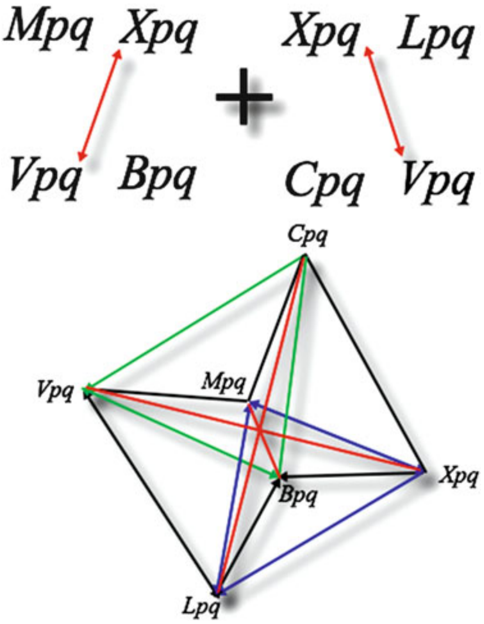


Fig. 7 WP2+SP2=D2

As in the case of $D1$ the main change with respect with the squares lies in the immediate inference rules, now we analyze this issue. The first anomaly is present in the simple conversion rules, in $D1$ we have four rules and here we have only two. In the second place, the $D1$ has six rules of conversion *per accidents* and the octahedron $D2$ hast only four. In the case of the obversion rule we have the same number in the two octahedra but, we have the same situation as in the $D1$, namely, the potential rule schemes that fails in one assignation, as we see below:

$(Bpq \to CpNq) = 0 \text{ iff } p = q = 1$
$(Cpq \to BpNq) = 0 \text{ iff } p = q = 0$
$(Lpq \to MpNq) = 0 \text{ iff } p = 1, q = 0$
$(Mpq \to LpNq) = 0 \text{ iff } p = 0, q = 1$
$(Vpq \to CpNq) = 0 \text{ iff } p = q = 1$
$(Xpq \to LpNq) = 0 \text{ iff } p = q = 0$
$(Lpq \to XpNq) = 0 \text{ iff } p = 1, q = 0$
$(Cpq \to VpNq) = 0 \text{ iff } p = 0, q = 1$

We will also give a justification of this facts in the next section. Following with contraposition, the octahedron $D2$ satisfy two more rules that the octahedron $D1$, in this sense we get four rules. And finally the octahedron $D2$ has three rules while the octahedron $D1$ only has two. To end this section we present the list of rules of $D2$ octahedra, and in the next section we try to solve the problems generated by these structures:

1. $Xpq \to Xqp$
2. $Vpq \to Vqp$
3. $Mpq \to Vqp$
4. $Xpq \to Bqp$
5. $Xpq \to Cqp$
6. $Lpq \to Vqp$
7. $Mpq \to XpNq$
8. $XpNq \to MpNq$
9. $Vpq \to BpNq$
10. $Bpq \to VpNq$
11. $Mpq \to MNqNp$
12. $Bpq \to BNqNp$
13. $Lpq \to LNqNp$
14. $Cpq \to CNqNp$
15. $Mpq \to BNpq$
16. $Xpq \to VNpq$
17. $Lpq \to CNpq$

4 Solving the Difficulties of the Octahedra: $D1 + D2 =$ Hexagonal Bipyramid of Opposition

In the previous section we have displayed the construction of two octahedra that extend Williamson's squares and preserve the propositional reconstruction of the traditional square of opposition. Despite being a conservative extension the octahedra they have some difficulties relative to the validity of the immediate inference rules and the symmetry of both. In this section we discuss these facts that cause problems and do not allow us to reconstruct faithfully the traditional square in terms of propositional logic; then, based on the analysis we argue in favor of the construction of a more complex structure that connects all the previous polyhedra. This structure is an Hexagonal Bipyramid. The novelty with respect of its construction could be summarize in the following points: (1) with this analysis we can establish some relevant properties needed to understand the restrictions of the rules of traditional square of opposition, (2) we will add a new restriction for a correct reconstruction of the square in terms of propositional logic, namely, the commutativity property; and finally (3) we define essential properties of the four corners of the square of opposition from the point of view of propositional logic. The last point will be emphasized in the final section in which we will apply all the results presented here to the traditional square of opposition.

We begin detailing the steps to form this structure and consequently we analyze the resulting rules. When we analyze these rules we present reasons for the exclusion of the formulas not satisfied in the octahedra and thus solve the problems of the previous section. With this solution we will undermine the asymmetry in the previous interpretation.

The reason for the asymmetry is again that the octahedra are intermediate points in building a more complex figure that more faithfully reconstructs both propositional interpretation of square as the Williamson's thesis. Initially, to show how to pass from the octahedra to the Hexagonal Bipyramid we need to transform the octahedra in hexagons as we see below (Fig. 8).

As we know, the octahedra are only a 3D-representation of a 2D-structure, namely the hexagon of opposition [4, p. 181], [2]. The choice of one representation over the other obeys heuristic questions, in the above case what guided our way to generate the three-

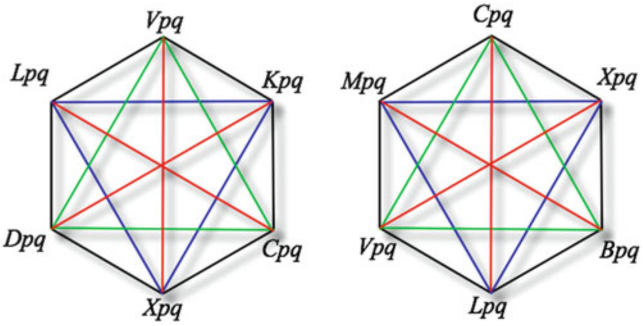


Fig. 8 HD1 and HD2

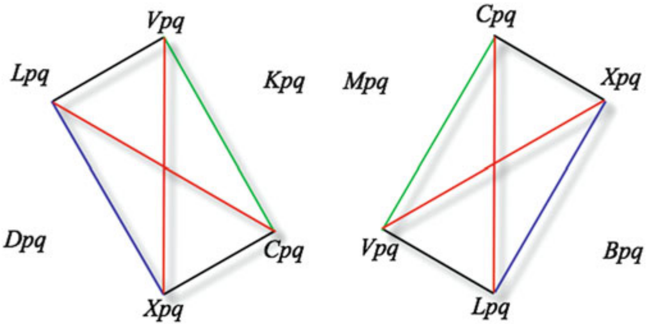


Fig. 9 SP2 in HD1 and HD2

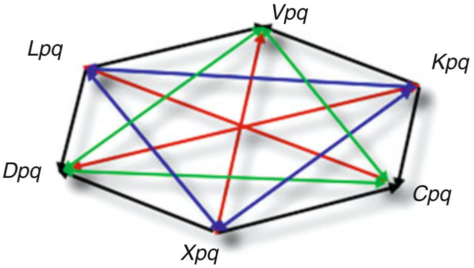


Fig. 10 Base=HD1

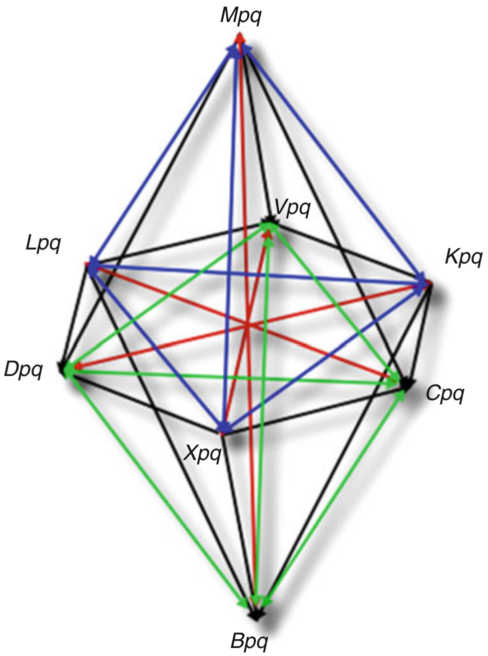
dimensional structure was to highlight two important facts: (1) the function of the spurious square, and (2) to display the asymmetry between the two octahedra, in the sense that the spurious square *SP*2 is in both octahedron and the spurious *SP*1 only the first. These facts are important now because for construction of the Hexagonal Bipyramid we need to consider again the function of the spurious squares. In this case we will take advantage of the visual characteristics of the hexagons to punctuate our thesis.

The feature that we wish to emphasize extracted from the three-dimensional analysis is the presence of the spurious square *SP*2 on both hexagons (octahedra), as we can see in Fig. 9.

We must highlight several facts that support our way of proceeding. As in the previous case, by joining two spurious squares with two genuine squares we take as a point of intersection an axis of contradictories, now what we need is another intersection point between the hexagons, and this is precisely the spurious square. The clue that led us to unite them was precisely the presence of this square on both structures generating asymmetry in the reconstruction.

To generate the Hexagonal Bipyramid (2*PH*) we take as the base the hexagon as shown in Fig. 10. Now, to complete the Hexagonal Bipyramid we need to remove from *HD*2 the vertices that are in the base, i.e., the spurious square *SP*2. This leaves us with a contradictory line going from *Mpq* to *Bpq*. The remainder is to complete the figure by

Fig. 11 2PH



taking the *MB* shaft and cutting the base by the center as shown in Fig. 11. The vertices of this axis works as the tips of the Hexagonal Bipyramid.

The following is to talk about the inference rules generated by this structure, and from this discussion we will study in depth the problems generated in the previous section and the solution that this structure provides. We must highlight several facts: (1) The operators of this figure are sufficient to generate a complete reconstruction of the traditional square from propositional logic; (2) the operators satisfy the constraints identified by Williamson; and (3) the use of these operators in specific vindicates the Williamson's thesis. In addition it should be noted a fact concerning operator properties: In addition to dividing the operators of this structure in "0-predominant" (universal) and "1-predominant" (particular), they can be subdivided into commutative and noncommutative. In this division lies the solution to the above problems and it is the key to understanding the properties of immediate inference rules.

We analyze the square with this distinction. Squares *WP*1 and *WP*2 satisfy a common feature, from the pair of universal operators one is commutative when the other is not, in *WP*1 the *L* is noncommutative and *K* is commutative, and in *WP*2 the *M* is noncommutative and *X* is commutative. The same in the particulars, in the first square *D* commutative and *C* noncommutative and in *WP*2 the *V* is commutative and *B* is not. This makes us suppose that in addition to the restrictions outlined by Williamson to get a correct reconstruction of the traditional square, we can add the following restriction: the commutative/noncommutative combination is distributed symmetrically on the square. In other words, it is not possible to have two adjacent commutative or noncommutative formulas in the square.

We can be more radical and specify this restriction for categorical formulas as follows. The A and the O corners must contain only a noncommutative operator and the E and I corner must contain only a commutative operator.[7] This restriction is preserved in all structures generated above, and from it we can now solve the problems encountered. Now we will analyze the rules that satisfies the Hexagonal Bipyramid and why only meets that set, consequently we explain why the other structures left out several potential rules. First, joining the two octahedra an intersection between the rules is generated because there are rules that both structures satisfy, we first present equipollence rules, which only undergo a change in the number. In each Williamson's squares there are sixteen equipollence rules, and in each octahedra there are thirty two rules, and now we have the following list of sixty four rules:

$$\begin{array}{l}
Lpq \equiv KpNq \equiv NCpq \equiv NDpNq \equiv NVNpq \equiv XNpq \equiv MNpNq \equiv NBNpNq \\
Kpq \equiv LpNq \equiv NDpq \equiv NCpNq \equiv NVNpNq \equiv XNpNq \equiv MNpq \equiv NBNpq \\
Dpq \equiv CpNq \equiv NKpq \equiv NLpNq \equiv VNpNq \equiv NXNpNq \equiv NMNpq \equiv BNpq \\
Cpq \equiv DpNq \equiv NLpq \equiv NKpNq \equiv VNpq \equiv NXNpq \equiv NMNpNq \equiv BNpNq \\
Vpq \equiv CNpq \equiv DNpNq \equiv NKNpNq \equiv NXpq \equiv NLNpq \equiv NMpNq \equiv BpNq \\
Xpq \equiv LNpq \equiv KNpNq \equiv NDNpNq \equiv NVpq \equiv NCNpq \equiv MpNq \equiv NBpNq \\
Mpq \equiv XpNq \equiv NBpq \equiv NVpNq \equiv NCNpNq \equiv LNpNq \equiv KNpq \equiv NDNpq \\
Bpq \equiv VpNq \equiv NMpq \equiv NXpNq \equiv CNpNq \equiv LNpNq \equiv NKNpq \equiv DNpq
\end{array}$$

Now we continue with the rules of immediate inference. The first group comprises the simple conversion rules. We have previously said that this rule is generated only between E and I of the traditional square. From our commutative analysis we can establish that the cause of this is that the formulas that can represent E or I corners are only formulas with commutative operator. Therefore, the operators susceptible to occupy one of those two corners are commutative, and consequently always preserve simple conversion. For this reason in the first octahedron there are more simple conversion rules that in the second, because the first octahedron has more universal commutative operators. In the second place the conversion rule, as we said states that a universal formula implies his subaltern with subject and predicate interchanged. The reason that there are formulas that do not satisfy this rule is that as the formula involved should be subaltern, one must be commutative if the other is not. In the Hexagonal Bipyramid we have twelve rules, nine present in the octahedra six on the first and four in the second, with a repeated rule present in both, and in addition to these, three new that resulted from the union of the two octahedra.

The next rule is the obversion. In this case there are several facts that highlight. First, we have said that this rule is generated between pairs of contrary or subcontrary formulas. Also, there are twenty four potential rules. Considering contrary and subcontrary relations

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[7] This restriction does not exclude the categorical notation or the first order interpretation of the square, and in the last section we will see why.

in the Hexagonal Pyramid we obtain the following list, of which only the first eight are satisfied rules.

1. $Lpq \rightarrow KpNq$
2. $Kpq \rightarrow LpNq$
3. $Dpq \rightarrow CpNq$
4. $Cpq \rightarrow DpNq$
5. $Mpq \rightarrow XpNq$
6. $Xpq \rightarrow MpNq$
7. $Vpq \rightarrow BpNq$
8. $Bpq \rightarrow VpNq$
9. $Kpq \rightarrow XpNq$
10. $Xpq \rightarrow KpNq$
11. $Vpq \rightarrow DpNq$
12. $Dpq \rightarrow VpNq$
13. $Vpq \rightarrow CNqNp$
14. $Xpq \rightarrow LNqNp$
15. $Lpq \rightarrow XNqNp$
16. $Cpq \rightarrow VNqNp$
17. $Bpq \rightarrow CpNq$
18. $Cpq \rightarrow BpNq$
19. $Lpq \rightarrow MpNq$
20. $Mpq \rightarrow LpNq$
21. $Mpq \rightarrow KNqNp$
22. $Kpq \rightarrow MNqNp$
23. $Bpq \rightarrow DNqNp$
24. $Dpq \rightarrow BNqNp$

The remaining are some of those mentioned above that are excluded from the octahedra and generate a pattern on an assignment that makes false (9–20). The second important fact is that there are two connected reasons that cause the last group of formulas are excluded, on the one hand that the operators must satisfy the adjacency of commutativity, so we can not find combinations of rules in which there are two commutative operators or two noncommutative. Although we found relations between commutative and noncommutative in the Hexagonal Bipyramid, this fact is justified because of the spurious squares connect the genuine ones. And this brings us to the second reason, the formulas excluded from this rule belongs to spurious squares. The formulas 9–12 belong to $SP1$ the 13–16 to $SP2$, and the remaining are not in any of the squares presented so far, and that is due to the fact that there are two new spurious squares that result of the union of the two octahedra (Fig. 12).

These squares are only present in the Hexagonal Bipyramid because of its vertices are scattered on both octahedra. They are analogous to the above in the following sense. $SP1$ and $SP3$ are spurious because the former is composed of commutative operators and the second noncommutative operators, these are spurious because they cancel commutativity adjacency. On the other hand, the $SP2$ and $SP4$ are spurious because they do not satisfy equipollence. This is how the problems of the octahedra are cleared and asymmetry is solved. Finally we analyze contraposition and inversion rules.

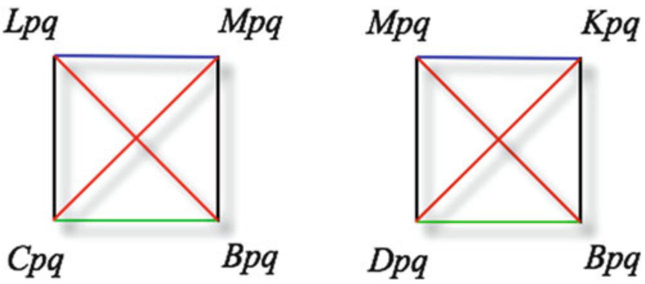


Fig. 12 SP3 and SP4

The contraposition rules are only satisfied by noncommutative operators, therefore there are only four, and that explains why in the octahedra are only two in the first and four in the second; the asymmetry is explained by the predominance of commutative operators in $D1$ and the prevalence of non-commutative $D2$. Finally, inversion rules are satisfied between pairs of contradictory operators, and in this case there is no difficulty, leaving us with the following list of rules. We continue in the las section with the interpretation of this facts in the traditional square.

1. $Kpq \rightarrow Kqp$
2. $Dpq \rightarrow Dqp$
3. $Xpq \rightarrow Xqp$
4. $Vpq \rightarrow Vqp$
5. $Lpq \rightarrow Dqp$
6. $Lpq \rightarrow Vqp$
7. $Kpq \rightarrow Cqp$
8. $Kpq \rightarrow Vqp$
9. $Xpq \rightarrow Dqp$
10. $Xpq \rightarrow Cqp$
11. $Mpq \rightarrow Vqp$
12. $Xpq \rightarrow Bqp$
13. $Lpq \rightarrow Vqp$
14. $Mpq \rightarrow Dqp$
15. $Mpq \rightarrow Cqp$
16. $Kpq \rightarrow Bqp$
17. $Lpq \rightarrow KpNq$
18. $Kpq \rightarrow LpNq$
19. $Dpq \rightarrow CpNq$
20. $Cpq \rightarrow DpNq$
21. $Mpq \rightarrow XpNq$
22. $Xpq \rightarrow MpNq$
23. $Vpq \rightarrow BpNq$
24. $Bpq \rightarrow VpNq$
25. $Lpq \rightarrow LNqNp$

26. $Cpq \rightarrow CNqNp$
27. $Mpq \rightarrow MNqNp$
28. $Bpq \rightarrow BNqNp$
29. $Lpq \rightarrow CNpq$
30. $Kpq \rightarrow DNpq$
31. $Xpq \rightarrow VNpq$
32. $Mpq \rightarrow BNpq$

5 From Bipyramid to Octagon of Opposition

In this section we discuss the final part of the analysis with reference to the first square presented: *TS*1. The thesis that we defend to close is related to the bond that—according to Williamson [5, p. 500]—Łukasiewicz established between logic of terms and propositional logic. To do this, we will present two ways to view the Hexagonal Bipyramid in which emphasis is placed on the Williamson's thesis as a unified way to present both squares. Our strategy will be to present the pyramid in traditional notation (A, E, I, O) and consequently order it to form a cube and an octagon, with reference to the two squares; finally we will use notation of first-order logic to show structural similarities and again we use the commutative interpretation to analyze the differences of each vertex. The following figure shows the Hexagonal Bipyramid with traditional notation (Fig. 13).

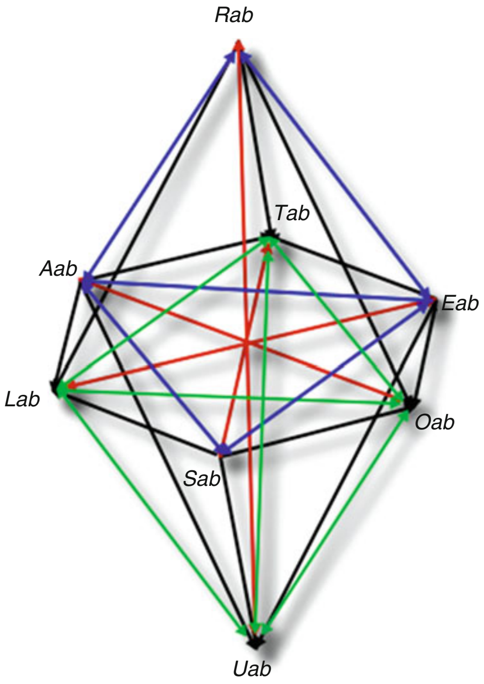


Fig. 13 Traditional 2PH

To better appreciate the link between the two squares we build a cube showing how they connect. This cube in turn can be transformed into a octagon which is simply the interpretation of two-dimensional cube. The cube shows how the two squares are connected from spurious square, in this representation becomes clear its function. The question now is how to interpret the Williamson's thesis from the relationship between these two squares (Fig. 14)?

Our position is that there are two squares, because of an important property of the operators, the inversion. Following to Gottshalk [3, p. 194] '[t]o invert a column of T's and F's is to turn the column upside down". The two squares *WP*1 and *WP*2 are inverse each other, and for that reason both separately satisfy the restrictions indicated for proper reconstruction of traditional logic, but also for that reason together satisfy the constraints. This octagon meets opposition relations in a different order than the other octagons, i.e. medieval octagons [1]; this is also due to inversion. For this reason, we can call this *The Inversion Octagon* (Fig. 15).

This octagon is the ultimate reconstruction of the traditional square, but still we can ask what about the remaining connectives of propositional logic, if we apply Williamson's thesis to get another octagon, this is also one that reconstructs the traditional theory of opposition? We believe that the answer is no because of the following three reasons: (1) the

Fig. 14 Cube of opposition

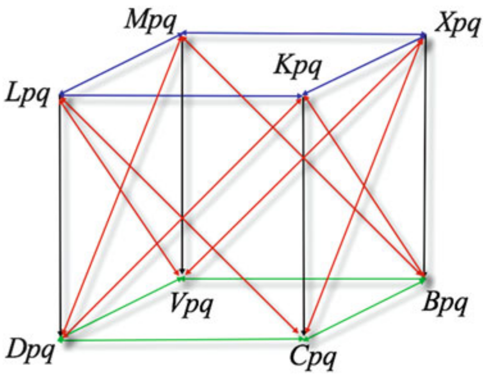
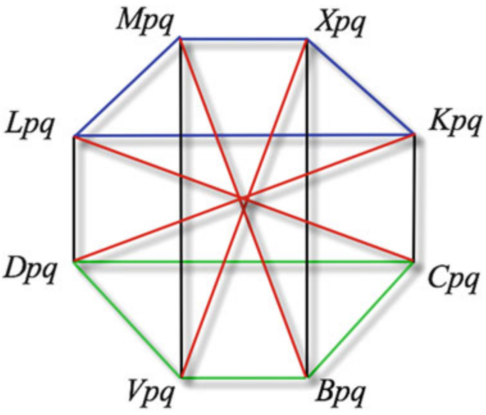


Fig. 15 Inversion octagon



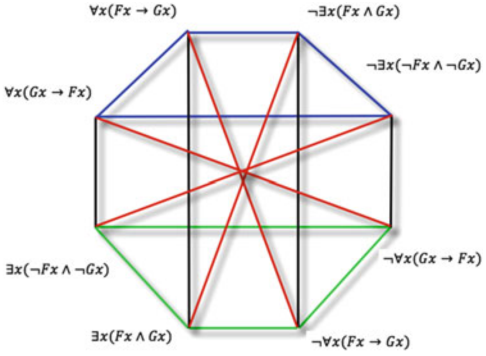


Fig. 16 First-order inversion octagon

remaining operators are not 0-predominant (therefore there is no universal operators), (2) the operators are not 1-predominant(therefore there in no particular operators), and (3) do not meet commutative adjacency. Now to conclude, we analyze this results in the first-order octagon of transposed squares (Fig. 16).

This octagon has the same properties of the previous one and therefore preserves all the equipollence and immediate inference rules presented. Also, is has the same constraints related with commutative adjacency, but in which sense this octagon preserves inversion? We believe that the octagon also satisfies inversion, but in different way depending on whether the formula is commutative or not. For example, take $\forall x(Fx \to Gx)$ we obtain its inverse only exchanging the F for the G; on the other side take $\exists x(Fx \wedge Gx)$ we obtain its inverse denying Fx and Gx. The first process is applied only to noncommutative formulas and the second to commutative ones. In both cases the inversion is satisfied in the sense that inversion may be defined as the negation of duality, in our octagon if we take again $\exists x(Fx \wedge Gx)$ we obtain its inverse changing the $\wedge$ for its dual $\vee$ and denying them, we obtain $\exists x\neg(\neg Fx \vee \neg Gx)$ which is equivalent to $\exists x(\neg Fx \wedge \neg Gx)$. the same with the remaining corners of the octagon. Finally, we think that this results vindicate the intuition of Łukasiewicz and Williamson [5, p. 500], namely:

> These results cast some light on a certain kind of connection between syllogistic and propositional logic. It has been stressed, especially by Łukasiewicz, that the procedures of traditional logic presuppose laws of propositional calculus. The analogies described above, however, rest on a direct comparison of the logic of terms and the logic of propositions; and they appear to suggest that syllogistic and propositional logic express, at some level, a common structure of reasoning.

6 Conclusion

We may summarize the main results in the following points: (1) Williamson's thesis serve us to generate many opposition structures that hold the constrains imposed in the paper to make a correct reconstruction of the traditional syllogistic logic in terms of proposition logic; (2) we emphasize that the commutativity property play a relevant role in the traditional presentation of the square, and therefore (3) we show the structural connection between these two structures.

Finally, we think that our interpretation of the connectives and quantifiers could be extend to analyze some relevant notions in logic, like the medieval distribution theory, the existential import, and the relation of the spurious square and the *disparate* in medieval octagons of opposition, but it remains open for further work.

Acknowledgements Many thanks to Yessica Ramos for their detailed revisions to previous versions of this paper, and I especially want to thank the two jurors who reviewed earlier versions of this work. Without his detailed comments would have been impossible to reach the conclusions presented here.

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Iconic and Dynamic Models to Represent "Distinctive" Predicates: The Octagonal Prism and the Complex Tetrahedron of Opposition

Ferdinando Cavaliere

Abstract The predications of the Blanché Hexagon, enriched by converses and negative terms, can be integrated into the 'Octagonal Prism' of Opposition, an exhaustive model drawn from Distinctive Predicate Calculus, here presented in iconic version. The 7 basic expressions, added to 9 cases without any existential presuppositions, are exhaustive and geometrically organizable in a Complex Tetrahedron of Opposition. This model has 'dynamic' features and a substructure, the 'Double Diamond', that is semantically interpretable in terms of synonymies that are gradually different, and can play an important role in theoretical and applied disciplines (e.g.: semantic search engines, translators).

Keywords Distinctive logic • Hexagon of opposition • Knowledge representation • Non-standard logic • Predicate logic • Synonymies • Tetrahedron of opposition

Mathematics Subject Classification Primary 03B65, Secondary 68T30, 03B20, 03B22, 03B60, 03B80

1 Distinctive Logic and the Octagonal Prism of Opposition

Taking an ordered pair of sets *ba*, we can use categorical predications to express the various possible cases. That is, we have:

- every b is a (Aba) universal affirmative
- no b is a (Eba) universal negative
- only some b is a (Yba) *distinctive* or *partial* or *exclusive* particular

The quantifier Y represents the intuitive natural language *some* and not the existential *some* of classical predicate logic. The latter means 'at least one, perhaps all', not excluding the universal quantifier, whereas the former stands for 'only some'. The partial quantifier presupposes the existence of at least two elements in the term that quantifies, the one for which the predicate delivers truth, the other for which it does not. The adjective *distinctive* alludes to the necessity to distinguish, in the subject of the particular, these two kinds of elements.

The three basic predications mentioned are mutually exclusive (incompatible) and jointly comprehensive. To these predications, Aba, Yba, Eba, called 'contrary', can be added the corresponding negations Oba, Uba, Iba, called 'sub-contrary'. The well-known

J.-Y. Béziau, G. Basti (eds.), *The Square of Opposition: A Cornerstone of Thought*, Studies in Universal Logic, DOI 10.1007/978-3-319-45062-9_16

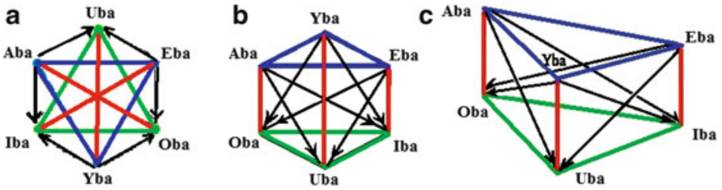
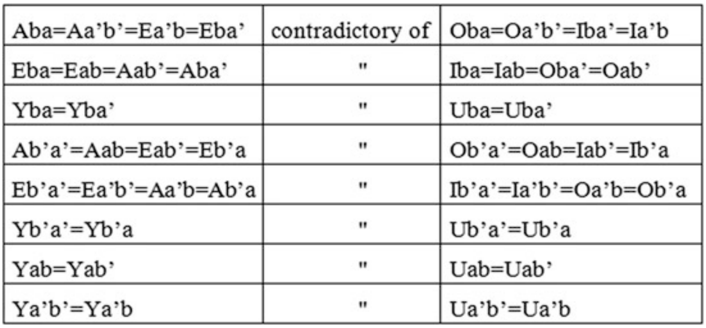


Fig. 1 Hexagons standard (**a**), modified (**b**), and prism (**c**) of opposition



Aba=Aa'b'=Ea'b=Eba'	contradictory of	Oba=Oa'b'=Iba'=Ia'b
Eba=Eab=Aab'=Aba'	"	Iba=Iab=Oba'=Oab'
Yba=Yba'	"	Uba=Uba'
Ab'a'=Aab=Eab'=Eb'a	"	Ob'a'=Oab=Iab'=Ib'a
Eb'a'=Ea'b'=Aa'b=Ab'a	"	Ib'a'=Ia'b'=Oa'b=Ob'a
Yb'a'=Yb'a	"	Ub'a'=Ub'a
Yab=Yab'	"	Uab=Uab'
Ya'b'=Ya'b	"	Ua'b'=Ua'b

Fig. 2 Immediate inferences

oppositional Hexagon of Blanché [1] graphically organizes these 6 predications and their relations of opposition, contradiction, subcontrariety, subordination (Fig. 1a).

We give here a representation of the oppositional hexagon different from the standard one (Fig. 1a, see Beziau [2]): here (Fig. 1b) the triangle of contraries (blue lines) forms the top triangle, that of subcontraries (green lines) the bottom triangle, and the contradictories (red lines) are shown as vertical connections of the three vertices of the two triangles, rather than as opposite vertices, as in the standard representation. The black arrows represent entailment relations.

The next step consists in re-fashioning the Hexagon in a tridimensional model as a Triangular Prism (Fig. 1c), which would make it a suitable extension of the model to encompass negative or complementary predicates (b′, a′) as well as inversions of the type a′b, ab′, etc. In this case the condition holds that none of the positive (b, a) or negative (b′, a′) terms has a null extension (and consequently, an extension that equals the universe).

This allows for a total of 24 combinations for A, E and Y forms. Some forms are equivalent. In the end, we are left with only 8 mutually irreducible primitive predications (Fig. 2, first column) plus 8 derived from them by negations (Fig. 2, last column).

The Triangular Prism is enriched with the new categoricals, whose equivalences are well detectable in the model of the Octagonal Prism (see Fig. 3, where black arrows are omitted for simplification).

Each triangle of contraries coincides with the mirror triangle of its categorical obversions. For example: the triangle Aba–Yba–Eba coincides with that of Eba′–Yba′–Aba′.

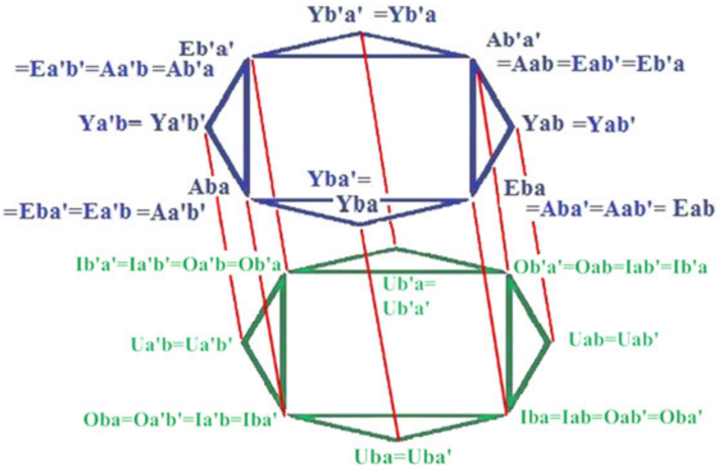


Fig. 3 Octagonal Prism

2 ‘Compound’ Development

In order to avoid redundancy, we focus our interest on the octagon formed by four triangles of contraries, ignoring those of subcontraries, that are not primitive. Little investigated, as far as we know, is the binary conjunction of the predications of these triangles. In the Fig. 4 we have connected by means of arrows the compatible categorical of the triangles Aba-Yba-Eba and Ab′a′-Yb′a′-Eb′a′ in two-by-two combinations.

At the end of the arrows we have placed the Venn diagram uniquely identified by the conjunctions given (in green the Universe of Discourse).[1]

[The pink ‘Double Diamond’ arrangement will be explained later].

The 7 diagrams thus obtained are exhaustive of the possible extensional relations between two classes, in addition considering their complementaries.

---

[1] Conjunction of categorical of two non-consecutive triangles in the Octagon of opposition:

	Aba	Yba	Eba
Ab'a'	Aba *Ab'a'	Yba *A b'a'	incompatible
Yb'a'	Aba * Yb'a'	Yba * Y b'a'	Eba * Y b'a'
Eb'a'	incompatible	Yba * E b'a'	Eba * E b'a'

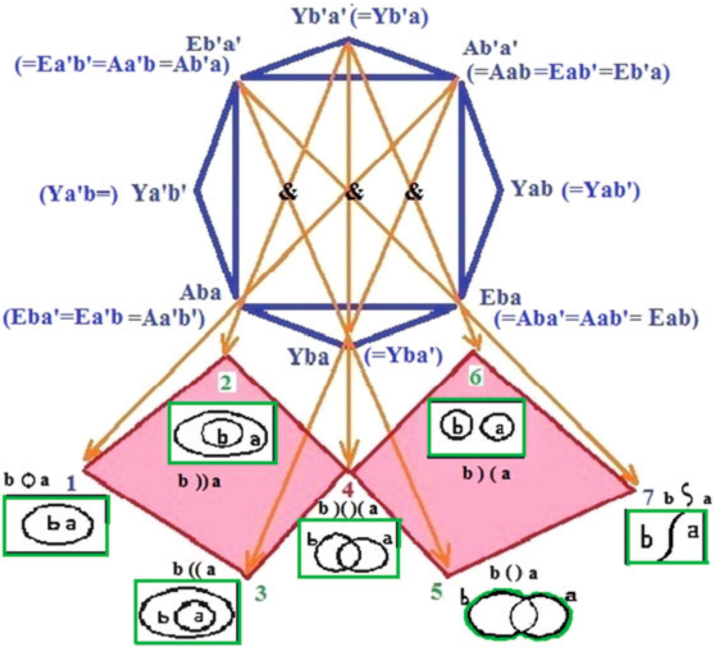


Fig. 4 Conjunctions

A similar result isn't obtained if we select two triangles that are consecutive because some conjunctions will not have a unique diagrammatic interpretation.[2]

Figure 5 shows how the compatible conjunctions of one categorical with a second that has its (positive) terms in inverse order, correspond to the 5 relations of Gergonne [3] (shown in red).

If we introduce negative terms, the Gergonne notation can distinguish the cases 5, 6 and 7, but cannot identify case 4.[3] Deductive systems such as that of Gergonne (see Faris [4]), freed from redundancies, give rise to a system based on 10 cases (Fig. 6).

As one can sees, the four X's relation are ambiguous. Of the 100 syllogistic combinations (10 × 10 base cases) 16 are inconclusive.

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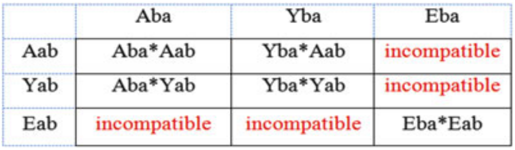
[2] Conjunction of categoricals of two consecutive triangles in the Octagon of opposition:



	Aba	Yba	Eba
Aab	Aba*Aab	Yba*Aab	incompatible
Yab	Aba*Yab	Yba*Yab	incompatible
Eab	incompatible	incompatible	Eba*Eab

[3] The following Figure shows that the Gergonne notation cannot identify case 4, other than by joining two of its relations X and using negative terms. The symbolism of the sets is even more costly than that of Gergonne.

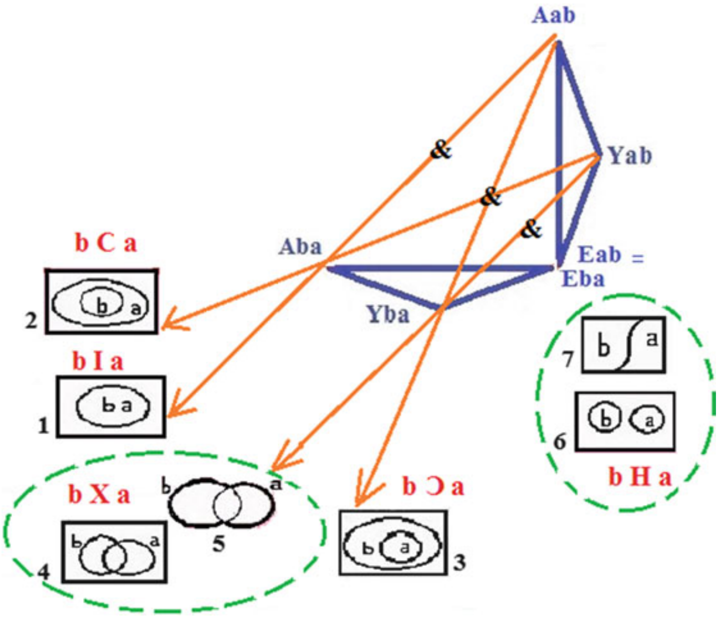
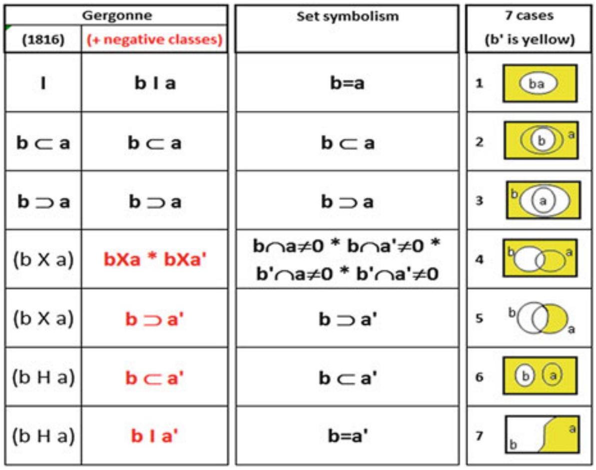


Fig. 5 Gergonne’s relations

In another study (Cavaliere [5]) it has been shown how the Gergonne relations, seen as double categoricals, can be interpreted as ‘Quantification of the Predicate’ (QoP), on which (freed from errors) were based the deductive systems of J. G. von Holland, C. E. Stanhope, G. Bentham, W. Hamilton [6], among others.



Gergonne		Set symbolism	7 cases
(1816)	(+ negative classes)		(b' is yellow)
I	b I a	b=a	1 ba
$b \subset a$	$b \subset a$	$b \subset a$	2 b a
$b \supset a$	$b \supset a$	$b \supset a$	3 b a
(b X a)	bXa * bXa'	$b \cap a \neq 0$ * $b \cap a' \neq 0$ * $b' \cap a \neq 0$ * $b' \cap a' \neq 0$	4 b a
(b X a)	$b \supset a'$	$b \supset a'$	5 b a
(b H a)	$b \subset a'$	$b \subset a'$	6 b a
(b H a)	b I a'	b=a'	7 b a

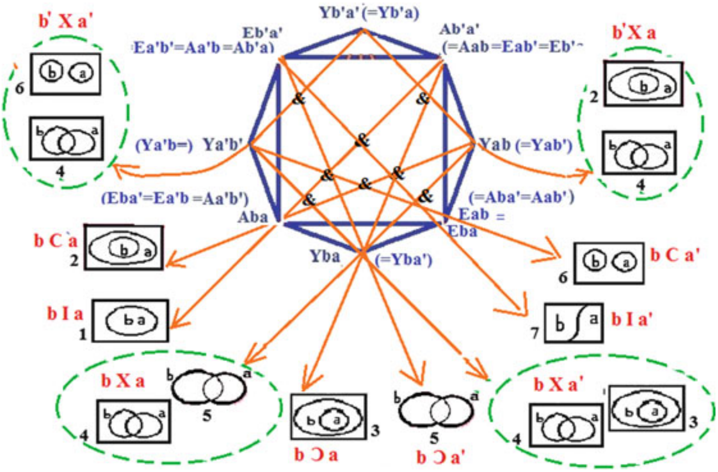
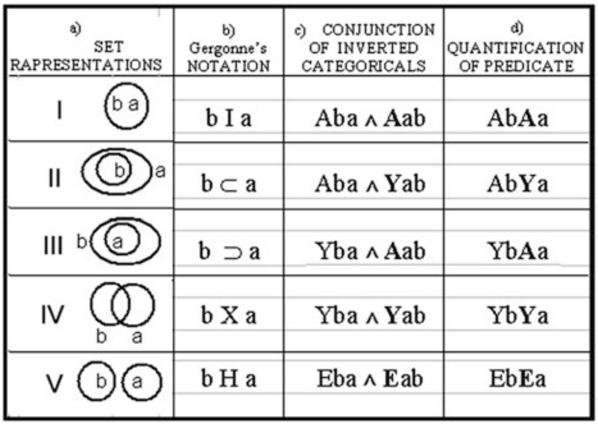


Fig. 6 Ten cases



a) SET RAPRESENTATIONS	b) Gergonne's NOTATION	c) CONJUNCTION OF INVERTED CATEGORICALS	d) QUANTIFICATION OF PREDICATE
I	b I a	Aba ∧ Aab	AbAa
II	b ⊂ a	Aba ∧ Yab	AbYa
III	b ⊃ a	Yba ∧ Aab	YbAa
IV	b X a	Yba ∧ Yab	YbYa
V	b H a	Eba ∧ Eab	EbEa

Fig. 7 Quantification of predicates

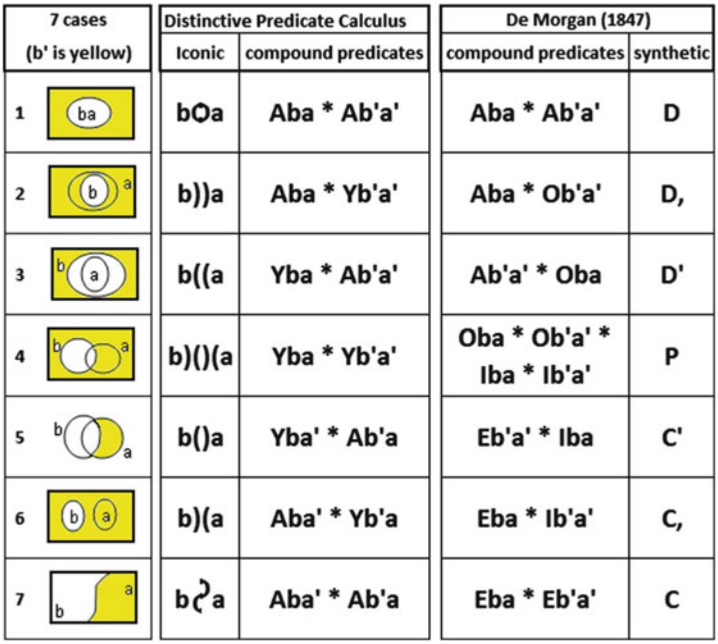
In Fig. 7, the transition is shown in the last two columns: the quantifier of the second categorical has been moved to the position before the predicate. E.g.:

All ***b*** are ***a*** and Only some ***a*** are ***b*** = All ***b*** are Only some ***a***.

The 7 cases were known to De Morgan [7] in the form of multiple conjunctions of classical categoricals, with which he built his 'Complex Syllogism'.

Using the particular quantifiers (I, O) instead of the partial (Y), he could not, however, express case 4 with only two predications (see Fig. 8), which complicated the evidence of the deductive rules.

De Morgan [8] also devised a notation system (the "spiculae" parenthesis) for syllogistic with quantification of the predicate, different from those of the authors mentioned above.



7 cases		Distinctive Predicate Calculus		De Morgan (1847)	
(b' is yellow)		Iconic	compound predicates	compound predicates	synthetic
1	ba	bOa	Aba * Ab'a'	Aba * Ab'a'	D
2	b a	b))a	Aba * Yb'a'	Aba * Ob'a'	D,
3	b a	b((a	Yba * Ab'a'	Ab'a' * Oba	D'
4	b a	b)()(a	Yba * Yb'a'	Oba * Ob'a' * Iba * Ib'a'	P
5	b a	b()a	Yba' * Ab'a	Eb'a' * Iba	C'
6	b a	b)(a	Aba' * Yb'a	Eba * Ib'a'	C,
7	b a	bʔa	Aba' * Ab'a	Eba * Eb'a'	C

Fig. 8 Seven cases

In the iconic version of our complex DPC (Cavaliere [9]), we liberally drew our inspiration (for cases 2, 3, 5 and 6) from such a system, simplifying it, but increasing its deductive power.[4]

---

[4] The code translation from grapheme to double predicate is as follows:

A..A becomes **(:)** *equals* **A..Y** becomes)) *enclosed in* **Y..Y** becomes)( )(*tetraconnects*

where the sequence of quantifiers refers to the complex predications of the table below, column IV, (in which the commas represent the second pair, complementary to the first). See Cavaliere [9].

I 7 Cases	II Explicit forms	III Equivalent forms	IV D7c
1	Aba * Eb′a	Aba * Ab′a′	AbaA″
2	Aba * Yb′a	Aba * Yb′a′	AbaY″
3	Yba * Eb′a	Ab′a′ * Yba	Ab′a′Y″
4	Yba * Yb′a	Yba * Yb′a′	YbaY″
5	Yba * Ab′a	Ab′a * Yba′	Ab′aY″
6	Eba * Yb′a	Aba′ * Yb′a	Aba′Y″
7	Eba * Ab′a	Aba′ * Ab′a	Aba′A″

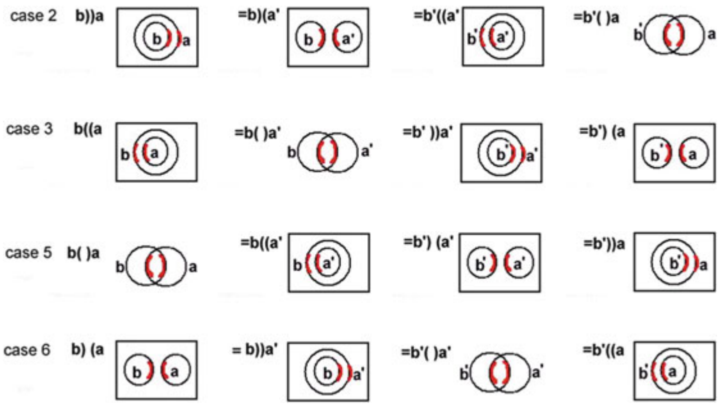


Fig. 9 Cases 2, 3, 5, 6

3 An Iconic Notation for the 7 Compound Cases

So we have created a notation with graphemes or parentheses, the latter called “closed”, if the parenthesis is concave towards its term, “open”, if it is convex. They are iconic for the diagrams corresponding to each relation (see Fig. 8, I and II columns).

The *immediate inference* rules are: the simple rule of *mirror* rotation of the *parenthesis* or other grapheme in conjunction with the inversion of the quality of the nearby term, e.g.

$$\mathrm{b}\,))\,\mathrm{a} = \mathrm{b}\,)(\,\mathrm{a}' = \mathrm{b}'\,((\,\mathrm{a}' = \mathrm{b}'\,()\,\mathrm{a}.$$

The (*equals*) relation ‘(:)’ consists of two hemicycles, the right and the left ones, each of which can refer to a term: if only one part rotates around its upper extremity, it results in a sort of ‘${}^{(}{}_{)}$’ or mirror ‘${}_{(}{}^{)}$’, (*integrates*) relation; if they both rotate the *equals* relation ‘(:)’ is restored. Instead, the rotation of (*tetraconnects*) relation)()(is not affected by free changes in the quality of the terms. This way complex distinctive predicates and immediate inference rules find easy translation into diagrams. Depending on the pair in question, we thus have four equivalent versions of diagrams and notation (Figs. 9 and 10).[5]

As regards mediated inferences of the Iconic DPC, see the deductive table of Fig. 11 that is equivalent to the one just presented in predicative forms, with the resolving algorithm in Cavaliere [9].

The compound DPC is more powerful and simpler than the Gergonne system [3] or QoP, which can, however, be derived from DPC. This incorporates classic syllogistic, including negative terms, and the Logic of Concepts (or Notions) of Vasil’ev [11], besides

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[5] Distinguishing negative and positive terms by the shape of the curve was proposed by C. S. Peirce, but only in his unpublished papers. See Moktefi and Pietarinen [10].

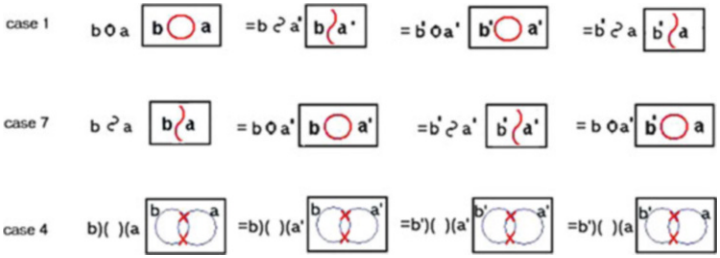


Fig. 10 Cases 1, 4, 7

		1	2	3	4	5	6	7
		b O a	b)) a	b((a	b)()(a	b( ) a	b) (a	b ₍⁾ a
1	a O c	b O c	b)) c	b((c	b)()(c	b( ) c	b) (c	b ₍⁾ c
2	a)) c	b)) c	b)) c	Ibc	Ycb	b ( ) c	Ib'c	b( ) c
3	a((c	b((c	Ib'c'	b((c	Yc'b	Ibc'	b) (c	b) (c
4	a)()(c	b)()(c	Yb'c	Ybc		Ybc	Yb'c	b)()(c
5	a ( ) c	b( ) c	Ib'c	b( ) c	Ycb	Ibc	b)) c	b)) c
6	a) (c	b) (c	b) (c	Ibc'	Yc'b	b((c	Ib'c'	b((c
7	a ₍⁾ c	b ₍⁾ c	b) (c	b( ) c	b)()(c	b((c	b)) c	b O c

Fig. 11 Deductive table

being, as far as we know, the first complete deductive system based on all the Hexagons retrievable from the categorical model of Blanché [1] (see Cavaliere [9]).

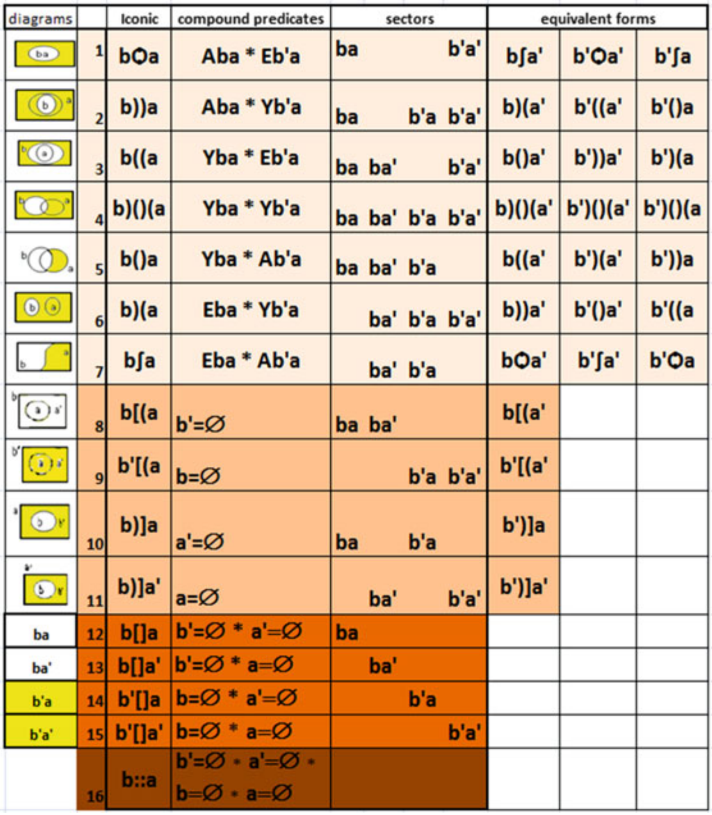
4 Existential Import and the Null Class

So far we have operated on the assumption that the classes treated were not empty. We now consider also the empty (or null) set.

In the enlargement of the bases of the system that we are building, a pair of properties is assigned to each element of the universe, whereby there are four possible choices: an element possesses both properties (ba), or neither (b′a′), or only the first of the two properties (ba′) or only the second (b′a). We can thus divide the universe into four sectors (ba, ba′, b′a, b′a′) each of which can be real (=have instantiations, representatives) or not.

In the Fig. 12, the sectors are specified for each diagrammatic situation, as well as the double predication form that describes it and the corresponding iconic notations.

For cases 8–15 a new symbolism has been introduced showing the relationship “[” (= “omnicomprehends”) that the universe has with a term. The deductive rule here stipulates that if the universal class omnicomprehends another class (not the universe) it omnicomprehends also the complementary class. The non-universal class is therefore always ‘closed’, in spite of the change of sign.



diagrams		Iconic	compound predicates	sectors	equivalent forms		
	1	bOa	Aba * Eb'a	ba b'a'	bʃa'	b'Oa'	b'ʃa
	2	b))a	Aba * Yb'a	ba b'a b'a'	b)(a'	b'((a'	b'()a
	3	b((a	Yba * Eb'a	ba ba' b'a'	b()a'	b'))a'	b')(a
	4	b)()(a	Yba * Yb'a	ba ba' b'a b'a'	b)()(a'	b')()(a'	b')()(a
	5	b()a	Yba * Ab'a	ba ba' b'a	b((a'	b')(a'	b'))a
	6	b)(a	Eba * Yb'a	ba' b'a b'a'	b))a'	b'()a'	b'((a
	7	bʃa	Eba * Ab'a	ba' b'a	bOa'	b'ʃa'	b'Oa
	8	b[(a	b'=∅	ba ba'	b[(a'		
	9	b'[(a	b=∅	b'a b'a'	b'[(a'		
	10	b)]a	a'=∅	ba b'a	b')]a		
	11	b)]a'	a=∅	ba' b'a'	b')]a'		
ba	12	b[]a	b'=∅ * a'=∅	ba			
ba'	13	b[]a'	b'=∅ * a=∅	ba'			
b'a	14	b'[]a	b=∅ * a'=∅	b'a			
b'a'	15	b'[]a'	b=∅ * a=∅	b'a'			
	16	b::a	b'=∅ * a'=∅ * b=∅ * a=∅				

Fig. 12 Cases with null class

If the second term is itself the universe, no immediate deductions can be made.

Finally, as a limiting case, when a term is empty, and so is its complement, the entire universe is empty (non-existent). The relation **::** ('double colon') expresses this 'disintegration'.

5 The Complex Tetrahedron of Opposition

An oppositional geometric model that can dynamically integrate the 7 cases of DD with the cases 8–16 is the Complex Tetrahedron (Fig. 13), provided with four 'actual' vertices plus twelve 'virtual' ones.[6]

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[6] Possible alternative to the Tetrahedron: Hasse diagram, Tesseract, Rhombic Dodecahedron (with the addition of the centre of gravity).

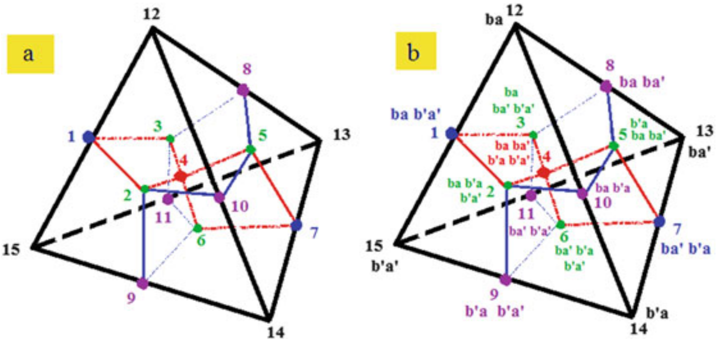


Fig. 13 Tetrahedron with diagrams (**a**) and with sectors (**b**)

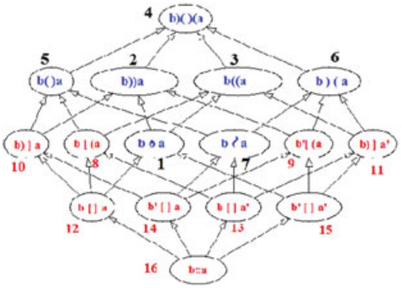
In Fig. 13a, each diagram finds its complementary on the surface of the Tetrahedron, exactly at the point opposite to it with respect to the center of gravity. The latter (red) represents the combination of all four sectors (case 4); it connects with the four midpoints (green) of the faces of the Tetrahedron, which represent the diagrams with three sectors (cases 2, 3, 5, 6).

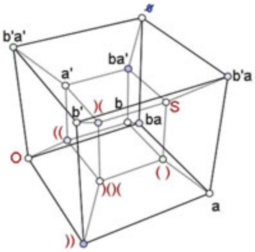
Each of these center-faces connects to three of the midpoints of the six edges of the Tetrahedron, which characterize the diagrams with only two sectors (of which two 'existential' diagrams or cases 1 and 7 (blue), and the remaining four with some empty term/class, amounting to the cases 8, 9, 10, 11 (purple)).

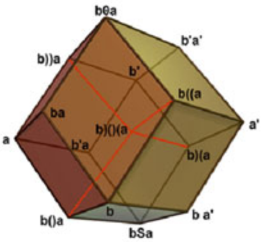
Each midpoint is connected with two of the four authentic vertices of the Tetrahedron representing the four cases with only one sector, that is, the cases 12, 13, 14 and 15 (in black).

These four vertices are connected to the outer space (or circumscribed sphere), which represents the empty universe, with 0 sectors (case 16) (in white). In summary:

- 1 case (center of gravity) with 4 sectors
- 4 cases (forming a small Tetrahedron) with 3 sectors
- 6 cases (forming an Octahedron) with 2 sectors
- 4 cases (forming a large Tetrahedron) with 1 sector
- 1 case (outer space) with 0 sectors







This tridimensional graph has ‘dynamic’ features: the existential presupposition decreases from the centre of gravity to outer space; this means that each transition from a ‘predicative’ vertex or node to a contiguous one corresponds to the subtraction or addition of a single diagrammatic logical subset or sector (Fig. 13b).

6 The Substructure of the ‘Double Diamond’

The Tetrahedron has the Double Diamond or DD, as a substructure. DD is a planar graph where each node represents a diagram, a double predication or an iconic relation (Fig. 14).

DD is inscribed into the Tetrahedron, half rotated 90° (Fig. 15, drawn by A. Moretti).

This spatial itinerary is organized as a twofold symmetry vertical and horizontal (Fig. 16).

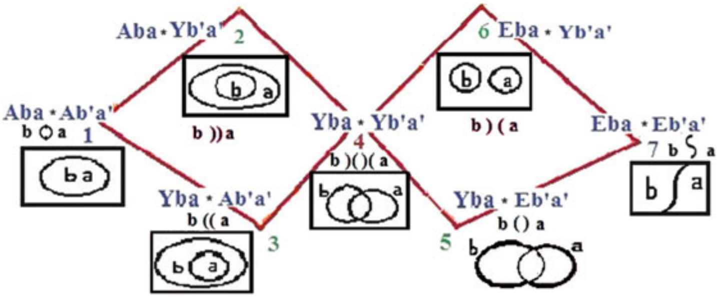


Fig. 14 The DD


2D bi-rhombus
the 3D twisted bi-rhombus
a tetrahedron emerges

Fig. 15 The DD twists

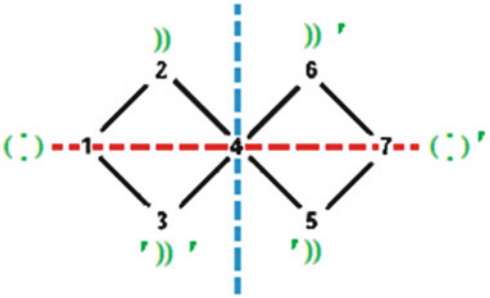


Fig. 16 Two asses of symmetry of DD

We see that, due to the immediate inference rules of the iconic relations, each complex predication placed on the left hand side of the vertical axis has a corresponding symmetrical (i.e. maintaining the same relational symbol) to the right, but with negation of the second term. The same applies proceeding from right to left. Given a complex predication, its symmetrical with respect to the horizontal axis, always maintaining the same symbol, sees the negation of both the first and the second term (or, equivalently, changes the direction of both brackets, while maintaining equal signs). Combining both symmetries we get, diagonally, expressions that differ in the sign of the first term.

The cases that are located on the axis itself, that are: 4 for the vertical axis, and 1, 4 and 7 for the horizontal one, being deprived of symbols such as parentheses, they apply the rules described above only for the part concerning the signs.

7 Interpretations and Applications of the Double Diamond

7.1 Predicative Oppositions

Thus organized, the DD assumes an important role in the theoretical metacontrol with regard to the past or present logical systems. For example, it includes the cases of the Triangle of Contraries of our Distinctive Calculus), that is equivalent to that of Vasil'ev's Logic of Concepts of 1910s years (see Vasil'ev [11]), as we can see in Fig. 17. In Fig. 18 we can see the 5 cases of Gergonne [3], analogous to those of QoP. The predications/relations in question are defined by the disjunction of the cases included in the sets shown.

The examples given all fall in the substructure of DD, but we can always, if need be, extend the sets under consideration to the vertices of Complex Tetrahedron that are free of existential presuppositions.

Fig. 17 Representation of Vasil'ev's logic of concepts

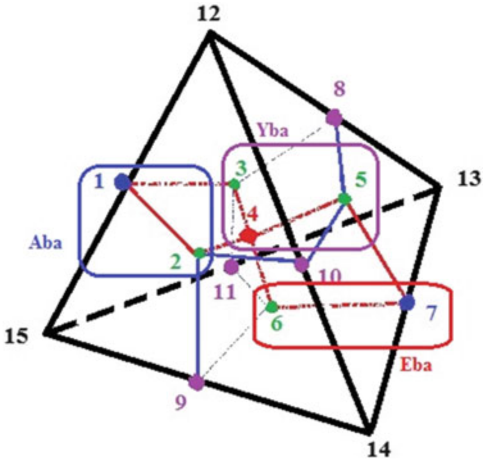
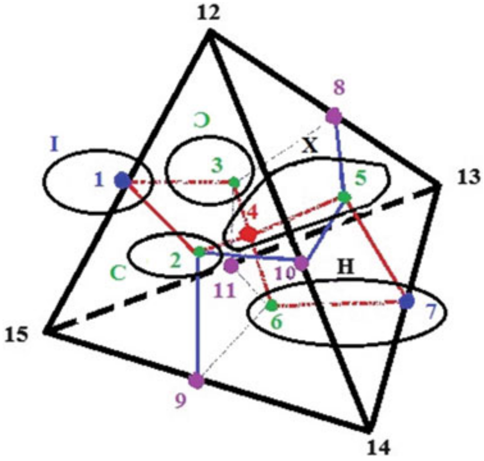


Fig. 18 Representation of Gergonne's relations



7.2 Intension (Connotation) as Opposed to Extension (Denotation)

The DD scheme lends itself to showing the relationship between an intensional (or connotational) and an extensional (or denotational) reading of concepts.[7]

So, if we consider the case 2 (or 5) descriptive of the extensional relations between the terms b and a, we will find the description of the intensional relationships between the same terms as in case 3 (or 6) and vice versa, being 2 (or 5) symmetric to 3 (or 6), respect to the horizontal axis. In cases 1, 4 and 7, the two descriptions coincide, as they are placed on the very symmetricity axis.

These considerations are a plausible hypothesis but not yet covered by a systematic check.

7.3 Similarity (Sameness) Versus Difference (Diversity)

The horizontal axis of DD shows how similarity and difference are inversely proportional concepts, when interpreted in logical terms, ordered in a linear sequence of four degrees. The classes shown in the diagrams 1–7 are defined in terms of the scale 'total-strong-weak-no Similarity' or, equivalently, 'nothing-weak-strong-total corresponding Difference' (Fig. 19).

---

[7] In a certain way, a concept can be defined as a set whose elements are its (essential) attributes or features. A proper subset will thus contain a smaller number of those very features. As one goes from a set to one of its proper subsets, certain features will get lost, which means a *generalization* or *abstraction* of the initial set. Such a restriction of features will correspond to an increase of referential extensions. The inverse procedure, the passing from a set to a superset, represents a *specialization* or *particularization*.

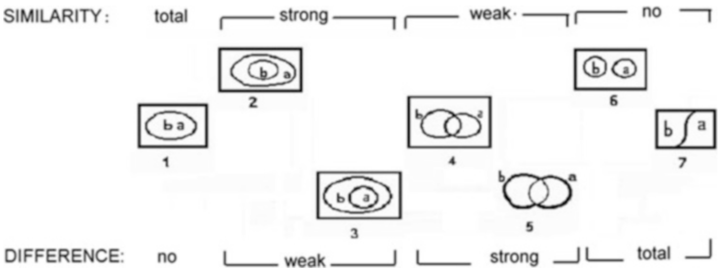


Fig. 19 Similarity scale

This interpretation may play an important theoretical role, because some scalar or fuzzy indefinable concepts from classical logic are logically illuminated by the DD model.

7.4 Synonymies

In a linguistic or semiotic context, DD gives a comprehensive survey of the possible semantic variations of the relation between two terms, showing up any conceptual gap that the disciplines mentioned have not captured. We can provide the framework for this match between the 7 cases and the types of synonymy.

The cases 1, 2, 3 and 7 are commonly known in the linguistic-semantic literature as well as in the dictionaries of synonyms and antonyms.

By analogy, we may complete the casuistry by coining new expressions such as *tetrameronymous* for (4), or *hypercomplement* for (5) or *hypocomplement* for (6) (see Fig. 20).

With expressions taken from ordinary language, a term may be, of a second one:

1. an equivalent 2. a restriction 3. an expansion 4. a limited connection 5. an integrative connection 6. a limited disconnection 7. an integrative disconnection.

7.5 Scientific and Cultural Applications: Exemples

The issue of synonymy is in turn important in the field of semantic technologies, such as data mining or ontologies. In a previous work the author has proposed the idea of a semantic search engine and translators based on the 7 types of synonymy (Cavaliere [12]).

Ignoring the specifications "integrative" and "limited" of the cases 4, 5, 6 and 7, let us return to the 5 Gergonne relations, which admit an interpretation in humanistic areas, for example in poetry (rhetorical figures), or in the library sciences (Fig. 21).

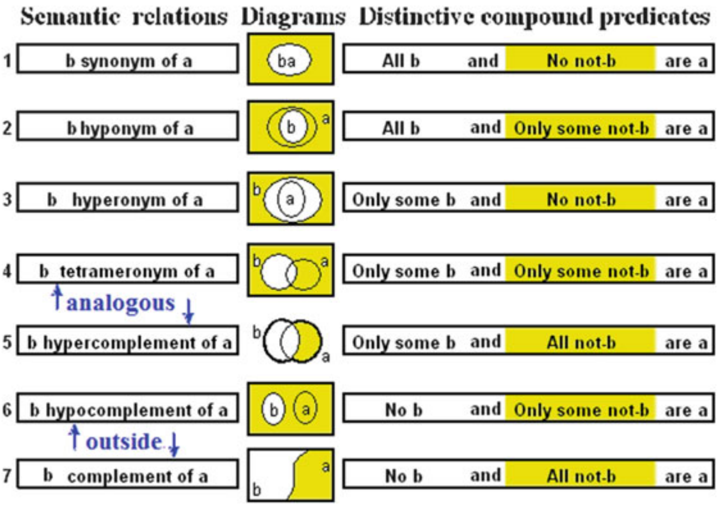
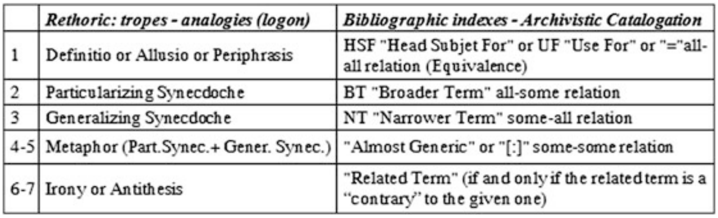


Fig. 20 Semantic cases



	Rethoric: tropes - analogies (logon)	*Bibliographic indexes - Archivistic Catalogation*
1	Definitio or Allusio or Periphrasis	HSF "Head Subjet For" or UF "Use For" or "="all-all relation (Equivalence)
2	Particularizing Synecdoche	BT "Broader Term" all-some relation
3	Generalizing Synecdoche	NT "Narrower Term" some-all relation
4-5	Metaphor (Part.Synec.+ Gener. Synec.)	"Almost Generic" or "[:]" some-some relation
6-7	Irony or Antithesis	"Related Term" (if and only if the related term is a "contrary" to the given one)

Fig. 21 Applications

7.6 Truth Values

The 7 cases can provide a classification of the main logical and philosophical orientations.

Positing that class b is that of all true predications, and class a that of all false ones, we recognize in case 7 the classical logic where every predications is true or false but not both (bivalent systems) (Fig. 22). In case 6 we can also see predications that are neither true nor false (trivalence or paracomplete or gap systems).

On the contrary, in case 5, predications appear that are both true and false (paraconsistent or glut systems). Case 4 presents all four combinations of predications described above (Four values of Belnap). In case 3 the idea that falsehood hides always truth, find a model in the Freudian ‘lapsus’. A model for case 2 is given by a philosophy that considers all truth false at the same time (that is, relative), but also admits the existence of pure (not-true) false predications (skepticism). As regard case 1, all propositions are either contradictory or meaningless (sophistics). Finally, it’s possible that some eastern philosophies (Buddhism, Zen, etc.) are related to the point of the Tetrahedron without existential presuppositions.

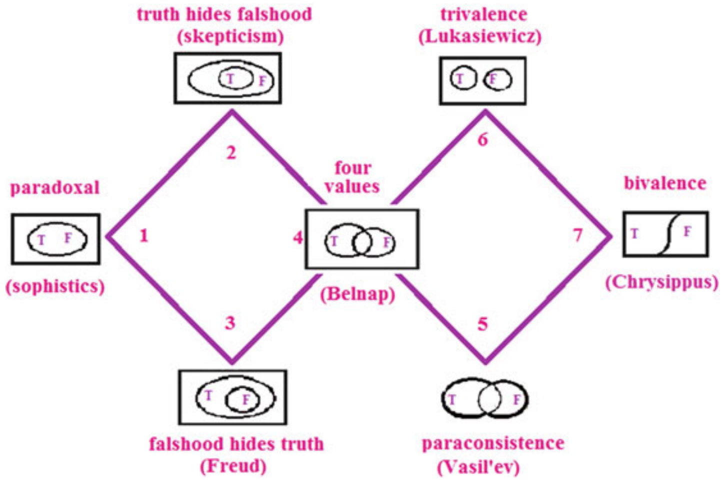


Fig. 22 Philosophies of truth

8 Possible Extensions of the Model

Theoretical extensions of the Complex Tetrahedron seem interesting in five directions:

(a) "Imaginary Logic" of Vasil'ev [11] or a similar paraconsistent logic.
(b) N-Oppositional Theory (NOT) of Moretti [13] and others.
(c) Identification of possible psycholinguistics sub-structures, such as the 'kite' model by Seuren and Jaspers [14] or other, in Seuren [15].
(d) Numerical predications with probabilistic or fuzzy applications (Cavaliere [16]).
(e) Ontologies, Semantic Web and Artificial Intelligence applications (Cavaliere [12]).

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The Exact Intuitionistic Meaning of the Square of Opposition

Joseph Vidal-Rosset

To David DeVidi and to Sean McLaughlin

Abstract This paper aims at providing a complete analysis of the intuitionistic version of the square of opposition and a reply to an article published by Mélès (Around and Beyond the Square of Opposition, ed. by J.-Y. Béziau, D. Jacquette (Studies in Universal Logic, Birkhaüser, 2012), pp. 201–218) on the same topic.

Keywords Intuitionistic logic • Square of opposition • Tableau methods

Mathematics Subject Classification (2000) Primary 03B20, 03F03; Secondary 03B10

1 The Classical Square of Opposition

I assume that the reader is familiar both with classical first order logic (for short, CFOL) and the standard symbolism adopted in this paper. Showing the discrepancies between, on the one hand, the square of opposition in CFOL and, on the other hand, the intuitionistic first order logic (for short, IFOL) is a good way to understand the latter; but before one must grasp the former. The well known classical square of opposition i.e. Fig. 1, page 292 is constructed by superimposing the two polygons of Fig. 2, page 292. Each valid implication in these polygons gives the meaning of what is either a subaltern proposition, or a contrary proposition, or contradictory proposition, or a sub-contrary proposition.

Definition 1.1 I call "negative" any formula obtained by prefixing one negation symbol to a formula of the square, i.e. $\neg(A)$, $\neg(E)$, $\neg(I)$, $\neg(O)$. A formula is "positive" if it is not negative: (A), (E), (I), (O) are positive.

Remark 1.2 There is no valid implication between two negative formulas, and there is only one validity only between a couple of positive formulas, i.e. in the subalternation case.

Remark 1.3 Most valid implications contain a positive and a negative formula.

Every theorem of this paper has been checked by IMOGEN i.e. Sean McLaughlin' theorem prover for intuitionistic First Order Logic (https://github.com/seanmcl/imogen).

J.-Y. Béziau, G. Basti (eds.), *The Square of Opposition: A Cornerstone of Thought*,
Studies in Universal Logic, DOI 10.1007/978-3-319-45062-9_17

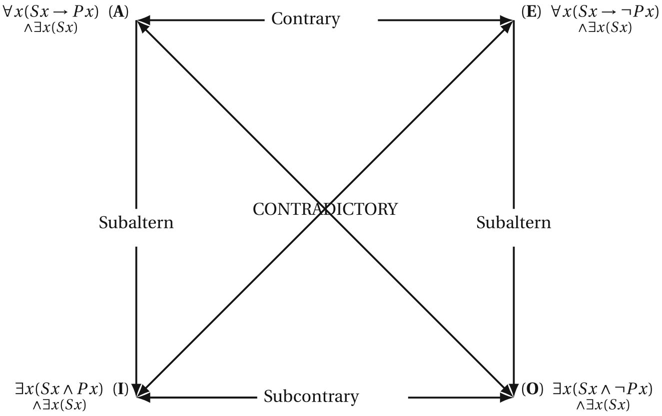


Fig. 1 Classical square of opposition

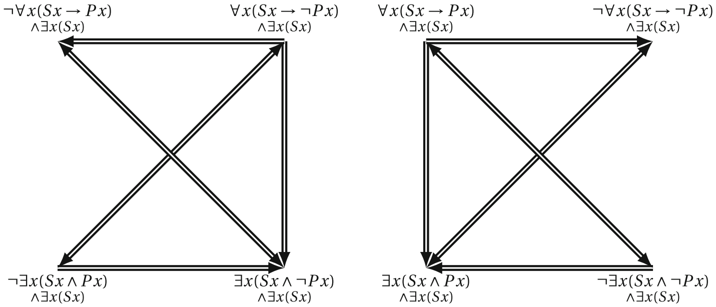


Fig. 2 Implications valid in CFOL

Remark 1.4 Contrariety and sub-contrariety relations are duals of each other: the former assumes a positive formula and concludes on a negative consequence, while the latter assumes a negative formula and concludes on a positive consequence. That explains why it is said, to explain the contrariety relation, that (A) and (E) cannot be both true but can be both false, and that (I) and (O) cannot be both false but can be both true, to explain the sub-contrariety relation.

Remark 1.5 The contradiction relation define four equivalences in the square, i.e.

1. $(A) \leftrightarrow \neg(O)$
2. $(O) \leftrightarrow \neg(A)$
3. $(E) \leftrightarrow \neg(I)$
4. $(I) \leftrightarrow \neg(E)$

Remark 1.6 The existential assumption $\exists x(Sx)$ is not necessary to prove a contradiction relation in the square; one needs the following equivalences to do it[1]:

$$(\neg\neg A \to A) \wedge (A \to \neg\neg A) \tag{1.1}$$

$$(\neg\forall x\neg(Fx) \to \exists x(Fx)) \wedge (\exists x(Fx) \to \neg\forall x\neg(Fx)) \tag{1.2}$$

$$(\neg\exists x\neg(Fx) \to \forall x(Fx)) \wedge (\forall x(Fx) \to \neg\exists x\neg(Fx)) \tag{1.3}$$

$$((A \to B) \to (\neg A \vee B)) \wedge ((\neg A \vee B) \to (A \to B)) \tag{1.4}$$

$$(\neg(A \wedge B) \to (\neg A \vee \neg B)) \wedge ((\neg A \vee \neg B) \to \neg(A \wedge B)) \tag{1.5}$$

$$(\neg(A \vee B) \to (\neg A \wedge \neg B)) \wedge ((\neg A \wedge \neg B) \to \neg(A \vee B)) \tag{1.6}$$

Example 1.7

$$\vdash_c \neg\forall x(Sx \to Px) \leftrightarrow \exists x(Sx \wedge \neg Px) \tag{1.7}$$

Proof A simple proof can be made on the basis of the classical equivalences above. By (1.3) and (1.4) one gets:

$$\neg\forall x(Sx \to Px) \leftrightarrow \neg\neg\exists x\neg(\neg Sx \vee Px) \tag{1.8}$$

By (1.1) and (1.6) one gets:

$$\neg\neg\exists x\neg(\neg Sx \vee Px) \leftrightarrow \exists x(\neg\neg Sx \wedge \neg Px) \tag{1.9}$$

By (1.1) one gets:

$$\exists x(\neg\neg Sx \wedge \neg Px) \leftrightarrow \exists x(Sx \wedge \neg Px) \tag{1.10}$$

---

[1] In this list, only (1.6) is an equivalence in IFOL: except this latter, all conjunctions, from (1.1) to (1.5) have a left component provable in CFOL but not in IFOL.

By transitivity of $\leftrightarrow$, from (1.8) to (1.10), one gets the conclusion:

$$\neg\forall x(Sx \to Px) \leftrightarrow \exists x(Sx \wedge \neg Px) \tag{1.11}$$

□

2 The Intuitionistic Square of Opposition

2.1 *Bell-DeVidi-Solomon Proof Method for Intuitionistic First Order Logic*

To understand the tree proof method for intuitionistic logic used in this paper, I reproduce in Table 1 the tree rules given by Bell et al. [1, p. 197, pp. 216–27], and also the transport and closure rules[2]:

2.2 *Mèlès' Incomplete Intuitionistic Square*

In a paper published in a volume edited by Béziau and Jacquette [2], Mélès [4, p. 207] writes that "in intuitionistic logic, the square is clearly incomplete" and illustrates immediately his claim by a figure that we reproduce in this paper in Fig. 3, page 295.

Unfortunately, Fig. 3 is misleading because it gives the impression that there is no square of opposition in IFOL. As we are going to see, Fig. 3 misses four provable implications in IFOL.

- First, in Fig. 3 the contradiction arrow from (A) to (O) must be restored. Indeed the implication

$$(A \wedge B) \to \neg(\neg A \vee \neg B) \tag{2.1}$$

is valid in intuitionistic logic, and also $\neg\exists x\neg(Ax)$ is intuitionistically deducible from $\forall x(Ax)$, and therefore[3]:

$$\vdash_i \forall x(Sx \to Px) \to \neg\exists x(Sx \wedge \neg Px) \tag{2.2}$$

---

[2] I do not mention the fork rule, because it is not used in this paper. I strongly recommend the reading of Bell et al.'s book.

[3] Symbols $\vdash_c$, $\nvdash_c$, $\vdash_i$, $\nvdash_i$ mean respectively: provable in classical logic, unprovable in classical logic, provable in intuitionistic logic, unprovable in intuitionistic logic. I use in this paper the algorithm of refutation trees defined by Bell, DeVidi and Solomon for intuitionistic logic [1, pp. 192–223] (see Sect. 2.1).

Table 1 Bell et al. tree rules for IFOL

	Disjunction	**Conjunction**	**Implication**
Affirmed	$A \vee B$ $\widehat{A \quad B}$	$A \wedge B$ A B	$A \to B$ $\widehat{?A \quad B}$
Unaffirmed	$?(A \vee B)\checkmark$ $?A$ $?B$	$?(A \wedge B)\checkmark$ $\widehat{?A \quad ?B}$	$\underline{?(A \to B)\checkmark}$ A $?B$

	Equivalence	**Negation**
Affirmed	$A \leftrightarrow B$ $\widehat{\begin{array}{cc} A & ?A \\ B & ?B \end{array}}$	$\neg A$ $?A$
Unaffirmed	$?(A \leftrightarrow B)\checkmark$ $\widehat{?(A \to B) \quad ?(B \to A)}$	$\underline{?\neg A\checkmark}$ A

rule **IUI**	rule **IEI**
$\forall x(Fx)$ Fc (for any appropriate c)	$\exists x(Fx)\checkmark$ Fc (c **new**)

rule **?UI**	rule **?EI**
$\underline{?\forall x(Fx)\checkmark}$ Fc (c **new**)	$?\exists x(Fx)$ Fc (for any appropriate c)

Transport rule: We are allowed to carry any statement not marked by "?" across any horizontal line (i.e. locality) introduced by the ? → and ?¬ rules.

Checkmark: Any checkmarked formula is deactivated.

Closure rule. A path is closed when (and only when) both P and $?P$ occur on it not separated by a horizontal line. When it is the case, the path is marked by ×. If all paths of a refutation tree $\mathscr{T}$ of a formula F are closed, $\mathscr{T}$ is closed and that shows that the assumption of the intuitionistic unprovability of F is a contradiction and that, therefore, F is intuitionistically valid.

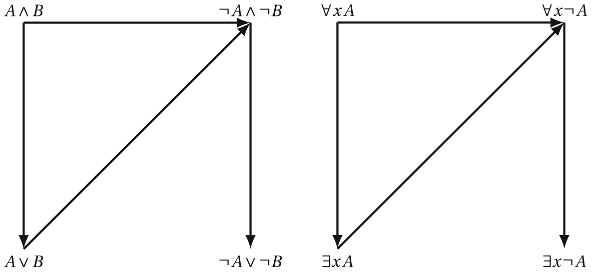
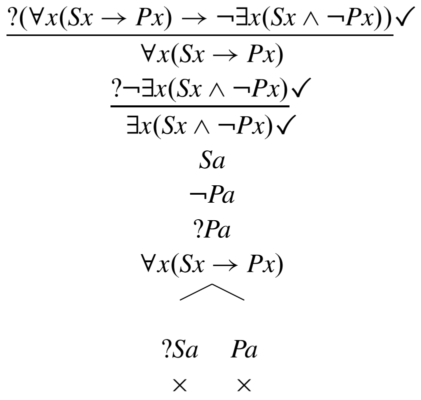


Fig. 3 Mélès' incomplete intuitionistic square

Proof

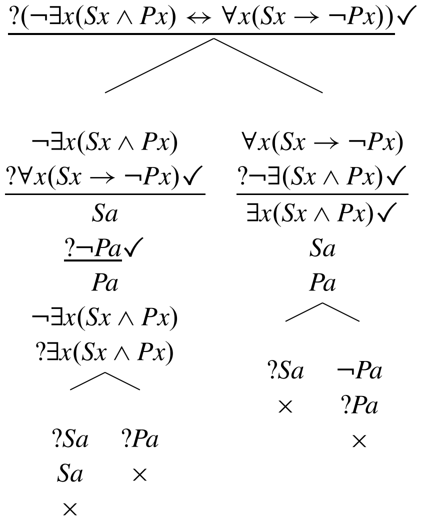


□

- Second, if one pays attention to the other contradiction relation of the square, i.e. between (I) and (E), Fig. 3 is correct but incomplete: it is true that $(A \vee B)$ entails $\neg(\neg A \wedge \neg B)$ and that the converse is not intuitionistically provable. But Fig. 3 does not show that an equivalence is intuitionistically provable between $\neg(A \vee B)$ and $(\neg A \wedge \neg B)$ and also between $\neg$ (I) and (E), i.e.

$$\vdash_i \neg\exists x(Sx \wedge Px) \leftrightarrow \forall x(Sx \rightarrow \neg Px) \tag{2.3}$$

Proof

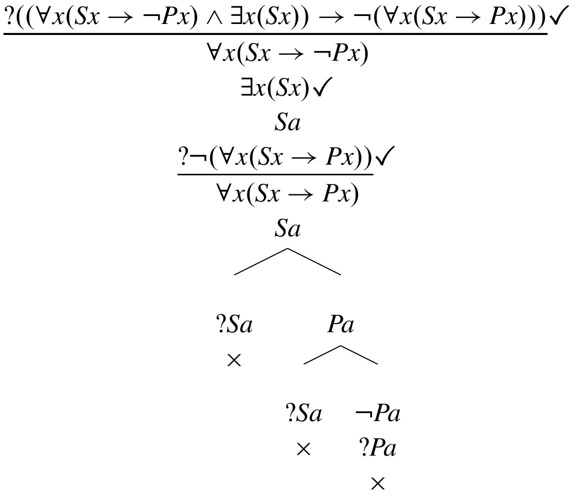


□

- Third, according to Fig. 3, the contrariety relation would be provable in intuitionistic logic only in the case of the implication $(A) \rightarrow \neg(E)$; $(E) \rightarrow \neg(A)$ would be intuitionistically unprovable. The following tree both refutes such a claim and proves this contrariety relation in IFOL:

$$\vdash_i (\forall x(Sx \rightarrow \neg Px) \wedge \exists x(Sx)) \rightarrow \neg\forall x(Sx \rightarrow Px) \tag{2.4}$$

Proof



□

- Fourth, sub-contrariety $\neg(I) \rightarrow (O)$ is provable in intuitionistic propositional logic as well as in IFOL[4]:

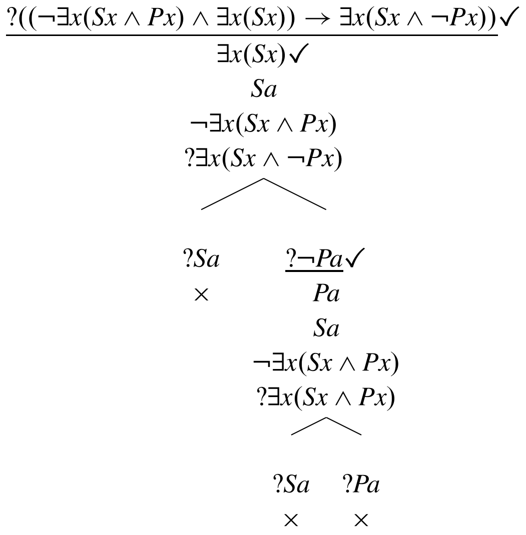
$$\neg(A \vee B) \rightarrow (\neg A \wedge \neg B), (\neg A \wedge \neg B) \rightarrow (\neg A \vee \neg B) \vdash_i \neg(A \vee B) \rightarrow (\neg A \vee \neg B) \tag{2.5}$$

$$\vdash_i (\neg\exists x(Sx \wedge Px) \wedge \exists x(Sx)) \rightarrow \exists x(Sx \wedge \neg Px) \tag{2.6}$$

---

[4] By transitivity of implication (2.5) is deducible from Fig. 3 ; but it is true that the other sub-contrariety relation i.e. $\neg(O) \rightarrow (I)$ is not intuitionistically provable. (See Table 2, page 302).

Proof



□

Consequently, we can assert a positive theorem in the next section.

2.3 A Proof That the Intuitionistic Square of Opposition Exists

Theorem 2.1 *The set of all implications intuitionistically provable in the classical square of opposition still defines a square.*

Proof By superimposing the polygons of Fig. 4. Note that all contrariety and contradictory relations with positive assumptions and negative consequences are provable as well as in the classical case, and because one sub-contrariety relation remains provable in IFOL, a square of opposition exists in IFOL. □

2.4 Do We Lose Something with the Intuitionistic Version of the Square?

As Fig. 4 shows, in IFOL the contradiction relations can no longer be expressed *via* the equivalences of Remark 1.5, except 3:

$$\vdash_i \neg\exists x(Sx \wedge Px) \leftrightarrow \forall x(Sx \rightarrow \neg Px) \tag{2.7}$$

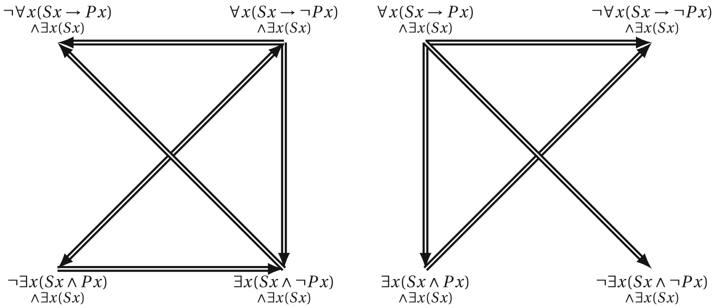


Fig. 4 Intuitionistic implications constructing the square

Comparing Figs. 4 and 2, it is clear that the beautiful duality of the classic square of opposition no longer exists in intuitionistic logic. So, in the polemic about classical logic vs. intuitionistic logic, we have to wonder if the square of opposition can be used as an argument on behalf of the conservative position. I am going to show that, in spite of the strength of classical logic, the reply to this question is negative.

One of the criticisms made against intuitionistic logic is that, being weaker than classical logic, intuitionistic logic prohibits some logical inferences as common as useful. When Mélès writes that there is an "intuitionistic gap in contradiction" [4, p. 208], it sounds like a similar reproach: it suggests that there are contradictory formulas in classical logic which are not identified as such in intuitionistic logic.[5] It is indeed the case and it is provable as follows.

Definition 2.2 A formula F is contradictory in FOL if and only if

$$F \vdash \bot \tag{2.8}$$

Theorem 2.3 *There are contradictory formulas in CFOL that cannot be considered as such in IFOL.*

Proof In the classical square of opposition, the implication $\neg(A) \to (O)$ is provable, therefore its negation is a contradiction, i.e.

$$\neg(\neg\forall x(Sx \to Px) \to \exists x(Sx \wedge \neg Px)) \vdash_c \bot \tag{2.9}$$

but in IFOL, Glivenko's theorem fails and the double negation of $\neg(A) \to (O)$ is not provable, i.e.

$$\neg(\neg\forall x(Sx \to Px) \to \exists x(Sx \wedge \neg Px)) \nvdash_i \bot \tag{2.10}$$

---

[5] In this context, a "gap" can mean a blank, a missing value.

Proof

$$
\begin{array}{c}
\underline{?(\neg(\neg\forall x(Sx \rightarrow Px) \rightarrow \exists x(Sx \wedge \neg Px)) \rightarrow \bot)}\checkmark \\
\neg(\neg\forall x(Sx \rightarrow Px) \rightarrow \exists x(Sx \wedge \neg Px)) \\
?\bot \\
\underline{?(\neg\forall x(Sx \rightarrow Px) \rightarrow \exists x(Sx \wedge \neg Px))}\checkmark \\
\neg\forall x(Sx \rightarrow Px) \\
?\exists x(Sx \wedge \neg Px) \\
\underline{?\forall x(Sx \rightarrow Px)}\checkmark \\
Sa \\
?Pa \\
\begin{array}{cc}
?Sa & \underline{?\neg Pa}\checkmark \\
\times & Pa
\end{array}
\end{array}
$$

□

Therefore, according to Definition 2.2 the formula

$$\neg(\neg\forall x(Sx \rightarrow Px) \rightarrow \exists x(Sx \wedge \neg Px)) \tag{2.11}$$

is a contradiction in CFOL but not in IFOL. □

It seems that the previous proof could be a logical argument against intuitionistic logic which would suffer of a sort of deficiency. But the embarrassment that theorem 2.3 can cause is dissipated by the following one:

Theorem 2.4 *A formula F is valid in CFOL if and only its Kuroda translation F^k is valid in IFOL.*

Proof See David et al. [3, pp. 155–157] □

There would be a real "intuitionistic gap in contradiction" only if there would be no intuitionistic translation of these formulas that are contradictory in CFOL but not IFOL. But we know that Gödel-Gentzen translation or Kuroda translation are secure means to translate any formula valid in CFOL into a formula valid in IFOL,[6] and therefore, any contradictory formula in CFOL can be translated by another contradictory formula in IFOL. For example, *via* Kuroda translation, the formula

$$\neg\forall x(Sx \rightarrow Px) \rightarrow \exists x(Sx \wedge \neg Px) \tag{2.12}$$

becomes

$$\neg\neg(\neg\forall x\neg\neg(Sx \rightarrow Px) \rightarrow \exists x(Sx \wedge \neg Px)) \tag{2.13}$$

---

[6] See David et al. [3, p. 156] and von Plato [5, p. 170].

Therefore the translation of the classical contradiction (2.11) is in IFOL the negation of (2.13) and the translation of (2.9) is

$$\neg(\neg\forall x\neg\neg(Sx \to Px) \to \exists x(Sx \wedge \neg Px)) \vdash_i \bot \tag{2.14}$$

Regarding the intuitionistic square, it is therefore possible to translate every valid implication of the classical square into its Kuroda expression, valid in turn in IFOL, and then

- The three equivalences involved by the contradictory relations and lost in IFOL are, thanks to Kuroda translation, redefined as follows:

$$\vdash_i \neg\neg(\forall x\neg\neg(Sx \to Px) \leftrightarrow \neg\exists x(Sx \wedge \neg Px)) \tag{2.15}$$

$$\vdash_i \neg\neg(\exists x(Sx \wedge \neg Px) \leftrightarrow \neg\forall x\neg\neg(Sx \to Px)) \tag{2.16}$$

$$\vdash_i \neg\neg(\exists x(Sx \wedge Px) \leftrightarrow \neg\forall x\neg\neg(Sx \to \neg Px)) \tag{2.17}$$

- once transformed *via* Kuroda translation, sub-contrary relation $\neg(O) \to (I)$ is valid in IFOL:

$$\vdash_i \neg\neg((\neg\exists x\neg\neg(Sx \wedge \neg Px) \wedge \exists x(Sx)) \to \exists x(Sx \wedge Px)) \tag{2.18}$$

Consequently, nothing of the classical square of opposition is really lost in IFOL. But it remains to show in conclusion why Kuroda's translation is far from trivial and why some classical implications in the square of opposition are no longer valid in intuitionistic logic.

3 Conclusion: What the Intuitionistic Square of Opposition Means Exactly

3.1 Kuroda Translation of the Classical Square

As David et al. [3, p. 156] masterfully point out, Kuroda translation is a rigorous expression, from an intuitionistic point of view, of the specificity of classical logic. Because this translation shows the *sufficient conditions* to prove a formula F in classical logic:

Table 2 The intuitionistic amputations inside the classical square

$$\nvdash_i \neg\forall x(Sx \to Px) \to \exists x(Sx \wedge \neg Px)$$

Proof.

$$\begin{array}{c}
\underline{?(\neg\forall x(Sx \to Px) \to \exists x(Sx \wedge \neg Px))\checkmark}\\
\neg\forall x(Sx \to Px)\\
?\exists x(Sx \wedge \neg Px)\\
?(Sa \wedge \neg Pa)\\
\underline{?\forall x(Sx \to Px)\checkmark}\\
\underline{?(Sb \to Pb)\checkmark}\\
Sb\\
?Pb
\end{array}$$

□

$$\nvdash_i \neg\forall x(Sx \to \neg Px) \to \exists x(Sx \wedge Px)$$

Proof.

$$\begin{array}{c}
\underline{?(\neg\forall x(Sx \to \neg Px) \to \exists x(Sx \wedge Px))\checkmark}\\
\neg\forall x(Sx \to Px)\\
?\exists x(Sx \wedge Px)\\
?(Sa \wedge Pa)\\
\underline{?\forall x(Sx \to \neg Px)\checkmark}\\
\underline{?(Sb \to \neg Pb)\checkmark}\\
Sb\\
\underline{?\neg Pb\checkmark}\\
Pb
\end{array}$$

□

$$\nvdash_i (\neg\exists x(Sx \wedge \neg Px)) \to (\forall x(Sx \to Px))$$

Proof.

$$\begin{array}{c}
\underline{?((\neg\exists x(Sx \wedge \neg Px)) \to (\forall x(Sx \to Px)))\checkmark}\\
\neg\exists x(Sx \wedge \neg Px)\\
\underline{?\forall(Sx \to Px)\checkmark}\\
\underline{?(Sa \to Pa)\checkmark}\\
Sa\\
?Pa\\
?\exists x(Sx \wedge \neg Px)\\
?(Sa \wedge \neg Pa)\\
\begin{array}{cc}
?Sa & \underline{?\neg Pa\checkmark}\\
\times & Pa
\end{array}
\end{array}$$

□

$$\nvdash_i (\neg\exists x(Sx \wedge \neg Px) \wedge \exists x(Sx)) \to \exists x(Sx \wedge Px)$$

Proof.

$$\begin{array}{c}
\underline{?((\neg\exists x(Sx \wedge \neg Px) \wedge \exists x(Sx)) \to \exists x(Sx \wedge Px))\checkmark}\\
\exists x(Sx)\checkmark\\
Sa\\
\neg\exists x(Sx \wedge \neg Px)\\
?\exists x(Sx \wedge \neg Px)\\
?\exists x(Sx \wedge Px)\\
\begin{array}{cc}
?Sa & ?Pa\\
\times & \begin{array}{cc}
?Sa & \underline{?\neg Pa\checkmark}\\
\times & Pa
\end{array}
\end{array}
\end{array}$$

□

1. using the rule of *reductio ad absurdum* on all sub-formulas of F preceded by $\forall$ and on F itself.[7]
2. making use of the *Ex Contradictione Quodlibet* rule i.e the intuitionistic rule $\bot E$[8]

3.2 The Intuitionistic Amputations in the Classical Square

By using Bell et al.'s proof method for IFOL, I am going to try to sum up the logical reasons of intuitionistic amputations in the classical square. The four classical implications in the square that are not provable in IFOL are contained Table 2 with their respective trees

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[7] Hence Kuroda's translation recipe: put a double negation just before F and just before each scope of universal quantifiers in F, e.g. if $\forall x(Fx)$ is in F, it is translated by $\forall x\neg\neg(Fx)$.

[8] Kuroda's translation is all what one needs as faithful translation of CFOL into IFOL. Gödel-Gentzen translation goes further because contrary to the former, it translates also classical logic into minimal logic, see [3, pp. 157–158] .

as countermodels.[9]

These amputations can be divided in two couples of formulas:

1. The trees of the pair $\{\neg(A) \rightarrow (O), \neg(E) \rightarrow (I)\}$ are countermodels showing that the assumption of the falsity of a universal statement does not analytically involve a counterexample disproving this latter.
2. The trees of the second pair, $\{\neg(O) \rightarrow (A), \neg(O) \rightarrow (I)\}$, are countermodels corresponding in IFOL to the intuitionistic refusal of reducing any double negation to a positive assertion.

The fact that these classical implications are no longer provable in IFOL can be explained by the independence of the connectives and quantifiers in intuitionistic logic, therefore that does not show that intuitionistic logic is weaker than classical logic, but that the former has stronger means of expressions than the latter.

Acknowledgements Many thanks to

- Baptiste Mélès both for the discussions we had about the intuitionistic version of the square of opposition and for his very useful remarks on a draft of my paper,
- Jean-Yves Béziau who gave to me the occasion of writing this paper,
- My institution, the Archives Poincaré, which funded my participation to the fourth conference on the square of opposition (Vatican, May 5–9, 2014),
- the referees of the final version of this paper.

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[9] To reply to a question asked by a referee, it is indeed possible to see this tree method for intuitionistic logic as an automated search of Kripke countermodels. A formula F is intuitionistically proved if and only if its refutation tree shows that it is not possible to give a Kripke countermodel for F, i.e. that the assumption that F is intuitionistically unprovable (i.e. ?F) leads to a contradiction.

Part VII
Applications of the Square

The Ontological Modal Collapse as a Collapse of the Square of Opposition

Christoph Benzmüller and Bruno Woltzenlogel Paleo

Abstract The *modal collapse* that afflicts Gödel's modal ontological argument for God's existence is discussed from the perspective of the modal square of opposition.

Keywords Higher-order logics • Interactive and automated theorem proving • Modal logics • Ontological argument

Mathematics Subject Classification (2000) Prim. 03B15; Sec. 68T15

1 Introduction

Attempts to prove the existence (or non-existence) of God by means of abstract, ontological arguments are an old tradition in western philosophy, with contributions by several prominent philosophers, including St. Anselm of Canterbury, Descartes and Leibniz. Kurt Gödel and Dana Scott studied and improved this argument, bringing it to a mathematically more precise form, as a chain of axioms, lemmas and theorems in a second-order modal logic [18, 26], shown in Fig. 1.

Gödel defines God as a being who possesses all *positive* properties and states a few reasonable (but debatable) axioms that such properties should satisfy. The overall idea of Gödel's proof is in the tradition of Anselm's argument, who defined God as an entity of which nothing greater can be conceived. Anselm argued that existence in the actual world would make such an assumed being even greater (more perfect), hence, by definition, God must exist. However, for Anselm existence was treated as a predicate and the possibility of God's existence was assumed as granted. These issues were criticized by Kant and Leibniz, respectively, and they were addressed in the work of Gödel.

Nevertheless, Gödel's work still leaves room for criticism. In particular, his axioms are so strong that, when assuming unrestricted comprehension principles,[1] they entail a *modal collapse* [27, 28]: everything that is the case is so necessarily. There has been an

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This work has been supported by the German National Research Foundation (DFG) under grant BE 2501/9-1,2.

[1] A possible direction to remedy modal collapse, as studied e.g. by Koons [21] is to impose restrictions on the domain of properties.

J.-Y. Béziau, G. Basti (eds.), *The Square of Opposition: A Cornerstone of Thought*, Studies in Universal Logic, DOI 10.1007/978-3-319-45062-9_18

A1 Either a property or its negation is positive, but not both:

$$\forall\varphi[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]$$

A2 A property necessarily implied by a positive property is positive:

$$\forall\varphi\forall\psi[(P(\varphi) \wedge \Box\forall x[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

T1 Positive properties are possibly exemplified:

$$\forall\varphi[P(\varphi) \rightarrow \Diamond\exists x\varphi(x)]$$

D1 A *God-like* being possesses all positive properties:

$$G(x) \equiv \forall\varphi[P(\varphi) \rightarrow \varphi(x)]$$

A3 The property of being God-like is positive:

$$P(G)$$

C Possibly, a God-like being exists:

$$\Diamond\exists x G(x)$$

A4 Positive properties are necessarily positive:

$$\forall\varphi[P(\varphi) \rightarrow \Box\ P(\varphi)]$$

D2 An *essence* of an individual is a property possessed by it and necessarily implying any of its properties:

$$\varphi\ ess\ x \equiv \varphi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\varphi(y) \rightarrow \psi(y)))$$

T2 Being God-like is an essence of any God-like being:

$$\forall x[G(x) \rightarrow G\ ess\ x]$$

D3 *Necessary existence* of an individual is the necessary exemplification of all its essences:

$$NE(x) \equiv \forall\varphi[\varphi\ ess\ x \rightarrow \Box\exists y\varphi(y)]$$

A5 Necessary existence is a positive property:

$$P(NE)$$

L1 If a god-like being exists, then necessarily a god-like being exists:

$$\exists x G(x) \rightarrow \Box\exists y G(y)$$

L2 If possibly a god-like being exists, then necessarily a god-like being exists:

$$\Diamond\exists x G(x) \rightarrow \Box\exists y G(y)$$

T3 Necessarily, a God-like being exists:

$$\Box\exists x G(x)$$

Fig. 1 Scott's version of Gödel's ontological argument [26]

impressive body of recent and ongoing work (cf. [1, 2, 11, 15–17, 19, 20, 28] and the references therein) proposing solutions for the modal collapse. The goal of this article is to discuss the modal collapse from the point of view of the modal square of opposition. Ontological arguments typically rely on an inversion of the normal direction of entailment in the modal square of opposition for one particular proposition (i.e. God's existence), and the modal collapse shows that this inversion in fact occurs for all propositions, resulting in a total collapse of the modal square of opposition.

2 A Collapse of the Modal Square

A crucial step of most ontological arguments is the claim that if God's existence is possible, then it is necessary. This is Lemma **L2** in Gödel's proof. In the modal square of opposition (Fig. 2), this is an unusual situation in which the **I** corner must imply and entail the **A** corner, in the particular case when ϕ is $\exists x G(x)$. Gödel's proof shows that his axioms are indeed strong enough to invert the direction of entailment for this choice of ϕ. This observation, however, immediately leads to the question whether the axioms are eventually even strong enough to enable the inverted entailment for any arbitrary sentence ϕ. That is essentially the question asked by Sobel [27], and his proof of the modal collapse (**MC**, cf. Fig. 3) provides an affirmative answer. It is possible to show that this form of the modal collapse entails (in modal logic ***K***) a collapse of the modal square (**MCs**), causing the subcontraries to entail (and even imply) their respective contraries. Normally, as shown in Fig. 2, in the modal square of opposition only the other direction of entailment holds: the contraries entail their subcontraries, assuming the *modal existential import* **ExImp** [14].

Moreover, in any modal logic where the axiom **T** holds (i.e. where the accessibility relation is reflexive), even a total collapse of the modalities (**MCt**) is entailed by **MC**. Interestingly, under this stronger form of modal collapse, the contraries entail their subcontraries even without the existential import.

Although Gödel's axioms lead to modal collapse, there are several variants (e.g. [1, 2, 11]) that are known to be immune to it. This means there must be at least one proposition ϕ such that the implication $\phi \to \Box\phi$ (from now on abbreviated as *collapse*(ϕ)) is not valid under the axioms and definitions used by the variant. But if the variant is sufficiently similar to Gödel's argument, also deriving Lemmas **L1** and **L2**, then *collapse*($\exists xG(x)$)

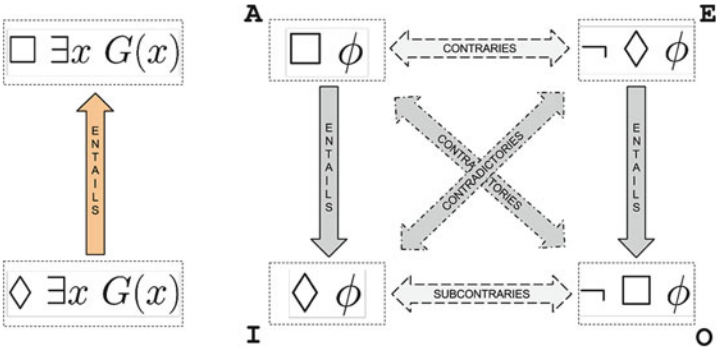

Fig. 2 Modal square of opposition

MC Everything that is the case is so necessarily: $\forall\phi[\phi \rightarrow \Box\phi]$

MCs Everything that is possible is necessary: $\forall\phi[\Diamond\phi \rightarrow \Box\phi]$

T Everything that is necessary is the case: $\forall\phi[\Box\phi \rightarrow \phi]$

ExImp (Modal Existential Import): $\Diamond\top$

AI Everything that is necessary is possible: $\forall\phi[\Box\phi \rightarrow \Diamond\phi]$

MCt Modalities collapse completely: $\forall\phi[(\phi \leftrightarrow \Box\phi) \wedge (\Diamond\phi \leftrightarrow \Box\phi)]$

Fig. 3 Modal collapse

A:D1 A *God-like* being necessarily possesses those and only those properties that are positive:

$$G_A(x) \equiv \forall\varphi[P(\varphi) \leftrightarrow \text{ffj}\varphi(x)]$$

A:MC The modal collapse happens for any positive properties applied to any god-like being:

$$\forall\varphi\forall x[(P(\varphi) \wedge G_A(x)) \rightarrow \mathit{collapse}(\varphi(x))]$$

A:MC1 The modal collapse does *not* happen for positive properties applied to arbitrary individuals (*counter-satisfiable*):

$$\forall\varphi\forall x[P(\varphi) \rightarrow \mathit{collapse}(\varphi(x))]$$

A:MC2 The modal collapse does *not* happen for an arbitrary properties applied to a god-like being (*counter-satisfiable*):

$$\forall\varphi\forall x[G_A(x) \rightarrow \mathit{collapse}(\varphi(x))]$$

Fig. 4 Restricted collapse for Anderson's emendation [1]

must be valid. Therefore, one may wonder how strong is their immunity to the modal collapse: is there any other proposition ϕ for which $\mathit{collapse}(\phi)$ is also valid?

For Anderson's emendation [1], for example, a form of the modal collapse (**A:MC**), restricted to positive properties applied to god-like beings, can be derived. The proof, under the modal logic ***K***, depends only on Anderson's alternative definition of god-like being (**A:D1**). This class of propositions for which the collapse occurs is tight: weaker restrictions (**A:MC1** and **A:MC2**), which could lead to larger classes, are counter-satisfiable (Fig. 4). These results hold under both constant and varying domain quantification, with possibilist and actualist quantifiers.

In any modal logic at least as strong as **K**, and even without relying on axioms specific to ontological arguments, it is easy to see (and even easier to check with an automated theorem prover) the following facts about classes of collapsing propositions:

1. Valid propositions are collapsing: if ϕ is valid, then $\mathit{collapse}(\phi)$ is valid.
2. The class of collapsing propositions is closed under logical equivalence: if $\mathit{collapse}(\phi)$ is valid and $\phi \leftrightarrow \phi'$ is valid, then $\mathit{collapse}(\phi')$ is valid.

3. The class of collapsing propositions is not generally closed under equi-validity: even if *collapse*(ϕ) is valid and ϕ and ϕ' are equi-valid, *collapse*(ϕ') may not be valid.
4. The class of collapsing propositions is not generally closed under implication: even if *collapse*(ϕ) is valid and $\phi \rightarrow \phi'$ is valid, *collapse*(ϕ') may not be valid.

An easy corollary of the second fact above is that any ontological argument relying on Lemmas **L1** and **L2** will necessarily lead to a modal collapse for all propositions that are logically equivalent to God's existence. The third and fourth facts indicate that characterizations of larger classes of propositions for which the modal collapse holds require using axioms specific to the variant of the ontological argument under consideration, as in the case of **A:MC**.

3 Final Remarks

All results announced in this note have been obtained experimentally using interactive and automated theorem provers and model finders [9, 10, 12, 13, 22]. The source codes of the experiments, as well as the resulting proofs and counter-models, are available in github.com/FormalTheology/GoedelGod/ in the files ModalCollapse.thy and `ModalSquareOfOpposition.thy` inside the folder Formalizations/Isabelle/Meta as well as in files inside the folder `Formalizations/Isabelle/Anderson`.

The technique enabling these experiments is the embedding of quantified modal logics into higher-order logics [3, 7, 8], for which automated theorem provers exist. This technique has already been successfully employed in the verification and reconstruction of Gödel's proof [4, 5, 24], and a detailed mathematical description is available in [6].

The modal collapse is an interesting example of philosophical controversy and dispute, to which we can apply Leibniz's idea of a *calculus ratiocinator* brought to reality in the form of contemporary automated theorem provers. A significant advantage provided by the use of computers is that all parameters (e.g. modal logic, domain conditions, semantics) under which the announced results hold must be explicitly specified in the source code. Consequently, the danger of misunderstandings is reduced. Current technology is increasingly ready to be embraced by those willing to practice computer-assisted theoretical philosophy [23, 25].

Ongoing and future work includes the computer-assisted study of the modal collapse in other variants of the ontological argument (e.g. [11, 17]). Furthermore, our experiments in Isabelle revealed a weakness of the current integration of the HOL-ATPs LEO-II and Satallax via Sledgehammer: most of the problems in our study solved by the two HOL-ATPs were still too hard to be reconstructed and verified by Isabelle's internal prover Metis. This points to relevant future work regarding the integration of HOL-ATPs in Isabelle.

Acknowledgements We would like to thank Paul Weingartner, André Fuhrmann and Melvin Fitting for several discussions about Gödel's ontological argument.

Note About Authorship Alphabetic order has been used for the authors' names. The extent and kind of contribution of each author cannot be inferred from the order.

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Fuzzy Eubouliatic Logic: A Fuzzy Version of Anderson's Logic of Prudence

Gert-Jan C. Lokhorst

Abstract Alan Ross Anderson was one of the first logicians who were interested in the logic of prudence and related concepts, such as caution. He called this area "eubouliatic logic," a term which has not become popular. Anderson made a distinction between four prudence-related concepts which can be placed in a square of opposition. Prudence and related concepts are nowadays often seen as fuzzy concepts. We investigate which consequences this has for the logic of these concepts.

Keywords Eubouliatic logic • Fuzzy logic • Modal logic • Prudence • Relevance logic • Square of opposition

Mathematics Subject Classification (2000) Primary 03B47; Secondary 03B52

1 Introduction

Alan Ross Anderson [1] was one of the first logicians who were interested in the logic of prudence and related concepts, such as caution. He called this area "eubouliatic logic" (from the Greek *euboulos*, meaning "prudent"). This appellation has not become very popular.

2 Relevant Logic

A *relevant* logic is a logic in which $A \to B$ is a theorem if and only if (i) A and B share a propositional variable or (meta-definable) propositional constant and (ii) either $\neg A$ or B is not a theorem [17]. Anderson's eubouliatic logic was based on relevant logic **R**. **R** is one of the best-known systems of relevant logic. The crucial difference between **R** and classical propositional logic is that **R** does not provide $A \to (B \to A)$ (the archetypical fallacy of relevance, "which would enable us to infer that Bach wrote the Coffee Cantata from the premiss that the Van Allen belt is doughnut-shaped—or indeed from any premiss you like" [2, Sect. 5.1]).

J.-Y. Béziau, G. Basti (eds.), *The Square of Opposition: A Cornerstone of Thought*, Studies in Universal Logic, DOI 10.1007/978-3-319-45062-9_19

Definition 2.1 **R** is axiomatized as follows [2, 3]:

R1: $A \to A$
R2: $(A \to B) \to ((B \to C) \to (A \to C))$
R3: $A \to ((A \to B) \to B)$
R4: $(A \to (A \to B)) \to (A \to B)$
R5: $(A \wedge B) \to A$
R6: $(A \wedge B) \to B$
R7: $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$
R8: $A \to (A \vee B)$
R9: $B \to (A \vee B)$
R10: $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$
R11: $(A \wedge (B \vee C)) \to ((A \wedge B) \vee C)$
R12: $(A \to \neg B) \to (B \to \neg A)$
R13: $\neg\neg A \to A$
R14: $A \leftrightarrow (\mathbf{t} \to A)$
R15: $A \to \mathbf{T}$
MP: From A and $A \to B$ to infer B
Adj: From A and B to infer $A \wedge B$
Definitions: $\mathbf{f} \overset{\text{df}}{=} \neg\mathbf{t}$, $\mathbf{F} \overset{\text{df}}{=} \neg\mathbf{T}$, $A \leftrightarrow B \overset{\text{df}}{=} (A \to B) \wedge (B \to A)$, $A_{\mathbf{t}} \overset{\text{df}}{=} A \wedge \mathbf{t}$.

3 Eubouliatic Logic

Definition 3.1 Anderson's eubouliatic logic $\mathbf{E_R}$ is **R** with a special constant **e** ("the good thing"), plus:

D1: $\mathcal{P}A \overset{\text{df}}{=} A \to \mathbf{e}$ ("A is prudent" means "A implies the good thing").
D2: $\mathcal{P}_wA$ ("it is imprudent that A"): A is not prudent: $\mathcal{P}_wA \overset{\text{df}}{=} \neg\mathcal{P}A$.
D3: $\mathcal{C}A$ ("it is cautious that A"): the negation of A is not prudent: $\mathcal{C}A \overset{\text{df}}{=} \neg\mathcal{P}\neg A$.
D4: $\mathcal{C}_wA$ ("it is incautious that A"): the negation of A is prudent: $\mathcal{C}_wA \overset{\text{df}}{=} \mathcal{P}\neg A$.
Ae: $\neg(\neg\mathbf{e} \to \mathbf{e})$.

Theorem 3.2 $\mathbf{E_R}$ *has the following theorems:*

1. $\mathcal{P}_wA \leftrightarrow \neg\mathcal{P}A$*:* $\mathcal{P}_wA$ *and* $\mathcal{P}A$ *are contradictories.*
2. $\mathcal{C}_wA \leftrightarrow \neg\mathcal{C}A$*:* $\mathcal{C}_wA$ *and* $\mathcal{C}A$ *are contradictories.*
3. $\mathcal{P}A \to \mathcal{C}A$ *("axiom of avoidance"):* $\mathcal{P}A$ *and* $\mathcal{C}A$ *are subalterns, in conformity with* [5, q. 49 art. 8] *and* [15, "prudence"].
4. $\mathcal{C}_wA \to \mathcal{P}_wA$*:* $\mathcal{C}_wA$ *and* $\mathcal{P}_wA$ *are subalterns.*
5. $\neg(\mathcal{P}A \wedge \mathcal{C}_wA)$*:* $\mathcal{P}A$ *and* $\mathcal{C}_wA$ *are contraries.*
6. $\mathcal{P}_wA \vee \mathcal{C}A$*:* $\mathcal{P}_wA$ *and* $\mathcal{C}A$ *are subcontraries.*

Proof From the definitions. □

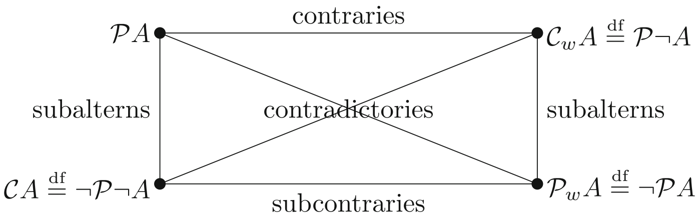

Fig. 1 Anderson [1, Fig. 8]

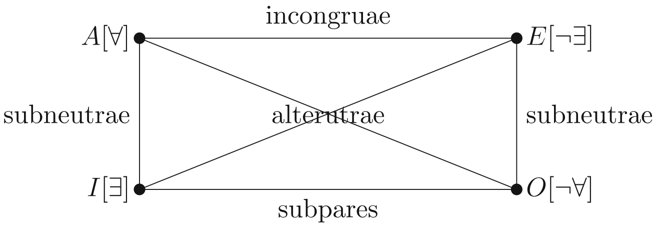

Fig. 2 First-order predicate calculus plus $\forall xFx \to \exists xFx$ [4, 8]

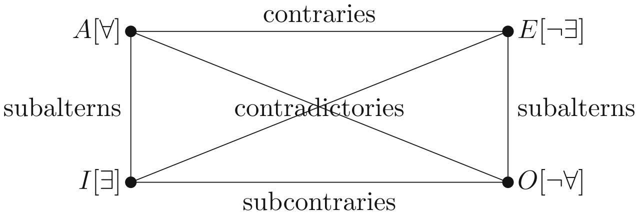

Fig. 3 First-order predicate calculus plus $\forall xFx \to \exists xFx$ [6], [1, Fig. 2]

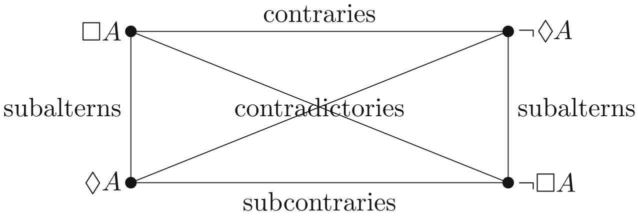

Fig. 4 Modal system **KD** [6], [1, Fig. 3]

These eubouliatic notions can be depicted in a square of opposition (Fig. 1). This square of opposition is the same as the square of opposition of Apuleius of Madaura (Fig. 2). The A E I O propositions are familiar from the medieval Aristotle tradition. This diagram is better known in the version of Boethius (Fig. 3). The modal square of opposition is similar (Fig. 4). The deontic square of opposition is also similar (Fig. 5). Anderson thought that $\mathcal{P}A$ could also be read as "it is safe that *A*." As we have argued elsewhere [11], "it is safe

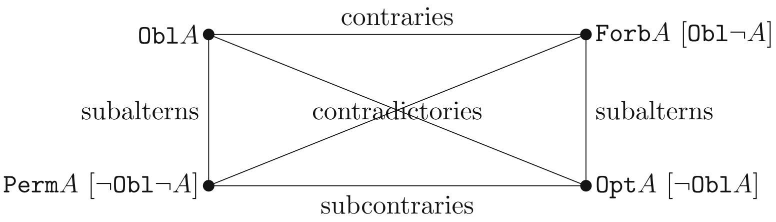

Fig. 5 Standard deontic logic. Leibniz [10], [1, Fig. 4]

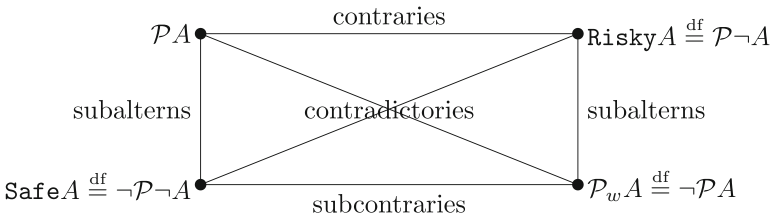

Fig. 6 Safety, risk, prudence and imprudence

that A" should be represented as $\mathcal{C}A$ rather than $\mathcal{P}A$. Figure 1 can therefore be relabeled as in Fig. 6.

Theorem 3.3 *The eubouliatic fragment of* $\mathbf{E_R}$ *(*$\mathbf{E_R}$ *without* $\mathbf{e}$*) can be axiomatized as* **R** *plus the following axioms:*

$\mathbf{E_R1}$: $(A \to B) \to (\mathcal{P}B \to \mathcal{P}A)$.
$\mathbf{E_R2}$: $A \to \mathcal{P}\mathcal{P}A$.
$\mathbf{E_R3}$: $\mathcal{P}A \to \neg\mathcal{P}\neg A$.

Proof For each derivation $A_1, \ldots, A_n$ define $\mathbf{e}$ as $\mathcal{P}\mathbf{t}$, where $\mathbf{t} \overset{\text{df}}{=} \bigwedge_{i=1}^{m}(p_i \to p_i)$ and p_1, $\ldots, p_m$ are the propositional variables occurring in $A_1, \ldots, A_n$ [11]. □

Theorem 3.4 $\mathbf{E_R}$ *does not provide* $\mathcal{P}p$.

Proof **R** does not provide $p \to (p \to p)$, as `MaGIC` [13] shows. □

4 Eubouliatic Logic is Not Satisfactory

Theorem 4.1 $\mathbf{E_R}$ *provides* $A \to \mathcal{P}\mathcal{P}A$.

Proof **R** provides $A \to ((A \to \mathbf{e}) \to \mathbf{e})$. □

This theorem is unacceptable. For example, if A stands for "there are nuclear reactors," then it says that if there are nuclear reactors, then it is prudent that it is prudent that there are nuclear reactors. This is unacceptable.

5 Modal Eubouliatic Logic

We therefore introduce a *modal* system of eubouliatic logic. $\Box A$ is read as "it is necessary that A."

Definition 5.1 **KDR** is **R** plus:

□1: $(\Box A \wedge \Box B) \to \Box(A \wedge B)$
□2: $\Box(A \to B) \to (\Box A \to \Box B)$
□D: $\Box A \to \neg\Box\neg A$
Nec: From A to infer $\Box A$.

Definition 5.2 Modal eubouliatic logic $\mathbf{E_{KDR}}$ is **KDR** plus the following definitions (instead of D1–D4):

D1*: $\mathcal{P}^*A \overset{\text{df}}{=} \Box(A \to \mathbf{e})$ ("A is necessarily prudent" means "it is necessary that A implies the good thing").
D2*: $\mathcal{P}^*_w A \overset{\text{df}}{=} \neg\mathcal{P}^*A$.
D3*: $\mathcal{C}^*A \overset{\text{df}}{=} \neg\mathcal{P}^*\neg A$.
D4*: $\mathcal{C}^*_w A \overset{\text{df}}{=} \mathcal{P}^*\neg A$.

Theorem 5.3 $\mathbf{E_{KDR}}$ *has the following theorems:*

$\mathbf{E_{KDR}}$**1:** $\Box(A \to B) \to (\mathcal{P}^*B \to \mathcal{P}^*A)$ *[from D1*, R2]*
$\mathbf{E_{KDR}}$**2:** $(\mathcal{P}^*A \wedge \mathcal{P}^*B) \to \mathcal{P}^*(A \vee B)$ *[from □1, R10]*
$\mathbf{E_{KDR}}$**3:** $\mathcal{P}^*A \to \neg\mathcal{P}^*\neg A$ *[from □D]*

Proof From the definitions. □

Theorem 5.4 $\mathbf{E_{KDR}}$ *provides neither* $A \to \mathcal{P}^*\mathcal{P}^*A$ *nor* $\mathcal{P}^*(A \to \mathcal{P}^*A)$.

Proof **KDR** provides neither $A \to \Box(\Box(A \to \mathbf{e}) \to \mathbf{e})$ nor $\Box(\Box A \to A)$, as `MaGIC` [13] shows. □

6 Linguistic Hedges

$\mathbf{E_{KDR}}$ is better than $\mathbf{E_R}$, but $\mathbf{E_R}$ and $\mathbf{E_{KDR}}$ have one major shortcoming: prudence seems to be a fuzzy concept, whereas $\mathbf{E_R}$ and $\mathbf{E_{KDR}}$ are not fuzzy.

The fuzziness of prudence is shown by the fact that "prudence" is typically used with "linguistic hedges," such as:

> sort of, kind of, loosely speaking, more or less, on the ... side (tall, fat, etc.), roughly, pretty (much), relatively, somewhat, rather, mostly, technically, strictly speaking, essentially, in essence, basically, principally, particularly, par excellence, largely, for the most part, very, especially, exceptionally, quintessential(ly), literally, often, more of a ... than anything else, almost, typically/typical, as it were, in a sense, in one sense, in a real sense, in an important sense, in a way, mutatis mutandis, in a manner of speaking, details aside, so to say, a veritable, a true, a real, a regular, virtually, all but

> technically, practically, all but a, anything but a, a self-styled, nominally, he calls himself a . . . , in name only, actually, really, be as much as . . . , -like, -ish, can be looked upon as, can be viewed as, pseudo-, crypto-, (he's) another (Caruso/Lincoln /Babe Ruth/. . .), is the . . . of—(e.g., America is the Roman Empire of the modern world, Chomsky is the De Gaulle of Linguistics, etc.) [9, p. 472].

These hedges imply that prudence *itself* is a fuzzy concept. Hedges applied to crisp, black/white concepts are simply redundant. This may explain why there are so many jokes about women who are "somewhat pregnant." "Pregnant" is crisp: a woman is either pregnant or not pregnant. This implies that a woman is somewhat pregnant (i.e., not nonpregnant) if and only if she is pregnant.

"Linguistic hedges" can, to some extent, be represented in **R**.

Definition 6.1

1. $\mathrm{vt}(A) \overset{\mathrm{df}}{=} A + A \overset{\mathrm{df}}{=} \neg A \to A$. vt($A$) is read as "$A$ is very true."
2. $\mathrm{st}(A) \overset{\mathrm{df}}{=} A \circ A \overset{\mathrm{df}}{=} \neg(A \to \neg A)$. st($A$) is read as "$A$ is slightly true."
3. $\mathrm{crisp}(A) \overset{\mathrm{df}}{=} (A \leftrightarrow \mathbf{T}) \vee (A \leftrightarrow \mathbf{F})$. crisp($A$) is read as "$A$ is crisp."

Theorem 6.2 **R** *provides all theorems mentioned in* [14] *and in the above, such as:*

H1 $vt(A) \leftrightarrow \neg st(\neg A)$
H2 $vt(A) \to A$
H3 $vt(A \to B) \to (vt(A) \to vt(B))$
H4 $vt(A \vee B) \to (vt(A) \vee vt(B))$
H5 $A \to st(A)$
H6 $vt(A \to B) \to (st(A) \to st(B))$
H7 $st(A \to B) \to (st(A) \to st(B))$
H8 $(A \to B) \to (st(A) \to st(B))$
H9 $\neg st(\mathbf{F})$
H10 $crisp(A) \to (st(A) \leftrightarrow A)$

Proof From the definitions. □

Theorem 6.3 **R** *provides none of the rules discussed in* [14], *such as* $\vdash A \Longrightarrow \vdash vt(A)$ *and* $\vdash \neg A \Longrightarrow \vdash \neg st(A)$.

Proof $\vdash \mathbf{t}$ by R14, but $\nvdash \mathrm{vt}(\mathbf{t})$, as `MaGIC` [13] shows. $\vdash \neg\mathbf{f}$ by R14, but $\nvdash \neg\mathrm{st}(\mathbf{f})$ by $\nvdash \mathrm{vt}(\mathbf{t})$ and H1. □

Note that axiom A**e** may be rewritten as st(¬**e**).

7 Fuzzy Relevant Logic

The "official" definition of fuzziness is more complicated, however.

Definition 7.1 We write $T \vdash_{\mathbf{L}} A$ for: A is derivable from T using the axioms and rules of system **L**.

Definition 7.2 $\mathbf{L}$ is a *weakly implicative* logic if and only if $\mathbf{L}$ provides [7, Def. 10, Lemma 3]:

Ref: $\vdash_{\mathbf{L}} A \to A$
MP: $A, A \to B \vdash_{\mathbf{L}} B$
WT: $A \to B, B \to C \vdash_{\mathbf{L}} A \to C$
Cng: $A \to B, B \to A \vdash_{\mathbf{L}} C \to C'$, where C' is a result of replacing some occurrence of A with B in C.

Definition 7.3 A weakly implicative logic $\mathbf{L}$ is *fuzzy* if and only if $\mathbf{L}$ is strongly complete with respect to the class of all linearly ordered $\mathbf{L}$-matrices [7, Def. 23].

Theorem 7.4 *A finitely axiomatizable weakly implicative logic* $\mathbf{L}$ *is fuzzy if and only if* $\mathbf{L}$ *has the prelinearity property:*

PP: *For each theory* T, $T \vdash_{\mathbf{L}} C$ *if and only if* $T, A \to B \vdash_{\mathbf{L}} C$ *and* $T, B \to A \vdash_{\mathbf{L}} C$.

Proof See [7, Theorem 3]. □

Definition 7.5 $\mathbf{LR}$ ($\mathbf{R}$ without distribution) is $\mathbf{R}$ without R11.

Theorem 7.6 (Relevant deduction theorem)

RDT: $T, A \vdash_{\mathbf{LR}} B$ *if and only if* $T \vdash_{\mathbf{LR}} A_{\mathbf{t}} \to B$.

Proof See [16, 19]. □

Theorem 7.7 *A finitely axiomatizable weakly implicative logic* $\mathbf{L}$ *that extends* $\mathbf{LR}$ *is fuzzy if and only if* $\mathbf{L}$ *provides:*

$\mathbf{PL_t}$*:* $(A \to B)_{\mathbf{t}} \vee (B \to A)_{\mathbf{t}}$ *(prelinearity).*

Proof "$\to$": we assume that $\mathbf{L}$ provides PP. We show that $\mathbf{L}$ provides $\mathrm{PL}_{\mathbf{t}}$.

1	$T \vdash_{\mathbf{L}} (A \to B)_{\mathbf{t}} \to \mathrm{PL}_{\mathbf{t}}$ and $T \vdash_{\mathbf{L}} (B \to A)_{\mathbf{t}} \to \mathrm{PL}_{\mathbf{t}}$	R8, R9, MP
2	$T, A \to B \vdash_{\mathbf{L}} \mathrm{PL}_{\mathbf{t}}$ and $T, B \to A \vdash_{\mathbf{L}} \mathrm{PL}_{\mathbf{t}}$	1, RDT
3	$T \vdash_{\mathbf{L}} \mathrm{PL}_{\mathbf{t}}$	2, PP.

"$\Longleftarrow$": we assume that $\mathbf{L}$ provides $\mathrm{PL}_{\mathbf{t}}$. We show that $\mathbf{L}$ provides PP.

1	$T, A \to B \vdash_{\mathbf{L}} C$ and $T, B \to A \vdash_{\mathbf{L}} C$	Hyp
2	$T \vdash_{\mathbf{L}} (A \to B)_{\mathbf{t}} \to C$ and $T \vdash_{\mathbf{L}} (B \to A)_{\mathbf{t}} \to C$	1, RDT
3	$T \vdash_{\mathbf{L}} \mathrm{PL}_{\mathbf{t}} \to C$	2, R10, MP, Adj
4	$T \vdash_{\mathbf{L}} C$	3, $\mathrm{PL}_{\mathbf{t}}$, MP
5	$[T, A \to B \vdash_{\mathbf{L}} C$ and $T, B \to A \vdash_{\mathbf{L}} C] \to T \vdash_{\mathbf{L}} C$	1, 4
6	$T \vdash_{\mathbf{L}} C \Longleftrightarrow [T, A \to B \vdash_{\mathbf{L}} C$ and $T, B \to A \vdash_{\mathbf{L}} C]$	5.

□

Definition 7.8 $\mathbf{FR}$ (fuzzy $\mathbf{R}$) is $\mathbf{LR}$ plus $\mathrm{PL}_{\mathbf{t}}$ [12, 16, 18, 19].

Theorem 7.9 $\vdash_{\mathbf{FR}}$ *R11.*

Proof By R1–R10, R14, $\mathrm{PL}_{\mathbf{t}}$, MP and Adj, as follows:

1	$\vdash_{\mathbf{LR}} (B \to C)_{\mathbf{t}} \to (B \to C)$	R5
2	$\vdash_{\mathbf{LR}} (B \to C)_{\mathbf{t}} \to \mathbf{t}$	R6
3	$\vdash_{\mathbf{LR}} \mathbf{t} \to (C \to C)$	R1, R14
4	$\vdash_{\mathbf{LR}} (B \to C)_{\mathbf{t}} \to (C \to C)$	2, 3, R2
5	$\vdash_{\mathbf{LR}} (B \to C)_{\mathbf{t}} \to ((B \vee C) \to C)$	1, 4, R10
6	$\vdash_{\mathbf{LR}} (B \to C)_{\mathbf{t}} \to ((A \wedge (B \vee C)) \to C)$	5, R5
7	$\vdash_{\mathbf{LR}} (B \to C)_{\mathbf{t}} \to (\mathrm{R}11)$	6, R9
8	$\vdash_{\mathbf{LR}} (C \to B)_{\mathbf{t}} \to \mathbf{t}$	R6
9	$\vdash_{\mathbf{LR}} \mathbf{t} \to (B \to B)$	R1, R14
10	$\vdash_{\mathbf{LR}} (C \to B)_{\mathbf{t}} \to (B \to B)$	8, 9, R2
11	$\vdash_{\mathbf{LR}} (C \to B)_{\mathbf{t}} \to (C \to B)$	R5
12	$\vdash_{\mathbf{LR}} (C \to B)_{\mathbf{t}} \to ((B \vee C) \to B)$	10, 11, R10
13	$\vdash_{\mathbf{LR}} (C \to B)_{\mathbf{t}} \to ((A \wedge (B \vee C)) \to (A \wedge B))$	12, R2
14	$\vdash_{\mathbf{LR}} (C \to B)_{\mathbf{t}} \to (\mathrm{R}11)$	13, R8
15	$\vdash_{\mathbf{LR}} ((B \to C)_{\mathbf{t}} \vee (C \to B)_{\mathbf{t}}) \to (\mathrm{R}11)$	7, 14, R10
16	$\vdash_{\mathbf{FR}} (B \to C)_{\mathbf{t}} \vee (C \to B)_{\mathbf{t}}$	$\mathrm{PL}_{\mathbf{t}}$
17	$\vdash_{\mathbf{FR}}$ R11	15, 16.

□

FR is therefore an extension of **R**.

The following table lists some properties of logics in the vicinity of **FR**.

name	definition	relevant?	fuzzy?	decidable?
RM	$\mathbf{R} + A \to (A \to A)$	no	yes	yes [2, Sect. 29.3.2]
FR	$\mathbf{LR} + \mathrm{PL}_{\mathbf{t}}$	yes	yes	unknown
R	$\mathbf{LR} + \mathrm{R}11$	yes	no	no [3, Sect. 65]
LR		yes	no	yes [3, Sect. 63.3]

It will be clear that $\vdash_{\mathbf{LR}} A \Longrightarrow \vdash_{\mathbf{R}} A \Longrightarrow \vdash_{\mathbf{FR}} A \Longrightarrow \vdash_{\mathbf{RM}} A$, but not conversely.

An algebraic Kripke-style semantics for **FR** is to be found in [16, 19].

8 Fuzzy Eubouliatic Logic

Definition 8.1 Fuzzy eubouliatic logic $\mathbf{E}_{\mathbf{FR}}$ is **FR** plus D1–D4, A**e**.

Theorem 8.2 $\mathbf{E}_{\mathbf{FR}}$ *provides* $A \to \mathcal{PP}A$.

Proof Same as the proof of Theorem 4.1. □

9 Modal Fuzzy Eubouliatic Logic

Definition 9.1 Modal fuzzy eubouliatic logic $\mathbf{E_{KDFR}}$ is **FR** plus $\Box 1$, $\Box 2$, $\Box D$, Nec, D1*–D4*.

Theorem 9.2 $\mathbf{E_{KDFR}}$ *provides neither* $A \to \mathcal{P}^*\mathcal{P}^*A$ *nor* $\mathcal{P}^*(A \to \mathcal{P}^*A)$.

Proof Same as the proof of Theorem 5.4. □

Theorem 9.3 $\mathbf{E_{KDFR}}$ *does not provide* $\mathcal{P}^*p$.

Proof **KDFR** plus $\Box A \leftrightarrow A$ does not provide $\Box(p \to (p \to p))$, as `MaGIC` [13] shows. □

$\mathbf{E_{KDFR}}$ is an extension of **R**, **FR**, **KDR** and $\mathbf{E_{KDR}}$. Therefore all observations on these systems made above also apply to $\mathbf{E_{KDFR}}$.

10 Conclusions

1. Anderson [1] remarked that he was "far from satisfied with [his] terminological choices." In contrast to Anderson, we are completely satisfied with our terminological choices because (i) we have given various references for the theorem that $\mathcal{P}A$ and $\mathcal{C}A$ are subalterns, and (ii) we have identified safety with caution rather than prudence.
2. Anderson's eubouliatic logic can be extended to a modal fuzzy eubouliatic logic. This does not affect the eubouliatic square of opposition.
3. Modal fuzzy eubouliatic logic makes it clear how the concepts of necessity, relevance, prudence, caution, imprudence, incautiousness, safety, risk and fuzziness are logically related to each other.

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Why Care beyond the Square? Classical and Extended Shapes of Oppositions in Their Application to "Introspective Disputes"

Sascha Benjamin Fink

Abstract So called "shapes of opposition"—like the classical square of opposition and its extensions—can be seen as graphical representations of the ways in which types of statements constrain each other in their possible truth values. As such, they can be used as a novel way of analysing the subject matter of disputes. While there have been great refinements and extensions of this logico-topological tool in the last years, the broad range of shapes of opposition are not widely known outside of a circle of specialists. This ignorance may lead to the presumption that the classical square of opposition fits all disputes. A broader view, which takes expanded shapes of opposition into account, may come to a more nuanced appraisal of possible disputes. Once we take other shapes of opposition into account, some alleged disputes may turn out to be *Schein*disputes. In order to do the wide range of linguistic expressions justice and to differentiate *Schein*disputes from real ones, a broader view is advised. To illustrate this point, I discuss the notion of "introspective disputes". These are commonly reconstructed as obeying the square, but are more aptly reconstructed with a more complex octagon. If we reconstruct these disputes based on Buridan's octagon, it becomes obvious that "introspective disputes" are likely *Schein*disputes.

Keywords Octagon of opposition • Buridan's octagon • Oblique terms • Genetive constructions • Introspection • Philosophy of mind • Scheindisputes

Mathematics Subject Classification (2000) Primary 03B65; Secondary 91F20

1 Introduction

Shapes of opposition, like the classical square, can be read as graphical representations of the ways in which types of statements constrain each other in their possible truth values. If we assent to a statement of one kind, p, we must, may, or cannot rationally assent to one of another kind, q, depending on whether the relations between p and q are contradictory, contrary, subaltern, or subcontrary. In the classical square, for example, if we assent to "All cats are black", we cannot assent to "There is a cat that is not black" and "All cats are not black"; but we must assent to "There is a cat that is black"; if we assent to "There is a black cat", we may (or may not) assent to "There is a cat that is not black". So shapes of opposition can be read as representing limitations of rational assent.

J.-Y. Béziau, G. Basti (eds.), *The Square of Opposition: A Cornerstone of Thought*, Studies in Universal Logic, DOI 10.1007/978-3-319-45062-9_20

Becoming and remaining in disputes has been one of the primary goals of education since before the enlightenment, and the classical square is possibly one of the oldest teaching tools when it comes to rational discursive proficiency. Its graphical way of representing makes complex interdependencies didactically accessible. It remains an invaluable tool for teaching philosophy and dialectics across a broad range of logics and topics.

Because shapes of opposition express where we may, must, or cannot assent (given our other commitments), they can be used to analyse disputes. We may also use it to distinguish real disputes from *Schein*- or pseudo-disputes. In a *genuine* dispute, a proponent *Pro* holds an opinion that p which is incompatible with the opinion that q, held by her opponent *Opp*. Even if p and q are not directly negations of another (either on the surface or on the syntactical level),[1] there can be disputes as long as p and q stand in contradictory and contrary relations. People rarely fight over subaltern opinions, as their compatibility is obvious even to the untrained eye. But some *pseudo*-disputes may arise over subcontrary opinions, because their compatibility is not always obvious: *Pro* might argue with *Opp* because *Pro* believes that one can like liquorice, and *Opp* that one may not like liquorice. An opera lover, who believe that some of Wagner's operas are worthwhile, may argue with a Verdi enthusiast stating that some of Wagner's operas are a waste of time. Pragmatic influences, biases, and presumptions may suggest disputes where, rationally, there are none.

Even though the square of opposition is widely taught, extensions as well as newer discussions and developments are hardly presented outside of the realms of specialists.[2] This might be a grave mistake because, due to a lack of acquaintance with other shapes of opposition, most people may unconsciously presume the classical square wherever they see fit. As a result, pseudo-disputes may be fuelled more than necessary. Some of these can be easily avoided by applying even the most basic extensions.

In the following, I illustrate the rise of such a *Schein*-dispute by making certain statements fit a square of opposition, however unnatural this is. My example is a group of disputes often called "phenomenological disputes" [16, 115], "introspective disagreements" [3, 34], or "introspective disputes". I suggest that a solution for these types of disputes can be derived by applying an obvious medieval extension of the square: Buridan's octagon for statements with oblique terms.

2 "Introspective Disputes"

In the 1990s, marked by widely recognised books by philosopher Daniel C. [8] and Nobel laureate Francis Crick [7], a new science started its ascent: an empirical research program on phenomenal consciousness. The phenomenal features of a mental state are those that

---

[1] For example, "Laura is rich" and "Laura is poor" are not syntactically or on the surface levels negations of another (because there is no "not" involved at all). Still, both sentences are contrary to another, because both can be simultaneously false: Laura could simply be doing alright.

[2] See e.g. [4, 5, 11, 19, 23, 24, 38], as well as the examples in this volume for specialists' papers on the shapes of opposition.

make it feel like something for the person whose mental state it is: A pain, an orgasm, tickling, the smell of Ylang-Ylang—all these experiences feel like something to the person being tickled, being in pain, having an orgasm, or smelling Ylang-Ylang. What makes these mental events feel like this or that are their phenomenal features.

Unlike other phenomena which are treated by science, phenomenal consciousness is only subjectively accessible: You and I may be in pain—but I can only access my pain *as pain* and you yours. *Your* pain, in contrast, is accessible *to me* only by your report, your behaviour, your physiological changes—but not *as an experience of pain with such-and-such specific phenomenal characteristics*. Ascribing specific phenomenal events to others is a matter of inference, not acquaintance. However, the phenomenal characteristics of the events in one's own mind can be directly grasped by introspection. Therefore, most adopt the creed by William James [15, 158] concerning phenomenality: "Introspective observation is what we have to rely on first and foremost and always."

However, introspection sometimes leads to widely diverging opinions. This divergence in opinions, where apparently opposing opinions are each justified by referring to one's introspective access to experiences, are widespread. I will call them *introspective disputes*. But if a method leads to widespread divergences in opinion, why should we deem that method to have any epistemic merit? Introspective disputes therefore motivate skepticism vis-à-vis introspection: There is no knowledge to be had by introspecting.

Introspective disputes are marked by three conditions. First, the disputants believe that their opinions, p and q, are incompatible; second, the method by which both p and q are mainly justified or purportedly based on is *introspection*—a non-inferential, internal, and direct form of acquaintance with the phenomenal aspects of a mental event;[3] third, the subject matter of these opinions are phenomenal features of mental events.[4]

Here is a *prima facie* paradigmatic case of such an introspective dispute: Horgan and Tienson [13] and Wilson [41] see themselves as having opposing opinions on whether intentional states or attitudes (beliefs, desires, wishes, hopes, etc.) have a distinct phenomenal character that outruns the phenomenal character of the sensory experiences that accompany them. On this, [13, 522f, my emphasis] write:

> Intentional states have a phenomenal character, and this phenomenal character is precisely the what-it's-like of experiencing a specific propositional-attitude type vis-à-vis a specific intentional content. [. . .] *Attentive introspection reveals that both the phenomenology of intentional content and the phenomenology of attitude type are phenomenal aspects of experience*, aspects that you cannot miss if you simply pay attention.

As a direct reply, Wilson [41, 415ff, my emphasis] counters:

> *When I engage in introspection* on the character of my experience, I find that it is thoroughly intentional, so thoroughly so that *it is hard to distinguish any purely qualitative, non-intentional remainder of the experience.* [. . .] In the spirit of Horgan's and Tienson's appeal for a reader to 'pay attention to your own experience', I have just done the decisive experiment: I thought first

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[3] Sometimes, the method is broadened: Reports that are deemed to express an introspectively formed opinion or questionnaires about experiences are sometimes seen as introspection as well.

[4] Some argue that more than the phenomenal aspects of experiences can be introspected. There might then be introspective disputes concerning attitudes, content, experience onset, and so on. I will focus only on introspective disputes concerning phenomenal aspects of experiences, as this subject matter provides the best examples for introspective disputes.

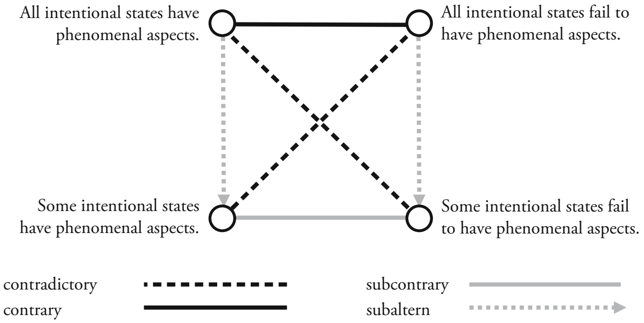

Fig. 1 Reconstruction of a *prima facie* introspective dispute with a square of opposition

> that George Bush is President of the United States, and had CNN-mediated auditory and visual phenomenology that focused on one of his speeches. I then took a short break, doodled a little, wandered around the room, and then had a thought with that very same content and . . . nothing.

One may easily construe this as an introspective dispute: Horgan's and Tienson's opinion is that intentional states have phenomenal aspects; Wilson's opinion is that intentional states have no phenomenal aspects. In this reconstruction, both opinions are contrary, and therefore cannot be simultaneously true. Additionally, both opinions are about the phenomenal aspects of experiences and both are explicitly justified by introspection. This suffices for being considered an "introspective dispute", if we presume the classical square of opposition (see Fig. 1).

There are numerous examples, which seem to follow a similar structure: If you look at a coin at an angle, does it look round or elliptical?[5] Moore [22, 30f] introspects it as elliptical but not as round, Peacocke [26, 98f] and Smith [39, 172] as round but not as elliptical.

Are conceptualisations part of our experiences? For example, do you experience a snowy landscape as consisting of *snow* or do you experience it simply as *a white expanse*? Siewert [37, 256] introspects that you do experience that cold white expanse before you as *being snow*, Dretske [9, 33] argues that you do not experience it as *snow*, but argues that you only conceptualise it as such.

Are perceptual expectancies part of the phenomenal character of an experience? Some say that they are [25, 960f], others fail to find such an expectancy-character in their experiences [27, 250].

Does phenomenal character present itself as nothing but representing the properties and objects of the external world? *Pure* representationalists [40, 160f] assent based on their introspection, *impure* representationalists or *anti*-representationalists claim that there is something in addition to the content of the experience available for introspection: some of their introspected experiences show a phenomenal character which is independent of the represented content [18, 277f].

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[5] See [34, Chap. 2] for a discussion with more historical references.

Do you experience only what you attend to or more? The abundancy-camp, [35, 137f and 37], holds that phenomenal character is independent of attention based on introspection; the sparsity-opposition holds that you only experience the few limited items that you attend to.[6]

Do experiences generally have an experienceable character of *for-me-ness*? Zahavi [42] holds that all experiences have such a character[7]; but this stands in tension with descriptions of long-term meditators, who report an absence of a self/other distinction in experience [29].[8]

Are dreams coloured or in black and white? Schwitzgebel [31, 32, 35] traced the wide divergence in opinions over time and area.

These are merely examples, but the list suffices to show that philosophy of mind is, at least *prima facie*, riddled with introspective disputes. If we presume that the statements discussed in these cases fit the classical *square* of opposition, we are forced to see these as genuine disputes.[9] Such introspective disputes are not only ubiquitous, but apparently also irresolvable—at least by introspection. And all other data (behaviour, report, physiology) does not seem to help us to decide these issues. If introspection leads to such widely diverging opinions, then how could one trust introspection? How could one defend it as having any epistemic merit?

3 The Threat of Introspective Disputes

Kriegel [16], Bayne and Spener [3] as well as Schwitzgebel [34] have diagnosed introspective disputes and suggested possible solutions or amendments. Other philosophers see the same issues, but take introspective disputes as a basis for defending hostile positions vis-à-vis introspection in general.

For example, Dennett [8, 44] has argued that a phenomenological or introspective method has failed because there is no agreement between its users about what does or does not hold for experiences.[10] Phenomenologists and introspectors defend incompatible

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[6] Armstrong [2, 300], for example, argues for this, and seems to justify that claim by introspection.

[7] See also Zahavi [43, 132]: “Whereas we live through a number of different experiences, the dimension of first-personal experiencing remains the same […] it may be described as an invariant dimension of first-personal givenness throughout the multitude of changing experiences.”

[8] See also Hume [14, Sect. VI].

[9] One may wonder how one justifies the general opinion by introspective acquaintance. But the method of eidetic or Phenomenological variation builds on one’s direct grasp of phenomenal experiences to justify such general statements. General statements about phenomenality can then be justified by one’s grasp of one’s experiences from the first-person perspective. So there are similar disputes in Phenomenology. For the purpose at hand, we may see such Phenomenological disputes as part of the larger class of introspective disputes.

[10] See also Dennett [8, 66]: “It is just astonishing to see how often ‘academic’ discussions of phenomenological controversies degenerate into desk-thumping cacophony, with everybody talking past everybody else. This is all the more surprising, in a way, because according to long-standing philosophical tradition, *we all agree* on what we find when we “look inside” at our own phenomenology. […] just

opinions, although they presume to have introspected tokens of the very same type of experience. (For example, some believe—purportedly based on introspection—that thoughts generally have phenomenal aspects, some hold that thoughts generally fail to have these phenomenal qualities.) If introspection were a reliable way to gain knowledge, how could it lead to such diverging, incompatible, but strongly held opinions? Controversy and contradiction among introspectors and Phenomenologists are indicators of something going wrong—for example: trusting introspection. Instead of using introspection, so Dennett's suggestion, we should use methods that are verifiable and reliable, e.g. behavioural psychology [8, 70ff].

Thomas Metzinger [20, 35] strikes a similar chord when he writes that Phenomenology failed to become an autonomous science of conscious experience, because the Phenomenological way of gaining data has no way of resolving conflicts between introspective reports or Phenomenological positions. He restates this point in *Being No One* [21, 591].[11] It is a mark of science, he claims, that *if* inconsistencies arise, scientific methodology suggests some way to overcome them. But for introspection or other first-person methods, nothing like this seems to be available—introspective disputes appear to be irresolvable. So introspection cannot be deemed scientific.

Eric Schwitzgebel [34, ix–xi] uses these variations in introspectively justified opinions as a jumping board for some skeptical musings:[12] Variations in opinion—given that there is no suiting variation in the subject matter—entails that somebody has a false opinion. But if we don't know who is wrong, we do not know if our decision to believe one person rather than her opponent leads us to knowledge. This inability to resolve introspective disputes raises suspicion concerning all claims of knowing by introspecting. It does not only mark introspection as *unscientific*, it invites a general skepticism vis-à-vis introspection [cf. 33].

One of the most violent reactions to such "introspective disputes" is phenomenal eliminativism: If these disputes stand irresolvably, then we can hardly claim that we know something about phenomenality. If there is nothing known (or known to be known, or proven to be known, or certain)—well, maybe there is nothing there to know.

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about every author who has written about consciousness has made what we might call the *first-person plural presumption*: Whatever mysteries consciousness may hold, *we* (you, gentle reader, and I) may speak comfortably together about our mutual acquaintances, the things we both find in our streams of consciousness. [...] This would be fine if it weren't for the embarrassing fact that controversy and contradiction bedevil the claims under these conditions of polite mutual agreement. We are fooling ourselves about something."

[11] "*The* epistemological problem regarding phenomenological, first-person approaches of "data generation" [meant as scare quotes] is that if *inconsistencies* in two individual "data sets" [scare quotes again] should appear, there is no way to settle the conflict. [...] This is a third defining characteristic of the scientific way of approaching reality: there are procedures to settle conflicts resulting from conflicting hypotheses. Epistemic progress continues." [21, 591].

[12] "I aim to persuade you that people in general know very little about what might seem to be obvious features of their stream of conscious experience [...] People often differ greatly in their judgments about their stream of experience (across cultures, between individuals within the same culture, or within the same individual over time). Sometimes, in such cases, it seems unlikely that their actual underlying experiences vary correspondingly. Consequently, some of their judgments—we don't necessarily know which ones—are probably wrong." [34, ix–x].

Maybe phenomenal experiences don't exist.[13] Persisting introspective disputes then invite eliminativism vis-à-vis phenomenal character.[14]

These arguments based on what appears to be introspective disputes are generally volatile: If there is a sufficient number of introspective disputes concerning specific aspects of phenomenality, they then pose a massive danger for the trustworthiness of introspection in general.

However, the situation need not be so dire. I think that what we take to be genuine introspective disputes are mere illusions—there are only *prima facie*-disputes. The illusion of introspective disputes can arise *even if introspection works perfectly.* Based on two universally accepted premises, I argue that we mistake statements that are merely subcontrary as being contrary or contradictory. These premises are, first, the thesis of universal phenomenal ownership (P1) and the thesis of introspective internalism (P2). If we take these two theses into account, it becomes obvious that a *square* of opposition is insufficient to model statements about phenomenal experiences. One has to switch to an *octagon*—more specifically: Buridan's Octagon for statements with oblique terms [28]. Reconstructed with this underlying shape, introspective disputes vanish. If they vanish, the skeptics' argument against introspection based on introspective disputes becomes vacuous.

4 Buridan's Octagon and Its Impact

With the implicit presumption of a square of opposition, we reconstruct the statements at play in "introspective disputes" as belonging to four basic types (compare Fig. 1), where we universally or particularly ascribe some phenomenal features F to mental events or not:

(A) All mental events (of type T) are F.
(E) All mental events (of type T) are not-F.
(I) Some mental event (of type T) is F.
(O) Some mental event (of type T) is not-F.

However, this reconstruction does not take into account two basic theses that are nearly universally accepted in the philosophy of mind. If we take them into account, they lead to

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[13] Kriegel [16, 122f] augurs this: "The above phenomenological disputes, and others like them, are disconcerting inasmuch as the Consciousness Studies community does not have accepted guidelines for adjudicating them. Phenomenological disputes have a way of leading to apparent deadlocks with remarkable immediacy. Disputants reach the foot-stomping stage of the dialectic more or less right after declaring their discordant positions. [...] The most violent reaction is to claim that there is no fact of the matter concerning these disputes."

[14] This is in accord with the presumption that a square of opposition is committed to existential import [1, II, 176, 20–21]. That is, if I were to claim that all goblins eat cheese and you claim that no goblin eats cheese, either statement commits us to the existence of goblins *if we presume that we two are in a dispute* [see also 30]. If we reject the presumption that we are in a dispute, we thereby reject the ontological commitment: If there are no goblins, we do not need to fight over their affinity to cheese. Rejecting the existence claim means that general statements can (under some views) be considered true, but vacuously so. This affects disputes based on contrary opinions.

the demise of genuine introspective disputes: because they are incompatible with genuine introspective disputes. These two theses are (P1) universal phenomenal ownership and (P2) introspective internalism.

For any mental event, there is somebody whose mind this mental event takes place in. Experiences are such mental events. Thus, there is an experiencer for every experience. This *experiencer* is often marked by a possessive phrase or a phrase with in a genitive form e.g. Mark's pain, John's orgasm, Luke's dream, etc. Even philosophers who are stout anti-physicalists and believe that there can be non-physical experiencers [10], they still pose somebody who experiences (even if there isn't some *body* corresponding to that experiencer). It is common ground that there are no free-floating experiences. So if we posit that there is some mental event e which has some phenomenal feature F, there must be some subject s in whose mind this mental event takes place. Just like for any dance there is a dancer, there is an experiencer for every experience. So the thesis of universal phenomenal ownership says that:

(P1) Every experience is *owned*.

While (P1) is a claim about experiences, introspective internalism is a thesis about the epistemic method under scrutiny—introspection:

(P2) Introspection is an *internal* process.

Say some s knows by introspection that some thought had phenomenal aspects. The corresponding belief created by such an introspective act and the subject matter of this knowledge (the thought in question) must be *in the same mind*. Thus, according to (P2), the *knower* s_k of some phenomenal fact about some mental event e_i, who gained that knowledge by introspection, must be identical with the *owner* s_o of the known event e_i. So if s_1 knows that some thought of s_2 had phenomenal aspects *by introspecting its phenomenal aspects*, then—by (P2)—$s_1 = s_2$. Just like testimony is an epistemic method that necessarily involves more than one person, introspection is necessarily a method that involves at most one person. Nobody can introspect someone else's experiences, so the common lore.[15]

If these two premises are accepted, as they usually are, then a square of opposition does not provide the right framework for reconstructing the statements at issue in an introspective dispute. We have to extend our tools beyond the square.

If all experiences are owned, then we ought to introduce another variable ranging over experiencers to grasp this fact. For each of these two variables, we have two quantifiers (*all* and *some*) that can range over them. As previously, we can ascribe or deny properties. This allows for $2 \times 2 \times 2$ different statement types, doubling the amount we can construct from four (in the square where we assume only one variable for experiences) to eight. These have the following form:

(AA) For all experiencers, all mental events (of type T) are F.
(AE) For all experiencers, all mental events (of type T) are not-F.

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[15] However, see [12] for a conflicting view.

(AI) For all experiencers, some mental event (of type T) is F.
(AO) For all experiencers, some mental event (of type T) is not-F.
(IA) For some experiencers, all mental events (of type T) are F.
(IE) For some experiencers, all mental events (of type T) are not-F.
(II) For some experiencers, some mental event (of type T) is F.
(IO) For some experiencers, some mental event (of type T) is not-F.

Terms in the genitive (like the one's we use in such possessive phrases like "Mark's pain") are *oblique* terms (as are accusative terms).[16] For such statements, John Buridan presented an *octagonal* shape in his *Summulae de Dialectica* [6] (see Fig. 2, reconstructed after [28, 14]).[17] In this shape, the statements at the corners correspond in form to the eight presented above.[18]

The switch from a square to an octagon exposes some ambiguity. An assertion like

(A') All dreams are coloured.

is in its surface structure ambiguous between

(AA') For all dreamers, all dreams are coloured.
(IA') For some dreamers,[19] all dreams are coloured.

In order to avoid confusion, one ought to make the experiencer variable explicit.

This switch from square to octagon thus necessitates a different reconstruction of the statements at issue in introspective disputes. This, in turn, suggests a different diagnosis, leading to a deflationary account of introspective disputes. Only the lowest four types of statements (IA, IE, II, and IO, see Fig. 2) can be justified by introspection if we accept introspective internalism (P2). Because I cannot introspect the experiences of another subject, I could only introspect *all* experiencers, if I were the only experiencer in existence.

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[16] Buridan's example is of a man's ass running, i.e. *omnis asinus hominis currit.* "*Hominis*" is here in the genitive.

[17] See especially [28] for a discussion of this form and its variations.

[18] In contrast to the reconstruction as an octagon, one might think that there should in principle be sixteen different types of statements and, therefore, a sixteen-sided figure: While I have introduced A...- and I...-statements (universally and particularly affirmative statements), I have ignored the possible forms of E...- or O...-statements (universally and particularly negative statements). Only a sixteen-sided figure would accommodate these additional eight statement-types.

Why are these omitted? Because each is equivalent to some of the forms mentioned. Consider the case of an AA-statement like "For all experiencers, all their thoughts have phenomenal aspects." This is equivalent to the EO-statement "For all experiencers, none of their thoughts fails to have phenomenal aspects." And consider an II-statement like "For some experiencer, some thought have phenomenal aspects." This is equivalent with the OE-statement that "For some experiencers, not all thoughts fail to have phenomenal aspects." The general rule for such quantifier-involving statements is: To find the equivalent statement to some $\Gamma\Delta$-statement, take the contrary or subcontrary type to Γ in the classical square (A transforms to E, I to O, and vice versa) and the contradictory form to Δ in the square (A transforms to O, I to E, and vice versa). This leads to a reduction of a hypothetical 16-sided figure to the octagon in the following way: AA≡EO; AE≡EI; AI≡EE; AO≡EO; IA≡OO; IE≡OI; II≡OE; IO≡OA. Thus, we may continue with the octagon as presented in Fig. 2.

[19] Especially the person whose opinion this is.

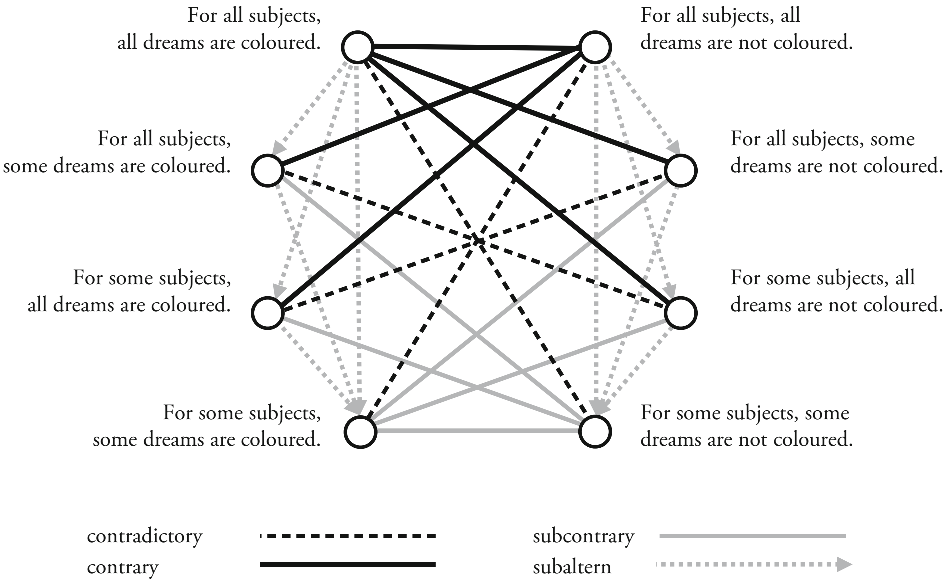

Fig. 2 Buridan's octagon of opposition, applied to an exemplary introspective dispute

A solipsist might be able to justify AA-, AE-, AI-, and AO-statements by introspection, but nobody else can. As long as we reject solipsism, only the lower four statements in Fig. 2 can be justified by introspection.

However, IA-, IE-, II-, and IO-statements can all be simultaneously true. Consider Alfie, Berta, Claire. Nothing speaks against it being the case that Alfie always dreams in colour, that Berta never dreams in colour, while Claire sometimes dreams in colour and sometimes not. In this population, IA-, IE-, II-, and IO-statements are simultaneously true. Among these introspectively justifiable statements, people may only be mistaken about being in a dispute.

Concerning the upper four statements (AA, AE, AI, and AO), which suffice for a dispute, one cannot justify them by introspection alone. One would need an additional method, probably an inductive step, to justify these statements.[20] But if disputes arise under these circumstances, one cannot directly blame their rise on introspection as skeptics like Dennett, Metzinger, and Schwitzgebel do. Induction is just as likely a source of error. If the skeptic bases her argument on these upper statement-types, her argument fails to be a direct argument against introspection. Someone fond of introspection may turn it to be an argument against induction.

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[20] Levin [17] argued that induction plays a major part of Phenomenological or eidetic variation, which is often used to justify such statements.

So *either* one justifies one's opinion solely by introspection—but then, the introspectors' opinions fail to be in the right oppositional relations for disputes, and so there are no introspective *disputes* anymore; *or* one justifies by a mixture of methods (e.g. introspection + induction) whereby disputes arise—but these are then no *introspective* disputes anymore, but introspective+x-disputes. Then, one cannot show that introspection, not the additional method x lead to this failure. So no matter how we turn, there are no pure introspective disputes given (P1) and (P2). "Introspective disputes" (in scare quotes) are either *Schein*-disputes or not based on introspection alone.

Thus, all the arguments against introspection based on "introspective disputes" (see Sect. 3) fail: The phenomenon we face—variations in introspection-based opinion—can arise *even if introspection were perfect and infallible*. All that is needed is some variation in the phenomenon, and this is quite likely when it comes to anything psychological: Some people may dream only in colour, some only in black-and-white just like some people like liquorice and some do not. Variation is natural.

5 Conclusion

I argued that if one sees divergences in introspection-based opinions as *disputes*, one reconstructs the statements accordingly under the presumption of a square of opposition. But this presumption is unjustified because nobody believes in free-floating experiences without an owner. So one has to introduce a possessive phrase. If we introduce a possessive phrase, we have to extend the square to Buridan's octagon for oblique terms. With one's presumptions corrected, the divergences in introspective-based opinion do not constitute a dispute given introspective internalism. Therefore, a range of arguments against introspection based on such alleged "disputes" fail to be substantive.

The steps made here are probably trivial for those trained with shapes of opposition. If so, this only underlines that we ought to popularise extensions of the square. Shapes of opposition make formal features of disputes easier accessible to people who think visually rather than formally. They therefore might be better tools for diagnosing and teaching important variations in dialectic structures in a way more accessible than calculus-style training. It stands to reason that if extensions of and alternatives to the square were more widely known, the square would stop to be a common unarticulated and unquestioned default presumption. Then, certain pseudo-disputes—like "introspective" ones—based on subcontrary opinions would become obvious: What looked like a dispute from one angle fails to be one if we look more carefully.[21]

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[21] This article is built on work published in my PhD-thesis (2015) at the University of Osnabrück. This work has been supported by a Lichtenberg scholarship. I am grateful to Karsten Engel and one anonymous reviewer for helpful comments, and also to the organisers of Square 2014 at the Vatican for a very enjoyable conference.

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